# On Aspects of Infinite Derivatives Field Theories & Infinite Derivative Gravity

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January 2018

A thesis submitted to Lancaster University for the degree of

Doctor of Philosophy in the Faculty of Science and Technology

# Abstract

Infinite derivative theory of gravity is a modification to the general theory of relativity. Such modification maintains the massless graviton as the only true physical degree of freedom and avoids ghosts. Moreover, this class of modified gravity can address classical singularities.

In this thesis some essential aspects of an infinite derivative theory of gravity are studied. Namely, we considered the Hamiltonian formalism, where the true physical degrees of freedom for infinite derivative scalar models and infinite derivative gravity are obtained. Furthermore, the Gibbons-Hawking-York boundary term for the infinite derivative theory of gravity was obtained. Finally, we considered the thermodynamical aspects of the infinite derivative theory of gravity over different backgrounds. Throughout the thesis, our methodology is applied to general relativity, Gauss-Bonnet and $f(R)$ theories of gravity as a check and validation.

*To my parents: Sousan and Siavash.*

# Acknowledgements

I am grateful and indebted to my parents; without their help I could never reach this stage of my life. They are the sole people whom supported me unconditionally all the way from the beginning. I hope I could make them proud.

I am thankful to all the people who have guided me through the course of my PhD, my supervisor: Dr Jonathan Gratus and my departmental collaborators: Prof Roger Jones, Dr Jaroslaw Nowak, Dr John McDonald and Dr David Burton.

I am grateful to my colleagues: Aindriú Conroy, James Edholm, Saleh Qutub and Spyridon Talaganis. I am specially thankful to Aindriú Conroy and Spyridon Talaganis, from whom I have learnt a lot. I shall thank Mikhail Goykhman for all the fruitful discussions and his valuable feedbacks.

I shall also thank the long list of my friends and specially Sofia who made Lancaster a joyful place for me, despite of the everlasting cold and rainy weather.

Last but not least, I shall also thank Dr Anupam Mazumdar for all the bitter experience. Nevertheless, I have learnt a lot about ethics and sincerity.

# Declaration

This thesis is my own work and no portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification at this or any other institute of learning.

*'I would rather have a short life with width rather than a narrow one with length."*
Avicenna

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- S. Talaganis and **A. Teimouri**, *"Hamiltonian Analysis for Infinite Derivative Field Theories and Gravity,"* arXiv:1701.01009 [hep-th].

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- **A. Teimouri**, S. Talaganis, J. Edholm and A. Mazumdar, *"Generalised Boundary Terms for Higher Derivative Theories of Gravity,"* JHEP **1608**, 144 (2016) [arXiv:1606.01911 [gr-qc]].

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- A. Conroy, A. Mazumdar and **A. Teimouri**, *"Wald Entropy for Ghost-Free, Infinite Derivative Theories of Gravity,"* Phys. Rev. Lett. **114**, no. 20, 201101 (2015), Erratum: [Phys. Rev. Lett. **120**, no. 3, 039901 (2018)] [arXiv:1503.05568 [hep-th]].

- A. Conroy, A. Mazumdar, S. Talaganis and **A. Teimouri**, *"Nonlocal gravity in D dimensions: Propagators, entropy, and a bouncing cosmology,"* Phys. Rev. D **92**, no. 12, 124051 (2015) [arXiv:1509.01247 [hep-th]].

- S. Talaganis and **A. Teimouri**, *"Rotating black hole and entropy for modified theories of gravity,"* arXiv:1705.02965 [gr-qc].

- **A. Teimouri**, *"Entropy of non-local gravity,"* arXiv:1705.11164 [gr-qc].

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# Chapter 1

# Introduction

General theory of relativity (GR), [1], can be regarded as a revolutionary step towards understanding one of the most controversial topics of theoretical physics: gravity. The impact of GR is outstanding. Not only does it relate the geometry of space-time to the existence of the matter in a very startling way, but it also passed, to this day, all experimental and observational tests it has undergone. However, like many other theories, GR is not perfect [2]. At classical level, it is suffering from black hole and cosmological singularities; and at quantum level, the theory is not renormalisable and also not complete in the ultraviolet (UV) regime. In other words, at short distances (high energies) the theory blows up.

It should be noted that non-renormalisability *is not* necessarily an indication that a theory is not UV complete. In fact, non-renormalisability can indicate a breakdown of perturbation theory at the energies of the order of the mass scale of the non-renormalisable operators, with a UV-complete but non-renormalisable theory at higher energies (*e.g.* loop quantum gravity). Renormalisability is however desirable as it allows the theory to be consistent and calculable at all energies, thus renormalisability may help to formulate of a UV complete theory, but its absence dos not necessarily mean the theory is UV incomplete.

As of today, obtaining a successful theory of quantum gravity [3, 4, 5, 6, 7] remains an open problem. At microscopic level, the current standard model (SM) of particle physics describes the weak, strong and electromagnetic interactions. The interactions in SM are explained upon quantisation of gauge field theories

[10]. On the other side of the spectrum, at macroscopic level, GR describes the gravitational interaction based on a classical gauge field theory. Yet, generalisation of the gauge field theory to describe gravity at the quantum level is an open problem. Essentially, quantising GR leads to a non-renormalisable theory [9]. On the other hand also, the generalisation of the SM, with the current understanding of the gauge groups, provides no description of gravity.

Renormalisation plays a crucial role in formulating a consistent theory of quantum gravity [8]. So far, efforts on this direction were not so successful. Indeed, as per now, quantum gravity is not renormalisable by power counting. This is to say that, quantum gravity is UV divergent. The superficial degree of divergence for a given Feynman diagram can be written as [170],

$$D = d + \left[n\left(\frac{d-2}{2}\right) - d\right]V - \left(\frac{d-2}{2}\right)N \tag{1.0.1}$$

where $d$ is the dimension of space-time, $V$ is the number of vertices, $N$ is the number of external lines in a diagram, and there are $n$ lines meeting at each vertex. The quantity that multiplies $V$ in above expression is just the dimension of the coupling constant (for example for a theory like $\lambda\phi^n$, where $\lambda$ is the coupling constant). There are three rules governing the renormalisability [170]:

1. When the coupling constant has positive mass dimension, the theory is super-renormalisable.

2. When the coupling constant is dimensionless the theory is renormalisable.

3. When the coupling constant has negative mass dimension the theory is non-renormalisable.

The gravitational coupling, which we know as the Newton's constant, $G_N = M_P^{-2}$, is dimensionful (where $M_P$ is the Planck mass) with negative mass dimension, whereas, the coupling constants of gauge theories, such as $\alpha$ of quantum electrodynamics (QED) [14], are dimensionless.

Moreover, in perturbation theory and in comparison with gauge theory, after each loop order, the superficial UV divergences in quantum gravity becomes

worse [11, 12, 13]. Indeed, in each graviton loop there are two more powers of loop momentum (that is to say that there are two more powers in energy expansion, *i.e.* 1-loop has order $(\partial g)^4$, 2-loop has order $(\partial g)^6$ and etc.), this is to atone dimensionally for the two powers of $M_P$ in the denominator of the gravitational coupling. Instead of the logarithmic divergences of gauge theory, that are renormalisable via a finite set of counterterms, quantum gravity contains an infinite set of counterterms. This makes gravity, as given by the Einstein-Hilbert (EH) action, an effective field theory, useful at scales only much less the the Planck mass.

Non-local theories may provide a promising path towards quantisation of gravity. Locality in short means that a particle is only affected by its neighbouring companion [10, 15]. Thus, non-locality simply means that a particle's behaviour is no longer constrained to its close neighbourhood but it also can be affected by interaction far away. Non-locality can be immediately seen in many approaches to quantise gravity, among those, string theory (ST) [16, 17, 20] and loop quantum gravity (LQG) [18, 19] are well known. Furthermore, in string field theory (SFT) [21, 22], non-locality presents itself, for instance in p-adic strings [23] and zeta strings [24]. Thus, it is reasonable to ask wether non-locality is essential to describe gravity.

ST [1] is known to treat the divergences and attempts to provide a finite theory of quantum gravity [16, 17]. This is done by introducing a length scale, corresponding to the string tension, at which particles are no longer point like. ST takes strings as a replacement of particles and count them as the most fundamental objects in nature. Particles after all are the excitations of the strings. There have been considerable amount of progress in unifying the fundamental forces in ST. This was done most successfully for weak, strong and electromagnetic forces. As for gravity, ST relies on supergravity (SUGRA) [25], to treat the divergences. This is due to the fact that supersymmetry soothes some of the UV divergences of quantum field theory, via cancellations between bosonic and fermionic loops, hence the UV divergences of quantum gravity become milder in

[1] It shall be mention that ST on its own has no problem in quantising gravity, as it is fundamentally a 2-dimensional CFT, which is completely a healthy theory. However, ST is known to work well only for small string coupling constant. Thus, ST successfully describes weakly interacting gravitons, but it is less well developed to describe strong gravitational field.

SUGRA. For instance, ST cures the two-loop UV divergences; comparing this with the UV divergences of GR at two-loop order shows that ST is astonishingly useful. However, SUGRA and supersymmetric theories in general have their own shortcomings. For one thing, SUGRA theories are not testable experimentally, at very least for the next few decades.

An effective theory of gravity, which one derives from ST (or otherwise) permits for higher-derivative terms. Before discussing higher derivative terms in the context of gravity one can start by considering a simpler problem of an effective field theory for a scalar field. In the context of ST, one may find an action of the following form,

$$S = \int d^D x \Big[\frac{1}{2}\phi K(\Box)\phi - V(\phi)\Big], \tag{1.0.2}$$

where $K(\Box)$ denotes the kinetic operator and it contains infinite series of higher derivative terms. The d'Alembertian operator is given by $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Finally, $V(\phi)$ is the interaction term. The choice of $K(\Box)$ depends on the model one studies, for instance in the p-adic [23, 42, 43, 44] or random lattice [41, 45, 46, 47, 48], the form of the kinetic operator is taken to be $K(\Box) = e^{-\Box/M^2}$, where $M^2$ is the appropriate mass scale proportional to the string tension. The choice of $K(\Box)$ is indeed very important. For instance for $K(\Box) = e^{-\Box/M^2}$, which is an entire function [69], one obtains a ghost free propagator. That is to say that there is no field with negative kinetic energy. In other words, the choice of an appropriate $K(\Box)$ can prevent introducing extra un-physical states in the propagator.

Furthermore, ST [21, 40] serves two types of perturbative corrections to a given background, namely the string loop corrections and the string world-sheet corrections. The latter is also known as alpha-prime ($\alpha'$) corrections. In terminology, $\alpha'$ is inversely proportional to the string tension and is equal to the sting length squared ($\alpha' = l_s^2$) and thus we shall know that it is working as a scale. Schematically the $\alpha'$ correction to a Lagrangian is given by,

$$L = L^{(0)} + \alpha' L^{(1)} + \alpha'^2 L^{(2)} + \cdots, \tag{1.0.3}$$

where $L^{(0)}$ is the leading order Lagrangian and the rest are the sub-leading corrections. This nature of the ST permits to have corrections to GR. In other words, it

had been suggested that a successful action of quantum gravity shall contain, in addition to the EH term, corrections that are functions of the metric tensor with more than two derivatives. The assumption is that these corrections are needed if one wants to cure non-renormalisability of the EH action [40]. An example of such corrections can be schematically written as,

$$l_s^2(a_1R^2 + a_2R_{\mu\nu}R^{\mu\nu} + a_2R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}) + \cdots \tag{1.0.4}$$

where $a_i$ are appropriate coefficients. After all, higher derivative terms in the action above, would have a minimal influence on the low energy regime and so the classical experiments remain unaffected. However, in the high energy domain they would dictate the behaviour of the theory. For instance, such corrections lead to stabilisation of the divergence structure and finally the power counting renormalisability. Moreover, higher derivative gravity focuses specifically on studying the problems of consistent higher derivative expansion series of gravitational terms and can be regarded as a possible approach to figure out the full theory of gravity.

In this thesis, we shall consider infinite derivatives theories [70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87]. These theories are a sub-class of non-local theories. In the context of gravity, infinite derivative theories are constructed by infinite series of higher-derivative terms. Those terms contain more than two derivatives of the metric tensor. Infinite derivative theories of gravity (IDG) gained an increasing amount of attention on recent years as they address the Big Bang singularity problem [53, 54, 55, 56, 57, 58] and they also have other interesting cosmological [59, 60, 61, 62, 63, 64, 65, 66] implementations. Particularly, as studied in [53], IDG can provide a cosmological non-singular bouncing solution where the Big Bang is replaced with Big Crunch. Subsequently, further progress made in [54, 55] to discover inflationary scenarios linked to IDG. Moreover, such IDG can modify the Raychaudari equations [67], such that one obtains a non-singular bouncing cosmology without violating the null energy conditions. Additionally, at microscopic level, one may consider small black holes with mass much smaller than the Planck mass and observe that IDG prevents singularities in the Newtonian limit where the gravitational potential is very weak [68].

After all, many infinite derivative theories were proposed in different contexts. [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]

It has been shown by Stelle [50, 51] that, gravitational actions which include terms quadratic in the curvature tensor are renormalisable. Such action was written as,

$$S = \int d^4x\sqrt{-g}[\alpha R + \beta R^2 + \gamma R_{\mu\nu}R^{\mu\nu}], \tag{1.0.5}$$

the appropriate choice of coupling constants, $\alpha, \beta$ and $\gamma$ leads to a renormalisable theory. Even though such theory is renormalisable, yet, it suffers from ghost. It shall be noted that one does not need to add the Riemann squared term to the action above, as in four dimensions it would be the Gauss-Bonnet theory,

$$S_{GB} = \int d^4x\sqrt{-g}[R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}], \tag{1.0.6}$$

which is an Euler topological invariant and it should be noted that such modification does not add any local dynamics to the graviton (because it is topological). For Stelle's action [50, 51] the GR propagator is modified schematically as,

$$\Pi = \Pi_{GR} - \frac{\mathcal{P}^2}{k^2 + m^2}, \tag{1.0.7}$$

where it can be seen that there is an extra pole with a negative residue in the spin-2 sector of the propagator (where $\mathcal{P}$ denotes spin projector operator). This concludes that the theory admits a massive spin-2 ghost. In literature this is known as the Weyl ghost. We shall note that the propagator in 4-dimensional GR is given by,

$$\Pi_{GR} = \frac{\mathcal{P}^2}{k^2} - \frac{\mathcal{P}^0_s}{2k^2}, \tag{1.0.8}$$

this shows that even GR has a negative residue at the $k^2 = 0$ pole, and thus a ghost. Yet this pole merely corresponds to the physical graviton and so is not harmful. The existence of ghosts is something that one shall take into account. At classical level they indicate that there is vacuum instability and at quantum level they indicate that unitarity is violated.

In contrast, there are other theories of modified gravity that may be ghost

free, yet they are not renormalisable. From which, $f(R)$ theories [88] are the most well known. The action for $f(R)$ theory is given by,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}f(R), \tag{1.0.9}$$

where $f(R)$ is the function of Ricci scalar. The most famous sub-class of $f(R)$ theory is known as Starobinsky model [89] which has implications in primordial inflation. Starobinsky action is given by,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}(M_P^2R + c_0R^2), \tag{1.0.10}$$

where $c_0$ is constant. The corresponding propagator is given by,

$$\Pi = \Pi_{GR} + \frac{1}{2}\frac{\mathcal{P}_s^0}{k^2+m^2}, \tag{1.0.11}$$

where there is an additional propagating degree of freedom in the scalar sector of the propagator, yet this spin-0 particle is not a ghost and is non-tachyonic for $m^2 \geq 0$.

[90] proposed a ghost-free tensor Lagrangian and its application in gravity. After that, progress were made by [53, 91, 92] to construct a ghost-free action in IDG framework. Such attempts were made to mainly address the cosmological and black hole singularities. Further development were made by [32, 33, 36, 37, 68, 93] to obtain a ghost-free IDG . This is to emphasis that ghost-freedom and renormalisability are important attributes when it comes to constructing a successful theory of quantum gravity.

*Infinite derivative theory of gravity*

Among various modifications of GR, infinite derivative theory of gravity is the promising theory in the sense that it is ghost-free, tachyonic-free and renormalisable, it also addresses the singularity problems. A covariant, quadratic in

curvature, asymptotically free theory of gravity which is ghost-free and tachyon-free around constant curvature backgrounds was proposed by [68].

We shall mention that asymptotic freedom means that the coupling constant decreases as the energy scale increases and vanishes at short distances. This is for the case that the coupling constant of the theory is small enough and so the theory can be dealt with perturbatively. As an example, QCD is an asymptotically free theory [10].

Finally, the IDG modification of gravity can be written as [93],

$$\begin{aligned} S &= S_{EH} + S_{UV} \\ &= \frac{1}{2}\int d^4x\sqrt{-g}\Big[M_P^2 R + RF_1(\bar{\Box})R + R_{\mu\nu}F_2(\bar{\Box})R^{\mu\nu} + R_{\mu\nu\lambda\sigma}F_3(\bar{\Box})R^{\mu\nu\lambda\sigma}\Big], \end{aligned} \tag{1.0.12}$$

where $S_{EH}$ denotes the Einstein-Hilbert action and $S_{UV}$ is the IDG modification of GR. In above notation, $M_P$ is the Planck mass, $\bar{\Box} \equiv \Box/M^2$ and $M$ is the mass scale at which the non-local modifications become important. The $F_i$'s are functions of the d'Alembertian operator and given by,

$$F_i(\bar{\Box}) = \sum_{n=0}^{\infty} f_{i_n}\bar{\Box}^n. \tag{1.0.13}$$

As we shall see later in section 5.2.2 one can perturb the action around Minkowski background. To do so, one uses the definition of the Riemannian curvatures in linearised regime and thus obtains,

$$\begin{aligned} S_{(2)} &= \frac{1}{32\pi G_N^{(D)}}\int d^Dx\Big[\frac{1}{2}h_{\mu\nu}\Box a(\bar{\Box})h^{\mu\nu} + h^\sigma_\mu b(\bar{\Box})\partial_\sigma\partial_\nu h^{\mu\nu} \\ &+ hc(\bar{\Box})\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1}{2}h\Box d(\bar{\Box})h + h^{\lambda\sigma}\frac{f(\bar{\Box})}{2\Box}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\mu\nu}\Big], \end{aligned} \tag{1.0.14}$$

where [94],

$$a(\bar{\Box}) = 1 + M_P^{-2}(\mathcal{F}_2(\bar{\Box})\Box + 4\mathcal{F}_3(\bar{\Box})\Box), \tag{1.0.15}$$

$$b(\bar{\Box}) = -1 - M_P^{-2}(\mathcal{F}_2(\bar{\Box})\Box + 4\mathcal{F}_3(\bar{\Box})\Box), \tag{1.0.16}$$

$$c(\bar{\Box}) = 1 - M_P^{-2}(4\mathcal{F}_1(\bar{\Box})\Box + \mathcal{F}_2(\bar{\Box})\Box), \tag{1.0.17}$$

$$d(\bar{\Box}) = -1 + M_P^{-2}(4\mathcal{F}_1(\bar{\Box})\Box + \mathcal{F}_2(\bar{\Box})\Box), \tag{1.0.18}$$

$$f(\bar{\Box}) = 2M_P^{-2}(2\mathcal{F}_1(\bar{\Box})\Box + \mathcal{F}_2(\bar{\Box})\Box + 2\mathcal{F}_3(\bar{\Box})\Box). \tag{1.0.19}$$

The field equations can be expressed in terms of the inverse propagator as,

$$\Pi^{-1\rho\sigma}_{\mu\nu} h_{\rho\sigma} = \kappa T_{\mu\nu}, \tag{1.0.20}$$

by writing down the spin projector operators in D-dimensional Minkowski space and then moving to momentum space the graviton propagator for IDG gravity can be obtained in the form of,

$$\Pi^{(D)}(-k^2) = \frac{\mathcal{P}^2}{k^2 a(-k^2)} + \frac{\mathcal{P}^0_s}{k^2[a(-k^2) - (D-1)c(-k^2)]}. \tag{1.0.21}$$

We note that, $\mathcal{P}^2$ and $\mathcal{P}^0_s$ are tensor and scalar spin projector operators respectively. Since we do not wish to introduce any extra propagating degrees of freedom apart from the massless graviton, we are going to take $f(\bar{\Box}) = 0$ that implies $a(\bar{\Box}) = c(\bar{\Box})$. Thus,

$$\Pi^{(D)}(-k^2) = \frac{1}{k^2 a(-k^2)} \left( \mathcal{P}^2 - \frac{1}{D-2} \mathcal{P}^0_s \right). \tag{1.0.22}$$

To this end, the form of $a(-k^2)$ should be such that it does not introduce any new propagating degree of freedom, it was argued in Ref. [53, 68] that the form of $a(\Box)$ should be an *entire function*, so as not to introduce any pole in the complex plane, which would result in additional degrees of freedom in the momentum space. In fact, the IDG action is ghost-free under the constraint [93],

$$2F_1(\bar{\Box}) + F_2(\bar{\Box}) + 2F_3(\bar{\Box}) = 0 \tag{1.0.23}$$

around Minkowski background. In other words, the above constraint ensures that the massless graviton remains the only propagating degree of freedom and

no extra degrees of freedom are being introduced. More specifically, if one chooses the graviton propagator to be constructed by an exponential of an entire function [53] [1],

$$a(-k^2) = e^{k^2/M^2}, \tag{1.0.24}$$

the propagators becomes (for $D = 4$),

$$\Pi(-k^2) = \frac{1}{k^2 e^{k^2/M^2}}\Big(\mathcal{P}^2 - \frac{1}{2}\mathcal{P}_s^0\Big) = \frac{1}{e^{k^2/M^2}}\Pi_{GR}. \tag{1.0.25}$$

Indeed, the choice of the exponential of an entire function prevents the production of new poles. For an exponential entire function, the propagator becomes exponentially suppressed in the UV regime while in the infrared (IR) regime one recovers the physical graviton propagator of GR [96, 118]. Recovery of GR at IR regime takes place as $k^2 \to 0$ that leads to $a(0) = 1$ and thus,

$$\lim_{k^2\to 0}\Pi(-k^2) = \frac{1}{k^2}\Big(\mathcal{P}^2 - \frac{1}{2}\mathcal{P}_s^0\Big), \tag{1.0.26}$$

which is the GR propagator. Furthermore, the IDG action given in (1.0.12) can resolve the singularities presented in GR, at classical level [53], upon choosing the exponential given in (1.0.24) (which represents the infinite derivatives); (consult appendix B). Such theory is known to also treat the UV behaviour, leading to the convergent of Feynman diagrams [96].

The IDG theory given by the action (1.0.12) is motivated and established fairly recently. Thus, there are many features in the context of IDG that must be studied. In this thesis we shall consider three important aspects of infinite derivative theories: The Hamiltonian analysis, the generalised boundary term and thermodynamical implications.

[1] The appearance of $a(-k^2)$ in the propagator definition is the consequence of having infinite derivative modification in the gravitational action. [53]

*Hamiltonian formalism*

Hamiltonian analysis is a powerful tool when it comes to studying the stabilities and instabilities of a given theory. It furthermore can be used to calculate the number of the degrees of freedom for the theory of interest. Stabilities of a theory can be investigated using the Ostrogradsky's theorem [98].

Let us consider the following Lagrangian density,

$$\mathcal{L} = \mathcal{L}(q, \dot{q}, \ddot{q}), \tag{1.0.27}$$

where "dot" denotes time derivative and so such Lagrangian density is a function of position, $q$, and its first and second derivatives, in this sense $\dot{q}$ is velocity and $\ddot{q}$ is acceleration. In order to study the classical motion of the system, the action must be stationary under arbitrary variation of $\delta q$. Hence, the condition that must be satisfied are given by the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) + \frac{d^2}{dt^2}\Big(\frac{\partial \mathcal{L}}{\partial \ddot{q}}\Big) = 0. \tag{1.0.28}$$

The acceleration can be uniquely solved by position and velocity if and only if $\frac{\partial^2 \mathcal{L}}{\partial \ddot{q}^2}$ is invertible. In other words, when $\frac{\partial^2 \mathcal{L}}{\partial \ddot{q}^2} \neq 0$, the theory is called non-degenerate. If $\frac{\partial^2 \mathcal{L}}{\partial \ddot{q}^2} = 0$, then the acceleration can not be uniquely determined. Indeed, non-degeneracy of the Lagrangian permits to use the initial data, $q_0, \dot{q}_0, \ddot{q}_0$ and $\dddot{q}_0$, and determine the solutions. Now let us define the following,

$$Q_1 = q, \qquad P_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \ddot{q}}\Big), \tag{1.0.29}$$

$$Q_2 = \dot{q}, \qquad P_2 = \frac{\partial \mathcal{L}}{\partial \ddot{q}}, \tag{1.0.30}$$

where $P_i$'s are the canonical momenta. In this representation the acceleration can be written in terms of $Q_1$, $Q_2$ and $P_2$ as $\ddot{q} = f(Q_1, Q_2, P_2)$. The corresponding

Hamiltonian density would then take the following form,

$$\mathcal{H} = P_1 Q_1 + P_2 f(Q_1, Q_2, P_2) - \mathcal{L}(Q_1, Q_2, f), \tag{1.0.31}$$

for such theory the vacuum decays into both positive and negative energy, and thus the theory is instable, this is because the Hamiltonian density, $\mathcal{H}$, is linear in the canonical momentum $P_1$, [98]. Such instabilities are called Ostrogradsky instability.

Higher derivative theories are known to suffer from such instability [20]. From the propagator analysis, the instabilities are due to the presence of ghost in theories that contain two or more derivatives. In gravity, the four-derivative gravitational action proposed by Stelle [50] is an example where one encounters Ostrogradsky instability.

Ostrogradsky instability is built upon the fact that for the highest momentum operator, which is associated with the highest derivative of the theory, the energy is given linearly, as opposed to quadratic. Yet in the case of IDG, one employs a specific expansion of infinite derivatives, which traces back to the Taylor expansion of $F(\bar{\Box})$, this leads to a graviton propagator that admits no ghost degree of freedom and thus one can overcome the instabilities. We mentioned that, IDG theory is ghost-free and there is no extra degrees of freedom. In this regard, we shall perform the Hamiltonian analysis for the IDG gravity [99] to make sure that the theory is not suffering from Ostrogradsky instability. Such analysis is performed in Chapter 3.

*Boundary term*

Given an action and a well posed variational principle, it is possible to associate a boundary term to the corresponding theory [101]. In GR, when varying the EH action, the surface contribution shall vanish if the action is to be stationary [102]. The surface contribution that comes out of the variation of the action is constructed by variation of the metric tensor (*i.e.* $\delta g_{\mu\nu}$) and variation of its derivatives (*i.e.* $\delta(\partial_\sigma g_{\mu\nu})$). However, imposing $\delta g_{\mu\nu} = 0$ and fixing the variation of the derivatives of the metric tensor are not enough to eliminate the surface contribution.

To this end, Gibbons, Hawking and York (GHY) [101, 102] proposed a modification to the EH action, such that the variation of the modification cancels the term containing $\delta(\partial_\sigma g_{\mu\nu})$ and so imposing $\delta g_{\mu\nu} = 0$ would be sufficient to remove the surface contribution. Such modification is given by,

$$S = S_{EH} + S_{GHY} \sim \int d^4x\sqrt{-g}R + 2\oint d^3x\sqrt{h}K, \tag{1.0.32}$$

where $S_{GHY}$ is the GHY boundary term, $K$ is the trace of the extrinsic curvature on the boundary and $h$ is the determinant of the induced metric defined on the boundary. Indeed, $S_{GHY}$ is essential to make the GR's action as given by the $S_{EH}$ stationary. It shall be noted that boundary terms are needed for those space-times that have well defined boundary. As an example, in the case of black holes, GHY term is defined on the horizon of the black hole (where the geometrical boundary of the black hole is located).

Additionally, $S_{GHY}$ possesses other important features. For instance, in Hamiltonian formalism, GHY action plays an important role when it comes to calculating the Arnowitt-Deser-Misner (ADM) energy [103]. Moreover, in Euclidean semiclassical approach, the black hole entropy is given entirely by the GHY term [104]. It can be concluded that, given a theory, obtaining a correct boundary is vital in understanding the physical features of the theory.

To understand the physics of IDG better, we shall indeed find the boundary term associated with the theory. Thus, in chapter 4 we generalise the GHY

boundary term for the IDG action [100] given by (1.0.12). The exitance of infinite series of covariant derivative in the IDG theory requires us to take a more sophisticated approach. To this end, ADM formalism and in particular coframe slicing was utilised. As we shall see later, our method recovers GR's boundary term when $\Box \to 0$.

*Thermodynamics*

Some of the most physically interesting solutions of GR are black holes. The laws that are governing the black holes' thermodynamics are known to be analogous to those that are obtained by the ordinary laws of thermodynamics. So far, only limited family of black holes are known, they are stationary asymptotically flat solutions to Einstein equations. These solutions are given by [105],

| | Non-rotating ($J = 0$) | Rotating ($J \neq 0$) |
|---|---|---|
| Uncharged ($Q = 0$) | Schwarzschild | Kerr |
| Charged ($Q \neq 0$) | Reissner-Nordström | Kerr-Newman |

where $J$ denotes the angular momentum and $Q$ is the electric charge. The reader shall note that a static background is a stationary one, and as a result a rotating solution is also stationary yet not static. Moreover, electrically charged black holes are solutions of Einstein-Maxwell equations and we will not consider them in this thesis.

Let us summarise the thermodynamical laws that are governing the black hole mechanics. The four laws of black hole thermodynamics are put forward by Bardeen, Carter, and Hawking [106]. They are:

1. *Zeroth law*: states that the surface gravity of a stationary black hole is uniform over the entire event horizon ($H$). *i.e.*

$$\kappa = const \quad \text{on} \quad H. \tag{1.0.33}$$

2. *First law*: states that the change in mass ($M$), charge ($Q$), angular momentum ($J$) and surface area ($A$) are related by:

$$\frac{\kappa}{8\pi}\delta A = \delta M + \Phi\delta Q - \Omega\delta J \tag{1.0.34}$$

where we note that $A = A(M, Q, J)$, $\Phi$ is the electrostatic potential and $\Omega$ is the angular velocity.

3. *Second law*: states that the surface area of a black hole can never decrease, *i.e.*

$$\delta A \geq 0 \tag{1.0.35}$$

given the null energy condition is satisfied.

4. *Third law*: states that the surface gravity of a black hole can not be reduced to zero within a finite advanced time, conditioning that the stress-tensor energy is bounded and satisfies the weak energy condition.

Hawking discovered that the quantum processes lead to a thermal flux of particles from black holes, concluding that they do indeed behave as thermodynamical systems [107]. To this end, it was found that black holes possesses a well defined temperature given by,

$$T = \frac{\hbar\kappa}{2\pi}, \tag{1.0.36}$$

this is known as the Hawking's temperature. Given this and the first law imply that the entropy of a black hole is proportional to the area of its horizon and thus the well known formula of [108],

$$S \equiv \frac{A}{4\hbar G_N}, \tag{1.0.37}$$

from the second law we must also conclude that the entropy of an isolated system can never decrease. It is important to note that Hawking radiation implies that the black hole area decreases which is the violation of the second law, yet one must consider the process of black hole evaporation as a whole. In other words,

the total entropy, which is the sum of the radiation of the black hole entropies, does not decrease.

So far we reviewed the entropy which corresponds to GR as it is described by the EH action. Deviation from GR and moving to higher order gravity means getting corrections to the entropy. Schematically we can write (for $f(R)$ and Lovelock entropies [109]),

$$S \sim \frac{A}{4G_N} + \text{higher curvature corrections}, \tag{1.0.38}$$

as such the first law holds true for the modified theories of gravity including the IDG theories. Yet in some cases the second law can be violated by means of having a decrease in entropy (for instance Lovelock gravity [109]). Indeed, to this day the nature of these violations are poorly understood. In other words it is not yet established wether $\delta(S_{BH} + S_{outside}) \geq 0$ holds true. The higher corrections of a given theory are needed to understand the second law better. To this end, we shall obtain the entropy for number of backgrounds and regimes [117, 118, 119, 120] in chapter 5 for IDG theories.

## 1.1 Summary of results in literature

In this part, we shall present series of studies made in the IDG framework, yet they are not directly the main focus of this thesis.

*UV quantum behaviour*

The perturbation around Minkowski background led to obtaining the linearised action and subsequently the linearised field equations for action (1.0.12), the relevant Bianchi identity was obtained and the corresponding propagator for the IDG action was derived. Inspired by this developments, an infinite derivative scalar toy model was proposed by [96]. Such action is given by (note that this action can be generalised to include quadratic terms as well, yet the purpose of

the study in [96] was to consider a toy model which can be handled technically),

$$S = \int d^4x \left[ \frac{1}{2}\phi\Box a(\bar{\Box})\phi + \frac{1}{4M_P}(\phi\partial_\mu\phi\partial^\mu\phi + \phi\Box\phi a(\bar{\Box})\phi - \phi\partial_\mu\phi a(\bar{\Box})\partial_\mu\phi) \right], \quad (1.1.39)$$

for above action, 1-loop and 2-loop computations were performed and it was found that counter terms can remove the momentum cut-off divergences. Thus, it was concluded that the corresponding Feynman integrals are convergent. It has been also shown by [96] that, at 2-loops the theory is UV finite. Furthermore, a method was suggested for rendering arbitrary n-loops to be finite. Also consult [94, 97]. It should be noted that (1.1.39) is a toy model with cubic interactions and considered in [96] due to its simplicity, however it is possible to consider quadratic interactions too and thus generalise the action.

*Scattering amplitudes*

One of the most interesting aspect of each theory in the view of high energy particle physics is studying the behaviour of the cross sections corresponding to the scattering processes [110]. A theory can not be physical if the cross section despairs at high energies. This is normally the case for theories with more than two derivatives. However, it has been shown by [95] that, infinite derivative scalar field theories can avoid this problem. This has been done by dressing propagators and vertices where the external divergences were eliminated when calculating the scattering matrix element. This is to say that, the cross sections within the infinite derivative framework remain finite.

*Field equations*

In [111], the IDG action given in (1.0.12) was considered. The full non-linear field equations were obtained using the variation principle. The corresponding Bianchi identities were verified and finally the linearised field equations were

calculated around Minkowski background. In similar fashion [15] obtained the linearised field equations around the de-Sitter (dS) background.

*Newtonian potential*

Authors of [68] studied the Newtonian potential corresponding to the IDG action, given in (1.0.12), in weak field regime. In linearised field equations taking $a(\bar{\Box}) = e^{-\bar{\Box}}$ leads to the following Newtonian potential (See Appendix B for derivation),

$$\Phi(r) = -\frac{\kappa m_g \mathrm{Erf}(\frac{Mr}{2})}{8\pi r}, \tag{1.1.40}$$

where $m_g$ is the mass of the object which generates the gravitational potential and $\kappa = 8\pi G_N$. In the limit where $r \to \infty$ one recovers the Minkowski space-time. In contrast, when $r \to 0$, the Newtonian potential becomes constant. This is where IDG deviates from GR for good, in other words, at short distances the singularity of the $1/r$ potential is replaced with a finite constant.

Similar progress was made by [112]. In the context of IDG the Newtonian potential was studied for a more generalised choice of entire function, *i.e.* $a(\bar{\Box}) = e^{\gamma(\bar{\Box})}$, where $\gamma$ is an entire function. It was shown that at large distances the Newtonian potential goes as $1/r$ and thus in agreement with GR, while at short distances the potential is non-singular.

Later on, [113] studied the Newtonian potential for a wider class of IDG. Such potentials were found to be oscillating and non-singular, a seemingly feature of IDG. [113] showed that for an IDG theory constrained to allow defocusing of null rays and thus the geodesics completeness, the Newtonian potential can be made non-singular and be in agreement with GR at large distances.

[113] concluded that, in the context of higher derivative theory of gravity, null congruences can be made complete, or can be made defocused upon satisfaction of two criteria at microscopic level: first, the graviton propagator shall have a scalar mode, comes with one additional root, besides the massless spin-2 and secondly, the IDG gravity must be, at least, ghost-free or tachyon-free.

*Singularities*

GR allows space-time singularity, in other words, null geodesic congruences focus in the presence of matter. [114] discussed the singularity freedom in the context of IDG theory. To this end, the Raychaudari equation corresponding to the IDG was obtained and the bouncing cosmology scenarios were studied. The latest progress in this direction outlined the requirements for defocusing condition for null congruences around dS and Minkowski backgrounds.

*Infrared modifications*

[115] considered an IDG action where the non-local modifications are accounted in the IR regime. The infinite derivative action considered in [115] contains an infinite power series of inverse d'Alembertian operators. As such they are given by,

$$G_i(\bar{\Box}) = \sum_{n=1}^{\infty} c_{i_n} \bar{\Box}^{-n}. \tag{1.1.41}$$

The full non-linear field equations for this action was obtained and the corresponding Bianchi identities were presented. The form of the Newtonian potential in this type of gravity was calculated. Some of the cosmological of implications, such as dark energy, of this theory were also studied [116].

## 1.2 Organisation of thesis

The content of this thesis is organised as follows:

**Chapter 2:** In this chapter the infinite derivative theory of gravity (IDG) is introduced and derived. This serves as a brief review on the derivation of

the theory which would be the focus point of this thesis.

**Chapter 3:** Hamiltonian analysis for an infinite derivative gravitational action, which is constructed by Ricci scalar and covariant derivatives, is performed. First, the relevant Hamiltonian constraints (*i.e.* primary/secondary and first class/ second class) are defined and a formula for calculating the number of degrees of freedom is proposed. Then, we applied the analysis to number of theories. For instance, a scalar field model and the well known $f(R)$ theory. In the case of gravity we employed ADM formalism and applied the regular Hamiltonian analysis to identify the constraints and finally to calculate the number of degrees of freedom.

**Chapter 4:** In this chapter the generalised GHY boundary term for the infinite derivative theory of gravity is obtained. First, the ADM formalism is reviewed and the coframe slicing is introduced. Next, the infinite derivative action is written in terms of auxiliary fields. After that, a generalised formula for obtaining the GHY boundary term is introduced. Finally, we employ the generalised GHY formulation to the infinite derivative theory of gravity and obtain the boundary term.

**Chapter 5:** In this chapter thermodynamical aspects of the infinite derivative theory of gravity are studied. We shall begin by reviewing the Wald's prescription on entropy calculation. Then, Wald's approach is used to obtain the entropy for IDG theory over a generic spherically symmetric background. Such entropy is then analysed in the weak field regime. Furthermore, the entropy of IDG action obtained over the $(A)dS$ background. As a check we used an approximation to recover the entropy of the well known Gauss-Bonnet theory from the $(A)dS$ background. We then study the entropy over a rotating background. This had been done by generalising the Komar integrals, for theories containing Ricci scalar, Ricci tensor and their derivatives. Finally, we shall obtain the entropy of a higher derivative gravitational theory where the action contains inverse d'Alembertian operators (*i.e.* non-locality).

**Conclusion:** In the final part of this thesis, we summarise the results of our study and discuss the findings. Furthermore, the future work is discussed in this section.

**Appendices:** We start by giving the notations, conventions and useful formulas relevant to this thesis. Furthermore, the detailed computations, relevant to each chapter, were presented so the reader can easily follow them.

# Chapter 2

# Overview of infinite derivative gravity

In this chapter we shall summarise the derivation of the infinite derivative gravitational (IDG) action around flat background. In following chapters we study different aspects of this gravitational action.

## 2.1 Derivation of the IDG action

The most general, quadratic in curvature, and generally covariant gravitational action in four dimensions [93] can be written as,

$$
\begin{aligned}
S &= S_{EH} + S_{UV}, && (2.1.1)\\
S_{EH} &= \frac{1}{2}\int d^4x\sqrt{-g}M_P^2 R, && (2.1.2)\\
S_{UV} &= \frac{1}{2}\int d^4x\sqrt{-g}\Big(R_{\mu_1\nu_1\lambda_1\sigma_1}\mathcal{O}^{\mu_1\nu_1\lambda_1\sigma_1}_{\mu_2\nu_2\lambda_2\sigma_2}R^{\mu_2\nu_2\lambda_2\sigma_2}\Big), && (2.1.3)
\end{aligned}
$$

where $S_{EH}$ is the Einstein-Hilbert action and $S_{UV}$ denotes the higher derivative modification of the GR in ultraviolet sector. The operator $\mathcal{O}^{\mu_1\nu_1\lambda_1\sigma_1}_{\mu_2\nu_2\lambda_2\sigma_2}$ retains general covariance.

Expanding (2.1.3), the total action becomes,

$$\begin{aligned} S &= \frac{1}{2}\int dx^4\sqrt{-g}\Big[M_P^2R + RF_1(\Box)R + RF_2(\Box)\nabla_\nu\nabla_\mu R^{\mu\nu} + R_{\mu\nu}F_3(\Box)R^{\mu\nu} \\ &+ R^\nu{}_\mu F_4(\Box)\nabla_\nu\nabla_\lambda R^{\mu\lambda} + R^{\lambda\sigma}F_5(\Box)\nabla_\mu\nabla_\sigma\nabla_\nu\nabla_\lambda R^{\mu\nu} + RF_6(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} \\ &+ R_{\mu\lambda}F_7(\Box)\nabla_\nu\nabla_\sigma R^{\mu\nu\lambda\sigma} + R^\rho{}_\lambda F_8(\Box)\nabla_\mu\nabla_\sigma\nabla_\nu\nabla_\rho R^{\mu\nu\lambda\sigma} \\ &+ R^{\mu_1\nu_1}F_9(\Box)\nabla_{\mu_1}\nabla_{\nu_1}\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} + R_{\mu\nu\lambda\sigma}F_{10}(\Box)R^{\mu\nu\lambda\sigma} \\ &+ R^\rho{}_{\mu\nu\lambda}F_{11}(\Box)\nabla_\rho\nabla_\sigma R^{\mu\nu\lambda\sigma} + R_{\mu\rho_1\nu\sigma_1}F_{12}(\Box)\nabla^{\rho_1}\nabla^{\sigma_1}\nabla_\rho\nabla_\sigma R^{\mu\rho\nu\sigma} \\ &+ R^{\nu_1\rho_1\sigma_1}_\mu F_{13}(\Box)\nabla_{\rho_1}\nabla_{\sigma_1}\nabla_{\nu_1}\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} \\ &+ R^{\mu_1\nu_1\rho_1\sigma_1}F_{14}(\Box)\nabla_{\rho_1}\nabla_{\sigma_1}\nabla_{\nu_1}\nabla_{\mu_1}\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma}\Big], \end{aligned} \tag{2.1.4}$$

it shall be noted that we performed integration by parts where it was appropriate. Also, $F_i$'s are analytical functions of d'Alembertian operator ($\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$). Around Minkowski background the operator would be simplified to: $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The functions $F_i$'s are given explicitly by,

$$F_i(\bar{\Box}) = \sum_{n=0}^{\infty} f_{i_n}\bar{\Box}^n, \tag{2.1.5}$$

where $\bar{\Box} \equiv \Box/M^2$. In this definition, $M$ is the mass-scale at which the non-local modifications become important at UV scale. Additionally, $f_{i_n}$ are the appropriate coefficients of the sum in (2.1.5).

Making use of the antisymmetric properties of the Riemann tensor,

$$R_{(\mu\nu)\rho\sigma} = R_{\mu\nu(\rho\sigma)} = 0, \tag{2.1.6}$$

and the Bianchi identity,

$$\nabla_\alpha R^\mu{}_{\nu\beta\gamma} + \nabla_\beta R^\mu{}_{\nu\gamma\alpha} + \nabla_\gamma R^\mu{}_{\nu\alpha\beta} = 0, \tag{2.1.7}$$

the action given in (2.1.4), reduces to,

$$\begin{aligned} S &= \frac{1}{2}\int dx^4\sqrt{-g}\Big[M_P^2R + RF_1(\Box)R + R_{\mu\nu}F_3(\Box)R^{\mu\nu} + RF_6(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} \\ &+ R_{\mu\nu\lambda\sigma}F_{10}(\Box)R^{\mu\nu\lambda\sigma} + R^{\nu_1\rho_1\sigma_1}_{\mu}F_{13}(\Box)\nabla_{\rho_1}\nabla_{\sigma_1}\nabla_{\nu_1}\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} \\ &+ R^{\mu_1\nu_1\rho_1\sigma_1}F_{14}(\Box)\nabla_{\rho_1}\nabla_{\sigma_1}\nabla_{\nu_1}\nabla_{\mu_1}\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma}\Big]. \end{aligned} \tag{2.1.8}$$

Due to the perturbation around Minkowski background, the covariant derivatives become partial derivatives and can commute around freely. As an example, (see Appendix D)

$$\begin{aligned} RF_6(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} &= \frac{1}{2}RF_6(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} \\ &+ \frac{1}{2}RF_6(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma}. \end{aligned} \tag{2.1.9}$$

By commuting the covariant derivatives we get,

$$\begin{aligned} RF_6(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} &= \frac{1}{2}RF_6(\Box)\nabla_\nu\nabla_\mu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} \\ &+ \frac{1}{2}RF_6(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma}. \end{aligned} \tag{2.1.10}$$

Finally, it is possible to relabel the indices and obtain,

$$RF_6(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma} = RF_6(\Box)\nabla_\nu\nabla_\mu\nabla_\lambda\nabla_\sigma R^{(\mu\nu)\lambda\sigma} = 0, \tag{2.1.11}$$

which vanishes due to antisymmetric properties of the Riemann tensor as mentioned in (2.1.7).

After all the relevant simplifications, we can write the IDG action as,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\Big(M_P^2R + RF_1(\bar{\Box})R + R_{\mu\nu}F_2(\bar{\Box})R^{\mu\nu} + R_{\mu\nu\lambda\sigma}F_3(\bar{\Box})R^{\mu\nu\lambda\sigma}\Big). \tag{2.1.12}$$

This is an infinite derivative modification to the GR.

# Chapter 3

# Hamiltonian analysis

In this chapter, we shall perform a Hamiltonian analysis on the IDG action given in (2.1.12). Due to the technical complexity, the analysis are being performed on a simpler version of this action by dropping the $R_{\mu\nu}F_2(\bar{\Box})R^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}F_3(\bar{\Box})R^{\mu\nu\lambda\sigma}$ terms. In our analysis, we obtain the true dynamical degrees of freedom. We shall note that not including $R_{\mu\nu}F_2(\bar{\Box})R^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}F_3(\bar{\Box})R^{\mu\nu\lambda\sigma}$ to the IDG action does not change the dynamics of the theory we are considering and thus the degrees of freedom would not be changed. Consult [96] for propagator analysis and the degrees of freedom. In fact, $R_{\mu\nu}F_2(\bar{\Box})R^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}F_3(\bar{\Box})R^{\mu\nu\lambda\sigma}$ terms exist as a matter of generality. We will proceed, by first shortly reviewing the Hamiltonian analysis, provide the definitions for *primary, secondary, first-class* and *second-class constraints* [123, 124, 125, 126, 127] and write down the formula for counting the number of degrees of freedom. We then provide some scalar toy models as examples and show how to obtain the degrees of freedom in those models. After setting up the preliminaries and working out the toy examples, we turn our attention to the IDG action and perform the analysis, finding the constraints and finally the number of degrees of freedom.

Hamiltonian analysis can be used as a powerful tool to investigate the stability and boundedness of a given theory. It is well known that, higher derivative theories, those that contain more than two derivatives, suffer from Ostrogradsky's instability [98]. Having infinite number of covariant derivatives in the IDG action however makes the Ostrogradsky's analysis redundant, as one can employ

a specific expansion which traces back to the Taylor expansion of $F(\bar{\Box})$ that leads to a ghost free graviton propagator.

In the late 1950s, the 3+1 decomposition became appealing; Richard Arnowitt, Stanley Deser and Charles W. Misner (ADM) [121, 122] have shown that it is possible to decompose four-dimensional space-time such that one foliates the arbitrary region $\mathcal{M}$ of the space-time manifold with a family of spacelike hypersurfaces $\Sigma_t$, one for each instant in time. In this chapter, we shall show how by using the ADM decomposition, and finding the relevant constraints, one can obtain the number of degrees of freedom. It will be also shown, that how the IDG action can admit finite/infinite number of the degrees of freedom.

## 3.1 Preliminaries

Suppose we have an action that depends on time evolution. We can write down the equations of motion by imposing the stationary conditions on the action and then use variational method. Consider the following action,

$$I = \int \mathcal{L}(q, \dot{q}) dt \,, \tag{3.1.1}$$

the above action is expressed as a time integral and $\mathcal{L}$ is the Lagrangian density depending on the position $q$ and the velocity $\dot{q}$. The variation of the action leads to the equations of motion known as Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \,, \tag{3.1.2}$$

we can expand the above expression, and write,

$$\ddot{q}\frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} - \dot{q}\frac{\partial^2 \mathcal{L}}{\partial q \partial \dot{q}} \,, \tag{3.1.3}$$

the above equation yields an acceleration, $\ddot{q}$, which can be uniquely calculated by position and velocity at a given time, if and only if $\frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial \dot{q}}$ is invertible. In other words, if the determinant of the matrix $\frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial \dot{q}} \neq 0$, *i.e.* non vanishing, then the

theory is called *non-degenerate*. If the determinant is zero, then the acceleration can *not* be uniquely determined by position and the velocity. The latter system is called *singular* and leads to *constraints* in the phase space [127, 128].

### 3.1.1 Constraints for a singular system

In order to formulate the Hamiltonian we need to first define the canonical momenta,

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \,. \tag{3.1.4}$$

The non-invertible matrix $\frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial \dot{q}}$ indicates that not all the velocities can be written in terms of the canonical momenta, in other words, not all the momenta are independent, and there are some relation between the canonical coordinates [123, 124, 125, 126, 127], such as,

$$\varphi(q,\ p) = 0 \iff \textit{primary constraints}\,, \tag{3.1.5}$$

known as *primary constraints*. Take $\varphi(q,\ p)$ for instance, if we have *vanishing canonical momenta*, then we have *primary constraints*. The *primary constraints* hold without using the equations of motion. The *primary constraints* define a submanifold smoothly embedded in a phase space, which is also known as the *primary constraint surface*, $\Gamma_p$. We can now define the Hamiltonian density as,

$$\mathcal{H} = p\dot{q} - \mathcal{L}\,. \tag{3.1.6}$$

If the theory admits *primary constraints*, we will have to redefine the Hamiltonian density, and write the *total* Hamiltonian density as,

$$\mathcal{H}_{tot} = \mathcal{H} + \lambda^a(q,p)\varphi_a(q,p)\,, \tag{3.1.7}$$

where now $\lambda^a(q,p)$ is called the *Lagrange multiplier*, and $\varphi_a(q,p)$ are linear combinations of the primary constraints [1]. The Hamiltonian equations of motion are the time evolutions, in which the Hamiltonian density remains invariant under arbitrary variations of $\delta p$, $\delta q$ and $\delta\lambda$ ;

$$\dot{p} = -\frac{\delta \mathcal{H}_{tot}}{\delta q} = \{q, \mathcal{H}_{tot}\}\,, \tag{3.1.8}$$

$$\dot{q} = -\frac{\delta \mathcal{H}_{tot}}{\delta p} = \{p, \mathcal{H}_{tot}\}\,. \tag{3.1.9}$$

As a result, the Hamiltonian equations of motion can be expressed in terms of the Poisson bracket. In general, for canonical coordinates, $(q^i,\ p_i)$, on the phase space, given two functions $f(q,\ p)$ and $g(q,\ p)$, the Poisson bracket can be defined as

$$\{f,g\} = \sum_{i=1}^{n} \Big( \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \Big)\,, \tag{3.1.10}$$

where $q_i$ are the generalised coordinates, and $p_i$ are the generalised conjugate momentum, and $f$ and $g$ are any function of phase space coordinates. Moreover, $i$ indicates the number of the phase space variables.

Now, any quantity is *weakly* vanishing when it is numerically restricted to be zero on a submanifold $\Gamma$ of the phase space, but does not vanish throughout the phase space. In other words, a function $F(p,q)$ defined in the neighbourhood of $\Gamma$ is called *weakly zero*, if

$$F(p,q)|_\Gamma = 0 \Longleftrightarrow F(p,q) \approx 0\,, \tag{3.1.11}$$

where $\Gamma$ is the *constraint surface* defined on a submanifold of the phase space. Note that the notation "$\approx$" indicates that the quantity is *weakly* vanishing; this

[1] We should point out that the total Hamiltonian density is the sum of the canonical Hamiltonian density and terms which are products of Lagrange multipliers and the primary constraints. The time evolution of the primary constraints, should it be equal to zero, gives the *secondary constraints* and those *secondary constraints* are evaluated by computing the Poisson bracket of the *primary constraints* and the total Hamiltonian density. In the literature, one may also come across the *extended* Hamiltonian density, which is the sum of the canonical Hamiltonian density and terms which are products of Lagrange multipliers and the first-class constraints, see [128].

is a standard Dirac's terminology, where $F(p,q)$ shall vanish on the constraint surface, $\Gamma$, but not necessarily throughout the phase space.

