Electoral competition with primaries and quality asymmetries

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Abstract

We introduce primaries –both closed and open– into a Downsian model of two-party electoral competition allowing the two candidates in each party's primary to differ in valence as well as in policy platform. The good news is that the introduction of either type of primaries acts as a stabilizing force since equilibria exist quite generally, serves as an arena for policy debates since all candidates propose differentiated platforms, and guarantees that each party's nominee is of higher quality than its primary opponent. Moreover, primaries tend to benefit the party whose median voter is closer to the overall median. The bad news is that the winner of the general election need not be the candidate with the highest overall quality since too competitive primaries can prove harmful. Given the differences between open and closed primaries, we show that the choice of primary type is particularly important and may determine the winner of the general election.

Keywords: Downsian model; Primaries; Valence; Open Primaries; Closed Primaries

Short title: Primaries with quality asymmetries

Supplementary material for this article is available in the appendix in the online edition.

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1 Introduction

Party primaries have become an increasingly common method of nominating candidates for a general election. In the US, since just after WWII, primaries are by and large conducted in the same manner as a general election and run by the same electoral authorities. In Europe and Latin America, primary elections are a more recent phenomenon, and primaries are generally run by the parties themselves. Several questions of interest naturally arise. For example, how does the introduction of primaries influence candidates' policy proposals? How does the introduction of primaries affect the outcome of the general election? If we allow candidates to vary both in policy positions and on a valence or quality dimension, how does the differentiation of candidates in the party primary on these two dimensions affect the party's performance in the general election? Also, how does the choice between a closed and an open primary type matter?

We answer such questions in the framework of a standard Downsian model of electoral competition by considering plurality two-party competition in a two-stage election (primary and general) where candidates choose and commit to a given platform prior to the primary election.[1] We first assume that both parties run closed primaries with two candidates competing in each primary. At the time of closed primaries, parties are treated as exogenous and differ in the preferences of their primary electorates. Candidates may differ both in policy platforms and in terms of a commonly valued and commonly known non-policy characteristic, which we may label a valence or quality dimension. By introducing this quality heterogeneity among candidates, voters who

[1] Standard assumptions of a Downsian model are office-motivated candidates, full commitment, and sincere voting among others. See Grofman (2004); Osborne (1993) for a list of assumptions that are generally perceived to best justify the description of a model as a Downsian one.

participate in the primaries and in the general election sincerely vote not only on the basis of policy proposals, but also on the basis of which candidate is considered to be "better" (for example, in terms of charisma, corruption allegations and experience).[2] Regarding candidates' behavior, we present all our formal results by positing that all candidates aim at maximizing their general election vote share in the absence of any kind of uncertainty.[3]

In our setup, we characterize the unique equilibrium of the game with several interesting properties arising. In equilibrium, each party's low valence candidate proposes a platform coinciding with the ideal policy of her party's median. Each party's high valence candidate is relatively more moderate than the low valence primary candidate and, targeting at the best electoral outcome in the general election, locates the closest possible to the society's median. The valence asymmetry between the two primary candidates ultimately determines how differentiated the two platforms are. The larger the advantage of one candidate is, the more she is capable of moving towards moderate policies and hence becoming more appealing in the general election while guaranteeing a primary victory. As far as valence is concerned, since high valence candidates always win their primary, our result is in line with recent empirical evidence showing that primaries tend to be effective at selecting high quality types (Hirano and Snyder, 2014). In terms of policy proposals, primaries serve as the arena for meaningful intra-party policy debates since primary candidates propose differentiated platforms. Nevertheless, the

[2] While in the general election there are no incentives for strategic voting, this is not the case in the primary. We discuss the effect of introducing a small share of strategic voters in the online appendix, and we argue that the main qualitative features of our equilibrium analysis are robust to such extension.

[3] Our results however are to some extent compatible with the alternative interpretation of candidates having some uncertainty regarding the location of the median voter in the general election when choosing their primary platforms so as to maximize the probability of winning the general election.

intensity of such debates is crucial in determining the winner of the general election and primaries can prove harmful to the party with the highest valence candidate. Our results show that the low valence general election candidate may win if she faced a weak party primary opponent, while the highest valence candidate could not propose a moderate enough platform because of tight primaries inside the losing party. Hence, our divergent equilibrium result shows that primaries may prove harmful to a party if they create too much competition during the nomination process. This *within-party competition effect* of primaries on electoral outcomes, is the first substantial result of our analysis and relates to the negative aspect of the divisive effect of primaries (Key, 1953; Agranov, 2016).

Second, our analysis suggests that primaries have a *matching effect*: candidates nominated by leftist (rightist) parties win more often when the society's median is leftist (rightist). This is a result of primaries making candidates more responsive to the policy preferences of their primary electorate rather than the general one. If for example the society becomes more leftist, the leftist party will win more often than before, since the high valence primary candidate of the rightist party cannot react to the median's shift. If the rightist high valence candidate were to propose leftist policies that would please the new median, this would potentially make her lose the primary. Notice that this otherwise intuitive feature of our equilibrium is, surprisingly, absent from most electoral competition models without primaries: models with office-motivated candidates usually generate equilibria in which candidates converge (either in deterministic or in probabilistic terms) and models with policy-motivated candidates often predict that candidates will locate equidistantly away from the society's median -and will hence tie- independently of whether the society's median is leftist or rightist (see, for example, Osborne and Slivinski 1996; Besley and Coate 1997; Ortuño-Ortín 1997; Saporiti 2014; Matakos et al. 2016). Overall, the matching effect may further strengthen the stability of the party system as it allows parties to form a more durable ideological framework. However, the asymmetry of party competition, with the party whose median voter is closer to the overall median being advantaged, allows us to recognize an important stylized fact about much political competition, namely that there may be (extended) periods during which one party is dominant (Merrill et al., 2008).

Third, the introduction of primaries in this intuitive setup extends our knowledge on equilibrium existence in modifications of the standard Downsian model. While without primaries a pure strategy Nash equilibrium in a similar two-party setup does not exist (Aragonès and Palfrey, 2002),[4] we show that with closed primaries *a unique equilibrium in pure strategies always exists*, and this is true for any distribution of voters' preferences. That is, while the standard Downsian model of electoral competition with valence asymmetries predicts that stability may (Ansolabehere and Snyder, 2000) or may not be reached (Aragonès and Palfrey, 2002), the introduction of primaries provides a clear *stabilizing effect*.[5]

[4]To be precise, Aragonès and Palfrey (2002) consider that the two heterogeneous candidates are win-motivated and hold imperfect information regarding voters' policy preferences. As it is argued in Aragonès and Xefteris (2017a), in these models, win-motivation with imperfect information about voters' preferences is technically *equivalent* to vote-share maximization and perfect information about voters' preferences. Equilibrium existence with win-motivation and perfect information about voters preferences equilibrium existence is rarely an issue (see, for instance, Ansolabehere and Snyder 2000).

[5]The search for stabilizing forces in multidimensional competition models has attracted previous attention and several proposals. Among others, Lin et al. (1999) consider probabilistic voting, Krasa and Polborn (2012); Dziubiński and Roy (2011); Aragonès and Xefteris (2017b) allow for differentiated candidates and Bräuninger (2007) for costly voting.

To the best of our knowledge, this is the first paper to point at these three effects of primaries simultaneously. The second and the third effect are obviously positive ones: primaries stabilize the electoral process and generate consistency between the party of the elected candidate and voters' policy preferences, and these in turn promote a sense of trust in the political system. The first effect, has, arguably, negative implications: a high valence candidate that faces hard within-party competition might end up losing the general election to a mediocre candidate that won in her party's primaries against a low quality opponent.

We propose several modifications to our analysis and find that our results are robust in a number of directions. First, we focus on open primaries. Once all candidates make their policy proposals, active citizens who are willing to participate in the procedure vote in the primary of the party in which their top-ranked candidate participates. This implies that by proposing moderate platforms, candidates not only increase their general election vote share, but also increase the amount of active voters participating in their party's primary. Hence, both parties' size as well as the ideal policy of the parties' medians are now endogenously determined and depend on the quality characteristics of all candidates. That is, in contrast to closed primaries where candidates gain nomination by focusing only on their party's median voter and the quality characteristics of their party's candidates, in open primaries attention is also paid to the quality characteristics of the other party given the endogenous sorting of voters across primaries. Interestingly, and despite the endogenous party formation described, the *stabilizing effect* of primaries prevails since for a large class of voters' distributions we still obtain a unique equilibrium in pure strategies such that: a) in some instances, the highest valence candidate does not emerge as the winner of the general election (*within-party competition effect of primaries*) and b) the winner of the leftist (rightist) primary is more likely to be the general election's winner when the society's median is leftist (rightist) (*matching effect of primaries*).

Finally, we investigate situations where a primary is held only in one of the two parties (either closed or open). This is of interest since incumbents often run for reelection without going through a nomination process and face a challenger who emerged from a primary. In our model, we assume that the position of the incumbent is fixed and that primary candidates in the opposition strategically choose their platforms (typically the incumbent has less flexibility than the challenger in credibly promising something different to the implemented policies). We find that when the incumbent implements socially detrimental (appealing) policies, the highest valence challenger is elected less (more) often under closed primaries than under open primaries. The reason why bad incumbents are less threatened by challengers that emerge from closed primaries than from open primaries, is that closed primaries hold candidates close to their party's median, who might be quite far from the society's one. Open primaries pose no such restriction and allow candidates to expand their primary electorate by moving towards the center. In other words, open primaries give incentives to high valence candidates of initially less moderate parties to move towards the center and, thus, to: a) win more often and, perhaps more importantly, b) make their parties more moderate by moving towards the centre and hence attracting new moderate voters for their primaries. These results point to an interesting effect of the organization of the party in opposition on the incumbent's decisions when the latter cares about reelection: incumbents have stronger incentives to implement moderate policies when the challenger's party holds open primaries than when it holds closed primaries.

Our paper complements the existing literature on primaries with valence asymmetries by adding several insightful new results.[6] Our work closely relates to Adams and Merrill (2008) since, to the best of our knowledge, this is the only other setup where all four primary candidates may differ in valence. Nevertheless, while Adams and Merrill (2008) focus on a probabilistic voting model we focus on a deterministic one.[7] As is well known, probabilistic and deterministic voting models with valence asymmetries deliver very diverse predictions on candidates' equilibrium behavior and as we show this is also true in the context of primaries.[8] The presence of a random element eventually leads to primary candidates proposing identical platforms, while the point of convergence might differ across parties (Adams and Merrill, 2008). On the contrary, we show that primary candidates run on different platforms giving back to primaries the element of an internal battlefield.

Hummel (2013) also employs a non probabilistic valence model, but, unlike us, he considers that: a) the

[6]Research on primaries without valence issues was presented among others in Owen and Grofman (2006); Meirowitz (2005); Coleman (1971); Aranson and Ordeshook (1972). For papers interested in the non commitment of primary winners and flip flopping between primary and general elections see Hummel (2010); Agranov (2016). For the effect of sequential primaries on electoral competition see Callander (2007); Deltas et al. (2016). For work on different ways of candidates' nomination including primaries see among others Crutzen et al. (2010); Jackson et al. (2007); Winer et al. (2014); Hortala-Vallve and Mueller (2015); Kselman (2015); Amorós et al. (2016); Buisseret and Van Weelden (2017).

[7]In a probabilistic voting model one votes for a certain candidate with a probability that is increasing in the utility that one derives from the election of this candidate. That is, one need not vote for the top-ranked candidate, although this is one's most probable action. In a deterministic voting model one always votes for the candidate offering the highest utility.

[8]For example, in standard two party competition models, while probabilistic voting models do not rule out convergent equilibria when valence asymmetries are not very large (see, for example, Schofield 2007), this never occurs in deterministic voting models (see, for example, Aragonès and Palfrey 2002).

higher (lower) valence candidate of the leftist party has precisely the same valence with the higher (lower) valence of the rightist party and b) voters may strategically decide not to support their top-ranked candidates. Similar to Hummel (2013) we show that high valence candidates propose more moderate policies than low valence ones. Nevertheless, by allowing all four candidates to differ in valence, we provide new results on the effect of primaries on the winner of the general election and demonstrate why and when the highest valence candidate may not win the general election. Takayama (2014) uses similar assumptions to those of Hummel (2013) and models primaries only in the challenger's party with overall three candidates of different valence. She shows that as the incumbent's valence increases, the qualifying challenger becomes more moderate. Our results under the presence of an incumbent are different and depend on the primary type. We show that if the party runs closed primaries, the policy proposed by the high valence challenger is not affected by the incumbent's characteristics. On the contrary, when the party organizes an open primary, the challenger becomes more moderate as the incumbent's valence decreases.

In Kartik and McAfee (2007), as in Hummel (2013), there are two levels of valence but different to the aforementioned papers high valence types are committed to an exogenous platform. In Serra (2011); Snyder and Ting (2011) both primaries and the general election function as a valence revelation mechanism and their focus is more on the adoption or not of primaries rather than on primary candidates' platforms proposals. In Andreottola (2016) only primaries serve as a valence revelation mechanism and in contrast to us his results show that the high valence primary candidate proposes more extreme platforms than the low valence candidate. For models of primary elections with endogenous valence see Serra (2010); Casas (2013).

The remainder is structured as follows: In Section 2 we present the model, in Section 3 we present our results for closed, open and one party primaries and in Section 4 we conclude. In the Online Appendix we discuss possible extensions and justifications of some of our main modeling elements and we present all the proofs.

2 The Model

The policy space is the $[0, 1]$ interval. We have a unit mass of general-election voters whose ideal policies are distributed according to an absolutely continuous, strictly increasing and twice differentiable distribution function $\Phi : [0, 1] \to [0, 1]$ with a unique median, $m \in (0, 1)$, defined by $\Phi(m) = \frac{1}{2}$. Two positive measure subsets of these voters form the two exogenously given parties and participate in a closed primary election where no other voters can participate. Let the median of the leftist party be the primary voter with ideal policy l and the median of the rightist party be the primary voter with ideal policy r with all l, r and m known and $0 < l < m < r < 1$.[9] Candidates A and B compete in the primary of the leftist party and candidates C and D compete in the primary of the rightist party. Each candidate $J \in \{A, B, C, D\}$ is characterized by a valence parameter $v_J \geq 0$ and strategically chooses and commits to an electoral platform $x_J \in S_J$, where $S_J = [0, m]$ if $J \in \{A, B\}$ and $S_J = [m, 1]$ if $J \in \{C, D\}$. We assume that $v_B > v_A$, $v_C > v_D$ and $v_B > v_C$. This ordering

[9]Notice that we impose very little structure on the precise kind of closed primaries that each party holds. While for example one party may run a (primary) election where only "core" party members are eligible, another party may run a primary open to all party members.

of valences assumes that B and C are the high valence candidates in each of the parties, places the highest valence candidate B in the leftist party, and provides equilibrium locations in order with candidates' "names".[10] We focus on the interesting case when valence differences are not very large (the exact formal constraints are presented in the statements of our propositions).[11]

The game has three stages. In stage one, all four candidates choose and announce their policy platforms simultaneously. In stage two, closed primary elections take place in each of the two parties.[12] In stage three, the general election takes place and each voter votes for one of the two primaries' winners. All ties, either in primaries or in the general election are broken with equiprobable draws.

