Two-Stage Trellis Decoding of RS Codes Based on the Shannon Product

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Abstract — **In this paper we present a technique for low-complexity decoding of Reed-Solomon (RS) codes. We describe a method for which the implementation complexity has been estimated when compared to standard Viterbi decoding over the minimal trellis. Finally we present some computer simulation results for decoding the (7, 5, 3) RS code.**

I. Introduction

Trellis design techniques for linear block codes have been under investigation since 1974 [1, 2, 3]. The problem returned to the public attention in 1988 when Forney [4] introduced the concepts of *coset codes* and *coset trellises*. Muder [5] proved that these trellises are *minimal* and that the number of states in the trellis diagram can be minimised by an appropriate reordering of symbols in the codeword. Such an optimum reordering has been obtained for some particular binary codes, [4, 6, 7, 9, 15, 16]. However, the general solution to this problem, as well as its extension for the non-binary codes remains unsolved and represents a complex analytical task. In this paper we introduce a new technique which allows an efficient design of minimal coset trellises of RS codes based on the Shannon product of trellises and propose a low- complexity trellis decoding technique that makes the implementation of the designed trellises feasible.

II. Shannon Product Of Trellises

Shannon [17] described the product of two channels which "corresponds to a situation where both channels are used each unit of time". We apply these results for the trellis design of linear block codes, in particular RS codes [18, 19].

Let $N(t) = [N_0, N_1, \ldots, N_{Nc}]$ be the state profile of the trellis T, $B(t) = [B_1, B_2, \ldots, B_{Nc}]$ be the branch profile of the trellis [19] and $L(t) = [l_1, l_2, \ldots, l_{Nc}]$ be the label size profile of the trellis, where B_j is a number of branches in the j-th depth of the trellis and l_j is the number of symbols being used for branch labelling at j-th depth. Given two trellises T and T' with N_c levels in each trellis, the Shannon product [18] $T * T'$ of these trellises is a trellis which at level t consists of $N_t N'_t$ nodes labelled by pairs $[S_i^{(t)}, S_j^{\prime(t)}]$, where $i = 1, 2, \ldots, N_t$ and $j = 1, 2, \ldots, N'_t$. Two nodes of adjacent levels are connected by branches labelled as

$$\omega_t^{\text{Sh}} = \omega_t \left[S_i^{(t-1)},\, S_p^{(t)}\right] + \omega'_t \left[S_j^{\prime\,(t-1)},\, S_r^{\prime\,(t)}\right] \qquad (1)$$

IFF trellis T has branch $\omega_t(S_i^{(t-1)}, S_j^{(t)})$ and trellis T' has branch $\omega'_t(S_i^{\prime(t-1)}, S_j^{\prime(t)})$ (addition in (1) is over $GF(q)$). Note, that from the definition we have the following state and branch profiles:

$$N_{\text{Sh}} = N(t)N'(t) \qquad (2)$$

$$B_{\text{Sh}} = B(t)B'(t) \qquad (3)$$

We define a sum $C + C'$ of codes C and C' as a set of $|C| * |C'|$ all possible sums $c + c'$, where c and c' are codewords from C and C', respectively: $c \in C$ and $c' \in C'$.

Proposition 1 (from [19]) *Consider two codes C and C' with the same codelength, n. Let T be a trellis of the code C and T' be a trellis of the code C'. The Shannon product, $T_{\text{Sh}} = T * T'$, of these trellises is the trellis of the code $C_{\text{Sh}} = C + C'$.*

III. Coset Trellises For RS Codes

A coset trellis represents a set of parallel sub-trellises, each one corresponding to one of the cosets of the basic code [4]. In order to design a minimal coset trellis we start with the calculation of the minimal number of states for every possible splitting point of the trellis [4]. At the next stage we choose the splitting points which have similar numbers of states and represent the generator matrix G in the following format:

$$G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix} = \begin{bmatrix} G_1 & G_2 & \ldots & G_{N_c-1} \end{bmatrix} \qquad (4)$$

where $G_i, i = 1, 2, \ldots, N_c - 1$, has l_i columns and k rows, and l_i corresponds to the splitting points obtained at the previous stage. Each row of G is used to design the trellis diagram of the $(n, 1, d)$ code over $GF(q)$ and the overall trellis diagram can be obtained as the Shannon product of k designed component trellises.

