Homotopy theory of moduli spaces

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Declaration

I hereby declare that this thesis is my own work, except where otherwise stated, and has not been submitted for the award of a higher degree elsewhere. The contents of Section 2 appear in Homology, Homotopy and Applications. The contents of Section 1 have been submitted for publication and the contents of Section 3 will be submitted for publication.

The contents of Section 1 and Section 2 were written solely by myself. The contents of Section 3 were the result of collaborative work with Christopher Braun. To elaborate on my contributions to Section 3 I include the following statement of Christopher Braun:

> The collaboration was suggested by James's PhD supervisor. The mathematics underlying this paper was worked out by regular discussions and meetings, as is usual for many mathematical collaborations, and thus credit should be shared equally in this aspect. The writing was done almost entirely by James.

Abstract

This thesis gathers three papers written by the author during PhD study at Lancaster University: [Mau15, Mau17, BM17]. In addition to these three papers, this thesis also contains two complementary sections. These two complementary sections are an introduction and a conclusion. The introduction discusses some recurring themes of the thesis and parts of the history leading to those results proven herein. The conclusion briefly comments on the outcomes of the research, its place in the current mathematical literature, and explores possibilities for further research.

A common theme of this thesis is the study of Maurer-Cartan elements and their moduli spaces, that is their study up to homotopy or gauge equivalence. Within the three papers cited above (and thus within this thesis), three different applications of Maurer-Cartan elements are demonstrated. The first application constructs certain moduli spaces as models for unbased disconnected rational topological spaces. The second application constructs certain moduli spaces as those governing formal algebraic deformation problems over, not necessarily local, commutative differential graded algebras. The third application uses the presentation of L_∞-algebras as solutions to the Maurer-Cartan equation in certain commutative differential graded algebras to construct minimal models of quantum L_∞-algebras. These quantum homotopy algebras arise as the 'higher genus' versions of classical (cyclic) homotopy algebras.

Contents

Introduction

The Maurer-Cartan equation is far reaching in the worlds of mathematics and physics. A physicist may be more familiar with the Maurer-Cartan equation when referred to as the (classical or quantum) master equation. For differential graded Lie algebras, the Maurer-Cartan equation appears to be a relatively simple object and, yet, it has many profound and deep applications. This is just part of its beauty, much like the equation $d^2 = 0$ in homological algebra. In mathematics, the Maurer-Cartan equation has ties with the study of topological spaces, deformation theory, and homotopy algebras, all of which are explored within this thesis. Of course, there exist many other examples where the Maurer-Cartan equation appears in mathematics. In Physics, a solution to the master equation describes a physically consistent state of a dynamic system. The links this thesis has to physics are largely in Section 3, where the Batalin-Vilkovisky formalism is used explicitly. The Batalin-Vilkovisky formalism first arose in physics in work of Batalin and Vilkovisky on the quantisation of gauge theories [BV81,BV83].

As with many topics in mathematics, one is often concerned with objects up to some notion of equivalence, perhaps weakened. For Maurer-Cartan elements, the standard choice of equivalence is that of gauge equivalence. The terminology is inspired by gauge fixing in physics: any two field configurations related by a gauge transformation are considered physically equivalent. It is a Theorem of Schlessinger and Stasheff [SS12] that the notions of gauge equivalence and homotopy equivalence coincide for Maurer-Cartan elements of pronilpotent differential graded Lie algebras. The latter notion of homotopy equivalence is often more natural in the settings considered within this thesis and is, therefore, the primary choice for the notion of equivalence of Maurer-Cartan elements.

Whilst the whole thesis is tied together via the theme of Maurer-Cartan elements, the first two thirds of this thesis could also be said to be concerned with the Koszul duality of commutative and Lie algebras. Koszul duality is a phenomenon in homological algebra that has many manifestations; another example is the classical Bernstein-Gelfand-Gelfand duality between the bounded derived categories of finitely generated graded modules over the symmetric and exterior algebras with dual vector spaces of generators [BGS96,BGG78]. Within this thesis, the manifestation of Koszul duality is as a Quillen equivalence of closed model structures on certain categories of generalised commutative and Lie algebras. The first result on the Koszul duality of commutative differential graded algebras and differential graded Lie algebras was proven by Quillen [Qui69]. Quillen had required (quite severe) restrictions on the grading of the objects considered. Hinich [Hin01] subsequently removed these grading restrictions. For Hinich to make his extensions, he needed to observe that it was necessary to refine the notion of weak equivalence; that is, Hinich worked with the finer notion of filtered quasi-isomorphisms as the weak equivalences in his model structure. Chuang, Lazarev, and Mannan [CLM14] extended the duality even further for pseudo-compact commutative differential graded algebras. For this extension it was necessary to use curved Lie algebras. Curved Lie algebras are a generalisation of differential graded Lie algebras that are known in the mathematical folklore, but appear sparsely in the literature. Curved Lie algebras, however, make a strong and important appearance within this thesis.

Koszul duality of differential graded algebras has also been studied and generalised over different operads: Positselski [Pos11], for example, studied the associative case. Further, one side of the equivalence proven by Positselski is a category consisting of curved objects. This suggests that in higher generalities, one side of a Quillen equivalence should be a category consisting of curved objects. This hypothesis is strengthened by the results of [CLM14] and the results of Sections 1 and 2 of this thesis.

As stated before, three different applications of Maurer-Cartan elements and their moduli spaces are demonstrated within the three main sections of this thesis. These three sections are briefly summarised now.

Section 1 constructs certain Maurer-Cartan moduli spaces as models for unbased disconnected rational topological spaces. Algebraic theories for connected spaces were first constructed in the two seminal papers of Quillen [Qui69] and Sullivan [Sul77]. Quillen related the theory of connected rational spaces to differential graded Lie algebras, whereas Sullivan had worked from the perspective of commutative differential graded algebras. Lazarev and Markl have recently constructed algebraic theories from both perspectives, c.f. [LM15]. The more general theory of Lazarev and Markl illuminates some differences between the two approaches that were not seen by the earlier approaches. For instance, the more general approach requires one to consider differential graded Lie algebras endowed with a linearly compact topology, whereas the commutative differential graded algebras remain discrete. As part of the construction of the models for unbased disconnected rational topological spaces contained within this thesis, a closed model structure on the category of pseudo-compact curved Lie algebras with curved morphisms is established. This closed model category is then shown to be Quillen equivalent to the known model structure on the category of unital commutative differential graded algebras of Lazarev and Markl. Thus, this equivalence of homotopy categories extends the well-known Koszul duality of differential graded Lie algebras and unital commutative differential graded algebras. Combining the newly proven Quillen equivalence with the Quillen equivalence between certain subcategories of topological spaces and unital commutative differential graded algebras of Lazarev and Markl, one can extend the equivalence to certain nice subcategories of disconnected topological spaces and curved Lie algebras.

Section 2 constructs certain Maurer-Cartan moduli spaces as those governing formal deformation functors over pseudo-compact, not necessarily local, unital commutative differential graded algebras. This is an extension to the often repeated slogan "in characteristic zero every deformation theory is governed by a differential graded Lie algebra." The ideas underlying the slogan initially arose in the work of Nijenhuis who noticed the similarities between the deformations of complex-analytic structures on compact manifolds and the deformations of associative algebras, see [NR66, NR67]. These ideas were further developed in communications of Quillen, Deligne, Drinfeld and others in the 1980s, see [Dri14], for example. More recently, these ideas have been even further developed by Kontsevich [Kon03], Schlessinger and Stasheff [SS12], Goldman and Millson [GM88], and many others. The general idea is to begin with a differential graded Lie algebra and perform a functorial procedure which constructs a deformation functor. Moreover, the procedure is supposed to respect the equivalence of differential graded Lie algebras, i.e. the procedure is supposed to be constructed in such a way that quasi-isomorphic differential graded Lie algebras result in isomorphic deformation functors. However, the slogan is not quite correct (nor sufficient for every situation) and the category of differential graded Lie algebras should be extended: the category of L_∞-algebras is considered one candidate offering an appropriate extension. In the steps towards constructing deformation functors over pseudo-compact unital commutative differential graded algebras, the category of marked curved Lie algebras is introduced and equipped with a closed model category structure. When working over an algebraically closed field, the model category of marked curved Lie algebras is shown to be Quillen equivalent to an existing model category structure on pseudo-compact unital commutative differential graded algebras, c.f. [CLM14]. Again, this Quillen equivalence extends the well-known Koszul duality of differential graded Lie algebras and unital commutative differential graded algebras. Having this Quillen equivalence in ones toolkit allows one to prove those deformation functors constructed from marked curved Lie algebras are representable in the homotopy category of pseudo-compact unital commutative differential graded algebras and are thus homotopy invariant.

Section 3 uses the presentation of L_∞-algebras as solutions to the Maurer-Cartan equation in certain commutative differential graded algebras to construct minimal models of quantum L_∞-algebras. These quantum homotopy algebras arise as the 'higher genus' versions of classical (cyclic) homotopy algebras. Amongst the main tools contained within this section (and of the Batalin-Vilkovisky formalism in general) are integrals over Lagrangian (or, more generally, isotropic) subspaces—these integrals stem from having a strong deformation retract of a vector space onto its homology and the Batalin-Vilkovisky formalism. The usefulness of these integrals is apparent from the fact they are known to map a formal function on a super vector space that solves the Maurer-Cartan equations to a formal function on the homology of this vector space that also solves the Maurer-Cartan equation. In fact, these integrals respect the homotopy of Maurer-Cartan elements. Rephrasing these sentences, one can present the integral as a morphism of quantum L-infinity structures on a vector space to quantum L-infinity structures on the homology which descends to the level of Maurer-Cartan moduli spaces. Using this, one can show this presentation of the integral as a morphism of Maurer-Cartan moduli spaces provides an explicit construction of the minimal model of a given quantum L-infinity algebra. Included in this section is a concise description of how to apply the Feynman diagram formalism to the formal integrals of this sections and write them as formal sums over stable graphs (which are briefly recalled).

Notations and conventions

Since this thesis gathers three papers into one thesis, notation varies slightly throughout the document. Although notation is largely consistent, certain symbols are reused in different sections with differing meanings. The author appreciates this may be confusing to the reader. Each paper, however, includes its own section titled 'Notations and conventions' and all additional symbols (except some standard notation) are defined as they arise. It is, therefore, hoped that the reader will read each paper without being confused by any clashes of notation or convention.

Section 1
Unbased rational homotopy theory: a Lie algebra approach

Abstract

In this paper an algebraic model for unbased rational homotopy theory from the perspective of curved Lie algebras is constructed. As part of this construction a model structure for the category of pseudo-compact curved Lie algebras with curved morphisms will be introduced; one which is Quillen equivalent to a certain model structure of unital commutative differential graded algebras, thus extending the known Quillen equivalence of augmented algebras and differential graded Lie algebras.

