# Approximate Zero Modes for the Pauli Operator on a Region

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## Abstract

Let $\mathcal{P}_{\Omega,tA}$ denoted the Pauli operator on a bounded open region $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary conditions and magnetic potential $A$ scaled by some $t > 0$. Assume that the corresponding magnetic field $B = \operatorname{curl} A$ satisfies $B \in L \log L(\Omega) \cap C^\alpha(\Omega_0)$ where $\alpha > 0$ and $\Omega_0$ is an open subset of $\Omega$ of full measure (note that, the Orlicz space $L \log L(\Omega)$ contains $L^p(\Omega)$ for any $p > 1$). Let $\mathsf{N}_{\Omega,tA}(\lambda)$ denote the corresponding eigenvalue counting function. We establish the strong field asymptotic formula

$$\mathsf{N}_{\Omega,tA}(\lambda(t)) = \frac{t}{2\pi} \int_\Omega |B(x)| \, dx \; + o(t)$$

as $t \to +\infty$, whenever $\lambda(t) = Ce^{-ct^\sigma}$ for some $\sigma \in (0,1)$ and $c, C > 0$. The corresponding eigenfunctions can be viewed as a localised version of the Aharonov-Casher zero modes for the Pauli operator on $\mathbb{R}^2$.

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## 1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open region and $A = (A_1, A_2) \in L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ a magnetic potential. The corresponding magnetic momentum operator is then $P_A = -i\nabla - A$, where $\nabla = (\nabla_1, \nabla_2)$ denotes the gradient operator on $\mathbb{R}^2$. We wish to consider the Pauli operator $\mathcal{P}_{\Omega,A}$ on $\Omega$ with magnetic potential $A$. For Dirichlet boundary conditions this can be defined as the non-negative operator $\mathcal{P}_{\Omega,A}$ associated to the closure of the form

$$\mathbf{p}_{\Omega,A}(u) = \|P_{A,+}u_+\|^2 + \|P_{A,-}u_-\|^2, \quad u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \in C_0^\infty(\Omega, \mathbb{C}^2), \qquad (1)$$

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where $P_{A,\pm} = P_{A,1} \pm iP_{A,2}$. Set $B = \text{curl}\, A = \nabla_1 A_2 - \nabla_2 A_1$, the magnetic field associated with the potential $A$ (initially defined as a distribution). A straightforward formal calculation leads to the Lichnerowicz formula

$$\mathcal{P}_{\Omega,A} = \begin{pmatrix} H_{\Omega,A} & 0 \\ 0 & H_{\Omega,A} \end{pmatrix} - \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}, \tag{2}$$

where $H_{\Omega,A} = P_{A,1}^2 + P_{A,2}^2$ is the magnetic Schrödinger operator. If we assume that $B$ belongs to the Orlicz space $L \log L(\Omega)$ then (2) can be rigorously justified and used to help show that $\mathcal{P}_{\Omega,A}$ has a compact resolvent and hence discrete spectrum (see Proposition 2.4). Enumerate the eigenvalues of $\mathcal{P}_{\Omega,A}$ (including multiplicities) as $0 \le \lambda_1(\mathcal{P}_{\Omega,A}) \le \lambda_2(\mathcal{P}_{\Omega,A}) \le \dots$, and introduce the corresponding counting function

$$\mathsf{N}_{\Omega,A}(\lambda) = \#\big\{ n \in \mathbb{N} : \lambda_n(\mathcal{P}_{\Omega,A}) \le \lambda \big\}, \quad \lambda \in \mathbb{R}.$$

We are interested in the behaviour of $\mathsf{N}_{\Omega,A}$ in the strong field regime. Fixing $A$ we consider $\mathsf{N}_{\Omega,tA}(\lambda(t))$ for the scaled potential $tA$ and $\lambda(t) \le O(t)$ in the limit $t \to +\infty$. A simple rescaling shows that this is equivalent to the semi-classical regime.

When $\lambda(t) = O(t)$ the quantity $\mathsf{N}_{\Omega,tA}(\lambda(t))$ obeys a natural Weyl type asymptotics. To state this precisely introduce auxiliary functions $\nu^-$ and $\nu^+$ which are, respectively, the maximal lower and minimal upper semi-continuous extensions of

$$\nu(b,\lambda) = \frac{|b|}{2\pi} \#\big\{ m \in \mathbb{Z} : 2|mb| \le \lambda \big\}, \quad \lambda, b \in \mathbb{R},\ b \ne 0,\ \lambda \notin 2|b|\mathbb{N}_0. \tag{3}$$

**Theorem 1.1.** *Suppose $B \in L \log L(\Omega) \cap C(\Omega_0)$ where $\Omega_0 \subseteq \Omega$ is open and $\Omega \setminus \Omega_0$ has zero (Lebesgue) measure. If $\lambda(t) = \Lambda t + o(t)$ for some $\Lambda \in \mathbb{R}$ then*

$$\liminf_{t\to\infty} \frac{1}{t} \mathsf{N}_{\Omega,tA}(\lambda(t)) \ge \int_\Omega \nu^-(B(x),\Lambda)\, dx$$

*and*

$$\limsup_{t\to\infty} \frac{1}{t} \mathsf{N}_{\Omega,tA}(\lambda(t)) \le \int_\Omega \nu^+(B(x),\Lambda)\, dx.$$

Numerous results similar or related to Theorem 1.1 have been obtained. Some of the earliest work ([3], [27]) looked at spectral asymptotics for magnetic (Schrödinger) bottles. While these works focused on a different class of operators the ideas of [3] in particular form the basis of our approach to Theorem 1.1 (see also [28]). For magnetic Schrödinger operators on a region various two term spectral asymptotic questions have been considered in both the Dirichlet and Neumann cases (see [14], [11], [4], [12] and references therein); whilst giving more precise details, these results also require greater regularity (and other conditions) on $\Omega$ and $B$.

In another direction, various authors have considered bound states of the Pauli operator with an additional electric potential. The presence of the latter distinguishes the strong field and semi-classical regimes, leading to multi-parameter problems. The semi-classical behaviour of sums of negative eigenvalues of the form

$\sum_n |\lambda_n|^\gamma$, $\gamma > 0$ was considered in [20], [8], [25] and [9] for example. These works all rely on a priori bounds on the eigenvalue sums which have the correct order in the parameters; typically Lieb-Thirring type inequalities have been developed for this purpose. However eigenvalue counting corresponds to the case $\gamma = 0$ (also known as the CLR inequality) and is always excluded in dimension 2.

The asymptotic bounds in Theorem 1.1 remain finite provided $B \in L^1(\Omega)$. We use the slightly stronger condition $B \in L \log L(\Omega)$ to obtain a priori bounds on $\mathsf{N}_{\Omega, tA}(\lambda(t))$ (covering the lack of a suitable Lieb-Thirring/CLR inequality); in turn these bounds are derived from estimates in [26] which don't extend to cover the $L^1$ case. The continuity condition $B \in C(\Omega_0)$ relates to our method for approximating $B$ locally by a constant field. While it is likely that at least the latter condition can be relaxed the optimal regularity condition for $B$ remains unclear.

The asymptotic lower and upper bounds given by Theorem 1.1 differ if the set

$$\big\{ x \in \Omega : 2m|B(x)| = \Lambda \text{ for some } m \in \mathbb{N}_0 \big\} \tag{4}$$

has non-zero measure. When $\Lambda \neq 0$ this is a non-generic situation for variable fields. On the other hand, when $\lambda(t) = o(t)$ (4) is the whole of $\Omega$ for any $B$; the lower bound in Theorem 1.1 then reduces to 0 while the upper bound becomes

$$\limsup_{t \to \infty} \frac{1}{t} \mathsf{N}_{\Omega, tA}(\lambda(t)) \leq \Phi_\Omega(|B|), \tag{5}$$

where

$$\Phi_\Omega(b) = \frac{1}{2\pi} \int_\Omega b(x) \, dx$$

is the *flux* of a magnetic field $b$ on $\Omega$ (see the end of Section 2 for some further details). It transpires that the upper bound gives the correct asymptotics for even sub-exponentially decaying $\lambda(t)$. Our main result is the following (in which $C^\alpha$ is used to denote the space of Hölder continuous functions).

**Theorem 1.2.** *Suppose $B \in L \log L(\Omega) \cap C^\alpha(\Omega_0)$ where $\alpha > 0$, $\Omega_0 \subseteq \Omega$ is open and $\Omega \setminus \Omega_0$ has zero (Lebesgue) measure. If $\lambda(t) \geq Ce^{-ct^\sigma}$ for some constants $\sigma \in (0,1)$ and $c, C > 0$ then*

$$\liminf_{t \to \infty} \frac{1}{t} \mathsf{N}_{\Omega, tA}(\lambda(t)) \geq \Phi_\Omega(|B|).$$

For strong fields this result guarantees the existence of approximately $\Phi_\Omega(|tB|)$ sub-exponentially small eigenvalues of the Pauli operator. The corresponding eigenfunctions, which we informally term *approximate zero modes*, can be viewed as a local version of the Aharonov-Casher zero modes. The latter are a dimension $\lfloor |\Phi_{\mathbb{R}^2}(tB)| \rfloor$ set of spin-definite zero energy bound states of the Pauli operator on $\mathbb{R}^2$; the spin is aligned with the dominate sign of $B$ (see [2] and [10]). In the strong field limit strong localisation should confine such states to regions where $B$ has its dominate sign; indeed such localisation of the Aharonov-Casher construction lies at the heart of our argument (see below and Section 4 for further details). A different manifestation of this localisation, relating to the ground state density of the Pauli operator on $\mathbb{R}^2$, was obtained in [7].

*Remark* 1.1. The magnetic potential $A$ (and hence the Pauli operator $\mathcal{P}_{\Omega,A}$) is not uniquely defined by $B$. If $\Omega$ is simply connected different choices of $A$ lead to unitarily equivalent Pauli operators (see Proposition 2.7) so the counting function $\mathsf{N}_{\Omega,A}$ will not depend on the particular choice of $A$. For more general regions this is no longer true; in this case our results hold independently of the choice of $A$.

*Remark* 1.2. It is possible to consider operators corresponding to non-Dirichlet boundary conditions. While there is a natural choice for a Neumann version of the magnetic Schrödinger operator (which has received particular attention in connection with the Ginzburg-Landau theory of superconductivity), it is less clear how one should define a Neumann version of the Pauli operator. One possibility would be to use the maximal closed extensions of $P_{A,\pm}$ in (1); however such an operator does not have a compact resolvent (even when $A \equiv 0$), leading to a very different class of spectral problems. Alternatively one could use (2) to define a "Neumann" Pauli operator in terms of the Neumann magnetic Schrödinger operator. With some additional restrictions on the regularity of $\Omega$ (such as having a Lipschitzian boundary) Theorems 1.1 and 1.2 can be extended to cover such operators (see Remark 2.6 for some further details). However the operator defined in this manner is not always non-negative (so cannot be the square of a Dirac operator). Only relatively crude estimates for the asymptotics of the size and number of negative eigenvalues follow from the immediate extensions to our results; in particular, for any $\varepsilon > 0$ the number of eigenvalues below $-\varepsilon t$ is $o(t)$ as $t \to +\infty$. Further work would be needed to determine whether (the majority of) those eigenvalues guaranteed by Theorem 1.2 have small absolute value (so can be regarded as belonging to approximate zero modes).

*Remark* 1.3. When $B$ is constant the spectra of the Pauli and magnetic Schrödinger operators on $\mathbb{R}^2$ reduces to a set Landau levels. For non-constant fields this level structure is destroyed, with the typical exception of the zero energy level of the Pauli operator (the Aharonov-Casher zero modes). On the other hand the lower and upper bounds given by Theorem 1.1 will differ for any $\Lambda$ corresponding to a Landau level generated by a value at which $B$ is locally constant on some region (these are precisely the $\Lambda$ for which the set in (4) has non-zero measure). It is likely that a version of Theorem 1.2 could be extended to such cases. A related problem of eigenvalue accumulation near Landau levels after perturbation by a decaying electric potential has been considered; see [6] and references therein.

Precise definitions and various preliminary results are collected in Section 2; in particular, the Lichnerowicz formula (2) is justified (Proposition 2.4), a priori bounds on $\mathsf{N}_{\Omega,A}(\lambda)$ are obtained (Proposition 2.5) and gauge transformations are discussed (Proposition 2.7).

The proof of Theorem 1.1 is given in Section 3. This follows a standard localisation type argument (c.f., [3], [25]) using a sequence of piecewise constant approximations to $B$ based on increasingly fine tilings of $\Omega_0$ by squares (Section 3.2). The corresponding approximation results for quadratic forms are obtained in Section 3.3 while the necessary eigenvalue counting function results for constant fields on a square are given in Section 3.1 (these are taken almost directly from [3]). The bounds in Theorem 1.1 are finally pieced together in Section 3.4.

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Theorem 1.2 is justified in Section 4 by initially reducing the problem to the case of fields of constant sign on a disc (Section 4.1). Suitable test functions on the disc can be constructed from holomorphic functions with the help of the "real gauge" transformation introduced in [2]; these functions need to be cut-off at the boundary, a process which ultimately leads to a spectral problem on the circle (Section 4.2, with further technical details in Sections 4.3 and 4.4).

## Notation

For a bounded open region $\Omega \subset \mathbb{R}^2$ we use $C(\Omega)$, $C^\alpha(\Omega)$ and $\mathcal{O}(\Omega)$ to denote the space of continuous, Hölder continuous and holomorphic functions on $\Omega$, without restriction on behaviour near the boundary; we replace $\Omega$ with $\overline{\Omega}$ to indicate uniform versions of the same spaces. For $k \in \mathbb{N}_0$ we use $C^{k,\alpha}(\Omega)$ and $W^{k,2}(\Omega)$ to denote the Hölder-Zygmund and Sobolev space consisting of functions with $k$ derivatives in $C^\alpha(\Omega)$ and $L^2(\Omega)$ respectively. The completion of $C_0^\infty(\Omega)$ in $W^{k,2}(\Omega)$ is denoted by $W_0^{k,2}(\Omega)$, while $C_0^k(\Omega)$ denotes the space of $k$-times continuously differentiable functions with compact support contained in $\Omega$. Unless otherwise indicated norms and inner-products are defined in the relevant $L^2$ sense.

