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Alexander Matros

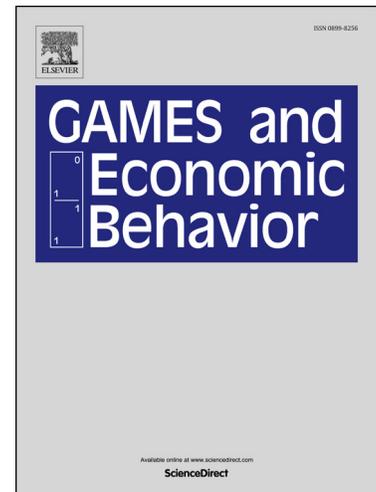
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## Lloyd Shapley and Chess with Imperfect Information<sup>1</sup>

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### Abstract

Anyone who has ever studied game theory knows the name Lloyd Shapley. Just recall Matching, Deferred-Acceptance Algorithm, Core, Market Games, Stochastic Games, Shapley value, and Shapley vector.<sup>3</sup> But Professor Shapley was also a great lover of chess with imperfect information. Upon our first encounter at Stony Brook in 1998, I was fortunate to investigate the chess problems he set before me. In this essay I analyze some of those problems, in commemoration of Lloyd Shapley's contributions to the study of chess and chess with imperfect information.

*JEL classification: C73*

*Key Words: Imperfect Information, Chess, Kriegspiel, Stochastic Games.*

“Talent hits a target no one else can hit. Genius hits a target no one else can see.”

Arthur Schopenhauer

### Preliminary: Chess with Imperfect Information

From the perspective of game theory, chess is not of great interest because it is ultimately a zero-sum game with perfect information: each player knows all previous moves, and may construct (so far only in theory) the complete game tree.<sup>4,5</sup> Therefore, using backward induction, it is mathematically feasible for any player to find a subgame perfect equilibrium.<sup>6</sup> Nevertheless, the vast number of possible decision nodes has thus far prevented even the most powerful computers from completing such an analysis.

An additional layer of complexity may be introduced to chess by limiting the information available to its players. Chess with imperfect information, or Kriegspiel, was introduced in

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<sup>3</sup> See also Serrano (2013, 2017) and [http://www.nobelprize.org/nobel\\_prizes/economic-sciences/laureates/2012/shapley-facts.html](http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2012/shapley-facts.html)

<sup>4</sup> Rules of chess can be found at <https://www.fide.com/component/handbook/?id=207&view=article>

<sup>5</sup> The “triviality” of chess was discussed in the literature. See, for example, Zermelo (1913), Ewerhart (2000), and Schwalbe and Walker (2001).

<sup>6</sup> Even though most people assume that chess is a finite game, this is formally not the case. See Ewerhart (2002).

Germany at the end of the 19th century,<sup>7</sup> and it is this more complex game which stimulated Professor Shapley's interest. In Kriegspiel each player can see their own pieces, but not those of their opponent. When it is a player's turn, she attempts a move, which the umpire will declare to be 'legal' or 'illegal'. If the move is illegal, the player tries again; if it is legal, that move stands. Because information is imperfect, the player cannot fully control the situation.<sup>8</sup>

On the occasion of my first meeting with Lloyd Shapley, I held an International Master title in chess, and so I challenged him to a game of Kriegspiel. In order to test my readiness, Professor Shapley first presented a lesson in chess with imperfect information.

### Lesson 1: How can you use your knowledge of game theory?

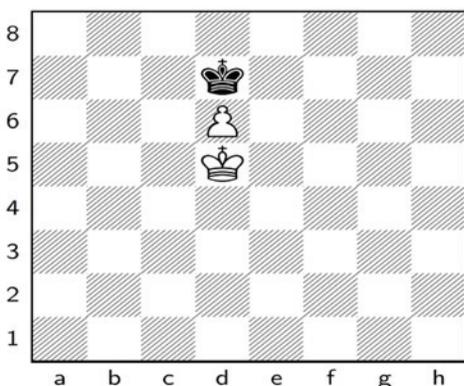


Diagram 1

Shapley set up the position on Diagram 1 and explained that both White and Black are aware that this position is common knowledge. The ultimate purpose of the exercise, meanwhile, is to demonstrate how White wins with probability 1 in Kriegspiel. I was quite astonished, because any chess beginner can easily make a draw (for Black) by playing **Kd7 -d8 - d7** ... and, at the right time, take the opposition on the eighth rank on **e8** and **c8**. However, it is not so straightforward in Kriegspiel. Let us see why. White plays **1. Ke5**. Black responds **1. ... Kd8** and after **2. Ke6** Black does not know whether the White king is on **c6** or **e6**. Black has to guess and the right move, **2. ... Ke8**, will be chosen with, for example, equal probability  $\frac{1}{2}$ . Next, White will try to play **3. Ke6 -**

<sup>7</sup> Nasar (1998) also noted that Kriegspiel was popular at Princeton when John Nash and Lloyd Shapley were Ph.D. students there.

<sup>8</sup> The rules of Kriegspiel are provided in Appendix C.

**e7**. If this move is legal, then after **d6 - d7 - d8Q** White promotes the queen and wins.<sup>9</sup> Nevertheless if the umpire declares that move **3. Ke6 - e7** is illegal, then Black guessed correctly, and White is forced back to the initial Diagram 1.

White repeats the same procedure again and again. After  $n$  times the probability that Black always guessed correctly is  $(1/2)^n$ .<sup>10</sup> Can Black do something else? Yes, Black can play **1. ... Ke8** and **2. ... Kd8** trying to play “for sure.” In this case White will never be able to play **3. Ke6 - e7** or **3. Kc6 - c7** and it might look like Black is always guessing correctly. Unfortunately, White wins with **3. d6 - d7** after **2. ... Kd8** whether the White king is on **c6** or **e6**. Thus, White has to proceed as follows: carry out the king maneuver described above with probability  $(1 - \epsilon)$  and play **d6 - d7** with probability  $\epsilon$ , where  $\epsilon$  is very close to zero. Thus I learned that the endgame with a king and pawn against a lone king is winning almost surely.<sup>11</sup>

After lesson 1, Shapley presumed that I could checkmate a lone king with a rook and king even in Kriegspiel,<sup>12</sup> and so he complicated the task.

### The Infinite Power of the Rook<sup>13</sup>

Consider the following situation.

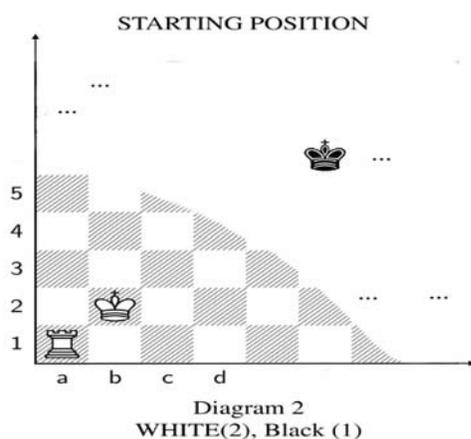
<sup>9</sup> We would assume that White can checkmate once the pawn is promoted to a queen, even though this task is not easy: White needs to avoid a stalemate or queen’s capture in Kriegspiel.

<sup>10</sup> Note that usual chess “three-fold repetition” rule is not applicable in Kriegspiel (Rule 15 in Appendix C), so that White can repeat the described maneuver indefinitely.

<sup>11</sup> See Appendix A for more details.