Introduction

Rational homotopy theory of connected spaces was developed by Quillen [27] from the viewpoint of differential graded Lie algebras and by Sullivan [29] from the viewpoint of commutative differential graded algebras. A standard reference for the correspondence between rational connected spaces—in both the pointed and unpointed cases—and commutative differential graded algebras is [4]. Section 1.5 relies heavily upon results of that paper. Within [4] a closed model category structure is constructed for the category of non-negatively graded unital commutative differential graded algebras. Further, it is shown that there exists a pair of Quillen adjoint functors between the category of non-negatively graded commutative differential graded algebras and the category of simplicial sets. These Quillen adjoint functors restrict to an equivalence on the homotopy categories of connected commutative differential graded algebras of finite type over $\mathbb{Q}$, and connected rational nilpotent simplicial sets of finite type over $\mathbb{Q}$. In the work of Lazarev and Markl [24], this correspondence between non-negatively graded unital commutative differential graded algebras and simplicial sets is generalised to create a disconnected rational homotopy theory by extending to the category of $\mathbb{Z}$-graded unital commutative differential graded algebras. Removing the restriction of non-negative grading, despite seeming relatively harmless, has profound consequences; for example, a commutative algebra concentrated in degree zero is necessarily cofibrant in the category of Bousfield and Gugenheim, but this is not always the case in the category of Lazarev and Markl. The latter category does, however, appear to be more natural as one can, for example, define Harrison-André-Quillen cohomology within it, cf. [3]. The theory of [24] relies upon a Quillen adjoint pair of functors between the categories of $\mathbb{Z}$-graded commutative differential graded algebras and simplicial sets. This functor gives rise to an adjunction on the level of homotopy categories and, moreover, restricts to an equivalence between the homotopy category of simplicial sets having a finite number of connected components, each being rational, nilpotent, and of finite type (over $\mathbb{Q}$) and a certain subcategory of the homotopy category of $\mathbb{Z}$-graded commutative differential graded algebras (given explicitly loc. cit.). Sections 1.4 and 1.5 rely on some of the work developed by Lazarev and Markl in their paper. Further, Lazarev and Markl constructed a second version of this disconnected rational homotopy theory: one formed using differential graded Lie algebras. This second version was performed by relating the homotopy theory of unital commutative differential graded algebras and differential graded Lie algebras. Relationships of this kind are often referred to as a Koszul duality, the theory of which was established by work of Quillen [27]. Quillen showed there existed a duality between differential graded counital cocommutative coalgebras and differential graded Lie algebras under (quite severe) restrictions on the grading of the objects considered. Subsequently, these restrictions were removed by Hinich, cf. [20].

The context of this paper is one in which attention is drawn to the categories of commutative differential graded algebras and pseudo-compact curved Lie algebras. A curved Lie algebra is said to be pseudo-compact if it is the inverse limit of an inverse system of finite dimensional nilpotent curved Lie algebras. This inverse limit is taken using strict morphisms, i.e. using morphisms that commute with the differentials and curvature elements. Herein it will be shown that there exists a Quillen equivalence between the categories of commutative differential graded algebras and pseudo-compact curved Lie algebras. This result is proven by a suitable adaptation of Hinich's methods [20] and influenced by the work of Positselski [26]. Consequently an algebraic model of unbased disconnected rational homotopy theory using pseudo-compact curved Lie algebras is obtained. The condition that the curved Lie algebras be pseudo-compact is necessary, for it cannot be removed even when restricted to connected spaces (or simplicial sets).

Numerous papers now discuss the homotopy theory of differential graded coalgebras over different operads. For example Positselski studied coassociative differential graded coalgebras [26]. However, the coalgebras were assumed to be conilpotent loc. cit., and thus the homotopy theory developed therein is not completely general. Furthermore, Positselski worked with curved objects suggesting that when discussing a Koszul duality in more general cases one side of the Quillen equivalence should be a category consisting of curved objects; a hypothesis that is strengthened by results of [12] and this paper.

Section 1.1 begins by defining and discussing some of the basic properties of curved Lie algebras and their morphisms, before moving on to discuss the category of pseudo-compact curved Lie algebras and filtrations of such objects. These filtrations are particularly important to this paper and the homotopy theory developed herein; the filtrations play an essential role in defining the correct notion of a weak equivalence in the category of pseudo-compact curved Lie algebras, cf. Section 1.4. Without these filtrations the usual notion of a weak-equivalence, i.e. a quasi-isomorphism, is simply not fine enough.

Section 1.3 introduces a pair of adjoint functors between the categories of pseudo-compact curved Lie algebras and unital commutative differential graded algebras; these functors are analogues of the Chevalley-Eilenberg and Harrison complexes (see [32] and [1,18] respectively) in homological algebra and are influenced by constructions of Positselski in the associative case, see [26].

A model structure for the category of pseudo-compact curved Lie algebras is defined in Section 1.4; one which, by the previously mentioned pair of adjoint functors, is Quillen equivalent to the existing model structure of unital commutative differential graded algebras given by Hinich [19]—the main result of the paper. As remarked above, the notion of a filtered quasi-isomorphism using the filtrations defined in Section 1.1 plays a fundamental role in the homotopy theory of pseudo-compact curved Lie algebras. The proof of this Quillen equivalence relies upon the duality for associative algebras contained within [26], as the functors defined in Section 1.3 can be 'embedded' into the adjunction given in op. cit., this is similar to the method used in [24].

In Section 1.5 this equivalence of homotopy categories is then applied, in a similar manner to Lazarev and Markl [24], in the construction of a disconnected rational homotopy theory from the perspective of pseudo-compact curved Lie algebras. This viewpoint results in a couple of corollaries. The first corollary shows that the Maurer-Cartan simplicial set functor commutes with coproducts when restricted to the correct subcategories of curved Lie algebras and simplicial sets; this result is an analogue of one proven in [24], but the proof herein is much more simple due to the material developed. The second corollary constructs Lie models for mapping spaces between two simplicial sets each composed of finitely many connected components where each component is rational, nilpotent, and of finite type. More precisely, it constructs the mapping space as the Maurer-Cartan simplicial set of a certain curved Lie algebra. This approach to calculating the mapping space is building upon previous results that used rational homotopy theory [5, 7, 16] and using Lie models [2, 8, 9]. The approach via the Maurer-Cartan simplicial set contained herein, however, lends itself to much more calculations than the previous results and is much like work of Lazarev [23]. In fact, under suitable conditions, one can combine the results [23, Theorem 8.1] and Corollary 1.89 to construct a model for the mapping space as a disjoint union of Maurer-Cartan simplicial sets.

Notation and conventions

Throughout this paper it is assumed that all commutative and Lie algebras are over a fixed base field, k, of characteristic 0. There are no further assumptions made about the base field until Section 1.5, when the field will necessarily be assumed to be $\mathbb{Q}$, the rational numbers. All unmarked tensors will be over the base field. Every graded space will be assumed to be $\mathbb{Z}$-graded, unless stated otherwise: Section 1.5, for example, will assume some algebras to be non-negatively graded. All commutative algebras possess a unit, unless stated otherwise, and likewise that all cocommutative coalgebras possess a counit.

The notation $\langle a, b, c \ldots \rangle$ will be understood to mean the graded k-vector space spanned by the basis vectors $a, b, c, \ldots$, where the degrees of $a, b, c, \ldots$ are specified. Given a graded vector space V, the notation $\hat{L}V$ will be understood to mean the completed free graded Lie algebra on the generators of V. More specifically, $\hat{L}V$ is the subspace of the completed tensor algebra $\prod_{i\geq 1} V^{\otimes i}$ spanned by Lie monomials.

This paper will often use an assortment of abbreviations, including: 'dg' for 'differential graded'; 'dgla' for 'differential graded Lie algebra'; 'cdga' for 'commutative differential graded algebra'; 'CMC' for 'closed model category' (in the sense of [27], for a review of this material consult [13]); 'LLP' for 'left lifting property'; and 'RLP' for 'right lifting property'.

All Lie algebras will be given the homological grading with lower indices, whereas commutative algebras will be given the cohomological grading with upper indices. There is, however, an important exception to this convention: the tensor product of a homologically graded Lie algebra with the cohomologically graded cdga Ω of the Sullivan-de Rham forms. In this context, Ω is considered as homologically graded using the relations $\Omega_i := \Omega^{-i}$ for each $i \geq 0$, ensuring the tensor product is a homologically graded curved Lie algebra. This convention is relevant in Section 1.5.

Although many of the Lie algebras in this paper (namely the curved Lie algebras) will not necessarily be complexes, they resemble them and possess an odd derivation often referred to as the differential. In the homological grading, the derivation possessed by a curved Lie algebra has degree -1. Given any homogeneous element, x, of some given graded algebra its degree is denoted by $|x|$. Therefore, in the homological setting a Maurer-Cartan element is of degree -1 and the curvature element is of degree -2.

Given a homologically graded space, V, define the suspension, ΣV, to be the homologically graded space using the convention $(\Sigma V)_i = V_{i-1}$. In the cohomological setting the suspension is defined by $(\Sigma V)^i = V^{i+1}$. When dealing with objects that are endowed with a topology (such as those that are pseudo-compact) taking the dual will be understood to mean taking the topological dual. In more detail, this is the functor that takes an object to the set of continuous morphisms from it to the ground field. Therefore, since any continuous functional on V^* must factor through a finite dimensional quotient, one has (by the well known property of finite dimensional spaces) the continuous functional corresponds to an element of V, i.e. within this paper it will always be the case that $V^{**} \cong V$. Applying the functor of linear discrete (or topological) duality takes homologically graded spaces to cohomologically graded ones, and vice versa—more precisely, denoting the dual by an asterisk, it can be seen that $(V_i)^* = (V^*)^i$. Note, a homologically graded space can be made into a cohomologically graded one (and vice versa), by setting $V_i = V^{-i}$. Moreover, ΣV^* will be used to denote $\Sigma(V^*)$, and with this notation there exists an isomorphism $(\Sigma V)^* \cong \Sigma^{-1} V^*$.

1.1 The category of curved Lie algebras

In this section the category of pseudo-compact curved Lie algebras with curved morphisms will be defined and some of its basic properties are discussed. Later, in Section 1.4.3, the category of pseudo-compact curved Lie algebras with curved morphisms will be shown to possess a model structure. Further, a pair of adjoint functors will be defined in Section 1.3 that induce a Quillen equivalence between this model category and the model category of unital cdgas given by Hinich [19], this is the content of Theorem 3.42.

Definition 1.1. *A curved Lie algebra is the triple $(\mathfrak{g}, d, \omega)$ where $\mathfrak{g}$ is a graded Lie algebra, d is a derivation of $\mathfrak{g}$ with $|d| = -1$ (known as the differential), and $\omega \in \mathfrak{g}$ with $|\omega| = -2$ (known as the curvature) such that:*

- *$d \circ d(x) = [\omega, x]$, for all $x \in \mathfrak{g}$;*
- *$d\omega = 0$.*

Remark 1.2. Notice that the term 'differential' is an abuse of notation as the differential of a curved Lie algebra need not square to zero. Accordingly, some authors prefer to use the term predifferential. Within this paper, the term 'differential' will be used, despite the abuse of notation.

Note that a curved Lie algebra with zero curvature is exactly a dgla. As with dglas, Maurer-Cartan elements play a key role in the theory of curved Lie algebras.

Definition 1.3. *A MC (Maurer-Cartan) element of a curved Lie algebra, $(\mathfrak{g}, d, \omega)$ is an element $\xi \in \mathfrak{g}$ with $|\xi| = -1$ such that the Maurer-Cartan equation,*

$$\omega + d\xi + \frac{1}{2}[\xi, \xi] = 0,$$

is satisfied.

Remark 1.4. If $\omega = 0$ (i.e. $\mathfrak{g}$ is a differential graded Lie algebra) then the classical MC equation is recovered:

$$d\xi + \frac{1}{2}[\xi, \xi] = 0.$$

It will be common practice within this paper to shorten the notation of a curved Lie algebra, $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$, to only its underlying graded Lie algebra, $\mathfrak{g}$, where there is no ambiguity in doing so. In the shortened case, the differential and curvature will be denoted by the obvious subscript.

Definition 1.5. *A curved Lie algebra is said to be pseudo-compact if it is an inverse limit of finite dimensional, nilpotent curved Lie algebras.*

Here it is important to note that there exists a natural topology on any given pseudo-compact curved Lie algebra (or more generally pseudo-compact vector space) generated by subspaces of finite codimension, which is induced by taking the inverse limit. Therefore, the bracket and differential of a pseudo-compact vector space will be continuous with respect to this topology. Moreover, when considering morphisms between pseudo-compact spaces they will always be assumed to continuous with respect to this topology. In particular, when taking the dual of such spaces it will be understood that it is the topological dual being taken as opposed to the linear discrete one. For more details on pseudo-compact algebras see [14, Section 3], [22, Appendix], and [30, Section 4].