The open disc with radius $R > 0$ and centre $a \in \mathbb{R}^2$ is denoted $\mathbb{D}_R(a)$. When $a = 0$ or $R = 1$ these values are omitted; in particular, $\mathbb{D}$ is the open unit disc. We also set $(x)_+ = \max\{x, 0\}$, the positive part of $x \in \mathbb{R}$.

General positive constants are denoted by $C$, with numerical subscripts used to keep track of particular constants in subsequent discussions.

## 2 Preliminaries

Let $A \in L_{\mathrm{loc}}^2(\Omega, \mathbb{R}^2)$. Consider the Dirac operator, initially defined by

$$\mathcal{D}_A u = \sigma.P_A u = \begin{pmatrix} 0 & P_{A,-} \\ P_{A,+} & 0 \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{6}$$

for $u \in C_0^\infty(\Omega, \mathbb{C}^2)$. The operator $\mathcal{D}_A$ is densely defined and symmetric, hence closable; by a slight abuse of notation we will also denote the closure by $\mathcal{D}_A$.

*Remark* 2.1. Alternatively we can proceed by considering the operators $P_{A,\pm}$ separately. Initially densely defined on $C_0^\infty(\Omega)$ these operators satisfy $P_{A,\pm} \subseteq P_{A,\mp}^*$ and are hence closable. Using the same notation for the closures (6) then holds for all $u \in \mathrm{Dom}(\mathcal{D}_A) = \mathrm{Dom}(P_{A,+}) \times \mathrm{Dom}(P_{A,-})$.

Define a quadratic form by

$$\mathbf{p}_{\Omega,A}(u) = \|\mathcal{D}_A u\|^2 = \|P_{A,+} u_+\|^2 + \|P_{A,-} u_-\|^2, \quad u \in \mathrm{Dom}(\mathcal{D}_A).$$

Since $\mathcal{D}_A$ is a closed operator $\mathbf{p}_{\Omega,A}$ is a closed non-negative quadratic form. The *Pauli operator* on $\Omega$ with magnetic potential $A$ and Dirichlet boundary conditions is defined to be the corresponding self-adjoint operator given by the representation theorem; we'll use the notation $\mathcal{P}_{\Omega,A}$.

*Remark* 2.2 (Case $A \equiv 0$). Since $\Omega$ is bounded $(\|\nabla_1 u\|^2 + \|\nabla_2 u\|^2)^{1/2}$ gives an equivalent norm on the Sobolev space $W_0^{1,2}(\Omega)$ (see [1]). Also, for $u \in C_0^\infty(\Omega)$,

$$\|P_{0,\pm} u\|^2 = \|\nabla_1 u\|^2 + \|\nabla_2 u\|^2 \mp i \int_\Omega \left( \nabla_1 \overline{u}\, \nabla_2 u - \nabla_2 \overline{u}\, \nabla_1 u \right) = \|\nabla_1 u\|^2 + \|\nabla_2 u\|^2.$$

Completion then gives $\mathrm{Dom}(\mathcal{D}_0) = W_0^{1,2}(\Omega, \mathbb{C}^2)$ with $\mathbf{p}_{\Omega,0}(u) \asymp \|u\|^2_{W_0^{1,2}(\Omega,\mathbb{C}^2)}$.

*Remark* 2.3. Since $P_{A,\pm} \overline{u} = -\overline{P_{-A,\mp} u}$ when $u \in C_0^\infty(\Omega)$ we get $\mathbf{p}_{\Omega,A}(\mathcal{J}u) = \mathbf{p}_{\Omega,-A}(u)$ for all $u \in \mathrm{Dom}(\mathbf{p}_{\Omega,A}) = \mathrm{Dom}(\mathbf{p}_{\Omega,-A})$, where $\mathcal{J}$ is the anti-linear isometric involution on $L^2(\Omega, \mathbb{C}^2)$ defined by

$$\mathcal{J}u = \begin{pmatrix} \overline{u_-} \\ \overline{u_+} \end{pmatrix}, \quad u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \in L^2(\Omega, \mathbb{C}^2).$$

It follows that $\mathcal{J}\mathcal{P}_{\Omega,A}\mathcal{J} = \mathcal{P}_{\Omega,-A}$, and so $\mathcal{P}_{\Omega,A}$ and $\mathcal{P}_{\Omega,-A}$ have the same spectrum.

In order to make use of results for Schrödinger operators we will need a rigorous form of the Lichnerowicz formula (2). We begin by introducing the magnetic Schrödinger operator in a way that parallels our introduction of the Pauli operator.

For $l = 1, 2$ we initially define the operator $P_{A,l} = -i\nabla_l - A_l$ on $C_0^\infty(\Omega)$. This operator is densely defined and symmetric, hence closable; by a slight abuse of notation we will also denote the closure by $P_{A,l}$. Setting

$$\mathcal{H}_A(\Omega) = \mathrm{Dom}(P_{A,1}) \cap \mathrm{Dom}(P_{A,2}),$$

the quadratic form defined by

$$\mathbf{h}_{\Omega,A}(u) = \|P_{A,1} u\|^2 + \|P_{A,2} u\|^2, \quad u \in \mathcal{H}_A(\Omega)$$

is closed and non-negative. The *magnetic Schrödinger operator* on $\Omega$ with magnetic potential $A$ and Dirichlet boundary conditions is defined to be the corresponding self-adjoint operator given by the representation theorem; we'll use the notation $H_{\Omega,A}$.

*Remark* 2.4 (Case $A \equiv 0$). It is straightforward to see that $\mathcal{H}_0(\Omega) = W_0^{1,2}(\Omega)$ with $\mathbf{h}_{\Omega,0}(u) \asymp \|u\|^2_{W_0^{1,2}(\Omega)}$ (c.f., Remark 2.2). Furthermore $H_{\Omega,0} = -\Delta_\Omega$, the laplacian on $\Omega$ with Dirichlet boundary conditions.

We need to add a scalar potential to the operator $H_{\Omega,A}$. Since $H_{\Omega,A}$ is a semi-bounded self-adjoint operator this can be done conveniently via the standard KLMN construction if the scalar potential is relatively form bounded with respect to $H_{\Omega,A}$ with relative bound less than 1 (see [22], for example). If $V \in L^1_{\mathrm{loc}}(\Omega)$ is a real-valued function then the form given by

$$\mathbf{v}(u) = \langle u, Vu \rangle \tag{7}$$

is certainly defined for $u \in C_0^\infty(\Omega)$. To extend $\mathbf{v}$ to $\mathcal{H}_0(\Omega) = W_0^{1,2}(\Omega)$ we need to restrict $V$ to the Orlicz space $L \log L(\Omega)$. More precisely, introduce the $N$-function

$$\mathscr{A}(t) = (t+1)\log(t+1) - t, \quad t \geq 0;$$

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we then define $L \log L(\Omega)$ to be the Orlicz space $L_{\mathscr{A}}(\Omega)$ (see [1]). It is straightforward to check that $L^p(\Omega) \subset L_{\mathscr{A}}(\Omega) \subset L^1(\Omega)$ for any $p > 1$. Now suppose $V \in L_{\mathscr{A}}(\Omega)$. By [26, Lemma 2.1] (see also Remark 2.5 below) $\mathbf{v}$ given by (7) then defines a bounded form on $W_0^{1,2}(\Omega)$, while the corresponding operator $T_V$ is compact. Viewing $T_V$ as multiplication by $V$ acting as a mapping $W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$, it follows that $V$ is relatively form compact and hence infinitesimally form bounded with respect to $H_{\Omega,0}$. The KLMN construction can then be used to define $H_{\Omega,0} - V$. Since the Dirichlet laplacian $-\Delta_\Omega = H_{\Omega,0}$ has a compact resolvent (see [23], for example) the infinitesimal form boundedness of $V$ implies $H_{\Omega,0} - V$ also has a compact resolvent (see [17]).

*Remark* 2.5. The results we need from [26] are mostly stated from the case of Neumann boundary conditions under the assumption that $\Omega$ has a Lipschitzian boundary. However it is easy to see that they also hold in the Dirichlet case for arbitrary bounded $\Omega$.

The results of the above discussion can be generalised to include a magnetic potential $A$ with the help of the diamagnetic inequality; a convenient form of the latter can be found in [15].

**Proposition 2.1.** *Let $A \in L_{\mathrm{loc}}^2(\Omega)$ and $V \in L_{\mathscr{A}}(\Omega)$. Then (multiplication by) $V$ is an infinitesimally form bounded perturbation of $H_{\Omega,A}$. Furthermore the semibounded self-adjoint operator $H_{\Omega,A} - V$ (resulting from the KLMN construction) has a compact resolvent.*

*Proof.* The discussion proceeding the result covers the case $A \equiv 0$. Using [15, Theorem 3.3; see also Remark 3.4(i)] it follows that $V$ is infinitesimally form bounded with respect to $H_{\Omega,A}$, while, for $t > 0$,

$$e^{-t(H_{\Omega,A} - V)} \preccurlyeq e^{-t(H_{\Omega,0} - V)} \tag{8}$$

(where $S \preccurlyeq T$ means that $S$ is dominated by $T$). However if $S \preccurlyeq T$ and $T$ is compact then $S$ must also be compact (see [5], [21]), while for a semi-bounded self-adjoint operator $Q$, $e^{-Q}$ is compact iff $Q$ has a compact resolvent. It follows that $H_{\Omega,A} - V$ has a compact resolvent. ∎

Our a priori bounds for the counting function of the Pauli operator can be obtained from suitable bounds on the number of negative eigenvalues of $H_{\Omega,A} - V$. The latter will be obtained through a two step process; results from [26] allow us to estimate the counting function for $H_{\Omega,0} - V$ under the condition that $V \in L_{\mathscr{A}}(\Omega)$, while the techniques of [24] allow us to use the diamagnetic inequality (see (8)) to generalise to $H_{\Omega,A} - V$.

**Proposition 2.2.** *Let $A \in L_{\mathrm{loc}}^2(\Omega)$ and $V \in L_{\mathscr{A}}(\Omega)$. Then*

$$\#\{\lambda_n(H_{\Omega,A} - V) \leq 0\} \leq C_1 \|V\|_{L_{\mathscr{A}}(\Omega)}.$$

*Proof.* Since the positive and negative parts of any $V \in L_{\mathscr{A}}(\Omega)$ also belong to $L_{\mathscr{A}}(\Omega)$, while the addition of a positive scalar potential can only raise eigenvalues, it suffices to prove the result assuming $V \geq 0$.

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Now $(H_{\Omega,0})^{1/2} : \mathcal{H}_0(\Omega) \to L^2(\Omega)$ is an isomorphism (this is equivalent to the fact that $\mathbf{h}_{\Omega,0}(u) \asymp \|u\|^2_{W_0^{1,2}(\Omega)}$ on $\mathcal{H}_0(\Omega) = W_0^{1,2}(\Omega)$; see Remark 2.4). Thus the expression $S_V = ((H_{\Omega,0})^{-1/2})^* T_V (H_{\Omega,0})^{-1/2}$ defines a non-negative self-adjoint operator on $L^2(\Omega)$. By [26, Corollary 2.3] $T_V$ and hence $S_V$ belong to the weak first Schatten class with $\|S_V\|_{1,w} \le C_{1,1} \|V\|_{L_{\mathscr{A}(\Omega)}}$ for some constant $C_{1,1}$; in other words

$$0 \le \lambda_n(S_V) \le C_{1,1} \|V\|_{L_{\mathscr{A}(\Omega)}} \, n^{-1}, \quad n \in \mathbb{N}.$$

A standard Birman-Schwinger type argument then gives

$$\#\{\lambda_n(H_{\Omega,0} - \gamma V) \le 0\} \le C_{1,1} \gamma \|V\|_{L_{\mathscr{A}}(\Omega)}, \quad \gamma \ge 0.$$

Denote the right hand side as $\mu(\gamma)$ and let $\widehat{\mu}$ be the Laplace transform of $\mu$; in particular $\gamma^{-1} \widehat{\mu}(\gamma^{-1}) = \mu(\gamma)$. Now (8) (with $V \equiv 0$) gives $e^{-tH_{\Omega,A}} \preccurlyeq e^{-tH_{\Omega,0}}$. Using [24, Theorem 3] we then obtain

$$\#\{\lambda_n(H_{\Omega,A} - \gamma V) \le 0\} \le e\gamma^{-1}\widehat{\mu}(\gamma^{-1}) = C_1 \gamma \|V\|_{L_{\mathscr{A}}(\Omega)}, \quad \gamma \ge 0,$$

where $C_1 = eC_{1,1}$. ∎

We can now compare the Pauli operator with the magnetic Schrödinger operator. We begin by looking at the corresponding forms. For any $A \in L^2_{\text{loc}}(\Omega)$ and $u \in C_0^\infty(\Omega)$ we can define $\mathbf{b}(u)$ to be the distribution $B = \nabla_1 A_2 - \nabla_2 A_1$ acting on the test function $|u|^2 \in C_0^\infty(\Omega)$. If $B \in L^1_{\text{loc}}(\Omega)$ then

$$\mathbf{b}(u) = \int_\Omega B|u|^2 = \langle u, Bu \rangle;$$

that is, $\mathbf{b}$ is just the form associated with the operator of multiplication by $B$.