<sup>12</sup> King and Rook vs King is the most widely studied ending in Kriegspiel. However, it is not simple by any means. See Boyce’s (1981) and Ciancarini and Favini’s (2009) algorithms for solving this problem.

<sup>13</sup> This problem was created by Lloyd Shapley in 1960. It is described *verbatim* from Shapley (1987). Shapley’s description is in *italic*. See problem 12 in Appendix B.



1. *The board is a quarter plane.*
2. *White's king and rook start as shown on Diagram 2.*
3. *Black places his king on any legal square, unknown to White.*
4. *White then plays to win with probability 1.*

Shapley, aware that I still lacked preparation, did not test me, and just showed the solution consisting of several steps.

Step 1. Find the Black king (give a check).

The goal is to give a check without losing the rook. If this goal is achieved, then White moves to Step 2. First, White makes sure that the Black king is not on files "a", "b", "c", or on the ranks "1", "2", "3". White plays **Ra1 - a3 - b3 - c3 - c1 - c2 - c3**. If White gives a check, then White proceeds to Step 2. Otherwise, White learns that the Black king is above the third rank and to the right of file "c". Now, White plays **Rc3 - c1**. Note that after this move, the Black king is above the 2-nd rank and to the right of file "c". White selects the next 10 files to the right of file "c" and places the rook on one of these files with an equal probability of 1/10. If this is not a check, then White plays **Rc1** on the next move. Again, White inspects the first three ranks **Rc1 - c2 - c3 - c1**. Now, White selects the next 20 files to the right of file "c" and places the rook on one of these files with an equal probability of 1/20. This procedure is continued for 30, 40, and so on files. Using this algorithm, White makes a check with probability 1. After the check, White knows that the Black king is at a particular file.

Step 2. The exact position of the Black king.

The goal is to find boundaries for the Black king. White moves the rook 100 files to the right of the file with the Black king and begins the familiar procedure. White's goal is to locate the Black king's rank (give a check). Thus, White selects the first 10 ranks and places the rook on one of

these ranks with an equal probability of  $1/10$ . If White gives a check, then White proceeds to Step 3. Otherwise, White selects the first 20 ranks and places the rook on one of these ranks with an equal probability of  $1/20$ . This procedure is continued for 30, 40, and so on ranks. If after 97 moves White still has not given a check, then White moves its rook 100 files to the right and continues the same algorithm. Using this algorithm, White makes a check with probability 1. After the check, White knows that the Black king is inside of a particular rectangle.

Step 3. Checkmate on the finite board.

White moves the rook 100 ranks above the position of the Black king. Then, White moves its king along the main diagonal for the next 98 moves. After that, White moves its rook 100 files to the right and again moves its king along the main diagonal for the next 98 moves and so on. This is the case when the White king moves almost twice as fast as its Black counterpart. As a result, the position on Diagram 3 will be reached.

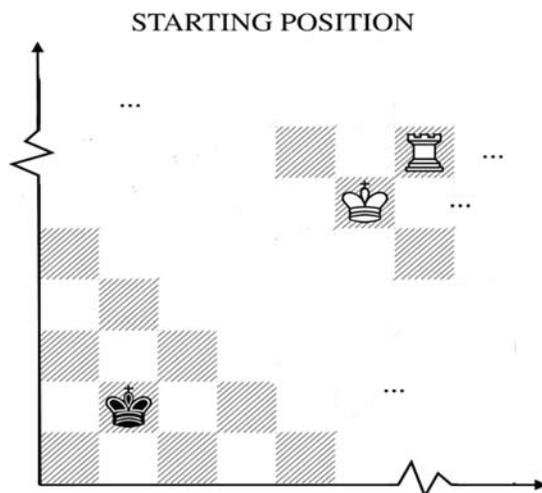


Diagram 3  
WHITE(2), Black (1)

Lloyd declared that I should be able to checkmate a lone king on a finite board after all the previous steps in Kriegspiel.<sup>14</sup> Following these two examples, Professor Shapley asked me to solve the problem presented in Diagram 4.

<sup>14</sup> This problem is not easy. See footnote 12.

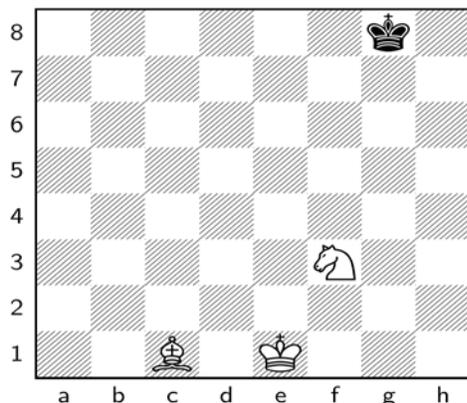
The Wide Open Spaces<sup>15</sup>

Diagram 4

The task requires White to checkmate in 22 moves from the present position.<sup>16</sup>

*“For initial shock value this endgame is unequalled. In fact, most people, seeing the problem for the first time, are doubly astonished. How can a bishop and knight in such an open position possibly force a Kriegspiel win -- in any number of moves? And even granting the premise of the problem, how can any human solver produce an exact strategic plan 22 moves deep? The mind boggles.*

*Actually, things are not all that bad. Some frantic fence - building by White at the outset reduces Black's running room, and it only remains for the solver to discover the little finesses required to break down Black's defensive perimeter. The actual kill, once the K has been driven into the corner, takes only four or five moves.*

*While the sharpest line of play does take just 22 moves anything under 30 means that you have essentially mastered the problem. Remember that you will need several moves after the K is cornered (see above), so don't waste time looking at variations that let Black run into the center.”*

My chess knowledge proved useful here, since once **Nf3 – e5** and **Bc1 – g5** are played, White receives a textbook position where a checkmate is achieved almost without any problems, even with imperfect information.

The next problem, however, was more complicated.

<sup>15</sup> This problem was created by Lloyd Shapley. It is described *verbatim* from Shapley (1987). The year of its creation is missing. See problem 8 in Appendix B.

<sup>16</sup> A task to give checkmate with a bishop and knight is difficult even in chess. It becomes a very difficult problem in Kriegspiel. See Ferguson (1992). Ferguson (1992) is the only paper (I found) which also mentions Shapley (1987).

### Just Horsing Around<sup>17</sup>

*“This problem makes no pretense of realism. But there is a pleasing pattern in the prescribed moves and a nice echo in the solution. One feature of this problem may disturb the solver at first. It seems that no clue whatever hints at the location of the Black King. Why, it might be just about anywhere!”*

The present position is on Diagram 5a, and the position nine moves earlier is on Diagram 5b. The task is to reconstruct the game from the given information below and find how White gives checkmate in 2 moves from the present position. Note that White has nine men while Black has only six men; the number of men is written in brackets next to White and Black on the diagrams. These numbers can play a crucial role in the problem.

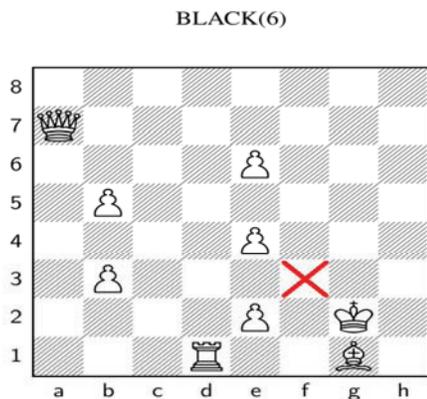


Diagram 5a  
WHITE(9) mates in 2.  
X site of White's try.