The utility of a voter with ideal policy $i \in [0,1]$ when candidate $J \in \{A, B, C, D\}$ is elected in office (or else, wins in the general election) is given by

$$u_i(x_J, v_J) = -|i - x_J| + v_J$$

in line with literature on electoral competition among candidates of unequal valence (see for example, Groseclose 2001; Aragonès and Palfrey 2002). Voters are sincere both in primaries and in the general election and vote for the candidate that offers them the highest utility. We assume that when some voters are indifferent among a number of candidates, they evenly split among them.

[10] One can show that our equilibrium results hold when allowing one or more equalities but at a considerable cost in the proof length.

[11] Indeed, most of the literature focuses in characterizing equilibria for this scenario (e.g. Groseclose 2001; Aragonès and Palfrey 2002) as when valence differences are very large electoral competition might become trivial.

[12] Our analysis would carry through if closed primaries in each party took place sequentially.

Since voters' behavior in stages two and three is essentially mechanic, one may define the expected vote share of candidate J in the general election as

$$P_J(x_J, x_{-J} : v, \Phi, l, r)$$

where x_{-J} is the vector of platforms of the other candidates and $v = (v_A, v_B, v_C, v_D)$. Candidates are Downsian, that is, they maximize expected vote shares in the general election. The equilibrium concept we employ is Nash equilibrium in pure strategies, that in this setup is a vector $\hat{x} = (\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D)$ such that for every $J \in \{A, B, C, D\}$ it is true that $P_J(\hat{x}_J, \hat{x}_{-J} : v, \Phi, l, r) \geq P_J(\acute{x}_J, \hat{x}_{-J} : v, \Phi, l, r)$ for any $\acute{x}_J \in S_J$.[13]

3 Results

Before presenting our main results, let us define two concepts of crucial relevance. In equilibrium each primary is won by the high valence candidate (B wins the leftist primary and C wins the rightist primary). We refer to the valence difference in the general election as the "toughness" faced by B (defined as $T_G = -(v_B - v_C)$). Similarly, we refer to the valence difference in each party as the "toughness" candidates B and C face in their primary elections respectively (defined as $T_L = -(v_B - v_A)$ and $T_R = -(v_C - v_D)$). Notice that given our restrictions on candidates' valence characteristics, "toughness" takes negative values, approaching zero when both candidates are of almost equal valence representing the "toughest" of all cases.

[13] Maximization of the general election vote share is one of the possibilities when introducing office motives and since we focus on pure strategies, it is a refinement of the following ordered pair of objectives: a) a candidate prefers all outcomes of the game in which she is the winner of the general election to any other outcome and b) among all outcomes in which a candidate does not win the general election, this candidate prefers the outcomes in which she wins her party's primaries. Recall also the alternative interpretation of our model hinted in Footnote 3 according to which Φ may be viewed as the candidates' beliefs regarding the location of m and candidates' objective as maximizing the probability of winning the election (we elaborate on this in the online appendix).

Proposition 1 *If valence differences are not very large, there exists a unique Nash equilibrium and it is such that in each party: a) the low valence candidate proposes the platform preferred by the party's median voter, b) the high valence candidate proposes a more moderate platform than the one preferred by the party's median voter, and c) the high valence candidate (i.e., B in the leftist party and C in the rightist one) wins the primary. Formally, if $l < m - v_B < m + v_B < r$ then there exists a unique Nash equilibrium $\hat{x}$ and is such that $\hat{x}_A = l, \hat{x}_B = l - T_L, \hat{x}_C = r + T_R$ and $\hat{x}_D = r$.*

All proofs can be found in the online appendix

The existence of a unique Nash equilibrium points at the *stabilizing* effect of primaries on electoral competition. That is, while in the absence of primaries a clear prediction is hard to be derived –either the model does not admit an equilibrium in pure strategies (Aragonès and Palfrey, 2002) or it admits a continuum of equilibria (Ansolabehere and Snyder, 2000)–, this is no longer true when the two candidates have been selected through a primary race. Moreover, valence asymmetries in primaries create interesting electoral dynamics and in equilibrium lead to the divergence of proposed platforms in the primary race (in line with Hummel 2013 and in contrast to Adams and Merrill 2008) with general election candidates locating somewhere between their parties' and the general election median (a platform ordering supported in the literature (Coleman, 1971; Aranson and Ordeshook, 1972; Burden, 2001; Adams et al., 2005)). In specific, the low valence candidate locates exactly at the party's median, while the high valence candidate is more moderate and locates closer to the society's median.[14] How far towards the society's median the high valence candidate is able to move depends on the "toughness" of the primary. The higher the valence asymmetry inside a party, the more the winning candidate can converge towards the society's median, thus improving her future performance in the general election. On the contrary, if both candidates are of almost equal valence, then the high valence candidate is not able to differentiate much and this may have a negative impact in her performance in the general election. In the characterized equilibrium, in each primary, all more extreme party members than the party's median are indifferent between the two candidates and therefore split between the two primary candidates. All more moderate party members than the party's median however strictly prefer the high valence over the low valence one and vote for her. Hence, the high valence candidate wins the primary for sure obtaining the support of three quarters of party members. Regarding the winner of the general election, the following Corollary indicates that any of the qualifying candidates may win.

Corollary 1 *Candidate B's prospects in the general election are benefitting from a moderate leftist party (i.e., high l), an extreme rightist party (i.e., high r), a leftist median voter (i.e., low m), and a "tough" primary in the rightist party (i.e. high T_R), while harmed by a "tough" leftist primary and general election (i.e., high T_L and T_G). The reverse holds for candidate C. Formally, candidate B wins the general election if $T_G + T_L <$*

[14]Recall that candidates in the leftist (rightist) party are constrained to propose platforms to the left (right) of the median. Parties however may impose different constraints on candidates' platforms (e.g., through a prerequisite of a minimum number of party officials' endorsements and/or resolutions of party summits regarding the party's flexibility on certain policy issues). Fortunately, all the arguments supporting the existence of the equilibrium, $\hat{x} = (l, l + v_B - v_A, r - v_C + v_D, r)$, continue to hold for all alternative strategy sets $\{\tilde{S}_A, \tilde{S}_B, \tilde{S}_C, \tilde{S}_D\}$, as long as $\hat{x}_J \in \tilde{S}_J$ for each $J \in \{A, B, C, D\}$. That is, as long as the constraints on candidates' platforms set by the party, allow a candidate to locate sufficiently close to the party's median, our equilibrium continues to exist (even in the extreme case in which parties do not constrain candidates at all). Of course the uniqueness arguments that we develop may not extend to all conceivable alternative strategy sets.

$l+r-2m+T_R$, candidate C wins the general election if $T_G+T_L>l+r-2m+T_R$ and each of B and C win with equal probability if $T_G+T_L=l+r-2m+T_R$.

The winner of the general election ultimately depends on parties' and the society's medians as well as the valence asymmetries determining the "toughness" of both primaries and the general election. We further explain our results focusing on the winning prospects of the highest valence candidate B with symmetric arguments holding for the prospects of candidate C. Our results indicate that B wins the general election when the aggregate toughness she faces in the primary and general election is low "enough" (i.e., $T_G+T_L<l+r-2m+T_R$). This condition illustrates that B is favoured by a moderate leftist party (large l), an extreme rightist party (large r), and a tough primary in the rightist party (large T_R). This condition also points at the *matching effect of primaries* since the leftist party wins more often as the society becomes more leftist (i.e., small m). On the contrary, B's election prospects are harmed by a tough general election (large T_G) since voters compare the valence characteristics of the two general election candidates and B loses more often as her advantage compared to C gets smaller. Finally, a tough primary (large T_L) is also harmful for B since the presence of an almost equally competent primary opponent obliges her to remain close to the party median so as to guarantee nomination thus harming her general election performance (*within-party competition effect*). This last effect also points to the fact that parties may not benefit when choosing both their primary candidates from a pool of highly competent members and some heterogeneity proves desirable. Ideally, when the time of primaries arrives, parties would

opt for a competition between their most competent member and a low valence internal opponent so as to have good chances in the general election.

3.1 Open primaries

Having obtained a very general result for closed primaries a natural question is what occurs when parties hold open primaries. Open primaries are an increasingly popular method used by several parties to select their nominees.[15] We therefore extend our setup allowing voters to decide in which primary to participate once all four candidates announce their platforms.[16]

For the analysis of open primaries some further assumptions are necessary. While again, candidates A and B constitute the candidates of the leftist party and candidates C and D constitute the candidates of the rightist party, we assume that only a subset of voters are "active" and participate in the primaries. Let active voters have ideal policies distributed according to any continuous log-concave distribution F with a unique median m^a.[17] These "active" voters participate in the primary of the party where the candidate that gives them the highest utility is competing and vote for that candidate. Hence, the distribution of active voters across parties and therefore the location of the primary median voters are endogenously determined and depend on candidates'

[15] In the US around one third of the states hold an open primary. In Europe the socialist parties of France, Greece and Italy run open primaries for their leaders as it is the case for the conservatives in the UK for some parliamentary candidates. The European Green Party ran a paneuropean open primary for the 2014 EU election. In Latin America open primaries take place in Argentina. For mixed empirical evidence on the effect of primaries on political competition see Kaufmann et al. (2003); Gerber and Morton (1998); Kanthak and Morton (2011).

[16] Our open primaries model has a similar spirit with the literature modeling endogenous parties (e.g., Eguia 2011a,b, 2012; Gomberg et al. 2016, 2004; Baron 1993).

[17] We consider that a continuous distribution function F is log-concave if $\frac{\partial^2 \ln F(x)}{\partial x^2} < 0$ and $\frac{\partial^2 \ln[1-F(x)]}{\partial x^2} < 0$ for every $x \in (0, 1)$. That is, the notion of log-concavity that we employ implies that F is strictly increasing and twice differentiable in its support too. While log-concavity of the distribution of voters' ideal policies is a general assumption and widely used in the political economy literature, our definition is weaker compared to the "standard" one assuming a log-concave density function (see Bagnoli and Bergstrom (2005) for further properties of log-concave distributions).

policy proposals and valence characteristics. As far as candidates are concerned, we assume that each candidate $J \in \{A, B, C, D\}$ strategically chooses and commits to an electoral platform $x_J \in S_J$, where $S_J = [0, m^a]$ if $J \in \{A, B\}$ and $S_J = [m^a, 1]$ if $J \in \{C, D\}$ while valence differences are again not very large. Finally, if no voters participate in one of the two primaries (i.e., when all active voters prefer the candidate(s) of one party compared to those of the other) we assume that each of the candidates qualifies to the general election with equal probability. The following Proposition characterizes the equilibrium:

Proposition 2 *If valence differences are not very large, there exists a unique Nash equilibrium and it is such that parties and their medians are uniquely defined and in each party: a) the low valence candidate proposes the platform preferred by the party's median voter, b) the high valence candidate proposes a more moderate platform than the one preferred by the party's median voter, and c) the high valence candidate (i.e., B in the leftist party and C in the rightist one) wins the primary. Formally, for every F there exists $\tilde{v}_B > 0$ such that for every $v_B \in (0, \tilde{v}_B)$: a) the endogenous party medians $(l^*, r^*) \in (0,1)^2$ are the unique values that solve $2F(l^*) = F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2})$ and $2[1 - F(r^*)] = 1 - F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2})$, and b) there exists a unique Nash equilibrium $\hat{x}$ and is such that $\hat{x}_A = l^*, \hat{x}_B = l^* - T_L, \hat{x}_C = r^* + T_R$ and $\hat{x}_D = r^*$.*

The equilibrium structure is similar to the one in closed primaries with primary losers locating on parties' medians and primary winners diverging from the party median towards the median of the society. What is different compared to closed primaries is that now party medians (i.e., l^* and r^*) are endogenously determined and depend on all four values of valence characteristics (see Example 1) as well as the distribution of active voters. Again, candidates B and C propose the two most moderate platforms but now the indifferent voter between the two determines not only their vote shares in the general election but also the distribution of active voters across the two primaries. Since active voters decide to participate in the primary where they can identify the candidate that gives them the highest utility, all active voters on the left (right) of the indifferent voter in the general election participate in the primary of the leftist (rightist) party.

Notice that an equilibrium exists and is unique, guaranteeing the *stabilizing* effect of primaries even if voters can freely choose in which primary to participate. Unlike, though, the case of closed primaries in which the fixed party structure guarantees equilibrium existence for any distribution of voters, in open primaries existence is obtained by some mild restriction on such distribution (F being log-concave).[18] All these suggest, that indeed the stabilizing effect of primaries holds even with open primaries for a very general class of preference profiles, but, as expected, it is weaker compared to the closed primaries case. We note though that the fact that open primaries might stabilize electoral competition for such a general class of preference profiles, is more surprising, at least to us, than the fact that they are less prone to lead to stability compared to closed primaries. At first sight, one could expect that the dynamics that lead to the existence of an equilibrium when primary electorates are fixed, would disappear once we considered that parties are endogenous.

[18]This restriction is necessary because an equilibrium may not exist when F is too convex around the location of the indifferent voter in the general election $\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2}+\frac{v_B-v_C}{2}$. If for example F is log-convex about this point (where the notion of log-convexity is symmetric to the one of log-concavity), a slight transition of B from her equilibrium platform $l^*+v_B-v_A$ to $l^*+v_B-v_A+\varepsilon$, brings in the leftist primary many new supporters of B. Hence, B still wins in the primary and improves her performance in the general election.