**Lemma 2.3.** *Let* $A \in L^2_{\text{loc}}(\Omega)$. *If* $u \in C_0^\infty(\Omega, \mathbb{C}^2)$ *then*

$$\mathbf{p}_{\Omega,A}(u) = \mathbf{h}_{\Omega,A}(u_+) - \mathbf{b}(u_+) + \mathbf{h}_{\Omega,A}(u_-) + \mathbf{b}(u_-). \tag{9}$$

*Proof.* If $v \in C_0^\infty(\Omega)$ then $\|P_{A,\pm}v\|^2 = \|P_{A,1}v\|^2 + \|P_{A,2}v\|^2 \mp 2\,\text{Im}\langle P_{A,1}v, P_{A,2}v\rangle$ while

$$
\begin{aligned}
2\,\text{Im}\langle P_{A,1}v, P_{A,2}v \rangle = \int_\Omega \Big[ &-i\,\nabla_1 \overline{v}\,\nabla_2 v - \nabla_1 \overline{v}\, A_2 v + A_1 \overline{v}\, \nabla_2 v - iA_1 A_2 |v|^2 \\
&+ i\,\nabla_2 \overline{v}\, \nabla_1 v - A_2 \overline{v}\, \nabla_1 v + \nabla_2 \overline{v}\, A_1 v + iA_2 A_1 |v|^2 \Big] \\
= \int_\Omega \Big[ &-A_2\,\nabla_1 |v|^2 + A_1\,\nabla_2 |v|^2 \Big].
\end{aligned}
$$

The final expression is just the distribution $\nabla_1 A_2 - \nabla_2 A_1 = B$ acting on $|v|^2$. The result now follows from the definitions of $\mathbf{p}_{\Omega,A}$, $\mathbf{h}_{\Omega,A}$ and $\mathbf{b}$. ∎

If $B \in L_{\mathscr{A}}(\Omega)$ the operators $H_{\Omega,A} \mp B$ can be defined as discussed above. The corresponding forms $\mathbf{h}_{\Omega,A} \mp \mathbf{b}$ have core $C_0^\infty(\Omega)$, while $C_0^\infty(\Omega, \mathbb{C}^2)$ is a core for $\mathbf{p}_{\Omega,A}$. The previous result then gives $\text{Dom}(\mathbf{p}_{\Omega,A}) = \mathcal{H}_A(\Omega, \mathbb{C}^2)$, with (9) extending to all $u \in \mathcal{H}_A(\Omega, \mathbb{C}^2)$. The operator identity (2) now follows, allowing Proposition 2.1 to be applied to $\mathcal{P}_{\Omega,A}$; we summarise what we need as follows.

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**Proposition 2.4.** *Suppose $A \in L^2_{\text{loc}}(\Omega)$ with $B \in L_{\mathscr{A}}(\Omega)$. Then the Lichnerowicz formula (2) holds as an operator identity for the Pauli and magnetic Schrödinger operators with Dirichlet boundary conditions. Furthermore $\mathcal{P}_{\Omega,A}$ has a compact resolvent and hence discrete spectrum.*

Proposition 2.2 can now be used to obtain a priori bounds on $\mathsf{N}_{\Omega,A}(\lambda)$. However we will need uniform versions of these bounds for sub-regions of $\Omega$. If $\Omega' \subseteq \Omega$ is open we can restrict $A$ to $\Omega'$ and consider the Pauli operator $\mathcal{P}_{\Omega',A}$ with corresponding counting function $\mathsf{N}_{\Omega',A}(\lambda)$. Using $\chi_{\Omega'}$ to denote the characteristic function for $\Omega'$ Proposition 2.4 and a simple variational argument then give

$$\mathsf{N}_{\Omega',A}(\lambda) \leq \#\{\lambda_n(H_{\Omega,A} - (\lambda + B)\chi_{\Omega'}) \leq 0\} + \#\{\lambda_n(H_{\Omega,A} - (\lambda - B)\chi_{\Omega'}) \leq 0\},$$

for any $\lambda \in \mathbb{R}$. However $\|(\lambda \pm B)\chi_{\Omega'}\|_{L_{\mathscr{A}}(\Omega)} \leq \|B\|_{L_{\mathscr{A}}(\Omega')} + |\lambda| \|1\|_{L_{\mathscr{A}}(\Omega')}$, so Proposition 2.2 now completes the following.

**Proposition 2.5.** *Suppose $A \in L^2_{\text{loc}}(\Omega)$ with $B \in L_{\mathscr{A}}(\Omega)$. Then*

$$\mathsf{N}_{\Omega',A}(\lambda) \leq 2C_1\big(\|B\|_{L_{\mathscr{A}}(\Omega')} + |\lambda|\|1\|_{L_{\mathscr{A}}(\Omega')}\big)$$

*for any open $\Omega' \subseteq \Omega$; the constant $C_1$ may depend on $\Omega$ but not on $\Omega'$.*

The magnetic potentials $A, A' \in L^2_{\text{loc}}(\Omega)$ are *gauge equivalent* if $A' = A + \nabla\psi$ for some $\psi \in W^{1,2}_{\text{loc}}(\Omega)$. It follows that $\operatorname{curl} A' = \operatorname{curl} A$ (as distributions), so $A'$ and $A$ generate the same magnetic field. The converse is not generally true; a topological condition on $\Omega$ is also required. The following is a particular case of [19, Lemma 1.1].

**Lemma 2.6.** *Suppose $\Omega$ is simply connected. If $A, A' \in L^2_{\text{loc}}(\Omega)$ satisfy $\operatorname{curl} A' = \operatorname{curl} A$ (as distributions) then there exists $\psi \in W^{1,2}_{\text{loc}}(\Omega)$ with $A' = A + \nabla\psi$.*

Now suppose $A, A' \in L^2_{\text{loc}}(\Omega)$ are gauge equivalent and $V \in L_{\mathscr{A}}(\Omega)$ (so that the operators $H_{\Omega,A} - V$ and $H_{\Omega,A'} - V$ correspond to the closures of the semi-bounded forms $\mathbf{h}_{\Omega,A} - \mathbf{v}$ and $\mathbf{h}_{\Omega,A'} - \mathbf{v}$ on $C_0^\infty(\Omega)$). Choosing $\psi \in W^{1,2}_{\text{loc}}(\Omega)$ with $A' = A + \nabla\psi$, the argument given for the proof of [19, Theorem 1.2] then shows the unitary operator $U_\psi$ of multiplication by $e^{i\psi}$ gives a unitary equivalence $H_{\Omega,A'} - V = U_\psi(H_{\Omega,A} - V)U_\psi^*$ (note that [19, Theorem 1.2] is stated for $\Omega = \mathbb{R}^2$ and only assumes $\operatorname{curl} A = \operatorname{curl} A'$; however the former is only used to guarantee the existence of $\psi$, after which the proof easily adapts to cover arbitrary $\Omega$). Coupled with Proposition 2.4 and Lemma 2.6 we arrive at the following.

**Proposition 2.7.** *Suppose $A, A' \in L^2_{\text{loc}}(\Omega)$ satisfy $A' = A + \nabla\psi$ for some $\psi \in W^{1,2}_{\text{loc}}(\Omega)$ (which follows from the condition $\operatorname{curl} A' = \operatorname{curl} A$ when $\Omega$ is simply connected). Also suppose $B \in L_{\mathscr{A}}(\Omega)$. Then $U_\psi$ gives the unitary equivalence $\mathcal{P}_{\Omega,A'} = U_\psi\mathcal{P}_{\Omega,A}U_\psi^*$; in particular, $\mathcal{P}_{\Omega,A}$ and $\mathcal{P}_{\Omega,A'}$ have the same spectrum.*

*Remark* 2.6 (Maximal operators). For $l = 1, 2$ the operator $P_{A,l}$ is the minimal closed extension of the magnetic momentum operator initially defined on $C_0^\infty(\Omega)$. The corresponding maximal closed extension thus satisfies $P_{A,l}^{\max} = P_{A,l}^*$. The closed non-negative quadratic form $\mathbf{h}_{\Omega,A}^{\max}(u) = \|P_{A,1}^{\max}u\|^2 + \|P_{A,2}^{\max}u\|^2$ can be used to define the magnetic Schrödinger operator on $\Omega$ with magnetic potential $A$ and Neumann boundary conditions (see [15] for further discussion of this operator).

Neumann versions of Propositions 2.1 and 2.2 are possible if we assume $\Omega$ has some additional regularity. In all cases it is sufficient to assume the existence of a linear extension operator which is continuous as a map $W^{k,2}(\Omega) \to W^{k,2}(\mathbb{R}^2)$ for $k = 0, 1$ (such operators exist if $\Omega$ has a Lipschitzian boundary). Assuming $\Omega$ satisfies such a condition and $B \in L_{\mathscr{A}}(\Omega)$ we can use (2) to define a "Neumann" Pauli operator from the Neumann magnetic Schrödinger operator; denote this operator by $\mathcal{P}_{\Omega,A}'$, with corresponding form $\mathbf{p}_{\Omega,A}'$. It is then possible to extend Theorems 1.1 and 1.2 to cover $\mathcal{P}_{\Omega,A}'$, for the most part simply by using a combination of variational arguments (note that $\mathbf{p}_{\Omega,A}'$ is an extension of $\mathbf{p}_{\Omega,A}$) and straightforward modifications to the given proofs; the most notable exception is Proposition 2.5 and its application where, to retain uniformity in $\Omega'$, one is forced to consider operators with mixed boundary conditions (Neumann on $\partial\Omega' \cap \partial\Omega$ and Dirichlet on $\partial\Omega' \setminus \partial\Omega$).

We complete this section by considering some basic properties of the auxiliary functions $\nu^{\pm}$. If $b \neq 0$ then (3) gives $\nu(b, \lambda) = 0$ when $\lambda < 0$, while

$$\nu(b, \lambda) = (2m + 1)\frac{|b|}{2\pi} \leq \frac{1}{2\pi}(\lambda + |b|)$$

when $2m|b| < \lambda < 2(m + 1)|b|$ for some $m \in \mathbb{N}_0$. It follows that $\nu$ is locally bounded, so $\nu^{\pm}$ are well defined locally bounded functions on $\mathbb{R}^2$. Furthermore

$$0 \leq \nu^{\pm}(b, \lambda) \leq \frac{1}{2\pi}(|b| + |\lambda|), \quad b, \lambda \in \mathbb{R}. \tag{10}$$

In particular (10) ensures the integrals appearing in Theorem 1.1 are finite whenever $B \in L^1(\Omega)$. We further note that $\nu^{\pm}$ are homogeneous of degree 1 while, for any $b, \lambda \in \mathbb{R}$, we have $\nu^{\pm}(b, \lambda) = 0$ if $\lambda < 0$, $\nu^{\pm}(0, \lambda) = \lambda/(2\pi)$ if $\lambda \geq 0$ ($\nu^{\pm}$ are actually continuous at $b = 0$), $\nu^{-}(b, 0) = 0$ and $\nu^{+}(b, 0) = |b|/(2\pi)$. The final identity reduces the upper bound in Theorem 1.1 to (5) when $\Lambda = 0$.

# 3 General Asymptotics

## 3.1 Constant field on a square

For $R > 0$ and $b \in \mathbb{R}$ let $\mathcal{P}_{R,b}$ denote a Pauli operator on the square $(0, R)^2$ with Dirichlet boundary conditions and corresponding to a constant magnetic field $b$. One choice for the magnetic potential is $A(x) = b(-x_2, x_1)/2$, while $(0, R)^2$ is simply connected so Proposition 2.7 shows that any other choice leads to a unitarily equivalent operator. Thus the eigenvalue counting function

$$\mathsf{N}_{R,b}(\lambda) = \#\left\{\lambda_n(\mathcal{P}_{R,b}) \leq \lambda\right\}$$

(counting with multiplicity) depends only on $R$, $b$ and $\lambda$. We can estimate $\mathsf{N}_{R,b}(\lambda)$ using the auxiliary function introduced in (3).

**Proposition 3.1.** *For any $\lambda, b \in \mathbb{R}$ and $\rho \in (0,1)$ we have*

$$R^2(1-\rho)^2\,\nu^+\!\big(b, \lambda - C_2 R^{-2}\rho^{-2}\big) \le \mathsf{N}_{R,b}(\lambda) \le R^2\,\nu^+(b, \lambda),$$

*where $C_2$ can be chosen as an absolute constant.*

*Proof.* Let $H_{R,|b|}$ denote a Dirichlet magnetic Schrödinger operator on $(0,R)^2$ corresponding to the constant field $|b|$, and let $\mathsf{n}_{R,|b|}(\lambda) = \#\{\lambda_n(H_{R,|b|}) \le \lambda\}$ denote the associated eigenvalue counting function (including multiplicity). Using Remark 2.3 and Proposition 2.4 we then get

$$\mathsf{N}_{R,b}(\lambda) = \mathsf{N}_{R,|b|}(\lambda) = \mathsf{n}_{R,|b|}(\lambda + |b|) + \mathsf{n}_{R,|b|}(\lambda - |b|).$$

On the other hand, [3, Theorem 3.1] gives an absolute constant $C_2$ such that

$$R^2(1-\rho)^2\,\mu\big(|b|, \lambda - C_2 R^{-2}\rho^{-2}\big) \le \mathsf{n}_{R,|b|}(\lambda) \le R^2\,\mu(|b|, \lambda),$$

where $\mu(|b|, \lambda) = 0$ for $\lambda < 0$, $\mu(0, \lambda) = \lambda/(4\pi)$ for $\lambda \ge 0$, and

$$\mu(|b|, \lambda) = \frac{|b|}{2\pi}\,\#\big\{m \in \mathbb{N}_0 : (2m+1)|b| \le \lambda\big\}$$

when $b \ne 0$. Since $\mu(|b|, \lambda + |b|) + \mu(|b|, \lambda - |b|) = \nu^+(b, \lambda)$ the result follows. ∎

## 3.2 Localisation

We want to approximate the field $B \in L_{\mathscr{A}}(\Omega) \cap C(\Omega_0)$ by a sequence of fields which take constant values on squares within $\Omega_0$. This approximation can only be made sufficiently good where $B$ is continuous (continuity is used when making the corresponding approximation to the potential; see Lemma 3.3). In turn this necessitates a degree of delicacy in the choice of the squares and the rate at which they approach the boundary of $\Omega_0$ (see Lemma 3.2).

