

# THE ALGEBRAS OF BOUNDED OPERATORS ON THE TSIRELSON AND BAERNSTEIN SPACES ARE NOT GROTHENDIECK SPACES

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ABSTRACT. We present two new examples of reflexive Banach spaces  $X$  for which the associated Banach algebra  $\mathcal{B}(X)$  of bounded operators on  $X$  is not a Grothendieck space, namely  $X = T$  (the Tsirelson space) and  $X = B_p$  (the  $p^{\text{th}}$  Baernstein space) for  $1 < p < \infty$ .

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

A *Grothendieck space* is a Banach space  $X$  for which every weak\*-convergent sequence in the dual space  $X^*$  converges weakly. The name originates from a result of Grothendieck, who showed that  $\ell_\infty$ , and more generally every injective Banach space, has this property. Plainly, every reflexive Banach space  $X$  is a Grothendieck space because the weak and weak\* topologies on  $X^*$  coincide. By the Hahn–Banach theorem, the class of Grothendieck spaces is closed under quotients, and hence in particular under passing to complemented subspaces.

Substantial efforts have been devoted to the study of Grothendieck spaces over the years, especially in the case of  $C(K)$ -spaces. Notable achievements include the constructions by Talagrand [16] (assuming the Continuum Hypothesis) and Haydon [10] of compact Hausdorff spaces  $K_T$  and  $K_H$ , respectively, such that the Banach spaces  $C(K_T)$  and  $C(K_H)$  are Grothendieck, but  $C(K_T)$  has no quotient isomorphic to  $\ell_\infty$ , while  $C(K_H)$  has the weaker property that it contains no subspace isomorphic to  $\ell_\infty$ . However,  $K_H$  has the significant advantage over  $K_T$  that it exists within ZFC. Haydon, Levy and Odell [11, Corollary 3F] have subsequently shown that Talagrand’s result cannot be obtained within ZFC itself because under the assumption of Martin’s Axiom and the negation of the Continuum Hypothesis, every non-reflexive Grothendieck space has a quotient isomorphic to  $\ell_\infty$ . Brech [4] has pursued these ideas even further using forcing to construct a model of ZFC in which there is a compact Hausdorff space  $K_B$  such that  $C(K_B)$  is a Grothendieck space and has density strictly smaller than the continuum, so in particular no quotient of  $C(K_B)$  is isomorphic to  $\ell_\infty$ . In another direction, Bourgain [3] has shown that  $H^\infty$  is a Grothendieck space.

Nevertheless, a general structure theory of Grothendieck spaces is yet to materialize, and many fundamental questions about the nature of this class remain open. Diestel [8,

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§3] produced an expository list of such questions in 1973. It is remarkable how few of these questions that have been resolved in the meantime.

We shall make a small contribution towards the resolution of the seemingly very difficult problem of describing the Banach spaces  $X$  for which the associated Banach algebra  $\mathcal{B}(X)$  of bounded operators on  $X$  is a Grothendieck space by proving the following result.

**Theorem 1.1.** *Let  $X$  be either the Tsirelson space  $T$  or the  $p^{\text{th}}$  Baernstein space  $B_p$ , where  $1 < p < \infty$ . Then the Banach algebra  $\mathcal{B}(X)$  is not a Grothendieck space.*

For details of the spaces  $T$  and  $B_p$ , we refer to Section 2. We remark that these spaces are not the first examples of reflexive Banach spaces  $X$  for which  $\mathcal{B}(X)$  is known not to be a Grothendieck space due to the following result of the second-named author [13].

**Theorem 1.2** (Kania). *Let  $X = \left(\bigoplus_{n \in \mathbb{N}} \ell_q^n\right)_{\ell_p}$ , where  $1 < p < \infty$  and either  $q = 1$  or  $q = \infty$ . Then the Banach algebra  $\mathcal{B}(X)$  is not a Grothendieck space.*

This paper is organized as follows: in Section 2, we prove Theorem 1.1, followed by a discussion of the key difference between the two cases (see Proposition 2.7 and the paragraph preceding it for details), before we conclude with a short section listing some related open problems.

## 2. THE PROOF OF THEOREM 1.1

The proof that  $\mathcal{B}(T)$  is not a Grothendieck space relies on abstracting the strategy used to prove Theorem 1.2. The following notion will play a key role in this approach. Let  $E$  be a Banach space with a normalized, 1-unconditional basis  $(e_n)_{n=1}^{\infty}$  (where ‘1-unconditional’ means that  $\left\|\sum_{j=1}^n \alpha_j \beta_j e_j\right\| \leq \max_{1 \leq j \leq n} |\alpha_j| \cdot \left\|\sum_{j=1}^n \beta_j e_j\right\|$  for each  $n \in \mathbb{N}$  and all choices of scalars  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ .) The  $E$ -direct sum of a sequence  $(X_n)_{n=1}^{\infty}$  of Banach spaces is given by

$$\left(\bigoplus_{n \in \mathbb{N}} X_n\right)_E = \left\{ (x_n) : x_n \in X_n \ (n \in \mathbb{N}) \text{ and the series } \sum_{n=1}^{\infty} \|x_n\| e_n \text{ converges in } E \right\}.$$

