Inhomogeneous Spatially Dispersive Electromagnetic Media

Jonathan Gratus[1,2] **and Matthew McCormack**[1,2]

[1]Physics Department, Lancaster University, LA1 4YB, UK
[2]The Cockcroft Institute, Sci-Tech Daresbury, Keckwick Lane
Daresbury, Warrington, WA4 4AD, UK

Abstract— Two key types of inhomogeneous spatially dispersive media are described, both based on a spatially dispersive generalisation of the single resonance model of permittivity. The boundary conditions for two such media with different properties are investigated using Lagrangian and distributional methods. Wave packet solutions to Maxwell's equations, where the permittivity varies and is periodic in the medium, are then found.

1. INTRODUCTION

All media are, at least to some extent, both temporally and spatially dispersive [1–5]. A temporally dispersive medium takes time to respond to an electromagnetic signal. A spatially dispersive medium responds not only to a signal at a particular point, but to signals in the neighbourhood of that point. Likewise all media are inhomogeneous, both on the macroscopic scale due to the finite nature of any sample of material, and on the microscopic scale.

Metamaterials [6–10] offer a way of constructing spatially dispersive media with desired electromagnetic properties. Compact, high gradient, accelerators have a wide range of applications in academia, industry, energy and health. A dielectric wakefield accelerator [6] uses electrons to create a field in a dielectric which in turn accelerates further electrons. Using a spatially dispersive dielectric with a periodic inhomogeneity, requires a knowledge about which electromagnetic fields propagate in such dielectric and how the fields pass through the vacuum-dielectric boundary.

For this article we will assume that there is a linear constitutive relationship between the polarization field $\mathbf{P}(t,\mathbf{x}) = \mathbf{D}(t,\mathbf{x}) - \mathbf{E}(t,\mathbf{x})$ and electric field $\mathbf{E}(t,\mathbf{x})$. All media respond linearly for sufficiently small electromagnetic fields and ultimately all media, including the vacuum, are non linear for sufficiently high fields. To simplify the analysis we make the following assumptions.

- There is no magnetization so that $\mathbf{H} = \mathbf{B}$.
- All fields are functions of time t and one spatial coordinate $x = x_1$, thus independent of x_2, x_3. In frequency domain, $k_2 = k_3 = 0$ and we set $k = k_1$.
- The electric and polarization fields are transverse so that $E_1(t,x) = 0$, $P_1(t,x) = 0$ and $B_1(t,x) = 0$. This assumption automatically satisfies the two non-dynamic source free Maxwell's equations.
- We choose a linearly polarized wave so that in the ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) frame $\mathbf{E}(t,x) = E(t,x)\mathbf{e}_2$, $\mathbf{P}(t,x) = P(t,x)\mathbf{e}_2$ and $\mathbf{B}(t,x) = B(t,x)\mathbf{e}_3$.

The relationship between $P(t,x)$ and $E(t,x)$ analysed here is a generalisation of the single resonance model of permittivity. The first generalisation is a simple extension to make the medium spatially dispersive, achieved by introducing a finite propagation speed β,

$$\tilde{P}(\omega,k) = \frac{\tilde{E}(\omega,k)}{\omega^2 - 2i\lambda\omega + (\alpha^2 - \lambda^2) - \beta^2|k|^2} \tag{1}$$

where $\tilde{P}(\omega,k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(\omega t + kx)} P(t,x)dtdx$ is the Fourier transform of $P(t,x)$. Media with such a constitutive relation can be constructed using a cross wire configuration [10].

The second generalisation is to allow the quantities λ, α and β to depend on position x. To do this we replace the frequency-wave number relation (1) with the PDE in space and time given by

$$-\frac{1}{(2\pi)^2}\frac{\partial^2 P}{\partial t^2} + \frac{2\lambda(x)}{2\pi}\frac{\partial P}{\partial t} + \big(\alpha(x)^2 - \lambda(x)^2\big)P + \frac{\partial}{\partial x}\Big(\frac{\beta(x)^2}{(2\pi)^2}\frac{\partial P}{\partial x}\Big) = E \tag{2}$$

It is easy to see that if λ, α and β are constants then the Fourier transform of (2) reproduces (1). We consider two types of inhomogeneity.