Remark 1.6. The definition of a pseudo-compact curved Lie algebra is similar to the definition of a pronilpotent Lie algebra, but more restrictive since every Lie algebra in the inverse system is assumed to be finite dimensional

Proposition 1.7. *Take a pseudo-compact curved Lie algebra, $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$, where $\mathfrak{g} = \varprojlim_i \mathfrak{g}_i$, and a unital cdga, A, both homologically graded. The completed tensor product*

$$\mathfrak{g}\hat{\otimes}A = \varprojlim_i \mathfrak{g}_i \otimes A$$

possesses a well defined pseudo-compact curved Lie algebra structure given as follows: the curvature is defined by $\omega_{\mathfrak{g}}\hat{\otimes}1$; the differential is defined on elementary tensors by $d(x\hat{\otimes}a) = d_{\mathfrak{g}}x\hat{\otimes}a + (-1)^{|x|}x\hat{\otimes}d_A a$; and the bracket is defined on elementary tensors by $[x\hat{\otimes}a, y\hat{\otimes}b] = [x,y]\hat{\otimes}(-1)^{|y||a|}ab$.

Proof. This is standard and straightforward. $\square$

It will be common for the adjective 'completed' to be dropped and $\hat{\otimes}$ to be referred to as the tensor product.

Definition 1.8. *A curved morphism of curved Lie algebras is defined to be the pair*

$$(f, \alpha) \colon (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}) \to (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}}),$$

where $f \colon \mathfrak{g} \to \mathfrak{h}$ is a morphism of graded Lie algebras and $\alpha \in \mathfrak{h}$ with $|\alpha| = -1$ such that:

- $d_{\mathfrak{h}}f(x) = f(d_{\mathfrak{g}}x) - [\alpha, f(x)]$, *for all* $x \in \mathfrak{g}$;
- $\omega_{\mathfrak{h}} = f(\omega_{\mathfrak{g}}) - d_{\mathfrak{h}}\alpha - \frac{1}{2}[\alpha, \alpha]$.

The image of an element $x \in \mathfrak{g}$ under the action of the curved morphism (f, α) is defined to be $f(x) + \alpha \in \mathfrak{h}$.

The composition of two curved morphisms, (f, α) and (g, β), (when such a composition exists) is defined as follows:

$$(f, \alpha) \circ (g, \beta) = (f \circ g, \alpha + f(\beta)).$$

A morphism with $\alpha = 0$ is said to be strict.

Remark 1.9. Note that a curved morphism will map $0_{\mathfrak{g}} \mapsto \alpha$. In fact, a curved morphism (f, α) is equivalent to the composition $(id, \alpha) \circ (f, 0)$.

Remark 1.10. In the case of a strict morphism it can be readily seen that the morphism is simply a graded Lie algebra morphism that respects the differentials and the image of the curvature of the domain is the curvature of the codomain. These morphisms are exactly those of [12]. Therefore, the α part of a curved morphism can be seen to act as an obstruction to the differentials commuting with the graded Lie algebra morphism and to the graded Lie algebra morphism preserving the curvature.

Just as with dgla morphisms, curved Lie algebra morphisms preserve MC elements.

Proposition 1.11. *Given a curved Lie algebra morphism*

$$(f, \alpha) \colon (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}) \to (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$$

and a MC element $\xi \in \mathfrak{g}$, the element $f(\xi) + \alpha \in \mathfrak{h}$ solves the MC equation. $\square$

Proposition 1.12. *Given a morphism of curved Lie algebras, $(f, \alpha)\colon (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}) \to (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$, there exists an inverse morphism $(f, \alpha)^{-1}\colon (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}}) \to (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$ such that:*

- $(f, \alpha) \circ (f, \alpha)^{-1} = (\mathrm{id}_{\mathfrak{g}}, 0)$, *and*
- $(f, \alpha)^{-1} \circ (f, \alpha) = (\mathrm{id}_{\mathfrak{h}}, 0)$,

if, and only if, f is an isomorphism of graded Lie algebras. Further, given an inverse graded Lie algebra morphism f^{-1} of f, the inverse of (f, α) is given by $(f^{-1}, -f^{-1}(\alpha))$.

Proof. If the graded Lie algebra morphism f is invertible, then clearly $(f^{-1}, -f^{-1}(\alpha))$ gives a two sided inverse of the morphism (f, α). Conversely, if (f, α) is invertible with inverse $(f, \alpha)^{-1} = (g, \beta)$ then

- $(f, \alpha) \circ (f, \alpha)^{-1} = (f \circ g, f(\beta) + \alpha) = (\mathrm{id}_{\mathfrak{g}}, 0)$, and
- $(f, \alpha)^{-1} \circ (f, \alpha) = (g \circ f, g(\alpha) + \beta) = (\mathrm{id}_{\mathfrak{h}}, 0)$.

From these equations it is clear that g must be a two sided graded Lie algebra inverse for f. Additionally, it can easily be seen that $\beta = -g(\alpha)$. □

It is important to note that a curved Lie algebra may be isomorphic to one with zero curvature (i.e. a dgla). To see this, let $\mathrm{ad}_{\xi}(-) = [\xi, -]$ be the adjoint action and take the curved isomorphism

$$(\mathrm{id}, -\xi)\colon (\mathfrak{g}, d, \omega) \to \left(\mathfrak{g}, d + \mathrm{ad}_{\xi}, \omega + d\xi + \frac{1}{2}[\xi, \xi]\right),$$

which has inverse (id, ξ). The curvature of the codomain is zero precisely when ξ is a MC element of $(\mathfrak{g}, d, \omega)$. Whence, the resulting curved Lie algebra will have zero curvature if, and only if, the element ξ belongs to the set of MC elements of $(\mathfrak{g}, d, \omega)$. In fact, these morphisms actually correspond to twisting by the element ξ, and such a twisting is denoted $\mathfrak{g}^{\xi}$. Twists of dgla are one way in which curved Lie algebras arise in mathematics. For more details see [6, 11], but note the notion of a curved morphism is not used in either citation.

Definition 1.13. *The category whose objects are pseudo-compact curved Lie algebras and morphisms are given by the continuous (with respect to the topology induced in taking the inverse limit) curved morphisms between them will be referred to as the category of pseudo-compact curved Lie algebras and will be denoted by $\hat{\mathscr{L}}$.*

In [3] it is shown that the functor of linear duality establishes an antiequivalence between the categories of pseudo-compact Lie algebras and conilpotent Lie coalgebras, where pseudo-compact Lie algebras were referred to as pronilpotent Lie algebras. This term, however, is not ideal, cf. [24, Remark 7.2]. Despite this, pseudo-compact curved Lie algebras are pronilpotent in the classical sense, i.e. they are isomorphic to an inverse limit of nilpotent curved Lie algebras.

Proposition 1.14. *For $\mathfrak{g} \in \hat{\mathscr{L}}$, let $\mathfrak{g} = F_1\mathfrak{g} \supseteq F_2\mathfrak{g} \supseteq \dots$ denote the lower central series. Then $\mathfrak{g} = \varprojlim_i \frac{\mathfrak{g}}{F_i\mathfrak{g}}$.*

Proof. For $\mathfrak{g} \in \hat{\mathscr{L}}$ one has, by definition, that $\mathfrak{g} = \varprojlim_n \mathfrak{g}_n$, where each $\mathfrak{g}_n$ is a finite dimensional curved Lie algebra. Now, the filtered limit of finite dimensional vector spaces is exact (since any filtered system of finite dimensional vector spaces satisfies the Mittag-Leffler condition see [32, Section 3.5]) and so by appropriate usage of exact sequences one arrives at $F_i(\mathfrak{g}_n) \cong \varprojlim_n F_i(\mathfrak{g}_n)$, for $i \geq 1$, and $\mathfrak{g}/F_i\mathfrak{g} \cong \varprojlim_n \mathfrak{g}_n/F_i(\mathfrak{g}^n)$. By definition, the finite dimensional curved Lie algebra $\mathfrak{g}_n$ is nilpotent, and so $\varprojlim_i \mathfrak{g}_n/F_i(\mathfrak{g}) \cong \mathfrak{g}_n$. Whence,

$$\varprojlim_i \mathfrak{g}/F_i\mathfrak{g} \cong \varprojlim_i \varprojlim_n \mathfrak{g}_n/F_i(\mathfrak{g}^n) \cong \varprojlim_n \varprojlim_i \mathfrak{g}_n/F_i(\mathfrak{g}^n) \cong \varprojlim_n \mathfrak{g}_n = \mathfrak{g}. \qquad \square$$

1.2 Filtrations

This section briefly discusses filtrated curved Lie algebras and some basic properties of these filtrations that will be insisted upon. Filtrations for cdgas are also (even more briefly) discussed. For more details regarding filtrations consult [32, Section 5.4]. It should be noted that a more general notion of filtration is treated in loc. cit.

1.2.1 Filtrations of curved Lie algebras

Filtrations play an important role in the model structure for curved Lie algebras (see Section 1.4), and the key details are briefly discussed here.

Definition 1.15. *A curved Lie algebra, $(\mathfrak{g}, d, \omega)$, is said to be filtered when equipped with a (descending) filtration denoted by $\{F_i\mathfrak{g}\}_{i\in\mathbb{N}}$ corresponding to a tower*

$$\mathfrak{g} = F_1\mathfrak{g} \supseteq F_2\mathfrak{g} \supseteq F_3\mathfrak{g} \supseteq \dots$$

of subspaces $F_i\mathfrak{g}$ for all $i \in \mathbb{N}$ such that the filtration respects the bracket and differential, i.e.

$$[F_i\mathfrak{g}, F_j\mathfrak{g}] \subseteq F_{i+j}\mathfrak{g} \text{ and } d(F_i\mathfrak{g}) \subseteq F_i\mathfrak{g}.$$

Notice here that only positively indexed filtrations of curved Lie algebras are considered. This is an important point as even the seemingly harmless step of allowing non-negatively indexed filtrations will drastically change the meaning of a filtered quasi-isomorphism (see Definition 1.27).

Definition 1.16. *Given a descending filtration $\{F_i\mathfrak{g}\}_{i\in\mathbb{N}}$ of a curved Lie algebra $\mathfrak{g}$, the associated graded algebra, denoted $\mathrm{gr}_F\,\mathfrak{g}$, is the algebra given by the sum*

$$\bigoplus_{i\in\mathbb{N}} \frac{F_i\mathfrak{g}}{F_{i+1}\mathfrak{g}}.$$

Proposition 1.17. *The associated graded algebra of a filtered curved Lie algebra inherits the bracket and differential.*

Proof. It is a straightforward consequence of the fact that the filtration respects the bracket and differential. □

Definition 1.18. *Given a filtered pseudo-compact curved Lie algebra, $\mathfrak{g}$, its filtration is said to be complete if*

$$\mathfrak{g} = \varprojlim_i \frac{\mathfrak{g}}{F_i\mathfrak{g}}.$$

Remark 1.19. The completeness condition is precisely that of $\mathfrak{g}$ being pronilpotent, i.e. an inverse limit of nilpotent curved Lie algebras.