This is a Banach space with respect to the coordinate-wise defined operations and the norm

$$\|(x_n)\| = \left\| \sum_{n=1}^{\infty} \|x_n\| e_n \right\|.$$

Analogously, we write  $\left(\bigoplus_{n \in \mathbb{N}} X_n\right)_{\ell_{\infty}}$  for the Banach space of uniformly bounded sequences  $(x_n)$  with  $x_n \in X_n$  for each  $n \in \mathbb{N}$ , equipped with the coordinate-wise defined operations and the norm  $\|(x_n)\| = \sup_n \|x_n\|$ .

The main property of the  $E$ -direct sum that we shall require is that every uniformly bounded sequence  $(U_n)_{n=1}^{\infty}$  of operators, where  $U_n \in \mathcal{B}(X_n)$  for each  $n \in \mathbb{N}$ , induces a ‘diagonal operator’  $\text{diag}(U_n)$  on  $\left(\bigoplus_{n \in \mathbb{N}} X_n\right)_E$  by the definition

$$\text{diag}(U_n)(x_n)_{n=1}^{\infty} = (U_n x_n)_{n=1}^{\infty},$$

and  $\|\text{diag}(U_n)\| = \sup_n \|U_n\|$ .

We shall also use the following result of W. B. Johnson [12].

**Theorem 2.1** (Johnson). *The Banach space  $(\bigoplus_{n \in \mathbb{N}} \ell_1^n)_{\ell_\infty}$  contains a complemented copy of  $\ell_1$ . Hence a Banach space that contains a complemented copy of  $(\bigoplus_{n \in \mathbb{N}} \ell_1^n)_{\ell_\infty}$  is not a Grothendieck space.*

**Lemma 2.2.** *Let  $X = (\bigoplus_{n \in \mathbb{N}} X_n)_E$ , where  $E$  is a Banach space with a normalized, 1-unconditional basis and  $(X_n)$  is a sequence of Banach spaces. Then  $\mathcal{B}(X)$  contains a complemented subspace which is isometrically isomorphic to  $(\bigoplus_{n \in \mathbb{N}} X_n)_{\ell_\infty}$ .*

*Proof.* We may suppose that  $X_n$  is non-zero for each  $n \in \mathbb{N}$ . Let  $n \in \mathbb{N}$ , and take  $w_n \in X_n$  and  $f_n \in X_n^*$  such that  $\|w_n\| = \|f_n\| = 1 = \langle w_n, f_n \rangle$ . For  $x_n \in X_n$ , the rank-one operator given by

$$x_n \otimes f_n: y \mapsto \langle y, f_n \rangle x_n, \quad X_n \rightarrow X_n,$$

has the same norm as  $x_n$ , so the map

$$\Delta: (x_n) \mapsto \text{diag}(x_n \otimes f_n), \quad \left( \bigoplus_{n \in \mathbb{N}} X_n \right)_{\ell_\infty} \rightarrow \mathcal{B}(X), \quad (2.1)$$

is an isometry. It is clearly linear, and therefore the image of  $\Delta$  is a subspace of  $\mathcal{B}(X)$  which is isometrically isomorphic to  $(\bigoplus_{n \in \mathbb{N}} X_n)_{\ell_\infty}$ . This subspace is complemented in  $\mathcal{B}(X)$  because  $\Delta$  has a bounded, linear left inverse, namely the map given by

$$U \mapsto (Q_n U J_n w_n)_{n=1}^\infty, \quad \mathcal{B}(X) \rightarrow \left( \bigoplus_{n \in \mathbb{N}} X_n \right)_{\ell_\infty},$$

where  $J_n: X_n \rightarrow X$  and  $Q_n: X \rightarrow X_n$  denote the  $n^{\text{th}}$  coordinate embedding and projection, respectively.  $\square$

**Corollary 2.3.** *Let  $X$  be a Banach space which contains a complemented subspace that is isomorphic to  $(\bigoplus_{n \in \mathbb{N}} \ell_1^{m_n})_E$  for some unbounded sequence  $(m_n)$  of natural numbers and some Banach space  $E$  with a normalized, 1-unconditional basis. Then  $\mathcal{B}(X)$  is not a Grothendieck space.*