Theorem 2 *The designed trellis is minimal coset trellis.*

Example 3 *Let our aim be to design a coset trellis for the (7, 3, 5) RS code with symbols taken from $GF(2^3)$. The generator matrix of the code is as follows:*

$$G = \begin{bmatrix} \alpha^3 & \alpha & 1 & \alpha^3 & 1 & 0 & 0 \\ 0 & \alpha^3 & \alpha & 1 & \alpha^3 & 1 & 0 \\ 0 & 0 & \alpha^3 & \alpha & 1 & \alpha^3 & 1 \end{bmatrix} \qquad (5)$$

Following the procedure described by Forney [4], the state profile for every splitting point of the trellis can be obtained as $N_{\mathrm{synd}} = [1, 8, 64, 512, 512, 64, 8, 1]$. *It is apparent that for a given (7, 3, 5) RS code one can design a number of different (but isomorphic) minimal trellises. One of these trellises may have 3 depths with the following state and label size profiles:*

$$N^1 = [1, 8, 8, 1] \quad (6)$$
$$L^1 = [1, 5, 1] \quad (7)$$

while the other trellis may have 3 depths with the following state and label size profiles:

$$N^1 = [1, 64, 64, 1] \quad (8)$$
$$L^1 = [2, 3, 2] \quad (9)$$

In our example we choose the latter trellis, thus the generator matrix of the code we represent in the following format:

$$\begin{aligned} G &= \begin{bmatrix} G_1 & G_2 & \dots & G_{N_c-1} \end{bmatrix} \\ &= \begin{bmatrix} \alpha^3 & \alpha & 1 & \alpha^3 & 1 & 0 & 0 \\ 0 & \alpha^3 & \alpha & 1 & \alpha^3 & 1 & 0 \\ 0 & 0 & \alpha^3 & \alpha & 1 & \alpha^3 & 1 \end{bmatrix} \end{aligned} \quad (10)$$

The overall trellis diagram, T, can be obtained as the Shannon product of 3 trellises, $T = T_1 * T_2 * T_3$, each one pertaining to a (7, 1, 5) code, generated by its corresponding row of G. These trellises are presented in Figure 1, and the overall trellis diagram is shown in Figure 2. As follows from this figure, the minimal coset trellis of the (7, 3, 5) RS code consists of 8 identical, parallel sub-trellises that differ only in their labelling and each such sub-trellis has 8 states and 3 depths.