Definition 1.20. *A filtration is said to be Hausdorff if*

$$\bigcap_{i\geq 1} F_iA = 0.$$

Definition 1.21. *Let $\mathfrak{g}$ be a pseudo-compact curved Lie algebra. The filtration induced by the lower central series of $\mathfrak{g}$ is the filtration inductively given by*

$$F_1\mathfrak{g} = \mathfrak{g}, \quad F_2\mathfrak{g} = [F_1\mathfrak{g}, \mathfrak{g}], \quad F_3\mathfrak{g} = [F_2\mathfrak{g}, \mathfrak{g}], \quad \ldots, \quad F_{i+1}\mathfrak{g} = [F_i\mathfrak{g}, \mathfrak{g}], \quad \ldots$$

Proposition 1.22. *The filtration given by the lower central series of a curved Lie algebra (above) respects the bracket and differential. Moreover, the filtration is complete and Hausdorff.*

Proof. The fact that the filtration respects the differential and bracket are quick checks. The final statement is a direct consequence of Proposition 1.14. □

Definition 1.23. *Complete (and thus Hausdorff) filtrations that respect the bracket and differential subject to the additional condition that $d^2 = 0$ in the associated graded objects (i.e. the associated graded algebras are complexes) are said to be admissible.*

Proposition 1.24. *The associated graded curved Lie algebra of a filtered curved Lie algebra, where the filtration is given by the lower central series, will be a true complex, i.e. $d^2 = 0$ on the associated graded.*

Proof. To see that $d^2 = 0$ on the associated graded notice that the adjoint action of the curvature will increase the filtration degree whereas the action of d fixes the filtration degree. □

Combining Proposition 1.22 and Proposition 1.24, one immediately sees the following.

Proposition 1.25. *The filtration given by the lower central series of a curved Lie algebra is an admissible filtration.* □

Definition 1.26. *A morphism of filtered curved Lie algebras is a morphism of curved Lie algebras that is compatible with the filtrations in the sense that $(f, \alpha)F_i\mathfrak{g} \subseteq F_i\mathfrak{h}$.*

As one would expect, a morphism of filtered curved Lie algebras induces a well-defined morphism of the associated graded objects.

The objects of $\hat{\mathscr{L}}$ do not form complexes (as it is not necessary that $d^2 = 0$) and as such there is no natural definition of a quasi-isomorphism. It is, however, possible to define the notion of a filtered quasi-isomorphism. Filtered quasi-isomorphisms are of particular importance when defining the weak equivalences of the model structure in Section 1.4.3.

Definition 1.27. *Let $\mathfrak{g}, \mathfrak{h} \in \hat{\mathscr{L}}$ be endowed with admissible filtrations, both denoted F. A filtered morphism $(f, \alpha)\colon \mathfrak{g} \to \mathfrak{h}$ in $\hat{\mathscr{L}}$ is said to be a filtered quasi-isomorphism if the induced morphism $\mathrm{gr}_F(f, \alpha)\colon \mathrm{gr}_F\, \mathfrak{g} \to \mathrm{gr}_F\, \mathfrak{h}$ is a quasi-isomorphism of dgla, i.e. $gr_F(f, \alpha)$ induces an isomorphism on the level of homology.*

Remark 1.28. Notice that it is important the filtrations be positively indexed, because even the seemingly innocent adaptation to allow non-negatively indexed filtrations could allow for the case when the filtrations are concentrated solely in degree 0 and in this case a filtered quasi-isomorphism would collapse to just a quasi-isomorphism. It is necessary that the notion of a weak equivalence in the model category of pseudo-compact curved Lie algebras be finer than a quasi-isomorphism, and so only positively indexed filtrations are considered.

It should be remarked here that no claim is made about the closure of filtered quasi-isomorphisms under composition.

1.2.2 Filtrations of cdgas

The above constructions for curved Lie algebras have—where applicable—a counterpart for cdgas whose definitions follow easily from those given above. Three key differences between the filtrations of cdgas used within this paper and the filtrations curved Lie algebras are that filtrations of cdgas are ascending, have no restriction on the indexing, and are cocomplete. No other assumptions are made about the filtrations on cdgas. For example, a filtration for a cdga need not be complete. An ascending filtration of a cdga A

$$F_1 A \subseteq F_2 A \subseteq F_3 A \subseteq \dots$$

is called cocomplete (or exhaustive) if $A = \varinjlim F_i A$. These differences reflect the contravariant duality between cdgas and curved Lie algebras. It will only be necessary to consider filtrations of cdgas in a couple of proofs, and in particular the proof of Lemma 1.61. This is quite different to the fundamental role admissible filtrations play in defining the weak equivalences of pseudo-compact curved Lie algebras.

1.3 Analogues of the Chevalley-Eilenberg and Harrison complexes

The category of unital cdgas with the standard cdga morphisms will be denoted by $\mathscr{A}$. Within this section a pair of contravariant functors will be defined (extending those of [24] and influenced by work of [26]). These functors will be shown to be adjoint, providing the base for the Quillen equivalence proven in Theorem 1.65.

In the following, a contravariant functor $\mathcal{L}\colon \mathscr{A} \to \hat{\mathscr{L}}$ will be constructed. Several ideas of Positselski [26] feature heavily in this construction. First, note that given a unital cdga, $A \in \mathscr{A}$, the underlying field is a subspace as $k = k \cdot 1 \subseteq A$, and therefore it is possible to take a linear retraction $\epsilon\colon A \to k$ (which may not commute with the differential or multiplication). Setting A_+ to be the kernel of this map, as vector spaces it is evident that $A = A_+ \oplus k$. Note, if A is augmented then ϵ can be chosen to be an augmentation (i.e. ϵ is a morphism of dg algebras) and A_+ is the augmentation ideal, i.e. the decomposition holds on the level of cdga. Note that because ϵ is not necessarily a morphism of dg algebras, the differential $d\colon A_+ \to A$ and the multiplication $m\colon A_+ \otimes A_+ \to A$ can be split as $d = (d_+, d_k)$ and $m = (m_+, m_k)$. Where

$$d_+\colon A_+ \to A_+; \qquad\qquad m_+\colon A_+ \otimes A_+ \to A_+;$$
$$d_k\colon A_+ \to k; \text{ and} \qquad\qquad m_k\colon A_+ \otimes A_+ \to k.$$

Definition 1.29. *Given a unital cdga $A \in \mathscr{A}$ and a linear retraction $\epsilon\colon A \to k$ with kernel A_+, let $\mathcal{L}(A)$ denote the pseudo-compact curved Lie algebra given by $(\hat{L}\Sigma A_+^*, d_+^* + m_+^*, d_k^* + m_k^*)$. Here the derivations d_+^* and m_+^* are the extensions of the duals to the whole curved Lie algebra (given by the Leibniz rule); the notation used here (somewhat abusively) is the same for both.*

Remark 1.30. The pseudo-compact curved Lie algebra $\mathcal{L}(A)$ has zero curvature if the linear retraction ϵ is a morphism of unital cdga, i.e. if A is augmented with augmentation ϵ. This is because the k parts of the morphisms vanish.

One could consider the category of unital cdgas, $\mathscr{A}$, as the category of pairs (A, ϵ), where A is a unital cdga and $\epsilon\colon A \to k$ is a retraction. This is not necessary as the following result shows.

Proposition 1.31. *Given a unital cdga A, the pseudo-compact curved Lie algebra $(\hat{L}\Sigma A_+^*, d_+^* + m_+^*, d_k^* + m_k^*)$ depends on the choice of linear retraction $\epsilon\colon A \to k$ up to isomorphism.*

Proof. A different choice of retraction is given by $\epsilon'(b) = \epsilon(b) + x(b)$, where $x \in A_+^*$ has degree 0. This leads to the isomorphism of pseudo-compact curved Lie algebras $(\mathrm{id}, x) : \hat{L}(\Sigma A_+^*) \to \hat{L}(\Sigma A_+^*)$ (or a twisting), since x will have degree minus one in $\hat{L}(\Sigma A_+^*)$. □

Let $A, B \in \mathscr{A}$, and A_+ and B_+ be the kernels of a pair of linear retractions on A and B respectively, then given a morphism $f\colon A \to B$ of $\mathscr{A}$ the linear morphism $f\colon A_+ \to B$ can be split as $f = (f_+, f_k)$ where

$$f_+\colon A_+ \to B_+, \text{ and}$$
$$f_k\colon A_+ \to k.$$

Taking the duals of these morphisms, one obtains the linear morphisms:

$$f_+^*\colon B_+^* \to A_+^*, \text{ and}$$

$$f_k^*\colon k \to A_+^*.$$

Clearly f_+^* can be extended to a graded Lie algebra morphism $f_+^*\colon \mathcal{L}(B) \to \mathcal{L}(A)$, denoted the same by an abuse of notation. Additionally, notice that it is possible to consider f_k^* as a degree -1 element of $\mathcal{L}(A)$. With these observations in mind the following proposition is made.

Proposition 1.32. *Given a morphism, $f\colon A \to B$ of $\mathscr{A}$, the morphism*

$$(f_+^*, f_k^*)\colon \mathcal{L}(B) \to \mathcal{L}(A))$$

constructed above is a well defined curved Lie algebra morphism.

Proof. The proof amounts to chasing the definitions. □

Remark 1.33. As with objects, the morphism $\mathcal{L}(f)$ depends on the retractions chosen for the cdgas A and B. If either retraction is changed, however, then the obtained morphism $\mathcal{L}(f)$ is changed by a pre- and/or post-composition with an isomorphism.

Definition 1.34. *Let $\mathcal{L}\colon \mathscr{A} \to \hat{\mathscr{L}}$ be the contravariant functor that sends a cdga A to $\mathcal{L}(A)$ as in Definition 1.29 and sends a morphism f to (f_+^*, f_k^*) as in Proposition 1.32.*

The functor $\mathcal{L}$ will be shown to provide one half of an adjoint pair and so to complete the pair of functors it is necessary to describe a contravariant functor going in the reverse direction.

Definition 1.35. *The contravariant functor $\mathcal{C}\colon \hat{\mathscr{L}} \to \mathscr{A}$ is given by the following. The underlying graded space is given by the symmetric algebra of the suspension of the continuous dual of $\mathfrak{g}$, i.e. $S\Sigma\mathfrak{g}^*$. This becomes a unital cdga with the concatenation product and the differential made of three parts coming from the duals of the curvature, the differential and the bracket of $\mathfrak{g}$ made into derivations and extended via the Leibniz rule; cf. [12].*

Given a morphism $(f,\alpha)\colon \mathfrak{g} \to \mathfrak{h}$ in $\hat{\mathscr{L}}$ associate to it the morphism

$$\mathcal{C}(f,\alpha)\colon \mathcal{C}(\mathfrak{h}) \to \mathcal{C}(\mathfrak{g})$$

given by $\Sigma(f^* \oplus \alpha^*)\colon \Sigma\mathfrak{h}^* \to \Sigma\mathfrak{g}^*$ extended as a morphism of cdgas.

Remark 1.36. The functor $\mathcal{C}$ is an analogue of the Chevalley-Eilenberg construction in homological algebra.

Notice that in Definition 1.35, the uncompleted symmetric algebra is taken and not the completed one. This is because $\mathfrak{g}$ is already complete in some sense.

If $(\mathfrak{g}, d, 0)$ has zero curvature then the cdga obtained by applying the contravariant functor $\mathcal{C}$ is augmented, cf. [24] since it is then precisely the same construction. It can, therefore, be understood that the curvature acts as an obstruction for the cdga $\mathcal{C}(\mathfrak{g})$ to be augmented, since the natural choice for augmentation fails to be a dg algebra morphism. More precisely, the part of the differential coming from the curvature maps into k and not $\mathfrak{g}$. Another reason to see why an augmentation fails to arise in the curved setting is that a MC element for a dgla corresponds to an augmentation of $S\Sigma\mathfrak{g}^*$. In the uncurved case the zero element is always a MC element and there is always an augmentation. In the case of a curved Lie algebra, however, there need not be any solutions to the MC equation.