*Proof.* Let  $Y = (\bigoplus_{n \in \mathbb{N}} \ell_1^{m_n})_E$ . Lemma 2.2 implies that  $\mathcal{B}(Y)$  contains a complemented copy of  $(\bigoplus_{n \in \mathbb{N}} \ell_1^{m_n})_{\ell_\infty}$ , which is isomorphic to  $(\bigoplus_{n \in \mathbb{N}} \ell_1^n)_{\ell_\infty}$  by Pełczyński's decomposition method, and therefore  $\mathcal{B}(Y)$  is not a Grothendieck space by Theorem 2.1. The assumption means that we can find bounded operators  $U: X \rightarrow Y$  and  $V: Y \rightarrow X$  such that  $UV = I_Y$ . This implies that  $\mathcal{B}(X)$  contains a complemented copy of  $\mathcal{B}(Y)$  because the operator  $R \mapsto URV$ ,  $\mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ , is a left inverse of  $S \mapsto VSU$ ,  $\mathcal{B}(Y) \rightarrow \mathcal{B}(X)$ . Therefore  $\mathcal{B}(X)$  is not a Grothendieck space.  $\square$

Following Figiel and Johnson [9], we use the term 'the Tsirelson space' and the symbol  $T$  to denote the dual of the reflexive Banach space originally constructed by Tsirelson [17] with the property that it does not contain any of the classical sequence spaces  $c_0$  and  $\ell_p$  for  $1 \leq p < \infty$ . We refer to [6] for an attractive introduction to the Tsirelson space, including

background information, its formal definition and a comprehensive account of what was known about it up until the late 1980's.

The following notion plays a key role in the study of the Tsirelson space (and in the definition of the Baernstein spaces, to be given below).

**Definition 2.4.** A non-empty, finite subset  $M$  of  $\mathbb{N}$  is (*Schreier-*)*admissible* if  $|M| \leq \min M$ , where  $|M|$  denotes the cardinality of  $M$ .

*Proof of Theorem 1.1 for  $X = T$ .* The unit vector basis is a normalized, 1-unconditional basis for  $T$ . We shall denote it by  $(t_n)_{n=1}^\infty$  throughout this proof. Take natural numbers  $1 = m_1 \leq k_1 < m_2 \leq k_2 < m_3 \leq k_3 < \dots$ , and set

$$M_n = [m_n, m_{n+1}) \cap \mathbb{N} \quad \text{and} \quad F_n = \text{span}\{t_j : j \in M_n\} \quad (n \in \mathbb{N}),$$

so that  $(F_n)$  is an unconditional finite-dimensional Schauder decomposition of  $T$ . For  $x_n \in F_n$  ( $n \in \mathbb{N}$ ), [5, Corollary 7(i)] shows that the series  $\sum_{n=1}^\infty x_n$  converges in  $T$  if and only if the series  $\sum_{n=1}^\infty \|x_n\| t_{k_n}$  converges in  $T$ , and when they both converge, the norms of their sums are related by

$$\frac{1}{3} \left\| \sum_{n=1}^\infty \|x_n\| t_{k_n} \right\| \leq \left\| \sum_{n=1}^\infty x_n \right\| \leq 18 \left\| \sum_{n=1}^\infty \|x_n\| t_{k_n} \right\|.$$

Consequently  $T$  is 54-isomorphic to the direct sum  $(\bigoplus_{n \in \mathbb{N}} F_n)_E$ , where  $E$  denotes the closed linear span of  $\{t_{k_n} : n \in \mathbb{N}\}$ . Taking  $m_n = 2^{n-1}$  for each  $n \in \mathbb{N}$ , we see that the sets  $M_n$  are admissible, which implies that  $F_n$  is 2-isomorphic to  $\ell_1^{m_n}$  for each  $n \in \mathbb{N}$ . Hence  $T$  is 108-isomorphic to  $(\bigoplus_{n \in \mathbb{N}} \ell_1^{m_n})_E$ , and the conclusion follows from Corollary 2.3.  $\square$

We shall now turn our attention to the Baernstein spaces  $B_p$  for  $1 < p < \infty$ . As the name suggests, they originate in the work of Baernstein, who introduced the space that we call  $B_2$  in [2], while the variant for general  $p$  is due to Seifert [15]. These spaces can be viewed as a natural precursor of the Tsirelson space, as the account in [6, Chapter 0] highlights.

The main reason for our interest in the Baernstein spaces in the present context is that while the conclusion of Theorem 1.1 remains valid for them, the method of proof that we used to establish Theorem 1.1 for the Tsirelson space does not carry over. We shall return to this point in Proposition 2.7 below, after we have given the formal definition of the Baernstein spaces and shown how to deduce Theorem 1.1 for them.