In Section 3 we consider a simple boundary between two homogeneous regions. For spatially dispersive media the standard boundary conditions for Maxwell's equations are insufficient to completely specify the solutions for outgoing waves in terms of the incoming waves. The additional equations are called "additional boundary conditions" (ABC) and have often drawn controversy [6–10, 12–14]. For the boundary between a spatially dispersive medium and the vacuum the standard ABC are given by Pekar [12]. In this article, we consider the boundary conditions between two spatially dispersive regions. We consider two methods for deriving the boundary conditions for Maxwell's equations for non-spatially dispersive media: One is the distributional or pill box method, the other is to use a Lagrangian method to derive natural boundary conditions. We show these give the same boundary conditions, which reduce to Pekar's boundary condition in the limit where $\beta \to 0$ in one of the regions.

In Section 4, we consider a periodic structure where α and λ are periodic with the same period. We assume that these quantities take the form of a constant term plus a small periodic inhomogeneity, e.g., $\alpha(x) = \alpha_0 + \alpha_1 \cos x$ where the α_1 is small. The goal in this section is to find solutions to the source free Maxwell equations, i.e., the dispersion relations. However since the medium is inhomogeneous it is not possible to find single mode solutions of the form $e^{2\pi i(\omega t + kx)}$ and is therefore necessary to look for packet solutions of the form $e^{2\pi i\omega t}\hat{P}(x)$. In the following we present an analytic form for an approximate solution for $\hat{P}(x)$. In addition we present a numerical method for finding the permitted frequencies and $\hat{P}(x)$.

In Section 5, we give a discussion of the implications of this research.

2. MAXWELL'S EQUATIONS AND STATIONARY MEDIA

From $\mathbf{E}(t,x) = E(t,x)\mathbf{e}_2$ etc., the source free dynamical Maxwell's equations become $E' = -\dot{B}$ and $B' = -(\dot{E} + \dot{P})$ which we can combine to give $E'' = \ddot{E} + \ddot{P}$. Taking the Fourier transform with respect to t gives

$$(2\pi)^{-2}\hat{E}'' = -\omega^2\left(\hat{E} + \hat{P}\right) \tag{3}$$

where $\hat{P}(\omega, x) = \int_{-\infty}^{\infty} e^{-2\pi i\omega t} P(t,x)dt$. In most cases, in the following, we will not explicitly write the ω argument in $\hat{E}$ and $\hat{P}$.

The Fourier transform of the constitutive relation (2) is

$$(2\pi)^{-2}\left(\beta^2(x)\hat{P}'\right)' + L(x)\hat{P} = \hat{E} \quad \text{where} \quad L(x) = (\omega + i\lambda(x))^2 + \alpha(x)^2 \tag{4}$$

3. ADDITIONAL BOUNDARY CONDITIONS FOR SPATIALLY DISPERSIVE MEDIA

In this section we set the media parameters $\alpha(x)$, $\beta(x)$, $\lambda(x)$ to be piecewise constant: $\alpha(x) = \theta(-x)\alpha_{\mathbf{L}} + \theta(x)\alpha_{\mathbf{R}}$ etc., so that $L(x) = \theta(-x)L_{\mathbf{L}} + \theta(x)L_{\mathbf{R}}$, for media constants $\alpha_{\boldsymbol{\mu}}, \beta_{\boldsymbol{\mu}}, \lambda_{\boldsymbol{\mu}} \geq 0$ with $\boldsymbol{\mu} = \mathbf{L}, \mathbf{R}$. Here $\theta(x)$ is the Heaviside step function $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$.

Eliminating $\hat{E}$ from (3) and (4) we have a fourth order ODE for $\hat{P}(x)$. For each region this is solved by

$$\hat{E}(x) = A^+_{\boldsymbol{\mu}} e^{2\pi i k^+_{\mu} x} + A^-_{\boldsymbol{\mu}} e^{-2\pi i k^+_{\mu} x} + B^+_{\boldsymbol{\mu}} e^{2\pi i k^-_{\mu} x} + B^-_{\boldsymbol{\mu}} e^{-2\pi i k^-_{\mu} x} \tag{5}$$

where

$$k^{\pm}_{\boldsymbol{\mu}} = \frac{\sqrt{\beta^2_{\boldsymbol{\mu}}\omega^2 + L_{\boldsymbol{\mu}} \pm \sqrt{(\beta^2_{\boldsymbol{\mu}}\omega^2 - L_{\boldsymbol{\mu}})^2 + 4\beta^2_{\boldsymbol{\mu}}\omega^2}}}{\sqrt{2}\beta_{\boldsymbol{\mu}}} \tag{6}$$