As in closed primaries and Corollary 1, any of the two general election candidates may win the election and the condition such that one or the other candidate wins is similar. The highest valence candidate B wins the general election as long as the aggregate "toughness" she faces is lower than a given threshold (i.e., $T_G + T_L < l^* + r^* - 2m + T_R$). Such threshold now clearly depends on the endogenous location of the primary median voters (i.e., l^* and r^*). Candidate B benefits if the endogenously formed leftist party is relatively moderate while the endogenously formed rightist party is relatively extreme. Similar to closed primaries, the highest valence candidate B is harmed if the primaries in the leftist party are much tougher in terms of valence than the ones of the rightist party (*within-party competition effect*). Moreover, the *matching effect of primaries* where the leftist candidate benefits from a leftist electorate is still present since candidate B wins more often as the society becomes more leftist (i.e., small m).[19]

Remember that the main difference between open and closed primaries is that while in closed primaries candidates propose platforms that depend exclusively on the exogenous location of the party's median voter and the valence characteristics of the party's candidates (Proposition 1), in open primaries proposed platforms depend on the endogenous location of the party's median voter and therefore the valence characteristics of all four candidates (Proposition 2). That is, in open primaries valence characteristics in each party also affect

[19] To see that this is not an artefact of having distinct primary and general election electorates, let for example $\Phi = F = \frac{2-a+ax}{2}x$, where $a \in [-2, 2]$. This is a simple class of distributions with linear densities where $a = -2$ corresponds to the triangular distribution with a peak at zero, $a = 0$ corresponds to the uniform distribution as in Example 1, and $a = 2$ corresponds to the triangular distribution with a peak at one. One can show that the matching effect is strongly present: when the society is left leaning the nominee of the leftist party enjoys an electoral advantage and vice versa. Formally, when most voters are leftist ($a < 0$ or $m < \frac{1}{2}$), then $\lim_{v_B \to 0}(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2}) \in (m, \frac{1}{2})$ and when most voters are rightist ($a > 0$ or $m > \frac{1}{2}$), then $\lim_{v_B \to 0}(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2}) \in (\frac{1}{2}, m)$.

nomination in the other party, a feature absent in closed primaries. The following example illustrates such interaction in open primaries.

Example 1 *From Proposition 2 we know that parties' median voters* (l^*, r^*) *are the unique values that solve* $2F(l^*) = F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2})$ *and* $2[1-F(r^*)] = 1 - F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2})$*. Let active voters be uniformly distributed (i.e.,* $F(x) = x$*). Then these two equations simplify to*

$$2l^* = \frac{l^* + v_B - v_A + r^* - v_C + v_D}{2} + \frac{v_B - v_C}{2},$$

$$2(1-r^*) = 1 - \frac{l^* + v_B - v_A + r^* - v_C + v_D}{2} + \frac{v_B - v_C}{2}.$$

Solving these two equations with respect to l^* *and* r^* *we identify the location of the primary median voter in each party as* $l^* = \frac{1}{4} - \frac{1}{2}v_A + v_B - v_C + \frac{1}{2}v_D$ *and* $r^* = \frac{3}{4} - \frac{1}{2}v_A + v_B - v_C + \frac{1}{2}v_D$*. In the unique equilibrium platforms are hence,*

$$\hat{x}_A = l^* = \frac{1}{4} - \frac{1}{2}v_A + v_B - v_C + \frac{1}{2}v_D,$$

$$\hat{x}_B = l^* - T_L = \frac{1}{4} - \frac{3}{2}v_A + 2v_B - v_C + \frac{1}{2}v_D,$$

$$\hat{x}_C = r^* + T_R = \frac{3}{4} - \frac{1}{2}v_A + v_B - 2v_C + \frac{3}{2}v_D,$$

$$\hat{x}_D = r^* = \frac{3}{4} - \frac{1}{2}v_A + v_B - v_C + \frac{1}{2}v_D.$$

As this example shows, all equilibrium platforms depend on all valence characteristics and illustrate forces that push parties to the extremes or the centre. Ceteris paribus, an increase in the valence of either B *or* D *move platforms to the right. On the contrary, an increase in the valence of either* A *or* C *move platforms to the left. That is, as a nominee's valence increases (i.e.,* B*'s or* C*'s), not only she moves to moderate grounds* *but also makes the other party propose extreme platforms. On the contrary, an increase in the valence of a losing primary contender's (i.e., A's or D's) not only ties the whole party to the extremes, but also permits the opposing party to propose moderate platforms.*

Given the discussed differences in platform proposals across primary types, one may wonder which primary type delivers higher social welfare (defined as the sum of individual utilities). Our analysis indicates that in the most symmetric scenario ($\Phi(x) = F(x)$ symmetric about $1/2$, and $F(l) = 1 - F(r) = 1/4$) the society is better off when parties hold open primaries. This is true because while in this symmetric case the winner (and hence her valence) is not affected by the primary type, the winner's platform is closer to the median's preferred policy when primaries are open.[20] Of course, since l and r need not coincide with the first and third quartile of the active voters' distribution in the open primary, one can think of cases where closed primaries are better for the society: these would involve exclusion of extreme voters from the open primary (i.e., l and r are close to m^a). When extremist voter participation in closed primaries is significant though (this is most often the case), open primaries seem to better serve the centrists' voters interests as they allow the potential winner to move closer to their preferences.

[20]Given that voters' utilities are linear in policy and the winner's valence is not affected by the primary type, the primary type that delivers higher welfare is the one that brings the winner's platform closer to the median's preferred policy. To see why in this symmetric scenario the winner is not affected by the primary type one can refer to Example 1 (i.e., F uniform and hence $l = 1/4$ and $r = 3/4$) while also assuming that the general electorate is uniformly distributed. Computing candidates' vote shares it turns out that under both open and closed primaries B wins the election if $T_G + T_L < T_R$ (i.e., B faces relatively less aggregate toughness than C does). Now comparing the winner's proposed platform in open primaries (Example 1, $\hat{x}_B = \frac{1}{4} - \frac{3}{2}v_A + 2v_B - v_C + \frac{1}{2}v_D$) or in closed primaries (Proposition 1, $\hat{x}_B = \frac{1}{4} + v_B - v_A$), indeed, the one proposed in open primaries is more moderate also whenever $T_G + T_L < T_R$. Therefore, while the winner is not affected by the primary type, indeed open primaries deliver higher welfare than closed ones.

3.2 One party primaries

In reality not both parties need to hold a primary before the general election. A typical situation of interest for the absence of primaries in one party is when an incumbent runs for reelection. Let the incumbent be candidate C with valence v_C and an ideal policy x_C that is known and fixed.[21] The leftist candidates A and B may run in a closed or open primary and while we still assume that $v_B > v_A$ we do not require that $v_B > v_C$ permitting the incumbent to be the highest valence candidate. As when both parties hold a primary, we assume that each candidate $J \in \{A, B\}$ strategically chooses and commits to an electoral platform $x_J \in S_J$ (where $S_J = [0, m]$ in closed primaries and $S_J = [0, m^a]$ in open primaries) and a Nash equilibrium in pure strategies is a vector $\hat{x} = (\hat{x}_A, \hat{x}_B)$ such that none of the two candidates has incentives to deviate. Let us start by presenting the results when the leftist party runs a closed primary.

Proposition 3 *If the leftist party runs a closed primary to challenge an incumbent and valence differences are not very large, there exists a unique Nash equilibrium and it is such that in the challenger's party: a) the low valence candidate proposes the platform preferred by the party's median voter, b) the high valence candidate proposes a more moderate platform than the one preferred by the party's median voter, and c) the high valence candidate (i.e., B) wins the primary. Formally, if $m + max\{v_B, v_C\} < x_C$ then there exists a unique Nash equilibrium $\hat{x}$ and is such that $\hat{x}_A = l$ and $\hat{x}_B = l - T_L$.*

Proposition 3 suggests that when the incumbent is challenged by a party organizing a closed primary, both

[21] Formally, let $x_C > m > l$ for the case of closed primaries and $x_C > m^a$ for the case of open primaries.

primary candidates follow the same strategies as when both parties hold closed primaries (i.e., $\hat{x}_A = l$ and $\hat{x}_B = l - T_L$). Notice that only the "toughness" of the primary race -and not the incumbent's characteristics- determines how more moderate is the high valence candidate compared to the low valence one and the party's median. Of course the incumbent's characteristics play a crucial role in determining the winner of the general election. The following corollary summarizes how.

Corollary 2 *Candidate B's prospects in the general election are benefitting from a moderate leftist party (i.e., high l), an extreme incumbent (i.e., high x_C), and a leftist median voter (i.e., low m), while harmed by a "tough" leftist primary and general election (i.e., high T_L and T_G). The reverse holds for candidate C. Formally, candidate B wins the general election if $T_G + T_L < l + x_C - 2m$, candidate C wins the general election if $T_G + T_L > l + x_C - 2m$ and each of B and C with equal probability if $T_G + T_L = l + x_C - 2m$.*

As Corollary 2 indicates, whether the leftist challenger succeeds in replacing the incumbent depends on the ideology of the leftist median, the ideal policy of the incumbent, as well as the "toughness" of the primary and of the general election in terms of valence. The aggregate "toughness" condition such that the challenger B wins (i.e., $T_G + T_L < l + x_C - 2m$) implies that the incumbent is of course harmed by her own extreme policies and low quality. Additionally, the challenger increases her chances to successfully replace the incumbent when she emerges from a moderate leftist party with non-competitive primaries, and when the median voter in the general election is leftist.

Notice that in closed primaries candidates aim at winning nomination focus only on their party's median. This explains why the challenger's proposed platform is not affected by the incumbent's platform. As we describe in the following proposition this is no longer true when the party in opposition holds an open primary. Let as before F denote the distribution of active voters. The natural way of extending sincere voting in this one-party primary scenario is to let active voters who like the most either candidate A or B participate in the primary of the leftist party supporting their favorite candidate, while active voters that like the most the incumbent do not participate in the primary. Again, if none of the active voters participate in the leftist primary we assume that A and B qualify to the general election with equal probability.

Proposition 4 *If the leftist party runs an open primary to challenge an incumbent and valence differences are not very large, there exists a unique Nash equilibrium and it is such that the challenger's party and its median are uniquely defined and: a) the low valence candidate proposes the platform preferred by the party's median voter, b) the high valence candidate proposes a more moderate platform than the one preferred by the party's median voter, and c) the high valence candidate (i.e., B) wins the primary. Formally, for every F there exists $v^{max} > 0$ such that for every $max\{v_B, v_C\} \in (0, v^{max})$: a) the endogenous party median $l^* \in (0, m^a)$ is the unique value that solves $2F(l^*) = F(\frac{l^*+v_B-v_A+x_C}{2} + \frac{v_B-v_C}{2})$, and b) there exists a unique Nash equilibrium $\hat{x}$ and is such that $\hat{x}_A = l^*, \hat{x}_B = l^* - T_L$.*

Proposition 4 presents a similar equilibrium structure as Proposition 3 with l^* denoting the location of the median of the endogenously formed leftist party. The condition providing such location permits us interesting insights on the effect of the incumbents' characteristics (i.e., x_C and v_C) on the platforms proposed in the leftist primary and participation in the latter. As it turns out, the leftist party tends to be "large" and hence more moderate when the incumbent implements extreme policies or is of low valence (formally l^* is strictly increasing in x_C and strictly decreasing in v_C). Similarly, the leftist party tends to propose moderate platforms when B is of high valence and A is of low valence (formally l^* is strictly increasing in v_B and strictly decreasing in v_A).[22] The aggregate "toughness" threshold condition such that the winning leftist candidate B also wins the general election follows the above intuition and is similar to when party B runs a closed primary and Corollary 2 (i.e., candidate B wins the general election if $T_G + T_L < l^* + x_C - 2m$).

The following example illustrates equilibrium proposals for the leftist candidates when primary voters are uniformly distributed.

Example 2 *Let "active" voters be uniformly distributed across the policy space. If the leftist party runs an open primary, the median voter in the endogenously formed party is given by* l^* *such that* $2l^* = \frac{l^*+v_B-v_A+x_C}{2} + \frac{v_B-v_C}{2}$ *(given that* F *is the uniform distribution). That is,* $l^* = \frac{1}{3}(x_C - v_A + 2v_B - v_C)$*. As Proposition 4 indicates, the low valence candidate* A *proposes platform* $\hat{x}_A = \frac{1}{3}(x_C - v_A + 2v_B - v_C)$*, while the high valence candidate* B *is more moderate by a distance* $v_B - v_A$ *and equilibrium platform* $\hat{x}_B = \frac{1}{3}(x_C - 4v_A + 5v_B - v_C)$*.*

In contrast to closed primaries, all platforms depend on the valence characteristics of all candidates in-

[22] Given the equilibrium condition $2F(l^*) = F(\frac{l^*+v_B-v_A+x_C}{2} + \frac{v_B-v_C}{2})$, the log-concavity of F suffices to obtain the aforementioned comparative statics of l^* with respect to all valence characteristics.

cluding those of the incumbent as well as the implemented policy with $\hat{x}_A$ and $\hat{x}_B$ increasing in x_C and v_B and decreasing in v_A and v_C as commented before. If we also assume that the society is uniformly distributed (i.e., ideologies' of the general electorate are evenly distributed across then policy space or formally stated $\Phi(x) = x$) then the location of the indifferent voter and hence B's vote share under an open primary is given by:

$$\frac{\hat{x}_B + x_C}{2} + \frac{v_B - v_C}{2} = \frac{2}{3}(2v_B - v_A + x_C - v_C)$$

Let us now go back to closed primaries. From Proposition 3, we know the proposed platforms are given by $\hat{x}_A = l$ and $\hat{x}_B = l + v_B - v_A$. When the society is uniformly distributed the location of the indifferent voter and hence B's vote share under a closed primary is given by:

$$\frac{l + v_B - v_A + x_C}{2} + \frac{v_B - v_C}{2}$$

Notice now that the location of the indifferent voter in the general election varies across the two primary types, and hence the selection of one system over the other clearly affects the electoral outcome and possibly the winner of the election. Comparing the location of the indifferent voter, we know that the vote share of the challenger's party is larger under an open primary rather than under an open primary if and only if

$$\frac{2}{3}(2v_B - v_A + x_C - v_C) > \frac{l + v_B - v_A + x_C}{2} + \frac{v_B - v_C}{2}.$$

This last condition can be simplified in a more intuitive manner as $T_L + T_G < x_C - 3l$. Simply put, this condition is equivalent to $l^* > l$ meaning that if the challenger were to run an open primary then the endogenous *median would be more moderate than the party's median voter if it were to run a closed primary.*

Open or Closed primary in the challenger's party?