In the remainder of this section, we fix a number  $p \in (1, \infty)$ . For  $x = (\alpha_n)_{n=1}^\infty \in c_{00}$  (the vector space of finitely supported scalar sequences),  $k \in \mathbb{N}$  and  $N_1, \dots, N_k \subseteq \mathbb{N}$ , let

$$\nu_p(x; N_1, \dots, N_k) = \left( \sum_{j=1}^k \mu(x, N_j)^p \right)^{\frac{1}{p}}, \quad \text{where} \quad \mu(x, N_j) = \sum_{n \in N_j} |\alpha_n|.$$

Given two non-empty subsets  $M$  and  $N$  of  $\mathbb{N}$ , where  $M$  is finite, we use the notation  $M < N$  to indicate that  $\max M < \min N$ . The  $p^{\text{th}}$  *Baernstein space*  $B_p$  can now be defined as the

completion of  $c_{00}$  with respect to the norm

$$\|x\|_{B_p} = \sup\{\nu_p(x; N_1, \dots, N_k) : k \in \mathbb{N} \text{ and } N_1 < N_2 < \dots < N_k \text{ are admissible subsets of } \mathbb{N}\}. \quad (2.2)$$

As we have already mentioned, Baernstein introduced the Banach space  $B_2$  in [2] and observed that it is reflexive and the unit vector basis of  $c_{00}$ , which we shall here denote by  $(b_n)_{n=1}^\infty$ , is a normalized, 1-unconditional basis for it. His proofs carry over immediately to general  $p$ . We write  $(b_n^*)_{n=1}^\infty$  for the sequence of coordinate functionals corresponding to the basis  $(b_n)_{n=1}^\infty$ .

In analogy with the proof of Theorem 1.1 for the Tsirelson space given above, we shall define  $m_n = 2^{n-1}$  and  $M_n = [m_n, m_{n+1}) \cap \mathbb{N}$  for each  $n \in \mathbb{N}$ , and we shall then consider the finite-dimensional blocking

$$F_n = \text{span}\{b_j : j \in M_n\} \quad (n \in \mathbb{N}) \quad (2.3)$$

of the unit vector basis for  $B_p$ . For later reference, we remark that  $F_n$  is isometrically isomorphic to  $\ell_1^{m_n}$  because  $M_n$  is admissible for each  $n \in \mathbb{N}$ .

Let  $(\bigoplus_{n \in \mathbb{N}} F_n)_{c_{00}}$  denote the vector space of sequences  $(x_n)$  such that  $x_n \in F_n$  for each  $n \in \mathbb{N}$  and  $x_n = 0$  eventually. For  $(x_n) \in (\bigoplus_{n \in \mathbb{N}} F_n)_{c_{00}}$ , we set

$$\Delta(x_n) = \sum_{n=1}^{\infty} x_n \otimes b_{m_{n+1}-1}^*, \quad (2.4)$$

where we note that the sum is finite, so that  $\Delta(x_n)$  defines a finite-rank operator on  $B_p$ .

**Lemma 2.5.** *For each  $(x_n) \in (\bigoplus_{n \in \mathbb{N}} F_n)_{c_{00}}$ ,*

$$\max_{n \in \mathbb{N}} \|x_n\|_{B_p} \leq \|\Delta(x_n)\| \leq \sqrt[p]{3} \max_{n \in \mathbb{N}} \|x_n\|_{B_p}.$$

*Proof.* The lower bound is clear because, for each  $k \in \mathbb{N}$ ,  $b_{m_{k+1}-1}$  is a unit vector in  $B_p$ , and  $\Delta(x_n)b_{m_{k+1}-1} = x_k$ .

It suffices to establish the upper bound in the case where  $\max_{n \in \mathbb{N}} \|x_n\|_{B_p} = 1$ . Then  $\mu(x_n, N) \leq 1$  for each  $n \in \mathbb{N}$  and  $N \subseteq \mathbb{N}$  because the support of  $x_n$  is admissible.

For  $N \subseteq \mathbb{N}$ , we introduce the set

$$\text{uep}(N) = \{m_{k+1} - 1 : k \in \mathbb{N}, N \cap M_k \neq \emptyset\},$$

which consists of the upper end points (hence the acronym ‘uep’) of the intervals  $M_k$  that  $N$  intersects. For later reference, we remark that  $\text{uep}(N)$  is admissible whenever  $N$  is admissible because

$$\min \text{uep}(N) \geq \min N \geq |N| \geq |\text{uep}(N)|.$$

Suppose that  $N \subseteq \mathbb{N}$  is non-empty and finite, and take a non-empty, finite subset  $K$  of  $\mathbb{N}$  such that  $\text{uep}(N) = \{m_{k+1} - 1 : k \in K\}$ . Then  $N$  is the disjoint union of the family

$\{N \cap M_k : k \in K\}$ , and for each  $y \in B_p$ , we have

$$\begin{aligned} \mu(\Delta(x_n)y, N) &= \sum_{k \in K} \mu(\Delta(x_n)y, N \cap M_k) = \sum_{k \in K} |\langle y, b_{m_{k+1}-1}^* \rangle| \mu(x_k, N \cap M_k) \\ &\leq \sum_{k \in K} |\langle y, b_{m_{k+1}-1}^* \rangle| = \mu(y, \text{uep}(N)). \end{aligned} \quad (2.5)$$