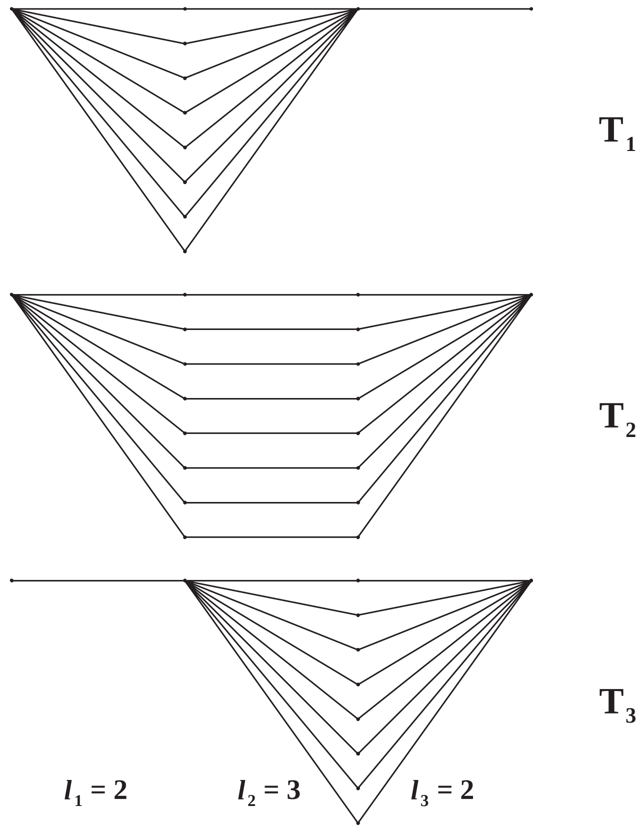

Fig. 1: Component Trellises for the (7, 3, 5) RS code.

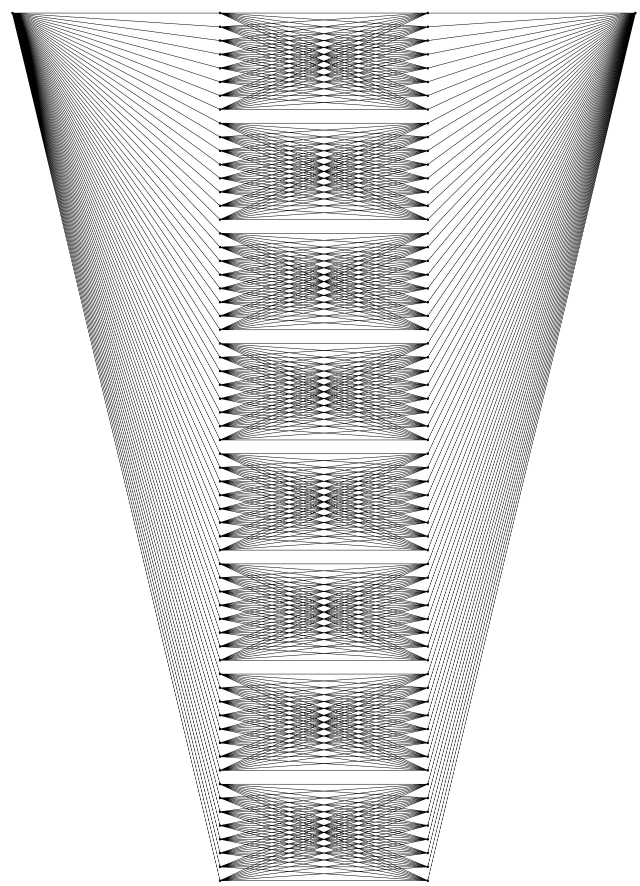

Fig. 2: Trellis Diagram of the (7, 3, 5) RS code.

IV. Two-Stage Trellis Decoding For RS Codes

Although the designed coset trellises are isomorphic to the minimal trellises, for long RS codes the trellis becomes unfeasible due to its considerable complexity and storage requirements. Recently the *two-stage sub-optimum trellis decoding* technique has been proposed for low complexity trellis decoding of binary codes [21, 22]. We propose a novel two-stage trellis decoding algorithm applicable to RS codes which allows the reduction of decoding complexity without significant loss of the decoding performance. The decoding procedure consists of two major steps:

1. Identify in which sub-trellis the maximum-likelihood path lies.
2. Apply the Viterbi decoding algorithm only to the sub-trellis indicated at step 1.

With reference to Proposition 1, we define C to be a coset of the RS code, thus the codewords of C' are the coset leaders of the RS code C_{Sh}. To identify in which sub-trellis (coset) the maximum-likelihood path lies we are unable to simply decode C' over its trellis, T', since the received word also contains symbols from C. Instead we decode each information symbol in C' separately. In general k recieved symbols are required to find an estimate of an information symbol X_j, $j = 1, 2, \ldots, k$, since there are k unknowns (the k information symbols). We thus form a set of independent equations which are a weighted sum of upto k symbols from the recieved word. By evaluating this set of equations and choosing the most probable we decode one information symbol. This is repeated for all information symbols in C'. The set of decoded information symbols identifies the sub-trellis of C_{Sh} to decode. Viterbi decoding of the chosen sub-trellis is performed in the normal way. Since the prediction of which sub-trellis to decode is itself subject to errors we can improve the decoder performance by decoding the best i, $i = 1, 2, \ldots q^{\frac{k-1}{2}}$, sub-trellises. In this case the chosen codeword is the one with highest confidence from the output of stage 2.