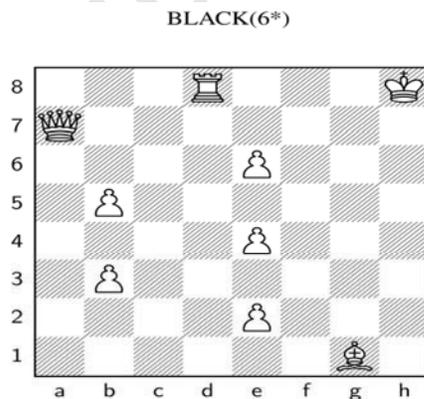


Diagram 5b  
WHITE(9)  
\*NOTE: All 8 Black pawns have been captured

Intervening moves and announcements

White	Black
1. <b>Rd8 – d1</b> “Black” “ch long & knight” <sup>18</sup>	4. <b>Kh6 - h5</b> “Black” “ch knight”
2. <b>Kh8 - h7</b> “Black” “ch knight”	5. <b>Kh5 - h4</b> “Black” “ch knight”
3. <b>Kh7 - h6</b> “Black” “ch knight”	6. <b>Kh4 - h3</b> “Black” “ch knight”
	7. <b>Kh3 - h2</b> “Black” “ch knight”
	8. <b>Kh2 - h1</b> “Black” “ch knight”

<sup>17</sup> This problem was created by Lloyd Shapley in 4/23/1968. It is described in his manuscript Shapley (1987). See problem 4 in Appendix B.

<sup>18</sup> This means “check on the long diagonal and by a knight.”

9. **Kh1 - h2** “no”

**Kh1 - g2** “Black” “try **f3**”<sup>19</sup>

10. ??? Task: White gives a checkmate in 2 moves.

Shapley, regarding my feeble attempts to help, began to demonstrate the solution himself. Double check after the 1st move is possible only if Black has its knight on **e5** and a bishop or queen on **a1**, **b2**, or **c3** in the initial position. Note that file **d** must be free because White’s first move was **1. Rd8 – d1**. Given that one Black knight cannot give

**8. Kh2 - h1 Nf5 - g3** + (now, of course, **9. Kh1 - h2** is not possible)

**9. Kh1 - g2.**

Here Shapley drew my attention to the fact that after the 3rd, 5th, and 7th Black moves there was no announcement that there was the possibility of taking on **f7**, **f5**, and **f3**, respectively. That led to the idea that all relevant pawns had to be pinned. We also had to remember that Black had only 6 pieces, postulating that Black should have two knights, a queen, and two rooks (in order to pin three pawns) in the initial position. Therefore, the Black rooks had to be on **b6** and **b4** because White pawns on **b3** and **b5** did not receive the possibilities to capture on move 9. Furthermore, the Black queen could only be on **b2** because of the double check on the first move.

Then Shapley questioned, “Where is the Black king in the initial position?” Despite analyzing the entirety of the chessboard, I failed to locate a square for the Black king after the first move **1. Rd8 - d1** (without announcements). Nonetheless, I was impressed by Shapley’s creativity when he placed the Black king on **f8** and the second Black knight on **e8**, suggesting that Black played **4. ... Ne8 – g7+** instead of **4. ... Ng3+**. Everything works now!

Ultimately, White has to checkmate in two moves. This exercise is simple after the previous tasks. Since the umpire announced that it is possible to take on **f3** on move 10, the Black

checks so that the White king goes through the entire file **h**, Black has to have two knights. Together we were able to arrange the Black knights so as to satisfy all the checks on 1 - 9 moves. So,

**1. ... Ne5 - f7++** (double check)

**2. Kh8 - h7 Nf7 - g5+**

**3. Kh7 - h6 Ng5 - f7+**

**4. Kh6 - h5 Ng3+** (this is a new knight)

**5. Kh5 - h4 Ng3 - f5 +**

**6. Kh4 - h3 Nf7 - g5 +**

**7. Kh3 - h2 Ng5 - f3 +**

<sup>19</sup> This means “White has a try on **f3**.”

queen had to move somewhere on move 9 while there is no pin. Let us try to play **10. Rd1 - d8**. If this is a checkmate, then we are done. The alternative possibility of the White rook's capture, however, implies that Black played **10. ... Qb2 - f6** and **11. ... Qf6xd8**. Therefore, White gives a checkmate with **12. Qa7 - f7X**. Finally, if White's move **10. Rd1 - d8** is 'illegal,' then this indicates that Black played **10. ... Qb2 - d4**. Hence, White gives a checkmate with **11. Qa7 - f7X**.

As a result, Lloyd was pleased by my progress and agreed to play.

### **Lesson 2: No Luck**

The main problem for Kriegspiel is to find an umpire. Unfortunately, we could not find one on the spot and thereby had to play Blitz.<sup>20</sup> Given my significant age advantage and my past professional chess career, Professor Shapley had slim chances for success in our two-game match. Due to Lloyd's competitive persona, the result of our match left him in unmitigated disappointment.<sup>21</sup>

### **Penultimate Lesson: Success**

#### **Ray's Send-off**

The next morning prior to the start of the first lecture, Shapley asked me to solve yet another problem, in which players start from a known position on Diagram 6 and the task is to find how White wins with probability arbitrarily close to 1.

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<sup>20</sup> Blitz is chess with a short time control. Each of us had 5 minutes for the whole game.

<sup>21</sup> Nasar (1998) also mentioned that Lloyd Shapley was very competitive.

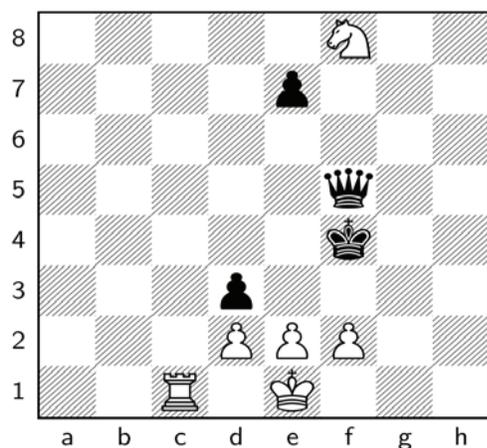


Diagram 6

Here is Shapley's description.<sup>22</sup> *"For a change, here's a problem where the players start from a known position. This is also the first problem in which probabilities enter the picture - and in a relatively sophisticated way. The "value of the game" is a win for White, but White has no strategy, pure or mixed, that achieves it, even in expected value. The situation responsible for this phenomenon is one that can arise rather easily in king-and-pawn endings in regular Kriegspiel. But: in this problem it is hardly more than an afterthought to the tactical maneuvering that is the main point of interest. This problem was composed (More precisely, it was adapted from a published chess problem whose author was not recorded. Does anyone recognize it?) as a parting gift to Ray Fulkerson on his departure from Rand for Cornell. It was physically engraved on an aluminum Kriegspiel screen/tray, the rest of whose surfaces were covered with his friends' signatures interspersed with pithy Kriegspiel sayings: "When in doubt, push past", "A cheap tactical maneuver!" "The canonical try", "Troop B", "Mark your ambiguities", "Hell no!" and the like."*<sup>23</sup>

I quickly found the solution, but various nuances appeared on the main line.<sup>24, 25</sup> So, **1. Rc1 - c4** (check on the rank). Now, if Black plays **1. ... Qf5 - e4**, the umpire has no announcements.

<sup>22</sup> This problem was created by Lloyd Shapley in May 1968. It is described in his manuscript Shapley (1987). See problem 6 in Appendix B.