Since $\lambda_{\boldsymbol{\mu}} > 0$ the waves are damped in the direction of motion, (see Figure 1). Maxwell's equations give us two boundary conditions

$$\left[\hat{E}\right] = 0 \quad \text{and} \quad \left[\hat{E}'\right] = 0 \tag{7}$$

where $[\hat{E}]$ is the discontinuity $[\hat{E}] = \lim_{x\to 0+}\hat{E}(x) - \lim_{x\to 0-}\hat{E}(x)$. However, we need two additional boundary conditions for $[\hat{P}]$ and $[\hat{P}']$. In the usual scattering problem we prescribe the incoming wave amplitudes $\{A^-_{\mathbf{L}}, B^-_{\mathbf{L}}, A^+_{\mathbf{R}}, B^+_{\mathbf{R}}\}$ and we calculate the outgoing wave amplitudes $\{A^+_{\mathbf{L}}, B^+_{\mathbf{L}}, A^-_{\mathbf{R}}, B^-_{\mathbf{R}}\}$.

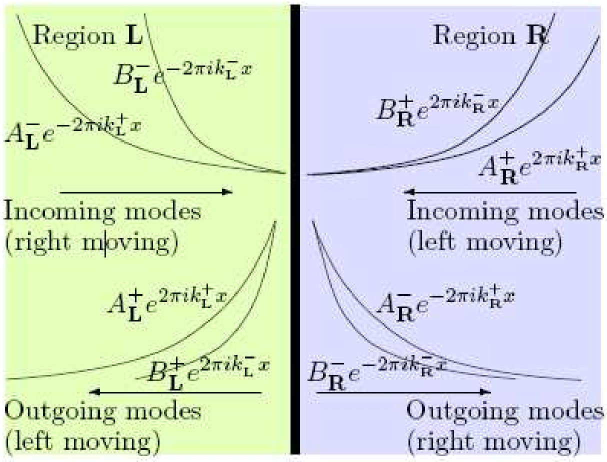

Figure 1: Incoming and outgoing modes, given in (5).

3.1. Lagrangian Formulation of Boundary Conditions

Due to the damping term, it is non trivial to formulate a Lagrangian which gives rise to both Maxwell's equations and the constitutive relation (2). However since we are interested in the boundary conditions, is it sufficient to use the Fourier transform Equations (3) and (4). These can be derived by varying the action

$$\mathcal{S}[\hat{E},\hat{P}] = \int \mathcal{L}\left(\hat{E},\hat{E}',\hat{P},\hat{P}',x\right)dx \tag{8}$$

where

$$\mathcal{L}\left(\hat{E},\hat{E}',\hat{P},\hat{P}',x\right) = \frac{1}{2}\left(\frac{(\hat{E}')^2}{(2\pi)^2\omega^2} - \hat{E}^2 + \frac{\beta(x)^2}{(2\pi)^2}(\hat{P}')^2 - L(x)\hat{P}^2\right) - \hat{E}\hat{P} \tag{9}$$

Varying (8) with respect to $\hat{E}$ and $\hat{P}$ away from the boundary yields (3) and (4) respectively. In order to obtain the boundary conditions we must consider variations with support which includes the boundary $x = 0$. It is necessary to assume $\hat{E}$ and $\hat{P}$ are continuous, i.e.,

$$\left[\hat{E}\right] = 0 \quad \text{and} \quad \left[\hat{P}\right] = 0 \tag{10}$$

Varying (8) with respect to $\hat{P}$ then gives

$$\begin{aligned}\delta_{\hat{P}}\mathcal{S} &= \int_{-\infty}^{\infty}\delta_{\hat{P}}\mathcal{L}dx = \int_{-\infty}^{\infty}\left(\frac{\beta(x)^2}{(2\pi)^2}\hat{P}'\delta\hat{P}' - L(x)\hat{P}\delta\hat{P} - \hat{E}\delta\hat{P}\right)dx\\ &= \int_{-\infty}^{0}\frac{d}{x}\left(\frac{\beta(x)^2}{(2\pi)^2}\hat{P}'\delta\hat{P}\right) + \int_{0}^{\infty}\frac{d}{x}\left(\frac{\beta(x)^2}{(2\pi)^2}\hat{P}'\delta\hat{P}\right) = -\left[\frac{\beta(x)^2}{(2\pi)^2}\hat{P}'\right]\delta\hat{P}\end{aligned}$$