Proposition 1.37. *The contravariant functor $\mathcal{C}\colon \hat{\mathscr{L}} \to \mathscr{A}$ is right adjoint to the contravariant functor $\mathcal{L}\colon \mathscr{A} \to \hat{\mathscr{L}}$.*

Proof. In order to prove the proposition it is sufficient to exhibit the following isomorphism:

$$\mathrm{Hom}_{\mathscr{A}}(\mathcal{C}(\mathfrak{g}), A) \cong \mathrm{Hom}_{\hat{\mathscr{L}}}(\mathcal{L}(A), \mathfrak{g}),$$

for any curved Lie algebra $\mathfrak{g}$ and any unital cdga A. To this end, assume

$$f\colon S\Sigma\mathfrak{g}^* \to A$$

is a morphism of unital cdgas. This morphism is uniquely determined by the linear morphism

$$f\colon \Sigma\mathfrak{g}^* \to A,$$

which in turn defines f_+ and f_k since $A = A_+ \oplus k$. Dualising, the linear morphisms

$$\begin{aligned} &f_+^*\colon \Sigma A_+^* \to \mathfrak{g}, \text{ and} \\ &f_k^*\colon \Sigma k \to \mathfrak{g} \end{aligned}$$

are obtained.

By extending $f_+^*\colon \Sigma A_+ \to \mathfrak{g}$ as a graded Lie algebra morphism to $\hat{L}\Sigma A_+$ and combining it with f_k^* the curved Lie algebra morphism $(f_+^*, f_k^*)\colon \mathcal{L}(A) \to \mathfrak{g}$ is obtained. It is straightforward, although slightly tiresome, to do the calculations. Hence, one side of the adjunction is proven. Now, assume that

$$(f, \alpha)\colon \mathcal{L}(A) \to \mathfrak{g}$$

is a curved Lie algebra morphism. The graded Lie algebra morphism, f, is uniquely determined by the underlying linear morphism

$$f\colon \Sigma A_+^* \to \mathfrak{g}$$

which induces $f_+\colon \Sigma\mathfrak{g}^* \to A_+$: this gives one component. The second comes from first considering α as the morphism $\alpha\colon \Sigma^{-1}k \to \mathfrak{g}$. Then by dualising and taking the suspension one obtains $f_k = \Sigma\alpha^*\colon \Sigma\mathfrak{g} \to k$. This construction yields a linear morphism $(f_+^*, f_k^*)\colon \Sigma\mathfrak{g} \to A$ which can be extended to morphism of commutative algebras

$$\mathcal{C}(\mathfrak{g}) \to A.$$

The final morphism also commutes with the differentials, which is a quick check. Hence the other side of the adjunction has been proven. $\square$

1.4 Model category of curved Lie algebras

Here it will be demonstrated that the category $\hat{\mathscr{L}}$ can be endowed with the structure of a model category with weak equivalences given by filtered quasi-isomorphisms. In addition, this model structure is Quillen equivalent to the model structure for unital cdga given in [19]. This equivalence will be shown using similar methods to [24]. More precisely, the proof of the equivalence will employ the Quillen equivalence that exists upon associative dg local algebras and pseudo-compact curved associative algebras (see [26]), as well as the primitive elements and universal enveloping algebra functors. First, though, it must be proven that $\hat{\mathscr{L}}$ possesses all small limits and colimits.

1.4.1 Limits and colimits

There does not exist an initial object in the category of pseudo-compact curved Lie algebras with curved morphisms. The closest object to an initial object is the curved Lie algebra freely generated by a single element (the curvature) of degree -2 with zero differential, i.e. $(\hat{L}\langle\omega\rangle, 0, \omega)$. There clearly will be a morphism from this object to every other object. Such a morphism, however, is not necessarily unique. Nevertheless, in the category of curved Lie algebras with strict morphisms this object is the initial object. Thus, it is necessary to formally add an initial object to the category $\hat{\mathscr{L}}$. From here on let $\hat{\mathscr{L}}_*$ denote the category of pseudo-compact curved Lie algebras and curved morphisms with a formal initial object added. The category $\hat{\mathscr{L}}_*$ does possess a terminal object, namely the zero curved Lie algebra $(0, 0, 0)$.

Proposition 1.38. *Here the product over a finite set will be described; the general case follows in a straightforward fashion from this description. The product over the set $I = \{i_1, i_2, \dots i_n\}$ indexing pseudo-compact curved Lie algebras, $(\mathfrak{g}_{i_j}, d_{i_j}, \omega_{i_j})$ for $1 \leq j \leq n$, is denoted by $\prod_{i\in I} \mathfrak{g}_i$ and given by the Cartesian product of underlying sets with bracket given by*

$$[(x_{i_1}, \dots, x_{i_n}), (x'_{i_1}, \dots, x'_{i_n})] = ([x_{i_1}, x'_{i_1}]_{i_1}, \dots, [x_{i_n}, x'_{i_n}]_{i_n}),$$

where $[,]_{i_j}$ is the bracket of $\mathfrak{g}_{i_j}$, differential given by

$$d(x_{i_1}, \dots, x_{i_n}) = (d_{i_1}x_{i_1}, \dots, d_{i_n}x_{i_n}),$$

and curvature given by

$$(\omega_{i_1}, \dots, \omega_{i_n}).$$

The projection morphisms are the obvious ones onto each factor, i.e. $\pi_{i_j} \colon \prod_{i\in I} \mathfrak{g}_i \to \mathfrak{g}_{i_j}$.

Proof. A straightforward check. $\square$

Proposition 1.39. *The equaliser of two curved morphisms $(f, \alpha), (g, \beta) \colon \mathfrak{g} \to \mathfrak{h}$ is given by the initial object if $\alpha \neq \beta$ and by $\{x \in \mathfrak{g} \colon f(x) = g(x)\}$ if $\alpha = \beta$.*

Proof. If $\alpha \neq \beta$ then 0 is not in the equaliser, because $f(0)-\alpha \neq g(0)-\beta$. Therefore, the initial object is the only object satisfying the conditions of the equaliser.

If $\alpha = \beta$, then it is a straightforward exercise to show that the space $\{x \in \mathfrak{g} \colon f(x) = g(x)\}$ respects the differential and bracket inherited from $\mathfrak{g}$. □

Proposition 1.40. *The coproduct in the category of pseudo-compact curved Lie algebras is easiest to describe in the binary case: given two pseudo-compact curved Lie algebras,* $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$ *and* $(\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$*, the coproduct* $\mathfrak{g} \coprod \mathfrak{h}$ *has underlying graded Lie algebra* $\mathfrak{g} * \mathfrak{h} * \hat{L}\langle z \rangle$*, where* $*$ *is the free product and* z *is a formal element of degree minus one. The differential is given by the rules:* $d|_{\mathfrak{g}} = d_{\mathfrak{g}}$*,* $d|_{\mathfrak{h}} = d_{\mathfrak{h}} - ad_z$ *and* $dz = \omega_{\mathfrak{g}} - \omega_{\mathfrak{h}} + \frac{1}{2}[x,x]$*. The resulting space has curvature equal to that of* $\mathfrak{g}$*. The two inclusion morphisms are given by*

$$(id_{\mathfrak{g}}, 0) \colon \mathfrak{g} \hookrightarrow \mathfrak{g} \coprod \mathfrak{h}$$

and

$$(id_{\mathfrak{h}}, z) \colon \mathfrak{h} \hookrightarrow \mathfrak{g} \coprod \mathfrak{h}.$$

Proof. It is straightforward to see that $\mathfrak{g} \coprod \mathfrak{h}$ is a well defined pseudo-compact curved Lie algebra. To demonstrate the required universal property, take a pseudo-compact curved Lie algebra $\mathfrak{x}$ and morphisms

$$(f_{\mathfrak{g}}, \alpha) \colon \mathfrak{g} \to \mathfrak{x}$$

and

$$(f_{\mathfrak{h}}, \beta) \colon \mathfrak{h} \to \mathfrak{x}.$$

Define a morphism $(f, \alpha) : \mathfrak{g} \coprod \mathfrak{h} \to \mathfrak{x}$ by $f|_{\mathfrak{g}} = f_{\mathfrak{g}}$, $f|_{\mathfrak{h}} = f_{\mathfrak{h}}$, and $f(z) = \beta - \alpha$. Again it is straightforward to show that (f, α) is a well defined morphism and the diagram

$$\begin{tikzcd}
& \mathfrak{x} & \\
\mathfrak{g} \arrow[ur, "(f_{\mathfrak{g}}, \alpha)"] \arrow[r] & \mathfrak{g} \coprod \mathfrak{h} \arrow[u, "(f, \alpha)"'] & \mathfrak{h} \arrow[ul, "(f_{\mathfrak{h}}, \beta)"'] \arrow[l]
\end{tikzcd}$$

commutes. Uniqueness of this construction is a quick check. □

Remark 1.41. Fix $\mathfrak{h} \in \hat{\mathscr{L}}_*$. Given $\mathfrak{g} \in \hat{\mathscr{L}}_*$, the functor assigning $\mathfrak{g} \mapsto \mathfrak{g} \coprod \mathfrak{h}$ is not exact. This is easily seen as the terminal object is not preserved.

Remark 1.42. Since $\coprod$ is a coproduct in the category $\hat{\mathscr{L}}_*$ there exists a curved isomorphism

$$\mathfrak{g} \coprod \mathfrak{h} \cong \mathfrak{h} \coprod \mathfrak{g}.$$

The isomorphism is strictly curved as the two resulting curved Lie algebras are related by a twist. Explicitly the morphism is given by (f, z), where f is the identity on $\mathfrak{g}$ and $\mathfrak{h}$, and maps z to $-z$. The inverse morphism has the identical action.

The (categorical) coproduct of Proposition 1.40 is similar to the (non-categorical) disjoint product of [24]. Informally, the coproduct can be thought of as taking the disjoint union of the two spaces, formally adding a MC base point (i.e. a solution the Maurer-Cartan equation) and then twisting the copy of $\mathfrak{h}$ with this base point to flatten its curvature.

Proposition 1.43. *The coequaliser of two curved morphisms $(f, \alpha), (g, \beta) \colon \mathfrak{g} \to \mathfrak{h}$ is the quotient of $\mathfrak{h}$ by the ideal generated by $f(x) - g(x)$ and $\alpha - \beta$, for all $x \in \mathfrak{g}$.*

Proof. A painless check. □

Proposition 1.44. *The category $\hat{\mathscr{L}}_*$ has all small limits and colimits.*

Proof. The category $\hat{\mathscr{L}}_*$ has an initial object, a terminal object, all products, all equalisers, all coproducts, and all coequalisers, therefore it has all small limits and small colimits, cf. [25, Chapter V]. □

1.4.2 Duality for associative dg algebras

It is now necessary to recall the definitions of the cobar constructions in the associative case. These constructions can be found, for example, in [26], where pseudo-compact local associative algebras were studied in the dual setting as conilpotent coassociative coalgebras.

Definition 1.45. *A curved associative algebra is a graded algebra with an odd derivation, d, called the differential and an element of degree -2, ω, called the curvature, such that:*

- $d^2 = \mathrm{ad}_\omega$, *and*
- $d(\omega) = 0$.

Definition 1.46. *The category of associative dg algebras with dg algebra morphisms will be denoted $\mathcal{A}ss$.*

A curved associative algebra is said to be pseudo-compact if is isomorphic to an inverse limit of finite dimensional nilpotent curved associative algebras. The category of pseudo-compact local associative curved algebras with continuous curved associative algebra morphisms will be denoted $\widehat{\mathcal{CA}ss}$.

Note that, just like for $\hat{\mathscr{L}}_*$, one has to formally add an initial object to $\widehat{\mathcal{CA}ss}$ for it to possess all limits. The category $\widehat{\mathcal{CA}ss}$ with an initial object formally added will be denoted $\widehat{\mathcal{CA}ss}_*$.

The following is a result of Hinich and Jardine, cf. [19, 21].

Theorem 1.47. *The category $\mathcal{A}ss$ possesses a model structure with the class of weak equivalences given by quasi-isomorphisms.* □

The following is a result of Positselski, cf. [26, Section 9].