For each $\delta > 0$ set
$$\Omega_\delta = \big\{x \in \Omega_0 : \overline{\mathbb{D}}_\delta(x) \subset \Omega_0\big\}.$$

Clearly $\Omega_\delta$ is open, $\Omega_\delta \subset\subset \Omega_{\delta'}$ whenever $0 \le \delta' < \delta$ (recall that $\Omega_0$ is bounded) and

$$\Omega_0 = \bigcup_{\delta > 0} \Omega_\delta. \tag{11}$$

Since $B \in C(\Omega_0)$ it follows that $B \in C(\overline{\Omega_\delta})$ for any $\delta > 0$.

**Lemma 3.2.** *We can find a strictly increasing sequence $(k_l)_{l \in \mathbb{N}_0}$ in $\mathbb{N}$ and, for each $k \ge k_0$, a finite indexing set $J_k$ and collection of disjoint open squares $S^{k,j}$, $j \in J_k$, of side length $2^{-k}$ with the following properties: setting*

$$\Omega^k = \operatorname{int} \overline{\bigcup_{j \in J_k} S^{k,j}} \tag{12}$$

*then, for each $l \in \mathbb{N}_0$ and $k_l \le k < k_{l+1}$,*

*(i)* $\Omega_{2^{-l+1}} \subseteq \Omega^k \subset\subset \Omega_{2^{-l}}$.

*(ii) For any $x, y \in \Omega^k$ with $|x - y| \leq 2^{-k-1/2}$ we have $|B(x) - B(y)| \leq 2^{-l}$.*

Each of the squares $S^{k,j}$ will be a translate of $(0, 2^{-k})^2$. We will use $\widehat{S}^{k,j}$ and $\overline{S}^{k,j}$ to denote the corresponding translates of $[0, 2^{-k})^2$ and $[0, 2^{-k}]^2$; thus $\overline{S}^{k,j}$ is just the closure of $S^{k,j}$ while $S^{k,j} \subset \widehat{S}^{k,j} \subset \overline{S}^{k,j}$. The set $\Omega^k$ is essentially the union of the squares $S^{k,j}$, $j \in J_k$, together with any edges lying between two squares. More precisely

$$\Omega^k = \operatorname{int} \bigcup_{j \in J_k} \overline{S}^{k,j} = \operatorname{int} \bigcup_{j \in J_k} \widehat{S}^{k,j} \, ;$$

in particular, each $x \in \Omega^k$ belongs to $\widehat{S}^{k,j}$ for a unique $j \in J_k$.

*Proof of Lemma 3.2.* For each $k \in \mathbb{N}_0$ and $\delta > 0$ set

$$d_k(\delta) = \sup\big\{|B(x) - B(y)| : x, y \in \Omega_\delta, \ |x - y| \leq 2^{-k-1/2}\big\}.$$

Since $B \in C(\overline{\Omega_\delta})$ we have $d_k(\delta) \to 0$ as $k \to \infty$ (for fixed $\delta$). Hence we can find a strictly increasing sequence $(k_l)_{l \in \mathbb{N}_0}$ in $\mathbb{N}$ with $d_{k_l}(2^{-l}) \leq 2^{-l}$ and $k_l > l$ for each $l \in \mathbb{N}_0$.

Let $k \geq k_0$ and choose $l \in \mathbb{N}_0$ so that $k_l \leq k < k_{l+1}$. Consider the tiling of $\mathbb{R}^2$ by copies of the square $[0, 2^{-k})^2$ which have been translated so that the corners lie on points of the lattice $(2^{-k}\mathbb{Z})^2$. Let $\widehat{S}^{k,j}$ for $j \in J_k$ denote the collection of squares from this tiling whose closure lies entirely within $\Omega_{2^{-l}}$. Set $S^{k,j} = \operatorname{int}(\widehat{S}^{k,j})$ for $j \in J_k$ and define $\Omega^k$ by (12). Clearly $\Omega^k \subset\subset \Omega_{2^{-l}}$. Now suppose $x \in \Omega_{2^{-l+1}}$, so $\overline{\mathbb{D}}_{2^{-l+1}}(x) \subset \Omega_0$. Let $\overline{S}$ be the closure of any square from the tiling with $x \in \overline{S}$ and let $y \in \overline{S}$. Then $|x - y| \leq 2^{-k+1/2} < 2^{-l}$ (since $k \geq k_l > l$) so $\overline{\mathbb{D}}_{2^{-l}}(y) \subset \overline{\mathbb{D}}_{2^{-l+1}}(x) \subset \Omega_0$ and hence $y \in \Omega_{2^{-l}}$. Thus $\overline{S} \subset \Omega_{2^{-l}}$ and so $S \in \{S^{k,j} : j \in J_k\}$. It follows that $x \in \Omega^k$. Finally, if $x, y \in \Omega^k$ with $|x - y| \leq 2^{-k-1/2}$ then $x, y \in \Omega_{2^{-l}}$ with $|x - y| \leq 2^{-k_l-1/2}$ (since $k \geq k_l$), so

$$|B(x) - B(y)| \leq d_{k_l}(2^{-l}) \leq 2^{-l}$$

(recall the defining properties of $k_l$). ∎

By Lemma 3.2(i) and (11) we get $\bigcap_{k \geq k_0}(\Omega \setminus \overline{\Omega^k}) = \bigcap_{\delta > 0}(\Omega \setminus \Omega_\delta) = \Omega \setminus \Omega_0$. Since $|\Omega| < \infty$ and $|\Omega \setminus \Omega_0| = 0$ it follows that

$$\big|\Omega \setminus \overline{\Omega^k}\big| \to 0 \text{ as } k \to \infty. \tag{13}$$

For $k \geq k_0$ set $\beta_k = 2^{-l}$ where $l \in \mathbb{N}_0$ is maximal such that $k_l \leq k$. Lemma 3.2 implies $(\beta_k)_{k \geq k_0}$ is a non-increasing positive sequence with $\beta_k \to 0$ as $k \to \infty$ while, for each $k \geq k_0$,

$$|B(x) - B(y)| \leq \beta_k \text{ whenever } x, y \in \Omega^k \text{ with } |x - y| \leq 2^{-k-1/2}. \tag{14}$$

For $k \geq k_0$ and $j \in J_k$ set $b^{k,j} = B(x)$ where $x$ is the centre of the square $S^{k,j}$.

**Lemma 3.3.** *For any $k \geq k_0$ and $j \in J_k$ we can find a potential $A^{k,j} \in L^2_{\mathrm{loc}}(S^{k,j})$ with $\mathrm{curl}\, A^{k,j} = b^{k,j}$ and $\|A - A^{k,j}\|_{L^\infty(S^{k,j})} \leq \alpha_k$ where $\alpha_k = 2^{-k-3/2}\beta_k$.*

In particular, the potential $A^{k,j}$ generates the constant field $b^{k,j}$ on $S^{k,j}$.

*Proof.* Let $k \geq k_0$ and $j \in J_k$. For convenience centre $S^{k,j}$ at the origin and set

$$\widetilde{A}_1^{k,j}(x) = -\frac{1}{2}\int_0^{x_2} B(x_1,t)\,dt \quad \text{and} \quad \widetilde{A}_2^{k,j}(x) = \frac{1}{2}\int_0^{x_1} B(t,x_2)\,dt.$$

Then $\widetilde{A}^{k,j}$, $\nabla_2 \widetilde{A}_1^{k,j}$ and $\nabla_1 \widetilde{A}_2^{k,j}$ are all continuous on $S^{k,j}$ (since $B$ is continuous) while $\mathrm{curl}\, \widetilde{A}^{k,j} = B = \mathrm{curl}\, A$. By Lemma 2.6 we can then find $\psi^{k,j} \in W^{1,2}_{\mathrm{loc}}(S^{k,j})$ with $A - \widetilde{A}^{k,j} = \nabla \psi^{k,j}$ on $S^{k,j}$. Now set

$$A_1^{k,j}(x) = \nabla_1 \psi^{k,j} - \frac{1}{2}b^{k,j}x_2 \quad \text{and} \quad A_2^{k,j}(x) = \nabla_2 \psi^{k,j} + \frac{1}{2}b^{k,j}x_1.$$

Then $A^{k,j} \in L^2_{\mathrm{loc}}(S^{k,j})$ with $\mathrm{curl}\, A^{k,j} = b^{k,j} = B(0)$. For $x \in S^{k,j}$ (14) leads to

$$\left|A_1(x) - A_1^{k,j}(x)\right| = \left|\frac{1}{2}\int_0^{x_2}(B(x_1,t) - B(0))\,dt\right| \leq 2^{-k-2}\beta_k$$

(note that $|x_2| \leq 2^{-k-1}$). Clearly a similar estimate holds for $A_2 - A_2^{k,j}$. $\blacksquare$

Also let $\chi_j$ denote the characteristic function of the set $\widehat{S}^{k,j} \cap \Omega^k$, restricted to $\Omega$. Define a piecewise constant field $B^k : \Omega \to \mathbb{R}$ by

$$B^k = \sum_{j \in J_k} b^{k,j}\chi_j. \tag{15}$$

The approximation $B^k$ converges to $B$ pointwise on $\Omega_0$ as $k \to \infty$; it will be helpful to combine this convergence with the Fatou-Lebesgue theorem as follows.

**Lemma 3.4.** *Suppose $k(t) \in \mathbb{N}$ and $\Gamma(t) \in \mathbb{R}$ for each $t > 0$ with $k(t) \to \infty$ and $\Gamma(t) \to \Gamma$ as $t \to \infty$. Then*

$$\liminf_{t \to \infty} \int_\Omega \nu^+\big(B^{k(t)},\Gamma(t)\big) \geq \int_\Omega \nu^-(B,\Gamma)$$

*and*

$$\limsup_{t \to \infty} \int_\Omega \nu^+\big(B^{k(t)},\Gamma(t)\big) \leq \int_\Omega \nu^+(B,\Gamma).$$

*Proof.* Let $k \geq k_0$. If $x \in \Omega^k$ then $x \in \widehat{S}^{k,j}$ for some $j \in J_k$ and so $B^k(x) = b^{k,j} = B(x_0)$ where $x_0$ denotes the centre of $S^{k,j}$. However $|x - x_0| \leq 2^{-k-1/2}$ so

$$|B^k(x) - B(x)| = |B(x_0) - B(x)| \leq \beta_k \tag{16}$$

by (14). Since $\beta_k \leq 1$ it follows that $|B^k(x)| \leq |B(x)| + 1$. This estimate is also valid when $x \notin \Omega^k$ since $B^k(x) = 0$ in this case. For any $\lambda \in \mathbb{R}$ (10) now gives

$$0 \leq \nu^+(B^k,\lambda) \leq \frac{1}{2\pi}\big(|B| + 1 + |\lambda|\big).$$

If $x \in \Omega_0$ then (11) and Lemma 3.2(i) imply $x \in \Omega^k$ for all sufficiently large $k$, so $B^k(x) \to B(x)$ as $k \to \infty$ by (16). Since $|\Omega \setminus \Omega_0| = 0$ it follows that $B^k$ converges to $B$ pointwise almost everywhere on $\Omega$ as $k \to \infty$. The result now follows from the Fatou-Lebesgue theorem (recall that $\nu^- \le \nu^+$ while $\nu^-$ and $\nu^+$ are lower and upper semi-continuous respectively). $\blacksquare$

## 3.3 Quadratic form estimates

Recall the notation introduced in Lemma 3.2. For each $k \ge k_0$ and $\delta > 0$ set

$$\widetilde{R}_\delta^k = \bigcap_{j \in J_k} \bigcup_{x \in \mathbb{R}^2 \setminus S^{k,j}} \mathbb{D}_{2^{-k}\delta}(x) \quad \text{and} \quad R_\delta^k = \widetilde{R}_\delta^k \cap \Omega.$$

Thus $R_\delta^k$ is an open subset of $\Omega$ which contains all of $\Omega \setminus \Omega^k$, together with a $2^{-k}\delta$-neighbourhood of the boundary of each square $S^{k,j}$, $j \in J_k$. In particular, any point of $R_\delta^k \cap \overline{\Omega^k}$ must lie in $\overline{S}^{k,j}$ for some $j \in J_k$, at a distance of less than $2^{-k}\delta$ from the boundary (of $S^{k,j}$). Since $S^{k,j}$ has side length $2^{-k}$ it follows that $\left|R_\delta^k \cap \overline{\Omega^k}\right| \le |J_k| 2^{-2k+2}\delta$. However $|J_k| \le 2^{2k}|\Omega|$ (since the disjoint squares $S^{k,j}$, $j \in J_k$ are all contained in $\Omega$) so

$$|R_\delta^k| \le \left|\Omega \setminus \overline{\Omega^k}\right| + \left|R_\delta^k \cap \overline{\Omega^k}\right| \le \left|\Omega \setminus \overline{\Omega^k}\right| + 4|\Omega|\delta. \tag{17}$$

We will need a partition of unity which is subordinate to the cover of $\mathbb{R}^2$ given by $\widetilde{R}_\delta^k$ and $S^{k,j}$ for $j \in J_k$. Using a standard construction we can find $\phi \in C^\infty(\mathbb{R}^2)$ and $\psi_j \in C_0^\infty(S^{k,j})$ for $j \in J_k$ so that

$$\phi^2 + \sum_{j \in J_k} \psi_j^2 = 1 \quad \text{and} \quad |\nabla\phi|^2 + \sum_{j \in J_k} |\nabla\psi_j|^2 \le C_3 2^{2k} \delta^{-2}, \tag{18}$$

where the constant $C_3$ can be chosen independently of $k$ and $\delta$ (note that, $\nabla\psi_j$ is non-zero only in a $2^{-k}\delta$-neighbourhood of the boundary of $S^{k,j}$). Also recall the approximating magnetic potential $A^{k,j}$ introduced in Lemma 3.3.