Now let  $N_1 < N_2 < \dots < N_k$  be admissible subsets of  $\mathbb{N}$  for some  $k \in \mathbb{N}$ . We aim to show that  $3\|y\|_{B_p}^p$  is an upper bound of the quantity

$$\nu_p(\Delta(x_n)y; N_1, \dots, N_k)^p = \sum_{j=1}^k \mu(\Delta(x_n)y, N_j)^p. \quad (2.6)$$

By adding at most two extra sets beyond the final set  $N_k$ , we may suppose that  $k$  is a multiple of 3. Moreover, we may suppose that there is no  $j \in \{1, \dots, k-1\}$  such that  $N_j \cup N_{j+1} \subseteq M_h$  for some  $h \in \mathbb{N}$ . Indeed, if there is, we can replace the sets  $N_j$  and  $N_{j+1}$  with their union  $N_j \cup N_{j+1}$ , which is still admissible, and this change will not decrease the value of (2.6) because

$$\begin{aligned} \mu(\Delta(x_n)y, N_j)^p + \mu(\Delta(x_n)y, N_{j+1})^p &\leq (\mu(\Delta(x_n)y, N_j) + \mu(\Delta(x_n)y, N_{j+1}))^p \\ &= \mu(\Delta(x_n)y, N_j \cup N_{j+1})^p. \end{aligned}$$

Set  $r_j = \min \text{uep}(N_j)$  and  $s_j = \max \text{uep}(N_j)$  for each  $j \in \{1, \dots, k\}$ . Then we have  $r_1 \leq s_1 \leq r_2 \leq s_2 \leq \dots \leq r_k \leq s_k$  because  $N_1 < N_2 < \dots < N_k$ . A much less obvious fact is that  $s_j < r_{j+3}$  for each  $j \in \{1, \dots, k-3\}$  (provided that  $k > 3$ ). To verify this, we assume the contrary, so that  $s_j = r_{j+3} = m_{h+1} - 1$  for some  $j \in \{1, \dots, k-3\}$  and  $h \in \mathbb{N}$ . Then it follows that  $N_{j+1}$  and  $N_{j+2}$  are both contained in  $M_h$ , which contradicts the above assumption. In other words, we have

$$\text{uep}(N_1) < \text{uep}(N_4) < \dots < \text{uep}(N_{k-2}), \quad \text{uep}(N_2) < \text{uep}(N_5) < \dots < \text{uep}(N_{k-1})$$

and

$$\text{uep}(N_3) < \text{uep}(N_6) < \dots < \text{uep}(N_k),$$

where each of the sets  $\text{uep}(N_j)$  is admissible. Hence, using (2.5), we conclude that

$$\begin{aligned} \nu_p(\Delta(x_n)y; N_1, \dots, N_k)^p &\leq \sum_{j=1}^k \mu(y, \text{uep}(N_j))^p \\ &= \nu_p(y; \text{uep}(N_1), \text{uep}(N_4), \dots, \text{uep}(N_{k-2}))^p \\ &\quad + \nu_p(y; \text{uep}(N_2), \text{uep}(N_5), \dots, \text{uep}(N_{k-1}))^p \\ &\quad + \nu_p(y; \text{uep}(N_3), \text{uep}(N_6), \dots, \text{uep}(N_k))^p \\ &\leq 3\|y\|_{B_p}^p, \end{aligned}$$

as required.  $\square$

**Corollary 2.6.** *The ideal  $\mathcal{K}(B_p)$  of compact operators on  $B_p$  contains a complemented subspace that is isomorphic to  $(\bigoplus_{n \in \mathbb{N}} \ell_1^n)_{c_0}$ .*

*Proof.* The map

$$\Delta: (x_n) \mapsto \Delta(x_n), \quad \left( \bigoplus_{n \in \mathbb{N}} F_n \right)_{c_{00}} \rightarrow \mathcal{K}(B_p), \quad (2.7)$$

given by (2.4) is linear, and Lemma 2.5 implies that it is bounded with respect to the norm  $\|(x_n)\|_\infty = \max_{n \in \mathbb{N}} \|x_n\|_{B_p}$  on its domain, so it extends uniquely to a bounded operator

$$\Delta: \left( \bigoplus_{n \in \mathbb{N}} F_n \right)_{c_0} \rightarrow \mathcal{K}(B_p).$$

For  $n \in \mathbb{N}$ , let  $Q_n: B_p \rightarrow F_n$  be the canonical basis projection of  $B_p$  onto  $F_n$ , and define

$$\Theta(U) = (Q_n U b_{m_{n+1}-1})_{n=1}^\infty \quad (U \in \mathcal{K}(B_p)).$$