Theorem 1.48. *The category $\widehat{\mathcal{CA}ss_*}$ possesses a model structure, where the weak equivalences are given by the minimal class of morphisms containing all of the filtered quasi-isomorphisms and satisfying the two out of three property.* $\square$

Definition 1.49. *Let $\hat{\mathcal{B}}\colon \mathcal{A}ss \to \widehat{\mathcal{CA}ss_*}$ be the contravariant functor assigning to an associative dg algebra the pseudo-compact associative curved algebra $\hat{\mathcal{B}}(A)$ whose underlying graded algebra is $\hat{\mathrm{T}}\Sigma A_+^*$, where A_+ is the kernel of a linear retraction $A \to k$. The differential is induced from the multiplication and differential in the same way as Definition 1.29.*

Just as with the functor $\mathcal{L}$ given in Section 1.3, the resulting pseudo-compact associative curved algebra under the functor $\hat{\mathcal{B}}$ has zero curvature if, and only if, the linear retraction is a true augmentation, i.e. a dg algebra morphism.

Definition 1.50. *Let $\mathcal{B}\colon \widehat{\mathcal{CA}ss_*} \to \mathcal{A}ss$ be the contravariant functor assigning to a pseudo-compact associative curved algebra the associative dg algebra $\mathcal{B}(A)$ whose underlying graded algebra is $\mathrm{T}\,\Sigma A_+^*$, where A_+ is again the kernel of a linear retraction $A \to k$. The differential is induced in the same way as in Definition 1.34.*

Remark 1.51. It is important to notice that given an associative algebra, A, the pseudo-compact associative algebra $\hat{\mathcal{B}}(A)$ is in fact a Hopf algebra, since it is a tensor algebra. The space of algebra generators in $\hat{\mathcal{B}}(A)$ contains only primitive elements. Moreover, if the multiplication of the associative algebra A is commutative, then the differential of $\hat{\mathcal{B}}(A)$ maps the space of algebra generators to primitives. Thus, for a cdga A, $\hat{\mathcal{B}}(A)$ is a dg Hopf algebra.

Much like in [24], the reason for recalling the definitions of the associative case is that the contravariant functors $\mathcal{C}$ and $\mathcal{L}$ can be 'embedded' into the following adjunction proven by Positselski [26].

Theorem 1.52. *The contravariant functors $\mathcal{B}$ and $\hat{\mathcal{B}}$ are adjoint. Moreover, they induce a Quillen equivalence.*

Proof. Pseudo-compact local associative curved algebras are dual to conilpotent coassociative curved coalgebras and thus it follows from [26, Section 6]. $\square$

Let $U\colon \hat{\mathscr{L}}_* \to \widehat{\mathcal{CA}ss_*}$ be the universal enveloping algebra functor, given analogously to the classical construction. Let $\mathrm{Prim}\colon \widehat{\mathcal{CA}ss_*} \to \hat{\mathscr{L}}_*$ be the primitive elements functor, given analogously to the classical construction.

The contravariant functors $\mathcal{L}$ and $\mathcal{C}$ can be seen to fit into the following 'commutative diagram' of functors:

$$\begin{tikzcd}
\mathcal{A}ss \arrow[r, shift left, "\hat{\mathcal{B}}"] & \widehat{\mathcal{A}ss_*} \arrow[l, shift left, "\mathcal{B}"] \arrow[d, shift left, "\mathrm{Prim}"] \\
\mathscr{A} \arrow[u, hook, "\text{forgetful}"] \arrow[r, shift left, "\mathcal{L}"] & \hat{\mathscr{L}}_*, \arrow[l, shift left, "\mathcal{C}"] \arrow[u, shift left, "U"]
\end{tikzcd}$$

The following two propositions show the necessary 'commutativity' of this diagram for the purposes of this paper, and the proofs follow in a straightforward manner from [24, Proposition 9.5].

Proposition 1.53. *Given a unital cdga, A, there is a natural isomorphism of pseudo-compact curved Lie algebras* $\mathrm{Prim}(\hat{\mathcal{B}}(A)) \cong \mathcal{L}(A)$. □

Proposition 1.54. *Given a pseudo-compact curved Lie algebra, $\mathfrak{g}$, the differential graded algebras $\mathcal{B}(U(\mathfrak{g}))$ and $\mathcal{C}(\mathfrak{g})$ are quasi-isomorphic.* □

1.4.3 Model structure

The category $\hat{\mathscr{L}}_*$ will now be endowed with a model structure that will be shown to be Quillen equivalent (via the contravariant functors defined in Section 1.3) to the model category of unital cdgas given by Hinich [19]. For completeness, first recall the model structure given by Hinich.

Theorem 1.55 (Hinich)**.** *The category of unital cdgas is a closed model category where a morphism is*

- *a weak equivalence if, and only if, it is a quasi-isomorphism;*
- *a fibration if, and only if, it is surjective;*
- *a cofibration if, and only if, it has the LLP with respect to all acyclic fibrations.*

□

Definition 1.56. *A morphism $(f, \alpha)\colon \mathfrak{g} \to \mathfrak{h}$ in $\hat{\mathscr{L}}_*$ is called*

- *a weak equivalence if, and only if, it belongs to the minimal class of morphisms containing the filtered quasi-isomorphisms and satisfy the two out of three property;*
- *a fibration if, and only if, the underlying graded Lie algebra morphism, f, is a surjective morphism;*
- *a cofibration if, and only if, it has the LLP with respect to all acyclic fibrations.*

Provided with this definition, some preliminary results will be discussed before showing that $\hat{\mathscr{L}}_*$ is in fact a model category with this model structure. First it is helpful to look at some useful facts regarding filtered quasi-isomorphisms and the units of the adjunction given in Section 1.3.

Proposition 1.57. *Given a filtered quasi-isomorphism $(f, \alpha)\colon \mathfrak{g} \to \mathfrak{h}$, the induced morphism*

$$\mathcal{C}(f, \alpha)\colon \mathcal{C}(\mathfrak{h}) \to \mathcal{C}(\mathfrak{g})$$

is a quasi-isomorphism of $\mathscr{A}$. Conversely, given a quasi-isomorphism $g\colon A \to B$ of $\mathscr{A}$ the induced morphism

$$\mathcal{L}(g)\colon \mathcal{L}(B) \to \mathcal{L}(A)$$

is a filtered quasi-isomorphism.

Proof. In the first instance, there exist filtrations on $\mathfrak{g}$ and $\mathfrak{h}$ such that $\mathrm{gr}_F(f,\alpha)$ is a quasi-isomorphism. These filtrations induce increasing and cocomplete filtrations upon $\mathcal{C}(\mathfrak{g})$ and $\mathcal{C}(\mathfrak{h})$. Therefore, $\mathrm{gr}_F\, \mathcal{C}(f,\alpha)$ is a quasi-isomorphism and—since the homology of chain complexes commutes with filtered colimits—$\mathcal{C}(f,\alpha)$ is a quasi-isomorphism.

Now for the converse statement. After applying the contravariant functor $\mathcal{L}$, take the filtrations induced by the lower central series. Since the brackets are built freely they preserve quasi-isomorphisms, whence $\mathcal{L}(g)$ is a filtered quasi-isomorphism. $\square$

Proposition 1.58.

- *Given a unital cdga, A, the morphism $i_A\colon \mathcal{CL}(A) \to A$ is a quasi-isomorphism.*
- *Given a pseudo-compact curved Lie algebra, $\mathfrak{g}$, the morphism $i_{\mathfrak{g}}\colon \mathcal{LC}(\mathfrak{g}) \to \mathfrak{g}$ is a filtered quasi-isomorphism.*

Proof. By Proposition 1.54 $\mathcal{C}(\mathcal{L}(A)) \simeq \mathcal{B}U(\mathcal{L}(A))$. Now, $\mathcal{B}U\mathcal{L}(A) = \mathcal{B}\hat{\mathcal{B}}(A)$ which is quasi-isomorphic to A by [26, Section 9], considering A as an associative dg algebra. Thus the first statement is proven.

To prove the second statement, consider the natural filtration by the lower central series on $\mathfrak{g}$ and the filtration it induces upon $\mathcal{LC}(\mathfrak{g})$; denote these two filtrations by F. Therefore, it is sufficient to the show that the morphism $\mathcal{LC}(\mathrm{gr}_F\,\mathfrak{g}) \to \mathrm{gr}_F\,\mathfrak{g}$ is a quasi-isomorphism of dgla—this follows from [24, Proposition 9.10]. $\square$

Lemma 1.59. *The contravariant functor $\mathcal{L}$ sends fibrations to cofibrations and cofibrations to fibrations.*

Proof. Given a fibration $f\colon A \to B$ of unital cdgas, to show that $\mathcal{L}(f)$ is a cofibration it is necessary to find a lift in each diagram of the form

$$\begin{tikzcd} \mathcal{L}(B) \arrow[r] \arrow[d, "\mathcal{L}(f)"'] & \mathfrak{g} \arrow[d, "(\phi,\alpha)"] \\ \mathcal{L}(A) \arrow[r] \arrow[ur, dashed] & \mathfrak{h}, \end{tikzcd}$$

where the morphism $(\phi,\alpha)\colon \mathfrak{g} \to \mathfrak{h}$ is an acyclic fibration of pseudo-compact curved Lie algebras. This is equivalent to seeking a lift in each diagram of the following form

$$\begin{tikzcd} \mathcal{C}(\mathfrak{h}) \arrow[r] \arrow[d, "\mathcal{C}(\phi,\alpha)"'] & A \arrow[d, "f"] \\ \mathcal{C}(\mathfrak{g}) \arrow[r] \arrow[ur, dashed] & B. \end{tikzcd}$$

This follows from the adjunction of the contravariant functors $\mathcal{L}$ and $\mathcal{C}$, see Proposition 1.37. Now, by assumption, the morphism f is a fibration. Further, by Proposition 1.57, the morphism $\mathcal{C}(\phi,\alpha)$ is a weak equivalence. Therefore, it suffices show that the morphism $\mathcal{C}(\phi,\alpha)$ is a cofibration. Let $\{F_i\mathfrak{g}\}_{i\in\mathbb{N}}$ denote the lower central series and let the kernel of (ϕ,α) be denoted by $K \subseteq \mathfrak{g}$. Therefore, the tower

$$\mathfrak{h} \cong \mathfrak{g}/(F_1\mathfrak{g} \cap K) \xleftarrow{\pi_1} \mathfrak{g}/(F_2\mathfrak{g} \cap K) \xleftarrow{\pi_2} \mathfrak{g}/(F_3\mathfrak{g} \cap K) \xleftarrow{\pi_3} \cdots$$

is obtained. In Proposition 1.14 it was shown that $\mathfrak{g} = \varprojlim_i \mathfrak{g}/F_i\mathfrak{g}$ for any $\mathfrak{g} \in \hat{\mathscr{L}}_*$. The limit of the above tower is, therefore, simply $\mathfrak{g}$ and hence the colimit of

$$\mathcal{C}(\mathfrak{h}) \hookrightarrow \mathcal{C}(\mathfrak{g}/(F_1\mathfrak{g} \cap K)) \hookrightarrow \mathcal{C}(\mathfrak{g}/(F_2\mathfrak{g} \cap K)) \hookrightarrow \mathcal{C}(\mathfrak{g}/(F_3\mathfrak{g} \cap K)) \hookrightarrow \cdots$$

is $\mathcal{C}(\mathfrak{g})$. It is thus sufficient to prove that each of the morphisms $\mathcal{C}(\pi_n)$ for $n \in \mathbb{N}$ are cofibrations in $\mathscr{A}$. First note that for each $n \geq 1$, $\ker(\pi_n) = (F_n\mathfrak{g} \cap K)/(F_{n+1}\mathfrak{g} \cap K)$. From here it is straightforward to see that, just as in [20, Section 5.2.2], $\mathcal{C}(\pi_n)$ is a standard cofibration obtained by adding free generators to $\mathcal{C}(\mathfrak{g}/(F_n\mathfrak{g} \cap K))$.