Since this vanishes for all variations $\delta\hat{P}$, one has

$$\left[\beta(x)^2\hat{P}'\right] = 0 \tag{11}$$

Similarly varying (8) with respect to $\hat{E}$ gives

$$[\hat{E}'] = 0. \tag{12}$$

In the limiting case $\beta_{\mathbf{L}} \to 0$, the left hand region is only temporally dispersive. One must make a choice about the behaviour of $\{A_{\mathbf{L}}^{+}, A_{\mathbf{L}}^{-}, B_{\mathbf{L}}^{+}, B_{\mathbf{L}}^{-}\}$ in this limit. For a certain choice the boundary conditions (7), (10) and (11) reduce to Pekar's ABC

$$\left[\hat{E}\right] = 0, \quad \left[\hat{E}'\right] = 0 \quad \text{and} \quad \left[\hat{P}\right] = 0 \tag{13}$$

3.2. Distributional Method of Boundary Conditions

Given smooth functions $f_{\mathbf{L}}(x)$ and $f_{\mathbf{R}}(x)$ and setting $f(x) = \theta(-x)f_{\mathbf{L}}(x)+\theta(x)f_{\mathbf{R}}(x)$, then one has $f''(x) = \delta'(x)[f]+2\delta(x)[f']+\theta(-x)f''_{\mathbf{L}}(,x)+\theta(x)f''_{\mathbf{R}}(x)$. In order to avoid multiplying distributions we must assume P is continuous and set $P'(x) = \theta(-x)P'_{\mathbf{L}}(x) + \theta(x)P'_{\mathbf{R}}(x)$. Substituting this into (3) and (4) again implies (10), (11) and (12).

$$\left[\hat{E}\right] = 0, \quad \left[\hat{P}\right] = 0, \quad \left[\hat{E}'\right] = 0 \quad \text{and} \quad \left[\hat{P}'\right] = 0 \tag{14}$$

In the limit $\beta_{\mathbf{L}} \to 0$, for appropriate choices, these again reduce to Pekar's ABC (12).

4. PERIODIC MEDIA

In this section we investigate media where the constitutive quantity $L(x)$ in (4) is periodic $L(x+1) = L(x)$ and β is constant. We assume that the amplitude of the inhomogeneity is dominated by the first mode, that is

$$L(\omega, x) = L_0(\omega) + 2\Lambda(\omega)\cos(2\pi x) \tag{15}$$

Taking the Fourier transforms of (3) and (4) with respect to x we get $(\omega^2 - k^2)\tilde{E}(k) = -\omega^2\tilde{P}(k)$ and $-k^2\beta^2\tilde{P} + (\tilde{L} * \tilde{P})(k) = \tilde{E}(k)$. Combining these into a single equation gives

$$\left(\tilde{L} * \tilde{P}\right)(k) = \left(\beta^2 k^2 - \frac{\omega^2}{\omega^2 - k^2}\right)\tilde{P}(k) \tag{16}$$

We look for periodic solutions of the form

$$\hat{P}(x) = \sum_{m=-\infty}^{\infty} P_m e^{2\pi i m x} \tag{17}$$

whose Fourier transform $\tilde{P}(k)$ consists of a series of delta functions $\tilde{P}(k) = \sum_{m=-\infty}^{\infty} P_m\delta(k - m)$. Substituting (16) and (14) into (15) yields the difference equation

$$\Lambda P_{k-1} + f_k P_k + \Lambda P_{k+1} = 0 \quad \text{where} \quad f_k = \frac{\beta^2 k^4 + \omega^2 + \left(\omega^2 - k^2\right)L_0(\omega) - \beta^2\omega^2 k^2}{\omega^2 - k^2} \tag{18}$$

Observe that having higher order modes in $L(\omega, x)$ will result in more terms in (17). We can trivially solve (17) simply by arbitrarily fixing P_0 and P_1 and then using (17) to solve for all subsequent P_k. However, in general, this will lead to P_k's which diverge $|P_k| \to \infty$ as $k \to \pm\infty$. The Fourier transform of this would therefore be a non-smooth wave which may not even be continuous. Therefore for physical solutions, we demand that $|P_k| \to 0$ as $k \to \pm\infty$. As we see below we can obtain approximate analytic solutions for small Λ. In addition we also give a numerical method for finding arbitrary solutions.