**Proposition 3.5.** *Let* $t > 0$ *and* $\varepsilon \in (0,1)$. *Then*

$$\mathbf{p}_{\Omega,tA}(u) \le (1-\varepsilon)^{-1} \sum_{j \in J_k} \mathbf{p}_{S^{k,j},tA^{k,j}}(u_j) + \varepsilon^{-1} t^2 \alpha_k^2 \|u\|^2 \tag{19}$$

*whenever* $u = \sum_{j \in J_k} u_j$ *with* $u_j \in C_0^\infty(S^{k,j}, \mathbb{C}^2)$, $j \in J_k$. *On the other hand,*

$$\mathbf{p}_{\Omega,tA}(u) \ge \mathbf{p}_{R_\delta^k,tA}(\phi u) + (1+\varepsilon)^{-1} \sum_{j \in J_k} \mathbf{p}_{S^{k,j},tA^{k,j}}(\psi_j u) - \left(\varepsilon^{-1} t^2 \alpha_k^2 + C_3 2^{2k} \delta^{-2}\right) \|u\|^2 \tag{20}$$

*for any* $u \in \mathcal{H}_{tA}(\Omega, \mathbb{C}^2)$.

*Proof.* Firstly let $j \in J_k$ and suppose $w \in C_0^\infty(S^{k,j}, \mathbb{C}^2)$. Then

$$\mathbf{p}_{S^{k,j},tA}(w) = \|\mathcal{D}_{tA}w\|^2 = \left\|\mathcal{D}_{tA^{k,j}}w - t\sigma.(A - A^{k,j})w\right\|^2.$$

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Now $|\sigma.(A - A^{k,j})\,\xi| = |A - A^{k,j}|\,|\xi|$ for any $\xi \in \mathbb{C}^2$, so Lemma 3.3 gives

$$\|\sigma.(A - A^{k,j})w\|^2 \leq \|A - A^{k,j}\|^2_{L^\infty(S^{k,j})}\|w\|^2 \leq \alpha_k^2\|w\|^2.$$

Basic norm estimates then lead to

$$(1 + \varepsilon)^{-1}\mathbf{p}_{S^{k,j},tA^{k,j}}(w) - \varepsilon^{-1}t^2\alpha_k^2\|w\|^2$$
$$\leq \mathbf{p}_{S^{k,j},tA}(w) \leq (1 - \varepsilon)^{-1}\mathbf{p}_{S^{k,j},tA^{k,j}}(w) + \varepsilon^{-1}t^2\alpha_k^2\|w\|^2. \qquad (21)$$

Taking completions extends this estimate to any $w \in \mathcal{H}_{tA^{k,j}}(S^{k,j}, \mathbb{C}^2)$.

If $u = \sum_{j \in J_k} u_j$ with $u_j \in C_0^\infty(S^{k,j}, \mathbb{C}^2)$, $j \in J_k$, then

$$\|u\|^2 = \sum_{j \in J_k}\|u_j\|^2 \quad \text{and} \quad \mathbf{p}_{\Omega,tA}(u) = \sum_{j \in J_k}\mathbf{p}_{S^{k,j},tA}(u_j)$$

since the $S^{k,j}$'s are disjoint; (19) now follows from the second estimate in (21).

Now suppose $v \in C_0^\infty(\Omega)$. Enlarge $J_k$ to $J_k'$ to include an index for $\phi$ and let $j \in J_k'$. Since $P_{tA}(\psi_j v) = \psi_j P_{tA}v - i(\nabla\psi_j)v$ the first part of (18) gives

$$\sum_{j \in J_k'}|P_{tA,l}(\psi_j v)|^2 = |P_{tA,l}v|^2 + \sum_{j \in J_k'}(\nabla_l\psi_j)^2\,|v|^2$$

for $l = 1, 2$. Integration over $\Omega$ and the second part of (18) then lead to

$$\mathbf{h}_{\Omega,tA}(v) \geq \sum_{j \in J_k'}\mathbf{h}_{\Omega,tA}(\psi_j v) - C_3 2^{2k}\delta^{-2}\|v\|^2.$$

An easy calculation also gives $\mathbf{b}(v) = \sum_{j \in J_k'}\mathbf{b}(\psi_j v)$. Hence

$$\mathbf{p}_{\Omega,tA}(u) \geq \mathbf{p}_{R_\delta^k,tA}(\phi u) + \sum_{j \in J_k}\mathbf{p}_{S^{k,j},tA}(\psi_j u) - C_3 2^{2k}\delta^{-2}\|u\|^2 \qquad (22)$$

for any $u \in C_0^\infty(\Omega, \mathbb{C}^2)$. For such $u$ (20) now follows from the first estimate in (21) and the fact that $\sum_{j \in J_k}\|\psi_j u\|^2 = \|u\|^2 - \|\phi u\|^2 \leq \|u\|^2$. Taking completions then gives (20) for all $u \in \mathcal{H}_{tA}(\Omega, \mathbb{C}^2)$. ∎

Standard variational arguments allow us to use the quadratic form estimates of Proposition 3.5 to obtain corresponding bounds on eigenvalue counting functions.

**Corollary 3.6.** *Let $t > 0$ and $\varepsilon \in (0, 1)$. For any $\lambda$ we have*

$$\mathsf{N}_{\Omega,tA}(\lambda) \geq \sum_{j \in J_k}\mathsf{N}_{S^{k,j},tA^{k,j}}\big((1 - \varepsilon)\big(\lambda - \varepsilon^{-1}t^2\alpha_k^2\big)\big)$$

*and*

$$\mathsf{N}_{\Omega,tA}(\lambda) \leq \mathsf{N}_{R_\delta^k,tA}\big(\lambda + \varepsilon^{-1}t^2\alpha_k^2 + C_3 2^{2k}\delta^{-2}\big)$$
$$+ \sum_{j \in J_k}\mathsf{N}_{S^{k,j},tA^{k,j}}\big((1 + \varepsilon)\big(\lambda + \varepsilon^{-1}t^2\alpha_k^2 + C_3 2^{2k}\delta^{-2}\big)\big).$$

## 3.4 Proof of Theorem 1.1

Write $\lambda(t) = (\Lambda + \gamma(t))t$ with $\Lambda \in \mathbb{R}$ and $\gamma(t) \to 0$ as $t \to +\infty$. For $t > 0$, $k \geq k_0$ and $\varepsilon, \delta, \rho \in (0,1)$ we can combine Corollary 3.6, Proposition 3.1, the homogeneity of $\nu^+$ and (15) to get

$$\frac{1}{t}\mathsf{N}_{\Omega,tA}(\lambda(t)) \geq \sum_{j \in J_k} 2^{-2k}(1-\rho)^2 \, \nu^+\big(b^{k,j}, (1-\varepsilon)\Gamma_k^-(t)\big)$$

$$= (1-\rho)^2 \int_\Omega \nu^+\big(B^k, (1-\varepsilon)\Gamma_k^-(t)\big) \qquad (23)$$

with $\Gamma_k^-(t) = \Lambda + \gamma(t) - \varepsilon^{-1}t\alpha_k^2 - C_2(1-\varepsilon)^{-1}t^{-1}2^{2k}\rho^{-2}$. Similarly

$$\frac{1}{t}\mathsf{N}_{\Omega,tA}(\lambda(t)) \leq \frac{1}{t}\mathsf{N}_{R_\delta^k,tA}\big(t\Gamma_k^+(t)\big) + \int_\Omega \nu^+\big(B^k, (1+\varepsilon)\Gamma_k^+(t)\big) \qquad (24)$$

with $\Gamma_k^+(t) = \Lambda + \gamma(t) + \varepsilon^{-1}t\alpha_k^2 + C_3 t^{-1}2^{2k}\delta^{-2}$.

Next recall that $(\beta_k)_{k \geq k_0}$ is a non-increasing positive sequence with $\beta_k \to 0$ as $k \to \infty$. Thus $(2^{2k}\beta_k^{-1})_{k \geq k_0}$ is an unbounded increasing sequence; it follows that we can define an unbounded non-decreasing function by setting

$$k(t) = \min\big\{k \geq k_0 : 2^{2k}\beta_k^{-1} \geq t\big\}$$

for any $t > 0$. Note that, if $t > 2^{2k_1}\beta_{k_1}^{-1}$ then $t \in \big(2^{2k(t)-2}\beta_{k(t)-1}^{-1}, 2^{2k(t)}\beta_{k(t)}^{-1}\big]$ so

$$t\alpha_{k(t)}^2 = t2^{-2k(t)-3}\beta_{k(t)}^2 \leq 2^{-3}\beta_{k(t)} \quad \text{and} \quad t^{-1}2^{2k(t)} \leq 4\beta_{k(t)-1}.$$

Hence $t\alpha_{k(t)}^2$, $t^{-1}2^{2k(t)} \to 0$ as $t \to \infty$. It follows that, for fixed $\varepsilon$, $\delta$ and $\rho$,

$$\Gamma_{k(t)}^-(t), \, \Gamma_{k(t)}^+(t) \to \Lambda \text{ as } t \to \infty. \qquad (25)$$

Now put $k = k(t)$ in (23). Lemma 3.4 and (25) then give

$$\liminf_{t \to \infty} \frac{1}{t}\mathsf{N}_{\Omega,tA}(\lambda(t)) \geq (1-\rho)^2 \int_\Omega \nu^-\big(B, (1-\varepsilon)\Lambda\big)$$

for any $\varepsilon, \rho \in (0,1)$. Taking $\varepsilon, \rho \to 0^+$ (together with the Fatou-Lebesgue theorem and lower semi-continuity of $\nu^-$) now leads to the lower bound in Theorem 1.1.

To obtain the upper bound firstly apply Proposition 2.5 to get the bound

$$\frac{1}{t}\mathsf{N}_{R_\delta^{k(t)},tA}\big(t\Gamma_{k(t)}^+(t)\big) \leq 2C_1\big(\|B\|_{L_\mathscr{A}(R_\delta^{k(t)})} + \big|\Gamma_{k(t)}^+(t)\big|\|1\|_{L_\mathscr{A}(R_\delta^{k(t)})}\big). \qquad (26)$$

Now $B, 1 \in L_\mathscr{A}(\Omega)$ while (17) and (13) give $\big|R_\delta^{k(t)}\big| \leq \big|\Omega \setminus \overline{\Omega^{k(t)}}\big| + 4|\Omega|\delta \to 4|\Omega|\delta$ as $t \to \infty$. Thus the right hand side of (26) must decay to 0 if we take $t \to \infty$ and then $\delta \to 0^+$. On the other hand Lemma 3.4 and (25) give

$$\limsup_{t \to \infty} \int_\Omega \nu^+\big(B^{k(t)}, (1+\varepsilon)\Gamma_{k(t)}^+(t)\big) \leq \int_\Omega \nu^+\big(B, (1+\varepsilon)\Lambda\big)$$

for any $\varepsilon \in (0,1)$. The upper bound in Theorem 1.1 now follows it we put $k = k(t)$ in (24), take $t \to \infty$ and then take $\delta, \varepsilon \to 0^+$.

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# 4 Approximate Zero Modes

## 4.1 Reduction to the disc

Most of the work in establishing Theorem 1.2 lies in establishing a version of this result for single signed fields on $\mathbb{D}$, the (open) unit disc in $\mathbb{R}^2$. We firstly quote this as a separate result and then show how the more general result follows.

**Theorem 4.1.** *Suppose $B \in C^\alpha(\overline{\mathbb{D}})$ for some $\alpha \in (0,1)$ and $B$ is single signed on $\mathbb{D}$. If $\lambda(t) \geq Ce^{-ct^\sigma}$ for some constants $\sigma \in (0,1)$ and $c, C > 0$ then*

$$\liminf_{t \to \infty} \frac{1}{t} \mathsf{N}_{\mathbb{D}, tA}(\lambda(t)) \geq \Phi_{\mathbb{D}}(|B|). \tag{27}$$

*Remark* 4.1 (General discs). Let $R > 0$ and suppose $B \in C^\alpha(\overline{\mathbb{D}}_R)$ is single signed and generated by the potential $A \in L^2_{\mathrm{loc}}(\mathbb{D}_R)$. Setting $A'(x) = RA(Rx)$ defines a potential $A' \in L^2_{\mathrm{loc}}(\mathbb{D})$ with associated field given by $B'(x) = R^2B(Rx)$; in particular $B' \in C^\alpha(\overline{\mathbb{D}})$ is single signed and $\Phi_{\mathbb{D}}(|B'|) = \Phi_{\mathbb{D}_R}(|B|)$. On the other hand, the expression $\mathcal{U}_R u(x) = Ru(Rx)$ defines a unitary map $\mathcal{U}_R : L^2(\mathbb{D}_R) \to L^2(\mathbb{D})$ with $\mathcal{U}_R \mathcal{P}_{\mathbb{D}_R, tA} \mathcal{U}_R^* = R^{-2}\mathcal{P}_{\mathbb{D}, tA'}$. Thus $\mathsf{N}_{\mathbb{D}_R, tA}(\lambda) = \mathsf{N}_{\mathbb{D}, tA'}(R^2\lambda)$ for any $\lambda$. It follows that Theorem 4.1 generalises to cover any disc in $\mathbb{R}^2$ (translation is clearly not an issue).