An interesting question then is, when would the party in opposition increase its vote share by running an open rather than a closed primary? The relevant condition ($T_L + T_G < x_C - 3l$) obtained in Example 2 indicates that this occurs when the aggregate toughness the leftist candidate B faces is low enough.[23] This may hold, if for example candidate B is a candidate of sufficiently high valence. Hence, one would expect that parties where a highly competent primary candidate competes may prefer open over closed primaries. This is because open primaries permit the highly competent candidate to "open" the party to the society and propose more moderate platforms than if she were to fight for nomination in a closed primary. In a similar spirit, the condition such that open primaries are preferred over closed ones is also more likely to hold when party members participating in the closed primary are relatively extreme (i.e., low l). This occurs because while under closed primaries the candidates would have to please an extreme primary electorate, open primaries permit them to move to a moderate ground and enrich the primary electorate with some moderate voters. Finally, parties in opposition are more likely to select their nominee through an open primary when the incumbent is implementing a relatively extreme policy (i.e., high x_C). In that instance, given that the implemented policies let many voters alienated, the opposition has incentives to open the primary to the society bringing in the party relatively moderate voters

[23] Remember that $T_L = -(v_B - v_A)$ denotes the "toughness" in the leftist primary (with $v_A < v_B$). Large values of T_L denote a very competitive primary that does not allow the primary winning candidate B moderate enough and become attractive in the general electorate. The "toughness" of the general election $T_G = -(v_B - v_C)$ also has a similar effect with larger values of T_G being detrimental for the leftist candidate. Note here that while $T_L < 0$ is still true, for the general election we have that $T_G < 0$ if $v_C < v_B$ and $T_G > 0$ if $v_C > v_B$.

that push the endogenously formed median of the party to a moderate policy ground.

Our example so far has illustrated how the challenger's vote share is higher under closed or open primaries depending on the aggregate toughness the leftist candidate B faces. The following table summarizes the winner of the election under all relevant scenarios and illustrates that the choice of primary type may actually determine the winner of the election and a wrong choice may prove detrimental for the challenger's party.

Aggregate Toughness ($T_L + T_G$)	Winner with Open	Winner with Closed $l < 0.25$	Winner with Closed $l > 0.25$
Very Low	B	B	B
Moderately Low	B	C	B
Moderately High	C	C	B
Very High	C	C	C

Table 1: Winner of the general election for open and closed primaries depending upon the challenger's party being extreme ($l < 0.25$) or moderate ($l > 0.25$) when the general electorate and active voters are uniformly distributed.

If the aggregate "toughness" B faces ($T_L + T_G$) is very low, the challenger always wins.[24] On the contrary, if the aggregate "toughness" is very high, the incumbent always remains in office. Nevertheless, for moderate levels of toughness the type of primary is crucial. Consider for instance that the leftist party is relatively extreme (i.e., $l < 0.25$) and that the aggregate toughness is moderately low. If the challenger emerges through a closed primary the incumbent remains in office, while if the challenger emerges from an open primary the incumbent is successfully replaced. The reason why the wrong choice of primary type may be detrimental is that under

[24] Formally, for $l < 0.25$ aggregate toughness is very low if $T_L + T_G < x_C - 1 + l$, moderately low if $x_C - 1 + l < T_L + T_G < x_C - 0.75$, moderately high if $x_C - 0.75 < T_L + T_G < x_C - 3l$ and very high if $T_L + T_G > x_C - 3l$. For $l > 0.25$ aggregate toughness is very low if $T_L + T_G < x_C - 3l$, moderately low if $x_C - 3l < T_L + T_G < x_C - 0.75$, moderately high if $x_C - 0.75 < T_L + T_G < x_C - 1 + l$ and very high if $T_L + T_G > x_C - 1 + l$.

a closed primary B is not able to propose moderate platforms since the party's median voter is quite to the left ($l < 0.25$) and a moderate platform would result in a lost closed primary. However, recall that aggregate toughness is moderately low (e.g., because B is quite talented) a fact that can be exploited by B under open primaries. As it turns out, open primaries permit B to move away from l by "opening" the party to the society and bringing the whole party to a moderate ground that eventually provides a victory in the general election. For similar reasons, a moderate leftist party facing moderately high aggregate toughness should not opt for an open primary but rather prefer a closed one given the moderate location of its median voter. To the degree that all parameters of interest (v_A, v_B, v_C, l and x_C) are measurable, the above table summarizes the empirically testable predictions of our model when a challenger aims at replacing an incumbent. Our results overall imply that the strategic choice of primary type by parties' leaders is of crucial importance since it may shape the nominee's success. In any case, parties seem to benefit by flexibility in their rule and should choose the latter on a case by case basis taking into consideration the characteristics of all candidates (those of the incumbent and the potential primary candidates).

3.3 When the two parties use different types of primaries

When both parties hold primaries, we have so far focused on situations where both parties hold the same primary type. Given that we have previously established how open and closed primaries differ when a challenger emerging from primaries faces an incumbent, a natural followup question is what occurs when one party holds a closed primary and the other holds an open primary. Our results permit us to discuss such situation. Let S^A denote the set of all active voters participating in primaries, with a subset of them S^C being eligible to participate in the closed primary. In the sincere setup we have been focusing, three natural cases emerge regarding how party members S^C behave with respect to the open primary. Party members eligible to vote in the closed primary still vote in a sincere manner and: a) participate only in the closed primary, b) participate in both primaries, or c) participate only in the one primary where they identify their preferred candidate.

If voters eligible to vote in the closed primary vote only in this primary, the situation is identical to the one we have described under the presence of an incumbent. In the closed primary, the low valence candidate will be locating at the party's median and the high valence candidate will be running on a more moderate platform. In the open primary, the two candidates will be focusing on the distribution of the remaining active voters ($S^A \setminus S^C$) and propose platforms as if they were facing an incumbent (i.e., the high valence candidate winning the closed primary). Hence, all intuition is as previously presented.

If voters eligible to vote in the closed primary are permitted to participate also in the open primary and do so, candidates in the closed primary will be behaving exactly as before, focusing only on the party's median and the "toughness" of the closed primary. Candidates in the open primary will be running in an open primary as if they were facing an incumbent but now they will be focusing on the median of the whole set of voters (S^A). Hence, while the closed primary voters (S^C) participating only in the closed primary affect the open primary exclusively through the location of their winning candidate, voting in both primaries also affects the open primary through the distribution of voters participating in the open primary.

Finally, if party members eligible to vote in the closed primary vote only in the primary with the candidate they prefer, the situation is similar to both parties running an open primary. The difference is that while candidates running in open primaries will be focusing on all active voters (S^A), candidates running in the closed primary will be focusing only on a subset of them (S^C).

4 Conclusion

Neo-Downsian modeling has generated a huge literature, with the initial simplifying assumptions of Downs' classic model of two party plurality competition over a single policy dimension enriched with more realistic assumptions, including multidimensionality, party primaries (varying from open to closed), and valence as a basis for voter choice, as well as extensions to multiparty competition under electoral rules other than first past the post. Here we have contributed to that tradition by seeking to develop a model of primary competition for plurality two-party elections that matches various stylized facts about the real world, including perhaps most notably a prediction of party differentiation in the general election, and allowing for a party whose support base is closer to the position of the overall median voter to be advantaged, rather than assuming that electoral competition leads to tweedledum-tweedledee politics with each party having an equal probability of victory. We have also allowed for differences across primary type and for different results when there is or is not an incumbent. Moreover, from a theoretical perspective, the equilibrium results we have complement the more common non-equilibrium results in multidimensional two-party competition.

We recognize that while we have made advances over previous models of party competition that include both primaries and valence, ours is far from the last word. In particular, models of candidate behavior, our own included, tend to focus on the perspective of a single candidate, and impute to that candidate office seeking and/or policy seeking goals. There are two important ways –each of which takes us into areas beyond the scope of the present article– in which that perspective could be modified in future work. First, we might add to prospective candidate's utility function a further consideration, namely their perception of the consequences of their candidacy on the success of their party in the general election. Second, we could move from the specifics of individual contests to ask about how political parties and interest groups affect the nomination process.

The conflict between what candidates want and what is in the overall interest of their party is especially sharp when it comes to legislative redistricting (Owen and Grofman, 1988). Looking at how this conflict is resolved in that domain gives us some ideas about how we might, in the future, model primary elections in a general equilibrium framework recognizing that individual election contests are embedded in a wider institutional setting. In the redistricting context, incumbents and challengers realize that the value of gaining office is enhanced if their party is the majority party. This may make incumbents more willing to "take a hit for the party", that is, accept some loss of certainty about their own re-election in return for increased chances of their party controlling the legislature. Of course, since a fundamental principle of politics is that no incumbent ever regards his district as safe enough, willingness to take a hit for the party may be limited. Relatedly, in the primary context, candidates for office may have an exaggerated sense of their own valence and thus, be unwilling to posit that their nomination in the primary will result in a loss for the party in the general election (cf. Uhlaner and Grofman 1986). Moreover, they may not be sophisticated enough to consider that their candidacy may affect the policies proposed by a competing primary contender in a way harming the party's general election chances even when it is that candidate who wins the primary and not themselves. Thus, while it certainly makes sense to allow for the possibility of candidates caring about overall consequences for their party in modeling platform choices, relying on this kind of altruism to deter primary challengers that can hurt the party in unintended ways may be unrealistic, especially once we take misperceptions into account.

But there is a second route by which conflicts between party interest and the interests of individual candidates get resolved, and that involves considerations of relative power. In the redistricting example, tradeoffs between overall party interests and candidate/office holder interests is largely resolved by the relative power of incumbents to control the process as opposed to that of other officials, such as governors, who might take a wider party-centric perspective. In the primary context, to really understand the dynamics of candidates' competition, we would need to move beyond our stylized framework and look behind the scenes at the recruitment of candidates (by parties and interest groups) and the nature of campaign support that primary candidates might expect to receive from the party (e.g., access to list of party donors) and from particular interest groups. In particular, there is universal consensus that "Money is the mother's milk of politics", as attested by both Democratic liberals such as the late Jesse Unruh of California, and contemporary Republican conservatives such as Rush Limbaugh. Thus, as candidates make their strategic decisions, they may be deterred to seek nomination by recognition of the fact that sources of monetary support are already committed to other candidates. The dynamics of party competition we described above show that, even though a primary challenger loses the primary, the mere fact of her candidacy may affect the ability of the party to win the general election, because of the impact the challenge has on the policy position taken by the candidate who does win the primary. However, even if individual candidates may have non-aligned interests with their party, actors such as party officials and interest groups with a longer term and strategic perspective may make decisions about who to support in the light of sophisticated calculations, seeking to deter challengers who might harm the party's general election chances and, if that fails, seeking to reinforce the primary chances of the preferred candidate.

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Electoral competition with primaries and quality asymmetries: Online Appendix

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Contents

1 Discussion

Throughout our formal discourse we have interpreted Φ as the distribution of general-election voters, and we have assumed that candidates have perfect information about it and choose their platforms so as to maximize their general election vote share. As we have noted in the Introduction, this setup is equivalent to one where candidates know the location of their parties' medians but face uncertainty regarding the distribution of the general electorate and choose platforms that maximize their probability of winning the general election.[1] Notice that this alternative interpretation is often reasonable since candidates may sometimes care just about winning rather than their total vote share, while the fact that candidates have better information about the preferences of their party's median compared to the eventual location of the median in the general election can be also justified. Indeed, there is often a long period between the primaries and the general election and, hence, assuming that at the time candidates choose their platforms they have more accurate expectations regarding the behavior of the electorate in primaries than in the general election seems reasonable. Keeping in mind the above interpretation, throughout this section we discuss some of our assumptions and possible relaxations of those that would not affect the substantial implications of our analysis.

1.1 Explaining losers' entry

In our model, there are two candidates –the low valence leftist candidate and the low valence rightist candidate– that get zero payoffs. Hence one could argue that their presence in the electoral race is not justified since they cannot ever qualify to the general election and enjoy a positive vote share: why should one declare candidacy if this does not improve one's payoffs? Indeed, these concerns are valid, but, fortunately, they can be easily addressed in our setup by assuming that candidates in each party represent different constituencies or ideological factions and are thus not indifferent to the implemented policies. This modification is largely in line with real world politics where parties are differentiated and hence attract different kinds of members, donors, activists, etc. That is, representation of heterogeneous constituencies or, simply, policy motivation, is not only an additional reasonable assumption, but fundamentally linked with the dynamic evolution of the party system when parties' nominees propose clearly distinct political platforms. Below we explain in detail, but without unnecessary

[1] Φ then describes candidates' beliefs about the median voter's ideal policy. Given the one and a half dimensional nature of our preferences (Groseclose, 2007), candidates' maximize their payoff by maximizing the probability that they are the median voter's top choice (when indifferent, a voter is assumed to break the tie with equiprobable draws).

formalities, how such secondary policy concerns may be integrated in the model and why they justify the entry of low valence primary candidates.

In specific, if we consider that candidates objectives are lexicographically ordered[2] –first, they care about winning and, then, about the implemented policy– and that the valence differences are sufficiently small, then our equilibrium, both in the closed and in the open primaries' case, is still an equilibrium even if in the beginning candidates were free to strategically decide whether they wish to be candidates or not. Obviously, the high valence candidates would still like to run because this gives them positive election probability, which is their primary goal. Let us hence focus on trying to understand the entry-related incentives of a low valence candidate –say the one of the leftist party– when she expects that all other candidates will enter. This low valence candidate knows that if she does not enter, the high valence candidate of her party will locate as right as possible (to better achieve her win related objectives) and, independently of where the candidates of the rightist party locate, will induce an expected policy outcome substantially to the right of m or m^{α} (for the closed and open primaries setup respectively). On the other hand, if this low valence candidate declares candidacy, she knows that the high valence candidate of her party will have to declare a policy close to the median of their primary electorate in order to qualify to the general election, and this will induce an expected implemented policy close to m (m^{α}). Hence, low valence candidates have good reasons to enter the race and crucially affect the electoral outcomes, even if winning is not a serious prospect for them. Importantly, strategic candidacy is not a trivial issue (see Dutta et al. 2001 for an excellent reference), and it is arguably very fortunate that our primaries' setup can provide clear intuition in support of candidates' entry choices.