To complete the proof of the statement, note the standard fact that cofibrations in unital cdga are monomorphisms and the contravariant functor $\mathcal{L}$ sends monomorphisms to epimorphisms, i.e. fibrations. $\square$

By [13, Theorem 9.7], once it has been shown that $\hat{\mathscr{L}}_*$ is a CMC with the model structure of Definition 1.56, Theorem 1.65 will have been proven. To this end, first it is noted that despite the functor $(-) \coprod \mathfrak{h}$, for some fixed $\mathfrak{h}$, not being exact (see Remark 1.41) it does have the following redeeming property.

Lemma 1.60. *Given a weak equivalence, $(f, \alpha)\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ of $\hat{\mathscr{L}}_*$, for a fixed $\mathfrak{h}$*

$$(f, \alpha) \coprod (\mathrm{id}_{\mathfrak{h}}, 0)\colon \mathfrak{g}_1 \coprod \mathfrak{h} \to \mathfrak{g}_2 \coprod \mathfrak{h}$$

is a weak equivalence too.

Proof. All curved Lie algebras are cofibrant, and coproducts preserve weak equivalences of cofibrant objects. Therefore, it is immediate. More explicitly, the morphism (f, α) is a weak equivalence and so there exist filtrations, F and G, on $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, such that the induced morphism on the associated graded algebras is a quasi-isomorphism. The filtrations F induces a filtration on the $\mathfrak{g}_1 \coprod \mathfrak{h}$ given by

$$\tilde{F}_i = \begin{cases} F_i\mathfrak{g}_1 \coprod \mathfrak{h} & \text{if } i = 1 \\ F_i\mathfrak{g}_1 \coprod 0 & \text{if } i > 1 \end{cases}.$$

Likewise, G induces a filtration, $\tilde{G}$, on $\mathfrak{g}_2 \coprod \mathfrak{h}$. The associated graded algebras of the coproducts are clearly quasi-isomorphic via the induced morphisms. $\square$

Now, enough auxiliary results have been developed to prove the remaining axioms of a model category, i.e. the lifting and the factorisation properties. The proofs of the lifting axioms rely on the next lemma which is proven with methods based upon that of [26] which in turn are based upon constructions originally performed in [20] in the proof of a similar lemma named op. cit. the 'Key Lemma'.

Lemma 1.61. *Let A be a unital cdga, $\mathfrak{g}$ be a pseudo-compact curved Lie algebra and $f\colon A \to \mathcal{C}(\mathfrak{g})$ be a surjective morphism. Consider the pushout*

$$\begin{tikzcd}
\mathcal{LC}(\mathfrak{g}) \arrow[r, "\mathcal{L}(f)"] \arrow[d, "i_{\mathfrak{g}}"'] & \mathcal{L}(A) \arrow[d, "j"] \\
\mathfrak{g} \arrow[r] & \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}.
\end{tikzcd}$$

Then the morphism $j\colon \mathcal{L}(A) \to \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}$ *is a weak equivalence.*

Proof. The morphism $j\colon \mathcal{L}(A) \to \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}$ is a weak equivalence if there exists filtrations on $\mathcal{L}(A)$ and $\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}$ such that j is a filtered quasi-isomorphism. To construct these filtrations, first filter $\mathfrak{g}$ by the natural filtration obtained by the lower central series. Denote this filtration, the filtration induced upon $\mathcal{C}(\mathfrak{g})$, and the filtration induced on A by the pre-images of the surjective morphism $A \to \mathcal{C}(\mathfrak{g})$ by F. Therefore, there exists a morphism $\mathcal{LC}(\operatorname{gr}_F \mathfrak{g}) \to \mathcal{L}(\operatorname{gr}_F A)$ which is a cofibrtation. So it is clear that $\operatorname{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}) = \mathcal{L}(\operatorname{gr}_F A) \coprod_{\mathcal{LC}(\operatorname{gr}_F \mathfrak{g})} \operatorname{gr}_F \mathfrak{g}$, i.e. take the pushout:

$$\begin{tikzcd}
\mathcal{LC}(\operatorname{gr}_F \mathfrak{g}) \arrow[r] \arrow[d] & \mathcal{L}(\operatorname{gr}_F A) \arrow[d] \\
\operatorname{gr}_F \mathfrak{g} \arrow[r] & \operatorname{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}).
\end{tikzcd}$$

Denote by n the positive grading induced by the indexing of the filtration F and introduce a filtration G on $\operatorname{gr}_F A$ by setting $G_0 \operatorname{gr}_F A = \operatorname{gr}_F A$ and $G_j \operatorname{gr}_F A$ the sum of the components of the ideal $\ker(\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g}))$ situated in the grading $n \geq j$. This would mean that the associated graded is equal to

$$\begin{aligned}
\operatorname{gr}_G \operatorname{gr}_F A = \frac{\operatorname{gr}_F A}{\oplus_{n\geq 1}(\ker(\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g})))_n} &\oplus \frac{\oplus_{n\geq 1}(\ker(\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g})))_n}{\oplus_{n\geq 2}(\ker(\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g})))_n} \\
&\oplus \frac{\oplus_{n\geq 2}(\ker(\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g})))_n}{\oplus_{n\geq 3}(\ker(\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g})))_n} \oplus \cdots \\
= \mathcal{C}(\operatorname{gr}_F \mathfrak{g}) \oplus (\ker(\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g})))_1 &\oplus (\ker(\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g})))_2 \oplus \cdots .
\end{aligned}$$

The filtration G is locally finite with respect to the grading n, but the differential may map $\operatorname{gr}_F A$ into the kernel of $\operatorname{gr}_F A \to \mathcal{C}(\operatorname{gr}_F \mathfrak{g})$, and so the goal of a filtered quasi-isomorphism has yet been achieved. Let G also denote the induced filtration upon $\mathcal{L}(\operatorname{gr}_F A)$ and hence $\mathcal{L}(\operatorname{gr}_F A) \coprod_{\mathcal{LC}(\operatorname{gr}_F \mathfrak{g})} \operatorname{gr}_F \mathfrak{g}$. Whence

$$\operatorname{gr}_G \operatorname{gr}_F \left(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \right) = \mathcal{L}(\operatorname{gr}_G \operatorname{gr}_F A) \coprod_{\mathcal{LC}(\operatorname{gr}_F \mathfrak{g})} \operatorname{gr}_F \mathfrak{g}.$$

Therefore, there are two gradings, n and j, upon $\operatorname{gr}_G \operatorname{gr}_F A$ and

$$\operatorname{gr}_G \operatorname{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}).$$

Thus, introduce a final filtration H by the rules $H_t \operatorname{gr}_G \operatorname{gr}_F A = \bigoplus_{n\geq 1, j\geq t} (\operatorname{gr}_G \operatorname{gr}_F A)_{n,j}$. This again induces a filtration upon $\mathcal{L}(\operatorname{gr}_F A) \coprod_{\mathcal{LC}(\operatorname{gr}_F \mathfrak{g})} \mathfrak{g}$. Whence

$$\operatorname{gr}_H \operatorname{gr}_G \operatorname{gr}_F \left(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \right) = \mathcal{L}(\operatorname{gr}_H \operatorname{gr}_G \operatorname{gr}_F A) \coprod_{\mathcal{LC}(\operatorname{gr}_F \mathfrak{g})} \operatorname{gr}_F \mathfrak{g}.$$

Now, $\mathcal{L}(\operatorname{gr}_H \operatorname{gr}_G \operatorname{gr}_F A) = \mathcal{LC}(\operatorname{gr}_F \mathfrak{g}) \coprod X$ where X is constructed by applying the contravariant functor $\mathcal{L}$ to the kernel components of $\operatorname{gr}_G \operatorname{gr}_F(A)$ and

$$\operatorname{gr}_H \operatorname{gr}_G \operatorname{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}) = \operatorname{gr}_F \mathfrak{g} \coprod X.$$

Further, since $\mathcal{LC}(\operatorname{gr}_F \mathfrak{g}) \to \operatorname{gr}_F \mathfrak{g}$ is a quasi-isomorphism, so is

$$\mathcal{L}(\operatorname{gr}_H \operatorname{gr}_G \operatorname{gr}_F A) \to \operatorname{gr}_H \operatorname{gr}_G \operatorname{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g})$$

and hence the statement has been proven. □

Lemma 1.62. *Given a morphism in the category of curved Lie algebras, $(f, \alpha)\colon \mathfrak{g} \to \mathfrak{h}$, it can be factorised as the composition of*

- *a cofibration followed by an acyclic fibration; and*
- *an acyclic cofibration followed by a fibration.*

Proof. To proceed, first consider the induced morphism $\mathcal{C}(f, \alpha)\colon \mathcal{C}(\mathfrak{h}) \to \mathcal{C}(\mathfrak{g})$. Since $\mathscr{A}$ is a model category it is possible factorise this morphism as

$$\mathcal{C}(\mathfrak{h}) \xrightarrow{j} A \xrightarrow{p} \mathcal{C}(\mathfrak{g}) ,$$

where j is a cofibration and p is a fibration in the category of unital cdga. Further, it is possible to choose either of the morphisms, j and p, to be a weak equivalence and doing so will specialise to one of the statements in the lemma. This factorisation in turn induces morphisms

$$\mathcal{LC}(\mathfrak{g}) \xrightarrow{\mathcal{L}(p)} \mathcal{L}(A) \xrightarrow{\mathcal{L}(j)} \mathcal{LC}(\mathfrak{h})$$

of curved Lie algebras. Therefore, taking the pushout

$$\begin{tikzcd} \mathcal{LC}(\mathfrak{g}) \arrow[r, "\mathcal{L}(p)"] \arrow[d, "i_{\mathfrak{g}}"'] & \mathcal{L}(A) \arrow[d, "(\tilde{\iota}, \gamma)"] \\ \mathfrak{g} \arrow[r, "(\iota, \beta)"'] & \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}, \end{tikzcd}$$

it can be seen that the morphism $(\iota, \beta)\colon \mathfrak{g} \to \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}$ is a cofibration, since it is obtained from a cobase change of the cofibration $\mathcal{L}(p)$. Further, there exists a morphism $(\rho, \epsilon)\colon \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \to \mathfrak{h}$ coming from the universal property of a pushout:

$$\begin{tikzcd}
\mathcal{LC}(\mathfrak{g}) \arrow[r, "\mathcal{L}(p)"] \arrow[d, "i_{\mathfrak{g}}"'] & \mathcal{L}(A) \arrow[d, "(\tilde{\iota}, \gamma)"] \arrow[ddr, bend left, "i_{\mathfrak{h}} \circ \mathcal{L}(j)"] & \\
\mathfrak{g} \arrow[r, "(\iota, \beta)"'] \arrow[drr, bend right, "(f, \alpha)"'] & \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \arrow[dr, dashed, "(\rho, \epsilon)"] & \\
 & & \mathfrak{h}.
\end{tikzcd}$$

Hence, (ρ, ϵ) is a fibration. Moreover, Lemma 1.61 shows that $(\tilde{\iota}, \gamma)$ is a weak equivalence.

The composition $(f, \alpha) = (\rho, \epsilon) \circ (\iota, \beta)$ provides the desired decompositions depending upon the choice of weak equivalence in the original decomposition.

For the first statement, it is necessary to show that (ρ, ϵ) is a weak equivalence. It is, therefore, sufficient to show that $i_{\mathfrak{h}} \circ \mathcal{L}(j)$ is a weak equivalence, since it then follows from the two of three property. To this end, choose j in the original factorisation to be a weak equivalence and the result follows.