*Proof of Theorem 1.2.* Set $\Omega_\pm = \{x \in \Omega_0 : \pm B(x) > 0\}$. Then $\Omega_+ \cup \Omega_-$ is open (as $B$ is continuous on $\Omega_0$), so the Vitali covering theorem (see [16], for example) allows us to find a countable sequence of mutually disjoint open discs $\mathbb{D}^1, \mathbb{D}^2, \cdots \subseteq \Omega_+ \cup \Omega_-$ with $|(\Omega_+ \cup \Omega_-) \setminus \Omega_{\mathbb{D}}| = 0$ where $\Omega_{\mathbb{D}} = \bigcup_{k \in \mathbb{N}} \mathbb{D}^k$. Since $B$ is continuous and non-zero on $\Omega_+ \cup \Omega_-$ it must be single signed on each $\mathbb{D}^k$. A straightforward variational argument also shows $\mathsf{N}_{\Omega, tA}(\lambda) \geq \sum_{k \in \mathbb{N}} \mathsf{N}_{\mathbb{D}^k, tA}(\lambda)$ for any $\lambda$. If $\lambda(t) \geq Ce^{-ct^\sigma}$ we can then apply Theorem 4.1 (see also Remark 4.1) and the superadditivity of $\liminf$ to get

$$\liminf_{t \to \infty} \frac{1}{t} \mathsf{N}_{\Omega, tA}(\lambda(t)) \geq \sum_{k \in \mathbb{N}} \Phi_{\mathbb{D}^k}(|B|) = \Phi_{\Omega_{\mathbb{D}}}(|B|).$$

However $\Phi_{\Omega_{\mathbb{D}}}(|B|) = \Phi_\Omega(|B|)$ since $|\Omega \setminus \Omega_0| = |(\Omega_+ \cup \Omega_-) \setminus \Omega_{\mathbb{D}}| = 0$ while $B = 0$ on $\Omega_0 \setminus (\Omega_+ \cup \Omega_-)$. ∎

## 4.2 Reduction to the circle

From Remark 2.3 it suffices to prove Theorem 4.1 in the case that $B$ is non-negative. Clearly we can also impose the flux normalisation condition

$$\Phi_{\mathbb{D}}(|B|) = \Phi_{\mathbb{D}}(B) = \frac{1}{2\pi} \int_{\mathbb{D}} B = 1 \tag{28}$$

(note that, Theorem 4.1 holds trivially when $B \equiv 0$). We will henceforth assume $B \in C^\alpha(\overline{\mathbb{D}})$ is non-negative on $\mathbb{D}$ and satisfies (28). Let

$$\beta = \|B\|_{L^\infty(\mathbb{D})}. \tag{29}$$

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Since $\mathbb{D}$ is simply connected Proposition 2.7 gives us the freedom to choose any magnetic potential $A \in L^2_{\mathrm{loc}}(\mathbb{D})$ whose associated field is $B$. A convenient choice can be made via the "scalar potential". Firstly let $\phi : \mathbb{D} \to \mathbb{R}$ be the solution of $\Delta\phi = B$ on $\mathbb{D}$, with $\phi = 0$ on $\partial\mathbb{D} = S^1$; such a solution exists, is unique and satisfies $\phi \in C^{2,\alpha}(\overline{\mathbb{D}})$ (see [13]). If we set $A = (-\nabla_2\phi, \nabla_1\phi) \in C^{1,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^2)$ then $A$ is a magnetic potential with associated field $\mathrm{curl}\, A = \nabla_1^2\phi + \nabla_2^2\phi = B$. Furthermore $\mp i\nabla_\pm\phi = \nabla_2\phi \mp i\nabla_1\phi = -(A_1 \pm iA_2)$ so

$$-i\, e^{\mp t\phi}\nabla_\pm\big(e^{\pm t\phi}\,\cdot\,\big) = -i\nabla_\pm \mp it\nabla_\pm\phi = P_{tA,\pm}\,. \tag{30}$$

For $u \in C^2_0(\mathbb{D}, \mathbb{C}^2)$ it follows that

$$\mathbf{p}_{\mathbb{D},tA}(u) = \big\| e^{-t\phi}\,\nabla_+(u_+ e^{t\phi}) \big\|^2 + \big\| e^{t\phi}\,\nabla_-(u_- e^{-t\phi}) \big\|^2.$$

Setting $v_\pm = u_\pm e^{\pm t\phi}$ we have $u \in C^2_0(\mathbb{D}, \mathbb{C}^2)$ iff $v \in C^2_0(\mathbb{D}, \mathbb{C}^2)$, while

$$\|u\|^2 = \int_{\mathbb{D}} |v_+|^2 e^{-2t\phi} + \int_{\mathbb{D}} |v_-|^2 e^{2t\phi} \tag{31}$$

and

$$\mathbf{p}_{\mathbb{D},tA}(u) = \int_{\mathbb{D}} |\nabla_+ v_+|^2 e^{-2t\phi} + \int_{\mathbb{D}} |\nabla_- v_-|^2 e^{2t\phi}. \tag{32}$$

It is straightforward to check that $C^2_0(\mathbb{D}, \mathbb{C}^2)$ is a core for the form $\mathbf{p}_{\mathbb{D},tA}$.

Using a variational argument we can establish Theorem 4.1 by constructing sufficiently large spaces of test functions $\mathcal{X}_t \subset C^2_0(\mathbb{D}, \mathbb{C}^2)$ for which

$$\mathbf{p}_{\mathbb{D},tA}(u) \le \lambda(t)\,\|u\|^2, \quad u \in \mathcal{X}_t.$$

By the strong maximum principle (see [13], for example) $\phi$ is strictly negative on $\mathbb{D}$. As $\lambda(t) \ll 1$ the exponential weights in (31) and (32) then encourage us to seek test functions with $v_- = 0$ and $\nabla_+ v_+ = 0$, at least away from the boundary $\partial\mathbb{D} = S^1$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the standard way we have $\nabla_+ = 2\bar{\partial}$, so $\nabla_+ v_+ = 0$ iff $v_+$ is in $\mathcal{O}(\mathbb{D})$ the set of holomorphic functions on $\mathbb{D}$. To get an element of $C^2_0(\mathbb{D})$ we multiply by a cut-off function $\chi \in C^\infty_0(\mathbb{D})$ (which should be $\mathbb{R}$-valued and differ from 1 only near $S^1$). If we take $v_+ \in \mathcal{O}(\mathbb{D})$ and set

$$u = \begin{pmatrix} \chi v_+ e^{-t\phi} \\ 0 \end{pmatrix} \in C^2_0(\mathbb{D}, \mathbb{C}^2), \tag{33}$$

then $\nabla_+(\chi v_+) = v_+ \nabla_+\chi$ and $|\nabla_+\chi|^2 = |\nabla\chi|^2$, so (31) and (32) become

$$\|u\|^2 = \int_{\mathbb{D}} |\chi|^2\,|v_+|^2 e^{-2t\phi} \tag{34}$$

and

$$\mathbf{p}_{\mathbb{D},tA}(u) = \int_{\mathbb{D}} |\nabla\chi|^2\,|v_+|^2 e^{-2t\phi}. \tag{35}$$

The remainder of our analysis will be focused near the boundary of $\mathbb{D}$ (on a neighbourhood of where $\nabla\chi \ne 0$). The information we need about $B$ is captured

by the boundary behaviour of $\phi$. Let $h$ denote the outward normal derivative of $\phi$ on $\partial \mathbb{D} = S^1$. Using polar coordinates $(r, \theta)$ on $\overline{\mathbb{D}}$ we have $h(\theta) = \nabla_r \phi(1, \theta)$, while $h \in C^{1,\alpha}(S^1)$ since $\phi \in C^{2,\alpha}(\overline{\mathbb{D}})$. As a consequence of the maximum principle $h$ is strictly positive (see [13, Lemma 3.4]); the quantity

$$\kappa = \max\{\|h\|_{L^\infty(S^1)}, \|1/h\|_{L^\infty(S^1)}\} \tag{36}$$

is thus finite and positive. The divergence theorem and condition (28) also give

$$\int_{S^1} h = \int_{\partial \mathbb{D}} \nabla_r \phi = \int_{\mathbb{D}} \operatorname{div} \nabla \phi = \int_{\mathbb{D}} B = 2\pi. \tag{37}$$

Let $H^2(S^1)$ denote the Hardy space on $S^1$. Each $f \in H^2(S^1)$ is the boundary trace of a unique function $Ef \in \mathcal{O}(\mathbb{D})$ ($E$ is just the usual identification of $H^2(S^1)$ with the Hardy space on $\mathbb{D}$). Now let $f \in H^2(S^1)$ and set $v_+ = Ef$. Using polar coordinates on $\overline{\mathbb{D}}$ define a function $w_f : [0, 1] \to [0, \infty)$ by

$$w_f(r) = \int_0^{2\pi} |v_+(r, \theta)|^2 \, e^{-2t\phi(r,\theta)} \, d\theta. \tag{38}$$

When $f \neq 0$ we can then set

$$\omega_f = \nabla_r \log(rw_f(r))\big|_{r=1} = \frac{\nabla_r w_f(1)}{w_f(1)} + 1.$$

To create a test function we still need to fix the cut-off function $\chi$. We want this to be radial (for convenience) and decaying in a layer of width $\delta > 0$ near the boundary of $\mathbb{D}$. Choose a smooth non-decreasing function $\rho : \mathbb{R} \to \mathbb{R}$ with $\rho = 0$ on $(-\infty, 0]$, $\rho = 1$ on $[1, \infty)$, and $|\nabla \rho| \leq 1/\sqrt{2}$. For $\delta \in (0, 1)$ and $r \in [0, 1]$ set

$$\rho_\delta(r) = \rho(\delta^{-1}(1 - r)). \tag{39}$$

**Proposition 4.2.** *Suppose* $0 \neq f \in H^2(S^1)$ *satisfies* $\omega_f \leq -6\beta t\delta$ *with* $t > 0$ *and* $\delta \in (0, 1/3]$. *Let* $u$ *be given by (33) where* $v_+ = Ef$ *and* $\chi(r, \theta) = \rho_\delta(r)$. *Then*

$$\mathbf{p}_{\mathbb{D}, tA}(u) \leq \frac{1}{4\delta^2} \exp\left[\omega_f \delta + 6\beta t\delta^2\right] \|u\|^2.$$

To use this estimate we need further information on the behaviour of $\omega_f$. A summary of the necessary information is contained in the next result.

**Proposition 4.3.** *Suppose* $\nu_t > 0$ *for* $t \geq 1$. *Then there exist constants* $C_{4,1}$, $C_{4,2}$ *and spaces* $X_t \subset H^2(S^1)$ *for* $t \geq 1$ *such that* $\omega_f \leq -\nu_t$ *for all* $0 \neq f \in X_t$ *and*

$$\dim X_t \geq t - C_{4,1}\nu_t - C_{4,2}t^{(1-2\alpha)_+}.$$

Propositions 4.2 and 4.3 are proved at the end of Sections 4.3 and 4.4 respectively.

*Remark* 4.2. The test functions given by (33) are purely spin-up (only the component $u_+$ is non-zero). To deal with the case $B \leq 0$ directly by an argument similar to that above we would need to consider purely spin-down test functions of the form

$$u = \begin{pmatrix} 0 \\ \chi v_- e^{t\phi} \end{pmatrix},$$

where $v_- \in \overline{\mathcal{O}}(\mathbb{D})$ (the set of anti-holomorphic functions on $\mathbb{D}$). Clearly the anti-linear isometry $\mathcal{J}$ from Remark 2.3 sends test functions of one type to the other.

*Proof of Theorem 4.1.* Choose $C_{5,1}$ and $C_{5,2}$ so that

$$t^{1-\sigma} e^{-ct^\sigma} \leq 4CC_{5,1}^2 \quad \text{and} \quad C_{5,1}(C_{5,2} - 6\beta C_{5,1}) = 2c \tag{40}$$

for all $t \geq 0$. Set $\delta_t = C_{5,1}t^{(\sigma-1)/2}$ and $\nu_t = C_{5,2}t^{(\sigma+1)/2}$. Suppose $t \geq t_0$, where $t_0 = \max\{1, (3C_{5,1})^{2/(1-\sigma)}\}$; thus $t \geq 1$ and $\delta_t \in (0, 1/3]$. Let $X_t \subset H^2(S^1)$ be as given by Proposition 4.3 and set

$$\mathcal{X}_t = \left\{ \begin{pmatrix} \rho_{\delta_t}(Ef)\, e^{-t\phi} \\ 0 \end{pmatrix} : f \in X_t \right\} \subset C_0^2(\mathbb{D}, \mathbb{C}^2).$$

Since $Ef = 0$ iff $f = 0$ we get

$$\dim \mathcal{X}_t = \dim X_t \geq t - C_{4,1}C_{5,2}t^{(\sigma+1)/2} - C_{4,2}t^{(1-2\alpha)_+}, \tag{41}$$

so $\liminf_{t\to\infty} t^{-1} \dim \mathcal{X}_t = 1$.

Let $0 \neq f \in X_t$ and set $u = \begin{pmatrix} \rho_{\delta_t}(Ef)e^{-t\phi} \\ 0 \end{pmatrix} \in \mathcal{X}_t$. Now

$$\nu_t \delta_t - 6\beta t \delta_t^2 = C_{5,1}C_{5,2}t^\sigma - 6\beta C_{5,1}^2 t^\sigma = 2ct^\sigma \quad \text{and} \quad \frac{1}{4\delta_t^2}e^{-ct^\sigma} = \frac{t^{1-\sigma}}{4C_{5,1}^2}e^{-ct^\sigma} \leq C$$

by (40), so Proposition 4.3 leads to $\omega_f \leq -\nu_t \leq -6\beta t \delta_t$. Proposition 4.2 then gives

$$\mathbf{p}_{\mathbb{D},tA}(u) \leq \frac{1}{4\delta_t^2} \exp\left[-\nu_t \delta_t + 6\beta t \delta_t^2\right] \|u\|^2 \leq Ce^{-ct^\sigma}\|u\|^2. \tag{42}$$

The result now follows from a variational argument. ∎

*Remark* 4.3. A more precise version of (the error term in) (27) can be obtained from (41); setting $C_{5,3} = \max\{1, 3C_{5,1}, C_{4,1}C_{5,2}\}$ we get

$$\mathsf{N}_{\mathbb{D},tA}(\lambda(t)) \geq t - C_{5,3}t^{(\sigma+1)/2} - C_{4,2}t^{(1-2\alpha)_+}$$

for all $t \geq 0$ (note that, the right hand side is negative when $t < t_0$).