When a theory admits *primary constraints*, we must ensure that the theory is consistent by essentially checking whether the primary constraints are preserved under time evolution or not. In other words, we demand that, on the constraint surface $\Gamma_p$,

$$\dot{\varphi}|_{\Gamma_p} = \{\varphi, \mathcal{H}_{tot}\}|_{\Gamma_p} = 0 \quad \Longleftrightarrow \dot{\varphi} = \{\varphi, \mathcal{H}_{tot}\} \approx 0\,. \tag{3.1.12}$$

That is,

$$\dot{\varphi} = \{\varphi, \mathcal{H}_{tot}\} \approx 0 \quad \Longrightarrow \textit{secondary constraint}\,. \tag{3.1.13}$$

By *demanding* that Eq. (3.1.12) (not identically) be zero on the constraint surface $\Gamma_p$ yields a *secondary constraint* [123, 129], and the theory is consistent. In case, whenever Eq. (3.1.12) fixes a Lagrange multiplier, then there will be no *secondary constraints*. The *secondary constraints* hold when the equations of motion are satisfied, but need not hold if they are not satisfied. However, if Eq. (3.1.12) is identically zero, then there will be no *secondary constraints*. All constraints (*primary and secondary*) define a smooth submanifold of the phase space called the *constraint surface*: $\Gamma_1 \subseteq \Gamma_p$. A theory can also admit *tertiary constraints*, and so on and so forth [128]. We can verify whether the theory is consistent by checking if the *secondary constraints* are preserved under time evolution or not.

Note that $\mathcal{H}_{tot}$ is the total Hamiltonian density defined by Eq. (3.1.7). To summarize, if a canonical momentum is vanishing, we have a *primary constraint*, while *enforcing* that the time evolution of the *primary constraint* vanishes on the constraint surface, $\Gamma_1$ give rise to a *secondary constraint*.

### 3.1.2 First and second-class constraints

Any theory that can be formulated in Hamiltonian formalism gives rise to Hamiltonian constraints. Constraints in the context of Hamiltonian formulation can be thought of as reparameterization; while the invariance is preserved [1]. The most

[1]For example, in the case of gravity, constraints are obtained by using the ADM formalism that is reparameterizing the theory under spatial and time coordinates. Hamiltonian constraints

important step in Hamiltonian analysis is the classification of the constrains. By definition, we call a function $f(p,q)$ to be *first-class* if its Poisson brackets with all other constraints vanish *weakly*. A function which is not *first-class* is called *second-class* [1]. On the constraint surface $\Gamma_1$, this is mathematically expressed as

$$\{f(p,q),\varphi\}|_{\Gamma_1} \approx 0 \quad \Longrightarrow \quad \textit{first-class}\,, \tag{3.1.14}$$

$$\{f(p,q),\varphi\}|_{\Gamma_1} \not\approx 0 \quad \Longrightarrow \quad \textit{second-class}\,. \tag{3.1.15}$$

We should point out that we use the "$\approx$" sign as we are interested in whether the Poisson brackets of $f(p,q)$ with all other constraints vanish on the constraint surface $\Gamma_1$ or not. Determining whether they vanish globally, *i.e.*, throughout the phase space, is not necessary for our purposes.

### 3.1.3 Counting the degrees of freedom

Once we have the physical canonical variables, and we have fixed the number of *first-class and/or second-class* constraints, we can use the following formula to count the number of the physical degrees of freedom [2], see [128],

$$\mathcal{N} = \frac{1}{2}(2\mathcal{A} - \mathcal{B} - 2\mathcal{C}) = \frac{1}{2}\mathcal{X} \tag{3.1.16}$$

where

- $\mathcal{N}$ = number of physical degrees of freedom
- $\mathcal{A}$ = number of *configuration space variables*
- $\mathcal{B}$ = number of *second-class constraints*
- $\mathcal{C}$ = number of *first-class constraints*
- $\mathcal{X}$ = number of *independent canonical variables*

generate time diffeomorphism, see [130].

[1] One should mention that the *primary/secondary* and *first-class/second-class* classifications overlap. A *primary constraint* can be *first-class or second-class* and a *secondary constraint* can also be *first-class* or *second-class*.

[2] Note that the phase space is composed of all positions and velocities together, while the configuration space consists of the position only.

## 3.2 Toy models

In this section we shall use Dirac's prescription and provide the relevant constraints for some toy models and then obtain the number of degrees of freedom. Our aim will be to study some very simplistic time dependent models before extending our argument to a covariant action.

### 3.2.1 Simple homogeneous case

Let us consider a very simple time dependent action,

$$I = \int \dot{\phi}^2 dt\,, \tag{3.2.17}$$

where $\phi$ is some time dependent variable, and $\dot{\phi} \equiv \partial_0 \phi$. For the above action the canonical momenta is [1]

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2\dot{\phi}\,. \tag{3.2.18}$$

If the canonical momenta is not vanishing, *i.e.* $p \neq 0$, then there is no constraints, and hence no classification, *i.e.* $\mathcal{B} = 0$ in Eq. (3.1.16), and so will be, $\mathcal{C} = 0$. The number of degrees of freedom is then given by the total number of the independent canonical variables:

$$\mathcal{N} = \frac{1}{2}\mathcal{X} = \frac{1}{2}(p,\, \phi) = \frac{1}{2}(1+1) = 1\,. \tag{3.2.19}$$

Therefore, this theory contains only one physical degree of freedom. A simple generalization of a time-dependent variable to infinite derivatives can be given

[1] We are working around Minkowski background with mostly plus, *i.e.*, $(-,+,+,+)$.

by:

$$\begin{aligned} I &= \int dt\phi\mathcal{F}\left(-\frac{\partial^2}{\partial t^2}\right)\phi \\ &= \int dt\left(c_0\phi^2 + c_1\phi\left(-\frac{\partial^2}{\partial t^2}\right)\phi + c_2\phi\left(-\frac{\partial^2}{\partial t^2}\right)^2\phi + c_3\phi\left(-\frac{\partial^2}{\partial t^2}\right)^3\phi + \cdots\right) \\ &= \int dt\left(c_0\phi^2 - c_1\phi\phi^{(2)} + c_2\phi\phi^{(4)} - c_3\phi\phi^{(6)} + \cdots\right), \end{aligned} \tag{3.2.20}$$

where $\phi = \phi(t)$, and $\mathcal{F}$ could take a form, like:

$$\mathcal{F}\left(-\frac{\partial^2}{\partial t^2}\right) = \sum_{n=0}^{\infty} c_n\left(-\frac{\partial^2}{\partial t^2}\right)^n . \tag{3.2.21}$$

The next step is to find the conjugate momenta, so that we can use the generalised formula [98],

$$\begin{aligned} p_1 &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\ddot{\phi}}\right) + \left(\frac{d}{dt}\right)^2\left(\frac{\partial\mathcal{L}}{\partial\dddot{\phi}}\right) - \cdots, \\ p_2 &= \frac{\partial\mathcal{L}}{\partial\ddot{\phi}} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dddot{\phi}}\right) + \left(\frac{d}{dt}\right)^2\left(\frac{\partial\mathcal{L}}{\partial\ddddot{\phi}}\right) - \cdots, \\ &\vdots \end{aligned} \tag{3.2.22}$$

Now the conjugate momenta for action Eq. (3.2.20) as,

$$\begin{aligned} p_1 &= c_1\dot{\phi} - c_2\phi^{(3)} + c_3\phi^{(5)} - c_4\phi^{(7)} + \cdots \\ p_2 &= -c_1\phi + c_2\phi^{(2)} - c_3\phi^{(4)} + c_4\phi^{(6)} - \cdots \\ p_3 &= -c_2\phi^{(1)} + c_3\phi^{(3)} - c_4\phi^{(5)} + \cdots \\ p_4 &= c_2\phi - c_3\phi^{(2)} + c_4\phi^{(4)} - \cdots \\ &\vdots \end{aligned} \tag{3.2.23}$$

and, so on and so forth. For Eq. (3.2.20), we can count the number of the degrees of freedom essentially by identifying the independent number of canonical

variables, that is,

$$\mathcal{N} = \frac{1}{2}\mathcal{X} = \frac{1}{2}(\phi,\ p_1,\ p_2,\ \cdots) = \frac{1}{2}(1+1+1+\cdots) = \infty\,. \tag{3.2.24}$$

An infinite number of canonical variables corresponding to an infinite number of time derivatives acting on a time-dependent variable leads to a theory that contains infinite number of degrees of freedom.

### 3.2.2 Scalar Lagrangian with covariant derivatives

As a warm up exercise, let us consider the following action,

$$I = \int d^4x \Big( c_0\phi^2 + c_1\phi\bar{\Box}\phi \Big)\,, \tag{3.2.25}$$

where $\phi$ is a generic scalar field of mass dimension 2; and $\bar{\Box} \equiv \Box/M^2$, where $M$ is the scale of new physics beyond the Standard Model, $\Box$ is d'Alembertian operator of the form $\Box = \eta^{\mu\nu}\nabla_\mu\nabla_\nu$, where $\eta_{\mu\nu}$ is the Minkowski metric, and $c_0,\ c_1$ are constants. We can always perform integration by parts on the second term, and rewrite

$$c_1\phi\bar{\Box}\phi = \frac{c_1}{M^2}\partial_\mu\phi\partial^\mu\phi = -\frac{c_1}{M^2}\partial_0\phi\partial^0\phi + \frac{c_1}{M^2}\partial_i\phi\partial^i\phi\,, \tag{3.2.26}$$

therefore the canonical momenta can be expressed, as

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 2\frac{c_1}{M^2}\dot{\phi}\,, \tag{3.2.27}$$

where we have used the notation $\dot{\phi} \equiv \partial_0\phi$, also note that $\partial_0\phi\partial^0\phi = -\partial_0\phi\partial_0\phi$. The next step is to write down the Hamiltonian density, as:

$$\begin{aligned}\mathcal{H} &= \pi\dot{\phi} - \mathcal{L} = 2\frac{c_1}{M^2}\dot{\phi}^2 - c_0\phi^2 - c_1\phi\bar{\Box}\phi \\ &= 2\frac{c_1}{M^2}\dot{\phi}^2 - c_0\phi^2 + \frac{c_1}{M^2}(-\dot{\phi}^2 + \partial_i\phi\partial^i\phi) \\ &= -c_0\phi^2 + \frac{c_1}{M^2}\dot{\phi}^2 + \frac{c_1}{M^2}\partial_i\phi\partial^i\phi\,.\end{aligned} \tag{3.2.28}$$

Again, if $\pi \neq 0$, or for instance, $c_1 \neq 0$, then there are no constraints. The number of degrees of freedom for the action will be given by,

$$\frac{1}{2}\mathcal{X} = \frac{1}{2}\{(p,\phi)\} = 1. \tag{3.2.29}$$

It can be seen from the examples provided that going to higher derivatives amounts to have infinite number of conjugate momenta and thus infinite number of degrees of freedom. In the next section we are going to construct an infinite derivative theory such that the number of the degrees of freedom are physical and finite.

## 3.3 Infinite derivative scalar field theory

Before considering any gravitational action, it is helpful to consider a Lagrangian that is constructed by infinite number of d'Alembertian operators, we build this action in Minkowski space-time,

$$I = \int d^4x\,\phi\mathcal{F}(\bar{\Box})\phi, \qquad \text{with:} \qquad \mathcal{F}(\bar{\Box}) = \sum_{n=0}^{\infty} c_n\bar{\Box}^n\,, \tag{3.3.30}$$

where $c_n$ are constants. Such action is complicated and thus begs for a more technical approach, we approach the problem by first writing an equivalent action

of the form,

$$I_{eqv} = \int d^4x A\mathcal{F}(\bar{\Box})A\,, \tag{3.3.31}$$

Where the auxiliary field, $A$, is introduced as an equivalent scalar field to $\phi$, this means that the equations of the motion for both actions ($I$ and $I_{eqv}$) are equivalent. In the next step, let us expand the term $\mathcal{F}(\bar{\Box})A$,

$$\mathcal{F}(\bar{\Box})A = \sum_{n=0}^{\infty} c_n\bar{\Box}^n A = c_0A + c_1\bar{\Box}A + c_2\bar{\Box}^2A + c_3\bar{\Box}^3A + \cdots \tag{3.3.32}$$

Now, in order to eliminate the contribution of $\bar{\Box}A$, $\bar{\Box}^2A$ and so on, we are going to introduce two auxiliary fields $\chi_n$ and $\eta_n$, where the $\chi_n$'s are dimensionless and the $\eta_n$'s have mass dimension 2 (this can be seen by parameterising $\bar{\Box}A$, $\bar{\Box}^2A$, $\cdots$). We show few steps here by taking some simple examples

- Let our action to be constructed by a single box only, then,

$$I_{eqv} = \int d^4x A\bar{\Box}A\,. \tag{3.3.33}$$

  Now, to eliminate $\bar{\Box}A$ in the term $A\bar{\Box}A$, we wish to add a following term in the above action,

$$\int d^4x\ \chi_1A(\eta_1 - \bar{\Box}A) = \int d^4x\left[\chi_1A\eta_1 + \eta^{\mu\nu}(\partial_\mu\chi_1A\partial_\nu A + \chi_1\partial_\mu A\partial_\nu A)\right]. \tag{3.3.34}$$

  where we derived above as follow:

$$\begin{aligned}
\chi_1A(\eta_1 - \Box A) &= \chi_1A\eta_1 - \chi_1A\Box A \\
&= \chi_1A\eta_1 - \eta^{\mu\nu}\chi_1A\partial_\mu\partial_\nu A \\
&= \chi_1A\eta_1 - \eta^{\mu\nu}\partial_\mu(\chi_1A\partial_\nu A) + \eta^{\mu\nu}\partial_\mu\chi_1A\partial_\nu A + \eta^{\mu\nu}\chi_1\partial_\mu A\partial_\nu A \\
&= \chi_1A\eta_1 + \eta^{\mu\nu}\partial_\mu\chi_1A\partial_\nu A + \eta^{\mu\nu}\chi_1\partial_\mu A\partial_\nu A\,,
\end{aligned} \tag{3.3.35}$$

  where it should be noted that we have dropped the total derivative and also

we have absorbed the factor of $M^{-2}$ into $\chi_1$ (the mass dimension of $\eta_1$ is modified accordingly). Therefore, here the d'Alembertian operator is not barred. Finally, we can write down the equivalent action in the following form,

$$I_{eqv} = \int d^4x \Big( A\eta_1 + \chi_1 A(\eta_1 - \bar{\Box} A) \Big) \,, \tag{3.3.36}$$

by solving the equation of motion for $\chi_1$, we obtain

$$\eta_1 = \bar{\Box} A \,, \tag{3.3.37}$$

and hence, Eqs. (3.3.33) and (3.3.36) are equivalent.

- Before generalising our method, let us consider the following,

$$I_{eqv} = \int d^4x \ \left[ A\bar{\Box} A + A\bar{\Box}^2 A \right] \,, \tag{3.3.38}$$

in order to eliminate the term $A\bar{\Box}^2 A$, we add the term

$$\int d^4x \ \chi_2 A(\eta_2 - \bar{\Box}\eta_1) = \int d^4x \Big[ \chi_2 A\eta_2 + \eta^{\mu\nu}(\partial_\mu \chi_2 A \partial_\nu \eta_1 + \chi_2 \partial_\mu A \partial_\nu \eta_1) \Big] \,. \tag{3.3.39}$$

We can rewrite action Eq. (3.3.38) as:

$$I_{eqv} = \int d^4x \Big( A(\eta_1 + \eta_2) + \chi_1 A(\eta_1 - \bar{\Box} A) + \chi_2 A(\eta_2 - \bar{\Box}\eta_1) \Big) \,. \tag{3.3.40}$$

Solving the equation of motion for $\chi_2$ yields $\eta_2 = \bar{\Box}\eta_1 = \bar{\Box}^2 A$.

Similarly, in order to eliminate the terms $A\bar{\Box}^n A$ and so on, we have to repeat the same procedure up to $\bar{\Box}^n$. Note that we have established this by solving the equation of motion for $\chi_n$, we obtain, for $n \geq 2$,

$$\eta_n = \bar{\Box}\eta_{n-1} = \bar{\Box}^n A. \tag{3.3.41}$$

Now, we can rewrite the action Eq. (3.3.31) as,

$$
\begin{aligned}
I_{eqv} &= \int d^4x \Bigg\{ A(c_0 A + \sum_{n=1}^{\infty} c_n \eta_n) + \chi_1 A(\eta_1 - \Box A) + \sum_{l=2}^{\infty} \chi_l A(\eta_l - \Box \eta_{l-1}) \Bigg\} \\
&= \int d^4x \Bigg\{ A(c_0 A + \sum_{n=1}^{\infty} c_n \eta_n) \\
&+ \eta^{\mu\nu}(A\partial_\mu \chi_1 \partial_\nu A + \chi_1 \partial_\mu A \partial_\nu A) + \eta^{\mu\nu} \sum_{l=2}^{\infty} (A\partial_\mu \chi_l \partial_\nu \eta_{l-1} + \chi_l \partial_\mu A \partial_\nu \eta_{l-1}) \\
&+ \sum_{l=1}^{\infty} A \chi_l \eta_l \Bigg\}, \\
&= \int d^4x \Bigg\{ A(c_0 A + \sum_{n=1}^{\infty} c_n \eta_n) + \sum_{l=1}^{\infty} A \chi_l \eta_l \\
&+ \eta^{00}(A\partial_0 \chi_1 \partial_0 A + \chi_1 \partial_0 A \partial_0 A) + \eta^{ij}(A\partial_i \chi_1 \partial_j A + \chi_1 \partial_i A \partial_j A) \\
&+ \eta^{00} \sum_{l=2}^{\infty} (A\partial_0 \chi_l \partial_0 \eta_{l-1} + \chi_l \partial_0 A \partial_0 \eta_{l-1}) + \eta^{ij} \sum_{l=2}^{\infty} (A\partial_i \chi_l \partial_j \eta_{l-1} + \chi_l \partial_i A \partial_j \eta_{l-1}) \Bigg\}.
\end{aligned}
\tag{3.3.42}
$$

where we have absorbed the powers of $M^{-2}$ into the $c_n$'s & $\chi_n$'s and the mass dimension of the $\eta_n$'s has been modified accordingly. Hence, the box operator is not barred. We shall also mention that in Eq. (3.3.42) we have decomposed the d'Alembertian operator to its components around the Minkowski background: $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \eta^{00}\partial_0\partial_0 + \eta^{ij}\partial_i\partial_j$, where the zeroth component is the time coordinate, and $\{i,\ j\}$ are the spatial coordinates running from 1 to 3. The conjugate momenta for the above action are given by:

$$
\begin{aligned}
p_A &= \frac{\partial \mathcal{L}}{\partial \dot{A}} = \Big[ -(A\partial_0 \chi_1 + \chi_1 \partial_0 A) - \sum_{l=2}^{\infty} (\chi_l \partial_0 \eta_{l-1}) \Big], \\
p_{\chi_1} &= \frac{\partial \mathcal{L}}{\partial \dot{\chi}_1} = -A\partial_0 A, \quad p_{\chi_l} = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_l} = -(A\partial_0 \eta_{l-1}), \\
p_{\eta_{l-1}} &= \frac{\partial \mathcal{L}}{\partial \dot{\eta}_{l-1}} = -(A\partial_0 \chi_l + \chi_l \partial_0 A).
\end{aligned}
\tag{3.3.43}
$$

where $\dot{A} \equiv \partial_0 A$. Therefore, the Hamiltonian density is given by

$$\begin{aligned}\mathcal{H} &= p_A\dot{A} + p_{\chi_1}\dot{\chi}_1 + p_{\chi_l}\dot{\chi}_l + p_{\eta_{l-1}}\dot{\eta}_{l-1} - \mathcal{L} \\ &= A(c_0 A + \sum_{n=1}^{\infty} c_n\eta_n) - \sum_{l=1}^{\infty} A\chi_l\eta_l \\ &- (\eta^{\mu\nu}A\partial_\mu\chi_1\partial_\nu A + \eta^{ij}\chi_1\partial_i A\partial_j A) \\ &- \eta^{\mu\nu}\sum_{l=2}^{\infty}(A\partial_\mu\chi_l\partial_\nu\eta_{l-1} + \chi_l\partial_\mu A\partial_\nu\eta_{l-1})\,. \end{aligned} \tag{3.3.44}$$

See Appendix E for the explicit derivation of (3.3.44). Let us recall the equivalent action (3.3.42) before integration by parts. That reads as,

$$I_{eqv} = \int d^4x \Bigg\{ A(c_0 A + \sum_{n=1}^{\infty} c_n\eta_n) + \chi_1 A(\eta_1 - \Box A) + \sum_{l=2}^{\infty}\chi_l A(\eta_l - \Box\eta_{l-1}) \Bigg\}\,; \tag{3.3.45}$$

we see that we have terms like :

$$\chi_1 A(\eta_1 - \Box A)$$

and

$$\chi_l A(\eta_l - \Box\eta_{l-1}), \quad for \;\; l \geq 2.$$

Additionally, we know that solving the equations of motion for $\chi_n$ leads to $\eta_n = \Box^n A$. Therefore, it shall be concluded that the $\chi_n$'s are the Lagrange multipliers, and not dynamical as a result. From the equations of motion, we get the following *primary constraints* [1]:

$$\begin{aligned}\sigma_1 &= \eta_1 - \Box A \approx 0\,, \\ &\vdots \\ \sigma_l &= \eta_l - \Box\eta_{l-1} \approx 0\,. \end{aligned} \tag{3.3.46}$$

[1] Let us note that $\Gamma_p$ is a smooth submanifold of the phase space determined by the *primary constraints*; in this section, we shall exclusively use the "$\approx$" notation to denote equality on $\Gamma_p$.

In other words, since $\chi_n$'s are the Lagrange multipliers, $\sigma_1$ and $\sigma_l$'s are *primary constraints*. The time evolutions of the $\sigma_n$'s fix the corresponding Lagrange multipliers $\lambda^{\sigma_n}$ in the total Hamiltonian (when we add the terms $\lambda^{\sigma_n}\sigma_n$ to the Hamiltonian density $\mathcal{H}$); therefore, the $\sigma_n$'s do not induce *secondary constraints*. As a result, to classify the above constraint, we will need to show that the Poisson bracket given by (3.1.10) is weakly vanishing:

$$\{\sigma_m, \sigma_n\}|_{\Gamma_p} = 0\,, \tag{3.3.47}$$

so that $\sigma_n$'s can be classified as *first-class constraints*. However, this depends on the choice of $\mathcal{F}(\Box)$, whose coefficients are hiding in $\chi$'s and $\eta$'s. It is trivial to show that, for this case, there is no *second-class constraint*, *i.e.*, $\mathcal{B} = 0$, as we do not have $\{\sigma_m, \sigma_n\} \not\approx 0$. That is, the $\sigma_n$'s are *primary, first-class constraints*. In our case, the number of phase space variables,

$$2\mathcal{A} \;\equiv\; 2 \times \Big\{(A, p_A), \underbrace{(\eta_1, p_{\eta_1}), (\eta_2, p_{\eta_2}), \cdots}_{n=1,\ 2,\ 3, \cdots \infty}\Big\} \equiv 2 \times (1 + \infty) = 2 + \infty\,. \tag{3.3.48}$$

For each pair, $(\eta_n,\ p_{\eta_n})$, we have assigned one variable, which is multiplied by a factor of 2, since we are dealing with field-conjugate momentum pairs, in the phase space. In the next section, we will fix the form of $\mathcal{F}(\bar{\Box})$ to estimate the number of *first-class constraints*, *i.e.*, $\mathcal{C}$ and, hence, the number of degrees of freedom. Let us also mention that the choice of $\mathcal{F}(\bar{\Box})$ will determine the number of solutions to the equation of motion for $A$ we will have, and consequently these solutions can be interpreted as *first-class constraints* which will determine the number of physical degrees of freedom, *i.e.* finite/infinite number of degrees of freedom will depend on the number of solutions of the equations of motion for $A$. See more detail on Appendix G.

### 3.3.1 Gaussian kinetic term and propagator

Let us now consider an example of infinite derivative scalar field theory, but with a Gaussian kinetic term in Eq. (3.3.30), *i.e.* by exponential of an entire function,

$$I_{eqv} = \int d^4x \; A\left(\Box e^{-\bar{\Box}}\right)A\,. \tag{3.3.49}$$

For the above action, the equation of motion for $A$ is then given by:

$$2\left(\Box e^{-\bar{\Box}}\right)A = 0\,. \tag{3.3.50}$$

We observe that there is a finite number of solutions; hence, there are also finitely many degrees of freedom [1]. In momentum space, we obtain the following solution,

$$k^2 = 0\,, \tag{3.3.51}$$

and the propagator will follow as [68, 93] :

$$\Pi(\bar{k}^2) \sim \frac{1}{k^2}e^{-\bar{k}^2}\,, \tag{3.3.52}$$

where we have used the fact that in momentum space $\Box \to -k^2$, and we have $\bar{k} \equiv k/M$. There are some interesting properties to note about this propagator:

- The propagator is suppressed by an *exponential of an entire function*, which has no zeros, poles. Therefore, the only dynamical pole resides at $k^2 = 0$, *i.e.*, the massless pole in the propagator, *i.e.*, degrees of freedom $\mathcal{N} = 1$. This is to say that, even though we have infinitely many derivatives, but there is only one relevant degrees of freedom that is the massless scalar field. In fact, there are no new dynamical degrees of freedom. Furthermore, in the UV the propagator is suppressed.

[1] Note that, for an infinite derivative action of the form $I_{eqv} = \int d^4x \; A\cos(\bar{\Box})A$, we would have an infinite number of solutions and, hence, infinitely many degrees of freedom. Note that the choice of $\cos(\bar{\Box})$ leads to infinite number of solutions due to the periodicity of the cosine function. In this footnote we take $\cos(\bar{\Box})$ to illustrate what it means by *bad choice* of $F(\bar{\Box})$.

- The propagator contains no *ghosts* (this is because an entire function does not give rise to poles in the infinite complex plane), which usually plagues higher derivative theories. By virtue of this, there is no analogue of Ostrógradsky instability at classical level. Given the background equation, one can indeed understand the stability of the solution.

The original action Eq. (3.3.49) can now be recast in terms of an equivalent action as:

$$I_{eqv} = \int d^4x \left[ A\big(\Box e^{-\bar{\Box}}\big) A + \chi_1 A(\eta_1 - \Box A) + \sum_{l=2}^{\infty} \chi_l A(\eta_l - \Box \eta_{l-1}) \right]. \tag{3.3.53}$$

We can now compute the number of the physical degrees of freedom. Note that the determinant of the phase-space dependent matrix $A_{mn} = \{\sigma_m, \sigma_n\} \neq 0$, so the $\sigma_n$'s do not induce further constraints, such as *secondary constraints*. Therefore [1],

$$\begin{aligned} &2\mathcal{A} \equiv 2 \times \left\{ (A, p_A), \underbrace{(\eta_1, p_{\eta_1}), (\eta_2, p_{\eta_2}), \cdots}_{n} \right\} = 2 \times (1 + \infty) = 2 + \infty \\ &\mathcal{B} = 0, \\ &2\mathcal{C} \equiv 2 \times (\sigma_n) = 2(\infty) = \infty, \\ &\mathcal{N} = \frac{1}{2}(2\mathcal{A} - \mathcal{B} - 2\mathcal{C}) = \frac{1}{2}(2 + \infty - 0 - \infty) = 1\,. \end{aligned} \tag{3.3.54}$$

As expected, the conclusion of this analysis yields exactly the same dynamical degrees of freedom as that of the Lagrangian formulation. The coefficients $c_i$ of $\mathcal{F}(\bar{\Box})$ are all fixed by the form of $\Box e^{-\bar{\Box}}$.

## 3.4 IDG Hamiltonian analysis

In this section we will take a simple action of IDG, and study the Hamiltonian density and degrees of freedom, we proceed by briefly recap the ADM formalism

[1] In this case and hereafter in this chapter, one shall include the $k^2 = 0$ solution when counting the number of degrees of freedom. This can be written in position space as $\Box A = 0$. Since $\Box A$ is already parameterised as $\eta_1$, the counting remains unaffected.

for gravity as we will require this in our analysis.

### 3.4.1 ADM formalism

One of the important concepts in GR is diffeomorphism invariance, *i.e.* when one transforms coordinates at given space-time points, the physics remains unchanged. As a result of this, one concludes that diffeomorphism is a local transformation. In Hamiltonian formalism, we have to specify the direction of time. A very useful approach to do this is ADM decomposition [121, 122], such decomposition permits to choose one specific time direction without violating the diffeomorphism invariance. In other words, choosing the time direction is nothing but gauge redundancy, or making sure that diffeomorphism is a local transformation. We assume that the manifold $\mathcal{M}$ is a time orientable space-time, which can be foliated by a family of space like hypersurfaces $\Sigma_t$, at which the time is fixed to be constant $t = x^0$. We then introduce an induced metric on the hypersurface as

$$h_{ij} \equiv g_{ij}|_t \,,$$

where the Latin indices run from 1 to 3 for spatial coordinates.

In $3+1$ formalism the line element is parameterised as,

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{3.4.55}$$

where $N$ is the *lapse function* and $N^i$ is the *shift vector*, given by

$$N = \frac{1}{\sqrt{-g^{00}}}, \qquad N^i = -\frac{g^{0i}}{g^{00}}\,. \tag{3.4.56}$$

In terms of metric variables, we then have

$$\begin{aligned} &g_{00} = -N^2 + h_{ij}N^iN^j, \qquad g_{0i} = N_i, \qquad g_{ij} = h_{ij}\,, \\ &g^{00} = -N^{-2}, \qquad g^{0i} = \frac{N^i}{N^2}, \qquad g^{ij} = h^{ij} - \frac{N^iN^j}{N^2}\,. \end{aligned} \tag{3.4.57}$$

Furthermore, we have a time like vector $n^\mu$ (*i.e.* the vector normal to the hypersurface) in Eq. (3.4.55), they take the following form:

$$n_i = 0, \quad n^i = -\frac{N^i}{N}, \quad n_0 = -N, \quad n^0 = N^{-1}\,. \tag{3.4.58}$$

From Eq. (3.4.55), we also have $\sqrt{-g} = N\sqrt{h}$. In addition, we are going to introduce a covariant derivative associated with the induced metric $h_{ij}$:

$$D_i \equiv e_i^\mu \nabla_\mu\,.$$

We will define the extrinsic curvature as:

$$K_{ij} = -\frac{1}{2N}\left(D_i N_j + D_j N_i - \partial_t h_{ij}\right)\,. \tag{3.4.59}$$

It is well known that the Riemannian curvatures can be written in terms of the 3+1 variables. In the case of scalar curvature we have [122]:

$$R = K_{ij}K^{ij} - K^2 + \mathcal{R} + \frac{2}{\sqrt{h}}\partial_\mu(\sqrt{h}n^\mu K) - \frac{2}{N\sqrt{h}}\partial_i(\sqrt{h}h^{ij}\partial_j N)\,, \tag{3.4.60}$$

where $K = h^{ij}K_{ij}$ is the trace of the extrinsic curvature, and $\mathcal{R}$ is scalar curvature calculated using the induced metric $h_{ij}$ [1].

One can calculate each term in (3.4.60) using the information about extrinsic curvature and those provided in (3.4.58). The decomposition of the d'Alembertian operator can be expressed as:

$$\begin{aligned}
\Box &= g^{\mu\nu}\nabla_\mu\nabla_\nu \\
&= (h^{\mu\nu} + \varepsilon n^\mu n^\nu)\nabla_\mu\nabla_\nu = (h^{ij}e_i^\mu e_j^\nu - n^\mu n^\nu)\nabla_\mu\nabla_\nu \\
&= h^{ij}D_iD_j - n^\nu\nabla_{\mathbf{n}}\nabla_\nu = \Box_{hyp} - n^\nu\nabla_{\mathbf{n}}\nabla_\nu\,,
\end{aligned} \tag{3.4.61}$$

[1] We note that the Greek indices are 4-dimensional while Latin indices are spatial and 3-dimensional.

where we have used the completeness relation for a space-like hypersurface, *i.e.* $\varepsilon = -1$, and we have also defined $\nabla_{\mathbf{n}} = n^{\mu}\nabla_{\mu}$.

### 3.4.2 ADM decomposition of IDG

Let us now take an IDG action. We will restrict ourselves to part of an IDG action which contains only the Ricci scalar,

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\Big[M_P^2 R + R\mathcal{F}(\bar{\Box})R\Big], \qquad \mathcal{F}(\bar{\Box}) = \sum_{n=0}^{\infty} f_n \bar{\Box}^n\,, \tag{3.4.62}$$

where $M_P$ is the 4-dimensional Planck scale, given by $M_P^2 = (8\pi G_N)^{-1}$, with $G_N$ is Newton's gravitational constant. The first term is Einstein Hilbert term, with $R$ being scalar curvature in four dimensions and the second term is the infinite derivative modification to the action, where $\bar{\Box} \equiv \Box/M^2$ , since $\Box$ has dimension mass squared and $\mathcal{F}(\bar{\Box})$ will be dimensionless. Note that $\Box$ is the 4-dimensional d'Alembertian operator given by $\Box = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$. Moreover, $f_n$ are the dimensionless coefficients of the series expansion.

Having the $3+1$ decomposition discussed in the earlier section, we rewrite our original action given in Eq.(3.4.62) in its equivalent form,

$$S_{eqv} = \frac{1}{2}\int d^4x\,\sqrt{-g}\Big[M_P^2 A + A\mathcal{F}(\bar{\Box})A + B(R-A)\Big]\,, \tag{3.4.63}$$

where we have introduced two scalar fields $A$ and $B$ with mass dimension two. Solving the equations of motion for scalar field $B$ results in $A = R$. The equations of motion for the original action, Eq.(3.4.62), are equivalent to the equations of motion for Eq. (3.4.63):

$$\delta S_{eqv} = \frac{1}{2}\delta\left\{\sqrt{-g}\Big[M_P^2 A + A\mathcal{F}(\bar{\Box})A + B(R-A)\Big]\right\} = 0 \Rightarrow R = A\,. \tag{3.4.64}$$

Following the steps of a scalar field theory, we expand $\mathcal{F}(\bar{\Box})A$,

$$\mathcal{F}(\bar{\Box})A = \sum_{n=0}^{\infty} f_n \bar{\Box}^n A = f_0 A + f_1 \bar{\Box} A + f_2 \bar{\Box}^2 A + f_3 \bar{\Box}^3 A + \cdots \tag{3.4.65}$$

As before, in order to eliminate $\bar{\Box}A,\ \bar{\Box}^2 A, \cdots$, we will introduce two new auxiliary fields $\chi_n$ and $\eta_n$ with the $\chi_n$'s being dimensionless and the $\eta_n$'s of mass dimension two.

- As an example, in order to eliminate $\bar{\Box}A$ in $A\bar{\Box}A$, we must

  add the following terms to Eq. (3.4.63):

$$\begin{aligned}
& \frac{1}{2}\int d^4x\sqrt{-g}\chi_1 A(\eta_1 - \bar{\Box}A) \\
= \ & \frac{1}{2}\int d^4x\sqrt{-g}[\chi_1 A\eta_1 - \chi_1 A\Box A] \\
= \ & \frac{1}{2}\int d^4x\sqrt{-g}[\chi_1 A\eta_1 - g^{\mu\nu}\chi_1 A\partial_\mu\partial_\nu A] \\
= \ & \frac{1}{2}\int d^4x\sqrt{-g}[\chi_1 A\eta_1 - g^{\mu\nu}\partial_\mu(\chi_1 A\partial_\nu A) \\
+ \ & g^{\mu\nu}\partial_\mu\chi_1 A\partial_\nu A + g^{\mu\nu}\chi_1\partial_\mu A\partial_\nu A] \\
= \ & \frac{1}{2}\int d^4x\sqrt{-g}[\chi_1 A\eta_1 + g^{\mu\nu}\partial_\mu\chi_1 A\partial_\nu A + g^{\mu\nu}\chi_1\partial_\mu A\partial_\nu A] \\
= \ & \frac{1}{2}\int d^4x\sqrt{-g}\Big[\chi_1 A\eta_1 + g^{\mu\nu}(\partial_\mu\chi_1 A\partial_\nu A + \chi_1\partial_\mu A\partial_\nu A)\Big]
\end{aligned} \tag{3.4.66}$$

  Solving the equation of motion for $\chi_1$; yields $\eta_1 = \bar{\Box}A$. In the about derivation we integrated by parts on the second step and have dropped the total derivative, also we have absorbed the factor of $M^{-2}$ into $\chi_1$(the mass dimension of $\eta_1$ is modified accordingly), hence, the d'Alembertian operator is not barred.

- For instance, in order to eliminate the term $A\bar{\Box}^2 A$, we add the term

$$\frac{1}{2}\int d^4x\ \chi_2 A(\eta_2 - \bar{\Box}\eta_1) = \frac{1}{2}\int d^4x\left[\chi_2 A\eta_2 + g^{\mu\nu}(\partial_\mu\chi_2 A\partial_\nu\eta_1 + \chi_2\partial_\mu A\partial_\nu\eta_1)\right]. \tag{3.4.67}$$

  Solving the equation of motion for $\chi_2$ yields; $\eta_2 = \bar{\Box}\eta_1 = \bar{\Box}^2 A$.

Similarly, in order to eliminate the terms $A\bar{\Box}^n A$ and so on, we have to repeat the same procedure up to $\bar{\Box}^n$. Again, we have shown that by solving the equations of motion for $\chi_n$, we obtain,

$$\eta_n = \bar{\Box}\eta_{n-1} = \bar{\Box}^n A, \quad for \quad n \geq 2.$$

Following the above steps, we can rewrite the action Eq. (3.4.63), as:

$$\begin{aligned}
S_{eqv} &= \frac{1}{2}\int d^4x\sqrt{-g}\Bigg\{A(M_P^2 + f_0 A + \sum_{n=1}^{\infty} f_n\eta_n) + B(R-A) + \chi_1 A(\eta_1 - \Box A) \\
&+ \sum_{l=2}^{\infty}\chi_l A(\eta_l - \Box\eta_{l-1})\Bigg\} \\
&= \frac{1}{2}\int d^4x\sqrt{-g}\Bigg\{A(M_P^2 + f_0 A + \sum_{n=1}^{\infty} f_n\eta_n) + B(R-A) \\
&+ g^{\mu\nu}(A\partial_\mu\chi_1\partial_\nu A + \chi_1\partial_\mu A\partial_\nu A) + g^{\mu\nu}\sum_{l=2}^{\infty}(A\partial_\mu\chi_l\partial_\nu\eta_{l-1} + \chi_l\partial_\mu A\partial_\nu\eta_{l-1}) \\
&+ \sum_{l=1}^{\infty} A\chi_l\eta_l\Bigg\},
\end{aligned} \tag{3.4.68}$$

where we have absorbed the powers of $M^{-2}$ into the $f_n$'s and $\chi_n$'s, and the mass dimension of $\eta_n$'s has been modified accordingly, hence, the box operator is not barred.

Note that the gravitational part of the action is simplified. In order to perform the ADM decomposition, let us first look at the $B(R-A)$ term, with the help of

Eq. (3.4.60) we can write:

$$B(R-A) = B\Big(K_{ij}K^{ij} - K^2 + \mathcal{R} - A\Big) - 2\nabla_{\mathbf{n}}BK - \frac{2}{\sqrt{h}}\partial_j(\partial_i(B)\sqrt{h}h^{ij}), \tag{3.4.69}$$

where from Eq. (3.4.60) we expanded the following terms,

$$\begin{aligned} & B\frac{2}{\sqrt{h}}\partial_\mu(\sqrt{h}n^\mu K) \\ &= \nabla_\mu\Big[B\frac{2}{\sqrt{h}}(\sqrt{h}n^\mu K)\Big] - (\nabla_\mu B)\frac{2}{\sqrt{h}}(\sqrt{h}n^\mu K) \\ &= \nabla_\mu\Big[2Bn^\mu K\Big] - 2(\nabla_\mu B)n^\mu K = -2n^\mu(\nabla_\mu B)K = -2\nabla_{\mathbf{n}}BK\,, \end{aligned} \tag{3.4.70}$$

and,

$$\begin{aligned} & -B\frac{2}{N\sqrt{h}}\partial_i(\sqrt{h}h^{ij}\partial_j N) \\ &= -\frac{2}{N\sqrt{h}}\partial_i(B(\sqrt{h}h^{ij}\partial_j N)) + \frac{2}{N\sqrt{h}}\partial_i(B)\sqrt{h}h^{ij}\partial_j N \\ &= \frac{2}{N\sqrt{h}}\partial_i(B)\sqrt{h}h^{ij}\partial_j N = \frac{2}{N\sqrt{h}}\partial_j(\partial_i(B)\sqrt{h}h^{ij}N) - \frac{2}{\sqrt{h}}\partial_j(\partial_i(B)\sqrt{h}h^{ij}) \\ &= -\frac{2}{\sqrt{h}}\partial_j(\partial_i(B)\sqrt{h}h^{ij})\,. \end{aligned} \tag{3.4.71}$$

Note that we have used $n^\mu\nabla_\mu \equiv \nabla_{\mathbf{n}}$ and dropped the total derivatives. Furthermore, we can use the decomposition of d'Alembertian operator, given in (3.4.61), and also in 3+1, we have $\sqrt{-g} = N\sqrt{h}$. Hence, the decomposition of the action

(3.4.68) becomes:

$$\begin{aligned} S_{eqv} &= \frac{1}{2}\int d^3x N\sqrt{h}\Bigg\{A(M_P^2 + f_0 A + \sum_{n=1}^{\infty} f_n\eta_n) + B\Big(K_{ij}K^{ij} - K^2 + \mathcal{R} - A\Big) \\ &- 2\nabla_{\mathbf{n}}BK - \frac{2}{\sqrt{h}}\partial_j(\partial_i(B)\sqrt{h}h^{ij}) \\ &+ h^{ij}(A\partial_i\chi_1\partial_j A + \chi_1\partial_i A\partial_j A) - (A\nabla_{\mathbf{n}}\chi_1\nabla_{\mathbf{n}}A + \chi_1\nabla_{\mathbf{n}}A\nabla_{\mathbf{n}}A) \\ &+ h^{ij}\sum_{l=2}^{\infty}(A\partial_i\chi_l\partial_j\eta_{l-1} + \chi_l\partial_i A\partial_j\eta_{l-1}) - \sum_{l=2}^{\infty}(A\nabla_{\mathbf{n}}\chi_l\nabla_{\mathbf{n}}\eta_{l-1} + \chi_l\nabla_{\mathbf{n}}A\nabla_{\mathbf{n}}\eta_{l-1}) \\ &+ \sum_{l=1}^{\infty}A\chi_l\eta_l\Bigg\}, \end{aligned} \tag{3.4.72}$$

where the Latin indices are spatial, and run from 1 to 3. Note that the $\chi$ fields were introduced to parameterise the contribution of $\bar{\Box}A$, $\bar{\Box}^2A$, $\cdots$, and so on. Therefore, $A$ and $\eta$ are auxiliary fields, which concludes that $\chi$ fields have no intrinsic value, and they are redundant. In other words they are Lagrange multiplier, when we count the number of phase space variables.

The same can not be concluded regarding the $B$ field, as it is introduced to obtain equivalence between scalar curvature, $R$, and $A$. Since $B$ field is coupled to $R$, and the scalar curvature is physical - we must count $B$ as a phase space variable. As we will see later in our Hamiltonian analysis, this is a crucial point while counting the number of physical degrees of freedom correctly. To summarize, as we will see, $B$ field is not a Lagrange multiplier, while $\chi$ fields are.

### 3.4.3 $f(R)$ gravity

Before proceeding further in our analysis and count the number of degrees of freedom for IDG, it is worth providing a well known example to test the machinery we build so far. To this end, let us consider the action for $f(R)$ gravity,

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R)\,, \tag{3.4.73}$$

where $f(R)$ is a function of scalar curvature and $\kappa = 8\pi G_N$. The equivalent action for above is then given by,

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big(f(A) + B(R-A)\Big)\,, \tag{3.4.74}$$

where again solving the equations of motion for $B$, one obtains $R = A$, and hence it is clear that above action is equivalent with Eq. (3.4.73). Using Eq. (3.4.69) we can decompose the action as,

$$S = \frac{1}{2\kappa}\int d^3x N\sqrt{h}\Big(f(A) + B\Big(K_{ij}K^{ij} - K^2 + \mathcal{R} - A\Big) - 2\nabla_{\mathbf{n}}BK \\ -\frac{2}{\sqrt{h}}\partial_j(\partial_i(B)\sqrt{h}h^{ij})\Big)\,. \tag{3.4.75}$$

Now that the above action is expressed in terms of $(h_{ab},\ N,\ N^i,\ B,\ A)$, and their time and space derivatives. We can proceed with the Hamiltonian analysis and write down the momentum conjugate for each of these variables:

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \sqrt{h}B(K^{ij} - h^{ij}K) - \sqrt{h}\nabla_{\mathbf{n}}Bh^{ij}, \qquad p_B = \frac{\partial\mathcal{L}}{\partial\dot{B}} = -2\sqrt{h}K, \\ p_A = \frac{\partial\mathcal{L}}{\partial\dot{A}} \approx 0, \qquad \pi_N = \frac{\partial\mathcal{L}}{\partial\dot{N}} \approx 0, \qquad \pi_i = \frac{\partial\mathcal{L}}{\partial\dot{N}^i} \approx 0\,. \tag{3.4.76}$$

where $\dot{A} \equiv \partial_0 A$ is the time derivative of the variable. We have used the "$\approx$" sign in Eq. (3.4.76) to show that $(p_A,\ \pi_N,\ \pi_i)$ are *primary constraints* satisfied on the constraint surface:

$$\Gamma_p = (p_A \approx 0,\ \pi_N \approx 0,\ \pi_i \approx 0).$$

$\Gamma_p$ is defined by the aforementioned *primary constraints*. For our purposes, whether the *primary constraints* vanish globally (which they do), *i.e.*, throughout the phase space, is irrelevant. Note that the Lagrangian density, $\mathcal{L}$, does not contain $\dot{A}$, $\dot{N}$ or $\dot{N}^i$, therefore, their conjugate momenta vanish identically.

We can define the Hamiltonian density as:

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} + p_B\dot{B} - \mathcal{L} \tag{3.4.77}$$

$$\equiv \; N\mathcal{H}_N + N^i\mathcal{H}_i\,, \tag{3.4.78}$$

where $\mathcal{H}_N = \dot{\pi}_N$, and $\mathcal{H}_i = \dot{\pi}_i$. By using Eq. (3.4.77), we can write

$$\begin{aligned}\mathcal{H}_N = &\frac{1}{\sqrt{h}B}\pi^{ij}h_{ik}h_{jl}\pi^{kl} - \frac{1}{3\sqrt{h}B}\pi^2 - \frac{\pi p_B}{3\sqrt{h}} + \frac{B}{6\sqrt{h}}p_B^2 \\ &-\sqrt{h}BR + \sqrt{h}BA + 2\partial_j[\sqrt{h}h^{ij}\partial_i]B + f(A)\,,\end{aligned} \tag{3.4.79}$$

and,

$$\mathcal{H}_i = -2h_{ik}\nabla_l\pi^{kl} + p_B\partial_i B\,. \tag{3.4.80}$$

Therefore, the total Hamiltonian can be written as,

$$H_{tot} = \int d^3x\,\mathcal{H} \tag{3.4.81}$$

$$= \int d^3x\left(N\mathcal{H}_N + N^i\mathcal{H}_i + \lambda^A p_A + \lambda^N\pi_N + \lambda^i\pi_i\right), \tag{3.4.82}$$

where $\lambda^A, \lambda^N, \lambda^i$ are Lagrange multipliers, and we have $G_A = \dot{p}_A$.

#### 3.4.3.1 Classification of constraints for $f(R)$ gravity

Having vanishing conjugate momenta means we can not express $\dot{A}$, $\dot{N}$ and $\dot{N}^i$ as a function of their conjugate momenta and hence $p_A \approx 0,\ \pi_N \approx 0$ and $\pi_i \approx 0$ are *primary constraints*, see (3.4.76). To ensure the consistency of the *primary constraints* so that they are preserved under time evolution generated by total Hamiltonian $H_{tot}$, we need to employ the Hamiltonian field equations and *enforce* that $\mathcal{H}_N$ and $\mathcal{H}_i$ be zero on the constraint surface $\Gamma_p$,

$$\dot{\pi}_N = -\frac{\delta\mathcal{H}_{tot}}{\delta N} = \mathcal{H}_N \approx 0, \qquad \dot{\pi}_i = -\frac{\delta\mathcal{H}_{tot}}{\delta N^i} = \mathcal{H}_i \approx 0\,, \tag{3.4.83}$$

such that $\mathcal{H}_N \approx 0$ and $H_i \approx 0$, and therefore they can be treated as *secondary constraints*.

Let us also note that $\Gamma_1$ is a smooth submanifold of the phase space determined by the *primary* and *secondary constraints*; hereafter in this section, we shall exclusively use the "$\approx$" notation to denote equality on $\Gamma_1$. It is usual to call $\mathcal{H}_N$ as the *Hamiltonian constraint*, and $\mathcal{H}_i$ as *diffeomorphism constraint*. Note that $\mathcal{H}_N$ and $\mathcal{H}_i$ are weakly vanishing *only* on the constraint surface; this is why the r.h.s of Eqs. (3.4.79) and (3.4.80) are not *identically* zero. If $\dot{\pi}_N = \mathcal{H}_N$ and $\dot{\pi}_i = \mathcal{H}_i$ were *identically* zero, then there would be no *secondary constraints*.

Furthermore, we are going to define $G_A$, and *demand* that $G_A$ be weakly zero on the constraint surface $\Gamma_1$,

$$G_A = \partial_t p_A = \{p_A, \mathcal{H}_{tot}\} = -\frac{\delta \mathcal{H}_{tot}}{\delta A} = -\sqrt{h} N (B + f'(A)) \approx 0\,, \tag{3.4.84}$$

which will act as a *secondary constraint* corresponding to *primary constraint* $p_A \approx 0$. Hence,

$$\Gamma_1 = (p_A \approx 0,\ \pi_N \approx 0,\ \pi_i \approx 0,\ G_A \approx 0,\ \mathcal{H}_N \approx 0,\ \mathcal{H}_i \approx 0).$$

Following the definition of Poisson bracket in Eq.(3.1.10), we can see that since the constraints $\mathcal{H}_N$ and $\mathcal{H}_i$ are preserved under time evolution, *i.e.*, $\dot{\mathcal{H}}_N = \{\mathcal{H}_N, \mathcal{H}_{tot}\}|_{\Gamma_1} = 0$ and $\dot{\mathcal{H}}_i = \{\mathcal{H}_i, \mathcal{H}_{tot}\}|_{\Gamma_1} = 0$, and they fix the Lagrange multipliers $\lambda^N$ and $\lambda^i$. That is, the expressions for $\dot{\mathcal{H}}_N$ and $\dot{\mathcal{H}}_i$ include the Lagrange multipliers $\lambda^N$ and $\lambda^i$; thus, we can solve the relations $\dot{\mathcal{H}}_N \approx 0$ and $\dot{\mathcal{H}}_i \approx 0$ for $\lambda^N$ and $\lambda^i$, respectively, and compute the values of the Lagrange multipliers. Therefore, we have no further constraints, such as *tertiary* ones and so on. We will check the same for $G_A$, that the time evolution of $G_A$ defined in the phase

space should also vanish on the constraint surface $\Gamma_1$,

$$\begin{aligned}
\dot{G}_A \equiv \{G_A, \mathcal{H}_{tot}\} &= \left(\frac{\delta G_A}{\delta N}\frac{\delta \mathcal{H}_{tot}}{\delta \pi_N} - \frac{\delta G_A}{\delta \pi_N}\frac{\delta \mathcal{H}_{tot}}{\delta N}\right) + \left(\frac{\delta G_A}{\delta N^i}\frac{\delta \mathcal{H}_{tot}}{\delta \pi_i} - \frac{\delta G_A}{\delta \pi_i}\frac{\delta \mathcal{H}_{tot}}{\delta N^i}\right) \\
&+ \left(\frac{\delta G_A}{\delta h_{ij}}\frac{\delta \mathcal{H}_{tot}}{\delta \pi^{ij}} - \frac{\delta G_A}{\delta \pi^{ij}}\frac{\delta \mathcal{H}_{tot}}{\delta h_{ij}}\right) + \left(\frac{\delta G_A}{\delta A}\frac{\delta \mathcal{H}_{tot}}{\delta p_A} - \frac{\delta G_A}{\delta p_A}\frac{\delta \mathcal{H}_{tot}}{\delta A}\right) \\
&+ \left(\frac{\delta G_A}{\delta B}\frac{\delta \mathcal{H}_{tot}}{\delta p_B} - \frac{\delta G_A}{\delta p_B}\frac{\delta \mathcal{H}_{tot}}{\delta B}\right) \\
&= \frac{\delta G_A}{\delta A}\frac{\delta \mathcal{H}_{tot}}{\delta p_A} + \frac{\delta G_A}{\delta B}\frac{\delta \mathcal{H}_{tot}}{\delta p_B} \\
&= N\left\{\frac{N}{3}\Big(2\pi - 2Bp_B\Big) - 2\sqrt{h}N^i\partial_i B - \sqrt{h}f''(A)\lambda^A\right\} \\
&\approx 0\,. \qquad (3.4.85)
\end{aligned}$$

The role of Eq. (3.4.85) is to fix the value of the Lagrange multiplier $\lambda^A$ as long as $f''(A) \neq 0$. We demand that $f''(A) \neq 0$ so as to avoid *tertiary constraints*. As a result, there are no *tertiary constraints* corresponding to $G_A$.