1.2 Strategic voters

Throughout our analysis we have considered that voters behave sincerely, in the sense that when encountered with a pair of candidates they always vote for the one they like best (taking in account both the candidates' policy proposals and their valences). Despite the fact, though, that this assumption captures the voting behavior of many individuals (Pons and Tricaud, 2017), it is fair to claim that a subset of voters might vote using different criteria. Indeed, while in the general election sincere voting is the only reasonable assumption since there are

[2] This is not necessary but greatly simplifies the involved arguments. One could consider instead that the candidates weight these two goals and care substantially more about winning, and end up with similar conclusions, but with a significant additional analytical cost.

only two candidates, things are more complicated in the primaries. It is possible that a voter might prefer one of the two primary candidates strictly to the other, but would rather see her least preferred candidate qualifying to the general election, since this might offer her a higher expected utility. Consider, for instance, that we are in the closed primaries framework, that $x_A = l$, $x_B = l + v_B - v_A + \varepsilon$ for some $\varepsilon > 0$ and that the candidates of the rightist party occupy their equilibrium positions (and hence candidate C is expected to qualify to the the general election). Notice that a primary voter i of the leftist party with ideal policy $x_i < l$ strictly prefers candidate A to candidate B, but, if ε is sufficiently small, she strictly prefers the lottery where B qualifies to compete in the general election against C

$$\Phi(\frac{x_B + v_B + x_C - v_C}{2})(-|x_B - x_i| + v_B) + [1 - \Phi(\frac{x_B + v_B + x_C - v_C}{2})](-|x_C - x_i| + v_C)$$

over the lottery where A qualifies to the general election

$$\Phi(\frac{x_A + v_A + x_C - v_C}{2})(-|x_A - x_i| + v_A) + [1 - \Phi(\frac{x_A + v_A + x_C - v_C}{2})](-|x_C - x_i| + v_C).$$

This is so, because as $\varepsilon \to 0$ candidates A and B offer voter i almost the same utility since $-|x_B - x_i| + v_B \to -|x_A - x_i| + v_A$, but the probability of B winning the general election conditional on having won the primary remains substantially larger than the probability of A winning the general election conditional on having won the primary. That is, if this voter behaves as if she were pivotal in the primary of the leftist party, she will vote for candidate B, and not for her favorite candidate, A. If this voter actually votes for her least preferred primary candidate applying this reasoning, then we say that this voter behaves strategically. We will now argue that the main features of our equilibrium analysis continue to hold if we add a small measure of strategic voters in the primary electorates.

Let us consider that a share of voters of the leftist primary of size $s \in [0, 1]$ are sincere and a share of size $1 - s$ are strategic. For simplicity, we consider that the leftist party holds a closed primary and that the rightist party nominates directly candidate C without a primary.[3] Moreover, we assume that the distributions of ideal policies of both kinds of primary voters are identical, strictly increasing and continuous; and define $l_s \in [l, m)$, such that, the sincere voters with ideal policies to the left of l_s are equal to half of the total primary electorate of

[3] Our arguments directly extend to the case in which both parties hold primaries.

the leftist party. Naturally, l_s need not exist when s is small, but it is guaranteed to exist when s is sufficiently close to one (we have that $l_s \to l$ when $s \to 1$). We will argue that if $x_A = l_s$ and $x_B = l_s + v_B - v_A$; and s is sufficiently close to one, then no candidate has any incentives to change her platform. That is, we will argue that introduction of a small share of strategic voters moves candidates to more central policies but does not upset the general structure of our equilibrium.

Indeed, candidate B wins the primary in the posited profile, and has no incentives to move: a) to the left since this will decrease her election probability in the general election (even if she still wins the primary), and b) to the right since, by doing that, A will be preferred by enough sincere voters to secure a majority and win the primary election. Similarly, A has no incentives to deviate: a) to the right as B will be preferred by enough sincere voters to secure B a majority and win the primary election,[4] and b) to the left, because if the deviation is small no strategic voter will vote for A and hence the sincere votes in favor of A will not be enough to win her the primary; and if the deviation is large, even if strategic voters vote for her, the measure of sincere voters that will vote for B will be enough to secure B a majority. To see why the latter is true consider that A deviates to $x'_A < l_s$ and compute the expected utilities of a strategic voter with ideal policy equal to x'_A conditional on A winning the primary and conditional on B winning the primary.[5] If A wins the primary, the voter expects

$$\Phi(\frac{x'_A + v_A + x_C - v_C}{2})v_A + [1 - \Phi(\frac{x'_A + v_A + x_C - v_C}{2})](-|x_C - x'_A| + v_C) \tag{1}$$

and if B wins the primary, the voter expects

$$\Phi(\frac{x_B + v_B + x_C - v_C}{2})(-|x_B - x'_A| + v_B) + [1 - \Phi(\frac{x_B + v_B + x_C - v_C}{2})](-|x_C - x'_A| + v_C). \tag{2}$$

Clearly, the incentives for the strategic voter with an ideology identical to the one proposed by A to vote for B come from the higher chances B has to win the general election compared to A (i.e., $\Phi(\frac{x'_A+v_A+x_C-v_C}{2}) > \Phi(\frac{x_B+v_B+x_C-v_C}{2})$ since $x_B > x_A$ and $v_B > v_A$). It is easy to see that if the deviation is "small", (2) is strictly

[4]Recall that $x_B = l_s + v_B - v_A$. Therefore as long as A deviates to any location $x'_A \in (l_s, x_B + v_B - v_A = l_s + 2(v_B - v_A))$ with $l_s + 2(v_B - v_A) \leq 1$ all sincere voters will vote for B and hence B will win the majority in the primary. All sincere voters will vote for B also if $l_s + 2(v_B - v_A) > 1$ since no deviation to the right of l_s makes a sincere voter vote for A. Finally, if $l_s + 2(v_B - v_A) \leq 1$ and A deviates to $x'_A \in [l_s + 2(v_B - v_A), 1]$ again B will be winning the primary. This is the case because B will be preferred by at least all sincere voters with ideal policies in $[0, l_s + 2(v_B - v_A))$, who constitute a clear majority.

[5]Notice that if a strategic voter with such preferences (i.e., voters' ideal point coincides with A's proposal) does not vote for A then no strategic voter does.

larger than (1) and hence the strategic voter will vote for B not making the deviation profitable (this is true when x'_A is smaller but sufficiently close to $x_B - v_B + v_A$, for any $x_B > v_B - v_A$). For the deviation to make the strategic voter to vote for A and that (1) to be larger than (2) it must be that x'_A is small and far enough from $x_B - v_B + v_A$. So if the share of strategic voters is sufficiently small (that is, $s \to 1$ and $l_s = x_B - v_B + v_A \to l$) we have that a strategic voter might vote for A only if x'_A is substantially smaller than l. But even in that case such deviation was not profitable fro A since the sincere voters that will vote for B will constitute a clear majority.

As we argued, strategic voting in primaries between candidates of different valences generates interesting dynamics promoting more moderate policies. But, perhaps more importantly, we have established that as long as sincere voting is the predominant voting behavior, the main features of the primaries' equilibrium analysis that we have conducted in the sincere voting environment continue to hold.

1.3 Robustness

Spatial models of electoral competition are known to provide different predictions depending on a number of factors including:[6] a) the number of competing candidates, b) candidates' objectives, c) voters' utility functions, d) the number of policy issues, e) the presence or not of valence differences, and f) whether the analysis focuses exclusively on pure strategies or permits the use of mixed ones. In the standard Downsian framework (two office motivated candidates and a unique policy issue), for instance, if one increases the policy issues form one to two, a pure strategy equilibrium rarely exists (Plott, 1967) and if one increases the candidates from two to three, then: a) when candidates are vote-share motivated, a pure strategy equilibrium does not exist (Shaked, 1982); while b) when candidates are win motivated, pure strategy equilibria exist –only in asymmetric strategies, though. Similarly if one adds valence asymmetries between the two candidates, pure strategy equilibria cease to exist (Aragonès and Palfrey, 2002) and the shape of voter's utility functions also drives candidates closer or farther apart (Kamada and Kojima, 2014). Notice that all the described changes in the model's predictions are substantial and not "technically" driven. They continue to hold even if we allow candidates to use ε-best responses (e.g., Palfrey 1984) or if we consider discrete policy spaces (e.g., Aragonès and Palfrey 2002) given that the continuity of the policy space might make a candidate not have a well defined best response function.[7]

[6] See Grofman (2004) for a summary of possible extensions of the original Downsian model.

[7] This is true even in the original unidimensional Downsian model with two vote-share maximizing candidates: unless the second candidate is located at the median voter's ideal policy, the first player has no well-defined best response!

Fortunately, the main implications of our primaries' analysis seem to be particularly resilient to a number of changes in the assumptions. Importantly, one can consider a multidimensional policy space and/or concave utility functions and expect similar dynamics to the ones presented in our model. Indeed, Ansolabehere and Snyder (2000) show that when two candidates of different valence compete in a multidimensional policy space and voters' have Euclidian preferences over policies then –if the valence difference is substantial but not necessarily extremely large– there exists a set of policy platforms that secure her election. In other words, if primaries are held in such a multidimensional environment, the high valence candidate can choose the primary winner location that will give her the largest probability of winning the general election –precisely as she does in our unidimensional setup. Similarly, for concave utility functions a set of winner locations for the high valence candidate in each primary election is guaranteed to exist and each of the high valence candidates will locate there trying to approach the other as much as possible.[8] Of course, in many variants of our original model with generalized policy spaces and voters' loss functions, it might be the case that the set of winner locations is not closed, and hence that best responses are not well defined. We stress though that such issues are only of technical nature and can be effectively dealt with by considering either ε-best responses (e.g., Palfrey 1984) or discrete policy spaces (e.g., Aragonès and Palfrey 2002). Indeed, if the sets of winner locations for the high valence candidates of the two parties are sufficiently apart, the only reasonable thing to expect is that high valence candidates will locate within this set and as "close" as possible to the set of winner locations of the other party. Naturally, these dynamics may collapse if these sets are close to each other and/or when valence differences between primary candidates are not very large. These arguments, though, strongly suggest that primaries are not a stabilizing force only in the context of the unidimensional model, and that their appeal is quite wider.

Also, if one attempts to consider alternative objective functions one is lead to similar conclusions. If, for example, candidates lexicographically maximize, first, the probability of winning the general election and, then, their primary vote share, it is evident that the low valence candidates will have incentives to marginally deviate –from their posited strategy in the closed primaries' equilibrium– towards the extremes to increase their primary vote share (since the probability that they will win the general election is zero). But if we consider, for example, a discrete policy space –composed of any number of policy alternatives– the resulting equilibrium profile will

[8] In the presence of valence asymmetries, convex utility functions might generate disconnected sets of supporters for a given candidate (e.g. both leftist and rightist extremists might prefer the first candidate, while moderate voters might prefer the second candidate). This would generate substantial changes in the incentive structure of electoral competition and is never the case with concave utilities.

be essentially identical to the one that we have characterized: the high valence primary candidate will locate to the most moderate location that guarantees that the median of her party will vote for her, and the low valence primary candidate of the leftist (rightist) party will propose the policy alternative that is to the left (right) of her party's median voter. Hence, despite the fact that providing formal results for all reasonable variations of the model is beyond the scope of the current analysis, we find strong indications that the substantial implications of the analysis qualify to alternative and more general settings.

Finally, even if one allows for mixed strategies it turns out that our unique pure strategy equilibrium describes, to all effects, the only reasonable outcome of the game.[9] Indeed, while mixed equilibria exist, they all involve the two high valence candidates selecting the pure strategy of the unique pure strategy equilibrium and the two low quality candidates using mixed strategies that assign sufficiently large probability to the ideal policy of their parties' medians. That is, outcome wise, they are equivalent: the winners of the primary elections and of the general election are unchanged. This is so because, as long as the low valence candidates locate at the pure strategy equilibrium location with large enough probability, the high valence candidates opt for the pure strategy equilibrium location. The reason why there are no "interesting" mixed equilibria (that is, equilibria in which the high valence candidates mix too) is the following: If a high valence candidate mixes in an equilibrium, the support of her mixed strategy should include only undominated policies, and this should rule out policies more extreme than her platform in the unique pure strategy equilibrium. Moreover, if a high valence candidate plays more moderate policies than her strategy in the unique pure equilibrium with positive probability, the equilibrium payoff of the low valence candidate of her party should be strictly positive (because otherwise she could deviate to the party median and secure a strictly positive expected vote share in the general election), and this suggests that the support of the mixed strategy of a low valence candidate cannot contain policies more extreme than the ideal policy of her party's median voter. Notice though that when the high valence candidate, e.g., B, finds herself located at the most moderate point of her mixed strategy's support, say $x'' > l$, then: a) in case the low valence candidate locates at $x'' - v_B + v_A$ or to its right (assume this happens with probability $p > 0$), B wins the primary election; and b) in case the low valence candidate does not locate at $x'' - v_B + v_A$ or to its right, the payoff of B is zero. But since A has never an incentive to locate at $x'' - v_B + v_A$ or to its right because she gets a payoff zero for all possible realizations of B's mixed strategy, it must be the case that

[9] The arguments regarding mixed strategy equilibria apply also to the main interpretation of our model presented in the manuscript where candidates maximize their general election vote share in the absence of any kind of uncertainty regarding the location of the median voter.

$p = 0$. This leads to the paradox of B receiving a zero payoff when the realization of her mixed strategy is the most moderate policy in its support. Hence, there is no equilibrium in which a high valence candidate places any positive probability to any policy other than the one used in the unique pure equilibrium of the game.

2 Proofs

Proof of Proposition 1. This is a four-player, asymmetric and discontinuous game and hence not only there is no standardized way to characterize a unique equilibrium but even existence of an equilibrium is not trivially guaranteed. We will establish our result first by characterizing properties of an equilibrium and identifying a unique strategy profile that satisfies all of them (Step 1: Characterization and uniqueness of an equilibrium). We will argue that this strategy profile is indeed a Nash equilibrium (Step 2: Existence of an equilibrium). Recall that for the case of closed primaries the result is stated for valence differences not being very large, that is v_B is such that $l < m - v_B < m + v_B < r$.

Step 1 In this part of the proof we establish a number of properties that a Nash equilibrium, $\hat{x}$, should satisfy, employing a series of Lemmas. All statements are presented assuming the existence of at least one equilibrium.

Lemma 1: In any Nash equilibrium, $\hat{x}$, B gets a strictly positive expected vote share in the general election.

Proof: Assume that in some equilibrium, $\hat{x}$, B gets an expected vote share equal to zero in the general election. Then, if B deviates to $x_B = l$, she wins her party's primary with certainty. This is so because: a) $v_B > v_A$ and b) for every $x_A > l - v_B + v_A$ it is true that $u_i(x_A, v_A) < u_i(l, v_B)$ for every $i < l + v_B - v_A$ and for every $x_A < l + v_B - v_A$ it is true that $u_i(x_A, v_A) < u_i(l, v_B)$ for every $i > l - v_B + v_A$. That is, for every $x_A \in [0, m]$ a majority of the primary voters of the leftist party votes for B. Moreover, since $v_B > \max\{v_A, v_C, v_D\}$, B is voted in the general election at least by all voters with an ideal policy in $(l-\varepsilon, l+\varepsilon)$ for some $\varepsilon > 0$. That is, her general election vote share is at least equal to $\Phi(l+\varepsilon) - \Phi(l-\varepsilon)$, which is strictly positive by the fact that Φ is strictly increasing. In other words, B has incentives to deviate from the posited strategy and, hence, a strategy profile such that B gets an expected vote share equal to zero in the general election, cannot be an equilibrium. $\square$

Lemma 2: In any Nash equilibrium, $\hat{x}$, B wins the primary of the leftist party with certainty.