<sup>23</sup> Here is information about Ray Fulkerson:

[https://ecommons.cornell.edu/bitstream/handle/1813/17887/Fulkerson\\_Delbert\\_Ray\\_1976.pdf;jsessionid=2877C78958F827FB0065C9BC475E0F12?sequence=2](https://ecommons.cornell.edu/bitstream/handle/1813/17887/Fulkerson_Delbert_Ray_1976.pdf;jsessionid=2877C78958F827FB0065C9BC475E0F12?sequence=2)

<sup>24</sup> The position on Diagram 6 can also be offered as a chess exercise under the task "White to move. Find the best continuation." I solved this chess problem where the solution is identical for Kriegspiel.

<sup>25</sup> Shapley gave me the position on Diagram 6 to solve. The White rook is at **c6** in his manuscript Shapley (1987), which does not affect the solution.

Therefore, **2. Rc4xe4** wins in this case. Otherwise if Black moves its king, then the umpire declares “try to capture on **d3**.” White plays **2. Rc4 - c5**. If the umpire announces “check on the rank”, this alludes that Black has played **1. ... Kf4 - e5** and, therefore, **3. Rc5xf5** wins. If there are no announcements after White’s move **2. Rc4 - c5**, then this indicates that Black has played **1. ... Kf4 - g5**. In this case if the umpire announces “White has lost a piece on **c5**”, then this means that Black has just played **2. ... Qf5xc5**. Therefore, White wins with **3. Nf8 - e6+** and **4. Ne6xc5**. Finally, if the White rook is not captured on **c5**, then White tries to play **3. Rc5 - f5** with the idea of capturing the queen. If this move is legal, then White takes the Black queen and the game is won. If White’s move is illegal, then White tries to choose **3. Rc5 - e5** with the same idea to capture the queen. If the umpire announces “check on the rank and Black has lost a piece on **e5**,” then this suggests that Black has played **2. ... Qf5 - e5**, White has just won the queen, and the rest is straightforward. If the umpire announces “Black has lost a pawn on **e5**,” then it is clear that Black has played **2. ... e7 - e5**. Now if White’s rook is still on the board after the next Black move, then White can capture the Black queen on **f5** with **4. Re5xf5** since the queen is pinned and cannot move. The umpire’s announcement “White has lost a piece on **e5**” implies move **3. ... Qf5xe5**, to which White replies **4. f2 - f4+**. If Black plays **4. ... Qe5xf4**, then the umpire declares “White has lost a pawn on **f4**, try to capture on **d3**.” White, of course, plays **5. Nf8 - e6+** and captures the queen on the next move **6. Ne6xf4**. After the Black king captures the White knight on move 6, White plays **7. e2xd3** and continues with the maneuver **Ke1 - f2 - e3** (or **e2**, if move **Kf2 - e3** is illegal, then White plays **Ke2 - e3** on the next move) and wins the game. Note that move **4. ... Kg5xf4** leads to the announcement “White has lost a pawn on **f4**.” Therefore, White plays **5. Nf8 - g6+** and captures the queen on the next move **6. Ng6xe5**. After the Black king captures the White knight on move 6, White plays **7. e2xd3** and Black can choose between **7. ... Ke5 - d4** and **7. ... Ke5 - d5**. If White correctly guesses the Black’s move, then White plays **8. Ke1 - e2** or **8. Ke1 - f2** respectively, which leads to an easy win for White. However if White’s guess is wrong, then White would lose the **d3** pawn, and the situation simplifies to the winning endgame: a king and pawn against a king (see Lesson 1).

Shapley discussed the solution once again and decided that I was prepared for a more advanced challenge.

**The Final Lesson: How I solved Shapley’s problem.**

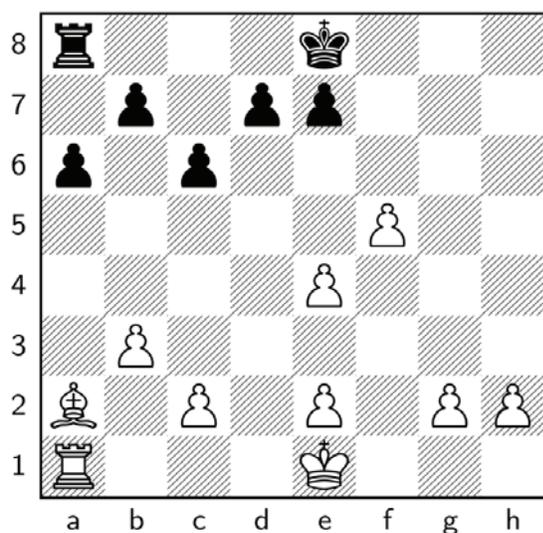


Diagram 7

Shapley set up the position on Diagram 7 and asked whether White could castle queenside. Before I could start, Lloyd added the second task: Suppose that the White pawn is on **f6** instead of **f5**, does that affect your decision?

Here, Shapley made a digression, lamenting that people do not like to solve problems. He had published these problems in a weekly newspaper, receiving numerous letters arguing the irrelevance of the position of the White pawns either on **f5** or **f6**. Many respondents also wrote that the problems did not have solutions. All in all, people wanted to see a published solution in the next newspaper issue, to which Shapley, disappointed by that attitude, insisted that the solution would not be published.

It was therefore clear that I had to solve this problem myself. At the same time, I was optimistic, because the task was about chess (and not Kriegspiel) and because I had been able to solve the previous problem. Therefore, I had to find a particular move order leading to the position on Diagram 7. What could be so unusual about the move order? What can be said about the position on Diagram 7? First, it is clear that the bishop on **a2** was promoted, because the bishop on **f1** did not move due to White's pawns on **e2** and **g2**. In order to promote a pawn to a light-color bishop, a White pawn had to reach either **e8** or **g8**. Then White moved its new bishop to **a2** and played **b3**. It looks simple, doesn't it? Let us check again. White has to promote its **a2** pawn to a light-color bishop, alluding that the pawn has to capture Black's pawns and pieces on squares **b3**, **c4**, **d5**, **e6**, **f7**, and **g8**. Because Black's pawn could only be captured on **f7**, White's pawn has to capture five Black pieces. Let us count. White can capture knight **b8**, queen **d8**, (Unfortunately, the bishop on **c8** does not help. Look at pawns on **b7** and **d7**.) knight **g8** (The bishop on **f8** has an opposite color.), and rook **h8**. Oops, there are just four pieces, which is not enough. We will need another Black

piece. Where can we get it? First, Black gives up its bishop **f8** on **e3** which leads to double **e** pawns. Second, Black promotes one of its pawns to a knight, for example, and forfeits it on **g8**. Thus, there are two promotions in this problem.

Let us count again. What White pieces can be captured by a Black pawn for its promotion? There are five pieces to capture: knight **b1**, bishop **c1**, queen **d1**, knight **g1**, and rook **h1**. Black could promote either **g7** or **h7** pawns, since White has to capture the **f7** pawn for its own pawn promotion. So far so good, but where can Black promote a pawn? Note that a Black pawn has to cross the second rank somewhere, which cannot be **h2**, **g2**, **e2**, and **c2**, because White has pawns on these squares. A Black pawn cannot appear on squares **f2** or **d2** because that would be a check, and the White king would have to move (and thus would not be able to castle later). Hence, there are just two possibilities: squares **a2** and **b2**. Note that neither pawn **g7** nor pawn **h7** could reach square **a2**. We are left only with square **b2**. Therefore, Black's **g7** pawn captures White pieces on **f6**, **e5**, **d4**, and **c3**. Then, White plays **b2 – b3** and Black captures a White piece on **b2**. After that, Black promotes its pawn to a knight on **b1**. This knight is later taken by the White pawn on **g8** when the new White bishop is promoted. This looks accurate, but the new White bishop should reach square **a2**, which is impossible with the White pawns **b3** and **c2**.