The next step in our Hamiltonian analysis is to classify the constraints. As shown above, we have 3 *primary constraints* for $f(R)$ theory. They are:

$$\pi_N \approx 0, \quad \pi_i \approx 0\,, \quad p_A \approx 0,$$

and there are three *secondary constraints*, that are:

$$\mathcal{H}_N \approx 0, \quad \mathcal{H}_i \approx 0, \quad G_A \approx 0.$$

Following the definition of Poisson bracket in Eq. (3.1.10), we have:

$$\begin{aligned}\{\pi_N,\pi_i\} &= \left(\frac{\delta\pi_N}{\delta N}\frac{\delta\pi_i}{\delta\pi_N}-\frac{\delta\pi_N}{\delta\pi_N}\frac{\delta\pi_i}{\delta N}\right)+\left(\frac{\delta\pi_N}{\delta N^i}\frac{\delta\pi_i}{\delta\pi_i}-\frac{\delta\pi_N}{\delta\pi_i}\frac{\delta\pi_i}{\delta N^i}\right)\\ &+ \left(\frac{\delta\pi_N}{\delta h_{ij}}\frac{\delta\pi_i}{\delta\pi^{ij}}-\frac{\delta\pi_N}{\delta\pi^{ij}}\frac{\delta\pi_i}{\delta h_{ij}}\right)+\left(\frac{\delta\pi_N}{\delta A}\frac{\delta\pi_i}{\delta p_A}-\frac{\delta\pi_N}{\delta p_A}\frac{\delta\pi_i}{\delta A}\right)\\ &+ \left(\frac{\delta\pi_N}{\delta B}\frac{\delta\pi_i}{\delta p_B}-\frac{\delta\pi_N}{\delta p_B}\frac{\delta\pi_i}{\delta B}\right)\approx 0\,. \end{aligned} \tag{3.4.86}$$

In a similar fashion, we can prove that:

$$\begin{aligned}&\{\pi_N,\pi_N\}=\{\pi_N,\pi_i\}=\{\pi_N,p_A\}=\{\pi_N,\mathcal{H}_N\}=\{\pi_N,\mathcal{H}_i\}=\{\pi_N,G_A\}\approx 0\\ &\{\pi_i,\pi_i\}=\{\pi_i,p_A\}=\{\pi_i,\mathcal{H}_N\}=\{\pi_i,\mathcal{H}_i\}=\{\pi_i,G_A\}\approx 0\\ &\{p_A,p_A\}=\{p_A,\mathcal{H}_N\}=\{p_A,\mathcal{H}_i\}\approx 0\\ &\{\mathcal{H}_N,\mathcal{H}_N\}=\{\mathcal{H}_N,\mathcal{H}_i\}=\{\mathcal{H}_N,G_A\}\approx 0\\ &\{\mathcal{H}_i,\mathcal{H}_i\}=\{\mathcal{H}_i,G_A\}\approx 0\\ &\{G_A,G_A\}\approx 0\,. \end{aligned} \tag{3.4.87}$$

The only non-vanishing Poisson bracket on $\Gamma_1$ is

$$\{p_A,G_A\}=-\frac{\delta p_A}{\delta p_A}\frac{\delta G_A}{\delta A}=-\frac{\delta G_A}{\delta A}=-\sqrt{h}Nf''(A)\not\approx 0\,. \tag{3.4.88}$$

Having $\{p_A,G_A\}\neq 0$ for $f''(A)\neq 0$ means that both $p_A$ and $G_A$ are *second-class constraints*. The rest of the constraints $(\pi_N,\pi_i,\mathcal{H}_N,\mathcal{H}_i)$ are to be counted as *first-class constraints*.

#### 3.4.3.2 Number of physical degrees of freedom in $f(R)$ gravity

Having identified the *primary and secondary constraints* and categorising them into *first and second-class constraints* [1], we can use the formula in (3.1.16) to

[1] Having *first-class and second-class constraints* means there are no arbitrary functions in the Hamiltonian. Indeed, a set of canonical variables that satisfies the constraint equations

count the number of the physical degrees of freedom. For $f(R)$ gravity, we have,

$$\begin{aligned} 2\mathcal{A} &= 2 \times \{(h_{ij}, \pi^{ij}), (N, \pi_N), (N^i, \pi_i), (A, p_A), (B, p_B)\} \\ &= 2(6+1+3+1+1) = 24, \\ \mathcal{B} &= (p_A, G_A) = (1+1) = 2, \\ 2\mathcal{C} &= 2 \times (\pi_N, \pi_i, \mathcal{H}_N, H_i) = 2(1+3+1+3) = 16, \\ \mathcal{N} &= \frac{1}{2}(24 - 2 - 16) = 3\,. \end{aligned} \tag{3.4.89}$$

Hence $f(R)$ gravity has 3 physical degrees of freedom in four dimensions; that includes the physical degrees of freedom for massless graviton and also an extra scalar degree of freedom [1].

Let us now briefly discuss few cases of interest:

- Number of degrees of freedom for $f(R) = R + \alpha R^2$:
  For a specific form of
  $$f(R) = R + \alpha R^2\,, \tag{3.4.90}$$
  where $\alpha = (6M^2)^{-1}$ to insure correct dimensionality. In this case we have,
  $$\{p_A, G_A\} = -\sqrt{h} N f''(A) = -2\sqrt{h} N \not\approx 0\,. \tag{3.4.91}$$
  The other Poisson brackets remain zero on the constraint surface $\Gamma_1$, and hence we are left with 3 physical degrees of freedom. These 3 degrees of are corresponding to the two polarised degrees of freedom for massless graviton, and one scalar mode. This extra degree of freedom which is a spin-0 particle is not a ghost and is non-tachyonic.

- Number of degrees of freedom for $f(R) = R$:
  For Einstein Hilbert action $f(R)$ is simply,
  $$f(R) = R\,, \tag{3.4.92}$$

determines the physical state.

[1] We may note that the Latin indices are running from 1 to 3 and are spatial. Moreover, $(h_{ij}, \pi^{ij})$ pair is symmetric therefore we get 6 from it.

for which,

$$\{p_A, G_A\} = -\sqrt{h}Nf''(A) \approx 0\,. \tag{3.4.93}$$

Therefore, in this case both $p_A$ and $G_A$ are *first-class constraints*. Hence, now our degrees of freedom counting formula in Eq. (3.1.16) takes the following form:

$$\begin{aligned}
2\mathcal{A} &= 2 \times \{(h_{ij}, \pi^{ij}), (N, \pi_N), (N^i, \pi_i), (A, p_A), (B, p_B)\} \\
&= 2(6+1+3+1+1) = 24, \\
\mathcal{B} &= 0, \\
2\mathcal{C} &= 2 \times (\pi_N, \pi_i, \mathcal{H}_N, H_i, p_A, G_A) = 2(1+3+1+3+1+1) = 20, \\
\mathcal{N} &= \frac{1}{2}(24 - 0 - 20) = 2\,,
\end{aligned} \tag{3.4.94}$$

which coincides with that of the spin-2 graviton as expected from the Einstein-Hilbert action.

### 3.4.4 Constraints for IDG

The action and the ADM decomposition of IDG has been explained explicitly so far. In this section, we will focus on the Hamiltonian analysis for the action of the form of Eq. (3.4.62). The first step is to consider Eq. (3.4.72), and obtain the conjugate momenta,

$$\begin{aligned}
&\pi_N = \frac{\partial \mathcal{L}}{\partial \dot{N}} \approx 0, \quad \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{N}^i} \approx 0, \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \sqrt{h}B(K^{ij} - h^{ij}K) - \sqrt{h}\nabla_{\mathbf{n}}Bh^{ij}, \\
&p_A = \frac{\partial \mathcal{L}}{\partial \dot{A}} = \sqrt{h}\Big[ -(A\nabla_{\mathbf{n}}\chi_1 + \chi_1\nabla_{\mathbf{n}}A) - \sum_{l=2}^{\infty}(\chi_l\nabla_{\mathbf{n}}\eta_{l-1})\Big], \quad p_B = \frac{\partial \mathcal{L}}{\partial \dot{B}} = -2\sqrt{h}K, \\
&p_{\chi_1} = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_1} = -\sqrt{h}A\nabla_{\mathbf{n}}A, \quad p_{\chi_l} = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_l} = -\sqrt{h}(A\nabla_{\mathbf{n}}\eta_{l-1}), \\
&p_{\eta_{l-1}} = \frac{\partial \mathcal{L}}{\partial \dot{\eta}_{l-1}} = -\sqrt{h}(A\nabla_{\mathbf{n}}\chi_l + \chi_l\nabla_{\mathbf{n}}A).
\end{aligned} \tag{3.4.95}$$

as we can see in this case, the time derivatives of the lapse, *i.e.* $\dot{N}$, and the shift function, $\dot{N}^i$, are absent. Therefore, we have two *primary constraints*,

$$\pi_N \approx 0, \quad \pi_i \approx 0\,. \tag{3.4.96}$$

The total Hamiltonian is given by:

$$H_{tot} = \int d^3x\, \mathcal{H} \tag{3.4.97}$$

$$= \int d^3x \left( N\mathcal{H}_N + N^i\mathcal{H}_i + \lambda^N\pi_N + \lambda^i\pi_i \right), \tag{3.4.98}$$

where $\lambda^N$and $\lambda^i$ are Lagrange multipliers and the Hamiltonian density is given by:

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} + p_A\dot{A} + p_B\dot{B} + p_{\chi_1}\dot{\chi}_1 + p_{\chi_l}\dot{\chi}_l + p_{\eta_{l-1}}\dot{\eta}_{l-1} - \mathcal{L} \tag{3.4.99}$$

$$= N\mathcal{H}_N + N^i\mathcal{H}_i\,, \tag{3.4.100}$$

using the above equation and after some algebra we have:

$$\begin{aligned}
\mathcal{H}_N = {} & \frac{1}{\sqrt{h}B}\pi^{ij}h_{ik}h_{jl}\pi^{kl} - \frac{1}{3\sqrt{h}B}\pi^2 - \frac{\pi p_B}{3\sqrt{h}} \\
& + \frac{B}{6\sqrt{h}}p_B^2 - \sqrt{h}BR + \sqrt{h}BA + 2\partial_j[\sqrt{h}h^{ij}\partial_i]B \\
& - \frac{1}{A\sqrt{h}}p_{\chi_1}(p_A - \frac{\chi_1}{A}p_{\chi_1}) - \frac{1}{A\sqrt{h}}\sum_{l=2}^{n} p_{\chi_l}(p_{\eta_{l-1}} - \frac{\chi_l}{A}p_{\chi_1}) \\
& - \sqrt{h}\sum_{l=1}^{n} A\chi_l\eta_l - \sqrt{h}\frac{1}{2}A(M_P^2 + f_0A + \sum_{n=1}^{\infty} f_n\eta_n) \\
& - \sqrt{h}h^{ij}(A\partial_i\chi_1\partial_jA + \chi_1\partial_iA\partial_jA) - \sqrt{h}h^{ij}\sum_{l=2}^{n}(A\partial_i\chi_l\partial_j\eta_{l-1} + \chi_l\partial_iA\partial_j\eta_{l-1})\,,
\end{aligned} \tag{3.4.101}$$

and,

$$\mathcal{H}_i = -2h_{ik}\nabla_l \pi^{kl} + p_A \partial_i A + p_{\chi_1}\partial_i \chi_1 + p_B \partial_i B + \sum_{l=2}^{n} (p_{\chi_l}\partial_i \chi_l + p_{\eta_{l-1}}\partial_i \eta_{l-1})\,. \tag{3.4.102}$$

As described before in Eq. (3.4.83), we can determine the *secondary constraints*, by:

$$\mathcal{H}_N \approx 0, \qquad \mathcal{H}_i \approx 0\,. \tag{3.4.103}$$

We can also show that, on the constraint surface $\Gamma_1$, the time evolutions $\dot{\mathcal{H}}_N = \{\mathcal{H}_N, \mathcal{H}_{tot}\} \approx 0$ and $\dot{\mathcal{H}}_i = \{\mathcal{H}_i, \mathcal{H}_{tot}\} \approx 0$ fix the Lagrange multipliers $\lambda^N$ and $\lambda^i$, and there will be no *tertiary constraints*.

#### 3.4.4.1 Classifications of constraints for IDG

As we have explained earlier, *primary and secondary constrains* can be classified into *first or second-class constraints*. This is derived by calculating the Poisson brackets constructed out of the constraints between themselves and each other. Vanishing Poisson brackets indicate *first-class constraint* and non vanishing Poisson bracket means we have *second-class constraint*.

For IDG action, we have two *primary constraints*: $\pi_N \approx 0$ and $\pi_i \approx 0$, and two *secondary constraints*: $\mathcal{H}_N \approx 0$, $H_i \approx 0$, therefore we can determine the classification of the constraints as:

$$\begin{aligned}
\{\pi_N, \pi_i\} &= \left(\frac{\delta\pi_N}{\delta N}\frac{\delta\pi_i}{\delta\pi_N} - \frac{\delta\pi_N}{\delta\pi_N}\frac{\delta\pi_i}{\delta N}\right) + \left(\frac{\delta\pi_N}{\delta N^i}\frac{\delta\pi_i}{\delta\pi_i} - \frac{\delta\pi_N}{\delta\pi_i}\frac{\delta\pi_i}{\delta N^i}\right) + \left(\frac{\delta\pi_N}{\delta h_{ij}}\frac{\delta\pi_i}{\delta\pi^{ij}} - \frac{\delta\pi_N}{\delta\pi^{ij}}\frac{\delta\pi_i}{\delta h_{ij}}\right) \\
&+ \left(\frac{\delta\pi_N}{\delta A}\frac{\delta\pi_i}{\delta p_A} - \frac{\delta\pi_N}{\delta p_A}\frac{\delta\pi_i}{\delta A}\right) + \left(\frac{\delta\pi_N}{\delta B}\frac{\delta\pi_i}{\delta p_B} - \frac{\delta\pi_N}{\delta p_B}\frac{\delta\pi_i}{\delta B}\right) + \left(\frac{\delta\pi_N}{\delta\chi_1}\frac{\delta\pi_i}{\delta p_{\chi_1}} - \frac{\delta\pi_N}{\delta p_{\chi_1}}\frac{\delta\pi_i}{\delta\chi_1}\right) \\
&+ \left(\frac{\delta\pi_N}{\delta\chi_l}\frac{\delta\pi_i}{\delta p_{\chi_l}} - \frac{\delta\pi_N}{\delta p_{\chi_l}}\frac{\delta\pi_i}{\delta\chi_l}\right) + \left(\frac{\delta\pi_N}{\delta\eta_{l-1}}\frac{\delta\pi_i}{\delta p_{\eta_{l-1}}} - \frac{\delta\pi_N}{\delta p_{\eta_{l-1}}}\frac{\delta\pi_i}{\delta\eta_{l-1}}\right) \approx 0\,.
\end{aligned} \tag{3.4.104}$$

In a similar manner, we can show that:

$$\begin{aligned}
&\{\pi_N, \pi_N\} = \{\pi_N, \pi_i\} = \{\pi_N, \mathcal{H}_N\} = \{\pi_N, \mathcal{H}_i\} \approx 0 \\
&\{\pi_i, \pi_i\} = \{\pi_i, \mathcal{H}_N\} = \{\pi_i, \mathcal{H}_i\} \approx 0 \\
&\{\mathcal{H}_N, \mathcal{H}_N\} = \{\mathcal{H}_N, \mathcal{H}_i\} \approx 0 \\
&\{\mathcal{H}_i, \mathcal{H}_i\} \approx 0\,.
\end{aligned} \tag{3.4.105}$$

Therefore, all of them $(\pi_N, \pi_i, \mathcal{H}_N, \mathcal{H}_i)$ are *first-class constraints*. We can established that by solving the equations of motion for $\chi_n$ yields

$$\eta_1 = \Box A, \quad \cdots, \quad \eta_l = \Box \eta_{l-1} = \Box^l A,$$

for $l \geq 2$. Therefore, we can conclude that the $\chi_n$'s are Lagrange multipliers, and we get the following *primary constraints* from equations of motion,

$$\begin{aligned}
\Xi_1 &= \eta_1 - \Box A = 0\,, \\
\Xi_l &= \eta_l - \Box \eta_{l-1} = 0\,,
\end{aligned} \tag{3.4.106}$$

where $l \geq 2$. In fact, it is sufficient to say that $\eta_1 - \Box A \approx 0$ and $\eta_l - \Box \eta_{l-1} \approx 0$ on a constraint surface spanned by *primary and secondary constraints*, *i.e.*, $(\pi_N \approx 0,\ \pi_i \approx 0,\ \mathcal{H}_N \approx 0,\ \mathcal{H}_i \approx 0,\ \Xi_n \approx 0)$. As a result, we can now show,

$$\{\Xi_n, \pi_N\} = \{\Xi_n, \pi_i\} = \{\Xi_n, \mathcal{H}_N\} = \{\Xi_n, \mathcal{H}_i\} = \{\Xi_m, \Xi_n\} \approx 0\,; \tag{3.4.107}$$

where we have used the notation $\approx$, which is a sufficient condition to be satisfied on the constraint surface defined by $\Gamma_1 = (\pi_N \approx 0,\ \pi_i \approx 0,\ \mathcal{H}_N \approx 0,\ \mathcal{H}_i \approx 0,\ \Xi_n \approx 0)$, which signifies that $\Xi_n$'s are now part of *first-class constraints*. We should point out that we have checked that the Poisson brackets of all possible pairs among the constraints vanish on the constraint surface $\Gamma_1$; as a result, there are no *second-class constraints*.

#### 3.4.4.2 Physical degrees of freedom for IDG

We can again use (3.1.16) to compute the degrees of freedom for IDG action (3.4.62). First, let us establish the number of the configuration space variables, $\mathcal{A}$. Since the auxiliary field $\chi_n$ are Lagrange multipliers, they are not dynamical and hence redundant, as we have mentioned earlier. In contrast we have to count the $(B,\ p_b)$ pair in the phase space as $B$ contains intrinsic value. For the IDG action Eq. (3.4.62), we have:

$$\begin{aligned} 2\mathcal{A} &\equiv 2\times\left\{(h_{ij},\pi^{ij}),(N,\pi_N),(N^i,\pi_i),(B,p_b),(A,p_A),\underbrace{(\eta_1,p_{\eta_1}),(\eta_2,p_{\eta_2}),\cdots}_{n=1,\ 2,\ 3,\cdots\infty}\right\} \\ &\equiv 2\times(6+1+3+1+1+\infty)=24+\infty\,, \end{aligned} \tag{3.4.108}$$

we have $(\eta_n,\ p_{\eta_n})$ and for each pair we have assigned one variable, which is multiplied by a factor of 2 since we are dealing with field-conjugate momentum pairs in the phase space. Moreover, as we have found from the Poisson brackets of all possible pairs among the constraints, the number of the *second-class constraints*, $\mathcal{B}$, is equal to zero. In the next sub-sections, we will show that the *correct* number of the *first-class constraints* depends on the choice of $\mathcal{F}(\bar{\Box})$.

### 3.4.5 Choice of $\mathcal{F}(\bar{\Box})$

In this sub-section, we will focus on an appropriate choice of $\mathcal{F}(\Box)$ for the action Eq.(3.4.62), such that the theory admits finite relevant degrees of freedom (this is what we want to satisfy when choosing the form of $\mathcal{F}(\Box)$). From the Lagrangian point of view, we could analyse the propagator of the action Eq.(3.4.62). It was found in Refs. [53, 68] that $\mathcal{F}(\bar{\Box})$ can take the following form,

$$\mathcal{F}(\bar{\Box})=M_P^2\frac{c(\bar{\Box})-1}{\Box}\,. \tag{3.4.109}$$

The choice of $c(\bar{\Box})$ determines how many roots we have and how many poles are present in the graviton propagator, see Refs. [53, 68, 93]. Here, we will consider two choices of $c(\bar{\Box})$, one which has infinitely many roots, and therefore infinite

poles in the propagator. For instance, we can choose

$$c(\bar{\Box}) = \cos(\bar{\Box})\,, \tag{3.4.110}$$

then the equivalent action would be written as:

$$S_{eqv} = \frac{1}{2}\int d^4x\,\sqrt{-g}\Big[M_P^2\Big(A + A\big(\frac{\cos(\bar{\Box})-1}{\Box}\big)A\Big) + B(R-A)\Big]\,. \tag{3.4.111}$$

By solving the equations of motion for $A$, and subsequently solving for $\cos(\bar{\Box})$ we get,

$$\cos(\bar{k}^2) = 1 - \frac{k^2(BM_P^{-2}-1)}{2A}\,, \tag{3.4.112}$$

where in the momentum space, we have ($\Box \to -k^2$, around Minkowski space), and also note $\bar{k} \equiv k/M$; where $B$ has mass dimension 2. From (F.0.10) in appendix F, we have that

$$B = M_P^2\left(1 + \frac{4A}{3k^2}\right)\,. \tag{3.4.113}$$

Therefore, solving $\cos(\bar{k}^2) = \frac{1}{3}$, we obtain infinitely many solutions. We observe that there is an infinite number of solutions; hence, there are also infinitely many degrees of freedom. These, infinitely many solutions can be written schematically as:

$$\begin{aligned}
&\Psi_1 = \Box A + a_1 A = 0\,,\\
&\Psi_2 = \Box A + a_2 A = 0\,,\\
&\Psi_3 = \Box A + a_3 A = 0\,,\\
&\vdots
\end{aligned} \tag{3.4.114}$$

or, in the momentum space,

$$\begin{aligned} -Ak^2 + Aa_1 = 0 \Rightarrow k^2 = a_1\,, \\ -Ak^2 + Aa_2 = 0 \Rightarrow k^2 = a_2\,, \\ -Ak^2 + Aa_3 = 0 \Rightarrow k^2 = a_3\,, \\ \vdots \end{aligned} \tag{3.4.115}$$

Now, acting the $\Box$ operators on Eq. (3.4.114), we can write

$$\begin{aligned} \Box\Psi_2 &= \Box^2 A + a_2\Box A\,, \\ \Box^2\Psi_3 &= \Box^3 A + a_3\Box^2 A\,, \\ &\vdots \\ \Box^{n-1}\Psi_n &= \Box^n A + a_n\Box^{n-1} A\,, \\ &\vdots \end{aligned} \tag{3.4.116}$$

As we saw earlier it is possible to parameterize the terms of the form $\Box A$, $\Box^2 A$, etc, by employing the auxiliary fields $\chi_l$, $\eta_l$, for $l \geq 1$. Therefore, we can write the solutions $\Psi_n$ as follows:

$$\begin{aligned} \Psi_1^{'} &= \eta_1 + a_1 A = 0\,, \\ \Psi_2^{'} &= \eta_2 + a_2\eta_1 = 0\,, \\ \Psi_3^{'} &= \eta_3 + a_3\eta_2 = 0\,. \\ &\vdots \end{aligned} \tag{3.4.117}$$

We should point out that we have acted the operator $\Box$ on $\Psi_2$, the operator $\Box^2$ on $\Psi_3$, etc. in order to obtain $\Psi_2^{'}$, $\Psi_3^{'}$, etc. As a result, we can rewrite the term $A + M_P^{-2} A\mathcal{F}(\bar{\Box})A$, as

$$A + M_P^{-2} A\mathcal{F}(\bar{\Box})A = a_0\Psi_1^{'}\prod_{n=2}^{\infty}\Box^{-n+1}\Psi_n^{'}\,. \tag{3.4.118}$$

We would also require $\phi_n$ auxiliary fields acting like Lagrange multipliers. Now, absorbing the powers of $M^{-2}$ into the coefficients where appropriate,

$$S_{eqv} = \frac{1}{2}\int d^4x\,\sqrt{-g}\Big[M_P^2 a_0\prod_{n=1}^{\infty}\psi_n + B(R-A) + \chi_1 A(\eta_1 - \Box A) \\ + \sum_{l=2}^{\infty}\chi_l A(\eta_l - \Box\eta_{l-1}) + \phi_1(\psi_1 - \Psi_1^{'}) + \sum_{n=2}^{\infty}\phi_n\big(\psi_n - \Box^{-n+1}\Psi_n^{'}\big)\Big]\,, \tag{3.4.119}$$

where $a_0$ is a constant and, let us define $\Phi_1 = \psi_1 - \Psi_1^{'}$ and, for $n \geq 2$, $\Phi_n = \psi_n - \Box^{-n+1}\Psi_n^{'}$. Then the equations of motion for $\phi_n$ will yield:

$$\Phi_n = \psi_n - \Box^{-n+1}\Psi_n^{'} = 0\,. \tag{3.4.120}$$

Again, it is sufficient to replace $\psi_n - \Box^{-n+1}\Psi_n^{'} = 0$ with $\psi_n - \Box^{-n+1}\Psi_n^{'} \approx 0$ satisfied at the constraint surface. As a result there are $n$ *primary constraints* in $\Phi_n$. Moreover, by taking the equations of motion for $\chi_n$'s and $\phi_n$'s simultaneously, we will obtain the *original* action, see Eq. (3.4.111). The time evolutions of the $\Xi_n$'s & $\Phi_n$'s fix the corresponding Lagrange multipliers $\lambda^{\Xi_n}$ & $\lambda^{\Phi_n}$ in the total Hamiltonian (when we add the terms $\lambda^{\Xi_n}\Xi_n$ & $\lambda^{\Phi_n}\Phi_n$ to the integrand in (3.4.98)); hence, the $\Xi_n$'s & $\Phi_n$'s do not induce *secondary constraints*.

Now, to classify these constraints, we can show that the following Poisson brackets involving $\Phi_n$ on the constraint surface ($\pi_N \approx 0, \pi_i \approx 0, \mathcal{H}_N \approx 0, \mathcal{H}_i \approx 0, \Xi_n \approx 0, \Phi_n \approx 0$) are satisfied[1]:

$$\{\Phi_n, \pi_N\} = \{\Phi_n, \pi_i\} = \{\Phi_n, \mathcal{H}_N\} = \{\Phi_n, \mathcal{H}_i\} = \{\Phi_m, \Xi_n\} = \{\Phi_m, \Phi_n\} \approx 0\,, \tag{3.4.121}$$

which means that the $\Phi_n$'s can be treated as *first-class constraints*. We should point out that we have checked that the Poisson brackets of all possible pairs among the constraints vanish on the constraint surface $\Gamma_1$; as a result, there are

[1]Let us note again that $\Gamma_1$ is a smooth submanifold of the phase space determined by the *primary* and *secondary constraints*; hereafter in this section, we shall exclusively use the "$\approx$" notation to denote equality on $\Gamma_1$.

no *second-class constraints*. Now, from Eq. (3.4.108), we obtain:

$$
\begin{aligned}
&2\mathcal{A} \equiv 2 \times \Big\{(h_{ij},\pi^{ij}),(N,\pi_N),(N^i,\pi_i),(B,p_b),(A,p_A),\underbrace{(\eta_1,p_{\eta_1}),(\eta_2,p_{\eta_2}),\cdots}_{n}\Big\}\\
&= 2\times(6+1+3+1+1+\infty) = 24+\infty\\
&\mathcal{B}=0,\\
&2\mathcal{C} \equiv 2\times(\pi_N,\pi_i,\mathcal{H}_N,\mathcal{H}_i,\Xi_n,\Phi_n) = 2(1+3+1+3+\infty+\infty) = 16+\infty+\infty,\\
&\mathcal{N} = \frac{1}{2}(2\mathcal{A}-\mathcal{B}-2\mathcal{C}) = \infty\,.
\end{aligned}
\tag{3.4.122}
$$

As we can see a injudicious choice for $\mathcal{F}(\bar{\Box})$ can lead to infinite number of degrees of freedom., and there are many such examples. However, our aim is to come up with a concrete example where IDG will be determined solely by massless graviton and at best one massive scalar in the context of Eq. (3.4.62).

### 3.4.6 $\mathcal{F}(e^{\bar{\Box}})$ and finite degrees of freedom

In the definition of $\mathcal{F}(\bar{\Box})$ as given in Eq. (3.4.109), if

$$
c(\bar{\Box}) = e^{-\gamma(\bar{\Box})}, \tag{3.4.123}
$$

where $\gamma(\bar{\Box})$ is an entire function, we can decompose the propagator into partial fractions and have just one extra pole apart from the spin-2 graviton. Consequently, in order to have just one extra degree of freedom, we have to impose conditions on the coefficient in $\mathcal{F}(\bar{\Box})$ series expansion (The reader may also consult Appendix G). Moreover, to avoid $\Box^{-1}$ terms appearing in the $\mathcal{F}(\bar{\Box})$, we must have that,

$$
c(\bar{\Box}) = \sum_{n=0}^{\infty} c_n \bar{\Box}^n\,, \tag{3.4.124}
$$

with the first coefficient $c_0 = 1$, therefore:

$$
\mathcal{F}(\bar{\Box}) = \Big(\frac{M_P}{M}\Big)^2 \sum_{n=0}^{\infty} c_{n+1}\bar{\Box}^n\,, \tag{3.4.125}
$$

Suppose we have $c(\bar{\Box}) = e^{-\bar{\Box}}$, then using Eq. (3.4.109) we have,

$$\mathcal{F}(\bar{\Box}) = \sum_{n=0}^{\infty} f_n \bar{\Box}^n \,, \tag{3.4.126}$$

where the coefficient $f_n$ has the form of,

$$f_n = \Big(\frac{M_P}{M}\Big)^2 \frac{(-1)^{n+1}}{(n+1)!} \,, \tag{3.4.127}$$

Indeed this particular choice of $c(\bar{\Box})$ is very well motivated from string field theory [53]. In fact the above choice of $\gamma(\bar{\Box}) = -\bar{\Box}$ contains at most one extra zero in the propagator corresponding to one extra scalar mode in the spin-0 component of the graviton propagator [68, 93]. We rewrite the action as:

$$S_{eqv} = \frac{1}{2}\int d^4x\,\sqrt{-g}\Big[M_P^2\Big(A + A\big(\frac{e^{-\bar{\Box}}-1}{\Box}\big)A\Big) + B(R-A)\Big] \,. \tag{3.4.128}$$

The equation of motion for $A$ is then:

$$M_P^2\Big(1 + 2\big(\frac{e^{-\bar{\Box}}-1}{\Box}\big)A\Big) - B = 0 \,. \tag{3.4.129}$$

In momentum space, we can solve the equation above:

$$e^{\bar{k}^2} = 1 - \frac{k^2(BM_P^{-2}-1)}{2A} \,, \tag{3.4.130}$$

where in the momentum space $\Box \to -k^2$ (on Minkowski space-time) and also $\bar{k} \equiv k/M$. From Eq. (F.0.15) in the appendix F, we have, $e^{\bar{k}^2} = \frac{1}{3}$, therefore solving Eq. (3.4.130), we obtain

$$B = M_P^2\left(1 + \frac{4A}{3k^2}\right) \,. \tag{3.4.131}$$

Note that we obtain only one extra solution (apart from the one for the massless spin-2 graviton). We observe that there is a finite number of real solutions; hence, there are also finitely many degrees of freedom. The form of the solution can be written schematically, as:

$$\Omega = \Box A + b_1 A = 0\,, \tag{3.4.132}$$

or, in the momentum space,

$$-Ak^2 + Ab_1 = 0 \Rightarrow k^2 = b_1\,, \tag{3.4.133}$$

Now, we can parameterize the terms like $\Box A$, $\Box^2 A$, etc. with the help of auxiliary fields $\chi_l$ and $\eta_l$, for $l \geq 1$. Therefore, equivalently,

$$\Omega^{'} = \eta_1 + b_1 A = 0\,. \tag{3.4.134}$$

Consequently, we can also rewrite the term $A\mathcal{F}(\bar{\Box})A$ with the help of auxiliary fields $\rho$ and $\omega$. Upon taking the equations of motion for the field $\rho$, one can recast $A + M_P^{-2} A\mathcal{F}(\bar{\Box})A = b_0\omega\ \mathcal{G}(A, \eta_1, \eta_2, \dots)$. Hence, we can recast the action, Eq. (3.4.128), as,

$$\begin{aligned} S_{eqv} = \frac{1}{2}\int d^4x\,\sqrt{-g}\Big[ & M_P^2 b_0\omega\ \mathcal{G}(A, \eta_1, \eta_2, \dots) + B(R - A) + \chi_1 A(\eta_1 - \Box A) \\ & + \sum_{l=2}^{\infty} \chi_l A(\eta_l - \Box\eta_{l-1}) + \rho\big(\omega - \Omega^{'}\big)\Big]\,, \end{aligned} \tag{3.4.135}$$

where $b_0$ is a constant, and we can now take $\rho$ as a Lagrange multiplier. The equation of motion for $\rho$ will yield:

$$\Theta = \omega - \Omega^{'} = 0\,. \tag{3.4.136}$$

Note that $\Theta = \omega - \Omega^{'} \approx 0$ will suffice on the constraint surface determined by *primary and secondary constraints* ($\pi_N \approx 0, \pi_i \approx 0, \mathcal{H}_N \approx 0, \mathcal{H}_i \approx 0, \Xi_n \approx 0, \Theta \approx 0$). As a result, $\Theta$ is a *primary constraint*. The time evolutions of the $\Xi_n$'s & $\Theta$ fix the corresponding Lagrange multipliers $\lambda^{\Xi_n}$ & $\lambda^{\Theta}$ in the total Hamiltonian

(when we add the terms $\lambda^{\Xi_n}\Xi_n$ & $\lambda^{\Theta}\Theta$ to the integrand in (3.4.98)); hence, the $\Xi_n$'s & $\Theta$ do not induce *secondary constraints*.

Furthermore, the function $\mathcal{G}(A, \eta_1, \eta_2, \dots)$ contains the root corresponding to the massless spin-2 graviton. Furthermore, taking the equations of motion for $\chi_n$'s and $\rho$ simultaneously yields the same equation of motion as that of in Eq. (3.4.128). The Poisson bracket of $\Theta$ with other constraints will give rise to

$$\{\Theta, \pi_N\} = \{\Theta, \pi_i\} = \{\Theta, \mathcal{H}_N\} = \{\Theta, \mathcal{H}_i\} = \{\Theta, \Xi_n\} = \{\Theta, \Theta\} \approx 0\,, \quad (3.4.137)$$

where $\approx$ would have been sufficient. This leads to $\Theta$ as a *first-class constraint*. Hence, we can calculate the number of the physical degrees of freedom as:

$$\begin{aligned}
2\mathcal{A} &\equiv 2 \times \left\{(h_{ij}, \pi^{ij}), (N, \pi_N), (N^i, \pi_i), (B, p_b), (A, p_A), \underbrace{(\eta_1, p_{\eta_1}), (\eta_2, p_{\eta_2}), \cdots}_{n}\right\} \\
&= 2 \times (6 + 1 + 3 + 1 + 1 + \infty) = 24 + \infty \\
\mathcal{B} &= 0, \\
2\mathcal{C} &\equiv 2 \times (\pi_N, \pi_i, \mathcal{H}_N, \mathcal{H}_i, \Xi_n, \Theta) = 2(1 + 3 + 1 + 3 + \infty + 1) = 18 + \infty, \\
\mathcal{N} &= \frac{1}{2}(2\mathcal{A} - \mathcal{B} - 2\mathcal{C}) = \frac{1}{2}(24 + \infty - 0 - 18 - \infty) = 3\,. \qquad (3.4.138)
\end{aligned}$$

This gives 2 degrees of freedom from the massless spin-2 graviton in addition to an extra degree of freedom as expected from the propagator analysis. This extra degree of freedom which is not a ghost or tachyon comes from the choice of $\mathcal{F}(\bar{\Box})$, in fact we have that $\mathcal{F}(\bar{\Box}) = M_P^2 \frac{e^{-\bar{\Box}}-1}{\Box}$ and this function admits one pole and hence one degree of freedom. This is shown explicitly on Appendix F. Moreover, note that the choice of $\mathcal{F}(\bar{\Box})$ is such that we avoid $\Box^{-1}$ terms.

## 3.5 Summary

In this chapter we used Hamiltonian analysis to study the number of the degrees of freedom for an infinite derivative theory of gravity (IDG). In this gravitational modification, IDG contains infinite number of covariant derivatives acting on the Ricci scalar.

In Lagrangian framework, the number of the degrees of freedom is determined from the propagator analysis. Particularly, it hinges on the number of the poles arising in the propagator. The results of this chapter support the original idea that both Lagrangian and Hamiltonian analysis will yield similar conclusions for infinite derivative theories with Gaussian kinetic term [53]. In case of IDG, one can study the scalar and the tensor components of the propagating degrees of freedom [68, 93], and for Gaussian kinetic term which determines $\mathcal{F}(\bar{\Box})$, there are only 2 dynamical degrees of freedom. In order to make sure that there are no poles other than the original poles (corresponding to the original degrees of freedom ) in the propagator, one shall demand that the propagator be suppressed by *exponential of an entire function*. An entire function does not have any poles in the finite complex plane This choice of propagator determines the kinetic term in Lagrangian for infinite derivative theories. For a scalar toy model the kinetic term becomes Gaussian, *i.e.*, $\mathcal{F} = \Box e^{-\bar{\Box}}$, while in gravity it becomes $\mathcal{F} = M_P^2 \Box^{-1}(e^{-\bar{\Box}} - 1)$.

From the Hamiltonian perspective, the essence of finding the dynamical degrees of freedom relies primarily on finding the total configuration space variables, and *first* and *second-class constraints*. As expected, infinite derivative theories will have infinitely many configuration space variables, and so will be *first* and *second-class constraints*. However, for a Gaussian kinetic term, $\mathcal{F} = M_P^2 \Box^{-1}(e^{-\bar{\Box}} - 1)$, the degrees of freedom are finite and the gravitational action we considered admits 3 relevant degrees of freedom, two for the massless graviton and an extra scalar mode that is not ghost or tachyonic.

# Chapter 4

# Boundary terms for higher derivative theories of gravity

In this chapter we wish to find the corresponding Gibbons-Hawking-York term for the most general quadratic in curvature gravity by using Coframe slicing within the ADM decomposition of space-time in four dimensions.

Irrespective of classical or quantum computations, one of the key features of a covariant action is to have a well-posed boundary condition. In particular, in the Euclidean path integral approach - requiring such an action to be stationary, one also requires all the boundary terms to disappear on any permitted variation. Another importance of boundary terms manifests itself in calculating the black hole entropy using the Euclidean semiclassical approach, where the entire contribution comes from the boundary term [104, 132, 133].

It is well known that the variation of the EH action leads to a boundary term that depends not just on the metric, but also on the derivatives of the metric. This is due to the fact that the action itself depends on the metric, along with terms that depend linearly on the second derivatives. Normally, in Lagrangian field theory, such linear second derivative terms can be introduced or eliminated, by adding an appropriate boundary term to the action. In gravity, the fact that the second derivatives arise linearly and also the existence of total derivative indicates that the second derivatives are redundant in the sense that they can

be eliminated by integrating by parts, or by adding an appropriate boundary term. Indeed, writing a boundary term for a gravitational action schematically confines the non-covariant terms to the boundary. For the EH action this geometrically transparent, boundary term is given by the Gibbons-Hawking-York (GHY) boundary term [102]. Adding this boundary to the bulk action results in an elimination of the total derivative, as seen for $f(R)$ gravity [134, 135].

In the Hamiltonian formalism, obtaining the boundary terms for a gravitational action is vital. This is due to the fact that the boundary term ensures that the path integral for quantum gravity admits correct answers. As a result, ADM showed [121] that upon decomposing space-time such that for the four dimensional Einstein equation we have three-dimensional surfaces (later to be defined as hypersurfaces) and one fixed time coordinate for each slices. We can therefore formulate and recast the Einstein equations in terms of the Hamiltonian and hence achieve a better insight into GR.

In the ADM decomposition, one foliates the arbitrary region $\mathcal{M}$ of the space-time manifold with a family of spacelike hypersurfaces $\Sigma_t$, one for each instant in time. It has been shown by the authors of [136] that one can decompose a gravitational action, using the ADM formalism and without necessarily moving into the Hamiltonian regime, such that we obtain the total derivative of the gravitational action. Using this powerful technique, one can eliminate this total derivative term by modifying the GHY term appropriately.

The aim of this chapter is to find the corresponding GHY boundary term for a covariant IDG. We start by providing a warm up example on how to obtain a boundary term for an infinite derivative, massless scalar field theory. We then briefly review the boundary term for EH term and introduce infinite derivative gravity. We shall set our preliminaries by discussing the time slicing and reviewing how one may obtain the boundary terms by using the $3+1$ formalism. We finally turn our attention to our gravitational action and find the appropriate boundary terms for such theory.

## 4.1 Warming up: Infinite derivative massless scalar field theory

Let us consider the following action of a generic scalar field $\phi$ of mass dimension 2:

$$S_\phi = \int d^4x\, \phi \Box^n \phi, \tag{4.1.1}$$

where $\Box = \eta^{\mu\nu}\nabla_\mu\nabla_\nu$, where $\eta_{\mu\nu}$ is the Minkowski metric [1] and $n \in \mathbb{N}_{>0}$. Generalising, we have that $\Box^n = \prod_{i=1}^n \eta^{\mu_i\nu_i}\nabla_{\mu_i}\nabla_{\nu_i}$. The aim is to find the total derivative term for the above action. We may vary the scalar field $\phi$ as: $\phi \to \phi + \delta\phi$. Then the variation of the action is given by

$$\begin{aligned} \delta S_\phi &= \int d^4x \left[\delta\phi\Box^n\phi + \phi\delta(\Box^n\phi)\right], \\ &= \int d^4x \left[\delta\phi\Box^n\phi + \phi\Box^n\delta\phi\right], \\ &= \int d^4x \left[(2\Box^n\phi)\delta\phi + X\right], \end{aligned} \tag{4.1.2}$$

where now $X$ are the $2n$ total derivatives:

$$\begin{aligned} X &= \int d^4x \left[\nabla_\mu(\phi\nabla^\mu\Box^{n-1}\delta\phi) - \nabla^\mu(\nabla_\mu\phi\Box^{n-1}\delta\phi)\right. \\ &+ \nabla_\lambda(\Box\phi\nabla^\lambda\Box^{n-2}\delta\phi) - \nabla^\lambda(\nabla_\lambda\phi\Box^{n-2}\delta\phi) + \cdots \\ &+ \left.\nabla_\sigma(\Box^{n-1}\phi\nabla^\sigma\delta\phi) - \nabla^\sigma(\nabla_\sigma\Box^{n-1}\phi\delta\phi)\right]. \end{aligned} \tag{4.1.3}$$

where "$\cdots$" in the above equation indicates the intermediate terms.

Let us now consider a more general case

$$S_\phi = \int d^4x\, \phi\mathcal{F}(\Box)\phi, \tag{4.1.4}$$

[1] The $\Box$ term comes with a scale $\Box/M^2$. In our notation, we suppress the scale $M$ in order not to clutter our formulae for the rest of this chapter.

where $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} c_n \Box^n$, where the '$c_n$'s are dimensionless coefficients. In this case the total derivatives are given by

$$X = \sum_{n=1}^{\infty} c_n \int d^4x \sum_{j=1}^{2n} (-1)^{j-1} \nabla_\mu (\nabla^{(j-1)} \phi \nabla^{(2n-j)} \delta\phi), \tag{4.1.5}$$

where the superscript $\nabla^{(j)}$ indicate the number of covariant derivatives acting to the right. Therefore, one can always determine the total derivative for any given action, and one can then preserve or eliminate these terms depending on the purpose of the study. In the following sections we wish to address how one can obtain the total derivative for a given gravitational action.

## 4.2 Introducing Infinite Derivative Gravity

The gravitational action is built up of two main components, the bulk part and the boundary part. In the simplest and the most well known case [102], for the EH action, the boundary term are the ones known as Gibbons-Hawking-York (GHY) term. We can write the total EH action in terms of the bulk part and the boundary part simply as, (See Appendix C for derivation),

$$\begin{aligned} S_G &= S_{EH} + S_B \\ &= \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^4x \sqrt{-g}\mathcal{R} + \frac{1}{8\pi G_N} \oint_{\partial\mathcal{M}} d^3y \, \varepsilon \, |h|^{1/2} \, K \,, \end{aligned} \tag{4.2.6}$$

where $\mathcal{R}$ is the Ricci-scalar, and $K$ is the trace of the extrinsic curvature with $K_{ij} \equiv -\nabla_i n_j$, $\mathcal{M}$ indicates the 4-dimensional region and $\partial\mathcal{M}$ denotes the 3-dimensional boundary region. $h$ is the determinant of the induced metric on the hypersurface $\partial\mathcal{M}$ and $\varepsilon = n^\mu n_\mu = \pm 1$, where $\varepsilon$ is equal to $-1$ for a spacelike hypersurface, and is equal to $+1$ for a timelike hypersurface when we take the metric signature is "mostly plus"; *i.e.* $(-,+,+,+)$. A unit normal $n_\mu$ can be introduced only if the hypersurface is not null, and $n^\mu$ is the normal vector to the hypersurface.

Indeed, one can derive the boundary term simply by using the variational principle. In this case the action is varied with respect to the metric, and it produces a total-divergent term, which can be eliminated by the variation of $S_B$, [102]. Finding the boundary terms for any action is an indication that the variation principle for the given theory is well posed.

As mentioned earlier on, despite the many successes that the EH action brought in understanding the universe in IR regime, the UV sector of gravity requires corrections to be well behaved. We shall recall the most general *covariant* action of gravity, which is quadratic in curvature,

$$\begin{aligned} S &= S_{EH} + S_{UV} \\ &= \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\Big[\mathcal{R} + \alpha\big(\mathcal{R}\mathcal{F}_1(\Box)\mathcal{R} + \mathcal{R}_{\mu\nu}\mathcal{F}_2(\Box)\mathcal{R}^{\mu\nu} \\ &\quad + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)\mathcal{R}^{\mu\nu\rho\sigma}\big)\Big], \quad \text{with} \quad \mathcal{F}_i(\Box) = \sum_{n=0}^{\infty} f_{i_n}\Box^n\,, \end{aligned} \tag{4.2.7}$$

where $\alpha$ is a constant with mass dimension $-2$ and the '$f_{i_n}$'s are dimensionless coefficients. For the full equations of motion of such an action, see [111]. The aim of this chapter is to seek the boundary terms corresponding to $S_{UV}$, while retaining the Riemann term.

## 4.3 Time Slicing

Any geometric space-time can be recast in terms of time like spatial slices, known as hypersurfaces. How these slices are embedded in space-time, determines the extrinsic curvature of the slices. One of the motivations of time slicing is to evolve the equations of motion from a well-defined set of initial conditions set at a well-defined spacelike hypersurface, see [137, 138].

### 4.3.1 ADM Decomposition

In order to define the decomposition, we first look at the foliation. Suppose that the time orientable space-time $\mathcal{M}$ is foliated by a family of spacelike hypersurfaces

$\Sigma_t$, on which time is a fixed constant $t = x^0$. We then define the induced metric on the hypersurface as $h_{ij} \equiv g_{ij}|_t$, where the Latin indices run from 1 to 3 [1]. Let us remind the line element as given in section 3.4.1, [139]:

$$ds^2 = -(N^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + h_{ij}dx^i dx^j \tag{4.3.8}$$

where

$$N = \frac{1}{\sqrt{-g^{00}}}$$

is the "lapse" function, and

$$\beta^i = -\frac{g^{0i}}{g^{00}}$$

is the "shift" vector.

In the above line element Eq. (4.3.8), we also have $\sqrt{-g} = N\sqrt{h}$. The induced metric of the hypersurface can be related to the 4 dimensional full metric via the completeness relation, where, for a spacelike hypersurface,

$$\begin{aligned} g^{\mu\nu} &= h^{ij}e_i^\mu e_j^\nu + \varepsilon n^\mu n^\nu \\ &= h^{\mu\nu} - n^\mu n^\nu \,, \end{aligned} \tag{4.3.9}$$

where $\varepsilon = -1$ for a spacelike hypersurface, and $+1$ for a timelike hypersurface, and

$$e_i^\mu = \frac{\partial x^\mu}{\partial y^i} \,, \tag{4.3.10}$$

are basis vectors on the hypersurface which allow us to define tangential tensors on the hypersurface[2]. We note '$x$'s are coordinates on region $\mathcal{M}$, while '$y$'s are coordinates associated with the hypersurface and we may also keep in mind that,

$$h^{\mu\nu} = h^{ij}e_i^\mu e_j^\nu \,, \tag{4.3.11}$$

[1] It should also be noted that Greek indices run from 0 to 3 and Latin indices run from 1 to 3, that is, only spatial coordinates are considered.

[2] We can use $h^{\mu\nu}$ to project a tensor $A_{\mu\nu}$ onto the hypersurface: $A_{\mu\nu}e_i^\mu e_j^\nu = A_{ij}$ where $A_{ij}$ is the three-tensor associated with $A_{\mu\nu}$.

where $h^{ij}$ is the inverse of the induced metric $h_{ij}$ on the hypersurface.

The change of direction of the normal $n$ as one moves on the hypersurface corresponds to the bending of the hypersurface $\Sigma_t$ which is described by the extrinsic curvature. The extrinsic curvature of spatial slices where time is constant is given by:

$$K_{ij} \equiv -\nabla_i n_j = \frac{1}{2N}\left(D_i\beta_j + D_j\beta_i - \partial_t h_{ij}\right) , \tag{4.3.12}$$

where $D_i = e_i^\mu \nabla_\mu$ is the intrinsic covariant derivative associated with the induced metric defined on the hypersurface, and $e_i^\mu$ is the appropriate basis vector which is used to transform bulk indices to boundary ones.

Armed with this information, one can write down the Gauss, Codazzi and Ricci equations, see [136]:

$$\begin{aligned}
\mathcal{R}_{ijkl} &\equiv K_{ik}K_{jl} - K_{il}K_{jk} + R_{ijkl} , && (4.3.13)\\
\mathcal{R}_{ijk\mathbf{n}} &\equiv n^\mu \mathcal{R}_{ijk\mu} = -D_i K_{jk} + D_j K_{ik} , && (4.3.14)\\
\mathcal{R}_{i\mathbf{n}j\mathbf{n}} &\equiv n^\mu n^\nu \mathcal{R}_{i\mu j\nu} \\
&= N^{-1}\left(\partial_t K_{ij} - \pounds_\beta K_{ij}\right) + K_{ik}K_j{}^k + N^{-1}D_iD_jN , && (4.3.15)
\end{aligned}$$

where in the left hand side of Eq. (4.3.13) we have the bulk Riemann tensor, but where all indices are now *spatial* rather than both spatial and temporal, and $R_{ijkl}$ is the Riemann tensor constructed purely out of $h_{ij}$, *i.e.* the metric associated with the hypersurface; and $\pounds_\beta$ is the Lie derivative with respect to shift[1].

#### 4.3.1.1 Coframe Slicing

A key feature of the 3+1 decomposition is the free choice of lapse function and shift vector which define the choice of foliation at the end. In this study we stick to the *coframe slicing*. The main advantage for this choice of slicing is the fact that the line element and therefore components of the infinite derivative function in our gravitational action will be simplified greatly. In addition, [140]

[1] We have $\pounds_\beta K_{ij} \equiv \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{jk}D_i\beta^k$.

has shown that such a slicing has a more transparent form of the canonical action principle and Hamiltonian dynamics for gravity. This also leads to a well-posed initial-condition for the evolution of the gravitational constraints in a vacuum by satisfying the Bianchi identities. In order to map the ADM line element into the coframe slicing, we use the convention of [140]. We define

$$\begin{aligned} \theta^0 &= dt\,, \\ \theta^i &= dx^i + \beta^i dt\,, \end{aligned} \tag{4.3.16}$$

where $x^i$ and $i = 1, 2, 3$ is the spatial and $t$ is the time coordinates.[1] The metric in the coframe takes the following form

$$ds^2_{\text{coframe}} = g_{\alpha\beta}\theta^\alpha\theta^\beta = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j\,, \tag{4.3.17}$$

where upon substituting Eq. (4.3.16) into Eq. (4.3.17) we recover the original ADM metric given by Eq. (4.3.8). In this convention, if we take $\mathbf{g}$ as the full space-time metric, we have the following simplifications:

$$g_{ij} = h_{ij}, \quad g^{ij} = h^{ij}, \quad g_{0i} = g^{0i} = 0. \tag{4.3.18}$$

The convective derivatives $\partial_\alpha$ with respect to $\theta^\alpha$ are

$$\begin{aligned} \partial_0 &\equiv \frac{\partial}{\partial t} - \beta^i\partial_i\,, \\ \partial_i &\equiv \frac{\partial}{\partial x^i}\,. \end{aligned} \tag{4.3.19}$$

For time-dependent space tensors $T$, we can define the following derivative:

$$\bar{\partial}_0 \equiv \frac{\partial}{\partial t} - \pounds_\beta\,, \tag{4.3.20}$$

where $\pounds_\beta$ is the Lie derivative with respect to the shift vector $\beta^i$. This is because the off-diagonal components of the coframe metric are zero, *i.e.*, $g_{0i} = g^{0i} = 0$.