Proof: Since by Lemma 1 in any Nash equilibrium B gets a strictly positive expected vote share in the general election, it must be the case that B advances to the general election with a strictly positive probability. Given that candidates are allowed to use only pure strategies, there are two cases: either B advances to the general election with certainty or with probability $\frac{1}{2}$. Candidate B advances to the general election with probability $\frac{1}{2}$ if and only if she receives exactly the same share of votes in the primaries of the leftist party as A. This may happen if and only if $|x_A - x_B| > v_B - v_A$ and either $\frac{x_A+v_A+x_B-v_B}{2} = l$ or $\frac{x_A-v_A+x_B+v_B}{2} = l$. In both cases the candidate located to the left of l expects a positive vote share conditional on qualifying to the general election (by the fact that valence differences are not very large) and can deviate marginally towards l, securing a sure win in the primaries and practically doubling her expected general election vote share. Hence, it cannot be that in equilibrium B advances to the general election with any probability smaller than 1. $\square$

The proof of the next lemma is similar to the proof of Lemma 1 –but not precisely identical. To make the proof as easy to follow as possible, we preferred to provide the whole line of reasoning in its support (which is, to a great extent, a repetition of the arguments in the proof of Lemma 1), rather than to just point out where one should make the minor modifications.

Lemma 3: In any Nash equilibrium, $\hat{x}$, C gets a strictly positive expected vote share in the general election.

Proof: Assume that in some equilibrium, $\hat{x}$, C gets a zero expected vote share in the general election. Then, if C deviates to $x_C = r$, she wins her party's primary with certainty. This is so because: a) $v_C > v_D$ and b) for every $x_D > r - v_C + v_D$ it is true that $u_i(x_D, v_D) < u_i(r, v_C)$ for every $i < r + v_C - v_D$ and for every $x_D < r + v_C - v_D$ it is true that $u_i(x_D, v_D) < u_i(r, v_C)$ for every $i > r - v_C + v_D$. That is, for every $x_D \in [m, 1]$ a majority of the primary voters of the rightist party votes for C. Moreover, since valence differences are not very large, C is voted in the general election at least by all voters with an ideal policy in $(r - \varepsilon, r + \varepsilon)$ for some $\varepsilon > 0$. That is, her general election vote share is at least equal to $\Phi(r + \varepsilon) - \Phi(r - \varepsilon)$, which is strictly positive by the fact that Φ is strictly increasing. In other words, C has incentives to deviate from the posited strategy and, hence, a strategy profile such that C gets an expected vote share equal to zero in the general election, cannot be an equilibrium. $\square$

The arguments supporting Lemma 4 are symmetric to the ones supporting Lemma 2 and are, hence, skipped.

Lemma 4: In any Nash equilibrium, $\hat{x}$, C wins the primary of the rightist party with certainty.

Lemma 5: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_A = \hat{x}_B - v_B + v_A$.

Proof: Consider first that, in equilibrium, $\hat{x}_A < \hat{x}_B - v_B + v_A$. Since by Lemma 2 it is the case that B wins with certainty the leftist party's primaries, it should be the case that $\frac{\hat{x}_A + v_A + \hat{x}_B - v_B}{2} < l$. By Lemmas 3 and 4 we know that C advances to the general election with certainty and that she receives there a strictly positive vote share. Hence, there exists $\varepsilon > 0$ such that if B deviates to $\hat{x}_B + \varepsilon$ it will still be the case that $\frac{\hat{x}_A + v_A + \hat{x}_B + \varepsilon - v_B}{2} < l$ (B wins the leftist party's primary with certainty) and moreover B will secure a strictly positive increase in her general election vote share.[10] So in equilibrium it cannot be the case that $\hat{x}_A < \hat{x}_B - v_B + v_A$. Now consider

[10]Since in any equilibrium, $\hat{x}$, C qualifies to the general election with certainty (Lemma 4) and expects a positive general election

that, in equilibrium, $\hat{x}_A > \hat{x}_B - v_B + v_A$. If $\hat{x}_A > \hat{x}_B + v_B - v_A$ then, given that B wins with certainty the leftist party's primaries (by Lemma 2), it should be the case that $\frac{\hat{x}_A - v_A + \hat{x}_B + v_B}{2} > l$. Again, there exists $\varepsilon > 0$ such that if B deviates to $\hat{x}_B + \varepsilon$ it will still be the case that $\frac{\hat{x}_A - v_A + \hat{x}_B + \varepsilon + v_B}{2} > l$ (B wins the leftist party's primaries with certainty) and moreover B will secure a strictly positive increase in her general election vote share. So in equilibrium it cannot be the case that $\hat{x}_A > \hat{x}_B - v_B + v_A$ either. $\square$

Lemma 6: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_A = l$.

Proof: Consider first that $\hat{x}_A < l$. Then by Lemma 5 it follows that $\hat{x}_B = \hat{x}_A + v_B - v_A < l + v_B - v_A$. This suggests that there exists $\varepsilon > 0$ such that if B deviates to $\hat{x}_B + \varepsilon$ it will be the case that $\frac{\hat{x}_A + v_A + \hat{x}_B + \varepsilon - v_B}{2} < l$ (B wins the leftist party's primary with certainty) and moreover B will secure a strictly positive increase in her general election vote share (applying the same reasoning as footnote 10). Hence, in equilibrium, it cannot be that $\hat{x}_A < l$. Now consider that $\hat{x}_A > l$. By Lemma 5 it follows that $\hat{x}_B = \hat{x}_A + v_B - v_A > l + v_B - v_A$. This suggests that A can deviate to $\hat{x}_A = l$, win her party's primary with certainty and secure a strictly positive vote share in the general election (by the fact that valence differences are not very large). Hence, in equilibrium, it cannot be that $\hat{x}_A > l$ either. $\square$

Lemma 7: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_B = l + v_B - v_A$.

Proof: This is a trivial implication of Lemmas 5 and 6. $\square$

Lemma 8: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_C = r - v_C + v_D$ and $\hat{x}_D = r$.

Proof: Consider a Nash equilibrium, $\hat{x}$. Since: a) by Lemma 4, C wins with certainty the primary of the rightist party, b) by Lemmas 6 and 7 $\hat{x}_A = l$ and $\hat{x}_B = l + v_B - v_A$, and c) valence differences are not very large; it must be the case that $\hat{x}_C \in [r - v_C + v_D, r + v_C - v_D]$, because when C makes such a choice, D does not qualify to the general election with a positive probability for any $x_D \in [m, 1]$. Otherwise –that is, if $\hat{x}_C \notin [r - v_C + v_D, r + v_C - v_D]$– D could locate at r, win the primary of the rightist party with certainty and secure a positive vote share in the general election. Moreover, by the fact that $\hat{x}_B < m$ and that valence differences are not very large, the general election vote share of C, conditional on C qualifying to the general election, is strictly decreasing on $[r - v_C + v_D, r + v_C - v_D]$. Hence, in any equilibrium, $\hat{x}$, it is the case that $\hat{x}_C = r - v_C + v_D$. Finally, if in equilibrium $\hat{x}_C = r - v_C + v_D$ and $\hat{x}_D \neq r$, then C can deviate to $\hat{x}_C = r - v_C + v_D - \varepsilon$ for some $\varepsilon > 0$, and thus, qualify to the general election with certainty and increase her

vote share (Lemma 3), we must have either $\hat{x}_B + v_B - v_C < \hat{x}_C$ or $\hat{x}_B + v_B - v_C = \hat{x}_C$. In both cases a marginal move of B towards the right induces an increase in her general election vote share: in the first case, only a marginal increase, and in the latter a substantial one.

general election vote share (since $\hat{x}_B < m$ and valence differences are not very large). So, if in equilibrium we have $\hat{x}_C = r - v_C + v_D$, we should also have $\hat{x}_D = r$. $\square$

Hence, by Lemmas 6, 7 and 8, it follows that there exists a unique strategy profile that is a candidate for an equilibrium in our game; $\hat{x} = (l, l + v_B - v_A, r - v_C + v_D, r)$.

Step 2 In this part of the proof we have to show that no candidate has incentives to deviate from $\hat{x} = (l, l + v_B - v_A, r - v_C + v_D, r)$. First of all, note that for this profile, in each primary the party's median voter is indifferent between the high and low valence candidate. This is also true for all more extreme voters than the median that split their support between the two primary candidates equally. All more more moderate voters than each party's median strictly prefer the high valence candidate over the low valence candidate and hence candidates B and C are the ones competing in the general election. Given that valence differences are not very large, in this strategy profile vote shares in the general election are hence $P_A(\hat{x}_A, \hat{x}_{-A} : v, \Phi, l, r) = P_D(\hat{x}_D, \hat{x}_{-D} : v, \Phi, l, r) = 0$ and $P_B(\hat{x}_B, \hat{x}_{-B} : v, \Phi, l, r) = 1 - P_C(\hat{x}_C, \hat{x}_{-C} : v, \Phi, l, r) = \Phi(\frac{\hat{x}_B + v_B + \hat{x}_C - v_C}{2})$.

Candidate A will have incentives to deviate if there exists $\acute{x}_A \in [0, m]$ such that $P_A(\acute{x}_A, \hat{x}_{-A} : v, \Phi, l, r) > 0$. But if A deviates to $\acute{x}_A < l$ then $\frac{\acute{x}_A + v_A + \hat{x}_B - v_B}{2} < l$ (B wins the primary of the leftist party with certainty) and hence $P_A(\acute{x}_A, \hat{x}_{-A} : v, \Phi, l, r) = 0$. If A deviates to $\acute{x}_A > l$ then at least all voters with ideal policies in $[0, l + v_B - v_A)$ will vote for candidate B in the primary of the leftist party and, hence, B will win the primary of the leftist party with certainty, inducing $P_A(\acute{x}_A, \hat{x}_{-A} : v, \Phi, l, r) = 0$. Therefore, candidate A has no incentives to deviate away from $\hat{x}_A = l$. Similar arguments rule out incentives for deviation away from $\hat{x}_D = r$ for candidate D.

If candidate B deviates to $\acute{x}_B < l + v_B - v_A$ then, even if she wins in the primary of the leftist party, she gets a vote share of $\Phi(\frac{\acute{x}_B + v_B + \hat{x}_C - v_C}{2})$ that is strictly smaller than $\Phi(\frac{\hat{x}_B + v_B + \hat{x}_C - v_C}{2})$. That is, her payoff is strictly smaller than $P_B(\hat{x}_B, \hat{x}_{-B} : v, \Phi, l, r)$. If candidate B deviates to $\acute{x}_B > l + v_B - v_A$ then $\frac{\hat{x}_A + v_A + \acute{x}_B - v_B}{2} > l$ and hence A wins the primary of the leftist party with certainty. That is, $P_B(\acute{x}_B, \hat{x}_{-B} : v, \Phi, l, r) = 0$. Therefore, candidate B has no incentives to deviate from $\hat{x}_B = l + v_B - v_A$. Similar arguments rule out incentives for deviation away from $\hat{x}_C = r - v_C + v_D$ for candidate C. $\blacksquare$

Proof of Corollary 1. In the unique equilibrium characterized in Proposition 1 B is the winner if $\Phi(\frac{\hat{x}_B + v_B + \hat{x}_C - v_C}{2}) > \frac{1}{2} \iff l + 2v_B - v_A + r - 2v_C + v_D > 2m$, C is the winner if $\Phi(\frac{\hat{x}_B + v_B + \hat{x}_C - v_C}{2}) < \frac{1}{2} \iff l + 2v_B - v_A + r - 2v_C + v_D < 2m$ and each of these candidates wins with probability $\frac{1}{2}$ if $\Phi(\frac{\hat{x}_B + v_B + \hat{x}_C - v_C}{2}) = \frac{1}{2} \iff$

$l + 2v_B - v_A + r - 2v_C + v_D = 2m.$ ■

Proof of Proposition 2. As in the Proof of Proposition 1 in Step 1 we identify a unique strategy profile that satisfies the equilibrium properties and then show in Step 2 that this profile is indeed an equilibrium. The proof is similar to Proposition 1, however further arguments are needed since by permitting open primaries each deviation implies a change in the primary electorates. Recall that in this case valence differences not being very large means that $v_B \in (0, \tilde{v}_B)$ for some $\tilde{v}_B > 0$.

Step 1 Additional to similar arguments as in the case of closed primaries we now start the proof by showing the existence of a unique pair $(l^*, r^*) \in (0,1)^2$. Again, in this Step all statements are presented assuming the existence of at least one equilibrium.

Lemma 9: For every log-concave, F, there exists $\tilde{v}_B > 0$ such that for every $v_B \in (0, \tilde{v}_B)$ a unique pair $(l^*, r^*) \in (0,1)^2$ exists such that: a) $2F(l^*) = F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2})$; b) $2[1-F(r^*)] = 1 - F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2})$; and c) $l^* < m^a - 2v_B < m^a + 2v_B < r^*$.

Proof: Since $m^a \in (0,1)$ there exists $\acute{v}_B > 0$ such that for every $v_B \in (0, \acute{v}_B)$ we have $0 < m^a - 2v_B < m^a + 2v_B < 1$. Consider that $v_B \in (0, \acute{v}_B)$ and let $q \in [0,1]$ and $(\bar{l}, \bar{r}) \in [0,1]^2$ be such that $2F(\bar{l}) = F(q)$ and $2[1 - F(\bar{r})] = 1 - F(q)$. When $q = 0$ we have $(\bar{l}, \bar{r}) = (0, m^a) \implies \frac{\bar{l}+v_B-v_A+\bar{r}-v_C+v_D}{2} + \frac{v_B-v_C}{2} = \frac{m^a+v_B-v_A-v_C+v_D}{2} + \frac{v_B-v_C}{2} > 0.$[11] When $q = 1$ we have $(\bar{l}, \bar{r}) = (m^a, 1) \implies \frac{m^a+1+v_B-v_A-v_C+v_D}{2} + \frac{v_B-v_C}{2} < 1.$[12] Moreover, given that both $\frac{\partial \bar{l}}{\partial q} \in (0,1)$ and $\frac{\partial \bar{r}}{\partial q} \in (0,1)$ - these derivatives are trivially positive and smaller than one due to log-concavity of F (see, Le Breton and Weber 2003, 2005)-, there exists a unique $q \in (0,1)$ for which $\frac{\bar{l}+v_B-v_A+\bar{r}-v_C+v_D}{2} + \frac{v_B-v_C}{2} = q$. That is, there exists a unique pair $(l^*, r^*) \in (0,1)^2$ for which $2F(l^*) = F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2})$ and $2[1-F(r^*)] = 1 - F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2} + \frac{v_B-v_C}{2})$. Notice that $\lim_{v_B \to 0} l^* = F^{-1}(\frac{1}{4}) \in (0, m^a)$. Hence, by continuity of $(l^*, r^*) \in (0,1)^2$ in candidates' quality parameters –as long as $v_B \in (0, \acute{v}_B)$– it follows that there exists $\tilde{v}_B > 0$ such that for every $v_B \in (0, \tilde{v}_B)$ we have $l^* < m^a - 2v_B < m^a + 2v_B < r^*$. □

Lemma 10: In any Nash equilibrium, $\hat{x}$, B gets a strictly positive expected vote share in the general election.