4.1. Approximate Analytic Wave Packet Solutions

We have found two approximate analytic solutions in the case when $\Lambda \ll L_0$, an even solution $(\omega^{(e)}, P^{(e)})$ and an odd solution $(\omega^{(o)}, P^{(o)})$. Since these are approximate solutions, we set the left hand side of (17) to $\mathcal{Q}_k$, that is $\mathcal{Q}_k = \Lambda P_{k-1} + f_k P_k + \Lambda P_{k+1}$ so that $\mathcal{Q}_k = 0$ is an exact solution to (17). By contrast we solve $\mathcal{Q}_k = O(\Lambda^p)$ for some order p which depends on k.

The even solution $\omega = \omega^{(e)}$ is given by

$$\left(f_1 - \Lambda^2\left(\frac{1}{f_2} + \frac{2}{f_0}\right)\right)\Big|_{\omega=\omega^{(e)}} = 0 \tag{19}$$

$P^{(e)}$ is then given by

$$P^{(e)}_{-1} = 1, \quad P^{(e)}_0 = -\frac{2\Lambda}{f_0}, \quad P^{(e)}_1 = 1 \quad \text{and} \quad P^{(e)}_m = \frac{(-\Lambda)^{|m|-1}}{\prod_{k=2}^{|m|} f_k} + O\left(\Lambda^{|m|+1}\right) \tag{20}$$

By direct substitution into $\mathcal{Q}_m$ shows $\mathcal{Q}_0 = 0$, $\mathcal{Q}_{\pm1} = O(\Lambda^4)$ and $\mathcal{Q}_m = O(\Lambda^{|m|+1})$ for $|m| \geq 2$.

The odd solution $\omega = \omega^{(o)}$ is given by

$$\left(f_1 - \frac{\Lambda^2}{f_2}\right)\Big|_{\omega=\omega^{(o)}} = 0 \tag{21}$$

and $P_m^{(o)}$ by

$$P_{-1}^{(o)} = -1, \quad P_0^{(o)} = 0, \quad P_1^{(o)} = 1 \quad \text{and} \quad P_m^{(o)} = \operatorname{sign}(m)\frac{(-\Lambda)^{|m|-1}}{\prod_{k=2}^{|m|} f_k} + O\left(\Lambda^{|m|+1}\right) \tag{22}$$

Again $\mathcal{Q}_0 = 0$, $\mathcal{Q}_{\pm1} = O(\Lambda^4)$ and $\mathcal{Q}_m = O(\Lambda^{|m|+1})$ for $|m| \geq 2$.

Depending on how L_0 and Λ depend on ω these are often the lowest two modes. The shape of these modes is given in Figure 2, using the numerical approach below. In this case the even mode $(\omega^{(e)}, P^{(e)})$ cannot be supported by the medium and is damped.

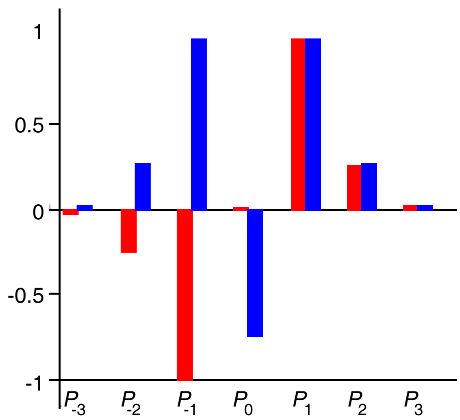

Figure 2: P_e (blue) and P_o (red) for $L_0 \equiv 1$, $\Lambda \equiv 0.75$. In this case $\omega^{(e)} = 0.753i$ and $\omega^{(o)} = 0.399$

4.2. Numerical Approaches

A numerical approximation scheme, which is valid if $\Lambda \not\ll L_0$ and gives packets in addition to (19)–(21), is as follows: Choose an integer $N \geq 2$. Then assume that $P_m \approx 0$ for $|m| > N$ thus truncating the infinite set of equation given by (17) to a set of $2N+1$ linear equations for $\{P_{-N}, \ldots, P_N\}$. Write this in matrix language $M\underline{b} = \underline{0}$ where M is a $(2N+1) \times (2N+1)$ matrix with $M_{k,k} = f_{k-N-1}$, $M_{k,k-1} = M_{k,k+1} = \Lambda$ and $\underline{b}_k = P_{k-N-1}$. Solve $\det(M) = 0$ to obtain values for ω. The corresponding null spaces give P_m.