I described my reasoning to Shapley and asked him whether there was indeed a solution. Shapley said that “there is a solution.” He provided encouragement, stating that I was on “the right track,” but had to find the right sequence of promotions. For some time (all night), I struggled with a solution, but no new ideas appeared. The next day was the last day of the conference, and I had to find a solution. The problem was not whether White could castle queenside with White pawns on **f5** or **f6**, but whether I could get the position on Diagram 7 in a legal way, when the Black pawn **g7** passes through square **b2**. Shapley hinted that castling queenside is possible with pawn **f6** but not with pawn **f5**.

Fortunately, I succeeded in constructing an idea before lunch. White has to capture rook **a8** and promote its pawn to a bishop, the new bishop returns to **a2**, while White plays **b2 – b3** and Black promotes its pawn on **b1** to a new rook. However, **b2 – b1R** will be a check! Note that the two remaining White pieces, bishop **a2** and rook **a1**, cannot capture the new Black rook. Moreover, White has no more pieces to block the check. What else can be done? Can White use Black pieces to block the check? Black only has its king. Therefore, after White has played **b2 – b3**, the Black

king moves to **a3**, and then Black plays **Ka3 – b2**, to which White replies with **Ba2 – b1** and the Black king moves to **c1**. Now, Black promotes **b2 – b1R** without check. Great news! The ultimate remaining problem, however, lies in transferring the Black king and rook back to the 8<sup>th</sup> rank. Although I revealed my progress to Shapley, he disclosed that the Black king maneuver failed to work.

And then I found it! I had completely forgotten about the Black dark-color bishop, which Black can give up on **e3** later on and use to block the first rank. This indicates that the White pawn is still on **d2**. Moreover, White had to play **f2 – f4** in order to give up rook **h1**. Only in this case, the Black pawn **g7** can reach **b2**. Let us count again. First, White's pawn **a2** captures rook **a8**, knight **b8**, queen **d8**, knight **g8**, rook **h8**, and pawn **f7** to promote to a new light-color bishop. Then this bishop moves to **a2**. White plays **b2 – b3**. Second, Black brings its dark-color bishop to **c1**. Third, White plays **f2 – f4** and transfers rook **h1**. Fourth, Black's **g7** pawn captures four White pieces and reaches **c3**. Fifth, White plays **Ba2 – b1** and Black captures the last White knight on **b2** with pawn **c3**. White plays **Bb1 – a2**. Black promotes its pawn **b2 – b1R**. White has no pieces to move, because the White king has to be at **e1** for the castle and the White rook and bishop cannot capture the new Black rook since this rook is present in Diagram 7. Therefore, White has to move a pawn, suggesting that White has the only move **f4 – f5**. Sixth, Black has to bring its rook back to **a8**. Thus, Black plays **Rb1 - b2**, to which White responds with **Ba2 – b1** and Black plays **Rb2 – a2**. Again, White has the only move **f5 – f6**! The rest is simple, as Black transfers its rook to **a8** and gives up its bishop on **e3**, allowing the position on Diagram 7 with the White pawn on **f6** to be reached. The grand design!

### Conclusion

Upon the conclusion of the conference, Shapley revealed that he collected interesting puzzles and problems in addition to designing his own. Professor Shapley gave me his manuscript (Shapley, 1987) during our next meeting in Stony Brook in 1999, where I discovered that, excluding the two positions on Diagrams 1 and 7, all the other problems had been published in that manuscript. Of course, he did not include solutions for any problem in the manuscript.

Lloyd Shapley's creativity in chess and chess with imperfect information is tremendously impressive. Moreover, I hypothesize that the study of Kriegspiel problems might have inspired

Shapley's seminal contribution to Stochastic games (Shapley, 1953). Certainly, the similarities are evident in Appendix A. Nonetheless, I dearly regret that I never asked Lloyd Shapley whether that hypothesis bears any relation to reality.

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ACCEPTED MANUSCRIPT



## Appendix B

The Invisible Chessboard<sup>26</sup>

Lloyd Shapley

June, 1987

## 1. IN THE YEAR OF OUR FORD. (L.R. FORD, 1956)

*A Kriegspiel “position” is not just a display of pieces on the two boards, describing one objective chess position and two subjective reconstructions. To the player, the position of the game is both less and more than what a kibitzer sees. Less...because half the playing field is out of sight; more...because the player carries in his head a remembrance of moves past, as well as announcements, inferences and impressions — a relevant record that is far richer, though less precise, than anything he could display on the board for the kibitzers’ edification. Just as poker is essentially a game of money-management, not card-play, so is Kriegspiel at heart a game of information-handling, not piece-manipulation. Unlike the chess player, who forever plans ahead, the Kriegspieler spends most of his thinking time to the reconstruction of past events.<sup>27</sup>*

*It perhaps follows that the best Kriegspiel problems ought pay as much attention to backward analysis as to forward play, somehow putting the solver into a state of partial information, say in the form of mass of clues and signals to be sorted out, or even by providing a complete synopsis of “the game thus far” from one player’s point of view, as in our first offering.*

*This ancient but elegant puzzle is due to Lester Ford, one of Rand’s early superstars (and not only at the Kriegspiel table). It enjoys the dubious distinction of publication in the RANDom NEWS, a short-lived and long-forgotten house organ. Strictly speaking the diagram on the next page is unnecessary, since the entire game (from Black’s point of view) is spelled out in the accompanying protocol. Black’s moves are admittedly rather silly, and when you’ve solved the problem you’ll see that White’s moves are even sillier. But chess problem settings have a long tradition of implausibility, to which Kriegspiel merely adds a new dimension.*

*As for the problem itself, today’s sophisticated Kriegspieler will soon unravel the “mystery of move nine” and expose White’s fatal indiscretion.<sup>28</sup> It is then child’s play to find the two-move coup de grace. (Adults may turn to the answers section.)*

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<sup>26</sup> I changed Shapley’s (old) descriptive chess notation to the (modern) algebraic one in Appendices B and C.

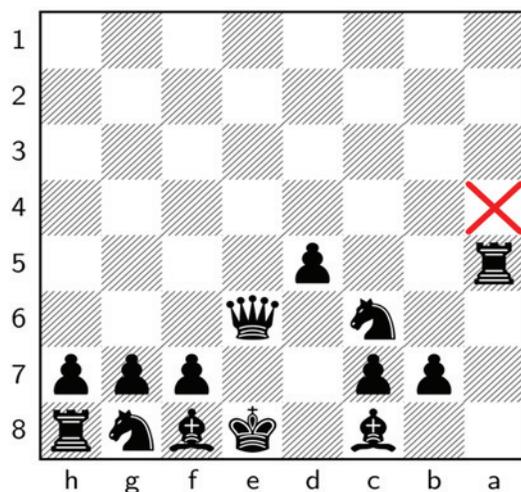
<sup>27</sup> *This is why chess skills are of limited value in playing kriegspiel, while the ability to keep track of one’s ambiguities is absolutely vital.*

<sup>28</sup> *Needless to say, his fate is richly deserved in this instance. But as the traditional “good guy” of chess problems, White will fare better in the sequel.*

## 1. THE MYSTERY OF MOVE 9.

Present position

WHITE(14)



Black (14) mates in 2

X site of pawn capture, move 10

Previous moves and announcements

WHITE	BLACK
1) ...	<b>e7 – e5</b>
2) ... “try d4”	<b>e5 x d4</b> “pawn gone d4”
3) ... “pawn gone d4”	<b>a7 – a5</b>
4) ...	<b>a5 – a4</b>
5) ...	<b>Nb8 – c6</b>
6) ...	<b>Ra8 – a5</b>
7) ...	<b>Qd8 – e7</b>
8) ...	<b>d7 – d5</b>
9) ... “try e6”	<b>Qe7 x e6</b> “pawn gone e6”
10) ... “pawn gone a4”	??