*Remark* 4.4. With some modifications to the proof we can extend Theorem 4.1 to obtain lower bounds for $\mathsf{N}_{\mathbb{D},tA}(\varepsilon)$ for fixed $\varepsilon > 0$. With $C_{6,1} = 2\sqrt{\varepsilon}$ and $C_{6,2} = 6\beta C_{6,1}^{-1}$ set $\delta_t = C_{6,1}^{-1}t^{-1/2}$ and $\nu_t = (C_{6,1}\log t + C_{6,2})t^{1/2}$ for $t \geq 0$.

Suppose $t \geq t_0$, where $t_0 = \max\{1, 9C_{6,1}^{-2}\}$; thus $t \geq 1$ and $\delta_t \in (0, 1/3]$. Let $X_t$ and $\mathcal{X}_t$ be as above. Now

$$\nu_t \delta_t - 6\beta t \delta_t^2 = C_{6,1} C_{6,1}^{-1} \log t + C_{6,2} C_{6,1}^{-1} - 6\beta C_{6,1}^{-2} = \log t \quad \text{and} \quad \frac{1}{4\delta_t^2} = \frac{C_{6,1}^2 t}{4} = \varepsilon t,$$

so the middle estimate in (42) gives $\mathbf{p}_{\mathbb{D}, tA}(u) \leq \varepsilon \|u\|^2$ for any $u \in \mathcal{X}_t$. On the other hand

$$\dim \mathcal{X}_t = \dim X_t \geq t - C_{4,1}(C_{6,1} \log t + C_{6,2})t^{1/2} - C_{4,2} t^{(1-2\alpha)_+}$$

for any $t \geq t_0$. Setting $C_{6,3} = \max\{1, C_{4,1}(C_{6,1} + C_{6,2}), 3/C_{6,1}\}$ we then obtain

$$\mathsf{N}_{\mathbb{D}, tA}(\varepsilon) \geq t - C_{6,3} t^{1/2} \log(t+2) - C_{4,2} t^{(1-2\alpha)_+}$$

for all $t \geq 0$ (note that, the right hand side is negative when $t < t_0$).

It is possible that the log is an artefact of our method and the second term in the asymptotics of $\mathsf{N}_{\mathbb{D}, tA}(\varepsilon)$ should be $O(t^{1/2})$, at least for sufficiently regular $B$.

## 4.3 Quadratic form estimates

The aim of this section is to prove Proposition 4.2. The presentation is simplified if we switch to polar coordinates; the magnetic momentum operators are then

$$P_r = -i\nabla_r - tA_r = -i\nabla_r + \frac{t}{r}\nabla_\theta \phi \quad \text{and} \quad P_\theta = -\frac{i}{r}\nabla_\theta - tA_\theta = -\frac{i}{r}\nabla_\theta - t\nabla_r \phi.$$

For functions $u, v$ defined on $\overline{\mathbb{D}}$ we will use $\langle u, v \rangle_{S^1}$ and $\|u\|_{S^1}$ to indicate the $L^2$-inner product and norm in the $S^1$ variable only; that is,

$$\langle u, v \rangle_{S^1} = \int_0^{2\pi} \overline{u(r, \theta)}\, v(r, \theta)\, d\theta \quad \text{and} \quad \|u\|_{S^1}^2 = \int_0^{2\pi} |u(r, \theta)|^2\, d\theta,$$

which depend on $r \in [0, 1]$.

Firstly we take a more detailed look at the function $w_f$ defined in (38).

**Lemma 4.4.** *Let $f \in H^2(S^1)$ and set $u = (Ef)\, e^{-t\phi}$. Then*

$$\nabla_r w_f = 2\langle u, P_\theta u \rangle_{S^1} \quad and \quad \nabla_r(r\nabla_r w_f) = 4r\|P_\theta u\|_{S^1}^2 - 2tr\langle u, Bu \rangle_{S^1}.$$

*Proof.* Firstly observe that for any functions $v, w$ on $\overline{\mathbb{D}}$ we have

$$\begin{aligned} \nabla_r \langle v, w \rangle_{S^1} &= \langle iP_r v, w \rangle_{S^1} + \langle v, iP_r w \rangle_{S^1} \\ &= \langle P_\theta v, w \rangle_{S^1} + \langle v, P_\theta w \rangle_{S^1} - i\langle Qv, w \rangle_{S^1} + i\langle v, Qw \rangle_{S^1}, \end{aligned} \quad (43)$$

where $Q = P_r + iP_\theta = e^{-i\theta} P_{tA,+}$. The second expression for $Q$ and (30) lead to

$$Qu = -ie^{-i\theta} e^{-t\phi}(\nabla_+(Ef)) = 0 \quad\quad (44)$$

(recall that $Ef \in \mathcal{O}(\mathbb{D})$). Now $w_f = \|u\|_{S^1}^2$ so (43) and (44) give

$$\nabla_r w_f = \langle P_\theta u, u \rangle_{S^1} + \langle u, P_\theta u \rangle_{S^1} = 2\langle u, P_\theta u \rangle_{S^1} \qquad (45)$$

as $P_\theta$ is symmetric with respect to $\langle \cdot, \cdot \rangle_{S^1}$. Using (43) and (44) again then gives

$$\nabla_r(r\nabla_r w_f)$$
$$= 2\langle P_\theta u, rP_\theta u \rangle_{S^1} + 2\langle u, P_\theta(rP_\theta u) \rangle_{S^1} - 2i\langle Qu, rP_\theta u \rangle_{S^1} + 2i\langle u, Q(rP_\theta u) \rangle_{S^1}$$
$$= 4r\|P_\theta u\|_{S^1}^2 + 2i\langle u, [Q, rP_\theta]u \rangle_{S^1}.$$

However $[Q, rP_\theta] = [P_r, rP_\theta] = -i\nabla_r(-rtA_\theta) + i\nabla_\theta(-tA_r) = itrB$. ∎

The formulae for the derivatives of $w_f$ given by Lemma 4.4 lead to the following.

**Lemma 4.5.** *Let* $0 \leq r_0 \leq 1$ *and* $f \in H^2(S^1)$. *If* $0 \leq b \leq -(\omega_f + 2\beta t(1 - r_0))$ *then* $e^{br}rw_f(r)$ *is decreasing for* $r \in [r_0, 1]$.

*Proof.* For any $b \in \mathbb{R}$ set $\ell_b(r) = \log(e^{br}rw_f(r)) = \log(w_f(r)) + \log r + br$. Then

$$\nabla_r(r\nabla_r \ell_b) = \frac{\nabla_r(r\nabla_r w_f)}{w_f} - r\frac{(\nabla_r w_f)^2}{w_f^2} + b$$

$$= \frac{r}{w_f^2}\left[4\|P_\theta u\|_{S^1}^2 \|u\|_{S^1}^2 - 4\langle u, P_\theta u \rangle_{S^1}^2\right] - 2tr\frac{\langle u, Bu \rangle_{S^1}}{\|u\|_{S^1}^2} + b$$

using Lemma 4.4 and the fact that $w_f = \|u\|_{S^1}^2$. Now $[\cdot] \geq 0$, $\langle u, Bu \rangle_{S^1} \leq \beta\|u\|_{S^1}^2$ (recall (29)) and $r \leq 1$. Hence $\nabla_r(r\nabla_r \ell_b) \geq -2\beta t + b$. Integrating from $r$ to 1 and using the fact that $\nabla_r \ell_b(1) = \omega_f + b$ we get

$$r\nabla_r \ell_b(r) \leq \nabla_r \ell_b(1) + \int_r^1 (2\beta t - b)\, dr = \omega_f + 2\beta t(1 - r) + br.$$

If $r_0 \leq r \leq 1$ and $0 \leq b \leq -(\omega_f + 2\beta t(1 - r_0))$ it follows that the right hand side is non-positive, and hence $\nabla_r \ell_b(r) \leq 0$. Thus $\exp(\ell_b(r)) = e^{br}rw_f(r)$ is decreasing for $r \in [r_0, 1]$. ∎

*Proof of Proposition 4.2.* Set $r_\delta = 1 - \delta$ and $b = -(\omega_f + 6\beta t\delta) \geq 0$, so $\rho_\delta(r) = 1$ for $r \leq r_\delta$ while $e^{br}rw_f(r)$ is decreasing for $r \in [1 - 3\delta, 1]$ by Lemma 4.5. Using (34) and (38) we then get

$$\|u\|^2 = \int_0^1 \rho_\delta(r)^2\, rw_f(r)\, dr \geq \int_{1-3\delta}^{r_\delta} rw_f(r)\, dr$$

$$\geq r_\delta w_f(r_\delta) \int_{1-3\delta}^{r_\delta} e^{b(r_\delta - r)}\, dr = r_\delta w_f(r_\delta)\frac{1}{b}\left(e^{2b\delta} - 1\right) \geq r_\delta w_f(r_\delta)\, 2\delta e^{b\delta}.$$

To estimate $\mathbf{p}_{\mathbb{D},tA}(u)$ note that $|\nabla\chi|^2 = |\nabla\rho_\delta|^2 \leq 1/(2\delta^2)$ and $\mathrm{supp}(\nabla\rho_\delta) \subseteq [r_\delta, 1]$, while $rw_f(r)$ is decreasing for $r \in [r_\delta, 1] \subset [1 - 3\delta, 1]$ by Lemma 4.5. Thus (35) and (38) lead to

$$\mathbf{p}_{\mathbb{D},tA}(u) = \int_0^1 |\nabla\rho_\delta(r)|^2\, rw_f(r)\, dr \leq \frac{1}{2\delta^2}\int_{r_\delta}^1 rw_f(r)\, dr \leq \frac{1}{2\delta}\, r_\delta w_f(r_\delta).$$

Combined with the previous estimate we then get $\mathbf{p}_{\mathbb{D},tA}(u) \leq (2\delta)^{-2}e^{-b\delta}\|u\|^2$. ∎

## 4.4 Analysis on the circle

Let $f \in H^2(S^1)$ and set $u = (Ef)\,e^{-t\phi}$. Then $u(1,\theta) = f(\theta)$ so $w_f(1) = \|f\|^2$. Also $A_\theta(1,\theta) = \nabla_r\phi(1,\theta) = h(\theta)$. Introducing the operator $T = -i\nabla - th$ on $S^1$, Lemma 4.4 now gives $\nabla_r w_f(1) = 2\langle f, Tf\rangle$. If $f \neq 0$ it follows that

$$\omega_f = 2\frac{\langle f, Tf\rangle}{\|f\|^2} + 1. \tag{46}$$

To construct the space $X_t$ in Proposition 4.3 we need to find $f \in H^2(S^1)$ for which $\omega_f$ is negative. In view of (46) this leads us to consider the spectral properties of the operator $T$ on $H^2(S^1)$. We begin by considering $T$ as an operator on $L^2(S^1)$ where a more explicit description is possible. Set

$$\eta(\theta) = \int_0^\theta h(\omega)\,d\omega$$

so $\eta(0) = 0$, $\eta(2\pi) = 2\pi$ (recall (37)) and $\nabla\eta = h$. Since $h$ and $1/h$ are both continuous and bounded away from 0 it follows that $\eta$ is a $C^1$-diffeomorphism of $S^1$. Thus $\mathcal{U}f = f \circ \eta$ defines a unitary map $\mathcal{U}$ from $L^2(S^1)$ with its usual inner-product to $L^2(S^1)$ with weighted inner-product $\langle\cdot,\cdot\rangle_h$ given by $\langle f,g\rangle_h = \langle f,hg\rangle = \langle hf,g\rangle$. Using $\|\cdot\|_h$ to denote the corresponding norm we have

$$\kappa^{-1}\|f\|^2 \le \|f\|_h^2 \le \kappa\|f\|^2 \tag{47}$$

for any $f \in L^2(S^1)$ (recall (36)). The image of the standard Fourier basis under $\mathcal{U}$ is $\{\xi_n : n \in \mathbb{Z}\}$ where $\xi_n = e^{in\eta}/\sqrt{2\pi}$ for $n \in \mathbb{Z}$. In particular, any $f \in L^2(S^1)$ can be written as $f = \sum_{n\in\mathbb{Z}}\gamma_n\xi_n$ for some constants $\gamma_n$ (given by $\gamma_n = \langle\xi_n,f\rangle_h$), whereupon $\|f\|_h^2 = \sum_{n\in\mathbb{Z}}|\gamma_n|^2$. Since $-i\nabla\xi_n = nh\,\xi_n$ we get

$$T\xi_n = (n-t)\,h\xi_n \tag{48}$$

and thus

$$\langle f, Tf\rangle = \sum_{n\in\mathbb{Z}}(n-t)|\gamma_n|^2. \tag{49}$$

For $M \ge 0$ let $Q_M$ denote the $\langle\cdot,\cdot\rangle_h$-orthogonal projection onto $\overline{\mathrm{Sp}}\{\xi_n : n > M\}$.