Proof: Assume that in some equilibrium, $\hat{x}$, B gets an expected vote share equal to zero in the general election. Then, if B deviates to $x_B = \hat{x}_A$, and given that $v_B > \max\{v_A, v_C, v_D\}$ she wins her party's primary with certainty and is voted in the general election at least by all voters with an ideal policy in $[\hat{x}_A, \hat{x}_A + \varepsilon)$ for

[11] This is implied by the fact that $m^a - 2v_B > 0$, $v_B > v_A > 0$, $v_C > v_D > 0$ and $v_B > v_C$.

[12] This is implied by the fact that $m^a + 2v_B < 1$, $v_B > v_A > 0$, $v_C > v_D > 0$ and $v_B > v_C$.

some $\varepsilon > 0$. $\square$

Lemma 11: In any Nash equilibrium, $\hat{x}_B > \hat{x}_A$.

Proof: Assume that in some equilibrium, $\hat{x}$, $\hat{x}_B \leq \hat{x}_A$. If $\hat{x}_B \leq \hat{x}_A$ are such that A wins the primary with certainty then it must be the case that $\hat{x}_B < \hat{x}_A$ and B has incentives to deviate to $\hat{x}_A$ that guarantees a primary victory for B and a positive expected vote share in the general election (Lemma 10). If $\hat{x}_B \leq \hat{x}_A$ are such that B wins the primary with certainty and $\hat{x}_D < \hat{x}_B + v_B - v_D$ and $\hat{x}_C < \hat{x}_B + v_B - v_C$ then for example C can deviate to $x_C = 1$ and increase her general election vote share. If at least one of $\hat{x}_D < \hat{x}_B + v_B - v_D$ and $\hat{x}_C < \hat{x}_B + v_B - v_C$ is violated then a deviation by B to $\hat{x}_B + \varepsilon$ for some $\varepsilon > 0$ guarantees a primary victory for B and an increase in the expected vote share in the general election. If $\hat{x}_B \leq \hat{x}_A$ are such that A and B tie in the primary then it must be the case that $\hat{x}_B < \hat{x}_A$ and a deviation by B to $\hat{x}_B + \varepsilon$ for some $\varepsilon > 0$ guarantees a primary victory for B, without reducing her general election vote share, therefore practically doubling her expected vote share in the general election. Hence there exists no equilibrium where $\hat{x}_B \leq \hat{x}_A$. $\square$

Lemma 12: In any Nash equilibrium, $\hat{x}$, B wins the primary of the leftist party with certainty.

Proof: Since in any Nash equilibrium, B gets a strictly positive expected vote share in the general election, it must be the case that B advances to the general election with a strictly positive probability. Given that candidates are allowed to use only pure strategies, there are two cases: either B advances to the general election with certainty or with probability $\frac{1}{2}$. Candidate B advances to the general election with probability $\frac{1}{2}$ if and only if she receives exactly the same share of votes in the primaries of the leftist party as A. Given that from Lemma 11 in equilibrium $\hat{x}_B > \hat{x}_A$ and the fact that A and B tie this may happen if and only if $\hat{x}_B > \hat{x}_A + v_B - v_A$.[13] Then there exists $\varepsilon > 0$ such that A deviates to $\hat{x}_A + \varepsilon$ hence wins the primary with certainty and marginally increases her general election vote share.[14] Hence, it cannot be that in equilibrium B advances to the general election with any probability smaller than 1. $\square$

Lemma 13: In any Nash equilibrium, $\hat{x}_D > \hat{x}_B + v_B - v_D$.

Proof: First, assume that in some equilibrium, $\hat{x}$, $\hat{x}_D < \hat{x}_B + v_B - v_D$ and one of the following three holds:

[13] This is true because if $\hat{x}_A < \hat{x}_B \leq \hat{x}_A + v_B - v_A$ the only way that A and B can tie is if no voters participate in the primary of the leftist party which never occurs given that $v_B > v_C$. Given that a positive measure of voters participates in the leftist primary then no tie can be sustained: If $\hat{x}_A < \hat{x}_B < \hat{x}_A + v_B - v_A$ all voters participating in the primary of the leftist party support B. If $\hat{x}_A < \hat{x}_B = \hat{x}_A + v_B - v_A$ again B qualifies with certainty since all voters to the left of $\hat{x}_A$ that participate in the leftist primary are indifferent between A and B and split their vote while all voters on the right of $\hat{x}_A$ that participate in the leftist primary vote for B.

[14] Notice that for A's general election vote share to increase it is sufficient that $\hat{x}_J \geq \hat{x}_A + \varepsilon + v_J - v_A$ for any $J = C, D$ which is always true given that $\hat{x}_B > \hat{x}_A + v_B - v_A$ and $\hat{x}_J > \hat{x}_B$ and $v_B > v_J$ for any $J = C, D$.

1. $\hat{x}_C < \hat{x}_B + v_B - v_C$: No voters participate in the open primary of the rightist party and hence each of C and D qualify to the general election with probability 1/2. However, C gets a zero expected vote share in the general election and hence has incentives to deviate for instance to $\hat{x}_B + v_B - v_C$ guaranteeing a primary victory and a positive expected vote share in the general election. Notice that this deviation is always feasible since by Lemma 9 we have that $m^a + 2v_B < 1$ and hence $m^a + v_B - v_C < 1$.

2. $\hat{x}_C = \hat{x}_B + v_B - v_C$: All voters participating in the open primary of the rightist party support candidate C and therefore C surely qualifies to the general election. In the general election all voters to the left of $\hat{x}_C$ vote for B and all voters to the right of $\hat{x}_C$ vote B or C with equal probability. Hence, C has incentives to the deviate to $\hat{x}_C + \varepsilon$ for some $\varepsilon > 0$ still guaranteeing a primary victory (D is still not voted by anyone in the primary) and practically doubling her general election vote share.

3. $\hat{x}_C > \hat{x}_B + v_B - v_C$: All voters participating in the open primary of the rightist party support candidate C and therefore C surely qualifies to the general election where she is encountering candidate B (Lemma 12). In the general election and since $\hat{x}_C > \hat{x}_B + v_B - v_C$, C is expecting a positive vote share. However, a deviation by C to $\hat{x}_C - \varepsilon$ for some $\varepsilon > 0$ is profitable since it still guarantees a primary victory and an increased vote share in the general election.

Hence, in equilibrium it can not be that $\hat{x}$, $\hat{x}_D > \hat{x}_B + v_B - v_D$.

Second, assume that in some equilibrium, $\hat{x}$, $\hat{x}_D = \hat{x}_B + v_B - v_D$ and one of the following three holds:

4. $\hat{x}_C < \hat{x}_B + v_B - v_C$: C does not get any votes in the primary and hence expects a zero vote share in the general election. A deviation of C to $\hat{x}_D$ guarantees a primary victory and a positive vote share.

5. $\hat{x}_C = \hat{x}_B + v_B - v_C$: Combining the fact that $\hat{x}_D = \hat{x}_B + v_B - v_D$ and $\hat{x}_C = \hat{x}_B + v_B - v_C$ we are in a situation where also $\hat{x}_D = \hat{x}_C + v_C - v_D$. In that instance $\hat{x}_A < \hat{x}_B < \hat{x}_C < \hat{x}_D$ with all voters on the right of $\hat{x}_D$ being indifferent among B,C and D. Given that by Lemma 11 $\hat{x}_A < \hat{x}_B$ and $v_A < v_B$ it can never be the case that all voters prefer A over B (and hence C and D) and therefor a positive measure of voters participates in the primary of the rightist party. That is, two thirds on the right of $\hat{x}_D$ (that are indifferent between C and D) and half of those between $\hat{x}_C$ and $\hat{x}_D$ (that prefer C over D) participate in the rightist primary. Therefor, candidate C wins the primary with certainty. Conditional on C qualifying to the general election, all general election voters on the right of $\hat{x}_C$ are indifferent between B and C

and therefore half of these voters constitute C's general election vote share. Hence, C has incentives to deviate to $\hat{x}_C + \varepsilon$ for some $\varepsilon > 0$ guaranteeing a) a sure primary victory, and b) a larger vote share in the general election. Points a) and b) together imply an increase in C's expected vote share in the general election and hence incentives to deviate.

6. $\hat{x}_C > \hat{x}_B + v_B - v_C$: If $\hat{x}_C$ and $\hat{x}_D$ are such that C wins the primary of the rightist party or C and D tie then a a deviation by C to $\hat{x}_C - \varepsilon$ for some $\varepsilon > 0$ is profitable since it guarantees a primary victory and an increased vote share in the general election. If $\hat{x}_C$ and $\hat{x}_D$ are such that D wins the primary of the rightist party then a a deviation by C to $\hat{x}_D = \hat{x}_B + v_B - v_D$ guarantees a primary victory for C and a positive expected vote share in the general election.

   Hence, in equilibrium it can not be that $\hat{x}$, $\hat{x}_D = \hat{x}_B + v_B - v_D$. Combined with the fact that $\hat{x}_D > \hat{x}_B + v_B - v_D$ can not be part of an equilibrium either we have shown that in equilibrium $\hat{x}_D < \hat{x}_B + v_B - v_D$.

   $\square$

Lemma 14: In any Nash equilibrium, $\hat{x}_C < \hat{x}_D$.

Proof: Assume that in some equilibrium, $\hat{x}$, $\hat{x}_C = \hat{x}_D$. Then given that in equilibrium by Lemma 13 $\hat{x}_D > \hat{x}_B + v_B - v_D$ some voters participate in the rightist primary that is won by C with certainty. Notice that $\hat{x}_D > \hat{x}_B + v_B - v_D$ also implies that C expects a positive vote share in the general election. C however has incentives to deviate to $\hat{x}_C - \varepsilon$ for some $\varepsilon > 0$ still winning the primary with certainty but also increasing her expected general election vote share. Hence it can never hold that $\hat{x}_C = \hat{x}_D$.

Assume now that in some equilibrium, $\hat{x}$, $\hat{x}_C > \hat{x}_D$. Recall that by Lemma 13 $\hat{x}_D > \hat{x}_B + v_B - v_D$. Hence, if C loses the primary election a deviation to $\hat{x}_D$ guarantees a victory in the rightist primary and a positive expected vote share in the general election. If C wins the primary election or there is a tie between C and D again deviation to $\hat{x}_D$ guarantees a primary victory and an increase in C's expected vote share in the general election. Hence it can never hold that $\hat{x}_C > \hat{x}_D$ either implying that in equilibrium $\hat{x}_C < \hat{x}_D$. $\square$

Lemma 15: In any Nash equilibrium, $\hat{x}$, C gets a strictly positive expected vote share in the general election.

Proof: Assume that in some equilibrium, $\hat{x}$, C gets a zero expected vote share in the general election. Since B competes in the general election with certainty, and $x_B \leq x_C$, C obtains a zero expected vote share if $\hat{x}_C < \hat{x}_B + v_B - v_C$. But this can not be an equilibrium since C has incentives to deviate. As we have shown in Lemma 13 in equilibrium $\hat{x}_D > \hat{x}_B + v_B - v_D$. Hence, a deviation of C to $x_C = \hat{x}_D$ guarantees a win in the rightist primary and a positive expected expected vote share in the general election. □

Lemma 16: In any Nash equilibrium, $\hat{x}$, $\hat{x}_C > \hat{x}_B + v_B - v_C$.

Proof: Notice that by Lemma 15 it follows that $\hat{x}_C \geq \hat{x}_B + v_B - v_C$. Hence to show that $\hat{x}_C > \hat{x}_B + v_B - v_C$ we need to prove that $\hat{x}_C = \hat{x}_B + v_B - v_C$ can never be part of an equilibrium. Assume that in equilibrium $\hat{x}_C = \hat{x}_B + v_B - v_C$. Then all voters on the right of $\hat{x}_C$ are indifferent between B and C. Hence, if C were to qualify on the general election its vote share would be equal to half of the voters on the right of B. Whether C qualifies to the general election depends on $\hat{x}_C$ and $\hat{x}_D$. If $\hat{x}_C$ and $\hat{x}_D$ are such that D qualifies with certainty to the general election then $\hat{x}_D > \hat{x}_C + v_C - v_D$.[15] In that case, C has incentives to deviate to $\hat{x}_D$ and qualify to the general election and obtain a positive expected vote share. If $\hat{x}_C$ and $\hat{x}_D$ are such that C qualifies with certainty then C has incentives to deviate to $\hat{x}_C + \varepsilon$ for some $\varepsilon > 0$ still winning the primary and almost doubling her expected general election vote share. Finally, if $\hat{x}_C$ and $\hat{x}_D$ are such that C qualifies with probability $1/2$ then $\hat{x}_D > \hat{x}_C + v_C - v_D$. In that case, the voters that participate in the primary of the rightist party are half of those between $\hat{x}_C$ and $\frac{\hat{x}_C + v_C + x_D - v_D}{2}$ (and all of them prefer C over D) and all voters on the right of $\frac{\hat{x}_C + v_C + x_D - v_D}{2}$ (and all of them prefer D over C). Then D has incentives to deviate to $\hat{x}_D - \varepsilon$ for some $\varepsilon > 0$ securing a primary victory while increasing her general election vote share. Hence, $\hat{x}_C = \hat{x}_B + v_B - v_C$ can not be part of an equilibrium proving that $\hat{x}_C > \hat{x}_B + v_B - v_C$. □

Lemma 17: In any Nash equilibrium, $\hat{x}$, C wins the primary of the rightist party with certainty.