[1] We note that in Eq. (4.3.16), the "$i$" for $\theta^i$ is just a superscript not a spatial index.

We shall see later on how this time slicing helps us to simplify the calculations when the gravitational action contains infinite derivatives.

#### 4.3.1.2 Extrinsic Curvature

A change in the choice of time slicing results in a change of the evolution of the system. The choice of foliation also has a direct impact on the form of the extrinsic curvature. In this section we wish to give the form of extrinsic curvature $K_{ij}$ in the coframe slicing. This is due to the fact that the definition of the extrinsic curvature is an initial parameter that describes the evolution of the system, therefore is it logical for us to derive the extrinsic curvature in the coframe slicing as we use it throughout the chapter. We use [140] to find the general definition for $K_{ij}$ in the coframe metric. In the coframe,

$$\gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + g^{\alpha\delta}C^{\epsilon}{}_{\delta(\beta}g_{\gamma)\epsilon} - \frac{1}{2}C^{\alpha}{}_{\beta\gamma}\,, \tag{4.3.21}$$

$$d\theta^{\alpha} = -\frac{1}{2}C^{\alpha}{}_{\beta\gamma}\theta^{\beta}\wedge\theta^{\gamma}\,, \tag{4.3.22}$$

where $\Gamma$ is the ordinary Christoffel symbol and "$\wedge$" denotes the exterior or wedge product of vectors $\theta$. By finding the coefficients $C$s and subsequently calculating the connection coefficients $\gamma^{\alpha}_{\beta\gamma}$, one can extract the extrinsic curvature $K_{ij}$ in the coframe setup. We note that the expression for $d\theta^{\alpha}$ is the Maurer-Cartan structure equation [144]. It is derived from the canonical 1-form $\theta$ on a Lie group $G$ which is the left-invariant $\mathfrak{g}$-valued 1-form uniquely determined by $\theta(\xi) = \xi$ for all $\xi \in \mathfrak{g}$.

We can use differential forms (See Appendix H) to calculate the $C$s, the coefficients of $d\theta$ where now we can write,

$$d\theta^{k} = -\left(\partial_i\beta^{k}\right)\theta^{0}\wedge\theta^{i} + \frac{1}{2}C^{k}{}_{ij}\theta^{j}\wedge\theta^{i}\,, \tag{4.3.23}$$

where $k = 1, 2, 3$. Now when we insert the $C$s from Appendix H,

$$d\theta^{1} = d\left(dx^{1} + \beta^{1}dt\right) = d\beta^{1}\wedge dt \tag{4.3.24}$$

and

$$\begin{aligned} d\theta^0 &= d(dt) = d^2(t) = 0 \\ d\theta^i &= d\beta^i \wedge dt. \end{aligned} \tag{4.3.25}$$

From the definition of $d\theta^\alpha$ in Eq. (4.3.22) and using the antisymmetric properties of the $\wedge$ product,

$$d\theta^\alpha = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\beta \wedge \theta^\gamma = \frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\gamma \wedge \theta^\beta = \frac{1}{2}C^\alpha{}_{\gamma\beta}\theta^\beta \wedge \theta^\gamma, \tag{4.3.26}$$

we get

$$C^\alpha{}_{\beta\gamma} = -C^\alpha{}_{\gamma\beta}\,. \tag{4.3.27}$$

Using these properties, we find that $C^m{}_{0i} = \frac{\partial\beta^m}{\partial x^i}$, $C^m{}_{ij} = 0$ and $C^0{}_{ij} = 0$. Using Eq. (4.3.21), we obtain that

$$\gamma^0_{ij} = -\frac{1}{2N^2}\Big(h_{il}\partial_j(\beta^l) + h_{jl}\partial_i(\beta^l) - \bar{\partial}_0 h_{ij}\Big)\,. \tag{4.3.28}$$

Since from Eq. (4.3.12)

$$K_{ij} \equiv -\nabla_i n_j = \gamma^\mu_{ij} n_\mu = -N\gamma^0_{ij}\,, \tag{4.3.29}$$

the expression for the extrinsic curvature in coframe slicing is given by:

$$K_{ij} = \frac{1}{2N}\Big(h_{il}\partial_j(\beta^l) + h_{jl}\partial_i(\beta^l) - \bar{\partial}_0 h_{ij}\Big)\,, \tag{4.3.30}$$

where $\partial_0$ is the time derivative and $\beta^l$ is the "shift" in the coframe metric Eq. (4.3.17).

#### 4.3.1.3 Riemann Tensor in the Coframe

The fact that we move from the ADM metric into the coframe slicing has the following implication on the form of the components of the Riemann tensor. Essentially, since in the coframe slicing in Eq. (4.3.17) we have $g^{0i} = g_{0i} = 0$, therefore we also have, $n^i = n_i = 0$. Hence the non-vanishing components of the Riemann tensor in the coframe, namely Gauss, Codazzi and Ricci tensor, become:

$$\begin{aligned}
\mathcal{R}_{ijkl} &= K_{ik}K_{jl} - K_{il}K_{jk} + R_{ijkl}\,, \\
\mathcal{R}_{0ijk} &= N(-D_k K_{ji} + D_j K_{ki})\,, \\
\mathcal{R}_{0i0j} &= N(\bar{\partial}_0 K_{ij} + N K_{ik}K_j{}^k + D_i D_j N\,),
\end{aligned} \tag{4.3.31}$$

with $\bar{\partial}_0$ defined in Eq. (4.3.20) and $D_j = e_j^\mu \nabla_\mu$. It can be seen that the Ricci equation, given in Eq. (4.3.13) is simplified in above due to the definition of Eq. (4.3.20). We note that Eq. (4.3.31) is in the coframe slicing, while Eqs. (4.3.13-4.3.15) are in the ADM frame only.

#### 4.3.1.4 D'Alembertian Operator in Coframe

Since we shall be dealing with a higher-derivative theory of gravity, it is therefore helpful to first obtain an expression for the $\Box$ operator in this subsection. To do so, we start off by writing the definition of a single box operator in the coframe, [146],

$$\begin{aligned}
\Box &= g^{\mu\nu}\nabla_\mu\nabla_\nu \\
&= (h^{\mu\nu} + \varepsilon n^\mu n^\nu)\nabla_\mu\nabla_\nu \\
&= -n^\mu n^\nu \nabla_\mu\nabla_\nu + h^{\mu\nu}\nabla_\mu\nabla_\nu \\
&= -n^0 n^0 \nabla_0\nabla_0 + h^{ij} e_i^\mu e_j^\nu \nabla_\mu\nabla_\nu \\
&= -\frac{1}{N^2}\nabla_0\nabla_0 + h^{ij} D_i D_j \\
&= -(N^{-1}\bar{\partial}_0)^2 + \Box_{hyp}\,,
\end{aligned} \tag{4.3.32}$$

where we note that the Greek indices run from 1 to 4 and the Latin indices run from 1 to 3 ($\varepsilon = -1$ for a spacelike hypersurface). We call the spatial box operator $\Box_{hyp} = h^{ij}D_iD_j$, which stands for "hypersurface" as the spatial coordinates are defined on the hypersurface meaning $\Box_{hyp}$ is the projection of the covariant d'Alembertian operator down to the hypersurface, *i.e.* only the tangential components of the covariant d'Alembertian operator are encapsulated by $\Box_{hyp}$. Also note that in the coframe slicing $g^{ij} = h^{ij}$. Generalising this result to the $n$th power, for our purpose, we get

$$\mathcal{F}_i(\Box) = \sum_{n=0}^{\infty} f_{i_n} \left[-(N^{-1}\bar{\partial}_0)^2 + \Box_{hyp}\right]^n \tag{4.3.33}$$

where the $f_{i_n}$s are the coefficients of the series.

## 4.4 Generalised Boundary Term

In this section, first we are going to briefly summarise the method of [136] for finding the boundary term. It has been shown that, given a general gravitational action

$$S = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\,\sqrt{-g}f(\mathcal{R}_{\mu\nu\rho\sigma})\,, \tag{4.4.34}$$

one can introduce two auxiliary fields $\varrho_{\mu\nu\rho\sigma}$ and $\varphi^{\mu\nu\rho\sigma}$, which are independent of each other and of the metric $g_{\mu\nu}$, while they have all the symmetry properties of the Riemann tensor $\mathcal{R}_{\mu\nu\rho\sigma}$. We can then write down the following equivalent action:

$$S = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\left[f(\varrho_{\mu\nu\rho\sigma}) + \varphi^{\mu\nu\rho\sigma}\left(\mathcal{R}_{\mu\nu\rho\sigma} - \varrho_{\mu\nu\rho\sigma}\right)\right]\,. \tag{4.4.35}$$

The reason we introduce these auxiliary fields is that the second derivatives of the metric appear only linearly in Eq. (4.4.35). Note that in Eq. (4.4.35), the terms involving the second derivatives of the metric are not multiplied by terms

of the same type, *i.e.* involving the second derivative of the metric, so when we integrate by parts once, we are left just with the first derivatives of the metric; we cannot eliminate the first derivatives of the metric as well - since in this study we are keeping the boundary terms. Note that the first derivatives of the metric are actually contained in these boundary terms if we integrate by parts twice, see our toy model scalar field theory example in Eqs. (4.1.2,4.1.3) [1]. Therefore, terms which are linear in the metric can be eliminated if we integrate by parts; moreover, the use of the auxiliary fields can prove useful in a future Hamiltonian analysis of the action.

From [136], we then decompose the above expression as

$$\varphi^{\mu\nu\rho\sigma}\left(\mathcal{R}_{\mu\nu\rho\sigma}-\varrho_{\mu\nu\rho\sigma}\right)=\phi^{ijkl}(\mathcal{R}_{ijkl}-\rho_{ijkl})-4\phi^{ijk}(\mathcal{R}_{ijk\mathbf{n}}-\rho_{ijk})-2\Psi^{ij}(\mathcal{R}_{i\mathbf{n}j\mathbf{n}}-\Omega_{ij})\,, \tag{4.4.36}$$

where

$$\mathcal{R}_{ijkl}\equiv\rho_{ijkl}\equiv\varrho_{ijkl},\qquad \mathcal{R}_{ijk\mathbf{n}}\equiv\rho_{ijk}\equiv n^{\mu}\varrho_{ijk\mu},\qquad \mathcal{R}_{i\mathbf{n}j\mathbf{n}}\equiv\Omega_{ij}\equiv n^{\mu}n^{\nu}\varrho_{i\mu j\nu} \tag{4.4.37}$$

are equivalent to the components of the Gauss, Codazzi and Ricci equations given in Eq. (4.3.13), also,

$$\phi^{ijkl}\equiv\varphi^{ijkl},\qquad \phi^{ijk}\equiv n_{\mu}\varphi^{ijk\mu},\qquad \Psi^{ij}\equiv-2n_{\mu}n_{\nu}\varphi^{i\mu j\nu}, \tag{4.4.38}$$

where $\phi^{ijkl}$, $\phi^{ijk}$ and $\Psi^{ij}$ are spatial tensors evaluated on the hypersurface. The equations of motion for the auxiliary fields $\varphi^{\mu\nu\rho\sigma}$ and $\varrho_{\mu\nu\rho\sigma}$ are, respectively given by [136],

$$\frac{\delta S}{\delta\varphi^{\mu\nu\rho\sigma}}=0\Rightarrow\varrho_{\mu\nu\rho\sigma}=\mathcal{R}_{\mu\nu\rho\sigma}\quad\text{and}\quad\frac{\delta S}{\delta\varrho_{\mu\nu\rho\sigma}}=0\Rightarrow\varphi^{\mu\nu\rho\sigma}=\frac{\partial f}{\partial\varrho_{\mu\nu\rho\sigma}}\,, \tag{4.4.39}$$

[1] This is because $\varrho_{\mu\nu\rho\sigma}$ and $\varphi^{\mu\nu\rho\sigma}$ are independent of the metric, and so although $f(\varrho_{\mu\nu\rho\sigma})$ can contain derivatives of $\varrho_{\mu\nu\rho\sigma}$, these are not derivatives of the metric. $\mathcal{R}_{\mu\nu\rho\sigma}$ contains a second derivative of the metric but this is the only place where a second derivative of the metric appears in Eq. (4.4.35)

where $\mathcal{R}_{\mu\nu\rho\sigma}$ is the four-dimensional Riemann tensor.

One can start from the action given by Eq. (4.4.35), insert the equation of motion for $\varphi^{\mu\nu\rho\sigma}$ and recover the action given by Eq. (4.4.34). It has been shown by [136] that one can find the total derivative term of the auxiliary action as

$$S = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x \left(\sqrt{-g}\mathcal{L} - 2\partial_\mu[\sqrt{-g}\, n^\mu K\cdot\Psi]\right) , \tag{4.4.40}$$

where $K = h^{ij}K_{ij}$, with $K_{ij}$ given by Eq. (4.3.30), and $\Psi = h^{ij}\Psi_{ij}$ , where $\Psi_{ij}$ is given in Eq. (4.4.43), are spatial tensors evaluated on the hypersurface $\Sigma_t$ and $\mathcal{L}$ is the Lagrangian density.

In Eq. (4.4.40), the second term is the total derivative. It has been shown that one may add the following action to the above action to eliminate the total derivative appropriately. Indeed $\Psi$ can be seen as a modification to the GHY term, which depends on the form of the Lagrangian density [136].

$$S_{GHY} = \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu n^\mu \Psi\cdot K\,, \tag{4.4.41}$$

where $n^\mu$ is the normal vector to the hypersurface and the infinitesimal vector field

$$d\Sigma_\mu = \varepsilon_{\mu\alpha\beta\gamma}e_1^\alpha e_2^\beta e_3^\gamma d^3y\,, \tag{4.4.42}$$

is normal to the boundary $\partial\mathcal{M}$ and is proportional to the volume element of $\partial\mathcal{M}$; in above $\varepsilon_{\mu\alpha\beta\gamma} = \sqrt{-g}[\mu\,\alpha\,\beta\,\gamma]$ is the Levi-Civita tensor and $y$ are coordinates intrinsic to the boundary [1], and we used Eq. (4.3.10). Moreover in Eq. (4.4.41), we have:

$$\Psi^{ij} = -\frac{1}{2}\frac{\delta f}{\delta\Omega_{ij}}\,, \tag{4.4.43}$$

where $f$ indicates the terms in the Lagrangian density and is built up of tensors $\varrho_{\mu\nu\rho\sigma}$, $\varrho_{\mu\nu}$ and $\varrho$ as in Eq. (4.4.35); $G_N$ is the universal gravitational constant and $\Omega_{ij}$ is given in Eq. (4.4.37). Indeed, the above constraint is extracted from the

[1] We shall also mention that Eq.(4.4.41) is derived from Eq.(4.4.40) by performing Stokes theorem, that is $\int_{\mathcal{M}} A^\mu{}_{;\mu}\sqrt{-g}\; d^dx = \oint_{\partial\mathcal{M}} A^\mu\; d\Sigma_\mu$, with $A^\mu = n^\mu K\cdot\Psi$.

equation of motion for $\Omega_{ij}$ in the Hamiltonian regime [136]. In the next section we are going to use the same approach to find the boundary terms for the most general, covariant quadratic order action of gravity.

## 4.5 Boundary Terms for Finite Derivative Theory of Gravity

In this section we are going to use the 3+1 decomposition and calculate the boundary term of the EH term $\mathcal{R}$, and

$$\mathcal{R}\Box\mathcal{R}, \quad \mathcal{R}_{\mu\nu}\Box\mathcal{R}^{\mu\nu}, \quad \mathcal{R}_{\mu\nu\rho\sigma}\Box\mathcal{R}^{\mu\nu\rho\sigma},$$

as prescribed in previous section, as a warm-up exercise.

We then move on to our generalised action given in Eq. (4.2.7). To decompose any given term, we shall write them in terms of their auxiliary field, therefore we have $\mathcal{R} = \varrho, \quad \mathcal{R}_{\mu\nu} \equiv \varrho_{\mu\nu}$, and $\mathcal{R}_{\mu\nu\rho\sigma} \equiv \varrho_{\mu\nu\rho\sigma}$, where the auxiliary fields $\varrho, \varrho_{\mu\nu}$ and $\varrho_{\mu\nu\rho\sigma}$ have all the symmetry properties of the Riemann tensor. We shall also note that the decomposition of the $\Box$ operator in 3+1 formalism in the coframe setup is given by Eq. (4.3.32).

### 4.5.1 $\mathcal{R}$

For the Einstein-Hilbert term $\mathcal{R}$, in terms of the auxiliary field $\varrho$ we find in Appendix I.1

$$\begin{aligned} f = \varrho &= g^{\mu\rho}g^{\nu\sigma}\varrho_{\mu\nu\rho\sigma} \\ &= \left(h^{\mu\rho} - n^\mu n^\rho\right)\left(h^{\nu\sigma} - n^\nu n^\sigma\right)\varrho_{\mu\nu\rho\sigma} \\ &= \left(h^{\mu\rho}h^{\nu\sigma} - n^\mu n^\rho h^{\nu\sigma} - h^{\mu\rho}n^\nu n^\sigma\right)\varrho_{\mu\nu\rho\sigma} \\ &= \left(\rho - 2\Omega\right), \end{aligned} \tag{4.5.44}$$

where $\Omega = h^{ij}\Omega_{ij}$ and we used $h^{ij}h^{kl}\rho_{ijkl} = \rho$, and $h^{ij}\rho_{i\nu j\sigma}n^\nu n^\sigma = h^{ij}\Omega_{ij}$ and $\varrho \equiv \mathcal{R}$ in the EH action and the right hand side of Eq. (4.5.44) is the $3+1$ decomposed form of the Lagrangian and hence $\rho$ and $\Omega$ are spatial. We may note that the last term of the expansion on the second line of Eq. (4.5.44) vanishes due to the symmetry properties of the Riemann tensor. Using Eq. (4.4.43), and calculating the functional derivative, we find

$$\Psi^{ij} = -\frac{1}{2}\frac{\delta f}{\delta \Omega_{ij}} = h^{ij}. \tag{4.5.45}$$

This verifies the result found in [136], and it is clear that upon substituting this result into Eq. (4.4.41), we recover the well known boundary for the EH action, as $K = h^{ij}K_{ij}$ and $\Psi \cdot K \equiv \Psi^{ij}K_{ij}$ where $K_{ij}$ is given by Eq. (4.3.30). Hence,

$$S_{GHY} \equiv S_0 = \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu n^\mu K\,, \tag{4.5.46}$$

where $d\Sigma_\mu$ is the normal to the boundary $\partial\mathcal{M}$ and is proportional to the volume element of $\partial\mathcal{M}$ while $n^\mu$ is the normal vector to the hypersurface.

### 4.5.2 $\mathcal{R}_{\mu\nu\rho\sigma}\Box\mathcal{R}^{\mu\nu\rho\sigma}$

Next, we start off by writing $\mathcal{R}_{\mu\nu\rho\sigma}\Box\mathcal{R}^{\mu\nu\rho\sigma}$ as its auxiliary equivalent $\varrho_{\mu\nu\rho\sigma}\Box\varrho^{\mu\nu\rho\sigma}$ to obtain

$$\begin{aligned}
\varrho_{\mu\nu\rho\sigma}\Box\varrho^{\mu\nu\rho\sigma} &= \delta^\alpha_\mu\delta^\beta_\nu\delta^\gamma_\rho\delta^\lambda_\sigma\varrho_{\alpha\beta\gamma\lambda}\Box\varrho^{\mu\nu\rho\sigma} \\
&= \Big[h^\alpha_\mu h^\beta_\nu h^\gamma_\rho h^\lambda_\sigma - \Big(h^\alpha_\mu h^\beta_\nu h^\gamma_\rho n^\lambda n_\sigma + h^\alpha_\mu h^\beta_\nu n^\gamma n_\rho h^\lambda_\sigma + h^\alpha_\mu n^\beta n_\nu h^\gamma_\rho h^\lambda_\sigma \\
&\quad + n^\alpha n_\mu h^\beta_\nu h^\gamma_\rho h^\lambda_\sigma\Big) + h^\alpha_\mu n^\beta n_\nu h^\gamma_\rho n^\lambda n_\sigma + h^\alpha_\mu n^\beta n_\nu n^\gamma n_\rho h^\lambda_\sigma + n^\alpha n_\mu h^\beta_\nu h^\gamma_\rho n^\lambda n_\sigma \\
&+ n^\alpha n_\mu h^\beta_\nu n^\gamma n_\rho h^\lambda_\sigma\Big]\varrho_{\alpha\beta\gamma\lambda}\left(-(N^{-1}\bar{\partial}_0)^2 + \Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\,,
\end{aligned} \tag{4.5.47}$$

where $\varrho_{\mu\nu\rho\sigma} = \delta^\alpha_\mu\delta^\beta_\nu\delta^\gamma_\rho\delta^\lambda_\sigma\varrho_{\alpha\beta\gamma\lambda}$ (where $\delta^\alpha_\mu$ is the Kronecker delta). This allowed us to use the completeness relation as given in Eq. (4.3.9). In Eq. (4.5.47), we used

the antisymmetry properties of the Riemann tensor to eliminate irrelevant terms in the expansion. From Eq. (4.5.47), we have three types of terms:

$$hhhh, \quad hhhnn, \quad hhnnnn.$$

The aim is to contract the tensors appearing in Eq. (4.5.47) and extract those terms which are $\Omega_{ij}$ dependent. This is because we only need $\Omega_{ij}$ dependent terms to obtain $\Psi^{ij}$ as in Eq. (4.4.43) and then the boundary as prescribed in Eq. (4.4.41).

A closer look at the expansion given in Eq. (4.5.47) leads us to know which term would admit $\Omega_{ij}$ type terms. Essentially, as defined in Eq. (4.4.37), $\Omega_{ij} = n^{\mu}n^{\nu}\varrho_{i\mu j\nu}$, therefore by having two auxiliary field tensors as $\varrho_{\alpha\beta\gamma\lambda}$ and $\varrho^{\mu\nu\rho\sigma}$ in Eq. (4.5.47) (with symmetries of the Riemann tensor) we may construct $\Omega_{ij}$ dependent terms. Henceforth, we can see that in this case the $\Omega_{ij}$ dependence comes from the *hhnnnn* term.

To see this explicitly, note that in order to perform the appropriate contractions in presence of the d'Alembertian operator, we first need to complete the contractions on the left hand side of the $\Box$ operator. We then need to commute the rest of the tensors by using the *Leibniz rule* to the right hand side of the components of the operator, *i.e.* the $\bar{\partial}_0$'s and the $\Box_{hyp}$, and only then do we obtain the $\Omega_{ij}$ type terms.

We first note that the terms that do not produce $\Omega_{ij}$ dependence are not involved in the boundary calculation, however they might form $\rho_{ijkl}$, $\rho_{ijk}$, or their contractions. These terms are equivalent to the Gauss and Codazzi equations as shown in Eq. (4.4.37), and we will address their formation in Appendix I.2. In addition, as we shall see, by performing the Leibniz rule one produces some associated terms, the $X_{ij}$'s, which appear for example in Eq. (4.5.48). Again we will keep them only if they are $\Omega_{ij}$ dependent, if not we will drop them.

**hhnnnn** terms: To this end we shall compute the *hhnnnn* terms, hence we commute the $h$'s and $n$'s onto the right hand side of the $\Box$ in the *hhnnnn* term

of Eq. (4.5.47):

$$
\begin{aligned}
&h^{\alpha}_{\mu}n^{\beta}n_{\nu}h^{\gamma}_{\rho}n^{\lambda}n_{\sigma}\varrho_{\alpha\beta\gamma\lambda}\left(-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{x}e^{\alpha}_{i}e^{x}_{\mu}\right)n^{\beta}n_{\nu}\left(h^{j}_{y}e^{\gamma}_{j}e^{y}_{\rho}\right)n^{\lambda}n_{\sigma}\varrho_{\alpha\beta\gamma\lambda}\left(-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{x}e^{x}_{\mu}\right)n_{\nu}\left(h^{j}_{y}e^{y}_{\rho}\right)n_{\sigma}\Omega_{ij}\left(-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\\
&=-N^{-2}\Omega_{ij}\Big\{\bar{\partial}^2_0\left(\Omega^{ij}\right)\\
&-\bar{\partial}_0\left[\varrho^{\mu\nu\rho\sigma}\bar{\partial}_0\left(\left[\left(h^{i}_{x}e^{x}_{\mu}\right)n_{\nu}\left(h^{j}_{y}e^{y}_{\rho}\right)n_{\sigma}\right]\right)\right]-\bar{\partial}_0\left(\left[\left(h^{i}_{x}e^{x}_{\mu}\right)n_{\nu}\left(h^{j}_{y}e^{y}_{\rho}\right)n_{\sigma}\right]\right)\bar{\partial}_0\left(\varrho^{\mu\nu\rho\sigma}\right)\Big\}\\
&+\Omega_{ij}\Big\{\Box_{hyp}\left[\Omega^{ij}\right]-D_a\left(D^a\left[e^{x}_{\mu}n_{\nu}e^{y}_{\rho}n_{\sigma}\right]h^{i}_{x}h^{j}_{y}\varrho^{\mu\nu\rho\sigma}\right)-D_a\left[e^{x}_{\mu}n_{\nu}e^{y}_{\rho}n_{\sigma}\right]D^a\left(h^{i}_{x}h^{j}_{y}\varrho^{\mu\nu\rho\sigma}\right)\Big\}\\
&=\Omega_{ij}\big(-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\big)\Omega^{ij}+\Omega_{ij}X^{ij}_{1}\\
&=\Omega_{ij}\Box\Omega^{ij}+\Omega_{ij}X^{ij}_{1}
\end{aligned}
\tag{4.5.48}
$$

where $\Omega_{ij}\equiv h_{ik}e^{k}_{\kappa}h_{jm}e^{m}_{\lambda}n_{\gamma}n_{\delta}\varrho^{\gamma\kappa\delta\lambda}=h_{ik}h_{jm}n_{\gamma}n_{\delta}\varrho^{\gamma k\delta m}$; we note that $X^{ij}_1$ only appears because of the presence of the $\Box$ operator.

$$
\begin{aligned}
X^{ij}_{1} &= N^{-2}(\bar{\partial}_0\left[\varrho^{\mu\nu\rho\sigma}\bar{\partial}_0\left(\left[\left(h^{i}_{x}e^{x}_{\mu}\right)n_{\nu}\left(h^{j}_{y}e^{y}_{\rho}\right)n_{\sigma}\right]\right)\right]+\bar{\partial}_0\left(\left[\left(h^{i}_{x}e^{x}_{\mu}\right)n_{\nu}\left(h^{j}_{y}e^{y}_{\rho}\right)n_{\sigma}\right]\right)\bar{\partial}_0\left(\varrho^{\mu\nu\rho\sigma}\right))\\
&- D_a\left(D^a\left[e^{x}_{\mu}n_{\nu}e^{y}_{\rho}n_{\sigma}\right]h^{i}_{x}h^{j}_{y}\varrho^{\mu\nu\rho\sigma}\right)-D_a\left[e^{x}_{\mu}n_{\nu}e^{y}_{\rho}n_{\sigma}\right]D^a\left(h^{i}_{x}h^{j}_{y}\varrho^{\mu\nu\rho\sigma}\right)\,.
\end{aligned}
\tag{4.5.49}
$$

The term $\Omega_{rs}X^{rs}_1$ will yield $X^{ij}_1$ when functionally differentiated with respect to $\Omega_{ij}$ as in Eq. (4.4.43). Also note $X^{ij}_1$ does not have any $\Omega^{ij}$ dependence. Similarly for the other $X$ terms which appear later in the chapter. We shall note that when we take $\Box=1$ in Eq. (4.5.47), we obtain,

$$
\begin{aligned}
&h^{\alpha}_{\mu}n^{\beta}n_{\nu}h^{\gamma}_{\rho}n^{\lambda}n_{\sigma}\varrho_{\alpha\beta\gamma\lambda}\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{x}e^{\alpha}_{i}e^{x}_{\mu}\right)n^{\beta}n_{\nu}\left(h^{j}_{y}e^{\gamma}_{j}e^{y}_{\rho}\right)n^{\lambda}n_{\sigma}\varrho_{\alpha\beta\gamma\lambda}\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{x}e^{x}_{\mu}\right)n_{\nu}\left(h^{j}_{y}e^{y}_{\rho}\right)n_{\sigma}\Omega_{ij}\varrho^{\mu\nu\rho\sigma}\\
&=\Omega_{ij}\left(h^{i}_{x}e^{x}_{\mu}\right)n_{\nu}\left(h^{j}_{y}e^{y}_{\rho}\right)n_{\sigma}\varrho^{\mu\nu\rho\sigma}\\
&=\Omega_{ij}\Omega^{ij}\,,
\end{aligned}
\tag{4.5.50}
$$

where we just contract the indices and we do **not** need to use the Leibniz rule as we can commute any of the tensors, therefore we do not produce any $X^{ij}$ terms

at all [1]. Finally, one can decompose Eq. (4.5.47) as

$$\varrho_{\mu\nu\rho\sigma}\Box\varrho^{\mu\nu\rho\sigma} = 4\Omega_{ij}\Box\Omega^{ij} + 4\Omega_{ij}X_1^{ij} + \cdots\,, \tag{4.5.51}$$

where "$\cdots$" are terms such as $\rho_{ijkl}\Box\rho^{ijkl}$, $\rho_{ijk}\Box\rho^{ijk}$ and terms that are not $\Omega^{ij}$ dependent and are the results of performing the Leibniz rule (see Appendix I.2). When we take $M^2 \to \infty$, *i.e.*, when we set $\Box \to 0$ (recall that $\Box$ has an associated mass scale $\Box/M^2$), which is also equivalent to considering $\alpha \to 0$ in Eq. (4.2.7), we recover the EH result.

When $\Box \to 1$, we recover the result for $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$ found in [136]. At both limits, $\Box \to 0$ and $\Box \to 1$, the $X_1^{ij}$ term is not present. To find the boundary term, we use Eq. (4.4.43) and then Eq. (4.4.41). We are going to use the Euler-Lagrange equation and drop the total derivatives as a result. We have,

$$\begin{aligned}
\Psi^{ij}_{\rm Riem} &= -\frac{1}{2}\frac{\delta f}{\delta\Omega_{ij}} = -\frac{4}{2}\frac{\delta(\Omega_{ij}\Box\Omega^{ij} + \Omega_{ij}X_1^{ij})}{\delta\Omega_{ij}} \\
&= -2\left\{\frac{\partial(\Omega_{ij}\Box\Omega^{ij})}{\partial\Omega_{ij}} + \Box\left(\frac{\partial(\Omega_{ij}\Box\Omega^{ij})}{\partial(\Box\Omega_{ij})}\right) + \frac{\partial(\Omega_{ij}X_1^{ij})}{\partial\Omega_{ij}}\right\} \\
&= -2(\Box\Omega^{ij} + \Box\Omega^{ij} + X_1^{ij}) = -4\Box\Omega^{ij} - 2X_1^{ij}\,.
\end{aligned} \tag{4.5.52}$$

Hence the boundary term for $\mathcal{R}_{\mu\nu\rho\sigma}\Box\mathcal{R}^{\mu\nu\rho\sigma}$ is,

$$S_1 = -\frac{1}{4\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu n^\mu K_{ij}(2\Box\Omega^{ij} + X_1^{ij}). \tag{4.5.53}$$

where $K_{ij}$ is given by Eq. (4.3.30).

[1] This is the same for $\Box^2$ and $\Box^n$.

### 4.5.3 $\mathcal{R}_{\mu\nu}\Box\mathcal{R}^{\mu\nu}$

We start by first performing the 3+1 decomposition of $\mathcal{R}_{\mu\nu}\Box\mathcal{R}^{\mu\nu}$ in its auxiliary form $\varrho_{\mu\nu}\Box\varrho^{\mu\nu}$,

$$\begin{aligned}
&\varrho_{\mu\nu}\Box\varrho^{\mu\nu} = g^{\rho\sigma}\varrho_{\rho\mu\sigma\nu}\Box g^{\mu\kappa}g^{\nu\lambda}g^{\gamma\delta}\varrho_{\gamma\kappa\delta\lambda} \\
&= \left(h^{\rho\sigma} - n^\rho n^\sigma\right)\left(h^{\mu\kappa} - n^\mu n^\kappa\right)\left(h^{\nu\lambda} - n^\nu n^\lambda\right)\left(h^{\gamma\delta} - n^\gamma n^\delta\right)\varrho_{\rho\mu\sigma\nu}\Box\varrho_{\gamma\kappa\delta\lambda} \\
&= \Big[h^{\rho\sigma}h^{\mu\kappa}h^{\nu\lambda}h^{\gamma\delta} - \big(n^\rho n^\sigma h^{\mu\kappa}h^{\nu\lambda}h^{\gamma\delta} + h^{\rho\sigma}n^\mu n^\kappa h^{\nu\lambda}h^{\gamma\delta} + h^{\rho\sigma}h^{\mu\kappa}n^\nu n^\lambda h^{\gamma\delta} \\
&+h^{\rho\sigma}h^{\mu\kappa}h^{\nu\lambda}n^\gamma n^\delta\big) + n^\rho n^\sigma h^{\mu\kappa}h^{\nu\lambda}n^\gamma n^\delta + h^{\rho\sigma}n^\mu n^\kappa n^\nu n^\lambda h^{\gamma\delta}\Big]\varrho_{\rho\mu\sigma\nu}\Box\varrho_{\gamma\kappa\delta\lambda}\,,
\end{aligned} \tag{4.5.54}$$

where we have used appropriate contractions to write the Ricci tensor in terms of the Riemann tensor. As before, we then used the completeness relation Eq. (4.3.9) and used the antisymmetric properties of the Riemann tensor to drop the vanishing terms. We are now set to calculate each term, which we do in more detail in Appendix I.3. Again our aim is to find the $\Omega_{ij}$ dependent terms, by looking at the expansion given in Eq. (4.5.54) and the distribution of the indices, the reader can see that the terms which are $\Omega_{ij}$ dependent are those terms which have *at least two nn*s contracted with one of the $\varrho$s such that we form $n^\mu n^\nu \varrho_{i\mu j\nu}$.

- <u>*hhhnn*</u> terms: We start with the *hhhnn* terms in Eq. (4.5.54). We calculate the first of these in terms of $\Omega_{ik}$ and $\rho^{ik}$ also by moving the '$h$'s and '$n$'s

onto the right hand side of the $\Box$,

$$
\begin{aligned}
& n^\rho n^\sigma h^{\mu\kappa} h^{\nu\lambda} h^{\gamma\delta} \varrho_{\rho\mu\sigma\nu} \Big( - \big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp} \Big) \varrho_{\gamma\kappa\delta\lambda} \\
& = n^\rho n^\sigma (h^{ij} e_i^\mu e_j^\kappa)(h^{kl} e_k^\nu e_l^\lambda)(h^{mn} e_m^\gamma e_n^\delta) \varrho_{\rho\mu\sigma\nu} \Big( - \big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp} \Big) \varrho_{\gamma\kappa\delta\lambda} \\
& = n^\rho n^\sigma (h^{ij} e_j^\kappa)(h^{kl} e_l^\lambda)(h^{mn} e_m^\gamma e_n^\delta) \varrho_{\rho i\sigma k} \Big( - \big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp} \Big) \varrho_{\gamma\kappa\delta\lambda} \\
& = \Omega_{ik} (h^{ij} e_j^\kappa)(h^{kl} e_l^\lambda)(h^{mn} e_m^\gamma e_n^\delta) \Big( - \big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp} \Big) \varrho_{\gamma\kappa\delta\lambda} \\
& = -N^{-2} \Omega_{ik} \Big\{ \bar{\partial}_0^2 (\rho^{ik}) - \bar{\partial}_0 \big( \varrho_{\gamma\kappa\delta\lambda} \bar{\partial}_0 [h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta] \big) \\
& - \bar{\partial}_0 [h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta] \bar{\partial}_0 \varrho_{\gamma\kappa\delta\lambda} \Big\} \\
& + \Omega_{ik} \Big\{ \Box_{hyp} (\rho^{ik}) - D_a \big( \varrho_{\gamma\kappa\delta\lambda} D^a [h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta] \big) \\
& - D_a [h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta] D^a \varrho_{\gamma\kappa\delta\lambda} \Big\} \\
& = \Omega_{ik} \Box \rho^{ik} + \Omega_{ik} X^{ik}_{2(a)} \,,
\end{aligned}
\tag{4.5.55}
$$

where the contraction is $h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta \varrho_{\gamma\kappa\delta\lambda} = h^{ij} h^{kl} \rho_{jl} = \rho^{ik}$, and

$$
\begin{aligned}
X^{ik}_{2(a)} = N^{-2} & \Big\{ \bar{\partial}_0 \big( \varrho_{\gamma\kappa\delta\lambda} \bar{\partial}_0 [h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta] \big) + \bar{\partial}_0 [h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta] \bar{\partial}_0 \varrho_{\gamma\kappa\delta\lambda} \Big\} \\
& - D_a \big( \varrho_{\gamma\kappa\delta\lambda} D^a [h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta] \big) - D_a [h^{ij} e_j^\kappa h^{kl} e_l^\lambda h^{mn} e_m^\gamma e_n^\delta] D^a \varrho_{\gamma\kappa\delta\lambda} \,.
\end{aligned}
\tag{4.5.56}
$$

- <u>*hhhnn*</u> trems: The next *hhhnn* term in Eq. (4.5.54) is

$$
\begin{aligned}
& h^{\rho\sigma} h^{\mu\kappa} h^{\nu\lambda} n^\gamma n^\delta \varrho_{\rho\mu\sigma\nu} \Big( - \big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp} \Big) \varrho_{\gamma\kappa\delta\lambda} \\
& = (h^{ij} e_i^\rho e_j^\sigma)(h^{kl} e_k^\mu e_l^\kappa)(h^{mn} e_m^\nu e_n^\lambda) n^\gamma n^\delta \varrho_{\rho\mu\sigma\nu} \Big( - \big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp} \Big) \varrho_{\gamma\kappa\delta\lambda} \\
& = \rho_{km} (h^{kl} e_l^\kappa)(h^{mn} e_n^\lambda) n^\gamma n^\delta \Big( - \big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp} \Big) \varrho_{\gamma\kappa\delta\lambda} \\
& = -N^{-2} \rho_{km} \Big\{ \bar{\partial}_0^2 (\Omega^{km}) - \bar{\partial}_0 \big( \varrho_{\gamma\kappa\delta\lambda} \bar{\partial}_0 [h^{kl} e_l^\kappa h^{mn} e_n^\lambda n^\gamma n^\delta] \big) - \bar{\partial}_0 [h^{kl} e_l^\kappa h^{mn} e_n^\lambda n^\gamma n^\delta] \bar{\partial}_0 \varrho_{\gamma\kappa\delta\lambda} \Big\} \\
& + \rho_{km} \Big\{ \Box_{hyp} (\Omega^{km}) - D_a \big( \varrho_{\gamma\kappa\delta\lambda} D^a [h^{kl} e_l^\kappa h^{mn} e_n^\lambda n^\gamma n^\delta] \big) - D_a [h^{kl} e_l^\kappa h^{mn} e_n^\lambda n^\gamma n^\delta] D^a \varrho_{\gamma\kappa\delta\lambda} \Big\} \\
& = \rho_{km} \Box \Omega^{km} + \cdots \,,
\end{aligned}
\tag{4.5.57}
$$

where we used $h^{kl}e^{\kappa}_{l}h^{mn}e^{\lambda}_{n}n^{\gamma}n^{\delta}\varrho_{\gamma\kappa\delta\lambda} = h^{kl}h^{mn}n^{\gamma}n^{\delta}\varrho_{\gamma l\delta n} = \Omega^{km}$ and we note that "$\cdots$" are extra terms which do not depend on $\Omega^{km}$.

- <u>*hhnnnn*</u> terms: The the next term in Eq. (4.5.54) is of the form *hhnnnn*:

$$
\begin{aligned}
&n^{\rho}n^{\sigma}h^{\mu\kappa}h^{\nu\lambda}n^{\gamma}n^{\delta}\varrho_{\rho\mu\sigma\nu}\left(-\left(N^{-1}\bar{\partial}_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda}\\
&=n^{\rho}n^{\sigma}(h^{ij}e^{\mu}_{i}e^{\kappa}_{j})(h^{kl}e^{\nu}_{k}e^{\lambda}_{l})n^{\gamma}n^{\delta}\varrho_{\rho\mu\sigma\nu}\left(-\left(N^{-1}\bar{\partial}_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda}\\
&=\Omega^{jl}e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}\left(-\left(N^{-1}\bar{\partial}_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda}\\
&=-N^{-2}\Omega^{jl}\Big\{\bar{\partial}_0^2(\Omega_{jl})-\bar{\partial}_0\big(\varrho_{\gamma\kappa\delta\lambda}\bar{\partial}_0[e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}]\big)-\bar{\partial}_0[e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}]\bar{\partial}_0\varrho_{\gamma\kappa\delta\lambda}\Big\}\\
&+\Omega^{jl}\Big\{\Box_{hyp}(\Omega_{jl})-D_a\big(\varrho_{\gamma\kappa\delta\lambda}D^a[e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}]\big)-D_a[e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}]D^a\varrho_{\gamma\kappa\delta\lambda}\Big\}\\
&=\Omega^{jl}\Box\Omega_{jl}+\Omega^{jl}X_{2(b)jl}\,,
\end{aligned}
\tag{4.5.58}
$$

where $e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}\varrho_{\gamma\kappa\delta\lambda} = n^{\gamma}n^{\delta}\varrho_{\gamma j\delta l} = \Omega_{jl}$, and

$$
\begin{aligned}
X_{2(b)jl} = N^{-2}&\Big\{\bar{\partial}_0\big(\varrho_{\gamma\kappa\delta\lambda}\bar{\partial}_0[e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}]\big)+\bar{\partial}_0[e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}]\bar{\partial}_0\varrho_{\gamma\kappa\delta\lambda}\Big\}\\
&-D_a\big(\varrho_{\gamma\kappa\delta\lambda}D^a[e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}]\big)-D_a[e^{\kappa}_{j}e^{\lambda}_{l}n^{\gamma}n^{\delta}]D^a\varrho_{\gamma\kappa\delta\lambda}\,.
\end{aligned}
\tag{4.5.59}
$$

- <u>*hhnnnn*</u> terms: Finally, the last *hhnnnn* terms in Eq. (4.5.54) is

$$
\begin{aligned}
&h^{\rho\sigma}n^{\mu}n^{\kappa}n^{\nu}n^{\lambda}h^{\gamma\delta}\varrho_{\rho\mu\sigma\nu}\left(-\left(N^{-1}\bar{\partial}_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda}\\
&=(h^{ij}e^{\rho}_{i}e^{\sigma}_{j})n^{\mu}n^{\kappa}n^{\nu}n^{\lambda}(h^{mn}e^{\gamma}_{m}e^{\delta}_{n})\varrho_{\rho\mu\sigma\nu}\left(-\left(N^{-1}\bar{\partial}_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda}\\
&=\Omega n^{\kappa}n^{\lambda}h^{mn}e^{\gamma}_{m}e^{\delta}_{n}\left(-\left(N^{-1}\bar{\partial}_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda}\\
&=-N^{-2}\Omega\Big\{\bar{\partial}_0^2(\Omega)-\bar{\partial}_0\big(\varrho_{\gamma\kappa\delta\lambda}\bar{\partial}_0[n^{\kappa}n^{\lambda}h^{mn}e^{\gamma}_{m}e^{\delta}_{n}]\big)-\bar{\partial}_0[n^{\kappa}n^{\lambda}h^{mn}e^{\gamma}_{m}e^{\delta}_{n}]\bar{\partial}_0\varrho_{\gamma\kappa\delta\lambda}\Big\}\\
&+\Omega\Big\{\Box_{hyp}(\Omega)-D_a\big(\varrho_{\gamma\kappa\delta\lambda}D^a[n^{\kappa}n^{\lambda}h^{mn}e^{\gamma}_{m}e^{\delta}_{n}]\big)-D_a[n^{\kappa}n^{\lambda}h^{mn}e^{\gamma}_{m}e^{\delta}_{n}]D^a\varrho_{\gamma\kappa\delta\lambda}\Big\}\\
&=\Omega\Box\Omega+\Omega X_{2(c)}\,,
\end{aligned}
\tag{4.5.60}
$$

where we used $n^\kappa n^\lambda h^{mn} e^\gamma_m e^\delta_n \varrho_{\gamma\kappa\delta\lambda} = h^{mn}\Omega_{mn} = \Omega$, and

$$X_{2(c)} = N^{-2}\Big\{\bar\partial_0\big(\varrho_{\gamma\kappa\delta\lambda}\bar\partial_0[n^\kappa n^\lambda h^{mn}e^\gamma_m e^\delta_n]\big) + \bar\partial_0[n^\kappa n^\lambda h^{mn}e^\gamma_m e^\delta_n]\bar\partial_0\varrho_{\gamma\kappa\delta\lambda}\Big\}$$
$$- D_a\big(\varrho_{\gamma\kappa\delta\lambda}D^a[n^\kappa n^\lambda h^{mn}e^\gamma_m e^\delta_n]\big) - D_a[n^\kappa n^\lambda h^{mn}e^\gamma_m e^\delta_n]D^a\varrho_{\gamma\kappa\delta\lambda}\,. \qquad (4.5.61)$$

Summarising this result, we can write Eq. (4.5.54), as

$$\begin{aligned}\varrho_{\mu\nu}\Box\varrho^{\mu\nu} &= \Omega(\Box\Omega + X_{2(c)}) + \Omega_{ij}(\Box\Omega^{ij} + X^{ij}_{2(b)}) - \rho_{ij}\Box\Omega^{ij} \\ &- \Omega_{ij}(\Box\rho^{ij} + X^{ij}_{2(a)}) + \cdots\,, \end{aligned} \qquad (4.5.62)$$

where "$\cdots$" are the contractions of $\rho_{ijkl}$ and $\rho_{ijk}$ (see Appendix I.3) and the terms that are the results of performing Leibniz rule, which have no $\Omega_{ij}$ dependence. When $\Box \to 1$, we recover the result for $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ found in [136].

At both limits, $\Box \to 0$ and $\Box \to 1$, the $X_2$ terms are not present. Obtaining the boundary term requires us to extract $\Psi^{ij}$ as it is given in Eq. (4.4.43). Hence the boundary for $\mathcal{R}_{\mu\nu}\Box\mathcal{R}^{\mu\nu}$ is given by,

$$\begin{aligned}S_2 = &-\frac{1}{8\pi G_N}\oint_{\partial\mathbb{M}} d\Sigma_\mu n^\mu\Big[K\Box\Omega + K_{ij}\Box\Omega^{ij} - K_{ij}\Box\rho^{ij}\Big] \\ &- \frac{1}{16\pi G_N}\oint_{\partial\mathbb{M}} d\Sigma_\mu n^\mu\Big[KX_{2(c)} + K_{ij}(X^{ij}_{2(b)} - X^{ij}_{2(a)})\Big]\,, \end{aligned} \qquad (4.5.63)$$

where $K \equiv h^{ij}K_{ij}$ and $K_{ij}$ is given by Eq. (4.3.30).

### 4.5.4 $\mathcal{R}\Box\mathcal{R}$

We do not need to commute any $h$'s, or $n$'s across the $\Box$ here, we can simply apply Eq. (4.5.44) to $\varrho\Box\varrho$, the auxiliary equivalent of the $\mathcal{R}\Box\mathcal{R}$ term:

$$\varrho\Box\varrho = (\rho - 2\Omega)\,\Box\,(\rho - 2\Omega)\,, \qquad (4.5.64)$$

whereupon extracting $\Psi^{ij}$ using Eq. (4.4.43), and using Eq. (4.4.41) as in the previous cases, we obtain the boundary term for $\mathcal{R}\Box\mathcal{R}$ to be

$$S_3 = -\frac{1}{4\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu\, n^\mu\Big[2K\Box\Omega - K\Box\rho\Big], \tag{4.5.65}$$

where $K \equiv h^{ij}K_{ij}$ and $K_{ij}$ is given by Eq. (4.3.30). Again when $\Box \to 1$, we recover the result for $\mathcal{R}^2$ found in [136].

### 4.5.5 Full result

Summarising the results of Eq. (4.5.53), Eq. (4.5.63) and Eq. (4.5.65), altogether we have

$$\begin{aligned}
S &= \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\Big[\varrho + \alpha\big(\varrho\Box\varrho + \varrho_{\mu\nu}\Box\varrho^{\mu\nu} + \varrho_{\mu\nu\rho\sigma}\Box\varrho^{\mu\nu\rho\sigma}\big) + \varphi^{\mu\nu\rho\sigma}\left(\mathcal{R}_{\mu\nu\rho\sigma} - \varrho_{\mu\nu\rho\sigma}\right)\Big] \\
&- \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu\, n^\mu\Big[-K + \alpha\big(-2K\Box\varrho + 4K\Box\Omega + K\Box\Omega + 4K_{ij}\Box\Omega^{ij} - K_{ij}\Box\rho^{ij} + K_{ij}\Box\Omega^{ij}\big)\Big] \\
&- \frac{1}{16\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu n^\mu\alpha\Big[KX_{2(c)} + K_{ij}(4X_1^{ij} + X_{2(b)}^{ij} - X_{2(a)}^{ij})\Big] \\
&= \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\Big[\varrho + \alpha\big(\varrho\Box\varrho + \varrho_{\mu\nu}\Box\varrho^{\mu\nu} + \varrho_{\mu\nu\rho\sigma}\Box\varrho^{\mu\nu\rho\sigma}\big) + \varphi^{\mu\nu\rho\sigma}\left(\mathcal{R}_{\mu\nu\rho\sigma} - \varrho_{\mu\nu\rho\sigma}\right)\Big] \\
&- \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu\, n^\mu\Big[-K + \alpha\big(-2K\Box\rho + 5K\Box\Omega + 5K_{ij}\Box\Omega^{ij} - K_{ij}\Box\rho^{ij}\Big] \\
&- \frac{1}{16\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu n^\mu\alpha\Big[KX_{2(c)} + K_{ij}(4X_1^{ij} + X_{2(b)}^{ij} - X_{2(a)}^{ij})\Big]\,.
\end{aligned} \tag{4.5.66}$$

This result matches with the EH action [136], when we take the limit $\Box \to 0$; that is, we are left with the same expression for boundary as in Eq. (4.5.46):

$$\begin{aligned}
S_{EH} &= \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\Big[\varrho + \varphi^{\mu\nu\rho\sigma}\left(\mathcal{R}_{\mu\nu\rho\sigma} - \varrho_{\mu\nu\rho\sigma}\right)\Big] \\
&+ \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu\, n^\mu K\,,
\end{aligned} \tag{4.5.67}$$

since the $X$-type terms are not present when $\Box \to 0$. When $\Box \to 1$, we recover the result for $\mathcal{R} + \alpha(\mathcal{R}^2 + \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma})$ found in [136]; that is, we are

left with

$$S = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\Big[\varrho + \alpha\big(\varrho^2 + \varrho_{\mu\nu}\varrho^{\mu\nu} + \varrho_{\mu\nu\rho\sigma}\varrho^{\mu\nu\rho\sigma}\big) + \varphi^{\mu\nu\rho\sigma}\left(\mathcal{R}_{\mu\nu\rho\sigma} - \varrho_{\mu\nu\rho\sigma}\right)\Big]$$
$$- \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu\, n^\mu\Big[-K + \alpha\big(-2K\rho + 5K\Omega + 5K_{ij}\Omega^{ij} - K_{ij}\rho^{ij}\Big]\,; \quad (4.5.68)$$

again the $X$-type terms are not present when $\Box \to 1$. We should note that the $X_1$ and $X_2$ terms are the results of having the covariant d'Alembertian operator so, in the absence of the d'Alembertian operator, one does not produce them at all and hence the result found in [136] is guaranteed.

We may now turn our attention to the $\mathcal{R}\Box^2\mathcal{R}, \mathcal{R}_{\mu\nu}\Box^2\mathcal{R}^{\mu\nu}$ and $\mathcal{R}_{\mu\nu\rho\sigma}\Box^2\mathcal{R}^{\mu\nu\rho\sigma}$. Here the methodology will remain the same. One first decomposes each term into its $3+1$ equivalent. Then one extracts $\Psi^{ij}$ using Eq. (4.4.43), and then the boundary terms can be obtained using Eq. (4.4.41). In this case we will have two operators, namely

$$\Box^2 = \Big(-\big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp}\Big)\Big(-\big(N^{-1}\bar{\partial}_0\big)^2 + \Box_{hyp}\Big) \quad (4.5.69)$$

This means that upon expanding to $3+1$, one performs the Leibniz rule twice and hence obtains eight total derivatives that do not produce any $\Omega_{ij}$s or its contractions that are relevant to the boundary calculations and hence must be dropped.