Since  $B_p$  is reflexive,  $(b_{m_{n+1}-1})_{n=1}^\infty$  is a weak-null sequence. Hence  $(U b_{m_{n+1}-1})_{n=1}^\infty$  is norm-null for each compact operator  $U$ , and therefore  $U \mapsto \Theta(U)$  defines a map from  $\mathcal{K}(B_p)$  into  $(\bigoplus_{n \in \mathbb{N}} F_n)_{c_0}$ . This map is clearly bounded and linear, and it is a left inverse of  $\Delta$ . Now the conclusion follows from the facts that  $F_n$  is isometrically isomorphic to  $\ell_1^{m_n}$  for each  $n \in \mathbb{N}$  and  $(\bigoplus_{n \in \mathbb{N}} \ell_1^{m_n})_{c_0}$  is isomorphic to  $(\bigoplus_{n \in \mathbb{N}} \ell_1^n)_{c_0}$ .  $\square$

*Proof of Theorem 1.1 for  $X = B_p$ .* The bidual of  $\mathcal{K}(B_p)$  is  $\mathcal{B}(B_p)$  because  $B_p$  is reflexive and has a basis. Hence, passing to the biduals in Corollary 2.6, we see that  $\mathcal{B}(B_p)$  contains a complemented subspace that is isomorphic to  $(\bigoplus_{n \in \mathbb{N}} \ell_1^n)_{\ell_\infty}$ , and the conclusion follows from Theorem 2.1 as before.  $\square$

Comparing the above proofs of Theorem 1.1 for the Tsirelson space on the one hand and the  $p^{\text{th}}$  Baernstein space on the other, we see that the former is significantly shorter and simpler. This is due to the fact that  $T$  is isomorphic to the  $E$ -direct sum of the blocks  $F_n$  for a suitably chosen Banach space  $E$  with a normalized, 1-unconditional basis. Indeed, this fact immediately allowed us to define the diagonal operator (2.1) for  $T$ , whereas it required substantial work to establish the boundedness of its counterpart, which is the bidual of the operator (2.7), for  $B_p$ . One may wonder whether this extra effort is really necessary. The following result addresses this question, showing that at least some argument is required.

**Proposition 2.7.** *There exists a uniformly bounded sequence  $(U_n)_{n=1}^\infty$  of rank-one operators, where  $U_n$  is defined on the subspace  $F_n$  of  $B_p$  given by (2.3), such that the corresponding diagonal map defined on the subspace  $\text{span} \bigcup_{n \in \mathbb{N}} F_n (= c_{00})$  of  $B_p$  by*

$$\text{diag}(U_n) \left( \sum_{j=1}^k x_j \right) = \sum_{j=1}^k U_j x_j \quad (k \in \mathbb{N}, x_1 \in F_1, \dots, x_k \in F_k) \quad (2.8)$$

*is not bounded with respect to the norm  $\|\cdot\|_{B_p}$ .*

It follows in particular that we cannot express  $B_p$  as the  $E$ -direct sum of the blocks  $F_n$  for any Banach space  $E$  with a normalized, 1-unconditional basis, and this choice of blocking shows that the counterpart of [5, Corollary 7(i)] fails for  $B_p$ . More generally, Proposition 2.7 implies that ‘diagonal operators’ need not exist if one replaces the  $E$ -direct sum  $X = (\bigoplus_{n \in \mathbb{N}} X_n)_E$  for a Banach space  $E$  with a normalized, 1-unconditional basis with a

Banach space  $X$  that merely has an unconditional finite-dimensional Schauder decomposition  $(X_n)_{n=1}^\infty$ .

In the proof of Proposition 2.7, we shall require the following elementary inequality.

**Lemma 2.8.** *Let  $a, b, c \in (0, \infty)$  with  $a > c$ , and let  $p \in (1, \infty)$ . Then*

$$a^p + (b + c)^p < (a + b)^p + c^p.$$

*Proof.* Consider the function  $f: (0, \infty) \rightarrow \mathbb{R}$  given by  $f(t) = (t+d)^p + 1 - t^p - (d+1)^p$ , where  $d \in (0, \infty)$  is a constant. Since  $f$  is differentiable with  $f'(t) = p(t+d)^{p-1} - pt^{p-1} > 0$  and  $f(1) = 0$ , we conclude that  $f(t) > 0$  for  $t > 1$ . Now the result follows by taking  $t = a/c > 1$  and  $d = b/c > 0$  in this inequality and rearranging it.  $\square$

*Proof of Proposition 2.7.* For each  $n \in \mathbb{N}$ , let  $U_n$  be the summation operator onto the first coordinate of  $F_n$ , that is,

$$U_n: \sum_{j \in M_n} \alpha_j b_j \mapsto \left( \sum_{j \in M_n} \alpha_j \right) b_{m_n}, \quad F_n \rightarrow F_n.$$

This is a rank-one operator which has norm 1 because  $F_n$  is isometrically isomorphic to  $\ell_1^{m_n}$ .