2. WHO? WHAT? WHEN? WHERE? HOW? (Iss 8/25/61)<sup>29</sup>

*The only legitimate sources of information in Kriegspiel are the official announcements. But the practical player takes advantage of many other cues and clues. He takes note of the referee's choice of words and all the little slips and pauses. He monitors the chatter of the kibitzers, witty and otherwise, and senses the hush that falls over the room when some great opportunity or disaster is near. He observes the tell-tale pattern of his opponent's "no" sequence, and tries to decode the swish and thump of pieces as they are being moved about on the opposite side of the screen.<sup>30</sup>*

*Unfortunately, the antiseptic setting of a written problem gives little opportunity to present such "soft" bits of information, from which the live game derives so much of its flavor.<sup>31</sup> In the next problem, we'll try to “soften up” the reader with a barrage of the hard stuff: surprise moves, fancy checks, and pawn tries that mysteriously appear and disappear. Perhaps these will induce a haze of happy befuddlement, not unlike the mental fog that often enwraps a real-time player overloaded with ambiguities in one of those quickie lunch-hour games. The difference is that now you have all the time you want to figure it out!*

*Rather than start at the beginning, as in No. 1, we pick up the action just six moves before the jumping-off point, making the somewhat unreal assumption that White, at that time, knew nothing*

<sup>29</sup> Iss means that Lloyd Stowell Shapley created this problem on 8/25/61.

<sup>30</sup> Before the high screen was introduced, Ray Fulkerson was the acknowledged master at reading eye, neck and shoulder movements across the screen. Les Ford, for his part, was the master at faking them.

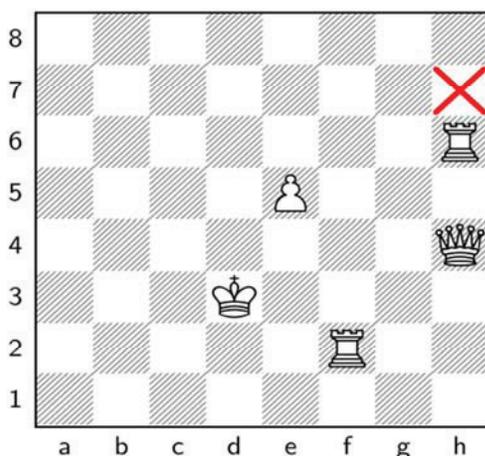
<sup>31</sup> But see Problem 11, where the referee makes one tiny little mistake, alas!

of Black's position except for the piece-count and the pawn-files. Never fear! The next six, fun-packed moves produce a wealth of intricately interlocking pieces of information - enough, indeed, for an excellent reconstruction when the jig-saw is finally put together.

One comment: The reader may wonder what significance could possibly attach to the "no" on Black's 6th. Surely White can win just: as quickly without hearing that "no" ...or can he?

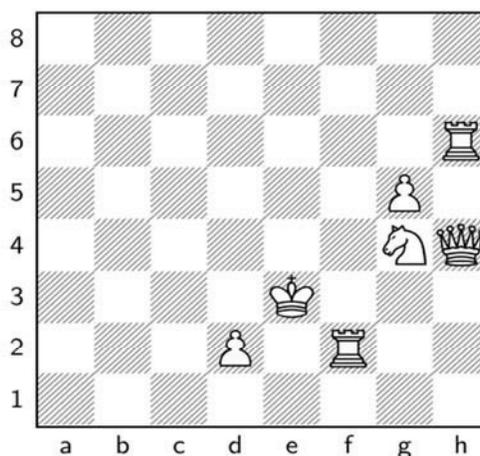
WHAT'S GOING ON HERE?

THE PRESENT POSITION  
BLACK (5)



WHITE (5) mates in 2  
X scene of the captures on move 6.

SIX MOVES EARLIER  
BLACK (7\*)



WHITE (7)  
\*Black has two pawns, known to be on the d and f files respectively.

*Intervening moves and announcements*

<i>White</i>	<i>Black</i>
1. $d2 - d3$ ...	"ch file, try $e4$ "
2. $d3xe4$ "piece $e4$ " ...	"ch knight, try $d5$ "
3. $Ke3 - d3$ ...	"tries $f5, f6$ "

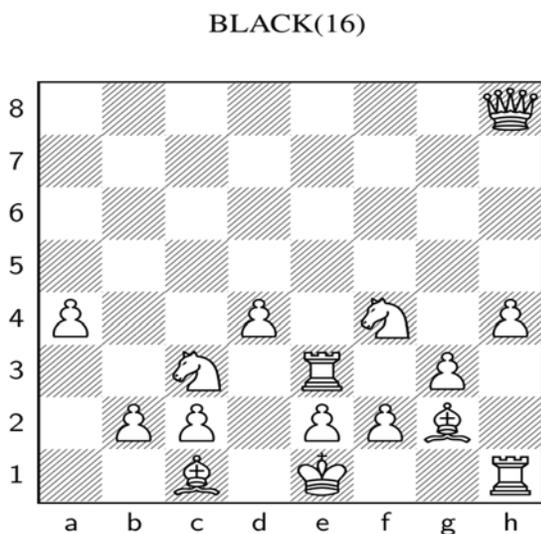
4. $e4 - e5$ "try $g4$ " ...	"piece $g4$ ; ch long"
5. $g5 - g6$ "ch short" ...	"tries $f6, h7$ "
6. $g6xh7$ "piece $h7$ " ...	"no"
7. ?? (mate in 2)	"pawn $h7$ "

### 3. THE VOICES OF SILENCE (Iss 11/15/68)

*No news is big news in No. 3. The silence that follows White's 14-th is absolutely deafening. In fact, the referee's inability to say "no" eventually gives the whole show away.*

*The setting is unusually realistic for a composed problem. White's opening is irreproachably "safe and sane", and even Black's eccentric plan of development is hardly any quirkier than many this author has witnessed in actual play.<sup>32</sup>*

*The purist may complain that the key-move is not unique. Yet the two solutions are such Siamese twins that it cannot fairly be said that the problem is "cooked".<sup>33</sup>*



### 5. TEN TEMPTING TRIES. (Iss/Joel Spencer, 7/20/66)

<sup>32</sup> *The author and Ray Fulkerson once thought to stage this game before an unsuspecting lunchtime audience, keeping even the referee in the dark. Alas, we never agreed who would play White!*

	White	...	Black
1.	d2 – d4		“Black”
2.	g2 – g3		“White”
3.	Bf1 – g2		“ ”
4.	h2 – h4		“ ”
5.	Ng1 – h3		“ ”
6.	Nh3 – f4		“ ”
7.	a2 – a4		“ ”
8.	Ra1 – a3		“ ”
9.	Qd1 – d3		“ ”
10.	Qd3 – g6		“ ”
11.	Ra3 – e3		“ ”
12.	Nb1 – c3		“ ”
13.	Qg6 – g8		“ ”
14.	Qg8 – h8		“ ”
15 a.	O-O		“no”
	b. <u>mate in 2</u>		

### 4. JUST HORSING AROUND (Iss 4/23/68)

This problem is described and considered on page 6.