### 4.5.6 Generalisation to IDG Theory

We may now turn our attention to the infinite derivative terms; namely, $\mathcal{R}\mathcal{F}_1(\Box)\mathcal{R}$, $\mathcal{R}_{\mu\nu}\mathcal{F}_2(\Box)\mathcal{R}^{\mu\nu}$ and $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)\mathcal{R}^{\mu\nu\rho\sigma}$. For such cases, we can write down the following relation (see Appendix I.4):

$$\begin{aligned} XD^{2n}Y &= D^{2n}(XY) - D^{2n-1}(D(X)Y) - D^{2n-2}(D(X)D(Y)) \\ &-D^{2n-3}(D(X)D^2(Y)) - \cdots - D(D(X)D^{2n-2}(Y)) \\ &-D(X)D^{2n-1}(Y)\,, \end{aligned} \quad (4.5.70)$$

where $X$ and $Y$ are tensorial structures such as $\varrho_{\mu\nu\rho\sigma}$, $\varrho_{\mu\nu}$, $\varrho$ and their contractions, while $D$ denotes any operators. These operators do not have to be differential operators and indeed this result can be generalised to cover the case where there are different types of operator and a similar (albeit more complicated) structure is recovered.

From (4.5.70), one produces $2n$ total derivatives, analogous to the scalar toy model case, see Eqs. (4.1.2,4.1.3). We can then write the $3+1$ decompositions for each curvature by generalising Eq. (4.5.53), Eq. (4.5.63) and Eq. (4.5.65) and writing $\mathcal{R}_{\mu\nu\rho\sigma}F_3(\Box)\mathcal{R}^{\mu\nu\rho\sigma}$, $\mathcal{R}_{\mu\nu}F_2(\Box)\mathcal{R}^{\mu\nu}$ and $\mathcal{R}F_1(\Box)\mathcal{R}$ in terms of their auxiliary equivalents $\varrho_{\mu\nu\rho\sigma}F_3(\Box)\varrho^{\mu\nu\rho\sigma}$, $\varrho_{\mu\nu}F_2(\Box)\varrho^{\mu\nu}$ and $\varrho F_1(\Box)\varrho$. Then

$$\varrho_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)\varrho^{\mu\nu\rho\sigma} = 4\Omega_{ij}\mathcal{F}_3(\Box)\Omega^{ij} + 4\Omega_{ij}X_1^{ij} + \cdots , \tag{4.5.71}$$

$$\begin{aligned}\varrho_{\mu\nu}\mathcal{F}_2(\Box)\varrho^{\mu\nu} &= \Omega\mathcal{F}_2(\Box)\Omega + \Omega_{ij}\mathcal{F}_2(\Box)\Omega^{ij} - \rho_{ij}\mathcal{F}_2(\Box)\Omega^{ij} \\ &- \Omega_{ij}\mathcal{F}_2(\Box)\rho^{ij} + \Omega_{ij}X_2^{ij} + \cdots ,\end{aligned} \tag{4.5.72}$$

$$\varrho\mathcal{F}_1(\Box)\varrho = (\rho - 2\Omega)\,\mathcal{F}_1(\Box)\,(\rho - 2\Omega) , \tag{4.5.73}$$

where we have dropped the irrelevant terms, as we did before, while $X_1^{ij}$ and $X_2^{ij}$ are the analogues of Eqs. (4.5.49), (4.5.56), (4.5.59), (4.5.61). We now need to use the generalised form of the Euler-Lagrange equations to obtain the $\Psi^{ij}$ in each case:

$$\begin{aligned}\frac{\delta f}{\delta\Omega_{ij}} &= \frac{\partial f}{\partial\Omega_{ij}} - \nabla_\mu\left(\frac{\partial f}{\partial(\nabla_\mu\Omega_{ij})}\right) + \nabla_\mu\nabla_\nu\left(\frac{\partial f}{\partial(\nabla_\mu\nabla_\nu\Omega_{ij})}\right) + \cdots \\ &= \frac{\partial f}{\partial\Omega_{ij}} + \sum_{n=1}^{\infty}\Box^n\left(\frac{\partial f}{\partial(\Box^n\Omega_{ij})}\right) ,\end{aligned} \tag{4.5.74}$$

where we have imposed that $\delta\Omega_{ij} = 0$ on the boundary $\partial\mathcal{M}$.

Hence, by using $\Omega = h^{ij}\Omega_{ij}$ and $\rho = h^{ij}\rho_{ij}$, we find in Appendix J that:

$$\begin{aligned}
\frac{\delta\big(\Omega\mathcal{F}(\Box)\Omega\big)}{\delta\Omega_{ij}} &= 2h^{ij}\mathcal{F}(\Box)\Omega, \qquad & \frac{\delta\big(\Omega_{ij}\mathcal{F}(\Box)\Omega^{ij}\big)}{\delta\Omega_{ij}} &= 2\mathcal{F}(\Box)\Omega^{ij} \\
\frac{\delta\big(\rho\mathcal{F}(\Box)\Omega\big)}{\delta\Omega_{ij}} &= h^{ij}\mathcal{F}(\Box)\rho, \qquad & \frac{\delta\big(\rho_{ij}\mathcal{F}(\Box)\Omega^{ij}\big)}{\delta\Omega_{ij}} &= \mathcal{F}(\Box)\rho^{ij} \\
\frac{\delta\big(\Omega\mathcal{F}(\Box)\rho\big)}{\delta\Omega_{ij}} &= h^{ij}\mathcal{F}(\Box)\rho, \qquad & \frac{\delta\big(\Omega_{ij}\mathcal{F}(\Box)\rho^{ij}\big)}{\delta\Omega_{ij}} &= \mathcal{F}(\Box)\rho^{ij}\,,
\end{aligned} \tag{4.5.75}$$

and so using Eq. (4.4.43), the $\Psi^{ij}$s are:

$$\begin{aligned}
\Psi^{ij}_{\rm Riem} &= -4\mathcal{F}_3(\Box)\Omega^{ij} - 2X_1^{ij} \\
\Psi^{ij}_{\rm Ric} &= \mathcal{F}_2(\Box)\rho^{ij} - h^{ij}\mathcal{F}_2(\Box)\Omega - \mathcal{F}_2(\Box)\Omega^{ij} - \frac{1}{2}X_2^{ij} \\
\Psi^{ij}_{\rm Scal} &= 2h^{ij}\mathcal{F}_1(\Box)\big(-2\Omega+\rho\big) \equiv 2h^{ij}\mathcal{F}_1(\Box)\varrho\,,
\end{aligned} \tag{4.5.76}$$

where we have used Eq. (4.5.44) in the last line. Finally, we can use Eq. (4.4.41) and write the boundary terms corresponding to our infinite-derivative action as,

$$\begin{aligned}
S_{tot} &= S_{gravity} + S_{boundary} \\
&= \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\Big[\varrho + \alpha\big(\varrho\mathcal{F}_1(\Box)\varrho + \varrho_{\mu\nu}\mathcal{F}_2(\Box)\varrho^{\mu\nu} \\
&+ \varrho_{\mu\nu\rho\sigma}\mathcal{F}_3(\Box)\varrho^{\mu\nu\rho\sigma}\big) + \varphi^{\mu\nu\rho\sigma}\left(\mathcal{R}_{\mu\nu\rho\sigma} - \varrho_{\mu\nu\rho\sigma}\right)\Big] \\
&+ \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} d\Sigma_\mu\, n^\mu\Big[K + \alpha\big(2K\mathcal{F}_1(\Box)\rho - 4K\mathcal{F}_1(\Box)\Omega \\
&- K\mathcal{F}_2(\Box)\Omega - K_{ij}\mathcal{F}_2(\Box)\Omega^{ij} + K_{ij}\mathcal{F}_2(\Box)\rho^{ij} - 4K_{ij}\mathcal{F}_3(\Box)\Omega^{ij} - 2X_1^{ij} - \frac{1}{2}X_2^{ij}\big)\Big]\,.
\end{aligned} \tag{4.5.77}$$

where $\Omega_{ij} = n^\gamma n^\delta \varrho_{\gamma i\delta j}$, $\Omega = h^{ij}\Omega_{ij}$, $\rho_{ij} = h^{km}\rho_{ijkm}$, $\rho = h^{ij}\rho_{ij}$, $K = h^{ij}K_{ij}$ and $K_{ij}$ is the extrinsic curvature given by Eq. (4.3.30). We note that when we decompose the $\Box$, after we perform the Leibniz rule enough times, we can reconstruct the $\Box$ in its original form, *i.e.* it is not affected by the use of the coframe. In this way, we can always reconstruct $\mathcal{F}_i(\Box)$. However, the form of the $X$-type terms will depend on the decomposition and therefore the use of the

coframe. In this regard, the $X$-type terms depend on the coframe but the $\mathcal{F}_i(\Box)$ terms do not.

## 4.6 Summary

This chapter generalised earlier contributions for finding the boundary term for a higher derivative theory of gravity. Our work focused on seeking the boundary term or GHY contribution for a covariant infinite derivative theory of gravity, which is quadratic in curvature.

Indeed, in this case some novel features distinctively filter through our analysis. Since the bulk action contains non-local form factors, $\mathcal{F}_i(\Box)$, the boundary action also contains the non-locality, as can be seen from our final expression Eq. (4.5.77). Eq. (4.5.77) *also* has a smooth limit when $M \to \infty$, or $\Box \to 0$, which is the local limit, and our results then reproduce the GHY term corresponding to the EH action, and when $\mathcal{F}_i(\Box) \to 1$, our results coincide with that of [136].

# Chapter 5

# Thermodynamics of infinite derivative gravity

In this chapter we will look at the thermodynamical aspects of the infinite derivative gravity (IDG). In particular, we are going to study the first law of thermodynamics [106] for number of cases. In other words, we are going to obtain the entropy of IDG and some other theories of modified gravity for static and spherically symmetric, $(A)dS$ and rotating background.

To proceed, we will briefly review how Wald [147, 148] derived the entropy from an integral over the Noether charge. We will use Wald's approach to find the entropy for static and spherically symmetric and $(A)dS$ backgrounds. For the rotating case, we are going to use the variation principle and obtain the generalised Komar integrals [149] for gravitational actions constructed by Ricci scalar, Ricci tensor and their derivatives. By using the Komar integrals we will obtain the energy and the angular momentum and finally the entropy using the first law. We finally shall use the Wald's approach for a non-local action containing inverse d'Alambertian operators and calculate the entropy in such case.

## 5.1 Wald's entropy, a brief review

The Bekenstein-Hawking [107, 108] law states that the entropy of a black hole, $S_{BH}$, is proportional to its horizon's area $A$ in units of Newton's constant. A black hole in Einstein's theory of gravity has entropy of,

$$S_{BH} = \frac{A}{4G_N}. \tag{5.1.1}$$

The above relation indicates that the entropy as it stands is geometrical and defined strictly by the black hole horizon. This relation shall satisfy the first law of black hole mechanics,

$$T_H dS = dM, \tag{5.1.2}$$

where $M$ is the conserved or the ADM mass, and $T_H = \kappa/2\pi$ is the Hawking temperature in terms of the surface gravity, $\kappa$. For the sake of simplicity, we assumed that no charge or rotation is involved with the black hole. We shall also note that the conserved mass and the surface gravity are well defined for a stationary black hole and thus their definitions are free of modification when considering various types of gravitational theories. The Wald entropy [147], $S_W$, is also a geometric entropy and interpreted by Noether charge for space-time diffeomorphisms. This entropy can be represented as a closed integral over a cross-section of the horizon, $\mathcal{H}$,

$$S_W = \oint_{\mathcal{H}} s_W dA\,, \tag{5.1.3}$$

where $s_W$ is the entropy per unit of horizon cross-sectional area. For a $D$-dimensional space-time with metric $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + \sum_{i,j=1}^{D-2}\sigma_{ij}dx_i dx_j$ , $dA = \sqrt{\sigma}dx_1 \dots dx_{D-2}$ .

In order to derive (5.1.3), we start by varying a Lagrangian density $L$ with respect to all fields $\{\psi\}$, which includes the metric. In compact presentation, (with all tensor indices suppressed),

$$\delta L \;=\; \mathcal{E}\cdot\delta\psi \;+\; d[\theta\,(\delta\psi)], \tag{5.1.4}$$

where $\mathcal{E} = 0$ are the equations of motion and the dot denotes a summation over all fields and contractions of tensor indices. Also, $d$ denotes a total derivative, so that $\theta$ is a boundary term.

We shall now introduce $\pounds_\xi$ to be a Lie derivative operating along some vector field $\xi$. Due to the diffeomorphism invariance of the theory we have,

$$\delta_\xi \psi = \pounds_\xi \psi,$$

and

$$\delta_\xi L = \pounds_\xi L = d\left(\xi \cdot L\right) \ .$$

with the help of the above identity and (5.1.4) we can identify the associated Noether current, $J_\xi$, as:

$$J_\xi \ = \ \theta\left(\pounds_\xi \psi\right) - \xi \cdot L \ . \tag{5.1.5}$$

In order to satisfy the equations of motion, *i.e.* $\mathcal{E} = 0$, we should have $dJ_\xi = 0$. This indicates that, there should be an associated potential, $Q_\xi$, such that $J_\xi = dQ_\xi$. Now, if $D$ is the dimension of the space-time and $\mathcal{S}$ is a $D-1$ hypersurface with a $D-2$ spacelike boundary $\partial\mathcal{S}$, then

$$\int_{\mathcal{S}} J_\xi \ = \ \int_{\partial\mathcal{S}} Q_\xi \ , \tag{5.1.6}$$

is the associated Noether charge.

Wald [147] proved in detail that the black holes' first law can be satisfied by defining the entropy in terms of a particular form of Noether charge. This is to choose surface $S$ as the horizon, $\mathcal{H}$, and the vector field, $\xi$, as the horizon Killing vector, $\chi$ (with appropriate normalisation to the surface gravity). Wald represented such entropy as [1] ,

$$S_W \ \equiv \ 2\pi \oint_{\mathcal{H}} Q_\chi \ . \tag{5.1.7}$$

[1]Note that since $\chi = 0$ on the horizon, the most right hand side term in (5.1.5) is not contributing to the Wald entropy. See [147].

To understand the charge let us begin with an example. Suppose we have a Lagrangian that is $\mathcal{L} = \mathcal{L}(g_{ab}, \mathcal{R}_{abcd})$ (This can be extended to the derivatives of the curvatures too),

$$L = \sqrt{-g}\mathcal{L}, \tag{5.1.8}$$

the variation of the above Lagrangian density is,

$$\delta L \;=\; -2\nabla_a \left(\mathcal{X}^{abcd}\nabla_c \delta g_{bd}\sqrt{-g}\right) \;+\; \cdots \;, \tag{5.1.9}$$

where dots indicates that we have dropped the irrelevant terms to the entropy. Moreover,

$$\mathcal{X}^{abcd} \;\equiv \frac{\partial \mathcal{L}}{\partial \mathcal{R}_{abcd}} \;. \tag{5.1.10}$$

The boundary can then be expressed as,

$$\theta \;=\; -2n_a \mathcal{X}^{abcd}\nabla_c \delta g_{bd}\sqrt{\gamma} \;+\; \cdots \;, \tag{5.1.11}$$

where $n^a$ is the unit normal vector and $\gamma_{ab}$ is the induced metric for the chosen surface $\mathcal{S}$. For an arbitrary diffeomorphism $\delta_\xi g_{ab} \;=\; \nabla_a \xi_b + \nabla_b \xi_a$ , the associated Noether current is given by

$$J \;=\; -2\nabla_a \left(\mathcal{X}^{abcd}\nabla_c \left(\nabla_b \xi_a + \nabla_a \xi_b\right) n_a \sqrt{h}\right) \;+\; \cdots \;. \tag{5.1.12}$$

We note that $h_{ab}$ is the induced metric corresponding to the $\partial\mathcal{S}$. Let us now assign the following: the horizon $\mathcal{S} \to \mathcal{H}$ and (normalised) Killing vector $\xi^a \to \chi^a$ ; so that $n_a\sqrt{h} \to \epsilon_a\sqrt{\sigma}$ with $\epsilon_a \equiv \epsilon_{ab}\chi^b$ , also we have $\epsilon_{ab} \equiv \nabla_a \chi_b$ as the binormal vector for the horizon. By noting that $\epsilon_{ab} \;=\; -\epsilon_{ba}$ and also using the symmetries of $\mathcal{X}^{abcd}$ which is due to the presence of Riemann tensor, we get

$$J \;=\; -2\nabla_b \left(\mathcal{X}^{abcd}\nabla_c \chi_d \epsilon_a \sqrt{\sigma}\right) \;+\; \cdots \;. \tag{5.1.13}$$

Finally the potential becomes:

$$Q \;=\; -\mathcal{X}^{abcd}\epsilon_{ab}\epsilon_{cd}\sqrt{\sigma} \;+\; \cdots \tag{5.1.14}$$

and so

$$S_W \;=\; -2\pi \oint_{\mathcal{H}} \mathcal{X}^{abcd}\epsilon_{ab}\epsilon_{cd}dA. \tag{5.1.15}$$

## 5.2 Spherically symmetric backgrounds

In this section we are going to use the Wald's approach [147] to obtain the entropy for number of cases where the space-time is defined by a spherically symmetric solution. In particular we will focus on a generic and homogenous spherically symmetric background. We then extend our calculations to the linearised limit and then to the $(A)dS$ backgrounds.

### 5.2.1 Generic static and spherically symmetric background

Let us recall the IDG action given by (2.1.12). In $D$-dimensions we can rewrite the action as:

$$\begin{aligned} I^{tot} &= \frac{1}{16\pi G_N^{(D)}}\int d^Dx\sqrt{-g}\big[R \\ &+ \alpha\big(R\mathcal{F}_1(\Box)R + R_{\mu\nu}\mathcal{F}_2(\Box)R^{\mu\nu} + R_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)R^{\mu\nu\lambda\sigma}\big)\big], \end{aligned} \tag{5.2.16}$$

where $G_N^{(D)}$ is the $D$-dimensional Newton's constant [1]; $\alpha$ is a constant [2] with dimension of inverse mass squared; and $\mu,\ \nu,\ \lambda,\ \sigma$ run from $0,\ 1,\ 2, \cdots D-1$. The form factors given by $\mathcal{F}_i(\Box)$ contain an infinite number of covariant derivatives, of the form:

$$\mathcal{F}_i(\Box) \equiv \sum_{n=0}^{\infty} f_{i_n}\left(\frac{\Box}{M^2}\right)^n, \tag{5.2.17}$$

with constants $f_{i_n}$, and $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ being the D'Alembertian operator. The reader should note that, in our presentation, the function $\mathcal{F}_i(\Box)$ comes with

[1] In $D$-dimensions $G_N^{(D)}$ has dimension of $[G_N^{(D)}] = [G_N^{(4)}]L^{D-4}$ where $L$ is unit length.

[2] Note that for an arbitrary choice of $\mathcal{F}(\Box)$ at action level, $\alpha$ can be positive or negative as one can absorbs the sign into the coefficients $f_{i_n}$ contained within $\mathcal{F}(\Box)$ to keep the overall action unchanged, however $\alpha$ has to be strictly positive once we impose ghost-free condition (to be seen later). [53]

an associated $D$-dimensional mass scale, $M \leq M_P = (1/\sqrt{(8\pi G_N^{(D)})})$, which determines the scale of non-locality in a quantum sense, see [96].

In the framework of Lagrangian field theory, Wald [147] showed that one can find the gravitational entropy by varying the Lagrangian and subsequently finding the Noether current as a function of an assigned vector field. By writing the corresponding Noether charge, it has been shown that, for a static black hole, the first law of thermodynamics can be satisfied and the entropy may be expressed by integrating the Noether charge over a bifurcation surface of the horizon. In so doing, one must choose the assigned vector field to be a horizon Killing vector, which has been normalised to unit surface gravity.

In order to compute the gravitational entropy of the IDG theory outlined above, we take a $D$-dimensional, static, homogenous and spherically symmetric metric of the form [141],

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{D-2}^2 \,. \tag{5.2.18}$$

For a spherically symmetric metric the Wald entropy given in (5.1.15) can be written as,

$$S_W = -2\pi \oint \frac{\delta \mathcal{L}}{\delta R_{abcd}} \epsilon_{ab}\epsilon_{cd} r^{D-2} d\Omega_{D-2}^2 \tag{5.2.19}$$

where we shall note that $\delta$ denotes the functional differentiation for a Lagrangian that not only does it include the metric and the curvature but also the derivatives of the curvature, *i.e.*

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\lambda\rho}, \nabla_{a_1} R_{\mu\nu\lambda\rho}, \ldots, \nabla_{(\alpha_1} \ldots \nabla_{\alpha_m)} R_{\mu\nu\lambda\rho}) \tag{5.2.20}$$

Note that the parentheses denote symmetrisation. Moreover, the integral in (5.2.19) is over $D-2$ dimensional space-like bifurcation surface. The $\epsilon_{ab}$ is the binormal vector to the bifurcation surface. This normal vector is antisymmetric under the exchange of $a \leftrightarrow b$ and normalised as $\epsilon_{ab}\epsilon^{ab} = -2$. For metric (5.2.18), the bifurcation surface is at $r = r_H$ and $t$ =constant. We note that $d\Omega_{D-2}^2$ is the

spherical element [1]. In the case of (5.2.18), the relevant Killing vector is $\partial_t$ and $\epsilon_{tr} = 1$. The $\epsilon$'s vanish for $a, b \neq t, r$. We finally write the Wald entropy as,

$$S_W = -8\pi \oint \frac{\delta \mathcal{L}}{\delta R_{rtrt}} r^{D-2} d\Omega^2_{D-2}\,. \tag{5.2.21}$$

Subsequently, we shall define the area of the horizon [139], that is,

$$A_H = \oint r^{D-2} d\Omega^2_{D-2} = \frac{2\pi^{n/2} r^{n-1}}{\Gamma(\frac{n}{2})}, \tag{5.2.22}$$

where $n = D-1$. As an example for a 4-dimensional metric of the form given by (5.2.18), we have,

$$A_H = \oint r^2 d\Omega^2_2 = \int_0^{2\pi} d\phi \int_0^{2\pi} r^2 \sin(\theta) d\theta = 4\pi r^2 \equiv \frac{2\pi^{3/2} r^2}{\Gamma(\frac{3}{2})}. \tag{5.2.23}$$

It is now possible to use the generalised Euler-Lagrange equation and calculate the functional differentiation given in (5.2.19). That is,

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta R_{abcd}} &= \frac{\partial \mathcal{L}}{\partial R_{abcd}} - \nabla_{\mu_1}\Big(\frac{\partial \mathcal{L}}{\partial(\nabla_{\mu_1} R_{abcd})}\Big) + \nabla_{\mu_1}\nabla_{\mu_2}\Big(\frac{\partial \mathcal{L}}{\partial(\nabla_{\mu_1}\nabla_{\mu_2} R_{abcd})}\Big) - \cdots \\ &+ (-1)^m \nabla_{(\mu_1 \cdots}\nabla_{\mu_m)} \frac{\partial \mathcal{L}}{\partial(\nabla_{(\mu_1 \cdots}\nabla_{\mu_m)} R_{abcd})}, \end{aligned} \tag{5.2.24}$$

where we shall note that parentheses denote symmetrisation. Now we are set to calculate the entropy for the IDG action, (5.2.16) via (5.2.21). To do so, we are

[1] In four dimensions we have $d\Omega^2_2 = d\theta + \sin^2\theta d\phi$.

required to calculate the quantity $\frac{\delta\mathcal{L}}{\delta R_{rtrt}}$. We have,

$$\begin{aligned}
\frac{\delta R}{\delta R_{abcd}} &= \frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc}), && (5.2.25)\\
\frac{\delta(R\mathcal{F}_1(\Box)R)}{\delta R_{abcd}} &= \mathcal{F}_1(\Box)(g^{ac}g^{bd} - g^{ad}g^{bc})R, && (5.2.26)\\
\frac{\delta(R_{\mu\nu}\mathcal{F}_2(\Box)R^{\mu\nu})}{\delta R_{abcd}} &= \frac{1}{2}\mathcal{F}_2(\Box)(g^{ac}R^{bd} - g^{ad}R^{bc}\\
&- g^{bc}R^{ad} + g^{bd}R^{ac}), && (5.2.27)\\
\frac{\delta(R_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)R^{\mu\nu\lambda\sigma})}{\delta R_{abcd}} &= 2\mathcal{F}_1(\Box)R^{abcd}, && (5.2.28)
\end{aligned}$$

by assigning $(a, b, c, d) \to (r, t, r, t)$ we obtain,

$$\begin{aligned}
\frac{\delta R}{\delta R_{rtrt}} &= \frac{1}{2}(g^{rr}g^{tt} - g^{rt}g^{tr}) = -\frac{1}{2}, && (5.2.29)\\
\frac{\delta(R\mathcal{F}_1(\Box)R)}{\delta R_{rtrt}} &= -\mathcal{F}_1(\Box)R, && (5.2.30)\\
\frac{\delta(R_{\mu\nu}\mathcal{F}_2(\Box)R^{\mu\nu})}{\delta R_{rtrt}} &= \frac{1}{2}\mathcal{F}_2(\Box)(g^{rr}R^{tt} + g^{tt}R^{rr}), && (5.2.31)\\
\frac{\delta(R_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)R^{\mu\nu\lambda\sigma})}{\delta R_{rtrt}} &= 2\mathcal{F}_1(\Box)R^{rtrt}, && (5.2.32)
\end{aligned}$$

where we note that $g^{tt}g^{rr} = -1$, and $g^{tr} = g^{rt} = 0$. See Appendix L for detailed derivation of the above functional differentiation. By using the Wald's formula given in (5.2.21) we have [118],

$$S_W = \frac{A_H}{4G_N^{(D)}}\big[1+\alpha(2\mathcal{F}_1(\Box)R-\mathcal{F}_2(\Box)\times(g^{rr}R^{tt}+g^{rr}R^{rr})-4\mathcal{F}_3(\Box)R^{rtrt})\big] \quad (5.2.33)$$

It is convenient, for illustrative purposes, to decompose the entropy equation into its $(r, t)$ and spherical components. For the metric given in (5.2.18) we denote the $r$ and $t$ directions by the indices $\{a, b\}$; and the spherical components by $\{m, n\}$. As such, we express the curvature scalar as follows

$$R = g^{\mu\nu}R_{\mu\nu} = g^{ab}R_{ab} + g^{mn}R_{mn}, \quad (5.2.34)$$

where $g_{ab}$ is a 2-dimensional metric tensor accounting for the $r, t$ directions and $g_{mn}$ is a $(D-2)$-dimensional metric tensor, corresponding to the angular components, such that

$$g^{\mu\nu}g_{\mu\nu} \equiv g^{ab}g_{ab} + g^{mn}g_{mn} = 2 + (D-2) = D. \tag{5.2.35}$$

Expanding the scalar curvature into Ricci and Riemann tensors, along with the properties of the static, spherically symmetric metric (5.2.18), allows us to express the relevant components of the entropy equation as follows:

$$g^{rr}R^{tt} + g^{rr}R^{rr} = -g_{tt}R^{tt} - g_{rr}R^{rr} = -g^{ab}R_{ab}. \tag{5.2.36}$$

Moreover,

$$g^{ab}R_{ab} = g^{ab}R^{\lambda}{}_{a\lambda b} = g^{ab}g^{\lambda\tau}R_{\tau a\lambda b} = g^{ab}g^{cd}R_{dacb} + g^{ab}g^{mn}R_{namb}, \tag{5.2.37}$$

we can write above as,

$$-g^{ab}g^{cd}R_{dacb} = -g^{ab}R_{ab} + g^{ab}g^{mn}R_{namb}, \tag{5.2.38}$$

by assigning the coordinates we have,

$$\begin{aligned} &-g^{rr}g^{rr}R_{rrrr} - g^{tt}g^{tt}R_{tttt} - g^{rr}g^{tt}R_{trtr} - g^{tt}g^{rr}R_{rtrt} \\ &= -g^{ab}R_{ab} + g^{ab}g^{mn}R_{namb}, \end{aligned} \tag{5.2.39}$$

given that for metric (5.2.18), $R_{rrrr} = R_{tttt} = 0$, $R_{trtr} = R_{rtrt}$ and $g^{tt}g^{rr} = -1$, we have,

$$2R_{rtrt} = -g^{ab}R_{ab} + g^{ab}g^{mn}R_{namb}, \tag{5.2.40}$$

or,

$$-4R_{rtrt} = 2g^{ab}R_{ab} - 2g^{ab}g^{mn}R_{manb}. \tag{5.2.41}$$

Substitution into Eq. (5.2.33), results in a decomposed $D$-dimensional entropy

equation for the action (5.2.16) in a static, spherically symmetric background :

$$\begin{aligned} S_W &= \frac{A_H}{4G_N^{(D)}}[1+\alpha(2\mathcal{F}_1(\Box)+\mathcal{F}_2(\Box)+2\mathcal{F}_3(\Box))g^{ab}R_{ab} \\ &+ 2\alpha(\mathcal{F}_1(\Box)g^{mn}R_{mn}-\mathcal{F}_3(\Box)g^{ab}g^{mn}R_{manb})]. \end{aligned} \tag{5.2.42}$$

### 5.2.2 Linearised regime

In this section we shall study an interesting feature of the entropy given in (5.2.42). To begin, let us consider the perturbations around $D$-dimensional Minkowski spacetime [1] with metric tensor $\eta_{\mu\nu}$, such that $\eta_{\mu\nu}\eta^{\mu\nu}=D$, and where the perturbations are denoted by $h_{\mu\nu}$ so that $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$. One should also note that we are using *mostly plus* metric signature convention.

The $\mathcal{O}(h^2)$ expressions for the Riemann tensor, Ricci tensor and curvature scalar in $D$-dimensions are given by [111, 141]:

$$\begin{aligned} R_{\mu\nu\lambda\sigma} &= \frac{1}{2}(\partial_{[\lambda}\partial_\nu h_{\mu\sigma]}-\partial_{[\lambda}\partial_\mu h_{\nu\sigma]}) \\ R_{\mu\nu} &= \frac{1}{2}(\partial_\sigma\partial_{(\nu}\partial^\sigma_{\mu)}-\partial_\mu\partial_\nu h-\Box h_{\mu\nu}) \\ R &= \partial_\mu\partial_\nu h^{\mu\nu}-\Box h. \end{aligned} \tag{5.2.43}$$

Thus, the IDG action given in (5.2.16) can be written as [94],

$$\begin{aligned} S_{(2)} &= \frac{1}{32\pi G_N^{(D)}}\int d^Dx\Big[\frac{1}{2}h_{\mu\nu}\Box a(\bar{\Box})h^{\mu\nu}+h^\sigma_\mu b(\bar{\Box})\partial_\sigma\partial_\nu h^{\mu\nu} \\ &+ hc(\bar{\Box})\partial_\mu\partial_\nu h^{\mu\nu}+\frac{1}{2}h\Box d(\bar{\Box})h+h^{\lambda\sigma}\frac{f(\bar{\Box})}{2\Box}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\mu\nu}\Big], \end{aligned} \tag{5.2.44}$$

where we have $\bar{\Box}\equiv\Box/M^2$. In above action we have [94],

$$R\mathcal{F}_1(\bar{\Box})R=\mathcal{F}_1(\bar{\Box})[h\Box^2h+h^{\lambda\sigma}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\mu\nu}-2h\Box\partial_\mu\partial_\nu h^{\mu\nu}], \tag{5.2.45}$$

[1] Later we shall see that in linearised regime we can take $f(r)=(1+2\Phi(r))$ and $f(r)^{-1}=(1-2\Psi(r))$, thus (5.2.18) will be of the form of (5.2.61).

$$R_{\mu\nu}\mathcal{F}_2(\bar{\Box})R^{\mu\nu} = \mathcal{F}_2(\bar{\Box})[\frac{1}{4}h\Box^2 h + \frac{1}{4}h^{\mu\nu}\Box^2 h^{\mu\nu} - \frac{1}{2}h^{\sigma}_{\mu}\Box\partial_\sigma\partial_\nu h^{\mu\nu} - \frac{1}{2}h\Box\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1}{2}h^{\lambda\sigma}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\mu\nu}], \tag{5.2.46}$$

$$R_{\mu\nu\lambda\sigma}\mathcal{F}_3(\bar{\Box})R^{\mu\nu\lambda\sigma} = \mathcal{F}_3(\bar{\Box})[h^{\mu\nu}\Box^2 h^{\mu\nu} - 2h^{\sigma}_{\mu}\Box\partial_\sigma\partial_\nu h^{\mu\nu} + h^{\lambda\sigma}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\mu\nu}]. \tag{5.2.47}$$

As a result, $a(\bar{\Box}), b(\bar{\Box}), c(\bar{\Box}), d(\bar{\Box})$ and $f(\bar{\Box})$ are given by [94],

$$a(\bar{\Box}) = 1 + M_P^{-2}(\mathcal{F}_2(\bar{\Box})\Box + 4\mathcal{F}_3(\bar{\Box})\Box), \tag{5.2.48}$$

$$b(\bar{\Box}) = -1 - M_P^{-2}(\mathcal{F}_2(\bar{\Box})\Box + 4\mathcal{F}_3(\bar{\Box})\Box), \tag{5.2.49}$$

$$c(\bar{\Box}) = 1 - M_P^{-2}(4\mathcal{F}_1(\bar{\Box})\Box + \mathcal{F}_2(\bar{\Box})\Box), \tag{5.2.50}$$

$$d(\bar{\Box}) = -1 + M_P^{-2}(4\mathcal{F}_1(\bar{\Box})\Box + \mathcal{F}_2(\bar{\Box})\Box), \tag{5.2.51}$$

$$f(\bar{\Box}) = 2M_P^{-2}(2\mathcal{F}_1(\bar{\Box})\Box + \mathcal{F}_2(\bar{\Box})\Box + 2\mathcal{F}_3(\bar{\Box})\Box). \tag{5.2.52}$$

It can be noted that,

$$a(\bar{\Box}) + b(\bar{\Box}) = 0, \tag{5.2.53}$$

$$c(\bar{\Box}) + d(\bar{\Box}) = 0, \tag{5.2.54}$$

$$b(\bar{\Box}) + c(\bar{\Box}) + f(\bar{\Box}) = 0, \tag{5.2.55}$$

$$a(\bar{\Box}) - c(\bar{\Box}) = f(\bar{\Box}). \tag{5.2.56}$$

By varying (5.2.44), one obtains the field equations, which can be represented in terms of the inverse propagator. By writing down the spin projector operators in $D$-dimensional Minkowski space and representing them in terms of the momentum space one can obtain the graviton $D$-dimensional propagator (around Minkowski

space) as [1],

$$\Pi^{(D)}(-k^2) = \frac{\mathcal{P}^2}{k^2 a(-k^2)} + \frac{\mathcal{P}_s^0}{k^2[a(-k^2) - (D-1)c(-k^2)]}. \tag{5.2.57}$$

We note that, $\mathcal{P}^2$ and $\mathcal{P}_s^0$ are tensor and scalar spin projector operators respectively. Since we do not wish to introduce any extra propagating degrees of freedom apart from the massless graviton, we are going to take $f(\bar{\Box}) = 0$. Thus,

$$\Pi^{(D)}(-k^2) = \frac{1}{k^2 a(-k^2)} \left( \mathcal{P}^2 - \frac{1}{D-2} \mathcal{P}_s^0 \right). \tag{5.2.58}$$

To this end, the form of $a(-k^2)$ should be such that it does not introduce any new propagating degree of freedom, and it was argued in Ref. [53, 68] that the form of $a(\Box)$ should be an *entire function*, so as not to introduce any pole in the complex plane, which would result in additional degrees of freedom in the momentum space.

Furthermore, the form of $a(-k^2)$ should be such that in the IR, for $k \to 0$, $a(-k^2) \to 1$, therefore recovering the propagator of GR in the $D$-dimensions. For $D = 4$, the propagator has the familiar $1/2$ factor in front of the scalar part of the propagator. One such example of an *entire function* is [53, 68]:

$$a(\bar{\Box}) = e^{-\bar{\Box}}, \tag{5.2.59}$$

which has been found to ameliorate the UV aspects of gravity while recovering the Newtonian limit in the IR. We conclude that choosing $f(\bar{\Box}) = 0$, yields $a(\bar{\Box}) = c(\bar{\Box})$ and therefore we get the following constraint:

$$2\mathcal{F}_1(\bar{\Box}) + \mathcal{F}_2(\bar{\Box}) + 2\mathcal{F}_3(\bar{\Box}) = 0. \tag{5.2.60}$$

At this point, the entropy found in (5.2.42) is very generic prediction for the IDG action. Indeed, the form of entropy is irrespective of the form of $a(\bar{\Box})$. Let us

[1] Obtaining the graviton propagator for the IDG action is not in the scope of this thesis. Such analysis have been done extensively and in detail by [94, 118].

assume that the $(t,\ r)$ component of the original spherically symmetric metric given by (5.2.18) takes the form:

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 - 2\Psi(r))dr^2 + r^2 d\Omega_2^2 \tag{5.2.61}$$

In fact, $\Phi$ and $\Psi$ are the two Newtonian potentials. Note that we now took $D = 4$ in the metric above for the sake of clarity. Considering the perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we have

$$h_{tt} = h^{tt} = -2\Phi, \quad h_{rr} = h^{rr} = -2\Psi, \tag{5.2.62}$$

$$h_{\theta\theta} = h^{\theta\theta} = 0, \quad h_{\phi\phi} = h^{\phi\phi} = 0. \tag{5.2.63}$$

As we are in the spherical coordinate we shall take the spherical form of the d'Alembertian operator,

$$\Box u = \frac{1}{r^2}\partial_r(r^2\partial_r u) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta u) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 u - \partial_t^2 u, \tag{5.2.64}$$

where $u$ is some variable at which we are operating the d'Alembertian operator at. However, since $\Phi$ and $\Psi$ are $r$-dependent, we are only left with the first term, *i.e.*

$$\Box u = \frac{1}{r^2}\partial_r(r^2\partial_r u). \tag{5.2.65}$$

Now let us take the Wald entropy found in (5.2.42) and calculate the relevant components in the linearised limit,

$$R_{ab} = R_{tt} + R_{rr}, \tag{5.2.66}$$

expanding the (5.2.43) gives,

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\mu h_\nu^\sigma + \partial_\nu\partial_\sigma h_\mu^\sigma - \partial_\nu\partial_\mu h - \Box h_{\mu\nu}), \tag{5.2.67}$$

hence,

$$\begin{aligned} R_{tt} &= \frac{1}{2}(\partial_\sigma\partial_t h_t^\sigma + \partial_t\partial_\sigma h_t^\sigma - \partial_t\partial_t h - \Box h_{tt}) \\ &= -\frac{1}{2}\Box h_{tt} = \Box\Phi = \Phi'' + \frac{2\Phi'}{r}, \end{aligned} \tag{5.2.68}$$

where 'prime' is differentiation with respect to $r$. Next we have,

$$\begin{aligned} R_{rr} &= \frac{1}{2}(\partial_\sigma\partial_r h_r^\sigma + \partial_r\partial_\sigma h_r^\sigma - \partial_r\partial_r h - \Box h_{rr}) \\ &= \frac{1}{2}(\partial_r\partial_r h_r^r + \partial_r\partial_r h_r^r - \partial_r\partial_r h - \Box h_{rr}) \\ &= \frac{1}{2}(2\partial_r^2 h_r^r - \partial_r\partial_r h - \Box h_{rr}) \\ &= \frac{1}{2}(2\partial_r^2(\eta^{rr}h_{rr}) - \partial_r^2(\eta^{tt}h_{tt} + \eta^{rr}h_{rr}) - \Box h_{rr}) \\ &= \frac{1}{2}(-4\Psi'' - 2\Phi'' + 2\Psi'' + 2\Psi'' + \frac{4\Psi'}{r}) \\ &= -\Phi'' + \frac{2\Psi'}{r}. \end{aligned} \tag{5.2.69}$$

We note that,

$$R_{mn} = R_{\theta\theta} + R_{\phi\phi} = 0. \tag{5.2.70}$$

Moving to the Riemann tensor, we have,

$$R_{\rho\mu\sigma\nu} = \frac{1}{2}(\partial_\sigma\partial_\mu h_{\rho\nu} + \partial_\nu\partial_\rho h_{\mu\sigma} - \partial_\nu\partial_\mu h_{\rho\sigma} - \partial_\sigma\partial_\rho h_{\mu\nu}), \tag{5.2.71}$$

and thus,

$$R_{rtrt} = \frac{1}{2}(\partial_r\partial_t h_{rt} + \partial_t\partial_r h_{tr} - \partial_t\partial_t h_{rr} - \partial_r\partial_r h_{tt}) = \Phi''. \tag{5.2.72}$$

From (5.2.41), it follows that,

$$\begin{aligned} g^{ab}g^{mn}R_{manb} &= 2R_{rtrt} + g^{ab}R_{ab} \\ &= 2R_{rtrt} + \eta^{tt}R_{tt} + \eta^{rr}R_{rr} \\ &= 2\Phi'' - \Phi'' - \frac{2\Phi'}{r} - \Phi'' + \frac{2\Psi'}{r} \\ &= -\frac{2\Phi'}{r} + \frac{2\Psi'}{r}. \end{aligned} \tag{5.2.73}$$

Now let us look back at the entropy equation given in (5.2.42) , and plug in the values,

$$S_W = \frac{A_H}{4G_N}\Big[1 - 2\Phi + 2\Psi - 4\alpha\mathcal{F}_3(\bar{\Box})\big(\frac{\Psi' - \Phi'}{r}\big)\Big], \tag{5.2.74}$$

where we used the constraint (5.2.60). Now if we take the Newtonian potentials to be equal, *i.e.* $\Phi(r) = \Psi(r)$,

$$S_W = \frac{A_H}{4G_N}. \tag{5.2.75}$$

To sum up we have shown that, the entropy for a spherically symmetric background in the linearised regime and within the IDG framework is given only by the area law. This is upon requiring that the massless graviton be the only propagating mode in the Minkowski background. In other words, we required in the linearised regime that $2\mathcal{F}_1(\bar{\Box}) + \mathcal{F}_2(\bar{\Box}) + 2\mathcal{F}_3(\bar{\Box}) = 0$.

### 5.2.3 $D$-Dimensional $(A)dS$ Entropy

We now turn our attention to another class of solutions which contain an horizon, such as the $(A)dS$ metrics [118], where the $D$-dimensional non-local action Eq. (5.2.16) must now be appended with a cosmological constant $\Lambda$ to ensure

that $(A)dS$ is a vacuum solution,

$$I^{tot} = \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \big[ R - 2\Lambda + \alpha \big( R\mathcal{F}_1(\Box)R + R_{\mu\nu}\mathcal{F}_2(\Box)R^{\mu\nu} + R_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)R^{\mu\nu\lambda\sigma} \big) \big] . \quad (5.2.76)$$

The cosmological constant is then given by

$$\Lambda = \pm \frac{(D-1)(D-2)}{2l^2} , \quad (5.2.77)$$

where the positive sign corresponds to $dS$, negative to $AdS$, and hereafter, the topmost sign will refer to $dS$ and the bottom to $AdS$. $l$ denotes the cosmological horizon. The $(A)dS$ metric can be obtained by taking

$$f(r) = \left( 1 \mp \frac{r^2}{l^2} \right) , \quad (5.2.78)$$

in Eq. (5.2.18). Recalling the $D$-dimensional entropy Eq. (5.2.42), we write,

$$\begin{aligned} S_W^{(A)dS} &= \frac{A_H^{(A)dS}}{4G_N^{(D)}} [1 + \alpha(2\mathcal{F}_1(\Box) + \mathcal{F}_2(\Box) + 2\mathcal{F}_3(\Box)) g^{ab} R_{ab} \\ &+ 2\alpha(\mathcal{F}_1(\Box) g^{\bar{m}\bar{n}} R_{\bar{m}\bar{n}} - \mathcal{F}_3(\Box) g^{ab} g^{\bar{m}\bar{n}} R_{\bar{m}a\bar{n}b})] , \end{aligned} \quad (5.2.79)$$

where now $A_H^{(A)dS} \equiv l^{D-2} A_{D-2}$, with $A_{D-2} = (2\pi^{\frac{D-1}{2}})/\Gamma[\frac{D-1}{2}]$. Given the $D$-dimensional definitions of curvature in $(A)dS$ background,

$$R_{\mu\nu\lambda\sigma} = \pm \frac{1}{l^2} g_{[\mu\lambda} g_{\nu]\sigma}, \quad R_{\mu\nu} = \pm \frac{D-1}{l^2} g_{\mu\nu}, \quad R = \pm \frac{D(D-1)}{l^2}, \quad (5.2.80)$$

simple substitution reveals the gravitational entropy in $(A)dS$ can be expressed as:

$$S_W^{(A)dS} = \frac{A_H^{(A)dS}}{4G_N^{(D)}} (1 \pm \frac{2\alpha}{l^2} \{ f_{1_0} D(D-1) + f_{2_0}(D-1) + 2f_{3_0} \}). \quad (5.2.81)$$

Note that $f_{i_0}$'s are now simply the leading constants of the functions $\mathcal{F}_i(\Box)$, due to the nature of curvature in $(A)dS$. In particular, in 4-dimensions, the combination $12f_{1_0}+3f_{2_0}+2f_{3_0}$ is very different from that of the Minkowski space constraint, see Eq. (5.2.60), required for the massless nature of a graviton around Minkowski. Deriving the precise form of the ghost-free constraint in $(A)dS$, is still an open problem for the action given by (5.2.76).

### 5.2.4 Gauss-Bonnet entropy in $(A)dS$ background

As an example, we will briefly check the entropy of Gauss-Bonnet gravity in $D$-dimensional $(A)dS$. Recalling that the Lagrangian for the Gauss Bonnet (GB) modification of gravity in four dimensions is given by,

$$\mathcal{L}_{GB} = \frac{\alpha}{16\pi G_N^{(D)}}\bigg(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}\bigg). \tag{5.2.82}$$

Hence, simply taking $f_{1_0} = f_{3_0} = 1$ and $f_{2_0} = -4$ in Eq. (5.2.81), recovers the $(A)dS$ entropy of the GB modification of gravity,

$$S_W^{(A)dS} = \frac{A_H^{(A)dS}}{4G_N^{(D)}}(1 \pm \frac{\alpha 2(D-2)(D-3)}{l^2}). \tag{5.2.83}$$

This result is also found by [150], showing the validity of our calculations. We note that the first term corresponds to the Einstein-Hilbert term. It shall be mentioned that $l$ denotes the cosmological radius. In the limit $l \to \infty$, the GB modification to entropy vanishes and this corresponds to the fact that the horizon is flat, so the GB term has no effect on the expression for the entropy, which is simply the area of the event horizon.

## 5.3 Rotating black holes and entropy of modified theories of gravity

In this section, we show how to obtain the Kerr entropy when the modification to the general relativity contains higher order curvatures up to Ricci tensor and also covariant derivatives by modifying the Komar integrals accordingly. We then obtain the entropy of the Kerr black hole for a number of modified theories of gravity. We show the corrections to the area law which occurs due to the modification of the general relativity and present an argument on how these corrections can be vanishing.

It is well established that the black holes behave as thermodynamical systems [106]. The first realisation of this fact was made by Hawking, [107]. It is discovered that quantum processes make black holes to emit a thermal flux of particles. As a result, it is possible for a black hole to be in thermal equilibrium with other systems. We shall recall the thermodynamical laws that govern black holes: The zeroth law states that the horizon of stationary black holes have a constant surface gravity. The first law states that when stationary black holes are being perturbed the change in energy is related to the change of area, angular momentum and the electric charge associated to the black hole. The second law states that, upon satisfying the null energy condition the surface area of the black hole can never decrease. This is the law which was realised by Hawking as the area theorem and showed that black holes radiate. Finally, the third law states that the black hole can not have vanishing surface gravity.

The second law of the black holes' thermodynamics requires an entropy for black holes. It was Hawking and Bekenstein, [108], who conjectured that black holes' entropy is proportional to the area of its event horizon divided by Planck length. Perhaps, this can be seen as one of the most striking conjectures in modern physics. Indeed, through Bekenstein bound, [151], one can see that the black hole entropy, as described by the area law, is the maximal entropy that can be achieved and this was the main hint that led to the holographic principle, [152].

The black hole entropy can be obtained through number of ways. For instance, Wald [147] has shown that the entropy for a spherically symmetric and stationary black hole can be obtained by calculating the Noether charge, see the previous sections of this chapter for this approach. Equivalently, one can obtain the change in mass and angular momentum by using the Komar formula and subsequently use the definition of the first law of the black holes' thermodynamics to obtain the entropy. Normally, obtaining the entropy for non-rotating black holes is very straightforward. In this case, one uses the Schwarzschild metric (for a charge-less case) and follows the Wald's approach to calculate the entropy. Also for rotating black holes that are described by Kerr metric one can simply use the Komar integrals to find the mass and angular momentum and finally obtain the entropy. However, when we deviate from Einstein's theory of general relativity obtaining the conserved charge and hence the entropy can be challenging. See [154, 155, 156, 157, 171] for advancement in finding the conserved charges.

In this section, we are going to briefly review the notion of Noether and Komar currents in variational relativity. We show how the two are identical and then we move to calculate the entropy of Kerr black holes for a number of examples, namely $f(R)$ gravity, $f(R, R_{\mu\nu})$ theories where the action can contain higher order curvatures up to Ricci tensor and finally higher derivative gravity. The entropy in each case is obtained by calculating the modified Komar integrals.

### 5.3.1 Variational principle, Noether and Komar currents

Variational principle is a powerful tool in physics. Most of the laws in physics are derived by using this rather simple and straightforward method. Given a gravitational Lagrangian,

$$L = L(g_{\mu\nu}, R_{\mu\nu}, \nabla_{a_1} R_{\mu\nu}, \ldots, \nabla_{(\alpha_1} \ldots \nabla_{\alpha_m)} R_{\mu\nu}), \tag{5.3.84}$$

where the Lagrangian is a constructed by the metric, Ricci tensors and its derivatives (This can be generalised to the case where Riemann tensors are involved, however that is beyond the scope of this section). Note that the parentheses denote symmetrisation. We can obtain the equations of motion by simply varying

the action with respect to the inverse metric, $g^{\mu\nu}$ and $R_{\mu\nu}$. In short form, this can be done by defining two *covariant momenta* [148]:

$$\pi_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}, \tag{5.3.85}$$

and,

$$\begin{aligned} P^{\mu\nu} &= \frac{\delta\mathcal{L}}{\delta R_{\mu\nu}} \\ &= \frac{\partial\mathcal{L}}{\partial R_{\mu\nu}} - \nabla_{\alpha_1}\frac{\partial\mathcal{L}}{\partial\nabla_{\alpha_1}R_{\mu\nu}} + \cdots + (-1)^m\nabla_{(\alpha_1}\ldots\nabla_{\alpha_m)}\frac{\partial\mathcal{L}}{\partial\nabla_{(\alpha_1}\ldots\nabla_{\alpha_m)}R_{\mu\nu}}. \end{aligned} \tag{5.3.86}$$

Thus the variation of the Lagrangian would be given by [153]:

$$\delta\mathcal{L} = \pi_{\mu\nu}\delta g^{\mu\nu} + P^{\mu\nu}\delta R_{\mu\nu}. \tag{5.3.87}$$

It is simple to see that in the example of Einstein Hilbert (EH) action, the first term admits the equations of motion (*i.e.* $\pi_{\mu\nu} = 0$) and the second term will be the boundary term. Since we are considering gravitational theories, the general covariance must be preserved at all time. In other words, the Lagrangian, $L$, is covariant with respect to the action under diffeomorphisms of space-time. Infinitesimally, the variation can be expressed as:

$$\delta_\xi L = d(i_\xi L) = \pi_{\mu\nu}\pounds_\xi g^{\mu\nu} + P^{\mu\nu}\pounds_\xi R_{\mu\nu}, \tag{5.3.88}$$

where $\delta_\xi$ denotes an infinitesimal variation of the gravitational action, $d$ is the exterior derivative, $i_\xi$ is interior derivative of forms along vector field $\xi$ and $\pounds_\xi$ is the Lie derivative with respect to the vector field. By expanding the Lie derivative of the Ricci tensor, and noting that:

$$\delta_\xi g_{\alpha\beta} = \pounds_\xi g_{\alpha\beta} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha, \tag{5.3.89}$$

the *Nother conserved current* can be obtained. The way this can be done for the EH action is demonstrated in Appendix M as an example. Furthermore, the conserved *Noether current associated to the general covariance of the Einstein-Hilbert action is identical to the generalised Komar current*. This can be seen explicitly in Appendix N. In general, we define the Komar current[1]as [153]:

$$\mathcal{U} = \nabla_\alpha \xi^{[\mu} P^{\nu]\alpha} ds_{\mu\nu}, \tag{5.3.90}$$

where $ds_{\mu\nu}$ denotes the surface elements for a given background and is the standard basis for $n-2$-forms over the manifold $M$ ($n = \dim(M)$).