Take natural numbers  $q \leq r$ , and set

$$y_{q,r} = \sum_{n=q}^r \frac{1}{n} x_n, \quad \text{where} \quad x_n = \frac{1}{2^{n-1}} \sum_{j \in M_n} b_j \in F_n \quad (n \in \mathbb{N}).$$

We aim to prove that

$$\|y_{q,r}\|_{B_p} = \left( \sum_{n=q}^r \frac{1}{n^p} \right)^{\frac{1}{p}}. \quad (2.9)$$

The right-hand side of (2.9) is certainly a lower bound on the norm of  $y_{q,r}$  because the sets  $M_q < M_{q+1} < \dots < M_r$  are admissible, so that

$$\|y_{q,r}\|_{B_p} \geq \nu_p(y_{q,r}; M_q, \dots, M_r) = \left( \sum_{n=q}^r \frac{1}{n^p} \right)^{\frac{1}{p}}. \quad (2.10)$$

The reverse inequality requires more work. We begin by observing that since  $y_{q,r}$  is finitely supported, the supremum in the definition (2.2) of the norm of  $y_{q,r}$  is attained, say

$$\|y_{q,r}\|_{B_p} = \nu_p(y_{q,r}; N_1, \dots, N_k), \quad (2.11)$$

where  $k \in \mathbb{N}$  and  $N_1 < \dots < N_k$  are admissible. We may suppose that  $\min N_1 \geq m_q$  and  $\max N_k < m_{r+1}$  because the support of  $y_{q,r}$  is contained in  $[m_q, m_{r+1})$ . We shall show that this additional assumption forces  $(N_1, \dots, N_k) = (M_q, \dots, M_r)$ , which in the light of (2.10) will ensure that (2.9) holds.

First, we note that  $\min N_1 = m_q$  because otherwise we would have  $\{m_q\} < N_1$  and

$$\nu_p(y_{q,r}; N_1, \dots, N_k) < \nu_p(y_{q,r}; \{m_q\}, N_1, \dots, N_k) \leq \|y_{q,r}\|_{B_p},$$

contrary to (2.11). Similar reasoning shows that

$$\max N_k = m_{r+1} - 1 \quad \text{and} \quad \max N_j + 1 = \min N_{j+1} \quad (j \in \{1, \dots, k-1\}).$$

Moreover, each of the sets  $N_1, \dots, N_k$  must be an interval: indeed, if  $N_j$  were not an interval for some  $j \in \{1, \dots, k\}$ , then the unique interval  $N'_j$  with  $\min N'_j = \min N_j$  and  $|N'_j| = |N_j|$  is admissible and satisfies  $\mu(y_{q,r}, N'_j) \geq \mu(y_{q,r}, N_j)$  because the coordinates of  $y_{q,r}$  in  $[m_q, m_{r+1})$  are decreasing and positive, and hence

$$\nu_p(y_{q,r}; N_1, \dots, N_k) < \nu_p(y_{q,r}; N_1, \dots, N_{j-1}, N'_j, \{\max N'_j + 1\}, N_{j+1}, \dots, N_k) \leq \|y_{q,r}\|_{B_p},$$

again contrary to (2.11). Thus we conclude that

$$\bigcup_{j=1}^k N_j = [m_q, m_{r+1}) \cap \mathbb{N} = \bigcup_{s=q}^r M_s.$$

Assume towards a contradiction that  $(N_1, \dots, N_k) \neq (M_q, \dots, M_r)$ . For each  $s \in \mathbb{N}$ ,  $M_s$  is maximal among all admissible intervals with minimum  $m_s$ , and so there must be some  $j \in \{2, \dots, k\}$  such that  $\min N_j \notin \{m_s : q < s \leq r\}$ . Let  $j_0$  be the smallest such  $j$ , so that  $\min N_{j_0-1} = m_s$  for some  $s \in \{q, \dots, r-1\}$ , but  $t = \min N_{j_0}$  is not of the form  $m_u$  for any  $u \in \mathbb{N}$ , which implies that  $t < m_{s+1}$  because  $N_{j_0-1}$  is admissible. Since  $N_{j_0}$  is also admissible, we have  $|N_{j_0}| \leq t$ . Hence the interval  $N''_{j_0} = N_{j_0} \cap [m_{s+1}, \infty) \subseteq M_{s+1}$  satisfies

$$|N''_{j_0}| = |N_{j_0}| - (m_{s+1} - t) \leq 2t - m_{s+1} = 2(t - m_s),$$

from which we deduce that

$$\mu(y_{q,r}, N''_{j_0}) = \frac{|N''_{j_0}|}{(s+1)2^s} \leq \frac{2(t - m_s)}{(s+1)2^s} < \frac{t - m_s}{s2^{s-1}} = \frac{|N_{j_0-1}|}{s2^{s-1}} = \mu(y_{q,r}, N_{j_0-1}). \quad (2.12)$$