Proof: Given that by Lemmas 11 and 14, $x_B > x_A$, $x_C < x_D$, B qualifies with certainty to the primary and C gets a strictly positive expected vote share in the general election the arguments proving this Lemma are symmetric to the ones of Lemma 12. □

Lemma 18: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_A = \hat{x}_B - v_B + v_A$.

Proof: Consider first that, in equilibrium, $\hat{x}_A < \hat{x}_B - v_B + v_A$. Since by Lemma 12 B wins with certainty the leftist party's primaries, it should be the case that $\frac{\hat{x}_A + v_A + \hat{x}_B - v_B}{2} < \tilde{l}$. By Lemmas 17 and 16 we know that C advances to the general election with certainty and that she receives there a strictly positive vote share. Hence, there exists $\varepsilon > 0$ such that if B deviates to $\hat{x}_B + \varepsilon$ it will still be the case that $\frac{\hat{x}_A + v_A + \hat{x}_B + \varepsilon - v_B}{2} < \tilde{l}$ (B wins the leftist party's primary with certainty) and moreover B will secure a strictly positive increase in her general

[15] For D to qualify with certainty there must be a positive measure of voters participating in the rightist primary (otherwise each of C and D qualify with equal probability). Notice that if $\hat{x}_C < \hat{x}_D < \hat{x}_C + v_C - v_D$ then D never qualifies to the general election with certainty since all primary voters vote for C. If $\hat{x}_C < \hat{x}_D = \hat{x}_C + v_C - v_D$ we are in situation 5 of Lemma 13 where C qualifies to the general election with certainty and hence again D can not qualify with certainty.

election vote share. So in equilibrium it cannot be the case that $\hat{x}_A < \hat{x}_B - v_B + v_A$. Now consider that, in equilibrium, $\hat{x}_A > \hat{x}_B - v_B + v_A$. Notice that in the primary of the leftist party A obtains zero votes and all primary voters (non-empty set given that v_B is the highest valence candidate) vote for candidate B. Hence, there exists $\varepsilon > 0$ such that if B deviates to $\hat{x}_B + \varepsilon$ it will still win the primary election with certainty while also increasing her general election vote share. So in equilibrium it cannot be the case that $\hat{x}_A > \hat{x}_B - v_B + v_A$ either. $\square$

By Lemmas 14, 16 and 18 it follows that in equilibrium only a voter with ideology $\frac{\hat{x}_B+v_B+\hat{x}_C-v_C}{2}$ is indifferent between participating in the leftist primary and the rightist one: all voters with an ideal policy to his left constitute the leftist primary electorate and all voters to his right constitute the rightist primary electorate. Hence, we can define the median of the leftist party as $\tilde{l}$ being the unique solution to $2F(\tilde{l}) = F(\frac{\hat{x}_B+v_B+\hat{x}_C-v_C}{2})$ and the median of the rightist party as $\tilde{r}$ being the unique solution to $2[1 - F(\tilde{r})] = 1 - F(\frac{\hat{x}_B+v_B+\hat{x}_C-v_C}{2})$.

Lemma 19: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_A = \tilde{l}$

Proof: Consider first that $\hat{x}_A < \tilde{l}$. Then by Lemma 18 it follows that $\hat{x}_B = \hat{x}_A + v_B - v_A < \tilde{l} + v_B - v_A$. This suggests that there exists $\varepsilon > 0$ such that if B deviates to $\hat{x}_B + \varepsilon$ it will be the case that $\frac{\hat{x}_A+v_A+\hat{x}_B+\varepsilon-v_B}{2} < \tilde{l}$ (B wins the leftist party's primary with certainty) and moreover B will secure a strictly positive increase in her general election vote share (Lemmas 13 and 16). Hence, in equilibrium, it cannot be that $\hat{x}_A < \tilde{l}$. Now consider that $\hat{x}_A > \tilde{l}$. By Lemma 18 it follows that $\hat{x}_B = \hat{x}_A + v_B - v_A > \tilde{l} + v_B - v_A$. This suggests that A can deviate to $\hat{x}_A = \tilde{l}$, win her party's primary with certainty and secure a strictly positive vote share in the general election. Hence, in equilibrium, it cannot be that $\hat{x}_A > \tilde{l}$ either. $\square$

Lemma 20: In equilibrium, $\hat{x}_A = l^*$, $\hat{x}_B = l^* + v_B - v_A$, $\hat{x}_C = r^* - v_C + v_D$ and $\hat{x}_D = r^*$.

Proof: Given that in equilibrium $\hat{x}_A = \tilde{l}$ and $\hat{x}_A = \tilde{l}+v_B-v_A$ (Lemmas 18 and 19) and that C qualifies to the general election with certainty (Lemma 17), it must be the case that the expected payoff of D is zero. Moreover, since $\hat{x}_C < \hat{x}_D$ (by Lemma 14), $\hat{x}_C$ must belong to $[\tilde{r}-v_C+v_D, \tilde{r}+v_C-v_D]$, otherwise: a) if $\hat{x}_C < \tilde{r}-v_C+v_D$, D could deviate to $\hat{x}_C + v_C - v_D + \epsilon$ for some $\epsilon > 0$, qualify to the general election with certainty and strictly increase his expected payoff; and b) if $\hat{x}_C > \tilde{r} + v_C - v_D$, D could deviate to $\hat{x}_C - v_C + v_D - \epsilon$ for some $\epsilon > 0$, qualify to the general election with certainty and strictly increase his expected payoff. Notice that for valence differences that are not very large, $\tilde{l}$ is always substantially smaller than m^a and hence $\tilde{l} + v_B - v_C < m^a$ and $m^a - v_C + v_D < \tilde{r}$. Finally, for $\hat{x}_C < \hat{x}_D$, $\tilde{r}$ and C's general election vote share (conditional on qualifying to the general election) increases continuously as C moves smoothly from $\hat{x}_C$ to m^a, so the only candidate for $\hat{x}_C = \tilde{r} - v_C + v_D$. From this it follows that the unique candidate for $\hat{x}_D = \tilde{r}$ (otherwise there exists $\epsilon > 0$ such that C increases her expected payoff by deviating to $\tilde{r} - v_C + v_D + \epsilon$). If $\hat{x}_B = \tilde{l} + v_B - v_A$ and $\hat{x}_C = \tilde{r} - v_C + v_D$ then the indifferent voter of the general election lies at $\frac{\tilde{l}+v_B-v_A+\tilde{r}-v_C+v_D}{2} + \frac{v_B-v_C}{2}$, and hence $\tilde{l} = l^*$ and $\tilde{r} = r^*$. $\square$

Step 2 Finally, we argue that any unilateral deviation from the posited profile is unprofitable. Given that this profile is such that A and D locate exactly at the median of their party and B and C as close to the society's median voter as possible (that is, as long as they win the primaries with certainty), nobody has an incentive to deviate to a more extreme policy. But deviations towards more moderate policies cannot be trivially ruled out. Assume that candidate B deviates towards the centre. Then, indeed she looses primary votes from the left but she gains primary votes from the right. For this deviation to be unprofitable we need that the gain from the right is smaller when compared to the loss from the left. Consider that B deviates to $x'_B \in (l^* + v_B - v_A, m^a]$. Since for valence differences not very large it is the case that $m^a + 2v_B < r^*$, it follows that the primary vote share of B is $V(x'_B) = \frac{F(\frac{x'_B+v_B+\hat{x}_C-v_C}{2})-F(\frac{x'_B-v_B+\hat{x}_A+v_A}{2})}{F(\frac{x'_B+v_B+\hat{x}_C-v_C}{2})}$. We notice that: a) $\lim_{x'_B \to (l^*+v_B-v_A)^+} V(x'_B) = \frac{F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2}+\frac{v_B-v_C}{2})-F(l^*)}{F(\frac{l^*+v_B-v_A+r^*-v_C+v_D}{2}+\frac{v_B-v_C}{2})} = \frac{1}{2}$; and b) $\frac{\partial V(x'_B)}{\partial x'_B} < 0$ for every $x'_B \in (l^* + v_B - v_A, m^a]$ due to log-concavity of F. In other words, if B deviates to any $x'_B \in (l^* + v_B - v_A, m^a]$, she loses from A in the primary and gets a payoff equal to zero. Similarly, one can show that A has no profitable deviations to the left of the posited strategy. The arguments that prove that C and D have no incentives to change policy platforms are symmetric. $\blacksquare$

Proof of Proposition 3.

This Proof is similar to the one of Proposition 1. Given that a primary election takes place only in the leftist party and C competes in the general election for sure some supporting arguments need some small modifications.

Lemma 21: In any Nash equilibrium, $\hat{x}$, B gets a strictly positive expected vote share in the general election.

Proof: The arguments needed to prove this Lemma are similar to the ones proving Lemma 1. If in equilibrium B gets an expected vote share equal to zero in the general election, B can deviate to $x_B = l$ and for the same arguments as in Lemma 1 win her her party's primary with certainty. However, given that $v_B > max\{v_A, v_C\}$ need not be true any longer, to show that for this deviation B gets a positive general election vote share where she competes against C, it is sufficient to show that $x_B = l \leq x_C + v_B - v_C$. The latter always holds when valence differences are not very large (i.e., $m + max\{v_B, v_C\} < x_C$). □

Lemma 22: In any Nash equilibrium, $\hat{x}$, B wins the primary of the leftist party with certainty.

Proof: The arguments needed to prove this Lemma are identical to the ones proving Lemma 2. □

Lemma 23: In any Nash equilibrium, $\hat{x}$, C gets a strictly positive expected vote share in the general election.

Proof: To show that C gets a strictly positive vote share against B in the general election it is sufficient to have that $x_C \geq \hat{x}_B + v_B - v_C$. The latter holds given that valence differences are not very large (i.e., $m + max\{v_B, v_C\} < x_C$) and $\hat{x}_B \in [0, m]$. □

Lemma 24: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_A = \hat{x}_B - v_B + v_A$.

Proof: The arguments needed to prove this Lemma are identical to the ones proving Lemma 5 (without the need to refer to the Lemma showing that C wins the rightist primary given that here C qualifies to the general election by assumption). □

Lemma 25: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_A = l$.

Proof: The arguments needed to prove this Lemma are identical to the ones proving Lemma 6. □

Hence, by Lemmas 24, and 25, it follows that there exists a unique strategy profile that is a candidate for an equilibrium in our game; $\hat{x} = (l, l + v_B - v_A)$.

The arguments of Step 2 of Proposition 1 proving that neither A nor B have incentives to deviate from the posited profile apply here as well and prove that $\hat{x} = (l, l + v_B - v_A)$ is indeed an equilibrium profile. ■

Proof of Corollary 2. In the unique equilibrium characterized in Proposition 3 B is the winner if $\Phi(\frac{\hat{x}_B+v_B+x_C-v_C}{2}) > \frac{1}{2} \iff l+2v_B-v_A+x_C-v_C > 2m$, C is the winner if $\Phi(\frac{\hat{x}_B+v_B+x_C-v_C}{2}) < \frac{1}{2} \iff l+2v_B-v_A+x_C-v_C < 2m$ and each of these candidates wins with probability $\frac{1}{2}$ if $\Phi(\frac{\hat{x}_B+v_B+x_C-v_C}{2}) = \frac{1}{2} \iff l+2v_B-v_A+x_C-v_C = 2m$.

■

Proof of Proposition 4.

This Proof is similar to the one of Proposition 2.

Lemma 26: For every log-concave, F, there exists $v^{max} > 0$ such that for every $max\{v_B, v_C\} \in (0, v^{max})$: a) $m^a + max\{v_b, v_c\} < x_c$, and b) there exists a unique $l^* \in (0, m^a)$ such that $2F(l^*) = F(\frac{l^*+v_B-v_A+x_C}{2} + \frac{v_B-v_C}{2})$.

Proof: Given the log-concavity of F uniqueness of l^* is guaranteed. To show that $l^* < m^a$ notice that as $max\{v_B, v_C\} \to 0$ we have that l^* is the solution of $2F(l^*) = \lim_{max\{v_B,v_C\}\to 0} F(\frac{l^*+v_B-v_A+x_C}{2} + \frac{v_B-v_C}{2}) = F(\frac{l^*+x_C}{2})$. Notice now that since $\frac{l^*+x_C}{2} < 1$ it must also be the case that $2F(l^*) < 1$ and hence $l^* < m^a$. □

Lemma 27: In any Nash equilibrium, $\hat{x}$, B gets a strictly positive expected vote share in the general election.

Proof: Assume that in some equilibrium, $\hat{x}$, B gets an expected vote share equal to zero in the general election. Then, if B deviates to $x_B = \hat{x}_A$, and given that $v_B > v_A$ she wins her party's primary with certainty if a non-empty set of voters participates in the leftist primary. To show that this is the case but also to prove that B gets a positive general election vote share for this deviation we need to show that $x_B = \hat{x}_A \leq x_C + v_B - v_C$. The latter always holds when valence differences are not very large (i.e., $m + max\{v_B, v_C\} < x_C$). □

Lemma 28: In any Nash equilibrium, $\hat{x}_B > \hat{x}_A$.

Proof: The proof is identical to the one of Lemma 11 (ignoring the arguments referring to candidate D). □

Lemma 29: In any Nash equilibrium, $\hat{x}$, B wins the primary of the leftist party with certainty.

Proof: The proof is identical to the one of Lemma 12. □

Lemma 30: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_A = \hat{x}_B - v_B + v_A$.

Proof: The proof is identical to the one of Lemma 18 with the following minor modifications: a) One does not need to refer to previous Lemmas to show that C advances to the general election since by assumption C is the incumbent, and b) to show that there exists a non-empty set of primary voters in the leftist primary the argument is not any longer that B is the highest valence candidate as in Lemma 12. Now this argument holds because valence differences are not very large (i.e., $m + max\{v_B, v_C\} < x_C$). □

Lemma 31: In any Nash equilibrium, $\hat{x}$, it is the case that $\hat{x}_A = l^*$.

Proof: The proof is identical to the one of Lemma 19 with the following minor modification: One does not need to refer to previous Lemmas to show that by the studied deviation B will secure a strictly positive increase in her election vote share. Now this argument holds because valence differences are not very large (i.e., $m + max\{v_B, v_C\} < x_C$). □

Hence, by Lemmas 30, and 31, it follows that there exists a unique strategy profile that is a candidate for an equilibrium in our game; $\hat{x} = (l^*, l^* + v_B - v_A)$.

The arguments of Step 2 of Proposition 2 proving that neither A nor B have incentives to deviate from the posited profile apply here as well and prove that $\hat{x} = (l^*, l^* + v_B - v_A)$ is indeed an equilibrium profile (recall that here valence difference not very large means that $m + max\{v_B, v_C\} < x_C$). ■

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