### 5.3.2 Thermodynamics of Kerr black hole

A solution to the Einstein field equations describing rotating black holes was discovered by Roy Kerr. This is a solution that only describes a rotating black hole without charge. Indeed, there is a solution for charged black holes (*i.e.* satisfies Einstein-Maxwell equations) known as Kerr-Newman. Kerr metric can be written in number of ways and in this section we are going to use the Boyer-Lindquist coordinate. The metric is given by [139]

$$ds^2 = -(1 - \frac{2Mr}{\rho^2})dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dtd\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \tag{5.3.91}$$

where,

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2, \quad \Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta. \tag{5.3.92}$$

[1]As a check it can be seen that for EH action we have,

$$P^{\alpha\beta}_{EH} = \sqrt{-g}g^{\alpha\beta}, \qquad \mathcal{U}_{EH} = \sqrt{-g}\nabla_\alpha \xi^{[\mu} g^{\nu]\alpha} ds_{\mu\nu}$$

which is exactly the same as what we obtain in Eq. (M.0.6).

The metric is singular at $\rho^2 = 0$. This singularity is real[1] and can be checked via Kretschmann scalar[2] [3]. The above metric has two horizons $r_\pm = m \pm \sqrt{m^2 - a^2}$. Furthermore, $a^2 \leq m^2$ is a length scale. Let us define the vector:

$$\xi^\alpha = t^\alpha + \Omega\phi^\alpha. \tag{5.3.93}$$

This vector is null at the event horizon. It is tangent to the horizon's null generators, which wrap around the horizon with angular velocity $\Omega$. Vector $\xi^\alpha$ is a Killing vector since it is equal to sum of two Killing vectors. After all, the event horizon of the Kerr metric is a Killing horizon. Using Eqs. (5.3.90) and (5.3.93) we can define the *Komar integrals* for the general Lagrangian (5.3.84) describing the energy and the angular momentum of the Kerr black hole as,

$$\mathcal{E} = -\frac{1}{8\pi} \lim_{S_t \to \infty} \oint_{S_t} \nabla_\lambda P^{\alpha\lambda} \xi^\beta_{(t)} ds_{\alpha\beta}, \tag{5.3.94}$$

$$\mathcal{J} = \frac{1}{16\pi} \lim_{S_t \to \infty} \oint_{S_t} \nabla_\lambda P^{\alpha\lambda} \xi^\beta_{(\phi)} ds_{\alpha\beta}, \tag{5.3.95}$$

where the integral is over $S_t$, which is a closed two-surfaces[4]. We shall note that $S_t$ is an $n-2$ surface. In above definitions $\xi^\beta_{(t)}$ is the space-time's time-like Killing vector and $\xi^\beta_{(\phi)}$ is the rotational Killing vector and they both satisfy the Killing's equation, $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$. Moreover, the sign difference in two definition has its root in the signature of the metric. In this thesis we are using mostly plus signature. The surface element is also given by,

$$ds_{\alpha\beta} = -2n_{[\alpha} r_{\beta]} \sqrt{\sigma} d\theta d\phi, \tag{5.3.96}$$

[1] This is different than the singularity at $\Delta = 0$ which is a coordinate singularity.

[2] The Kretschmann scalar for Kerr metric is given by: $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48M^2(r^2 - a^2\cos^2\theta)(\rho^4 - 16a^2r^2\cos^2\theta)}{\rho^{12}}$.

[3] We shall note that scalar curvature, $R$, and Ricci tensor, $R_{\mu\nu}$ are vanishing for the Kerr metric and only some components of the Riemann curvature are non-vanishing.

[4] Note that we can write $\lim_{S_t \to \infty} \oint_{S_t}$ as simply $\oint_{\mathcal{H}}$ where $\mathcal{H}$ is a two dimensional cross section of the event horizon.

where $n_\alpha$ and $r_\alpha$ are the time-like (*i.e.* $n_\alpha n^\alpha = -1$) and space-like (*i.e.* $r_\alpha r^\alpha = 1$) normals to $S_t$. For Kerr metric in Eq. (5.3.91) the normal vectors are defined as:

$$n_\alpha = (-\frac{1}{\sqrt{-g^{tt}}}, 0, 0, 0) = (-\sqrt{\frac{\rho^2\Delta}{\Sigma}}, 0, 0, 0), \tag{5.3.97}$$

$$r_\beta = (0, \frac{1}{\sqrt{g^{rr}}}, 0, 0) = (0, \sqrt{\frac{\rho^2}{\Delta}}, 0, 0). \tag{5.3.98}$$

Furthermore, the two dimensional cross section of the event horizon described by $t$ =constant and also $r = r_+$ (*i.e.* constant), hence, from metric in Eq. (5.3.91) we can extract the induced metric as:

$$\sigma_{AB} d\theta^A d\theta^B = \rho^2 d\theta^2 + \frac{\Sigma}{\rho^2} \sin^2\theta d\phi^2. \tag{5.3.99}$$

Thus we can write,

$$\sqrt{\sigma} = \sqrt{\Sigma} \sin\theta d\theta d\phi. \tag{5.3.100}$$

*First law* of black hole thermodynamics states that when a stationary black hole at manifold $\mathcal{M}$ is perturbed slightly to $\mathcal{M} + \delta\mathcal{M}$, the difference in the energy, $\mathcal{E}$, angular momentum, $\mathcal{J}_a$, and area, $\mathcal{A}$, of the black hole are related by:

$$\delta\mathcal{E} = \Omega^a \delta\mathcal{J}_a + \frac{\kappa}{8\pi}\delta\mathcal{A} = \Omega^a \delta\mathcal{J}_a + \frac{\kappa}{2\pi}\delta\mathcal{S}, \tag{5.3.101}$$

where $\Omega^a$ are the angular velocities at the horizon. We shall note that $\mathcal{S}$ is the associated entropy. $\kappa$ denotes the *surface gravity* of the Killing horizon and for the metric given in Eq. (5.3.91) the surface gravity is given by

$$\kappa = \frac{\sqrt{m^2 - a^2}}{2mr_+}. \tag{5.3.102}$$

The surface area [139] of the black hole is given by[1]:

$$\mathcal{A} = \oint_{\mathcal{H}} \sqrt{\sigma} d^2\theta, \tag{5.3.103}$$

where $d^2\theta = d\theta d\phi$. Now by using Eq. (5.3.100), the surface area can be obtained as,

$$\mathcal{A} = \oint_{\mathcal{H}} \sqrt{\sigma} d^2\theta = \int_0^{\pi} \sin(\theta) d\theta \int_0^{2\pi} d\phi (r_+^2 + a^2) = 4\pi(r_+^2 + a^2). \tag{5.3.104}$$

Modified theories of gravity were proposed as an attempt to describe some of the phenomena that Einstein's theory of general relativity can not address. Examples of these phenomena can vary from explaining the singularity to the dark energy. In the next subsections, we obtain the entropy of the Kerr black hole for number of these theories.

### 5.3.3 Einstein-Hilbert action

As a warm up exercise let us start the calculation for the most well knows case, where the action is given by:

$$S_{EH} = \frac{1}{2} \int d^4x \sqrt{-g} M_P^2 R, \tag{5.3.105}$$

[1]We shall note that $\mathcal{S} = \mathcal{A}/4$ (with $G = 1$) denotes the Bekenstein-Hawking entropy.

where $M_P^2$ is the Planck mass squared. For this case, as shown in footnote 1, the Komar integrals can be found explicitly as [153], (see Appendix O for derivation)

$$\begin{aligned}\mathcal{E} &= -\frac{1}{8\pi}\oint_{\mathcal{H}} \nabla^\alpha t^\beta ds_{\alpha\beta} \\ &= -\frac{1}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \\ &\left(\frac{1}{2}\sin(\theta)\left(a^2\cos(2\theta)+a^2+2r^2\right)\frac{8m\left(a^2+r^2\right)\left(a^2\cos(2\theta)+a^2-2r^2\right)}{\left(a^2\cos(2\theta)+a^2+2r^2\right)^3}\right) = m.\end{aligned} \tag{5.3.106}$$

We took $\xi^\alpha = t^\alpha$, where $t^\alpha = \frac{\partial x^\alpha}{\partial t}$; $x^\alpha$ are the space-time coordinates. So, for instance, $g_{\mu\nu}\xi^\mu\xi^\nu = g_{\mu\nu}t^\mu t^\nu = g_{tt}$, that is after the contraction of the metric with two Killing vectors, one is left with the $tt$ component of the metric. In similar manner, we can calculate the angular momentum as,

$$\begin{aligned}\mathcal{J} &= \frac{1}{16\pi}\oint_{\mathcal{H}} \nabla^\alpha \phi^\beta ds_{\alpha\beta} \\ &= \frac{1}{16\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \\ &\left(\frac{1}{2}\sin(\theta)\left(a^2\cos(2\theta)+a^2+2r^2\right)\right. \\ &\times \left.\frac{-8am\sin^2(\theta)\left(a^4-3a^2r^2+a^2(a-r)(a+r)\cos(2\theta)-6r^4\right)}{\left(a^2\cos(2\theta)+a^2+2r^2\right)^3}\right) = ma.\end{aligned} \tag{5.3.107}$$

Now given Eq. (5.3.101), we have,

$$\frac{\kappa}{2\pi}\delta\mathcal{S} = \delta\mathcal{E} - \Omega^a\delta\mathcal{J}_a = (1-\Omega a)\delta m - \Omega m\delta a. \tag{5.3.108}$$

By recalling the surface gravity from Eq. (5.3.102) we have,

$$\mathcal{S} = 2\pi m r_+. \tag{5.3.109}$$

which is a well known result.

### 5.3.4 $f(R)$ theories of gravity

There are numerous ways to modify the Einstein theory of general relativity, one of which is going to higher order curvatures. A class of theories which attracted attention in recent years is the $f(R)$ theory of gravity [88]. This type of theories can be seen as the series expansion of the scalar curvature, $R$, and one of the very important features of them is that they can avoid Ostrogradski instability. The action of this gravitational theory is generally given by:

$$S_{f(R)} = \frac{1}{2}\int d^4x\sqrt{-g}f(R), \tag{5.3.110}$$

where $f(R)$ is the function of scalar curvature and it can be of any order. In this case the Komar potential can be obtained by,

$$P^{\alpha\beta}_{f(R)} = \frac{\delta f(R)}{\delta R}\frac{\delta R}{\delta R_{\alpha\beta}} = \frac{1}{2}f'(R)\sqrt{-g}g^{\alpha\beta}, \tag{5.3.111}$$

and thus:

$$\mathcal{U}_{f(R)} = \frac{1}{2}f'(R)\sqrt{-g}\nabla_\alpha\xi^{[\mu}g^{\nu]\alpha}ds_{\mu\nu}. \tag{5.3.112}$$

This results in modification of the energy and angular momentum as (see Appendix P for validation),

$$\mathcal{E}_{f(R)} = -\frac{1}{8\pi}\oint_{\mathcal{H}} f'(R)\nabla^\alpha t^\beta ds_{\alpha\beta} = f'(R)m, \tag{5.3.113}$$

and

$$\mathcal{J}_{f(R)} = \frac{1}{16\pi}\oint_{\mathcal{H}} f'(R)\nabla^\alpha \phi^\beta ds_{\alpha\beta} = f'(R)ma. \tag{5.3.114}$$

We know that the $f(R)$ theory of gravity is essentially the power expansion in the scalar curvature,

$$f(R) = M_P^2 R + \alpha_1 R^2 + \alpha_2 R^3 + \cdots + \alpha_{n-1}R^n, \tag{5.3.115}$$

where $\alpha_i$ maintains the correct dimensionality, and thus,

$$f'(R) = M_P^2 + 2\alpha_1 R + 3\alpha_2 R^2 + \cdots + n\alpha_{n-1} R^{n-1}. \tag{5.3.116}$$

As a result, the entropy of $f(R)$ theory of gravity is given only by the Einstein Hilbert contribution,

$$\mathcal{S}_{f(R)} = \mathcal{S}_{EH} = 2\pi m r_+. \tag{5.3.117}$$

This is due to fact that the scalar curvature, $R$, is vanishing for the Kerr metric given in Eq. (5.3.91) and so only the leading term in Eq. (5.3.116) will be accountable.

### 5.3.5 $f(R, R_{\mu\nu})$

After considering the $f(R)$ theories of gravity, it is natural to think about the more general form of gravitational modification. In this case: $f(R, R_{\mu\nu})$, the action would contain terms like $R_{\mu\nu}R^{\mu\nu}$, $R^{\mu\alpha}R_{\alpha}{}^{\nu}R_{\nu\mu}$ and so on. Let us take a specific example of,

$$S_{R_{\mu\nu}} = \frac{1}{2}\int d^4x\sqrt{-g}(M_P^2 R + \lambda_1 R_{\mu\nu}R^{\mu\nu} + \lambda_2 R^{\mu\lambda}R_{\lambda}{}^{\nu}R_{\nu\mu}), \tag{5.3.118}$$

where $\lambda_1$ and $\lambda_2$ are coefficients of appropriate dimension (*i.e.* mass dimension $L^2$ and $L^4$ respectively where $L$ denotes length). The momenta would then be obtained as,

$$P^{\alpha\beta}_{R_{\mu\nu}} = \frac{\sqrt{-g}}{2}(M_P^2 g^{\alpha\beta} + 2\lambda_1 R^{\alpha\beta} + 3\lambda_2 R^{\beta\lambda}R^{\alpha}{}_{\lambda}). \tag{5.3.119}$$

As before, the only contribution comes from the EH term since the Ricci tensor is vanishing for the Kerr metric given in Eq. (5.3.91). So, without proceeding further, we can conclude that in this case the entropy is given by the area law only and with no correction.

### 5.3.6 Higher derivative gravity

Another class of modified theories of gravity are the higher derivative theories. We shall denote the action by $S(g, R, \nabla R, \nabla R_{\mu\nu}, \cdots)$. In this class, there are covariant derivatives acting on the curvatures. Moreover, there are theories that contain inverse derivatives acting on the curvatures [115]. These are known as non-local theories of gravity.

A well established class of higher derivative theory of gravity is given by [53] where the action contains infinite derivatives acting on the curvatures. It has been shown that having infinite derivatives can cure the singularity problem [68]. This is achieved by replacing the singularity with a bounce. Moreover, this class of theory preserves the ghost freedom. This is of a very special importance, since in other classes of modified gravity, deviating from the EH term and going to higher order curvature terms means one will have to face the ghost states. Having infinite number of derivatives makes it extremely difficult to find a metric solution which satisfies the equations of motion. Moreover, infinite derivative theory is associated with singularity freedom and Kerr metric is a singular one. As a result, in this section we wish to consider a finite derivative example as a matter of illustration, let us define the Lagrangian of the form:

$$\mathcal{L}_{HD} = \sqrt{-g}\left[M_P^2 R + R\mathcal{F}_1(\bar{\Box})R + R_{\mu\nu}\mathcal{F}_2(\bar{\Box})R^{\mu\nu}\right] , \tag{5.3.120}$$

where $\mathcal{F}_1(\bar{\Box}) = \sum_{n=1}^{m_1} f_{1_n}\bar{\Box}^n$, $\mathcal{F}_2(\bar{\Box}) = \sum_{n=1}^{m_2} f_{2_n}\bar{\Box}^n$ while $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is d'Alembertian operator and $\bar{\Box} = \Box/M^2$ to ensure the correct dimensionality. Note that $f_{i_n}$ are the coefficient of the expansion. We also note that $m_1$ and $m_2$ are some finite number. In this case, we have the Lagrange momenta as,

$$P_{HD}^{\alpha\beta} = \sqrt{-g}\left[M_P^2 g^{\alpha\beta} + 2f_{1_n}g^{\alpha\beta}\sum_{n=1}^{m_1}\bar{\Box}^n R + 2f_{2_n}\sum_{n=1}^{m_2}\bar{\Box}^n R^{\alpha\beta}\right] . \tag{5.3.121}$$

As mentioned previously for the Kerr metric: $R = R^{\alpha\beta} = 0$, this is to conclude that the only non-vanishing term which will contribute to the entropy will be the first term, in the above equation, which corresponds to the EH term in the action given in Eq. (5.3.120).

### 5.3.7 Kerr metric as and solution of modified gravities

After providing some examples of the modified theories of gravity, the reader might ask wether the Kerr metric is the solution of these theories. In this section we are going to address this issue by considering the higher derivative action. This is due to the fact that the higher derivative action given in (5.3.120) contains Ricci scalar and Ricci tensor and additionally their derivatives and hence the arguments can be applied to other theories provided in this section.

Let us consider (5.3.120), the equations of motion is given by [111],

$$
\begin{aligned}
&G^{\alpha\beta} + 4G^{\alpha\beta}\mathcal{F}_1(\bar{\Box})R + g^{\alpha\beta}R\mathcal{F}_1(\bar{\Box})R - 4\left(\nabla^{\alpha}\nabla^{\beta} - g^{\alpha\beta}\Box\right)\mathcal{F}_1(\bar{\Box})R \\
&-2\Omega_1^{\alpha\beta} + g^{\alpha\beta}(\Omega_{1\sigma}^{\sigma} + \bar{\Omega}_1) + 4R_{\mu}^{\alpha}\mathcal{F}_2(\bar{\Box})R^{\mu\beta} \\
&-g^{\alpha\beta}R_{\nu}^{\mu}\mathcal{F}_2(\bar{\Box})R_{\mu}^{\nu} - 4\nabla_{\mu}\nabla^{\beta}(\mathcal{F}_2(\bar{\Box})R^{\mu\alpha}) + 2\Box(\mathcal{F}_2(\bar{\Box})R^{\alpha\beta}) \\
&+2g^{\alpha\beta}\nabla_{\mu}\nabla_{\nu}(\mathcal{F}_2(\bar{\Box})R^{\mu\nu}) - 2\Omega_2^{\alpha\beta} + g^{\alpha\beta}(\Omega_{2\sigma}^{\sigma} + \bar{\Omega}_2) - 4\Delta_2^{\alpha\beta} = 0\,, \qquad (5.3.122)
\end{aligned}
$$

we have defined the following symmetric tensors [111]:

$$
\begin{aligned}
\Omega_1^{\alpha\beta} &= \sum_{n=1}^{\infty} f_{1_n} \sum_{l=0}^{n-1} \nabla^{\alpha}R^{(l)}\nabla^{\beta}R^{(n-l-1)}, \quad \bar{\Omega}_1 = \sum_{n=1}^{\infty} f_{1_n} \sum_{l=0}^{n-1} R^{(l)}R^{(n-l)}, \\
\Omega_2^{\alpha\beta} &= \sum_{n=1}^{\infty} f_{2_n} \sum_{l=0}^{n-1} R_{\nu}^{\mu;\alpha(l)}R_{\mu}^{\nu;\beta(n-l-1)}, \quad \bar{\Omega}_2 = \sum_{n=1}^{\infty} f_{2_n} \sum_{l=0}^{n-1} R_{\nu}^{\mu(l)}R_{\mu}^{\nu(n-l)}\,, \\
\Delta_2^{\alpha\beta} &= \frac{1}{2}\sum_{n=1}^{\infty} f_{2_n} \sum_{l=0}^{n-1} [R_{\sigma}^{\nu(l)}R^{(\beta|\sigma|;\alpha)(n-l-1)} - R_{\sigma}^{\nu;(\alpha(l)}R^{\beta)\sigma(n-l-1)}]_{;\nu}\,, \qquad (5.3.123)
\end{aligned}
$$

Also note that $R^{(m)} \equiv \Box^m R$ and that we absorbed the mass dimension in $f_{i_n}$'s where it was necessary. For the Kerr metric, the Ricci tensor and, consequently, the Ricci scalar vanish identically. Therefore, any quantity with covariant derivatives acting on the Ricci tensor and the Ricci scalar would also be equal to zero; as a result, each of the five quantities defined in (5.3.123) would also vanish. Hence, one may observe that each of the terms in the left-hand side of (5.3.122) becomes equal to zero. Thus, the Kerr metric satisfies the equation of motion (5.3.122), implying that the Kerr metric can be regarded as a solution of the action (5.3.120).

This has been shown explicitly by [159]. However, the same can not be said in the presence of the Riemann tensors.

## 5.4 Non-local gravity

For higher derivative theories of gravity, it is possible to write the action in terms of auxiliary fields. Doing so results in converting a non-local action [115] to a local one. We use this approach to find the entropy for a non-local gravitational action.

Indeed, Einstein's theory of general relativity can be modified in number of ways to address different aspects of cosmology. Non-local gravity is constructed by inversed d'Alembertian operators that are accountable in the IR regime. In particular, they could filter out the contribution of the cosmological constant to the gravitating energy density, possibly providing the key to solving one of the most notorious problems in physics [160], see also [163].

Moreover, such modification to the theory of general relativity arises naturally as quantum loop effect [115] and used initially by [64] to explain the cosmic acceleration. Non-local gravity further used to explain dark energy [116]. Since such gravity is associated with large distances, it is also possible to use it as an alternative to understand the cosmological constant [160]. Additionally, non-local corrections arise, in the leading order, in the context of bosonic string [161].

It is argued in [162] that, non-locality may have a positive rule in understanding the black hole information problem. Recently, the non-local effect was studied in the context of Schwarzschild black hole [163, 164]. In similar manner, the entropy of some non-local models were studied in [165].

### 5.4.1 Higher derivative gravity reparametrisation

Let us take the following higher derivative action,

$$I_0 + I_1 = \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \big[ R + RF(\bar{\Box}) R \big],$$
$$\text{with:} \quad F(\bar{\Box}) = \sum_{n=0}^{m} f_n \bar{\Box}^n, \tag{5.4.124}$$

where $G_N^{(D)}$ is the $D$ dimensional Newton's gravitational constant, $R$ is scalar curvature, $\Box = \nabla_\mu \nabla^\mu$ is the d'Alembertian operator and $\bar{\Box} \equiv \Box/M^2$, this is due to the fact that $\Box$ has dimension mass squared and we wish to have dimensionless $F(\bar{\Box})$, we shall note that $f_n$'s are dimensionless coefficients of the series expansion. In the above action we denoted the EH term as $I_0$. Finally, $m$ is some finite positive integer. The above action can be written as [99],

$$I_0 + \tilde{I}_1 = \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \Big[ R + \sum_{n=0}^{m} \Big( R f_n \eta_n + R \chi_n (\eta_n - \bar{\Box}^n R) \Big) \Big], \tag{5.4.125}$$

where we introduced two auxiliary fields $\chi_n$ and $\eta_n$. This is the method which we used in the Hamiltonian chapter. By solving the equations of motion for $\chi_n$, we obtain: $\eta_n = \bar{\Box}^n R$, and hence the original action given in Eq. (5.4.124 ) can be recovered. This equivalence is also noted in [99].

We are now going to use the Wald's prescription given in (5.2.21) over the spherically symmetric metric (5.2.18). We know from (5.2.33) that the entropy for action (5.4.124) is given by,

$$\mathcal{S}_0 + \mathcal{S}_1 = \frac{A_H}{4G_N^{(D)}} \Big( 1 + 2F(\bar{\Box}) R \Big), \tag{5.4.126}$$

where we denoted the entropy by $\mathcal{S}$. Now let us obtain the entropy for $\tilde{I}_1$ ,

following the entropy Eq. (5.2.21), we have,

$$\begin{aligned}\tilde{\mathcal{S}}_1 &= -\frac{A_H}{2G_N^{(D)}} \times \sum_{n=0}^{m}(-\frac{1}{2}f_n\eta_n - \frac{1}{2}\chi_n\eta_n + \chi_n\bar{\Box}^n R) \\ &= -\frac{A_H}{2G_N^{(D)}} \times \sum_{n=0}^{m}(-\frac{1}{2}f_n\eta_n - \frac{1}{2}\chi_n\eta_n + \chi_n\eta_n) \\ &= \frac{A_H}{4G_N^{(D)}} \times \sum_{n=0}^{m}(2f_n\eta_n) = \frac{A_H}{4G}(2F(\bar{\Box})R). \end{aligned} \tag{5.4.127}$$

Where we fixed the lagrange multiplier as $\chi_n = -f_n$. It is clear that both $I_1$ and $\tilde{I}_1$ are giving the same result for the entropy as they should. This is to verify that it is always possible to use the equivalent action and find the correct entropy. This method is very advantageous in the case of non-local gravity, where we have inversed operators.

Before proceeding to the non-local case let us consider $\tilde{I}_1$,

$$\tilde{I}_1 = \frac{1}{16\pi G_N^{(D)}} \int d^D x\sqrt{-g} \sum_{n=0}^{m} \Big( R f_n \eta_n + R\chi_n(\eta_n - \bar{\Box}^n R)\Big), \tag{5.4.128}$$

It is mentioned that solving the equations of motion for $\chi_n$ results in:

$$\eta_n \equiv \bar{\Box}^n R. \tag{5.4.129}$$

We shall mention that in order to form

$$F(\bar{\Box}) = \sum_{n=0}^{m} f_n \bar{\Box}^n, \tag{5.4.130}$$

in the second term of (5.4.128) we absorbed the $f_n$ into the Lagrange multiplier, $\chi_n$. Let us consider the fixing $\chi_n = -f_n$. We do so by substituting the value of

the Lagrange multiplier,

$$\begin{aligned}\tilde{I}_1 &= \frac{1}{16\pi G_N^{(D)}}\int d^Dx\sqrt{-g}\sum_{n=0}^{m}\Big(Rf_n\eta_n - Rf_n(\eta_n - \bar{\Box}^n R)\Big)\\ &= \frac{1}{16\pi G_N^{(D)}}\int d^Dx\sqrt{-g}\sum_{n=0}^{m}Rf_n\bar{\Box}^n R \equiv \frac{1}{16\pi G_N^{(D)}}\int d^Dx\sqrt{-g}RF(\bar{\Box})R.\end{aligned} \tag{5.4.131}$$

As expected $\tilde{I}_1$ and $I_1$ are again equivalent. Thus the fixation of the Lagrange multiplier is valid.

### 5.4.2 Non-local gravity's entropy

The non-local action can be written as,

$$\begin{aligned}&I_0 + I_2 = \frac{1}{16\pi G_N^{(D)}}\int d^Dx\sqrt{-g}\big[R + RG(\bar{\Box})R\big],\\ &\text{with:}\quad G(\bar{\Box}) = \sum_{n=0}^{m}c_n\bar{\Box}^{-n}.\end{aligned} \tag{5.4.132}$$

In this case the inversed d'Alembertian operators are acting on the scalar curvature. In order to localise the above action we are going to introduce two auxiliary fields $\xi_n$ and $\psi_n$ and rewrite the action in its local form as,

$$I_0 + \tilde{I}_2 = \frac{1}{16\pi G_N^{(D)}}\int d^Dx\sqrt{-g}\Big[R + \sum_{n=0}^{m}\Big(Rc_n\psi_n + R\xi_n(\bar{\Box}^n\psi_n - R)\Big)\Big]. \tag{5.4.133}$$

Solving the equations of motion for $\xi_n$, results in having:

$$\bar{\Box}^n\psi_n = R \quad \text{or} \quad \psi_n = \bar{\Box}^{-n}R. \tag{5.4.134}$$

Thus, the original action given in Eq. (5.4.132) can be recovered. This equivalence is also noted by [165, 166].

Finding the Wald's entropy for the non-local action as stands in (5.4.132) can be a challenging task, this is due to the fact that for an action of the form Eq. (5.4.132), the functional differentiation contains inversed operators acting on the scalar curvature and Wald's prescription for such case can not be applied. However, by introducing the equivalent action and localising the gravity as given in Eq. (5.4.133), one can obtain the entropy as it had been done in the previous case. We know that the contribution of the EH term to the entropy is $\mathcal{S}_0 = A_H/4G$. Thus we shall consider the entropy of $\tilde{I}_2$:

$$\begin{aligned}
\tilde{\mathcal{S}}_2 &= - \frac{A_H}{2G_N^{(D)}} \times \sum_{n=0}^{m} (-\frac{1}{2} c_n \psi_n - \frac{1}{2} \xi_n \bar{\Box}^n \psi_n + \xi_n R) \\
&= -\frac{A_H}{2G_N^{(D)}} \times \sum_{n=0}^{m} (-\frac{1}{2} c_n \psi_n - \frac{1}{2} \xi_n \bar{\Box}^n \psi_n + \xi_n \bar{\Box}^n \psi_n) \\
&= \frac{A_H}{4G_N^{(D)}} \times \sum_{n=0}^{m} (c_n \psi_n + c_n \bar{\Box}^n \psi_n) \\
&= \frac{A_H}{4G_N^{(D)}} \times \sum_{n=0}^{m} (c_n (\bar{\Box}^{-n} R) + c_n \bar{\Box}^n (\bar{\Box}^{-n} R)) \\
&= \frac{A_H}{4G_N^{(D)}} \times \sum_{n=0}^{m} (c_n \bar{\Box}^{-n} R + c_n R), \qquad (5.4.135)
\end{aligned}$$

where we took $\xi_n = -c_n$. Furthermore, we used the fact that $\bar{\Box}^n(\bar{\Box}^{-n} R) = R$, [115].

As before let us check the validity of the $\xi_n = -c_n$ by considering $\tilde{I}_2$,

$$\begin{aligned}
\tilde{I}_2 &= \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \sum_{n=0}^{m} \Big( R c_n \psi_n + R\xi_n (\bar{\Box}^n \psi_n - R) \Big) \\
&= \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \sum_{n=0}^{m} \Big( R c_n \psi_n - R c_n (\bar{\Box}^n \psi_n - R) \Big) \\
&= \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \sum_{n=0}^{m} \Big( R c_n \psi_n - R c_n (\psi_n - \bar{\Box}^{-n} R) \Big) \\
&\equiv \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} R G(\bar{\Box}) R, \qquad (5.4.136)
\end{aligned}$$

where used the property of (5.4.134) and recovered the non-local action in (5.4.132). Thus the fixation of $\xi_n = -c_n$ is valid.

## 5.5 Summary

In this chapter we have shown how Wald's approach can be used to find the entropy for a static, spherically symmetric metric. It is shown that deviation from GR results in having correction to the entropy. However, in the framework of IDG, we have shown that one can recover the area law by going to the linearised regime for a spherically symmetric background. Linearisation means perturbation around Minkowski background and obtaining a constraint, required to have the massless graviton as the only propagating degree of freedom.

We then used Wald's formulation of entropy to find the corrections around the $(A)dS$ background. We verified our result by providing the entropy of the Gauss-Bonnet gravity as an example. It has been shown that the constraint found in the linearised regime (around Minkowski background) is not applicable to the $(A)dS$ case. This is due to the fact that the form of the propagator for $(A)dS$ background is an open problem and thus there is no known ghost free constraint.

We continued our study to a rotating background described by the Kerr metric. For this case, we modified Komar integrals appropriately to calculate the entropy for the Kerr background in various examples. It has been shown that

deviating from the EH gravity up to Ricci tensor will have no effect in the amount of entropy, and the entropy is given solely by the area law. This is because the scalar curvature and Ricci tensor are vanishing for the Kerr metric given in Eq. (5.3.91) (see [159] on rigorous derivation of this).

In the presence of the Riemann tensor and its derivatives, the same conclusion can not be made. This is due to number of reasons: i) The Riemann tensor for the Kerr metric is non-vanishing, ii) In the presence of the Riemann tensor and its derivatives, the Ricci-flat ansatz illustrated by [159] may not hold and thus Kerr background may not be an exact solution to the modified theories of gravity. iii) Variation of the action and obtaining the appropriate form of Komar integral is technically demanding and requires further studies.

Finally, we have provided a method to obtain the entropy of a non-local action. Gravitational non-local action is constructed by inversed d'Alembertian operators acting on the scalar curvature. The Wald approach to find the entropy can not be applied to a non-local action. Thus, we introduced an equivalent action via auxiliary fields and localised the non-local action. We then obtained the entropy using the standard method provided by Wald. In the case of higher derivative gravity we have checked that both, the original higher derivative action and its equivalent action, are producing the same results for the entropy. However, such check can not be done in the non-local case. As a future work, it would be interesting to obtain the Noether charge such that one can calculate the entropy for a non-local action without the need of localisation and check wether the entropies agree after localisation.

# Chapter 6

# Conclusion

In this thesis some of the classical aspects of infinite derivative gravitational (IDG) theories were considered. The aim was mainly to build an appropriate machinery which can be used later to build upon and further understanding of infinite derivative theories of gravity. The main focus of this thesis was a ghost- and singularity-free infinite derivative theory of gravity. This theory is made up of covariant derivatives acting on Riemannian curvatures, we represented this infinite series of derivatives as function $F(\bar{\Box})$. We found that upon choosing specific form of $F(\bar{\Box})$, that is an exponential of an entire function, the theory is ghost free and singularity free.

*Outline of results*

In Chapter 3 we performed the Hamiltonian analysis for wide range of higher derivatives and infinite derivatives theories. We started our analysis by considering some toy models: homogeneous case and infinite derivative scalar model and then we moved on and applied the Hamiltonian analysis to infinite derivative gravitational theory (IDG). The aim of our analysis were to find the physical degrees of freedom for higher derivative theories from the Hamiltonian formalism. As for the IDG action, we truncated the theory such that the only modification would be $RF_1(\bar{\Box})R$ term. Such action is simpler than the general IDG containing

higher order terms such as $R_{\mu\nu}F_2(\bar{\Box})R^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}F_3(\bar{\Box})R^{\mu\nu\lambda\sigma}$. Adding higher curvature terms would lead to further complexities when it comes to the ADM decomposition and we shall leave this for future studies.

From Lagrangian formalism, the number of degrees of freedom is determined via propagator analysis. In other words, calculating number of degrees of freedom is associated with the number of poles arising in the propagator for a given theory. As for the case of IDG is it known that for a Gaussian kinetic term in the Lagrangian, the theory admits two dynamical degrees of freedom. This can be readily obtained by considering the spin-0 and spin-2 components of the propagator. In order to maintain the original dynamical degrees of freedom and avoiding extra poles (and thus extra propagating degrees of freedom), one shall demand that the propagator be suppressed by the exponential of an entire function. This is due to the fact that an entire function does not produce poles in the infinite complex plain. Thus, it is reasonable to modify the kinetic term in the Lagrangian for infinite derivative theories. Such modification in the case of scalar toy model would take the form of $F(\bar{\Box}) = \Box e^{-\bar{\Box}}$ and in the context of gravity the modification would be $F(\bar{\Box}) = M_P^2\Box^{-1}(e^{-\bar{\Box}} - 1)$. It is clear that one must expect the same physical results from Lagrangian and Hamiltonian analysis for a given theory.

To obtain the number of degrees of freedom in Hamiltonian regime, one starts with first identifying the configuration space variables and computing the first class and second class constraints. In the case of IDG, there exist infinite number of configuration space variables and thus first class and second class constraints. However, for a Gaussian kinetic term, $F(\bar{\Box})$, the number of degrees of freedom are finite. This holds for both scalar toy models and gravitational Hamiltonian densities.

In Chapter 4, the generalised Gibbons-Hawking-York (GHY) boundary term, for the IDG theory, was obtained. It has been shown that in order to find the boundary term for the IDG theory one shall use the ADM formalism and in particular coframe slicing to obtain the appropriate form of the extrinsic curvature. Moreover, in coframe slicing d'Alembertian operators are fairly easy to be

handled when in comes to commutation between derivatives and tensorial components. It should be noted that the conventional way of finding the surface contribution is using the variation principle. However, for an infinite derivative term that would not be a suitable approach. This is due to the fact that for a theory with $n$ number of covariant derivatives we will have $2n$ total derivatives, and clearly extracting a GHY type surface term to cancel these total derivatives is not a trivial task. Indeed, it is not clear, in the case of IDG, how from the variation principle one would be extracting a neat extrinsic curvature to cancel out the surface contribution. To this end, we took another approach, namely to recast the IDG action to an equivalent form where now we have auxiliary fields. We then decomposed the equivalent action and used it to calculate the generalised GHY term for the IDG theory. To validate our method it can be seen that, for the case of $\Box \to 0$ our result would recover the GR's boundary term as given by the GHY action and for $\Box \to 1$ one shall recover the well known results of Gauss-Bonnet gravity upon substituting the right coefficients.

In Chapter 5 we considered the thermodynamical aspects of the IDG theory. In GR it is well known that the entropy of a stationary black hole is given by the area law. Given different solutions to the Einstein-Hilbert action the area law would be modified yet the proportionality of the entropy to the area remains valid. The deviation from GR results in correction to the entropy. In the context of IDG we performed entropy calculation and obtained the corresponding corrections. We began our analysis by considering a static and spherically symmetric background. We shall note that in this metric we have not defined any value for $f(r)$. This is to keep the metric general. We then used Wald's description and obtained the corrections. We extended our discussion to the linearised regime and found that in the weak field limit the entropy of the IDG action is solely given by the area law and thus the higher order corrections are not affecting the entropy. It is important to remind that in order to achieve this result we imposed the constraint $2F_1(\bar{\Box}) + F_2(\bar{\Box}) + 2F_3(\bar{\Box}) = 0$, which ensures that the only propagating degree of freedom is the massless graviton. Moreover, we imposed $\Phi(r) = \Psi(r)$, in other words we demanded that the Newtonian potentials to be the same.

We then moved to the $(A)dS$ backgrounds and compute the entropy for the IDG theory using the Wald's approach and obtained the corrections. $(A)dS$ backgrounds admit constant curvatures. This leads to have constant corrections to the entropy. In this regime, we have shown that upon choosing the appropriate coefficients, the IDG entropy reduces to the corresponding Gauss-Bonnet gravity.

After, we turned our attention to a rotating background and used variational principle to find the generalised Komar integrals for theories that are constructed by the metric tensor, Ricci scalar, Ricci tensor and their derivatives. We then used the first law of thermodynamics and computed the entropy for number of cases: $f(R)$, $f(R, R_{\mu\nu})$ and finally higher derivative theories of gravity. We used the Ricci flatness ansatz found in [159] and concluded that for a rotating background described by the Kerr metric we have $R = R_{\mu\nu} = 0$ regardless of the modified theory of gravity one is considering; thus the only contribution to the entropy comes from the Einstein-Hilbert term. This holds true as long as we do not involve the Riemann contribution to the gravitational action. Furthermore, the generalised form of Komar integrals are unknown for the case where the gravitational action contains Riemann tensor and its derivatives.

Finally, we wrapped up the chapter by considering an infinite derivative action where we have inverse d'Alembertian operators ($\Box^{-1}$), the entropy of such action using the Wald approach can not be found. This is due to the fact that the functional differentiation is not known for inverse derivatives. For instance, terms like $\delta(R(\bar{\Box})^{-1}R)/\delta R_{\mu\nu\lambda\sigma}$ can not be differentiated using the normal Euler-Lagrange functional differentiation. The non-locality of such action can be localised by introducing auxiliary fields and rewriting the non-local action in its localised equivalent form. This allows to compute the entropy using the Wald's prescription. We verified our method for an already known theory where the Lagrangian density is given by $\mathcal{L} \sim R + RF_1(\bar{\Box})R$.

*Future work*

- IDG is now know to address the black hole singularity in the weak field

regime. It would be interesting to see wether the singularities can be avoided in the case of astrophysical black holes.

- The form of Wald entropy for non-local theories of gravity, where there is $\Box^{-1}$ in the action, is not known. It would be interesting to formulate the charge for such theories and to check wether they reproduce the same results as if one was to localise the theory by introducing auxiliary fields.

- Addressing the cosmological singularity issue in the presence of matter source is an open question. The exact cosmological solutions were only obtained in the presence of a cosmological constant. A realistic cosmological scenario must include an appropriate exit from the inflationary phase. So far such transition is unknown and any progress in that direction is useful.

- So far IDG is studied around Minkowski background. It would be interesting to discover the classical and quantum aspects of IDG over other backgrounds.

- Establishing unitarity within the framework of IDG is another open problem where a novel prescription shall be found. This would be a major step towards construction of a fully satisfactory theory of quantum gravity.

- Deriving IDG or any other modified theories of gravity from string theory is another open problem, despite the fact that they are stringy inspired. It is of great interest to know how any of the modified theories of gravity can be derived from string theory and not just by writing down an effective gravitational action.

- There are many other aspects of IDG which can be studied using the current knowledge built by others and us. An example of that would be obtaining the holographic entanglement entropy for an IDG in the context of AdS/CFT (see for instance [167, 168]). Another example would be studying the IDG action in other dualities such as Kerr/CFT, where one can gain great knowledge about the CFT in the context of higher derivative theories (see the Gauss-Bonnet example [169]).

- In the cosmological sense, non-local theories can be promising in studying some phenomenological aspects such as dark energy (see [116] as an example). It would be interesting to look at other phenomenological aspects of IDG.

- As for Hamiltonian formalism, it would be interesting to obtain the physical degrees of freedom for the full IDG action, containing the higher order curvatures. This would be a good check to make sure one recovers the same results from the Lagrangian analysis.

# Appendix A

## A.1 Useful formulas, notations and conventions

The metric signature used in this thesis is,

$$g_{\mu\nu} = (-, +, +, +). \tag{A.1.1}$$

In natural units ($\hbar = c = 1$). We also have,

$$M_P = \kappa^{-1/2} = \sqrt{\frac{\hbar c}{8\pi G_N^{(D)}}}, \tag{A.1.2}$$

where $M_P$ is the Planck mass and $G_N^{(D)}$ is Newton's gravitational constant in D-dimensional space-time.

The relevant mass dimensions are:

$$[dx] = [x] = [t] = M^{-1}, \tag{A.1.3}$$

$$[\partial_\mu] = [p_\mu] = [k_\mu] = M^1, \tag{A.1.4}$$

$$[velocity] = \frac{[x]}{[t]} = M^0. \tag{A.1.5}$$

As a result,

$$[d^4x] = M^{-4}. \tag{A.1.6}$$

The action is a dimensionless quantity:

$$[S] = [\int d^4x\mathcal{L}] = M^0. \tag{A.1.7}$$

Therefore,

$$[\mathcal{L}] = M^4. \tag{A.1.8}$$

## A.2 Curvature

Christoffel symbol is,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\tau}(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu}). \tag{A.2.9}$$

The Riemann tensor is,

$$R^{\lambda}{}_{\mu\sigma\nu} = \partial_\sigma\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\sigma\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\sigma\mu}, \tag{A.2.10}$$

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda}R^{\lambda}{}_{\sigma\mu\nu} = g_{\rho\lambda}(\partial_\mu\Gamma^{\lambda}_{\nu\sigma} - \partial_\nu\Gamma^{\lambda}_{\mu\sigma}), \tag{A.2.11}$$

$$R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda} = R_{\lambda\sigma\mu\nu}, \tag{A.2.12}$$

$$R_{\mu\nu\lambda\sigma} + R_{\mu\lambda\sigma\nu} + R_{\mu\sigma\nu\lambda} = 0. \tag{A.2.13}$$

The Ricci tensor is given by,

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\lambda\mu}, \tag{A.2.14}$$

The Ricci tensor associated with the Christoffel connection is symmetric,

$$R_{\mu\nu} = R_{\nu\mu}. \tag{A.2.15}$$

The Ricci scalar is given by,

$$R = R^{\mu}_{\ \mu} = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu}\partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial^\mu \Gamma^{\lambda}_{\mu\lambda} + g^{\mu\nu}\Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\nu\mu} - g^{\mu\nu}\Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\lambda\mu}. \quad \text{(A.2.16)}$$

The Weyl tensor is given by,

$$C^{\mu}_{\ \alpha\nu\beta} \equiv R^{\mu}_{\alpha\nu\beta} - \frac{1}{2}(\delta^\mu_\nu R_{\alpha\beta} - \delta^\mu_\beta R_{\alpha\nu} + R^\mu_\nu g_{\alpha\beta} - R^\mu_\beta g_{\alpha\nu}) + \frac{R}{6}(\delta^\mu_\nu g_{\alpha\beta} - \delta^\mu_\beta g_{\alpha\nu}), \quad \text{(A.2.17)}$$

$$C^{\lambda}_{\ \mu\lambda\nu} = 0. \quad \text{(A.2.18)}$$

The Einstein tensor is given by,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \quad \text{(A.2.19)}$$

Varying the Einstein-Hilbert action,

$$S_{EH} = \frac{1}{2}\int d^4x \sqrt{-g}(M_P^2 R - 2\Lambda), \quad \text{(A.2.20)}$$

where $\Lambda$ is the cosmological constant of mass dimension 4, leads to the Einstein equation,

$$M_P^2 G_{\mu\nu} + g_{\mu\nu}\Lambda = T_{\mu\nu}, \quad \text{(A.2.21)}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. When considering the perturbations around the Minkowski space-time, the cosmological constant is set to be zero.

The Bianchi identity is given by,

$$\nabla_\kappa R_{\mu\nu\lambda\sigma} + \nabla_\sigma R_{\mu\nu\kappa\lambda} + \nabla_\lambda R_{\mu\nu\sigma\kappa} = 0. \quad \text{(A.2.22)}$$

This results from the sum of cyclic permutations of the first three indices. We note that the antisymmetry properties of Riemann tensor allows this to be written as,

$$\nabla_{[\kappa} R_{\mu\nu]\lambda\sigma} = 0. \quad \text{(A.2.23)}$$

Contracting (A.2.22) with $g^{\mu\lambda}$ results in the contracted Bianchi identity,

$$\nabla_\kappa R_{\nu\sigma} - \nabla_\sigma R_{\nu\kappa} + \nabla^\lambda R_{\lambda\nu\sigma\kappa} = 0. \tag{A.2.24}$$

Contracting (A.2.24) with $g^{\nu\kappa}$, we obtain,

$$\nabla_\kappa R^\kappa_\sigma = \frac{1}{2}\nabla_\sigma R, \tag{A.2.25}$$

which similarly implies,

$$\nabla^\sigma \nabla_\kappa R^\kappa_\sigma = \frac{1}{2}\Box R, \tag{A.2.26}$$

and,

$$\nabla_\mu G^\mu_\nu = 0. \tag{A.2.27}$$

## A.3 Useful formulas

The commutation of covariant derivatives acting on a tensor of arbitrary rank is given by:

$$\begin{aligned}
[\nabla_\rho, \nabla_\sigma] X^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} &= -T_{\rho\sigma}{}^\lambda \nabla_\lambda X^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \\
&+ R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda\mu_2 \ldots \mu_k}{}_{\nu_1 \ldots \nu_k} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1\lambda \ldots \mu_k}{}_{\nu_1 \ldots \nu_k} + \cdots \\
&- R^\lambda{}_{\nu_1\rho\sigma} X^{\mu_1 \ldots \mu_k}{}_{\lambda\nu_2 \ldots \nu_l} - R^\lambda{}_{\nu_2\rho\sigma} X^{\mu_1 \ldots \mu_k}{}_{\nu_1\lambda \ldots \nu_l} - \cdots ,
\end{aligned} \tag{A.3.28}$$

where the Torsion tensor is given by,

$$T_{\mu\nu}{}^\lambda = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]}. \tag{A.3.29}$$

Covariant derivative action on a tensor of arbitrary rank is given by,

$$\begin{aligned}
\nabla_\sigma X^{\mu_1\mu_2 \ldots \mu_k}{}_{\nu_1\nu_2 \ldots \nu_l} &= \partial_\sigma X^{\mu_1\mu_2 \ldots \mu_k}{}_{\nu_1\nu_2 \ldots \nu_l} \\
&+ \Gamma^{\mu_1}_{\sigma\lambda} X^{\lambda\mu_2 \ldots \mu_k}{}_{\nu_1\nu_2 \ldots \nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} X^{\mu_1\lambda \ldots \mu_k}{}_{\nu_1\nu_2 \ldots \nu_l} + \cdots \\
&- \Gamma^\lambda_{\sigma\nu_1} X^{\mu_1\mu_2 \ldots \mu_k}{}_{\lambda\nu_2 \ldots \nu_l} - \Gamma^\lambda_{\sigma\nu_2} X^{\mu_1\mu_2 \ldots \mu_k}{}_{\nu_1\lambda \ldots \nu_l} - \cdots .
\end{aligned} \tag{A.3.30}$$

The Lie derivative along $V$ on some arbitrary ranked tensor is given by,

$$\begin{aligned}\pounds_V X^{\mu_1\mu_2\ldots\mu_k}_{\quad\nu_1\nu_2\ldots\nu_l} &= V^\sigma\partial_\sigma X^{\mu_1\mu_2\ldots\mu_k}_{\quad\nu_1\nu_2\ldots\nu_l} \\ &- (\partial_\lambda V^{\mu_1})X^{\lambda\mu_2\ldots\mu_k}_{\quad\nu_1\nu_2\ldots\nu_l} - (\partial_\lambda V^{\mu_2})X^{\mu_1\lambda\ldots\mu_k}_{\quad\nu_1\nu_2\ldots\nu_l} - \cdots \\ &+ (\partial_{\nu_1} V^\lambda)X^{\mu_1\mu_2\ldots\mu_k}_{\quad\lambda\nu_2\ldots\nu_l} + (\partial_{\nu_2} V^\lambda)X^{\mu_1\mu_2\ldots\mu_k}_{\quad\nu_1\lambda\ldots\nu_l} + \cdots . \quad \text{(A.3.31)}\end{aligned}$$

In similar manner the Lie derivative of the metric would be,

$$\pounds_V g_{\mu\nu} = V^\sigma\nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda)g_{\lambda\nu} + (\nabla_\nu V^\lambda)g_{\mu\lambda} = 2\nabla_{(\mu}V_{\nu)}. \quad \text{(A.3.32)}$$

Symmetric and anti-symmetric properties, respectively, are,

$$X_{(ij)k\ldots} = \frac{1}{2}(X_{ijk\ldots} + X_{jik\ldots}), \quad \text{(A.3.33)}$$

$$X_{[ij]k\ldots} = \frac{1}{2}(X_{ijk\ldots} - X_{jik\ldots}). \quad \text{(A.3.34)}$$

# Appendix B

# Newtonian potential

Let us consider the Newtonian potential in the weak-field regime. The Newtonian approximation of a perturbed metric for a static point source is given by the following line element,

$$\begin{aligned} ds^2 &= (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\ &= -[1 + 2\Phi(r)]dt^2 + [1 - 2\Psi(r)](dx^2 + dy^2 + dz^2) \end{aligned} \tag{B.0.1}$$

where the perturbation is given by,

$$h_{\mu\nu} = \begin{pmatrix} -2\Phi(r) & 0 & 0 & 0 \\ 0 & -2\Psi(r) & 0 & 0 \\ 0 & 0 & -2\Psi(r) & 0 \\ 0 & 0 & 0 & -2\Psi(r) \end{pmatrix}. \tag{B.0.2}$$

The field equations for the infinite derivative theory of gravity is given by,

$$\begin{aligned} -\kappa T_{\mu\nu} &= \frac{1}{2}[a(\bar{\Box})\Box h_{\mu\nu} + b(\bar{\Box})\partial_\sigma(\partial_\mu h^\sigma_\nu + \partial_\nu h^\sigma_\mu) + c(\bar{\Box})(\partial_\nu\partial_\mu h + \eta_{\mu\nu}\partial_\sigma\partial_\tau h^{\sigma\tau}) \\ &+ d(\bar{\Box})\eta_{\mu\nu}\Box h + \frac{f(\bar{\Box})}{\Box}\partial_\mu\partial_\nu\partial_\sigma\partial_\tau h^{\sigma\tau}], \end{aligned} \tag{B.0.3}$$

where $\kappa = 8\pi G_N^{(D)} = M_P^{-2}$ and $T_{\mu\nu}$ is the stress-energy tensor. By using the definitions of the linearised curvature, we can rewrite the field equations as,

$$\kappa T_{\mu\nu} = a(\bar{\Box})R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}c(\bar{\Box})R - \frac{f(\bar{\Box})}{2}\partial_\mu\partial_\nu R. \tag{B.0.4}$$

It is apparent how the modification of Einstein-Hilbert action changed the field equations. The trace and 00-component of the field equations are given by,

$$\begin{aligned} -\kappa T_{00} &= \frac{1}{2}[a(\bar{\Box}) - 3c(\bar{\Box})]R, \\ \kappa T_{00} &= a(\bar{\Box})R_{00} + \frac{1}{2}c(\bar{\Box})R, \end{aligned} \tag{B.0.5}$$

where $T_{00}$ gives the energy density. In the static, linearised limit, $\Box = \nabla^2 = \partial_i\partial^i$. In other words, the flat space d'Alembertian operator becomes the Laplace operator. This leads to,

$$\begin{aligned} -\kappa T_{00} &= (a(\bar{\Box}) - 3c(\bar{\Box}))(2\nabla^2\Psi - \nabla^2\Phi), \\ \kappa T_{00} &= (a(\bar{\Box}) - c(\bar{\Box}))\nabla^2\Phi + 2c(\bar{\Box})\nabla^2\Psi. \end{aligned} \tag{B.0.6}$$

We note that,

$$R = 2(2\nabla^2\Psi - \nabla^2\Phi), \quad R_{00} = \nabla^2\Phi. \tag{B.0.7}$$

The Newtonian potentials can be related as,

$$\nabla^2\Phi = -\frac{a(\bar{\Box}) - 2c(\bar{\Box})}{c(\bar{\Box})}\nabla^2\Psi, \tag{B.0.8}$$

and therefore,

$$\kappa T_{00} = \frac{a(\bar{\Box})(a(\bar{\Box}) - 3c(\bar{\Box}))}{a(\bar{\Box}) - 2c(\bar{\Box})}\nabla^2\Phi = \kappa m_g\delta^3(\vec{r}). \tag{B.0.9}$$

We note that $T_{00}$ is the point source and $T_{00} = m_g\delta^3(\vec{r})$, $m_g$ is the mass of the object generating the gravitational potential and $\delta^3$ is the three-dimensional Dirac

delta-function; this is given by,

$$\delta^3(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{ik\vec{r}}. \tag{B.0.10}$$

Thus, with noting that $\Box \to -k^2$, we can take the Fourier components of (B.0.9) and obtain,

$$\begin{aligned} \Phi(r) &= -\frac{\kappa m_g}{(2\pi)^3} \int_{-\infty}^{\infty} d^3k \frac{a(-\bar{k}^2) - 2c(-\bar{k}^2)}{a(-\bar{k}^2)(a(-\bar{k}^2) - 3c(-\bar{k}^2))} \frac{e^{ik\vec{r}}}{k^2} \\ &= -\frac{\kappa m_g}{2\pi^2 r} \int_0^{\infty} dk \frac{a(-\bar{k}^2) - 2c(-\bar{k}^2)}{a(-\bar{k}^2)(a(-\bar{k}^2) - 3c(-\bar{k}^2))} \frac{\sin(kr)}{k}, \end{aligned} \tag{B.0.11}$$

and,

$$\Psi(r) = -\frac{\kappa m_g}{2\pi^2 r} \int_0^{\infty} dk \frac{c(-\bar{k}^2)}{a(-\bar{k}^2)(a(-\bar{k}^2) - 3c(-\bar{k}^2))} \frac{\sin(kr)}{k}. \tag{B.0.12}$$

In order to avoid extra degrees of freedom in the scalar sector of the propagator and to maintain massless graviton as the only propagating degree of freedom, we shall set $a(\bar{k}^2) = c(\bar{k}^2)$, this leads to,

$$\Phi(r) = \Psi(r) = -\frac{\kappa m_g}{(2\pi)^2 r} \int_0^{\infty} dk \frac{\sin(kr)}{a(-\bar{k}^2)k}. \tag{B.0.13}$$

Taking $a(\bar{\Box}) = e^{-\bar{\Box}}$, we obtain,

$$\Phi(r) = \Psi(r) = -\frac{\kappa m_g}{(2\pi)^2 r} \int_0^{\infty} dk \frac{\sin(kr)}{e^{\frac{k^2}{M^2}} k} = -\frac{\kappa m_g \mathrm{Erf}[\frac{Mr}{2}]}{8\pi r}. \tag{B.0.14}$$

As $r \to \infty$, then $\mathrm{Erf}[\frac{Mr}{2}] \to 1$ and we recover the $-r^{-1}$ divergence of GR. When $r \to 0$, we have,

$$\lim_{r\to 0} \Phi(r) = \lim_{r\to 0} \Psi(r) = -\frac{\kappa m_g M}{8\pi^{3/2}} \tag{B.0.15}$$

which is constant. Thus the Newtonian potential is non-singular. See Fig. B.1.