The equations are evaluated by taking the hard-decision value of each received symbol. If we attempt to perform "soft" Galois Field operations by retaining some of the soft-decision information then the performance is further enhanced. Figure 3 shows the performance of our algorithm compared with uncoded and HDMLD. Note that decoding all 64 sub-trellises is equivalent to SDMLD. Figure 4 is the corresponding performance when the soft-decision data is incorporated into the GF operations.

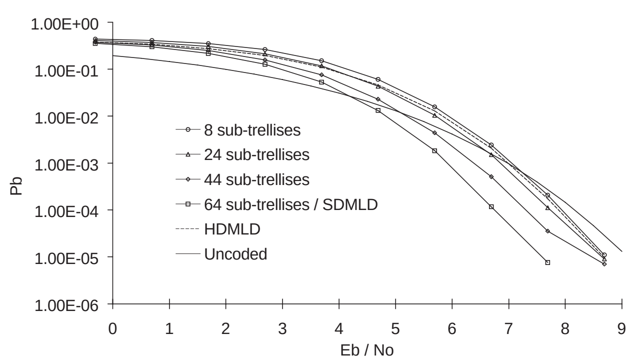

Fig. 3: Two-stage decoding performance for the (7, 3, 5) RS code using a hard-decision sub-trellis estimator.

We have estimated the decoder complexity (stage 1 using hard-decision values) for a system implemented on AT & T's DSP32C digital signal processor. This uses the approximate number of mathematical operations and their relative cost in terms of the number of CPU cycles required for their execution (see Table

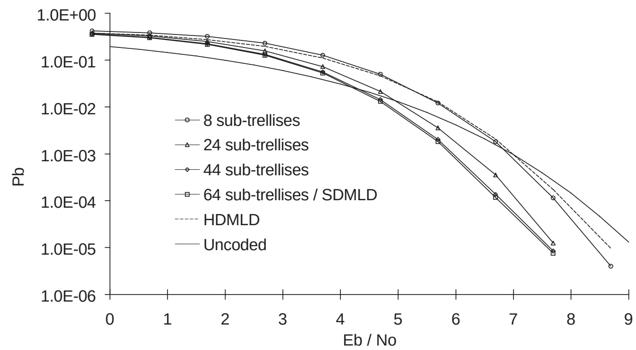

Fig. 4: Two-stage decoding performance for the (7, 3, 5) RS code using a soft-decision sub-trellis estimator.

1). Table 2 shows the complexity of the decoding algorithm for various numbers of sub-trellises decoded. The complexity of Viterbi decoding the full minimal trellis is 64537 cycles.

	+	+ (s.o.p.)[1]	*	==	== (b.p.)[2]
float	2	1	2	6	7
int	1	1	n/a	3	4
GF	1	1	5	n/a	n/a

Tab. 1: Relative complexity of algebraic operations.

Number of sub-trellises	Complexity (CPU cycles)
8	15299
16	29835
24	44371
32	58907
40	73443
48	87979
56	102515
64	117051

Tab. 2: Complexity versus number of sub-trellises decoded for the (7, 5, 3) RS code.

V. Conclusion

We have shown a method by which the decoder complexity can be reduced by taking advantage of the inherent regular structure of coset trellises. By varying the number of sub-trellises decoded the system can be made adaptive in response to the amount of channel noise.

[1] addition as part of a sum-of-products expression.

[2] comparisons whilst remembering the best path (used for converging branches on the trellis).

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