5. CONCLUSION AND DISCUSSION

In this article we address two key problems: What wave packets can propagate though a spatially dispersive medium and how wave packets behave as they pass through a boundary between two media.

We have given two methods of deriving the boundary conditions for a junction between two spatially dispersive regions. These two methods give the same boundary conditions. These boundary conditions are tied to the choice of differential equation and ultimately depend on the microstructure of the materials.

We have given approximate solutions to Maxwell's equations for a periodic spatially dispersive medium. Since we have only given two modes, it is natural explore the behaviour of a general mode. From the numerical approach it appears that the higher frequency resemble the case for homogeneous media. This is currently being explored.

ACKNOWLEDGMENT

The authors are grateful for the support provided by STFC (the Cockcroft Institute ST/G008248/1), EPSRC (the Alpha-X project EP/J018171/1) and the Lancaster University Faculty of Science and Technology studentship program. In addition the authors would like to thank David Burton (Physics Department, Lancaster University) and Adam Noble (Physics Department, Strathclyde University) for help in preparing this article, and the referee for making useful suggestions.

REFERENCES

1. Agranovich, V. M. and V. L. Ginzburg, *Spatial Dispersion in Crystal Optics and the Theory of Excitons*, Wiley-Interscience, New York, 1967.
2. Agarwal, G. S., D. N. Pattanayak, and E. Wolf, "Electromagnetic fields in spatially dispersive media," *Phys. Rev. B*, Vol. 10, No. 4, 1447–1475, 1974.
3. Belov, P. A., R. Marques, S. I. Maslovski, I. S. Nefedov, M. Silveirinha, C. R. Simovski, and S. A. Tretyakov, "Strong spatial dispersion in wire media in the very large wavelength limit," *Phys. Rev. B*, Vol. 67, No. 11, 113103, 2003.
4. Gratus, J. and R. W. Tucker, "Covariant constitutive relations and relativistic inhomogeneous plasmas," *J. Math. Phys.*, Vol. 52, 042901, 2011.
5. Gratus, J. and R. W. Tucker, "Covariant constitutive relations, landau damping and non-stationary inhomogeneous plasmas," *Progress In Electromagnetics Research M*, Vol. 13, 145–156, 2010.
6. Maslovski, S. I., T. A. Morgado, M. G. Silveirinha, C. S. R. Kaipa, and A. B. Yakovlev, "Generalized additional boundary conditions for wire media," *New Journal of Physics*, Vol. 12, No. 11, 113047, 2010.
7. Silveirinha, M. G., "Additional boundary condition for the wire medium," *IEEE Transactions on Antennas and Propagation*, Vol. 54, No. 6, 1766–1780, 2006.
8. Silveirinha, M. G., C. A. Fernandes, and J. R. Costa, "Additional boundary condition for a wire medium connected to a metallic surface," *New Journal of Physics*, Vol. 10, No. 5, 053011, 2008.
9. Silveirinha, M. G., "Additional boundary conditions for nonconnected wire media," *New Journal of Physics*, Vol. 11, No. 11, 113016, 2009.
10. Costa, J. T. and M. G. Silveirinha, "Macroscopic electromagnetic response of arbitrarily shaped spatially dispersive bodies formed by metallic wires," *Physical Review B*, Vol. 86, No. 7, 075129, 2012.
11. Rosenzweig, J. B., G. Andonian, P. Muggli, P. Niknejadi, G. Travish, O. Williams, K. Xuan, and V. Yakimenko, "High frequency, high gradient dielectric wakefield acceleration experiments at SLAC and BNL," *AIP Conference Proceedings*, Vol. 1299, No. 1, 364–369, 2010.
12. Pekar, S. I., "The theory of electromagnetic waves in a crystal in which excitons are produced," *J. Explt. Theoet. Phys. (U.S.S.R)*, Vol. 6, No. 33, 785–200, 1958.
13. Zeyher, R., J. L. Birman, and W. Brenig, "Spatial dispersion effects in resonant polariton scattering. I. Additional boundary conditions for polarization fields," *Phys. Rev. B*, Vol. 6, 4613–4616, 1972.
14. Henneberger, K., "Additional boundary conditions: An historical mistake," *Phys. Rev. Lett.*, Vol. 80, No. 13, 2889–2892, 1998.