**Lemma 4.6.** *Let $f \in L^2(S^1)$ and $M \ge 0$. Then*

$$\langle f, Tf\rangle \le (M-t)\|f\|_h^2 + \langle Q_M f, TQ_M f\rangle.$$

*Proof.* Write $f = \sum_{n\in\mathbb{Z}}\gamma_n\xi_n$ for some $\gamma_n$. Then $Q_M f = \sum_{n>M}\gamma_n\xi_n$ so

$$\langle f, Tf\rangle - \langle Q_M f, TQ_M f\rangle = \sum_{n\le M}(n-t)|\gamma_n|^2 \le (M-t)\sum_{n\in\mathbb{Z}}|\gamma_n|^2 = (M-t)\|f\|_h^2$$

with the help of (49). ∎

We shall now move our attention to consider $T$ acting on $H^2(S^1)$. Let $\Pi^+$ denote the orthogonal projection of $L^2(S^1)$ onto $H^2(S^1)$ and $\Pi^-$ its complement; that is

$$\Pi^+ f = \frac{1}{\sqrt{2\pi}} \sum_{k \geq 0} \widehat{f}(k)\, e^{ik\theta} \quad \text{and} \quad \Pi^- = I - \Pi^+,$$

where $\widehat{f}(k)$ denotes the $k$th Fourier coefficient of $f$. A key idea in our argument is the fact that, for large $n$, $\xi_n$ and $h\xi_n$ "almost" lie in the space $H^2(S^1)$ in the sense that $\Pi^- \xi_n$ and $\Pi^- h\xi_n$ become small. This is made more precise via the quantities

$$\alpha_m = \sum_{n > m} \|\Pi^- \xi_n\|^2 \quad \text{and} \quad \beta_m = \sum_{n > m} \|\Pi^- h\xi_n\|^2.$$

**Proposition 4.7.** *There exists a constant $C_7$ such that*

$$\alpha_m,\, \beta_m \leq C_7\, (m+1)^{-2\alpha}, \quad m \geq 0.$$

We shall consider families of diffeomorphisms of $S^1$ which are related to $\eta$. Firstly note that a positively oriented homeomorphism of $S^1$ can be viewed as a continuous strictly increasing map $\psi : \mathbb{R} \to \mathbb{R}$ which satisfies $\psi(\theta + 2\pi) = \psi(\theta) + 2\pi$. If $\psi$ is differentiable then $\nabla\psi : \mathbb{R} \to \mathbb{R}$ is $2\pi$-periodic and hence can be viewed as a map on $S^1$. It is straightforward to check that $\psi$ is a (positively oriented) $C^{2,\alpha}$-diffeomorphism of $S^1$ if $\nabla\psi \in C^{1,\alpha}(S^1)$ and $\nabla\psi$ is strictly positive; in this case $\nabla\psi$ and $\nabla\psi^{-1}$ are both uniformly bounded away from 0, while $\nabla\psi^{-1} \in C^{1,\alpha}(S^1)$.

*Proof of Proposition 4.7.* For each $\tau \in [0,1]$ set $\eta_\tau(\theta) = \tau\eta(\theta) + (1-\tau)\theta$. Then $\eta_\tau(0) = 0$, $\eta_\tau(2\pi) = 2\pi$ and $\nabla\eta_\tau = \tau h + (1-\tau)$. Thus $\nabla\eta_\tau \in C^{1,\alpha}(S^1)$ with $\nabla\eta_\tau \geq \tau\kappa^{-1} + (1-\tau) \geq \kappa^{-1}$, so $\nabla\eta_\tau$ is bounded away from 0 uniformly in $\tau$. It follows that $\eta_\tau$ is a $C^{2,\alpha}$-diffeomorphism of $S^1$. Setting

$$y_\tau = \frac{1}{\sqrt{2\pi}} \nabla\eta_\tau^{-1} = \frac{1}{\sqrt{2\pi}} \frac{1}{(\nabla\eta_\tau) \circ \eta_\tau^{-1}}$$

we get $y_\tau \in C^{1,\alpha}(S^1)$, while $\|y_\tau\|_{C^{1,\alpha}(S^1)}$ can be bounded uniformly for $\tau \in [0,1]$. Using standard estimates for the Fourier coefficients of functions in $C^{1,\alpha}(S^1)$ (see [18], for example) we can then find $C_{7,1}$ so that

$$|\widehat{y_\tau}(n)| \leq C_{7,1}|n|^{-1-\alpha}, \quad \tau \in [0,1],\ n \neq 0. \tag{50}$$

Now suppose $n \geq 0$ and $k > 0$. Set $\tau = n/(k+n) \in [0,1]$. Then

$$\widehat{\xi_n}(-k) = \frac{1}{2\pi} \int_0^{2\pi} e^{in\eta(\theta)+ik\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\nabla\eta_\tau(\theta)} e^{i(k+n)\eta_\tau(\theta)} \nabla\eta_\tau(\theta)\, d\theta$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} y_\tau(\omega)\, e^{i(k+n)\omega}\, d\omega = \widehat{y_\tau}(-(k+n)).$$

For any $m \geq 0$ we can now combine this with (50) to get

$$\alpha_m = \sum_{n > m} \sum_{k > 0} \left|\widehat{\xi_n}(-k)\right|^2 \leq C_{7,1}^2 \sum_{n > m} \sum_{k > 0} (k+n)^{-2-2\alpha} \leq \frac{C_{7,1}^2}{2\alpha}(m+1)^{-2\alpha}.$$

We can estimate $\beta_m$ using a similar argument. In particular, we need to consider the $C^{2,\alpha}$-diffeomorphisms of $S^1$ given by $\zeta_\tau(\theta) = \tau\eta^{-1}(\theta)+(1-\tau)\theta$, $\tau \in [0,1]$. Then

$$z_\tau = \frac{1}{\sqrt{2\pi}}\nabla\zeta_\tau^{-1}$$

is uniformly bounded in $C^{1,\alpha}(S^1)$, while for any $n \geq 0$ and $k > 0$ it can be shown that the $(-k)$th Fourier coefficient of $h\xi_n$ is just $\widehat{z_\tau}(-(k+n))$. The remainder of the argument to estimate $\beta_m$ proceeds exactly as for $\alpha_m$. ∎

Proposition 4.7 establishes that $\alpha_0$ is finite; it follows that $\alpha_m$ is non-increasing with $\lim_{m\to\infty}\alpha_m = 0$. A similar comment applies to $\beta_m$. Also recall that $Q_m$ denotes the $\langle\cdot,\cdot\rangle_h$-orthogonal projection onto $\overline{\mathrm{Sp}}\{\xi_n : n > m\}$.

**Lemma 4.8.** *Suppose $Q_m f = f$ for some $f \in L^2(S^1)$ and $m \geq 0$. If $\alpha_m \leq 1/(2\kappa^2)$ then $\|f\|_h^2 \leq 2\kappa^2\|\Pi^+ f\|^2$.*

*Proof.* Write $f = \sum_{n>m}\gamma_n\xi_n$ for some $\gamma_n$. Then

$$\|\Pi^- f\|^2 \leq \sum_{n>m}|\gamma_n|^2 \sum_{n>m}\|\Pi^-\xi_n\|^2 = \alpha_m\|f\|_h^2.$$

Since $f = \Pi^+ f + \Pi^- f$ is an orthogonal decomposition (47) now gives

$$\|f\|_h^2 \leq \kappa^2\|f\|^2 = \kappa^2(\|\Pi^+ f\|^2 + \|\Pi^- f\|^2) \leq \kappa^2\|\Pi^+ f\|^2 + \frac{1}{2}\|f\|_h^2.$$

The result follows. ∎

**Lemma 4.9.** *Suppose $M > 0$ and $f \in \mathrm{Sp}\{\xi_n : 0 < n \leq M\}$. Then*

$$\langle Q_M\Pi^+ f, TQ_M\Pi^+ f\rangle \leq M\big[\alpha_0\alpha_M\beta_0\beta_M\big]^{1/2}\|f\|_h^2.$$

*Proof.* Write $f = \sum_{0<n\leq M}\gamma_n\xi_n$ for some $\gamma_n$. Then

$$Q_M\Pi^+ f = \sum_{n'>M}\langle\xi_{n'},\Pi^+ f\rangle_h\,\xi_{n'} = \sum_{n'>M}\sum_{0<n\leq M}\gamma_n\langle\xi_{n'},\Pi^+\xi_n\rangle_h\,\xi_{n'}. \tag{51}$$

Noting that $\langle\xi_{n'},\Pi^+\xi_n\rangle_h = -\langle\xi_{n'},\Pi^-\xi_n\rangle_h$ when $n \neq n'$, (49) and (51) now give

$$\langle Q_M\Pi^+ f, TQ_M\Pi^+ f\rangle \leq \sum_{n'>M}(n'-t)\left|\sum_{0<n\leq M}\gamma_n\langle\xi_{n'},\Pi^-\xi_n\rangle_h\right|^2$$

$$\leq \sum_{n'>M} n'\left[\sum_{0<n\leq M}|\gamma_n|^2 \sum_{0<n\leq M}|\langle\xi_{n'},\Pi^-\xi_n\rangle_h|^2\right]$$

$$= \|f\|_h^2 \sum_{0<n\leq M}\sum_{n'>M} n'\,|\langle\xi_{n'},\Pi^-\xi_n\rangle_h|^2. \tag{52}$$

However, for any $n, n'$ we have $\langle \xi_{n'}, \Pi^- \xi_n \rangle_h = \langle \Pi^- h\xi_{n'}, \Pi^- \xi_n \rangle$ while

$$n' \overline{\langle \xi_{n'}, \Pi^- \xi_n \rangle_h} = \langle \Pi^- \xi_n, n' h\xi_{n'} \rangle = \langle \Pi^- \xi_n, -i\nabla \xi_{n'} \rangle$$
$$= \langle \Pi^-(-i\nabla)\xi_n, \xi_{n'} \rangle = n\langle \Pi^- h\xi_n, \Pi^- \xi_{n'} \rangle.$$

(note that, $\Pi^-$ is an $\langle \cdot, \cdot \rangle$-orthogonal projection). Hence

$$\sum_{0 < n \leq M} \sum_{n' > M} n' |\langle \xi_{n'}, \Pi^- \xi_n \rangle_h|^2 = \sum_{0 < n \leq M} \sum_{n' > M} n \langle \Pi^- h\xi_{n'}, \Pi^- \xi_n \rangle \langle \Pi^- h\xi_n, \Pi^- \xi_{n'} \rangle$$
$$\leq M \sum_{n > 0} \|\Pi^- \xi_n\| \, \|\Pi^- h\xi_n\| \sum_{n' > M} \|\Pi^- \xi_{n'}\| \, \|\Pi^- h\xi_{n'}\|$$
$$\leq M \left[ \alpha_0 \beta_0 \alpha_M \beta_M \right]^{1/2}.$$

The result now follows from (52). $\blacksquare$

**Proposition 4.10.** *Suppose* $0 \leq m < M \leq t$. *Let* $X = \mathrm{Sp}\{\xi_n : m < n \leq M\}$ *and* $X^+ = \Pi^+ X \subset H^2(S^1)$. *If* $\alpha_m \leq 1/(2\kappa^2)$ *then* $\dim X^+ = M - m$ *and*

$$\langle f, Tf \rangle \leq -\kappa^2 \big( t - M - 2M \left[ \alpha_0 \alpha_M \beta_0 \beta_M \right]^{1/2} \big) \|f\|^2, \quad f \in X^+.$$

*Proof.* Let $f \in X$ and set $f^+ = \Pi^+ f \in X^+$. If $f^+ = 0$ then $f = 0$ by Lemma 4.8; thus $\dim X^+ = \dim X = M - m$. On the other hand, combining (47) with Lemmas 4.6, 4.8 and 4.9 gives

$$\langle f^+, Tf^+ \rangle \leq (M - t)\|f^+\|_h^2 + \langle Q_M f^+, TQ_M f^+ \rangle$$
$$\leq \kappa^2 (M - t)\|f^+\|^2 + M \left[ \alpha_0 \alpha_M \beta_0 \beta_M \right]^{1/2} \|f\|_h^2$$
$$\leq \kappa^2 \big( M - t + 2M \left[ \alpha_0 \alpha_M \beta_0 \beta_M \right]^{1/2} \big) \|f^+\|^2,$$

as required. $\blacksquare$

*Proof of Proposition 4.3.* Choose $m \geq 0$ so that $\alpha_m \leq 1/(2\kappa^2)$ (which is possible by Proposition 4.7). Also let

$$\nu_{1,t} = \frac{1}{2\kappa^2}(\nu_t + 1) + 2C_7^2 \, t^{(1-2\alpha)_+}$$

and $M_t = \min\{n \in \mathbb{N}_0 : n \geq t - \nu_{1,t} - 1\}$; in particular, $M_t \geq t - \nu_{1,t} - 1$. Set

$$X_t = \Pi^+ \, \mathrm{Sp}\{\xi_n : m < n \leq M_t\} \subset H^2(S^1).$$

Proposition 4.10 gives $\dim X_t \geq M_t - m \geq t - (\nu_{1,t} + m + 1)$ (note that, $X_t = \{0\}$ when $M_t \leq m$). The required estimate for $\dim X_t$ now follows if we take $C_{4,1} = 1/(2\kappa^2)$ and $C_{4,2} = C_{4,1} + 2C_7^2 + m + 1$ (note that, $t^{(1-2\alpha)_+} \geq 1$ for $t \geq 1$).

Now let $0 \neq f \in X_t$. Then $1 \leq M_t \leq t - \nu_{1,t} \leq t$ (otherwise $X_t = \{0\}$), leading to $M_t(M_t + 1)^{-2\alpha} \leq M_t^{1-2\alpha} \leq t^{(1-2\alpha)_+}$. Propositions 4.10 and 4.7 then give

$$\langle f, Tf \rangle \leq \kappa^2 \big( M_t - t + 2M_t \left[ \alpha_0 \alpha_{M_t} \beta_0 \beta_{M_t} \right]^{1/2} \big) \|f\|^2$$
$$\leq \kappa^2 \big( M_t - t + 2C_7^2 \, M_t(M_t + 1)^{-2\alpha} \big) \|f\|^2$$
$$\leq \kappa^2 \big( -\nu_{1,t} + 2C_7^2 \, t^{(1-2\alpha)_+} \big) \|f\|^2 = -\frac{1}{2}(\nu_t + 1) \, \|f\|^2,$$

so $\omega_f \leq -\nu_t$ by (46). $\blacksquare$

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