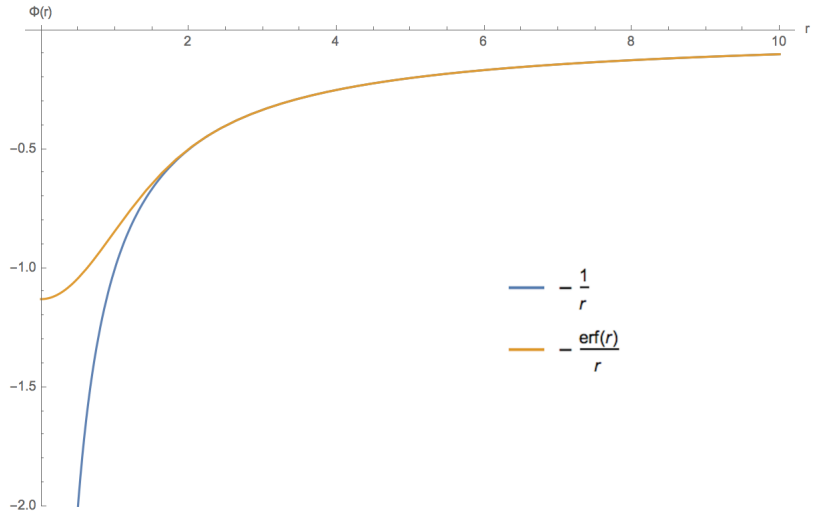


**Figure B.1:** Newtonian potentials. The orange line denotes the non-singular potential while the blue line indicates the original GR potential.

# Appendix C

# Gibbons-York-Hawking boundary term

Let us take the Einstein Hilbert action,

$$S = \frac{1}{2\kappa}\big(S_{EH} + S_{GYH}\big), \tag{C.0.1}$$

where,

$$S_{EH} = \int_{\mathcal{V}} d^4x\,\sqrt{-g}R, \tag{C.0.2}$$

$$S_{GYH} = 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}K, \tag{C.0.3}$$

Gibbons-York-Hawking boundary term is denoted by $S_{GYH}$. Also, $\kappa = 8\pi G_N$. We are considering the space-time as a pair $(\mathcal{M}, g)$ with $\mathcal{M}$ a four-dimensional manifold and $g$ a metric on $\mathcal{M}$. Thus, $\mathcal{V}$ is a hyper-volume on manifold $\mathcal{M}$, and $\partial\mathcal{V}$ is its boundary. $h$ the determinant of the induced metric, $K$ is the trace of the extrinsic curvature of the boundary $\partial\mathcal{V}$, and $\varepsilon$ is equal to $+1$ if $\partial\mathcal{V}$ is time-like and $-1$ if $\partial\mathcal{V}$ is space-like. We shall derive the $S_{GYH}$ in this section. To start we fix the following condition,

$$\delta g_{\alpha\beta}\Big|_{\partial\mathcal{V}} = 0, \tag{C.0.4}$$

We also have the following useful formulas,

$$\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}, \qquad \delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}, \tag{C.0.5}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}, \tag{C.0.6}$$

$$\delta R^{\alpha}_{\beta\gamma\delta} = \nabla_{\gamma}(\delta\Gamma^{\alpha}_{\delta\beta}) - \nabla_{\delta}(\delta\Gamma^{\alpha}_{\gamma\beta}), \tag{C.0.7}$$

$$\delta R_{\alpha\beta} = \nabla_{\gamma}(\delta\Gamma^{\gamma}_{\beta\alpha}) - \nabla_{\beta}(\delta\Gamma^{\gamma}_{\gamma\alpha}). \tag{C.0.8}$$

Let us now vary the Einstein-Hilbert action,

$$\delta S_{EH} = \int_{\mathcal{V}} d^4x \left(R\delta\sqrt{-g} + \sqrt{-g}\,\delta R\right). \tag{C.0.9}$$

The variation of the Ricci scalar is given by,

$$\delta R = \delta g^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}. \tag{C.0.10}$$

using the Palatini's identity (C.0.8),

$$\begin{aligned}\delta R &= \delta g^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}\big(\nabla_{\gamma}(\delta\Gamma^{\gamma}_{\beta\alpha}) - \nabla_{\beta}(\delta\Gamma^{\gamma}_{\alpha\gamma})\big),\\ &= \delta g^{\alpha\beta}R_{\alpha\beta} + \nabla_{\sigma}\big(g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})\big),\end{aligned} \tag{C.0.11}$$

where the metric compatibility indicates $\nabla_{\gamma}g_{\alpha\beta} \equiv 0$ and dummy indices were relabeled. Plugging back this result into the action variation, we obtain,

$$\begin{aligned}\delta S_{EH} &= \int_{\mathcal{V}} d^4x \left(R\delta\sqrt{-g} + \sqrt{-g}\,\delta R\right),\\ &= \int_{\mathcal{V}} d^4x \left(-\frac{1}{2}Rg_{\alpha\beta}\sqrt{-g}\,\delta g^{\alpha\beta} + R_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} + \sqrt{-g}\nabla_{\sigma}\big(g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})\big)\right),\\ &= \int_{\mathcal{V}} d^4x\,\sqrt{-g}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right)\delta g^{\alpha\beta} + \int_{\mathcal{V}} d^4x\sqrt{-g}\nabla_{\sigma}\big(g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})\big).\end{aligned} \tag{C.0.12}$$

We are going to name the divergence term as $\delta S_B$, *i.e.*

$$\delta S_B = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\,\nabla_\sigma\big(g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma})\big), \tag{C.0.13}$$

and define,

$$V^\sigma = g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma}), \tag{C.0.14}$$

yielding,

$$\delta S_B = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\,\nabla_\sigma V^\sigma. \tag{C.0.15}$$

The Gauss-Stokes theorem is given by,

$$\int_{\mathcal{V}} d^nx\,\sqrt{|g|}\nabla_\mu A^\mu = \oint_{\partial\mathcal{V}} d^{n-1}y\,\varepsilon\sqrt{|h|}n_\mu A^\mu, \tag{C.0.16}$$

where $n_\mu$ is the unit normal to $\partial\mathcal{V}$. W shall use this and rewrite the boundary term as,

$$\delta S_B = \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n_\sigma V^\sigma, \tag{C.0.17}$$

with $V^\sigma$ given in (C.0.14). The variation of the Christoffel symbol is given by,

$$\begin{aligned}\delta\Gamma^\sigma_{\beta\alpha} &= \delta\bigg(\frac{1}{2}g^{\sigma\gamma}\big[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}\big]\bigg),\\ &= \frac{1}{2}\delta g^{\sigma\gamma}\big[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}\big] + \frac{1}{2}g^{\sigma\gamma}\big[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})\big].\end{aligned} \tag{C.0.18}$$

From the boundary conditions $\delta g_{\alpha\beta} = \delta g^{\alpha\beta} = 0$. Thus,

$$\delta\Gamma^\sigma_{\beta\alpha}\Big|_{\partial\mathcal{V}} = \frac{1}{2}g^{\sigma\gamma}\big[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})\big], \tag{C.0.19}$$

and so,

$$V^\mu\Big|_{\partial\mathcal{V}} = g^{\alpha\beta}\bigg[\frac{1}{2}g^{\mu\gamma}\big[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})\big]\bigg] - g^{\alpha\mu}\bigg[\frac{1}{2}g^{\nu\gamma}\partial_\alpha(\delta g_{\nu\gamma})\bigg], \tag{C.0.20}$$

we can write

$$\begin{aligned} V_\sigma\Big|_{\partial\mathcal{V}} = g_{\sigma\mu}V^\mu\Big|_{\partial\mathcal{V}} &= g_{\sigma\mu}g^{\alpha\beta}\left[\frac{1}{2}g^{\mu\gamma}\big[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})\big]\right] \\ &\quad - g_{\sigma\mu}g^{\alpha\mu}\left[\frac{1}{2}g^{\nu\gamma}\partial_\alpha(\delta g_{\nu\gamma})\right] \\ &= \frac{1}{2}\delta^\gamma_\sigma g^{\alpha\beta}\big[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})\big] - \frac{1}{2}\delta^\alpha_\sigma g^{\nu\gamma}\big[\partial_\alpha(\delta g_{\nu\gamma})\big] \\ &= g^{\alpha\beta}\big[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})\big]. \end{aligned} \tag{C.0.21}$$

Let us now calculate the term $n^\sigma V_\sigma\big|_{\partial\mathcal{V}}$. We note that,

$$g^{\alpha\beta} = h^{\alpha\beta} + \varepsilon n^\alpha n^\beta, \tag{C.0.22}$$

then

$$\begin{aligned} n^\sigma V_\sigma\Big|_{\partial\mathcal{V}} &= n^\sigma(h^{\alpha\beta} + \varepsilon n^\alpha n^\beta)[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})], \\ &= n^\sigma h^{\alpha\beta}[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})], \end{aligned} \tag{C.0.23}$$

where we use the antisymmetric part of $\varepsilon n^\alpha n^\beta$ with $\varepsilon = n^\mu n_\mu = \pm 1$. Since $\delta g_{\alpha\beta} = 0$ on the boundary we have $h^{\alpha\beta}\partial_\beta(\delta g_{\sigma\alpha}) = 0$. Finally we have,

$$n^\sigma V_\sigma\Big|_{\partial\mathcal{V}} = -n^\sigma h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha}). \tag{C.0.24}$$

Hence, the variation of the Einstein-Hilbert action is,

$$\delta S_{EH} = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right)\delta g^{\alpha\beta} - \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma. \tag{C.0.25}$$

The variation of the Gibbons-York-Hawking boundary term is,

$$\delta S_{GYH} = 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}\delta K. \tag{C.0.26}$$

Using the definition of the trace of extrinsic curvature,

$$\begin{aligned} K &= \nabla_\alpha n^\alpha, \\ &= g^{\alpha\beta}\nabla_\beta n_\alpha, \\ &= (h^{\alpha\beta} + \varepsilon n^\alpha n^\beta)\nabla_\beta n_\alpha, \\ &= h^{\alpha\beta}\nabla_\beta n_\alpha, \\ &= h^{\alpha\beta}(\partial_\beta n_\alpha - \Gamma^\gamma_{\beta\alpha} n_\gamma), \end{aligned} \tag{C.0.27}$$

and subsequently its variation,

$$\begin{aligned} \delta K &= -h^{\alpha\beta}\delta\Gamma^\gamma_{\beta\alpha} n_\gamma, \\ &= -\frac{1}{2}h^{\alpha\beta}g^{\sigma\gamma}\big[\partial_\beta(\delta g_{\sigma\alpha}) + \partial_\alpha(\delta g_{\sigma\beta}) - \partial_\sigma(\delta g_{\beta\alpha})\big] n_\gamma, \\ &= -\frac{1}{2}h^{\alpha\beta}\big[\partial_\beta(\delta g_{\sigma\alpha}) + \partial_\alpha(\delta g_{\sigma\beta}) - \partial_\sigma(\delta g_{\beta\alpha})\big] n^\sigma, \\ &= \frac{1}{2}h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha}) n^\sigma, \end{aligned} \tag{C.0.28}$$

where we used $h^{\alpha\beta}\partial_\beta(\delta g_{\sigma\alpha}) = 0$, $h^{\alpha\beta}\partial_\alpha(\delta g_{\sigma\beta}) = 0$ on the boundary; The variation of the Gibbons-York-Hawking becomes,

$$\delta S_{GYH} = \oint_{\partial\mathcal{V}} d^3y\, \varepsilon\sqrt{|h|}h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma. \tag{C.0.29}$$

We see that the second term of (C.0.25) is matching with (C.0.29). In other words, this term exactly cancel the boundary contribution of the Einstein-Hilbert term.

# Appendix D

# Simplification example in IDG action

Let us consider the following terms from (2.1.4),

$$RF_1(\Box)R + RF_2(\Box)\nabla_\nu\nabla_\mu R^{\mu\nu} + R^\nu{}_\mu F_4(\Box)\nabla_\nu\nabla_\lambda R^{\mu\lambda}, \tag{D.0.1}$$

we can recast above as,

$$RF_1(\Box)R + \frac{1}{2}RF_2(\Box)\Box R + \frac{1}{2}R^\nu{}_\mu F_4(\Box)\nabla_\nu\nabla^\mu R, \tag{D.0.2}$$

by having the identity $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$ and also $\nabla_\nu\nabla_\mu R^{\mu\nu} = \frac{1}{2}\Box R$, which occurs due to contraction of Bianchi identity, we can perform integration by parts on the final term and obtain,

$$\begin{aligned} & RF_1(\Box)R + \frac{1}{2}RF_2(\Box)\Box R + \frac{1}{2}\nabla^\mu\nabla_\nu R^\nu{}_\mu F_4(\Box)R \\ = \quad & RF_1(\Box)R + \frac{1}{2}RF_2(\Box)\Box R + \frac{1}{4}RF_4(\Box)\Box R \\ \equiv \quad & RF_1(\Box)R, \end{aligned} \tag{D.0.3}$$

where we redefined the arbitrary function $F_1(\Box)$ to absorb $F_2(\Box)$ and $F_4(\Box)$. We simplified (2.1.4) in similar manner to reach (2.1.8).

# Appendix E

# Hamiltonian density

Hamiltonian density corresponding to action Eq. (3.3.44) is explicitly given by,

$$
\begin{aligned}
\mathcal{H} &= p_A\dot{A} + p_{\chi_1}\dot{\chi}_1 + p_{\chi_l}\dot{\chi}_l + p_{\eta_{l-1}}\dot{\eta}_{l-1} - \mathcal{L} \\
&= -(A\dot{\chi}_1\dot{A} + \dot{A}\chi_1\dot{A}) - \sum_{l=2}^{\infty}(\dot{A}\chi_l\dot{\eta}_{l-1}) \\
&- (A\dot{A})\dot{\chi}_1 - (A\dot{\eta}_{l-1})\dot{\chi}_l - (A\dot{\chi}_l\dot{\eta}_{l-1} + \chi_l\dot{A}\dot{\eta}_{l-1}) \\
&- \Bigg( A(f_0A + \sum_{n=1}^{\infty} f_n\eta_n) + \sum_{l=1}^{\infty} A\chi_l\eta_l \\
&- (A\partial_0\chi_1\partial_0A + \chi_1\partial_0A\partial_0A) + g^{ij}(A\partial_i\chi_1\partial_jA + \chi_1\partial_iA\partial_jA) \\
&- \sum_{l=2}^{\infty}(A\partial_0\chi_1\partial_0\eta_{l-1} + \chi_l\partial_0A\partial_0\eta_{l-1}) + g^{ij}\sum_{l=2}^{\infty}(A\partial_i\chi_l\partial_j\eta_{l-1} + \chi_l\partial_iA\partial_j\eta_{l-1})\Bigg) \\
&= -\sum_{l=2}^{\infty}(\dot{A}\chi_l\dot{\eta}_{l-1}) - (A\dot{A})\dot{\chi}_1 - (A\dot{\eta}_{l-1})\dot{\chi}_l + \Bigg( A(f_0A + \sum_{n=1}^{\infty} f_n\eta_n) - \sum_{l=1}^{\infty} A\chi_l\eta_l \qquad \text{(E.0.1)} \\
&- g^{ij}(A\partial_i\chi_1\partial_jA + \chi_1\partial_iA\partial_jA) - g^{ij}\sum_{l=2}^{\infty}(A\partial_i\chi_l\partial_j\eta_{l-1} + \chi_l\partial_iA\partial_j\eta_{l-1})\Bigg) \qquad \text{(E.0.2)} \\
&= A(f_0A + \sum_{n=1}^{\infty} f_n\eta_n) - \sum_{l=1}^{\infty} A\chi_l\eta_l \\
&- (g^{\mu\nu}A\partial_\mu\chi_1\partial_\nu A + g^{ij}\chi_1\partial_iA\partial_jA) - g^{\mu\nu}\sum_{l=2}^{\infty}(A\partial_\mu\chi_l\partial_\nu\eta_{l-1} + \chi_l\partial_\mu A\partial_\nu\eta_{l-1})\,. \qquad \text{(E.0.3)}
\end{aligned}
$$

# Appendix F

# Physical degrees of freedom via propagator analysis

We have an action of the form [53]

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left[M_P^2 R + R\mathcal{F}\left(\bar{\Box}\right)R\right] \tag{F.0.1}$$

or, equivalently,

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\left[M_P^2 A + A\mathcal{F}\left(\bar{\Box}\right)A + B(R-A)\right]\,. \tag{F.0.2}$$

$A$ and $B$ have mass dimension 2.

The propagator around Minkowski space-time is of the form [68, 93]

$$\Pi(-k^2) = \frac{\mathcal{P}^2}{k^2 a(-k^2)} + \frac{\mathcal{P}_s^0}{k^2(a(-k^2)-3c(-k^2))}\,, \tag{F.0.3}$$

where $a(\Box)=1$ and $c(\Box)=1+M_P^{-2}\mathcal{F}\left(\bar{\Box}\right)\Box$. Hence,

$$\Pi(-k^2) = \frac{\mathcal{P}^2}{k^2} + \frac{\mathcal{P}_s^0}{k^2(-2+3M_P^{-2}k^2\mathcal{F}(-k^2/M^2))} \tag{F.0.4}$$

We know that [68, 93]

$$\mathcal{F}(\bar{\Box}) = M_P^2 \frac{c(\bar{\Box}) - 1}{\Box} \,. \tag{F.0.5}$$

Only if $c(\Box)$ is the exponent of an entire function can we decompose into partial fractions and have just one extra pole.

The upshot is that, in order to have just one extra degree of freedom, we have to impose conditions on the coefficients in $\mathcal{F}\left(\bar{\Box}\right)$. In order to avoid $\Box^{-1}$ terms appearing in $\mathcal{F}(\bar{\Box})$, we must have that

$$c(\bar{\Box}) = \sum_{n=0}^{\infty} c_n \bar{\Box}^n \tag{F.0.6}$$

and $c_0 = 1$. Hence,

$$\mathcal{F}(\bar{\Box}) = \left(\frac{M_P}{M}\right)^2 \sum_{n=0}^{\infty} c_{n+1} \bar{\Box}^n \,. \tag{F.0.7}$$

To get infinitely many poles and, hence, degrees of freedom, one could have, for instance, that

$$c(\bar{\Box}) = \cos(\bar{\Box}) \,, \tag{F.0.8}$$

so that $c_0 = 1$. Then Eq. (F.0.2) becomes

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_P^2 A + M_P^2 A \left( \frac{\cos(\bar{\Box}) - 1}{\Box} \right) A + B(R - A) \right] . \tag{F.0.9}$$

Using (F.0.4), apart from the $k^2 = 0$ pole, we have poles when

$$\cos\left(\frac{k^2}{M^2}\right) = \frac{1}{3} \,. \tag{F.0.10}$$

Eq. (F.0.10) has infinitely many solutions due to the periodicity of the cosine function and, therefore, the propagator has infinitely many poles and, hence, degrees of freedom. We can write the solutions as $\bar{k}^2 = 2m\pi$, where $m = 0, 1, 2, \cdots$, one can also write:

$$\cos(\bar{k}^2) = \prod_{l=1}^{\infty} \left( 1 - \frac{4\bar{k}^4}{(2l-1)^2 \pi^2} \right) \tag{F.0.11}$$

or

$$\cos(\bar{\Box}) = \prod_{l=1}^{\infty} \left(1 - \frac{4\bar{\Box}^2}{(2l-1)^2\pi^2}\right) \tag{F.0.12}$$

Now, to get just one extra degrees of freedom, one can make, for instance, the choice $c(\bar{\Box}) = e^{-\bar{\Box}}$, then

$$\mathcal{F}(\bar{\Box}) = \sum_{n=0}^{\infty} f_n \bar{\Box}^n \,, \tag{F.0.13}$$

where

$$f_n = \left(\frac{M_P}{M}\right)^2 \frac{(-1)^{n+1}}{(n+1)!} \,. \tag{F.0.14}$$

Using (F.0.4), apart from the $k^2 = 0$ pole, we have poles when

$$e^{k^2/M^2} = \frac{1}{3} \,. \tag{F.0.15}$$

There is just one extra pole and, hence, degrees of freedom. In total, there are 3 degrees of freedom.

# Appendix G

# Form of $\mathcal{F}(\bar{\Box})$ and constraints

We have shown in section 3.4.6 that the primary constraints are built as follow:

$$\sigma_1 = \eta_1 - \Box A \approx 0, \cdots, \sigma_l = \eta_l - \Box \eta_{l-1} \approx 0. \tag{G.0.1}$$

We also mentioned that they are first class constraints due to their Poission brackets vanishing weakly. The number of the degrees of freedom are related to the form of $\mathcal{F}(\bar{\Box})$. Expanding on this, we shall consider the case when: $\mathcal{F}(\bar{\Box}) = \Box e^{-\bar{\Box}}$. Then for an action of the form,

$$S = \int d^4x A \mathcal{F}(\bar{\Box}) A. \tag{G.0.2}$$

The equation of motion for $A$ is given by

$$2\mathcal{F}(\bar{\Box})A = 0. \tag{G.0.3}$$

For the case when $\mathcal{F}(\bar{\Box}) = \Box e^{-\bar{\Box}}$, the equation of motion becomes,

$$\Theta = \eta_1 = 0. \tag{G.0.4}$$

Moreover, $\mathcal{F}(\bar{\Box})$ cannot be written in the form

$$\mathcal{F}(\bar{\Box}) = (\Box + m_2^2)(\Box + m_1^2)\mathcal{G}_1(\bar{\Box}), \tag{G.0.5}$$

where $m_1^2$, $m_2^2$ are arbitrarily chosen parameters and $\mathcal{G}_1$ has no roots. This is a constraint, which can be written as follows:

$$\Xi_2 = \mathcal{F}(\bar{\Box})A - (\Box + m_2^2)(\Box + m_1^2)\mathcal{G}_1(\bar{\Box})A \not\approx 0. \tag{G.0.6}$$

Moreover, we have the constraint

$$\Xi_3 = \mathcal{F}(\bar{\Box})A - (\Box + m_3^2)(\Box + m_2^2)(\Box + m_1^2)\mathcal{G}_2(\bar{\Box})A \not\approx 0, \tag{G.0.7}$$

where $m_1^2$, $m_2^2$, $m_3^2$ are arbitrarily chosen parameters and $\mathcal{G}_2$ has no roots. This goes on and on.

Regarding degrees of freedom and given that the constraints are first-class, we have,

$$\begin{aligned}
2\mathcal{A} &\equiv 2 \times \{(A, p_A), (\eta_1, p_{\eta_1}), (\eta_2, p_{\eta_2}), \dots\} = 2 \times (2 + \infty) = 4 + \infty, \\
\mathcal{B} &= 0, \\
2\mathcal{C} &\equiv 2 \times (\Theta, \Xi_2, \Xi_3, \dots) = 2(1 + \infty) = 2 + \infty, \\
\mathcal{N} &= \frac{1}{2}(2\mathcal{A} - \mathcal{B} - 2\mathcal{C}) = \frac{1}{2}(4 + \infty - 2 - \infty) = 1,
\end{aligned} \tag{G.0.8}$$

as expected.

A similar prescription can be applied in infinite derivative gravity. Now as a final clarification we shall reparametrise these constraints (*i.e.* $\Theta, \Xi_2, \Xi_3, \dots$), into $\sigma$'s. From (G.0.6) we have,

$$\Xi_2 = \mathcal{F}(\bar{\Box})A - (\Box + m_2^2)(\Box + m_1^2)\mathcal{R}_1(\bar{\Box})A \approx 0, \tag{G.0.9}$$

where $\mathcal{R}_1(\bar{\Box})$ has no roots and contains $\Box^{-1}$ terms. Then

$$\frac{\mathcal{F}(\bar{\Box})}{\mathcal{R}_1(\bar{\Box})}A \approx (\Box + m_2^2)(\Box + m_1^2)A = \eta_2 + (m_1^2 + m_2^2)\eta_1 + m_1^2 m_2^2 A \qquad \text{(G.0.10)}$$

Redefining $\eta_2$ and $\eta_1$ appropriately, (G.0.10) can be written in the form $\eta_2 - \Box\eta_1 = 0$. Similarly for $\Xi_3$ and so on. Hence, the constraints are equivalent.

# Appendix H

# $K_{ij}$ in the Coframe Metric

In this section we wish to use the approach of [140] and find the general definition for $K_{ij}$ in the coframe metric. Given,

$$\gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + g^{\alpha\delta}C^{\epsilon}{}_{\delta(\beta}g_{\gamma)\epsilon} - \frac{1}{2}C^{\alpha}{}_{\beta\gamma}\,, \tag{H.0.1}$$

$$d\theta^{\alpha} = -\frac{1}{2}C^{\alpha}{}_{\beta\gamma}\theta^{\beta}\wedge\theta^{\gamma}\,, \tag{H.0.2}$$

where $\Gamma$ is the ordinary Christoffel symbol, $\wedge$ is the ordinary wedge product and the $C$s are coefficients to be found. By comparing the values given in [140] with the ordinary Christoffel symbols, we can see that

$$C^{i}{}_{00} = C^{0}{}_{0i} = C^{0}{}_{i0} = C^{0}{}_{00} = 0\,,$$

$$C^{i}{}_{0k} = C^{i}{}_{k0} + 2\partial_k\beta^i\,,$$

$$C^{i}{}_{jk} = C_k{}^{i}{}_{j} + C_j{}^{i}{}_{k}\,, \tag{H.0.3}$$

Now in the coframe metric in Eq. (4.3.17),

$$\begin{aligned} & g^{0\delta} C^{\epsilon}{}_{\delta(i} g_{j)\epsilon} \,, \\ = \; & -\frac{1}{N^2} \left[ C^{\epsilon}{}_{0(i} g_{j)\epsilon} \right] \,, \\ = \; & -\frac{1}{2N^2} \left[ C^{\epsilon}{}_{0i} g_{j\epsilon} + C^{\epsilon}{}_{0j} g_{i\epsilon} \right] \,, \\ = \; & -\frac{1}{2N^2} \left[ C^{m}{}_{0i} g_{jm} + C^{m}{}_{0j} g_{im} \right] \,, \\ = \; & -\frac{1}{2N^2} \left[ C_{j0i} + C_{i0j} \right] \end{aligned} \tag{H.0.4}$$

and $C_{j0i} = g_{\alpha j} C^{\alpha}{}_{0i} = g_{jk} C^{k}{}_{0i}$. In the coframe and using the conventions in [140]

$$K_{ij} = -\nabla_i n_j = \frac{1}{2N} \left( D_i \beta_j + D_j \beta_i - \dot{h}_{ij} \right) \,, \tag{H.0.5}$$

and

$$\partial_0 h_{ij} = \partial_t h_{ij} - \beta^l \partial_l h_{ij} \,. \tag{H.0.6}$$

In general, for a $p$-form $\alpha$ and a $q$-form $\beta$,

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \,, \tag{H.0.7}$$

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta) \,. \tag{H.0.8}$$

Hence, if $p$ is odd,

$$\alpha \wedge \alpha = (-1)^{p^2} \alpha \wedge \alpha = -\alpha \wedge \alpha = 0 \,. \tag{H.0.9}$$

From Eq. (H.0.2) and Eq. (H.0.3) we can see that

$$\begin{aligned} d\theta^1 &= -\frac{1}{2}C^1{}_{\beta\gamma}\theta^\beta \wedge \theta^\gamma \,, \\ &= -\frac{1}{2}C^1{}_{0i}\theta^0 \wedge \theta^i - \frac{1}{2}C^1{}_{i0}\theta^i \wedge \theta^0 - \frac{1}{2}C^1{}_{ij}\theta^i \wedge \theta^j \,, \\ &= -\frac{1}{2}\left[C^1{}_{i0} + 2\partial_i\beta^1\right]\theta^0 \wedge \theta^i + \frac{1}{2}C^1{}_{i0}\theta^0 \wedge \theta^i + \frac{1}{2}C^1{}_{ij}\theta^j \wedge \theta^i \,, \\ &= -\left(\partial_i\beta^1\right)\theta^0 \wedge \theta^i + \frac{1}{2}C^1{}_{ij}\theta^j \wedge \theta^i \,. \end{aligned} \tag{H.0.10}$$

We get a similar result for $d\theta^2$ and $d\theta^3$, so we can say that

$$d\theta^k = -\left(\partial_i\beta^k\right)\theta^0 \wedge \theta^i + \frac{1}{2}C^k{}_{ij}\theta^j \wedge \theta^i \,, \tag{H.0.11}$$

where $k = 1, 2, 3$. Now from the definition of $\theta$ in Eq. (4.3.22),

$$d\theta^1 = d\left(dx^1 + \beta^1 dt\right) = d\beta^1 \wedge dt \,, \tag{H.0.12}$$

and

$$\begin{aligned} d\theta^0 &= d(dt) = d^2(t) = 0 \,, \\ d\theta^i &= d\left(dx^i + \beta^i dt\right) \,, \\ &= d\left(dx^i\right) + d\left(\beta^i \wedge dt\right) \,, \\ &= d\left(\beta^i \wedge dt\right) \,, \\ &= d\beta^i \wedge dt \,. \end{aligned} \tag{H.0.13}$$

Let us point out that $\beta^i dt = \beta^i \wedge dt$.

$$\begin{aligned}
\theta^0 \wedge \theta^i &= dt \wedge \left(dx^i + \beta^i \wedge dt\right) , \\
&= dt \wedge dx^i \\
\theta^i \wedge \theta^j &= \left(dx^i + \beta^i dt\right) \wedge \left(dx^j + \beta^j dt\right) , \\
&= dx^i \wedge dx^j + dx^i \wedge \left(\beta^j dt\right) + \left(\beta^i dt\right) \wedge dx^j + \left(\beta^i dt\right) \wedge \left(\beta^j \wedge dt\right) , \\
&= dx^i \wedge dx^j + dx^i \wedge \beta^j \wedge dt + \beta^i \wedge dt \wedge dx^j , \\
&= dx^i \wedge dx^j + \beta^j \wedge dx^i \wedge dt - \beta^i \wedge dx^j \wedge dt . \qquad \text{(H.0.14)}
\end{aligned}$$

Now using Eq. (H.0.11) and Eq. (H.0.13),

$$\begin{aligned}
d\theta^k &= d\beta^k \wedge dt , \\
&= \left(\frac{\partial \beta^k}{\partial x^1} dx^1 + \frac{\partial \beta^k}{\partial x^2} dx^2 + \frac{\partial \beta^k}{\partial x^3} dx^3\right) \wedge dt , \\
&= -\left(\partial_i \beta^k\right) dt \wedge dx^i - \frac{1}{2} C^k{}_{ij} \left[dx^i \wedge dx^j + \beta^j \wedge dx^i \wedge dt - \beta^i \wedge dx^j \wedge dt\right] , \qquad \text{(H.0.15)}
\end{aligned}$$

where $k = 1, 2, 3$. From the definition of $d\theta^\alpha$ in Eq. (H.0.2) and using the anti-symmetric properties of the $\wedge$ product from Eq. (H.0.9),

$$\begin{aligned}
d\theta^\alpha &= -\frac{1}{2} C^\alpha{}_{\beta\gamma} \theta^\beta \wedge \theta^\gamma , \\
&= -\frac{1}{2} C^\alpha{}_{\gamma\beta} \theta^\gamma \wedge \theta^\beta , \\
&= \frac{1}{2} C^\alpha{}_{\gamma\beta} \theta^\beta \wedge \theta^\gamma , \qquad \text{(H.0.16)}
\end{aligned}$$

and therefore

$$C^\alpha{}_{\beta\gamma} = -C^\alpha{}_{\gamma\beta} , \qquad \text{(H.0.17)}$$

we can then write

$$C^{\alpha}{}_{00} = C^{\alpha}{}_{11} = C^{\alpha}{}_{22} = C^{\alpha}{}_{33} = 0\,,$$
$$C^{\alpha}{}_{0i} = -C^{\alpha}{}_{i0}\,. \tag{H.0.18}$$

Combining Eq. (H.0.2), Eq. (H.0.13), Eq. (H.0.15) and utilising Eq. (H.0.17)

$$\begin{aligned}
0 &= d\theta^0 = -\frac{1}{2}C^0{}_{\beta\gamma}\theta^\gamma \wedge \theta^\beta\,,\\
&= -\frac{1}{2}C^0{}_{0i}\theta^0 \wedge \theta^i - \frac{1}{2}C^0{}_{i0}\theta^i \wedge \theta^0 - \frac{1}{2}C^0{}_{ij}\theta^i \wedge \theta^j (\text{for } i \neq j)\,,\\
&= -C^0{}_{0i}\theta^0 \wedge \theta^i - C^0{}_{ij}\theta^i \wedge \theta^j (\text{for } i < j)\,,\\
&= -C^0{}_{01}\theta^0 \wedge \theta^1 - -C^0{}_{02}\theta^0 \wedge \theta^2 - C^0{}_{03}\theta^0 \wedge \theta^3 - C^0{}_{12}\theta^1 \wedge \theta^2 - C^0{}_{13}\theta^1 \wedge \theta^3\\
&\quad -C^0{}_{23}\theta^2 \wedge \theta^3\,,\\
&= -C^0{}_{01}dt \wedge dx^1 - C^0{}_{02}dt \wedge dx^2 - C^0{}_{03}dt \wedge dx^3\,,\\
&\quad -C^0{}_{12}\left[dx^1 \wedge dx^2 + \beta^2 dx^1 \wedge dt - \beta^1 dx^2 \wedge dt\right]\,,\\
&\quad -C^0{}_{13}\left[dx^1 \wedge dx^3 + \beta^3 dx^1 \wedge dt - \beta^1 dx^3 \wedge dt\right]\,,\\
&\quad -C^0{}_{23}\left[dx^2 \wedge dx^3 + \beta^3 dx^2 \wedge dt - \beta^2 dx^3 \wedge dt\right]\,.
\end{aligned} \tag{H.0.19}$$

In order for this to be satisfied, each term must vanish separately as the $dx^\alpha \wedge dx^j$ are linearly independent and so the coefficient of each must be zero and thus $C^0{}_{12} = C^0{}_{13} = C^0{}_{23} = C^0{}_{01} = C^0{}_{02} = C^0{}_{03} = 0$ and thus $C^0{}_{\alpha\beta} = 0$. Similarly using Eqs. (H.0.2), (H.0.13), (H.0.15) and (H.0.17)

$$\begin{aligned}
d\beta^1 \wedge dt &= \frac{\partial\beta^1}{\partial dx^1}dx^1 + \frac{\partial\beta^1}{\partial dx^2}dx^2 + \frac{\partial\beta^1}{\partial dx^3}dx^3\,,\\
&= d\theta^1 = -C^1{}_{0i}\theta^0 \wedge \theta^i - C^1{}_{ij}\theta^i \wedge \theta^j\,,\\
&= -C^1{}_{01}dt \wedge dx^1 - C^1{}_{02}dt \wedge dx^2 - C^1{}_{03}dt \wedge dx^3\,,\\
&\quad -C^1{}_{12}\left[dx^1 \wedge dx^2 + \beta^2 dx^1 \wedge dt - \beta^1 dx^2 \wedge dt\right]\,,\\
&\quad -C^1{}_{13}\left[dx^1 \wedge dx^3 + \beta^3 dx^1 \wedge dt - \beta^1 dx^3 \wedge dt\right]\,,\\
&\quad -C^1{}_{23}\left[dx^2 \wedge dx^3 + \beta^3 dx^2 \wedge dt - \beta^2 dx^3 \wedge dt\right]\,.
\end{aligned} \tag{H.0.20}$$

Again, in order for this relation to be satisfied, $C^1{}_{12} = C^1{}_{13} = C^1{}_{23} = 0$ and

$C^1{}_{01} = \frac{\partial \beta^1}{\partial x^1}$, $C^1{}_{02} = \frac{\partial \beta^1}{\partial x^2}$, $C^1{}_{03} = \frac{\partial \beta^1}{\partial x^3}$. We deduce that $C^m{}_{0i} = \frac{\partial \beta^m}{\partial x^i}$, $C^m{}_{ij} = 0$ and $C^0{}_{ij} = 0$. Using Eq. (H.0.2) and that in the coframe $\Gamma^0_{ij} = \frac{1}{2}\frac{1}{N^2}\bar{\partial}_0 h_{ij}$, we obtain that

$$\gamma^0_{ij} = -\frac{1}{2N^2}\Big(h_{il}\partial_j(\beta^l) + h_{jl}\partial_i(\beta^l) - \bar{\partial}_0 h_{ij}\Big)\,. \tag{H.0.21}$$

Since from Eq. (4.3.12)

$$K_{ij} \equiv -\nabla_i n_j = \gamma^\mu_{ij} n_\mu = -N\gamma^0_{ij}\,, \tag{H.0.22}$$

Eq. (H.0.1) becomes

$$K_{ij} = \frac{1}{2N}(h_{il}\partial_j(\beta^l) + h_{jl}\partial_i(\beta^l) - \bar{\partial}_0 h_{ij})\,. \tag{H.0.23}$$

# Appendix I

# 3+1 Decompositions

## I.1 Einstein-Hilbert term

We can write the Einstein-Hilbert term $\mathcal{R}$ as its auxiliary equivalent $\varrho$. Then we can use the completeness relation Eq. (4.3.9) to show that

$$\begin{aligned}
\varrho &= g^{\mu\rho}g^{\nu\sigma}\varrho_{\mu\nu\rho\sigma}\,, \\
&= \left(h^{\mu\rho} - n^\mu n^\rho\right)\left(h^{\nu\sigma} - n^\nu n^\sigma\right)\varrho_{\mu\nu\rho\sigma}\,, \\
&= \left(h^{\mu\rho}h^{\nu\sigma} - n^\mu n^\rho h^{\nu\sigma} - h^{\mu\rho}n^\nu n^\sigma + n^\mu n^\rho n^\nu n^\sigma\right)\varrho_{\mu\nu\rho\sigma}\,, \\
&= \left(h^{\mu\rho}h^{\nu\sigma} - n^\mu n^\rho h^{\nu\sigma} - h^{\mu\rho}n^\nu n^\sigma\right)\varrho_{\mu\nu\rho\sigma}\,, \\
&= \left(\rho - 2\Omega\right)\,, \qquad \text{(I.1.1)}
\end{aligned}$$

noting that the term with four $n^\alpha$s vanishes due to the antisymmetry of the Riemann tensor in the first and last pair of indices (recall that $\varrho_{\mu\nu\rho\sigma}$ has the same symmetry properties as the Riemann tensor)

## I.2 Riemann Tensor

In this section we wish to show the contraction of the rest of the terms in Eq. (4.5.47) for the sake of completeness. We have, from $hhhh$,

$$
\begin{aligned}
&h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\gamma}_{\rho}h^{\lambda}_{\sigma}\varrho_{\alpha\beta\gamma\lambda}\left[-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\right]\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{\mu}e^{\alpha}_{i}\right)\left(h^{j}_{\nu}e^{\beta}_{j}\right)\left(h^{k}_{\rho}e^{\gamma}_{k}\right)\left(h^{l}_{\sigma}e^{\lambda}_{l}\right)\varrho_{\alpha\beta\gamma\lambda}\Big[-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\Big]\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{\mu}\right)\left(h^{j}_{\nu}\right)\left(h^{k}_{\rho}\right)\left(h^{l}_{\sigma}\right)\rho_{ijkl}\Big[-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\Big]\varrho^{\mu\nu\rho\sigma}\\
&=-N^{-2}\left[\left(h^{i}_{m}e^{m}_{\mu}\right)\left(h^{j}_{n}e^{n}_{\nu}\right)\left(h^{k}_{x}e^{x}_{\rho}\right)\left(h^{l}_{y}e^{y}_{\sigma}\right)\right]\rho_{ijkl}\Big[-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\Big]\varrho^{\mu\nu\rho\sigma}\\
&=-N^{-2}\rho_{ijkl}\Big\{\bar{\partial}^2_0\left(\rho^{ijkl}\right)\\
&-\bar{\partial}_0\left[\varrho^{\mu\nu\rho\sigma}\bar{\partial}_0\left(\left[\left(h^{i}_{m}e^{m}_{\mu}\right)\left(h^{j}_{n}e^{n}_{\nu}\right)\left(h^{k}_{x}e^{x}_{\rho}\right)\left(h^{l}_{y}e^{y}_{\sigma}\right)\right]\right)\right]\\
&-\bar{\partial}_0\left(\left[\left(h^{i}_{m}e^{m}_{\mu}\right)\left(h^{j}_{n}e^{n}_{\nu}\right)\left(h^{k}_{x}e^{x}_{\rho}\right)\left(h^{l}_{y}e^{y}_{\sigma}\right)\right]\right)\bar{\partial}_0\left(\varrho^{\mu\nu\rho\sigma}\right)\Big\}\\
&+\rho_{ijkl}\Big\{\Box_{hyp}\left[\rho^{ijkl}\right]-D_a\left(D^a\left[e^{m}_{\mu}e^{n}_{\nu}e^{x}_{\rho}e^{y}_{\sigma}\right]h^{i}_{m}h^{j}_{n}h^{k}_{x}h^{l}_{y}\varrho^{\mu\nu\rho\sigma}\right)\\
&-D_a\left[e^{m}_{\mu}e^{n}_{\nu}e^{x}_{\rho}e^{y}_{\sigma}\right]D^a\left(h^{i}_{m}h^{j}_{n}h^{k}_{x}h^{l}_{y}\varrho^{\mu\nu\rho\sigma}\right)\Big\}
\end{aligned}
\tag{I.2.2}
$$

which produced $\rho_{ijkl}\Box\rho^{ijkl}$ and the terms which are the results of Leibniz rule. Next in Eq. (4.5.47) is,

$$
\begin{aligned}
&h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\gamma}_{\rho}n^{\lambda}n_{\sigma}\varrho_{\alpha\beta\gamma\lambda}\left(-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{\mu}e^{\alpha}_{i}\right)\left(h^{j}_{\nu}e^{\beta}_{j}\right)\left(h^{k}_{\rho}e^{\gamma}_{k}\right)n^{\lambda}n_{\sigma}\varrho_{\alpha\beta\gamma\lambda}\left(-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{m}e^{m}_{\mu}\right)\left(h^{j}_{n}e^{n}_{\nu}\right)\left(h^{k}_{x}e^{x}_{\rho}\right)n^{\lambda}n_{\sigma}\varrho_{ijk\lambda}\left(-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\\
&=\left(h^{i}_{m}e^{m}_{\mu}\right)\left(h^{j}_{n}e^{n}_{\nu}\right)\left(h^{k}_{x}e^{x}_{\rho}\right)n_{\sigma}\rho_{ijk}\left(-(N^{-1}\bar{\partial}_0)^2+\Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\\
&=-N^{-2}\rho_{ijk}\Big\{\bar{\partial}^2_0\left(\rho^{ijk}\right)\\
&-\bar{\partial}_0\left[\varrho^{\mu\nu\rho\sigma}\bar{\partial}_0\left(\left[h^{i}_{m}e^{m}_{\mu}h^{j}_{n}e^{n}_{\nu}h^{k}_{x}e^{x}_{\rho}n_{\sigma}\right]\right)\right]-\bar{\partial}_0\left(\left[h^{i}_{m}e^{m}_{\mu}h^{j}_{n}e^{n}_{\nu}h^{k}_{x}e^{x}_{\rho}n_{\sigma}\right]\right)\bar{\partial}_0\left(\varrho^{\mu\nu\rho\sigma}\right)\Big\}\\
&+\rho_{ijk}\Big\{\Box_{hyp}\left[\rho^{ijk}\right]-D_a\left(D^a\left[e^{m}_{\mu}e^{n}_{\nu}e^{x}_{\rho}n_{\sigma}\right]h^{i}_{m}h^{j}_{n}h^{k}_{x}\varrho^{\mu\nu\rho\sigma}\right)\\
&-D_a\left[e^{m}_{\mu}e^{n}_{\nu}e^{x}_{\rho}n_{\sigma}\right]D^a\left(h^{i}_{m}h^{j}_{n}h^{k}_{x}\varrho^{\mu\nu\rho\sigma}\right)\Big\}
\end{aligned}
\tag{I.2.3}
$$

with $\rho_{ijk} \equiv n^\mu \rho_{ijk\mu}$. Here we produced $\rho_{ijk}\Box\rho^{ijk}$ and the extra terms which are the results of the Leibniz rule. Similarly we can find the contractions for different terms in Eq. (4.5.47).

## I.3 Ricci Tensor

In similar way as we did in the Riemann case we can find all the other contractions in the expansion of Eq. (4.5.54) which we omitted. They are:

$$
\begin{aligned}
&h^{\rho\sigma}h^{\mu\kappa}h^{\nu\lambda}h^{\gamma\delta}\varrho_{\rho\mu\sigma\nu}\Box\varrho_{\gamma\kappa\delta\lambda} \\
&= (h^{im}e^\rho_i e^\sigma_m)(h^{jn}e^\mu_j e^\kappa_n)(h^{kx}e^\nu_k e^\lambda_x)(h^{ly}e^\gamma_l e^\delta_y)\varrho_{\rho\mu\sigma\nu}\left(-\left(N^{-1}\bar\partial_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda} \\
&= (h^{jn}e^\kappa_n)(h^{kx}e^\lambda_x)(h^{ly}e^\gamma_l e^\delta_y)\rho_{jk}\left(-\left(N^{-1}\bar\partial_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda} \\
&= -N^{-2}\rho_{jk}\Big\{\bar\partial_0^2(\rho^{jk})-\bar\partial_0(\varrho_{\gamma\kappa\delta\lambda}\bar\partial_0[(h^{jn}e^\kappa_n)(h^{kx}e^\lambda_x)(h^{ly}e^\gamma_l e^\delta_y)]) \\
&-\bar\partial_0[(h^{jn}e^\kappa_n)(h^{kx}e^\lambda_x)(h^{ly}e^\gamma_l e^\delta_y)]\bar\partial_0\varrho_{\gamma\kappa\delta\lambda}\Big\} \\
&+\rho_{jk}\Big\{\Box_{hyp}(\rho^{jk})-D_a(\varrho_{\gamma\kappa\delta\lambda}D^a[(h^{jn}e^\kappa_n)(h^{kx}e^\lambda_x)(h^{ly}e^\gamma_l e^\delta_y)]) \\
&-D_a[(h^{jn}e^\kappa_n)(h^{kx}e^\lambda_x)(h^{ly}e^\gamma_l e^\delta_y)]D^a\varrho_{\gamma\kappa\delta\lambda}\Big\}
\end{aligned}
\tag{I.3.4}
$$

with $(h^{jn}e^\kappa_n)(h^{kx}e^\lambda_x)(h^{ly}e^\gamma_l e^\delta_y)\varrho_{\gamma\kappa\delta\lambda} = \rho^{jk}$. Above we produced $\rho_{jk}\Box\rho^{jk}$ plus other terms that are results of the Leibniz rule. And,

$$
\begin{aligned}
&h^{\rho\sigma}n^\mu n^\kappa h^{\nu\lambda}h^{\gamma\delta}\varrho_{\rho\mu\sigma\nu}\left(-\left(N^{-1}\bar\partial_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda} \\
&= (h^{ij}e^\rho_i e^\sigma_j)n^\mu n^\kappa(h^{kl}e^\nu_k e^\lambda_l)(h^{mn}e^\gamma_m e^\delta_n)\varrho_{\rho\mu\sigma\nu}\left(-\left(N^{-1}\bar\partial_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda} \\
&= n^\kappa(h^{kl}e^\lambda_l)(h^{mn}e^\gamma_m e^\delta_n)\rho_k\left(-\left(N^{-1}\bar\partial_0\right)^2+\Box_{hyp}\right)\varrho_{\gamma\kappa\delta\lambda} \\
&= -N^{-2}\rho_k\Big\{\bar\partial_0^2(\rho^k)-\bar\partial_0\big(\varrho_{\gamma\kappa\delta\lambda}\bar\partial_0[n^\kappa h^{kl}e^\lambda_l h^{mn}e^\gamma_m e^\delta_n]\big)-\bar\partial_0[n^\kappa h^{kl}e^\lambda_l h^{mn}e^\gamma_m e^\delta_n]\bar\partial_0\varrho_{\gamma\kappa\delta\lambda}\Big\} \\
&+\rho_k\Big\{\Box_{hyp}(\rho^k)-D_a\big(\varrho_{\gamma\kappa\delta\lambda}D^a[n^\kappa h^{kl}e^\lambda_l h^{mn}e^\gamma_m e^\delta_n]\big)-D_a[n^\kappa h^{kl}e^\lambda_l h^{mn}e^\gamma_m e^\delta_n]D^a\varrho_{\gamma\kappa\delta\lambda}\Big\}
\end{aligned}
\tag{I.3.5}
$$

where we used $n^\mu \varrho_{\mu k} = \rho_k$ and $n^\kappa h^{kl} e^\lambda_l h^{mn} e^\gamma_m e^\delta_n \varrho_{\gamma\kappa\delta\lambda} = n^\kappa h^{kl} \varrho_{\kappa l} = \rho^k$. We produced $\rho_k \Box \rho^k$ plus other terms that are results of the Leibniz rule. We may also note that one can write, $\rho_{ij} \equiv h^{kl}\rho_{ikjl}$, $\rho \equiv h^{ik}h^{il}\rho_{ijkl}$ and $\rho_i \equiv h^{jk}\rho_{jik}$.