Set  $L = [t, m_{s+1}) \cap \mathbb{N}$ . Then we have  $N_{j_0} = L \cup N''_{j_0}$  and  $M_s = N_{j_0-1} \cup L$ , with both unions being disjoint, so in the light of (2.12) we can apply Lemma 2.8 to obtain the following inequality

$$\begin{aligned} \mu(y_{q,r}, N_{j_0-1})^p + \mu(y_{q,r}, N_{j_0})^p &= \mu(y_{q,r}, N_{j_0-1})^p + (\mu(y_{q,r}, L) + \mu(y_{q,r}, N''_{j_0}))^p \\ &< (\mu(y_{q,r}, N_{j_0-1}) + \mu(y_{q,r}, L))^p + \mu(y_{q,r}, N''_{j_0})^p = \mu(y_{q,r}, M_s)^p + \mu(y_{q,r}, N''_{j_0})^p. \end{aligned}$$

Thus the admissible sets  $N_1 < \dots < N_{j_0-2} < M_s < N''_{j_0} < N_{j_0+1} < \dots < N_k$  satisfy

$$\nu_p(y_{q,r}; N_1, \dots, N_k) < \nu_p(y_{q,r}; N_1, \dots, N_{j_0-2}, M_s, N''_{j_0}, N_{j_0+1}, \dots, N_k) \leq \|y_{q,r}\|_{B_p},$$

once again contradicting (2.11). Therefore we must have  $(N_1, \dots, N_k) = (M_q, \dots, M_r)$ , and as already explained, (2.9) follows.

Direct application of the definitions shows that

$$\text{diag}(U_n)y_{q,r} = \sum_{n=q}^r \frac{1}{2^{n-1}n} U_n \left( \sum_{j \in M_n} b_j \right) = \sum_{n=q}^r \frac{1}{n} b_{m_n}. \quad (2.13)$$

In particular, choosing  $r = 2q$  and  $q \geq 3$ , we see that the support  $N = \{m_n : q \leq n \leq 2q\}$  of the vector on the right-hand side of (2.13) is admissible because

$$|N| = q + 1 \leq 2^{q-1} = \min N.$$

Hence on the one hand we have

$$\|\text{diag}(U_n)y_{q,2q}\|_{B_p} = \mu(\text{diag}(U_n)y_{q,2q}, N) = \sum_{n=q}^{2q} \frac{1}{n} \geq \frac{q+1}{2q} > \frac{1}{2} \quad (q \geq 3).$$

On the other hand, (2.9) shows that  $\|y_{q,2q}\|_{B_p} \rightarrow 0$  as  $q \rightarrow \infty$  because the series  $\sum_{n=1}^{\infty} 1/n^p$  converges. Therefore  $\text{diag}(U_n)$  cannot be bounded.  $\square$

### 3. OPEN PROBLEMS

Pfzner [14] showed that  $C^*$ -algebras have Pelczyński's property (V). This implies that von Neumann algebras are Grothendieck spaces, so in particular  $\mathcal{B}(H)$  is a Grothendieck space for each Hilbert space  $H$ . No other examples of infinite-dimensional Banach spaces  $X$  for which  $\mathcal{B}(X)$  is a Grothendieck space are known. In particular, the following questions remain open:

- Is  $\mathcal{B}(\ell_p)$  a Grothendieck space for  $p \in (1, \infty) \setminus \{2\}$ ? In this case, the problem is equivalent to deciding whether  $(\bigoplus_{n \in \mathbb{N}} \mathcal{B}(\ell_p^n))_{\ell_\infty}$  is a Grothendieck space because  $\mathcal{B}(\ell_p)$  and  $(\bigoplus_{n \in \mathbb{N}} \mathcal{B}(\ell_p^n))_{\ell_\infty}$  are isomorphic as Banach spaces (see [1, Theorem 2.1]; for  $p = 2$ , see also [7, p. 317], where the result is credited to Haagerup and Lindenstrauss).
- Is  $\mathcal{B}(X)$  a Grothendieck space for every weak Hilbert space  $X$ ?
- More generally, is  $\mathcal{B}(X)$  a Grothendieck space for every super-reflexive Banach space  $X$ ?
- Let  $X$  be a reflexive Banach space with an unconditional finite-dimensional Schauder decomposition  $(F_n)_{n=1}^{\infty}$  whose blocks  $F_n$  are uniformly isomorphic to  $\ell_1^{m_n}$ , where  $m_n = \dim F_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Is it true that  $\mathcal{B}(X)$  is not a Grothendieck space? If it is true, it would provide a unified proof of the two cases considered in Theorem 1.1.
- Is there a non-reflexive Banach space  $X$  for which  $\mathcal{B}(X)$  is a Grothendieck space? If such a space  $X$  exists, both  $X$  and  $X^*$  would necessarily be Grothendieck spaces (because  $\mathcal{B}(X)$  contains complemented copies of them), and therefore  $X$  would be non-separable. No non-reflexive Grothendieck space  $X$  for which  $X^*$  is also a Grothendieck space is known.

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