<sup>33</sup> *Cook (kook) vt.: To spoil (a problem) by finding two or more key moves. Webster's New International Dictionary, 2nd ed.*

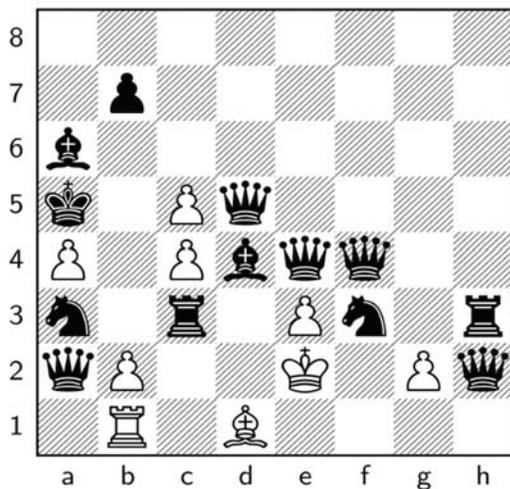
Here's another example of a single move releasing a flood of information. This time, however, the voice is far from silent. TEN (count 'em!) brand-new pawn tries appear out of the blue. White, naturally, resists all ten temptations and gallantly "pushes past" to victory.

Here's the sound track: "White to move." "Black to move." "White has tries on a3, b6, c3, d4, d5, f3, f4 and h3, not to mention the double try on b5." "Mate!"

This is of course just a stunt, not a real problem. But before turning the page, you might enjoy trying to figure out the position that permits this sequence of announcements.<sup>34</sup>

Is ten the maximum for a legal game? We think so.

BLACK(13)



WHITE(9)

Referee's announcements

"White" "Black"

"Tries at a3, b6, b5 (double), c3, d4, d5, f3, f4, h3."

"Mate!"

## 6. RAY'S SEND-OFF. (Iss 5/68)

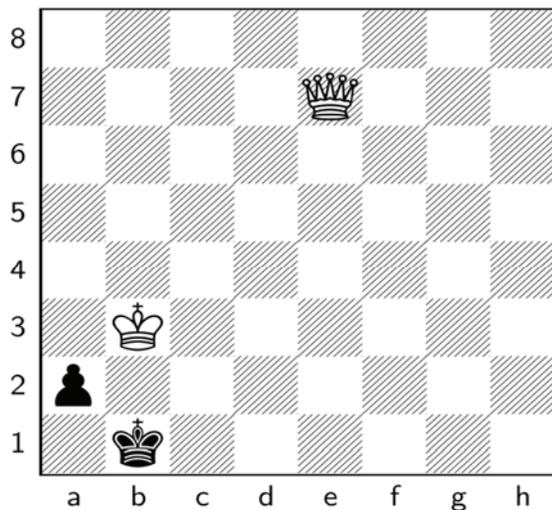
This problem is described and considered on page 10.

## 7. AN ENDGAME STUDY.

<sup>34</sup> Hint: Put White's king at his e2 and Black's king at a5. Then give White a rook and bishop in addition to the necessary pawns.

The present position

BLACK(2)



WHITE (2) to win.

*White, knowing the exact position, has brought his king to b3 and Black has just*

*made his move. If the referee says "White" or "Black promotes", then there is an obvious mate at e1. So we assume that the announcement is "Black promotes, check knight."*

*Now White must take care not to allow Black's king to escape the corner while the knight roams free, for in that case Black would have at least a small chance of picking off the queen. In fact, White has a strategy that wins in five moves or less, wherein he either checkmates the king or captures the unprotected knight without danger of stalemate. Can you find it?*

#### 8. THE WIDE OPEN SPACES. (lss)

This problem is described and considered on pages 5 and 6.

#### 9. THE VOICES OF SILENCE II. (lss 2/76)

*In No. 3, the "Silence" is broken at the very end – when White tries to castle and gets a "no". It seems that this was inevitable in a composition based on the "silence" theme. To give Black a free move at that critical point: - with all his men still on the board and in some state of development - would unravel White's reconstruction and make a forced mate in a reasonable number of moves unattainable.*

*Sometime later, however, a way was found to get rid of this thematic impurity. The idea was to get Black into such a tangled position that his final move, though silent, would not be "free". No. 9 is*

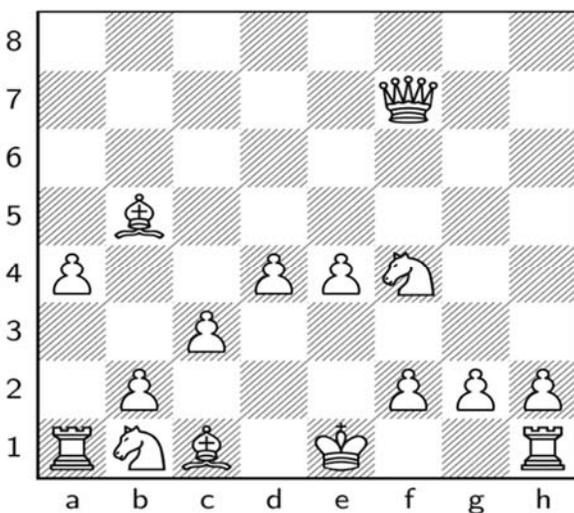
a realization of that idea. Indeed, the solver will eventually discover that Black's 14th move is uniquely determined, since all his other moves would trigger announcements.

The present setting is quieter than No. 3 in other respects as well. For one thing, there is no sudden burst of information at the end. White's dramatic 5-th move creates a little flurry, but any information gained is given plenty of time to disperse, and the remaining nine moves apparently tell us nothing new. The solution itself is also subdued. The key-move is unforcing and uninformative, and it turns out that White need pay no attention to the referee's voice as the winning strategy unfolds, except to listen for a possible "hell, no!"

## 9. TOTAL SILENCE.

### PRESENT POSITION

#### BLACK(16)



WHITE(16) mates in 2.

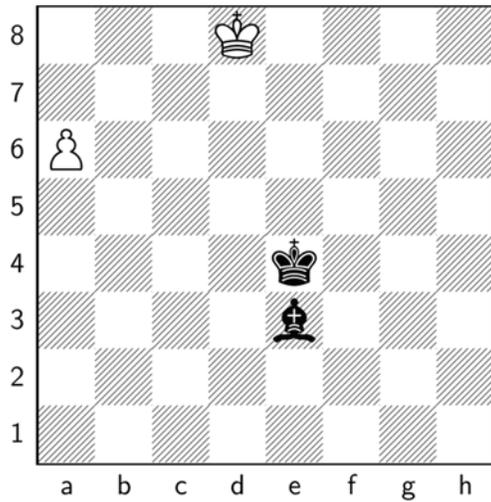
### Previous moves and announcements

	White		Black
1.	<i>e2 – e3</i>	"Black" ...	"White"
2.	<i>Ng1 – h3</i>	" ...	"
3.	<i>Nh3 – f4</i>	" ...	"
4.	<i>Qd1 – h5</i>	" ...	"
5.	<i>Qh5 – e8</i>	" ...	"
6.	<i>Qe8 – a4</i>	" ...	"
7.	<i>c2 – c3</i>	" ...	"
8.	<i>Qa4 – d1</i>	" ...	"
9.	<i>a2 – a4</i>	" ...	"
10.	<i>Qd1 – h5</i>	" ...	"
11.	<i>Qh5 – f7</i>	" ...	"
12.	<i>Bf1 – b5</i>	" ...	"
13.	<i>e3 – e4</i>	" ...	"
14.	<i>d3 – d4</i>	" ...	"
15.	??	(mate in 2)	