## I.4 Generalisation from $\Box$ to $\mathcal{F}(\Box)$

In Eq. (4.5.48) for $\Box^2$, we have,

$$
\begin{aligned}
&\Omega_{ij}\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\Box^2 \varrho^{\mu\nu\rho\sigma} \\
&= \Omega_{ij}\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\left(-(N^{-1}\bar{\partial}_0)^2 + \Box_{hyp}\right)\left(-(N^{-1}\bar{\partial}_0)^2 + \Box_{hyp}\right)\varrho^{\mu\nu\rho\sigma}\,, \\
&= N^{-4}\Omega_{ij}\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\bar{\partial}_0^4 \varrho^{\mu\nu\rho\sigma} \\
&-N^{-2}\Omega_{ij}\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\bar{\partial}_0^2 D_a D^a \varrho^{\mu\nu\rho\sigma} \\
&+\Omega_{ij}\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right] D_a D^a \left[-\left(N^{-1}\bar{\partial}_0\right)^2\right]\varrho^{\mu\nu\rho\sigma} \\
&+\Omega_{ij}\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right] D_a D^a D_b D^a \varrho^{\mu\nu\rho\sigma}\,.
\end{aligned}
\tag{I.4.6}
$$

As a general rule we can write,

$$
\begin{aligned}
&XDDDDY = D\left(XDDDY\right) - D(X)DDD(Y)\,, \\
&= D\left(D(XDD(Y)) - D(X)DD(Y)\right) - D(X)DDD(Y)\,, \\
&= DD(XDD(Y)) - D(D(X)DD(Y)) - D(X)DDD(Y)\,, \\
&= DD\left(D(XD(Y)) - D(X)D(Y)\right) - D(D(X)DD(Y)) - D(X)DDD(Y)\,, \\
&= DDD(XD(Y)) - DD(D(X)D(Y)) - D(D(X)DD(Y)) - D(X)DDD(Y)\,, \\
&= DDD\left(D(XY) - D(X)Y\right) - DD(D(X)D(Y)) - D(D(X)DD(Y)) \\
&-D(X)DDD(Y)\,, \\
&= DDDD(XY) - DDD(D(X)Y) - DD(D(X)D(Y))\,, \\
&-D(D(X)DD(Y)) - D(X)DDD(Y)\,,
\end{aligned}
\tag{I.4.7}
$$

where $X$ and $Y$ are some tensors and $D$ is some operator. Applying this we can write,

$$\begin{aligned}
&N^{-4}\Omega_{ij}\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\bar{\partial}_0^4\varrho^{\mu\nu\rho\sigma}\\
&=N^{-4}\Omega_{ij}\{\bar{\partial}_0^4(\Omega^{ij})-\bar{\partial}_0^4\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\varrho^{\mu\nu\rho\sigma}-\bar{\partial}_0^3\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\bar{\partial}_0\varrho^{\mu\nu\rho\sigma}\\
&-\bar{\partial}_0^2\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\bar{\partial}_0^2\varrho^{\mu\nu\rho\sigma}-\bar{\partial}_0\left[h^i_x e^x_\mu n_\nu h^j_y e^y_\rho n_\sigma\right]\bar{\partial}_0^3\varrho^{\mu\nu\rho\sigma}\}+\cdots\,,
\end{aligned}\tag{I.4.8}$$

where we dropped the irrelevant terms. We moreover can generalise the result of (I.4.7) and write,

$$\begin{aligned}
&XD^{2n}Y=D^{2n}(XY)-D^{2n-1}(D(X)Y)-D^{2n-2}(D(X)D(Y))\,,\\
&-D^{2n-3}(D(X)D^2(Y))-\cdots-D(D(X)D^{2n-2}(Y))-D(X)D^{2n-1}(Y)\,.
\end{aligned}\tag{I.4.9}$$

# Appendix J

# Functional Differentiation

Given the constraint equation

$$2\Psi^{ij} + \frac{\delta f}{\delta \Omega_{ij}} = 0, \tag{J.0.1}$$

suppose that $f = \Omega\mathcal{F}(\Box)\Omega$ and $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, where the coefficients $f_n$ are

massless [1]. Then, using the generalised Euler-Lagrange equations, we have in the

[1] Recall that the $\Box$ term comes with an associated scale $\Box/M^2$.

coframe (and imposing the condition that $\delta\Omega_{ij} = 0$ on the boundary $\partial\mathcal{M}$)

$$\begin{aligned}
\frac{\delta f}{\delta \Omega_{ij}} &= \frac{\partial f}{\partial \Omega_{ij}} - \nabla_\mu \left( \frac{\partial f}{\partial(\nabla_\mu \Omega_{ij})} \right) + \nabla_\mu \nabla_\nu \left( \frac{\partial f}{\partial(\nabla_\mu \nabla_\nu \Omega_{ij})} \right) + \cdots \\
&= \frac{\partial f}{\partial \Omega_{ij}} + \Box \left( \frac{\partial f}{\partial(\Box \Omega_{ij})} \right) + \Box^2 \left( \frac{\partial f}{\partial(\Box^2 \Omega_{ij})} \right) + \cdots \\
&= \frac{\partial f}{\partial \Omega_{ij}} + \sum_{n=1}^{\infty} \Box^n \left( \frac{\partial f}{\partial(\Box^n \Omega_{ij})} \right) \\
&= f_0 \frac{\partial(\Omega^2)}{\partial \Omega_{ij}} + f_1 \frac{\partial(\Omega \Box \Omega)}{\partial \Omega_{ij}} + f_1 \Box \left( \frac{\partial(\Omega \Box \Omega)}{\partial(\Box \Omega_{ij})} \right) + f_2 \Box^2 \left( \frac{\partial \Omega \Box^2 \Omega}{\partial(\Box^2 \Omega_{ij})} \right) + \cdots \\
&= 2 f_0 h^{ij} \Omega + f_1 h^{ij} \Box \Omega + f_1 \Box (h^{ij} \Omega) + \cdots \\
&= 2 f_0 h^{ij} \Omega + f_1 \Big[ h^{ij} \Box \Omega + (\Box \Omega)\, h^{ij} \Big] + \cdots \\
&= 2 f_0 h^{ij} \Omega + 2 f_1 h^{ij} \Box \Omega + \cdots \\
&= 2 h^{ij} \left( f_0 + f_1 \Box + \cdots \right) \Omega \\
&= 2 h^{ij} \mathcal{F}(\Box) \Omega \,, \qquad \text{(J.0.2)}
\end{aligned}$$

where we have used that $\Box g^{ij} = \Box h^{ij} = 0$. Note also that:

$$f_1 \Box \left( \frac{\partial(\Omega \Box \Omega)}{\partial(\Box \Omega_{ij})} \right) = f_1 \Box \Big( \frac{\partial(\Omega \Box [h^{mn} \Omega_{mn}])}{\partial(\Box \Omega_{ij})} \Big) \,. \qquad \text{(J.0.3)}$$

So we can summarise the results and write,

$$\begin{aligned}
\frac{\delta(\Omega \Box \Omega)}{\delta \Omega_{ij}} &= \frac{\partial(\Omega \Box \Omega)}{\partial \Omega_{ij}} + \Box \left( \frac{\partial(\Omega \Box \Omega)}{\partial(\Box \Omega_{ij})} \right) \\
&= h^{ij} \Box \Omega + \Box(h^{ij} \Omega) = \Big[ h^{ij} \Box \Omega + \Box \Omega h^{ij} \Big] = 2 h^{ij} \Box \Omega \,. \qquad \text{(J.0.4)}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta(\Omega_{ij} \Box \Omega^{ij})}{\delta \Omega_{ij}} &= \frac{\partial(\Omega_{ij} \Box \Omega^{ij})}{\partial \Omega_{ij}} + \Box \left( \frac{\partial(\Omega_{ij} \Box \Omega^{ij})}{\partial(\Box \Omega_{ij})} \right) \\
&= \Box \Omega^{ij} + \Box \Omega^{ij} = 2 \Box \Omega^{ij} \,. \qquad \text{(J.0.5)}
\end{aligned}$$

$$\frac{\delta(\rho\Box\Omega)}{\delta\Omega_{ij}} = \Box\left(\frac{\partial(\rho\Box\Omega)}{\partial(\Box\Omega_{ij})}\right) = \Box(\rho h^{ij}) = h^{ij}\Box\rho\,. \tag{J.0.6}$$

$$\frac{\delta(\rho_{ij}\Box\Omega^{ij})}{\delta\Omega_{ij}} = \Box\left(\frac{\partial(\rho_{ij}\Box\Omega^{ij})}{\partial(\Box\Omega_{ij})}\right) = \Box\rho^{ij}\,. \tag{J.0.7}$$

$$\frac{\delta(\Omega\Box\rho)}{\delta\Omega_{ij}} = \frac{\partial(\Omega\Box\rho)}{\partial\Omega_{ij}} = h^{ij}\Box\rho\,. \tag{J.0.8}$$

$$\frac{\delta(\Omega_{ij}\Box\rho^{ij})}{\delta\Omega_{ij}} = \frac{\partial(\Omega_{ij}\Box\rho^{ij})}{\partial\Omega_{ij}} = \Box\rho^{ij}\,. \tag{J.0.9}$$

and generalise this to:

$$\frac{\delta\big(\Omega\mathcal{F}(\Box)\Omega\big)}{\delta\Omega_{ij}} = 2h^{ij}\mathcal{F}(\Box)\Omega, \qquad \frac{\delta\big(\Omega_{ij}\mathcal{F}(\Box)\Omega^{ij}\big)}{\delta\Omega_{ij}} = 2\mathcal{F}(\Box)\Omega^{ij}\,, \tag{J.0.10}$$

$$\frac{\delta\big(\rho\mathcal{F}(\Box)\Omega\big)}{\delta\Omega_{ij}} = h^{ij}\mathcal{F}(\Box)\rho, \qquad \frac{\delta\big(\rho_{ij}\mathcal{F}(\Box)\Omega^{ij}\big)}{\delta\Omega_{ij}} = \mathcal{F}(\Box)\rho^{ij}\,, \tag{J.0.11}$$

$$\frac{\delta\big(\Omega\mathcal{F}(\Box)\rho\big)}{\delta\Omega_{ij}} = h^{ij}\mathcal{F}(\Box)\rho, \qquad \frac{\delta\big(\Omega_{ij}\mathcal{F}(\Box)\rho^{ij}\big)}{\delta\Omega_{ij}} = \mathcal{F}(\Box)\rho^{ij}\,. \tag{J.0.12}$$

# Appendix K

# Riemann tensor components in ADM gravity

Using the method of [145], we can find the Riemann tensor components. The Christoffel symbols for the ADM metric in Eq. (4.3.8) are

$$\begin{aligned}
\Gamma_{ij0} &= \Gamma_{i0j} = -NK_{ij} + D_j\beta_i \\
\Gamma_{ijk} &= {}^{(3)}\Gamma_{ijk} \\
\Gamma^0_{00} &= \frac{1}{N}\left(\dot{N} + \beta^i\partial_i N - \beta^i\beta^j K_{ij}\right) \\
\Gamma^0_{0i} = \Gamma^0_{i0} &= \frac{1}{N}\left(\partial_i N - \beta^j K_{ij}\right) \\
\Gamma^i_{0j} = \Gamma^i_{j0} &= -\frac{\beta^i\partial_j N}{N} - N\left(h^{ik} - \frac{\beta^i\beta^k}{N^2}\right)K_{kj} + D_j\beta^i \\
\Gamma^0_{ij} &= -\frac{1}{N}K_{ij} \\
\Gamma^i_{jk} &= {}^{(3)}\Gamma^i_{jk} + \frac{\beta^i}{N}K_{jk}
\end{aligned} \tag{K.0.1}$$

where $K_{ij}$ is the extrinsic curvature given by (4.3.12) and in the ADM metric, $N$ is the lapse, $\beta_i$ is the shift and $h_{ij}$ is the induced metric on the hypersurface.

Now we can find the Riemann tensor components

$$\begin{aligned}
\mathcal{R}_{ijkl} &= g_{i\rho}\partial_k\Gamma^\rho_{lj} - g_{i\rho}\partial_l\Gamma^\rho_{kj} + \Gamma_{ik\rho}\Gamma^\rho_{lj} - \Gamma_{il\rho}\Gamma^\rho_{kj} \\
&= -\beta_i\partial_k\left(\frac{1}{N}K_{jl}\right) + h_{im}\partial_k\left({}^{(3)}\Gamma^m_{jl} + \frac{\beta^m}{N}K_{jl}\right) - \frac{1}{N}K_{jl}\left(-NK_{ik} + D_k\beta_i\right) \\
&\quad + {}^{(3)}\Gamma_{ikm}\left({}^{(3)}\Gamma^m_{lj} + \frac{\beta^m}{N}K_{lj}\right) - (k \leftrightarrow l) \\
&= R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}
\end{aligned} \tag{K.0.2}$$

where $R_{ijkl}$ is the Riemann tensor of the induced metric on the hypersurface. Then

$$\begin{aligned}
n_\mu\mathcal{R}^\mu{}_{ijk} &= -N\left(\partial_j\Gamma^0_{ki} + \Gamma^0_{j\rho}\Gamma^\rho_{ki}\right) - (j \leftrightarrow k) \\
&= \partial_j K_{ki} + {}^{(3)}\Gamma^m_{ki}K_{jm} - (j \leftrightarrow k) \\
&= D_j K_{ki} - D_k K_{ji}
\end{aligned} \tag{K.0.3}$$

Relabelling the indices, we obtain that

$$n^\mu\mathcal{R}_{ijk\mu} = D_j K_{ki} - D_i K_{jk} \tag{K.0.4}$$

Finally, we have that

$$\begin{aligned}
n_\mu\mathcal{R}^\mu{}_{i0j} &= n_\mu\left(\partial_0\Gamma^\mu_{ji} - \partial_j\Gamma^\mu_{0i} + \Gamma^\mu_{0\rho}\Gamma^\rho_{ji} - \Gamma^\mu_{j\rho}\Gamma^\rho_{0i}\right) \\
&= \dot{K}_{ij} + D_iD_jN + NK_i{}^kK_{kj} - D_j\left(K_{ik}\beta^k\right) - K_{kj}D_i\beta^k
\end{aligned} \tag{K.0.5}$$

Hence

$$\begin{aligned} n^\mu n^\nu \mathcal{R}_{\mu i \nu j} &= n^0 n^\mu \mathcal{R}_{\mu i 0 j} + n^k n^\mu \mathcal{R}_{\mu i k j} \\ &= \frac{1}{N}\left(\dot{K}_{ij} + D_i D_j N + N K_i{}^k K_{kj} - D_j\left(K_{ik}\beta^k\right) - K_{kj} D_i \beta^k\right) \\ &\quad + \frac{\beta^k}{N^2}\left(D_j K_{ki} - D_k K_{ji}\right) \\ &= \frac{1}{N}\left(\dot{K}_{ij} + D_i D_j N + N K_i{}^k K_{kj} - \pounds_\beta K_{ij}\right) \end{aligned} \tag{K.0.6}$$

where $\pounds_\beta K_{ij} \equiv \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{jk} D_i \beta^k$. Therefore overall, we have

$$\mathcal{R}_{ijkl} \equiv K_{ik}K_{jl} - K_{il}K_{jk} + R_{ijkl}\,, \tag{K.0.7}$$

$$\mathcal{R}_{ijk\mathbf{n}} \equiv n^\mu \mathcal{R}_{ijk\mu} = D_j K_{ik} - D_i K_{jk}\,, \tag{K.0.8}$$

$$\mathcal{R}_{i\mathbf{n}j\mathbf{n}} \equiv n^\mu n^\nu \mathcal{R}_{i\mu j\nu} = N^{-1}\left(\partial_t K_{ij} - \pounds_\beta K_{ij}\right) + K_{ik}K_j{}^k + N^{-1} D_i D_j N\,, \tag{K.0.9}$$

# K.1 Coframe

Since in the coframe slicing Eq. (4.3.17) we have $g^{0i} = g_{0i} = 0$, therefore from $n^i = n_i = 0$ . Then the Christoffel symbols become [1]

$$\begin{aligned} \Gamma^0_{00} &= \frac{1}{2} g^{0\mu}\left(\bar{\partial}_0 g_{\mu 0} + \bar{\partial}_0 g_{0\mu} - \partial_\mu g_{00}\right) \\ &= \frac{1}{2} g^{00}\partial_0 g_{00} = \frac{1}{2}\left(-\frac{1}{N^2}\right)\partial_0\left(-N^2\right) \\ &= \frac{\partial_0 N}{N}, \end{aligned} \tag{K.1.10}$$

[1] In the coframe slicing when we write $\partial_\mu$ we mean that $\partial_\mu$ is $\bar{\partial}_0$ when $\mu = 0$ and $\partial_\mu$ is $\partial_i$ when $\mu = i$.

$$\begin{aligned}\Gamma^0_{0i} = \Gamma^0_{i0} &= \frac{1}{2}g^{0\mu}\left(\bar{\partial}_0 g_{\mu i} + \partial_i g_{\mu 0} - \partial_\mu g_{i0}\right) \\ &= \frac{1}{2}g^{00}\partial_i g_{00} = \frac{1}{2}\left(\frac{-1}{N^2}\right)\partial_i\left(-N^2\right) \\ &= \frac{\partial_i N}{N},\end{aligned} \tag{K.1.11}$$

$$\begin{aligned}\Gamma^i_{00} &= \frac{1}{2}g^{i\mu}\left(\bar{\partial}_0 g_{\mu 0} + \bar{\partial}_0 g_{\mu 0} - \partial_\mu g_{00}\right) \\ &= -\frac{1}{2}g^{ij}\partial_j g_{00} = -\frac{1}{2}h^{ij}\partial_j\left(-N^2\right) \\ &= N h^{ij}\partial_j N,\end{aligned} \tag{K.1.12}$$

$$\Gamma^i_{j0} = \frac{1}{2}g^{i\mu}\left(\bar{\partial}_0 g_{\mu j} + \partial_j g_{\mu 0} - \partial_\mu g_{j0}\right) \tag{K.1.13}$$

$$= \frac{1}{2}g^{ik}\left(\bar{\partial}_0 g_{kj}\right)$$

$$= \frac{1}{2}h^{ik}\bar{\partial}_0 h_{jk}, \tag{K.1.14}$$

$$\begin{aligned}\Gamma^0_{ij} &= \frac{1}{2}g^{0\mu}\left(\partial_j g_{\mu i} + \partial_i g_{\mu j} - \partial_\mu g_{ij}\right) \\ &= -\frac{1}{2}g^{00}\bar{\partial}_0 g_{ij} = -\frac{1}{2}\left(\frac{-1}{N^2}\right)\bar{\partial}_0 h_{ij} \\ &= \frac{1}{2}\frac{1}{N^2}\bar{\partial}_0 h_{ij},\end{aligned} \tag{K.1.15}$$

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2}g^{i\mu}\left(\partial_j g_{\mu k} + \partial_k g_{\mu j} - \partial_\mu g_{kj}\right) \\ &= \frac{1}{2}h^{il}\left(\partial_j h_{lk} + \partial_k h_{lj} - \partial_l h_{jk}\right).\end{aligned} \tag{K.1.16}$$

To summarise

$$\begin{aligned}
\Gamma^0_{00} &= \frac{\partial_0 N}{N}, & \Gamma^0_{0i} &= \frac{\partial_i N}{N}, \\
\Gamma^i_{00} &= N h^{ij}\partial_j N, & \Gamma^i_{j0} &= \frac{1}{2}h^{ik}\bar{\partial}_0 h_{jk}, \\
\Gamma^0_{ij} &= \frac{1}{2}\frac{1}{N^2}\bar{\partial}_0 h_{ij}, & \Gamma^i_{jk} &= \frac{1}{2}h^{il}\left(\partial_j h_{lk} + \partial_k h_{lj} - \partial_l h_{jk}\right).
\end{aligned} \tag{K.1.17}$$

Then using Eq. (4.3.21) and Eq. (4.3.22), we can find the $\gamma^\mu_{\nu\rho}$s, the analogues of the Christoffel symbols in the coframe.

$$\begin{aligned}
\gamma^i_{jk} = \Gamma^i_{jk}, \quad \gamma^i_{0k} = -NK^i{}_k, \quad \gamma^i_{j0} = -NK^i{}_k + \partial_j\beta^i, \quad \gamma^0_{ij} = -N^{-1}K_{ij}, \\
\gamma^i_{00} = N\partial^i N, \quad \gamma^0_{0i} = \gamma^0_{i0} = \partial_i \log N, \quad \gamma^0_{00} = \partial_0 \log N
\end{aligned} \tag{K.1.18}$$

Then using the same method as in Eq. (K.0.2),

$$\begin{aligned}
\mathcal{R}_{ijkl} &= g_{i\rho}\partial_k\gamma^\rho_{lj} - g_{i\rho}\partial_l\gamma^\rho_{kj} + \gamma_{ik\rho}\gamma^\rho_{lj} - \gamma_{il\rho}\gamma^\rho_{kj} \\
&= R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}
\end{aligned} \tag{K.1.19}$$

Next

$$\begin{aligned}
\mathcal{R}_{0ijk} &= -N^2\left(\partial_j\gamma^0_{ki} + \gamma^0_{j\rho}\gamma^\rho_{ki}\right) - (j \leftrightarrow k) \\
&= N\left(D_j K_{ki} - D_k K_{ji}\right)
\end{aligned} \tag{K.1.20}$$

Finally we have that in the coframe,

$$\begin{aligned}
\mathcal{R}_{0i0j} &= -N^2\left(\bar{\partial}_0\gamma^0_{ji} - \partial_j\gamma^0_{0i} + \gamma^0_{0\rho}\gamma^\rho_{ji} - \gamma^0_{j\rho}\gamma^\rho_{0i}\right) \\
&= N\left(\bar{\partial}_0 K_{ij} + NK_i{}^k K_{kj} + D_i D_j N\right)
\end{aligned} \tag{K.1.21}$$

Hence the non-vanishing components of the Riemann tensor in the coframe, namely the Gauss, Codazzi and Ricci tensor, become:

$$\begin{aligned}
\mathcal{R}_{ijkl} &= K_{ik}K_{jl} - K_{il}K_{jk} + R_{ijkl}\,, \\
\mathcal{R}_{0ijk} &= N(D_jK_{ki} - D_kK_{ji})\,, \\
\mathcal{R}_{0i0j} &= N(\bar{\partial}_0 K_{ij} + NK_{ik}K_j{}^k + D_iD_jN\,), \qquad \text{(K.1.22)}
\end{aligned}$$

where $K_{ij}$ is the extrinsic curvature of the hypersurface, given in the coframe by Eq. (4.3.30) and $R_{ijkl}$ is the Riemann tensor of the induced metric on the hypersurface.

# Appendix L

# Entropy and functional differentiation

For the scalar curvature which corresponds to the Einstein-Hilbert term we have,

$$\begin{aligned}\frac{\delta R}{\delta R_{\mu\nu\rho\sigma}} &= \frac{\delta(g^{\beta\xi}g^{\alpha\gamma}R_{\alpha\beta\gamma\xi})}{\delta R_{\mu\nu\rho\sigma}} \\ &= g^{\beta\xi}g^{\alpha\gamma}\delta^{[\mu}_{[\alpha}\delta^{\nu]}_{\beta]}\delta^{[\rho}_{[\gamma}\delta^{\sigma]}_{\xi]} \\ &= g^{\beta\xi}g^{\alpha\gamma}\delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta}\delta^{[\rho}_{\gamma}\delta^{\sigma]}_{\xi} \\ &= g^{\rho[\mu}g^{\nu]\sigma}. \end{aligned} \tag{L.0.1}$$

The next term we shall consider is $RF(\Box)R$, to do so we shall use the generalised Euler-Lagrange equation given in (5.2.24),

$$\begin{aligned}\frac{\delta(RF(\Box)R)}{\delta R_{\mu\nu\rho\sigma}} &= f_0\frac{\partial(R^2)}{\partial R_{\mu\nu\rho\sigma}} + f_1\frac{\partial(R\Box R)}{\partial R_{\mu\nu\rho\sigma}} \\ &+ f_1\Box\frac{\partial(R\Box R)}{\partial(\Box R_{\mu\nu\rho\sigma})} + f_2\Box^2\frac{\partial(R\Box^2 R)}{\partial(\Box^2 R_{\mu\nu\rho\sigma})} + \cdots, \end{aligned} \tag{L.0.2}$$

where $\cdots$ are the terms up to infinity. Term by term we have,

$$\begin{aligned} f_0\frac{\partial(R^2)}{\partial R_{\mu\nu\rho\sigma}} &= f_0\frac{\partial(g^{\beta\xi}g^{\alpha\gamma}g^{bd}g^{ac}R_{\alpha\beta\gamma\xi}R_{abcd})}{\partial R_{\mu\nu\rho\sigma}} \\ &= g^{\beta\xi}g^{\alpha\gamma}g^{bd}g^{ac}\delta^{[\mu}_{[\alpha}\delta^{\nu]}_{\beta]}\delta^{[\rho}_{[\gamma}\delta^{\sigma]}_{\xi]}R_{abcd} \\ &+ g^{\beta\xi}g^{\alpha\gamma}g^{bd}g^{ac}\delta^{[\mu}_{[a}\delta^{\nu]}_{b]}\delta^{[\rho}_{[c}\delta^{\sigma]}_{d]}R_{\alpha\beta\gamma\xi} \\ &= 2g^{\rho[\mu}g^{\nu]\sigma}R, \end{aligned} \tag{L.0.3}$$

$$\begin{aligned} f_1\frac{\partial(R\Box R)}{\partial R_{\mu\nu\rho\sigma}} &= f_1\frac{\partial(g^{ac}g^{bd}R_{abcd}\Box R)}{\partial R_{\mu\nu\rho\sigma}} \\ &= f_1 g^{ac}g^{bd}\delta^{[\mu}_{a}\delta^{\nu]}_{b}\delta^{[\rho}_{c}\delta^{\sigma]}_{d}\Box R \\ &= f_1 g^{\rho[\mu}g^{\nu]\sigma}\Box R, \end{aligned} \tag{L.0.4}$$

$$\begin{aligned} f_1\Box\frac{\partial(R\Box R)}{\partial(\Box R_{\mu\nu\rho\sigma})} &= f_1\Box\frac{\partial(g^{ac}g^{bd}R\Box R_{abcd})}{\partial(\Box R_{\mu\nu\rho\sigma})} \\ &= f_1\Box(g^{ac}g^{bd}\delta^{[\mu}_{a}\delta^{\nu]}_{b}\delta^{[\rho}_{c}\delta^{\sigma]}_{d}R) \\ &= f_1 g^{\rho[\mu}g^{\nu]\sigma}\Box R. \end{aligned} \tag{L.0.5}$$

Thus, we can summarise as,

$$\begin{aligned} \frac{\delta(RF(\Box)R)}{\delta R_{\mu\nu\rho\sigma}} &= 2g^{\rho[\mu}g^{\nu]\sigma}R + f_1 g^{\rho[\mu}g^{\nu]\sigma}\Box R + f_1 g^{\rho[\mu}g^{\nu]\sigma}\Box R + \cdots \\ &= 2g^{\rho[\mu}g^{\nu]\sigma}(f_0 + f_1\Box + f_2\Box^2 + \cdots)R = 2g^{\rho[\mu}g^{\nu]\sigma}F(\Box)R. \end{aligned} \tag{L.0.6}$$

In similar manner we shall consider the next term,

$$\begin{aligned} \frac{\delta(R_{\alpha\beta}F(\Box)R^{\alpha\beta})}{\delta R_{\mu\nu\rho\sigma}} &= f_0\frac{\partial(R_{\alpha\beta}R^{\alpha\beta})}{\partial R_{\mu\nu\rho\sigma}} + f_1\frac{\partial(R_{\alpha\beta}\Box R^{\alpha\beta})}{\partial R_{\mu\nu\rho\sigma}} \\ &+ f_1\Box\frac{\partial(R_{\alpha\beta}\Box R^{\alpha\beta})}{\partial(\Box R_{\mu\nu\rho\sigma})} + f_2\Box^2\frac{\partial(R_{\alpha\beta}\Box^2 R^{\alpha\beta})}{\partial(\Box^2 R_{\mu\nu\rho\sigma})} + \cdots, \end{aligned} \tag{L.0.7}$$

again, term by term we have:

$$\begin{aligned} f_0\frac{\partial(R_{\alpha\beta}R^{\alpha\beta})}{\partial R_{\mu\nu\rho\sigma}} &= f_0\frac{\partial(g^{\eta\zeta}g^{\gamma\lambda}g^{\alpha\kappa}g^{\beta\omega}R_{\eta\alpha\zeta\beta}R_{\gamma\kappa\lambda\omega})}{\partial R_{\mu\nu\rho\sigma}} \\ &= f_0 g^{\eta\zeta}g^{\gamma\lambda}g^{\alpha\kappa}g^{\beta\omega}\delta^{[\mu}_{[\eta}\delta^{\nu]}_{\alpha]}\delta^{[\rho}_{[\zeta}\delta^{\sigma]}_{\beta]}R_{\gamma\kappa\lambda\omega} \\ &+ f_0 g^{\eta\zeta}g^{\gamma\lambda}g^{\alpha\kappa}g^{\beta\omega}\delta^{[\mu}_{[\gamma}\delta^{\nu]}_{\kappa]}\delta^{[\rho}_{[\lambda}\delta^{\sigma]}_{\omega]}R_{\eta\alpha\zeta\beta} \\ &= 2f_0 g^{\eta\zeta}g^{\alpha\kappa}g^{\beta\omega}\delta^{[\mu}_{\eta}\delta^{\nu]}_{\alpha}\delta^{[\rho}_{\zeta}\delta^{\sigma]}_{\beta}R_{\kappa\omega} \\ &= 2f_0 g^{\kappa[\nu}g^{\mu][\rho}g^{\sigma]\omega}R_{\kappa\omega}, \end{aligned} \tag{L.0.8}$$

$$\begin{aligned} f_1\frac{\partial(R_{\alpha\beta}\Box R^{\alpha\beta})}{\partial R_{\mu\nu\rho\sigma}} &= f_1\frac{\partial(g^{\eta\zeta}g^{\gamma\lambda}g^{\alpha\kappa}g^{\beta\omega}R_{\eta\alpha\zeta\beta}\Box R_{\gamma\kappa\lambda\omega})}{\partial R_{\mu\nu\rho\sigma}} \\ &= f_1\frac{\partial(g^{\eta\zeta}g^{\alpha\kappa}g^{\beta\omega}R_{\eta\alpha\zeta\beta}\Box R_{\kappa\omega})}{\partial R_{\mu\nu\rho\sigma}} \\ &= f_1 g^{\eta\zeta}g^{\alpha\kappa}g^{\beta\omega}\delta^{[\mu}_{\eta}\delta^{\nu]}_{\alpha}\delta^{[\rho}_{\zeta}\delta^{\sigma]}_{\beta}\Box R_{\kappa\omega} \\ &= f_1 g^{\kappa[\nu}g^{\mu][\rho}g^{\sigma]\omega}\Box R_{\kappa\omega}, \end{aligned} \tag{L.0.9}$$

$$\begin{aligned} f_1\Box\frac{\partial(R_{\alpha\beta}\Box R^{\alpha\beta})}{\partial(\Box R_{\mu\nu\rho\sigma})} &= f_1\Box\frac{\partial(g^{\eta\zeta}g^{\gamma\lambda}g^{\alpha\kappa}g^{\beta\omega}R_{\eta\alpha\zeta\beta}\Box R_{\gamma\kappa\lambda\omega})}{\partial(\Box R_{\mu\nu\rho\sigma})} \\ &= f_1\Box(g^{\gamma\lambda}g^{\alpha\kappa}g^{\beta\omega}\delta^{[\mu}_{\gamma}\delta^{\nu]}_{\kappa}\delta^{[\rho}_{\lambda}\delta^{\sigma]}_{\omega}R_{\alpha\beta}) \\ &= f_1\Box(g^{\alpha[\nu}g^{\mu][\rho}g^{\sigma]\alpha\beta}R_{\alpha\beta}), \end{aligned} \tag{L.0.10}$$

thus:

$$\begin{aligned} \frac{\delta(R_{\alpha\beta}F(\Box)R^{\alpha\beta})}{\delta R_{\mu\nu\rho\sigma}} &= 2f_0 g^{\kappa[\nu}g^{\mu][\rho}g^{\sigma]\omega}R_{\kappa\omega} + 2f_1 g^{\kappa[\nu}g^{\mu][\rho}g^{\sigma]\omega}\Box R_{\kappa\omega} + \cdots \\ &= 2g^{\kappa[\nu}g^{\mu][\rho}g^{\sigma]\omega}(f_0 + f_1\Box + f_2\Box^2 + \cdots)R_{\kappa\omega} \\ &= 2g^{\kappa[\nu}g^{\mu][\rho}g^{\sigma]\omega}F(\Box)R_{\kappa\omega}. \end{aligned} \tag{L.0.11}$$

Finally we can consider the Riemann tensor contribution,

$$\begin{aligned}\frac{\delta(R_{\alpha\beta\gamma\eta}F(\Box)R^{\alpha\beta\gamma\eta})}{\delta R_{\mu\nu\rho\sigma}} &= f_0\frac{\partial(R_{\alpha\beta\gamma\eta}R^{\alpha\beta\gamma\eta})}{\partial R_{\mu\nu\rho\sigma}} + f_1\frac{\partial(R_{\alpha\beta\gamma\eta}\Box R^{\alpha\beta\gamma\eta})}{\partial R_{\mu\nu\rho\sigma}} \\ &+ f_1\Box\frac{\partial(R_{\alpha\beta\gamma\eta}\Box R^{\alpha\beta\gamma\eta})}{\partial(\Box R_{\mu\nu\rho\sigma})} + f_2\Box^2\frac{\partial(R_{\alpha\beta\gamma\eta}\Box^2 R^{\alpha\beta\gamma\eta})}{\partial(\Box^2 R_{\mu\nu\rho\sigma})} + \cdots,\end{aligned} \tag{L.0.12}$$

as before, we consider the leading order terms and then generalise the results:

$$\begin{aligned}f_0\frac{\partial(R_{\alpha\beta\gamma\eta}R^{\alpha\beta\gamma\eta})}{\partial R_{\mu\nu\rho\sigma}} &= f_0\frac{\partial(g^{\alpha\xi}g^{\beta\lambda}g^{\gamma\kappa}g^{\eta\omega}R_{\alpha\beta\gamma\eta}R_{\xi\lambda\kappa\omega})}{\partial R_{\mu\nu\rho\sigma}} \\ &= f_0 g^{\alpha\xi}g^{\beta\lambda}g^{\gamma\kappa}g^{\eta\omega}\delta^{[\mu}_{[\alpha}\delta^{\nu]}_{\beta]}\delta^{[\rho}_{[\gamma}\delta^{\sigma]}_{\eta]}R_{\xi\lambda\kappa\omega} \\ &+ f_0 g^{\alpha\xi}g^{\beta\lambda}g^{\gamma\kappa}g^{\eta\omega}\delta^{[\mu}_{[\xi}\delta^{\nu]}_{\lambda]}\delta^{[\rho}_{[\kappa}\delta^{\sigma]}_{\omega]}R_{\alpha\beta\gamma\eta} \\ &= 2f_0\delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta}\delta^{[\rho}_{\gamma}\delta^{\sigma]}_{\eta}R^{\alpha\beta\gamma\eta} = 2f_0R^{\mu\nu\rho\sigma},\end{aligned} \tag{L.0.13}$$

$$f_1\frac{\partial(R_{\alpha\beta\gamma\eta}\Box R^{\alpha\beta\gamma\eta})}{\partial R_{\mu\nu\rho\sigma}} = f_1\delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta}\delta^{[\rho}_{\gamma}\delta^{\sigma]}_{\eta}\Box R^{\alpha\beta\gamma\eta} = f_1\Box R^{\mu\nu\rho\sigma}, \tag{L.0.14}$$

$$\begin{aligned}f_1\Box\frac{\partial(R_{\alpha\beta\gamma\eta}\Box R^{\alpha\beta\gamma\eta})}{\partial(\Box R_{\mu\nu\rho\sigma})} &= f_1\Box\frac{\partial(g^{\alpha\xi}g^{\beta\lambda}g^{\gamma\kappa}g^{\eta\omega}R_{\alpha\beta\gamma\eta}\Box R_{\xi\lambda\kappa\omega})}{\partial(\Box R_{\mu\nu\rho\sigma})} \\ &= f_1\Box\frac{\partial(R^{\xi\lambda\kappa\omega}\Box R_{\xi\lambda\kappa\omega})}{\partial(\Box R_{\mu\nu\rho\sigma})} \\ &= f_1\Box(\delta^{[\mu}_{\xi}\delta^{\nu]}_{\lambda}\delta^{[\rho}_{\kappa}\delta^{\sigma]}_{\omega}R^{\xi\lambda\kappa\omega}) = f_1\Box R^{\mu\nu\rho\sigma}.\end{aligned} \tag{L.0.15}$$

We can conclude that,

$$\begin{aligned}\frac{\delta(R_{\alpha\beta\gamma\eta}F(\Box)R^{\alpha\beta\gamma\eta})}{\delta R_{\mu\nu\rho\sigma}} &= 2f_0R^{\mu\nu\rho\sigma} + f_1\Box R^{\mu\nu\rho\sigma} + f_1\Box R^{\mu\nu\rho\sigma} + \cdots \\ &= 2(f_0 + f_1\Box + f_2\Box^2 + \cdots)R^{\mu\nu\rho\sigma} = 2F(\Box)R^{\mu\nu\rho\sigma}.\end{aligned} \tag{L.0.16}$$

# Appendix M

# Conserved current for Einstein-Hilbert gravity

Given the EH action to be of the form,

$$S_{EH} = \frac{M_P^2}{2}\int d^4x\sqrt{-g}R. \tag{M.0.1}$$

we can imply the variation principle infinitesimally by writing,

$$\begin{aligned}\delta_\xi S_{EH} = \frac{M_P^2}{2}\int d^4x\delta_\xi(\sqrt{g}R) &= \frac{M_P^2}{2}\int d^4x\sqrt{g}\Big(G_{\mu\nu}\delta_\xi g^{\mu\nu} + g^{\mu\nu}\delta_\xi(R_{\mu\nu})\Big)\\ &= \frac{M_P^2}{2}\int d^4x\sqrt{g}\nabla_\alpha(\xi^\alpha R) = 0,\end{aligned} \tag{M.0.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and given by $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. The term involving the Einstein tensor can be expanded further as,

$$G_{\mu\nu}\delta_\xi g^{\mu\nu} = G_{\mu\nu}(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu) = 2G_{\mu\nu}\nabla^\mu\xi^\nu = \nabla_\mu(-2R^\mu_\nu + \delta^\mu_\nu R)\xi^\nu, \tag{M.0.3}$$

where we used Eq. (5.3.89) and performed integration by parts. Then we move on to the next term and expand it as,

$$g^{\mu\nu}\delta_\xi R_{\mu\nu} = (\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\delta_\xi g_{\mu\nu} = \nabla_\lambda\Big((g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\Big), \tag{M.0.4}$$

by substituting Eq's. (M.0.3) and (M.0.4) into (M.0.2) we obtain,

$$\delta_\xi S_{EH} = \frac{M_P^2}{2}\int d^4x\sqrt{-g}\nabla_\mu\Big( - 2R^\mu_\nu\xi^\nu + (g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_\nu(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\Big) = 0, \tag{M.0.5}$$

and hence for any vector field $\xi^\mu$ one obtains the conserved Nöether current,

$$J^\mu(\xi) = R^\mu_\nu\xi^\nu + \frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_\nu(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha) \equiv \nabla_\nu(\nabla^{[\mu}\xi^{\nu]}). \tag{M.0.6}$$

# Appendix N

# Generalised Komar current

It can be shown that the Nöether current that was obtained in Eq. (M.0.6) is identical to generalised Komar current via

$$\begin{aligned} J^{\mu}(\xi) &= \frac{1}{2}\nabla_{\nu}(\nabla^{\mu}\xi^{\nu} - \nabla^{\nu}\xi^{\mu}) = \nabla_{\nu}\nabla^{\mu}\xi^{\nu} - \frac{1}{2}\nabla_{\nu}(\nabla^{\nu}\xi^{\mu} + \nabla^{\mu}\xi^{\nu}) \\ &= [\nabla^{\nu}, \nabla^{\mu}]\xi_{\nu} + \nabla^{\mu}(\nabla_{\nu}\xi^{\nu}) - \frac{1}{2}\nabla_{\nu}(\nabla^{\nu}\xi^{\mu} + \nabla^{\mu}\xi^{\nu}) \\ &= R^{\mu}_{\nu}\xi^{\nu} + \frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_{\nu}(\nabla_{\alpha}\xi_{\beta} + \nabla_{\nu}\xi_{\beta}), \end{aligned} \tag{N.0.1}$$

where we used: $[\nabla^{\nu}, \nabla^{\mu}]\xi_{\nu} = R^{\mu}_{\lambda\mu\nu}\xi^{\lambda} = R_{\lambda\nu}\xi^{\lambda}$.

# Appendix O

# Komar integrals in Boyer-Linquist coordinate

In Boyer-Linquist coordinate the kerr metric is given by,

$$g_{\mu\nu} = \begin{pmatrix} \frac{2Mr}{r^2+a^2\cos^2(\theta)} - 1 & 0 & 0 & -\frac{2aMr\sin^2(\theta)}{r^2+a^2\cos^2(\theta)} \\ 0 & \frac{r^2+a^2\cos^2(\theta)}{a^2+r^2-2Mr} & 0 & 0 \\ 0 & 0 & r^2+a^2\cos^2(\theta) & 0 \\ -\frac{2aMr\sin^2(\theta)}{r^2+a^2\cos^2(\theta)} & 0 & 0 & \frac{\sin^2(\theta)\left(\left(a^2+r^2\right)^2-a^2\left(a^2+r^2-2Mr\right)\sin^2(\theta)\right)}{r^2+a^2\cos^2(\theta)} \end{pmatrix} \tag{O.0.1}$$

Given,

$$n^1 = (1,0,0,0), \qquad n^2 = (0,1,0,0). \tag{O.0.2}$$

the only surviving components of normal vectors would be,

$$n^1_{[\alpha}n^2_{\beta]} = n^1_{[t}n^2_{r]} = \frac{1}{2}(n^1_t n^2_r - n^1_r n^2_t) = \frac{1}{2}, \tag{O.0.3}$$

$$n^1_{[\alpha}n^2_{\beta]} = n^1_{[r}n^2_{t]} = \frac{1}{2}(n^1_r n^2_t - n^1_t n^2_r) = -\frac{1}{2}. \tag{O.0.4}$$

We now want to calculate the Komar integrals:

$$M = -\frac{1}{8\pi}\oint_{\mathcal{H}} \nabla^\alpha t^\beta ds_{\alpha\beta}, \tag{O.0.5}$$

lets us take:

$$\begin{aligned}
\nabla^\alpha t^\beta ds_{\alpha\beta} &= \sqrt{-g}\nabla^\alpha t^\beta n^1_{[\alpha} n^2_{\beta]} d\theta d\phi \\
&= \sqrt{-g}(g^{\alpha\lambda}\nabla_\lambda t^\beta) n^1_{[\alpha} n^2_{\beta]} d\theta d\phi \\
&= \sqrt{-g} g^{\alpha\lambda}(\partial_\lambda t^\beta + \Gamma^\beta_{\lambda\rho} t^\rho) n^1_{[\alpha} n^2_{\beta]} d\theta d\phi \\
&= \sqrt{-g} g^{\alpha\lambda}\Gamma^\beta_{\lambda\rho} t^\rho n^1_{[\alpha} n^2_{\beta]} d\theta d\phi \\
&= \sqrt{-g}\Big(g^{t\lambda}\Gamma^r_{\lambda t} n^1_{[t} n^2_{r]} + g^{r\lambda}\Gamma^t_{\lambda t} n^1_{[r} n^2_{t]}\Big) d\theta d\phi \\
&= \frac{1}{2}\sqrt{-g}\Big(g^{t\lambda}\Gamma^r_{\lambda t} - g^{r\lambda}\Gamma^t_{\lambda t}\Big) d\theta d\phi \\
&= \frac{1}{2}\sqrt{-g}\Big(g^{tt}\Gamma^r_{tt} + g^{t\phi}\Gamma^r_{\phi t} - g^{rr}\Gamma^t_{rt}\Big) d\theta d\phi.
\end{aligned} \tag{O.0.6}$$

We have:

$$g^{tt}\Gamma^r_{tt} + g^{t\phi}\Gamma^r_{\phi t} - g^{rr}\Gamma^t_{rt} = \frac{8m\left(a^2+r^2\right)\left(a^2\cos(2\theta)+a^2-2r^2\right)}{\left(a^2\cos(2\theta)+a^2+2r^2\right)^3}, \tag{O.0.7}$$

$$\sqrt{-g} = \frac{1}{2}\sin(\theta)\left(a^2\cos(2\theta)+a^2+2r^2\right). \tag{O.0.8}$$

Thus,

$$\begin{aligned}
M &= -\frac{1}{8\pi}\oint_{\mathcal{H}} \nabla^\alpha t^\beta ds_{\alpha\beta} \\
&= -\frac{1}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \\
&\quad \left(\frac{1}{2}\sin(\theta)\left(a^2\cos(2\theta)+a^2+2r^2\right)\frac{8m\left(a^2+r^2\right)\left(a^2\cos(2\theta)+a^2-2r^2\right)}{\left(a^2\cos(2\theta)+a^2+2r^2\right)^3}\right) = m.
\end{aligned} \tag{O.0.9}$$

Now let us look at the angular momentum:

$$J = \frac{1}{16\pi}\oint_{\mathcal{H}} \nabla^\alpha \phi^\beta ds_{\alpha\beta}, \tag{O.0.10}$$

$$\begin{aligned}
\nabla^\alpha \phi^\beta ds_{\alpha\beta} &= \sqrt{-g}\nabla^\alpha \phi^\beta n^1_{[\alpha} n^2_{\beta]} d\theta d\phi \\
&= \sqrt{-g}(g^{\alpha\lambda}\nabla_\lambda \phi^\beta) n^1_{[\alpha} n^2_{\beta]} d\theta d\phi \\
&= \sqrt{-g} g^{\alpha\lambda}(\partial_\lambda \phi^\beta + \Gamma^\beta_{\lambda\rho}\phi^\rho) n^1_{[\alpha} n^2_{\beta]} d\theta d\phi \\
&= \sqrt{-g} g^{\alpha\lambda}\Gamma^\beta_{\lambda\rho}\phi^\rho n^1_{[\alpha} n^2_{\beta]} d\theta d\phi \\
&= \sqrt{-g}\Big(g^{t\lambda}\Gamma^r_{\lambda\phi} n^1_{[t} n^2_{r]} + g^{r\lambda}\Gamma^t_{\lambda\phi} n^1_{[r} n^2_{t]}\Big) d\theta d\phi \\
&= \frac{1}{2}\sqrt{-g}\Big(g^{t\lambda}\Gamma^r_{\lambda\phi} - g^{r\lambda}\Gamma^t_{\lambda\phi}\Big) d\theta d\phi \\
&= \frac{1}{2}\sqrt{-g}\Big(g^{tt}\Gamma^r_{t\phi} + g^{t\phi}\Gamma^r_{\phi\phi} - g^{rr}\Gamma^t_{r\phi}\Big) d\theta d\phi,
\end{aligned} \tag{O.0.11}$$

we have:

$$\begin{aligned}
& g^{tt}\Gamma^r_{t\phi} + g^{t\phi}\Gamma^r_{\phi\phi} - g^{rr}\Gamma^t_{r\phi} \\
= \; & -\frac{8am\sin^2(\theta)\left(a^4 - 3a^2r^2 + a^2(a-r)(a+r)\cos(2\theta) - 6r^4\right)}{\left(a^2\cos(2\theta) + a^2 + 2r^2\right)^3}.
\end{aligned} \tag{O.0.12}$$

Hence,

$$\begin{aligned}
J &= \frac{1}{16\pi}\oint_{\mathcal{H}} \nabla^\alpha \phi^\beta ds_{\alpha\beta} \\
&= \frac{1}{16\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \\
&\quad \left(\frac{1}{2}\sin(\theta)\left(a^2\cos(2\theta) + a^2 + 2r^2\right)\right. \\
&\times \left.\frac{-8am\sin^2(\theta)\left(a^4 - 3a^2r^2 + a^2(a-r)(a+r)\cos(2\theta) - 6r^4\right)}{\left(a^2\cos(2\theta) + a^2 + 2r^2\right)^3}\right) = ma.
\end{aligned} \tag{O.0.13}$$

Note: $\xi^\alpha = t^\alpha + \Omega_H \phi^\alpha$.

# Appendix P

# $f(R)$ gravity conserved current

Variation of the $f(R)$ action given in (5.3.110) would follow as,

$$\begin{aligned}
\delta_\xi I &= \int d^4x \delta_\xi \Big(\sqrt{-g} f(R)\Big) \\
&= \int d^4x \delta_\xi \Big(\delta_\xi(\sqrt{-g}) f(R) + \sqrt{-g}\delta_\xi(f(R))\Big) \\
&= \int d^4x \sqrt{-g}\Big([f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R)]\delta_\xi g^{\mu\nu} + f'(R)g^{\mu\nu}\delta_\xi(R_{\mu\nu})\Big) \\
&= \int d^4x \sqrt{-g}\nabla_\alpha\Big(\xi^\alpha f(R)\Big).
\end{aligned} \tag{P.0.1}$$

Now let $\mathcal{G}_{\mu\nu} = f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R)$, then:

$$\begin{aligned}
\mathcal{G}_{\mu\nu}\delta_\xi g^{\mu\nu} &= \mathcal{G}_{\mu\nu}\pounds_\xi g^{\mu\nu} = \mathcal{G}_{\mu\nu}(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu) = 2\mathcal{G}_{\mu\nu}\nabla^\mu\xi^\nu \\
&= -2\nabla_\mu\mathcal{G}^{\mu\nu}\xi_\nu = -2\nabla_\mu\mathcal{G}^\mu_\nu\xi^\nu = \nabla_\mu(-2f'(R)R^\mu_\nu + \delta^\mu_\nu f(R)R)\xi^\nu.
\end{aligned} \tag{P.0.2}$$

And,

$$\begin{aligned} f'(R)g^{\mu\nu}\delta_\xi R_{\mu\nu} &= f'(R)(\nabla_\lambda(g^{\mu\nu}\delta_\xi\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta_\xi\Gamma^\lambda_{\lambda\mu})) \\ &= f'(R)\nabla_\lambda\big(g^{\mu\nu}\delta_\xi\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta_\xi\Gamma^\nu_{\nu\mu}\big) \\ &= f'(R)(\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\delta_\xi g_{\mu\nu} \\ &= f'(R)(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_\mu\nabla_\nu\delta_\xi g_{\alpha\beta} \\ &= f'(R)\nabla_\lambda\Big((g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu\delta_\xi g_{\alpha\beta}\Big) \\ &= f'(R)\nabla_\lambda\Big((g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\Big). \end{aligned} \tag{P.0.3}$$

Thus,

$$\begin{aligned} \delta_\xi I &= \int d^4x\sqrt{g}\Big(\mathcal{G}_{\mu\nu}\delta_\xi g^{\mu\nu} + g^{\mu\nu}\delta_\xi(R_{\mu\nu})\Big) \\ &= \int d^4x\sqrt{g}\Big(\nabla_\mu(-2f'(R)R^\mu_\nu\xi^\nu + \delta^\mu_\nu f(R)R\xi^\nu) \\ &+ f'(R)\nabla_\mu\Big((g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_\nu(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\Big)\Big) \\ &= \int d^4x\sqrt{g}\nabla_\alpha(\xi^\alpha f(R)) = 0. \end{aligned} \tag{P.0.4}$$

Which reduces to

$$\delta_\xi I_{EH} = \int d^4x\sqrt{g}\nabla_\mu\Big(-2f'(R)R^\mu_\nu\xi^\nu + f'(R)(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_\nu(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\Big) = 0. \tag{P.0.5}$$

Thus the conserved Noether current is:

$$J^\mu(\xi) = f'(R)R^\mu_\nu\xi^\nu + \frac{1}{2}f'(R)(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_\nu(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha) = f'(R)\nabla_\nu(\nabla^{[\mu}\xi^{\nu]}). \tag{P.0.6}$$

Moreover,

$$\begin{aligned} J^{\mu}(\xi) &= \frac{1}{2}f'(R)\nabla_{\nu}(\nabla^{\mu}\xi^{\nu} - \nabla^{\nu}\xi^{\mu}) = f'(R)\nabla_{\nu}\nabla^{\mu}\xi^{\nu} - \frac{1}{2}f'(R)\nabla_{\nu}(\nabla^{\nu}\xi^{\mu} + \nabla^{\mu}\xi^{\nu}) \\ &= f'(R)[\nabla^{\nu}, \nabla^{\mu}]\xi_{\nu} + f'(R)\nabla^{\mu}(\nabla_{\nu}\xi^{\nu}) - \frac{1}{2}f'(R)\nabla_{\nu}(\nabla^{\nu}\xi^{\mu} + \nabla^{\mu}\xi^{\nu}) \\ &= f'(R)R^{\mu}_{\nu}\xi^{\nu} + \frac{1}{2}f'(R)(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_{\nu}(\nabla_{\alpha}\xi_{\beta} + \nabla_{\nu}\xi_{\beta}). \end{aligned} \tag{P.0.7}$$

Now the Komar Integrals modified by,

$$M = -\frac{f'(R)}{8\pi}\oint_{\mathcal{H}} \nabla^{\alpha}t^{\alpha}ds_{\alpha\beta} = f'(R)m, \tag{P.0.8}$$

$$J = \frac{f'(R)}{16\pi}\oint_{\mathcal{H}} \nabla^{\alpha}\phi^{\alpha}ds_{\alpha\beta} = f'(R)ma. \tag{P.0.9}$$

As appeared in (5.3.113) and (5.3.114).

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