## 10. HIT THE MOVING TARGET (T. E. Ferguson, 7/76)

### THE PRESENT POSITION

BLACK(2)

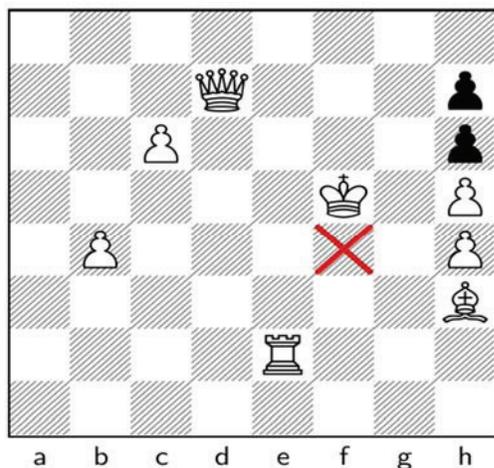
WHITE (2) wins with  $p > \frac{1}{2}$ 

*Both players are assumed to have exact information at the start, and it is White's move. Black can draw by picking off the pawn at a7, or by sacrificing the bishop on that square when his king is ready to move into c7. On the other hand, if the pawn is allowed to promote, the queen will win easily against the bishop.*

*The situation clearly calls for mixed strategies.*

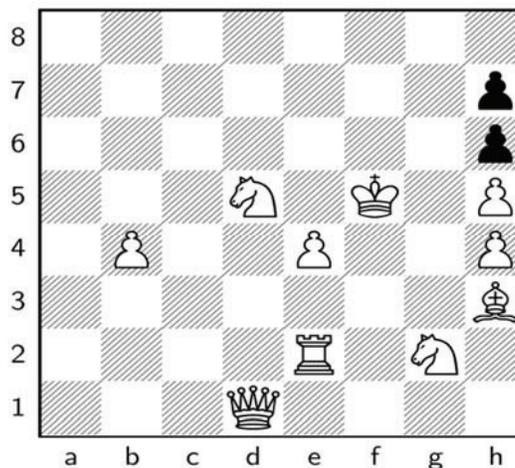
## 11. OOPS! (Iss 5/71)

THE PRESENT POSITION  
BLACK (8)



WHITE (8) mates in 6  
X site of Black's latest capture.

THE POSITION 4 MOVES AGO  
BLACK (10\*)



WHITE (10)  
\*Black is known to have 6 pieces and 4 pawns.

*Previous moves and announcements*

<i>White</i>	<i>Black</i>
1. Ng2 – f4 “try d5”	... “piece d5, try d5”
2. e4xd5 “piece d5, try d5”	... “try c6”
3. d5xc6 “pawn c6”	... “no”
	... “White”
4. Kf5 – f6 “no”	
Qd1 – d7 “try f4”	

But at this point, amid murmurs of discontent, Refreconsiders and admits that his “no” at move 3 was in error. The position is accordingly retracted, Black is told to try the move again, and play resumes with the corrected announcement –

3.	...	“White”
4. Qd1 – d7 “try f4”	...	“piece f4”

White starts to object, but is solemnly assured that apart from the false “no” there have been no mistakes. After pondering this for a while, White announces the mate in 6.

## 12. THE INFINITE POWER OF THE ROOK. (Iss c. 1960)

This problem is described and considered on pages 3 and 4.

## Appendix C

## KRIEGSPIEL RULES

Standard chess rules apply, with the following additions and elucidations.

1. Personnel – two players, referee, kibitzers.
2. Each player has a complete chess set (board, black and white men).
3. A player may freely rearrange the men of opposite color on his board - these men have no official role in the game. The players may not see each other's boards and men.
4. The referee monitors the game, and is required to announce the following:
  - a. Whose turn it is to move (called here the mover).
  - b. The square(s) on which the mover's pawns have currently valid options to make captures (called "tries"), thus, "White has a pawn try on e5," "Black has a double try on c3," etc.
  - c. Each rebuff ("no") experienced by the mover in attempting to move.
  - d. The fact that a capture has taken place, the category of the captured man according to the pawn/piece dichotomy, and its location. Thus, "Black has lost a piece on d8."
  - e. Checks, which are announced by whichever of the following is (are) correct...
    - i. Check on the long diagonal.
    - ii. Check on the short diagonal.
    - iii. Check on the rank (or "horizontal").
    - iv. Check on the file (or "vertical").
    - v. Check by a knight.  
(The diagonals considered are the pair which intersect at the king. Check by a pawn is announced as if it were a bishop or queen, i.e. without revealing that it is a pawn.)
  - f. Pawn promotions, but not where they occurred nor what piece was chosen.
  - g. Checkmate and stalemate.
5. The referee does not review announcements more than one move old, and does not recapitulate losses.
6. The referee does not rebuff in the usual manner attempts which are illegal *per se*, such as moving to or through a square occupied by one's own man, failing to remove a captured piece from the board, moving a rook diagonally, responding to a check by other than a king move or interposition or capture compatible with the announced character of the check, etc. The special rebuff "hell, no!" is used here.
7. Errors — It is the referee's function to communicate effectively with the players. For example, if a player misidentifies the square named in an announcement the referee will correct him, taking care not to disclose any significant information while doing so. Referee blunders range from trivial to catastrophic. The remedies include general reprimand, reverse play, and declaring the game void.
8. A player may, before moving, demand a count of the rebuffs ("no's") sustained by his opponent on the previous move. In fact, he may demand such information during the opponent's turn.
9. A player may attempt any move which is compatible with his own situation (men and deployment and with the referee's current announcement). In interpreting the word "compatible", the player is not presumed to remember previous plays or to make logical inferences.
10. A move is completed when a piece touches the board or a presumed enemy piece on a legally admissible square.

11. En passant options and captures are announced in the same manner as other options and captures are announced. The fact that they are en passant is not specified.
12. When a check exists, only those pawn options (“tries”) are announced which if taken will eliminate the check.
13. Kibitzers - The game is a spectator sport par excellence, and everything is done to keep it so. The kibitzers have the right to criticize the play, the players and the referee. However, the ethics of the situation require that the kibitzers never intentionally give useful information to the players. Probably the game breaks down as the number of kibitzers increases indefinitely - even with half a dozen; a pinned pawn has but small chance of not being found out in a false try situation.
14. It is considered ethical for a player to capitalize on blunders and all unsolicited information received from referee and kibitzers - indeed, he may attempt to solicit “information” from his opponent or otherwise heckle him.
15. Draws - A player may offer a draw immediately after making any move that is not rebuffed. The offer must be accepted or rejected before the next move is attempted. A player may not claim a draw unilaterally on the presumption of a twice-repeated position a “perpetual check,” an absence of mating strength or a failure to make timely progress in the end game. But the referee, on his own initiative and discretion, may declare a game drawn for any of these reasons except “twice-repeated position.” (A “position” in Kriegspiel is deemed to include all the information gathered by the players from the beginning, and hence is never repeated.)

- based on KRIEGSSFIEL RULES AT RAND by J. D. Williams, 4/17/50