For the second factorisation, choose p to be the weak equivalence in the original factorisation. This ensures that (ι, β) is a weak equivalence because the rest of the morphisms in the commutative square $\mathcal{L}(p)$, $i_{\mathfrak{g}}$, and $(\tilde{\iota}, \gamma)$ are weak equivalences. $\square$

Remark 1.63. It is possible that the factorisations proven to exist in Lemma 1.62 could be given more directly. One (standard) method would be to define analogues of the n-disk and n-sphere in the category $\hat{\mathscr{L}}_*$. The author notes he has not investigated this approach.

One lift is given by the definition of the classes of morphism in the model structure. Therefore, it is only necessary to prove that all cofibrations have the left lifting property with respect to all acyclic fibrations.

Lemma 1.64. *Given a commutative diagram of the form*

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[d, "(f, \alpha)"'] & \mathfrak{a} \arrow[d, "(\phi, \beta)"] \\
\mathfrak{h} \arrow[r] & \mathfrak{b},
\end{tikzcd}$$

where f is an acyclic cofibration and ϕ is a fibration, there exists a morphism (or lift) $\mathfrak{h} \to \mathfrak{a}$ such that the diagram still commutes, i.e. acyclic cofibrations have the LLP with respect to all fibrations.

Proof. First, using the proof of Lemma 1.62 (and borrowing notation from the proof) one has $(f, \alpha) = (\rho, \epsilon) \circ (\iota, \beta)$, where $(\rho, \epsilon)\colon \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \to \mathfrak{h}$ is an acyclic fibration and $(\iota, \beta)\colon \mathfrak{g} \to \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}$ is an acyclic cofibration. Note that both (ρ, ϵ) and (ι, β) are acyclic by the 2 of 3 property.

Since (f, α) is a cofibration it has, by definition, the LLP with respect to all acyclic fibrations. In particular, the following commutative diagram exists:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "{(\iota,\beta)}"] \arrow[d, "{(f,\alpha)}"'] & \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \to \mathfrak{h} \arrow[d, "{(\rho,\epsilon)}"] \\
\mathfrak{h} \arrow[ur, "{(h,\zeta)}"] \arrow[r, equal] & \mathfrak{h},
\end{tikzcd}$$

which implies that (f, α) is a retract of (ι, β), as the following diagram shows:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, equal] \arrow[d, "{(f,\alpha)}"'] & \mathfrak{g} \arrow[r, equal] \arrow[d, "{(\iota,\beta)}"] & \mathfrak{g} \arrow[d, "{(f,\alpha)}"] \\
\mathfrak{h} \arrow[r, "{(h,\zeta)}"'] & \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \to \mathfrak{h} \arrow[r, "{(\rho,\gamma)}"'] & \mathfrak{h}.
\end{tikzcd}$$

Therefore, if (ι, β) has the LLP with respect to all fibrations, so must (f, α). Since (ι, β) is obtained by a cobase change of $\mathcal{L}(p)$, it suffices to show that $\mathcal{L}(p)$ has the LLP with respect to all fibrations. To this end, begin with the following diagram:

$$\begin{tikzcd}
\mathcal{LC}(\mathfrak{g}) \arrow[r] \arrow[d, "\mathcal{L}(p)"'] & \mathfrak{x} \arrow[d, "{(\varphi,\kappa)}"] \\
\mathcal{L}(A) \arrow[r] & \mathfrak{y},
\end{tikzcd}$$

where (φ, κ) is a fibration of curved Lie algebras. Using the adjunction between the contravariant functors $\mathcal{C}$ and $\mathcal{L}$ (Proposition 1.37), finding a lift in the above diagram is equivalent to finding a lift in the following diagram:

$$\begin{tikzcd}
\mathcal{C}(\mathfrak{y}) \arrow[r] \arrow[d, "{\mathcal{C}(\varphi,\kappa)}"'] & A \arrow[d, "p"] \\
\mathcal{C}(\mathfrak{x}) \arrow[r] & \mathcal{C}(\mathfrak{g}).
\end{tikzcd}$$

There exists a lift $H\colon \mathcal{C}(\mathfrak{x}) \to A$ in this diagram because p is an acyclic fibration, $\mathcal{C}(\varphi, \kappa)$ is a cofibration (see Lemma 1.59), and the category $\mathscr{A}$ is a model category. Thus, the proof is complete. $\square$

Theorem 1.65. *The category of curved Lie algebras with curved morphisms defines a model category with the model structure defined in Definition 1.56. Moreover, it is Quillen equivalent to the model category of unital commutative differential graded algebras.*

Proof. It follows from the preceding results. □

1.5 Unbased rational homotopy theory

For the remainder of the paper it is assumed that the ground field is the rational numbers, $\mathbb{Q}$. Within this section, results contained in [4] and [24] will be used alongside the Quillen equivalence of Theorem 1.65 to construct a disconnected rational homotopy theory using the category of pseudo-compact curved Lie algebras.

The category of simplicial sets will be denoted by $\mathscr{S}$. Recall that, the category $\mathscr{S}$ possesses a well known model structure where the weak equivalences are weak homotopy equivalences. More details can be found, for instance, in [4] and [13]. With this model structure all simplicial sets are cofibrant and the fibrant objects are the Kan complexes.

First some definitions are recalled.

Definition 1.66. *A group, G, is uniquely divisible if the equation $x^r = g$ has a unique solution $x \in G$ for each $g \in G$ and $r > 1$.*

Definition 1.67. *A connected Kan complex, X, is said to be*

- *nilpotent if for any choice of base vertex*
  - *its fundamental group, $\pi_1(X)$, is nilpotent and*
  - *every other homotopy group, $\pi_i(X)$, is acted on nilpotently the fundamental group, in the sense that the sequence $G_{0,i} = \pi_i(X)$, $G_{k+1,i} \colon = \{gn - n \colon n \in N_{k,i}, g \in \pi_1(X)\}$ terminates;*
- *rational if each of its homotopy groups are uniquely divisible.*

Definition 1.68. *A cdga is said to be connected if it is concentrated in non-negative degrees and equal to the ground field, $\mathbb{Q}$, in degree 0. Similarly, a cdga X is said to be homologically connected if $H^0(X) = \mathbb{Q}$ and $H^i(X) = 0$ for all $i < 0$. Extending this notion, a cdga X is said to be homologically disconnected cdga if $H^0(X)$ is isomorphic to a finite product of copies of $\mathbb{Q}$ and $H^i(X) = 0$ for all $i < 0$.*

Remark 1.69. It is worth noting that a homologically disconnected cdga may in fact be homologically connected.

Definition 1.70. *Given some category $X(= \hat{\mathscr{L}}_*, \mathscr{A}, \mathscr{S})$, let $\mathrm{ho}(X)$ denote the homotopy category of X. Given some category $X(= \mathscr{A}, \mathscr{S})$, let X^c denote the full subcategory of connected objects and let X^{dc} denote the full subcategory of objects with finitely many connected components.*

For example, the objects of the category $\mathrm{ho}(\mathscr{S}^c)$ are connected Kan complexes.

Definition 1.71. *A cdga $A \in \mathscr{A}$ is called a Sullivan algebra if its underlying graded algebra is the free commutative graded algebra $\wedge V$ on a graded vector space V satisfying the following: V is the union of an increasing series of graded subspaces $V_0 \subset V_1 \subset \dots$, with $d \equiv 0$ on V_0 and $d(V_k)$ is contained in $\wedge(V_{k-1})$.*

A Sullivan algebra is called minimal if the image of the differential d is contained in $\wedge^+(V)^2$, where $\wedge^+(V)$ is the direct sum of the positive degree subspaces of $\wedge(V)$.

Recall that every non-negatively graded homologically connected cdga admits a minimal model, cf. [4, Section 7]. That is, every non-negatively graded homologically connected cdga is weakly equivalent to a minimal algebra. Moreover, the minimal algebra is unique up to isomorphism.

Definition 1.72.

- *A cofibrant homologically connected cdga is said to be of finite type over $\mathbb{Q}$ if its minimal model has finitely many generators in each degree; and*
- *nilpotent connected Kan complex is said to be of finite type over $\mathbb{Q}$ if its homology groups with coefficients in $\mathbb{Q}$ are finite dimensional vector spaces over $\mathbb{Q}$.*

The adjective 'finite type' will be understood to mean 'finite type over $\mathbb{Q}$'.

Definition 1.73. *The prefix $\mathrm{fN}\mathbb{Q}-$ applied to a category of simplicial sets will denote the full subcategory composed of rational, nilpotent objects of finite type over $\mathbb{Q}$, and the prefix $\mathrm{f}\mathbb{Q}-$ applied to a category of algebras will denote the full subcategory of objects of finite type over $\mathbb{Q}$.*

For example, the category $\mathrm{fN}\mathbb{Q}-\mathrm{ho}(\mathscr{S}^c)$ denotes the full subcategory of $\mathrm{ho}(\mathscr{S})$ of rational, nilpotent Kan complexes of finite type.

Any non-negatively graded homologically disconnected cdga is isomorphic to a finite product of homologically connected cdgas, cf. [24, Theorem B]. Further, the homologically disconnected cdga is said to be of finite type if each connected cdga in the finite product is of finite type, see Proposition 4.4 in op. cit.

Let $\mathscr{A}_{\geq 0}$ denote the category of non-negatively graded cdgas. Recall, from [4, Theorem 9.4], there exists a pair of adjoint functors

$$F\colon \mathscr{A}_{\geq 0} \leftrightarrows \mathscr{S} : \Omega,$$

that induce an equivalence of the homotopy categories $\mathrm{f}\mathbb{Q}-\mathrm{ho}\mathscr{A}^c_{\geq 0}$ and $\mathrm{fN}\mathbb{Q}-\mathrm{ho}\mathscr{S}^c$. This is the so-called Sullivan-de Rham equivalence. Here Ω is the de Rham functor [4, Section 2] and F is the functor given by $X \mapsto F(X, \mathbb{Q})$ where $F(X, \mathbb{Q})$ is the function space of [4, Section 5.1] (called the Bousfield-Kan functor). Note there is also an analogue of this result in the pointed case contained in op. cit. Combining the equivalence of these homotopy categories with [24, Proposition 3.5]—where the authors extend the existence of minimal models to arbitrary homologically connected cdgas to prove the categories $\mathrm{ho}\,\mathscr{A}^c_{\geq 0}$ and $\mathrm{ho}\,\mathscr{A}^c$ are equivalent—it follows that the categories $\mathrm{fN}\mathbb{Q}-\mathrm{ho}\mathscr{S}^c$ and $\mathrm{f}\mathbb{Q}-\mathrm{ho}\mathscr{A}^c$ are equivalent. Further, one has the following theorem of Lazarev and Markl [24, Theorem C].

Theorem 1.74 (Lazarev-Markl)**.** *The three categories* $\mathrm{fN}\mathbb{Q}{-}\mathrm{ho}\mathscr{S}^{dc}$*,* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}_{\geq 0}$*, and* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}$ *are equivalent.*

It will be the aim of the rest of this section to describe a subcategory $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}^{dc}_*$ of $\mathrm{ho}(\hat{\mathscr{L}}_*)$ equivalent to the categories $\mathrm{fN}\mathbb{Q}{-}\mathrm{ho}\mathscr{S}^{dc}$, $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}_{\geq 0}$, and $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}$, extending the theorem of Lazarev and Markl.

Proposition 1.75. *For any homologically connected cdga, A, the curved Lie algebra $\mathcal{L}(X)$ is weakly equivalent to a dgla, i.e. a curved Lie algebra with zero curvature.*

Proof. First note that any minimal algebra has a unique augmentation, thus being endowed with an augmentation means that this minimal algebra corresponds to a dgla under the contravariant functor $\mathcal{L}$. Therefore, there exists a filtered quasi-isomorphism $\mathcal{L}(A) \to \mathcal{L}(M_A)$ since the contravariant functor $\mathcal{L}$ preserves weak equivalences, where M_A is a minimal model for A. □

Proposition 1.76. *For any homologically disconnected cdga, A, the curved Lie algebra $\mathcal{L}(A)$ is weakly equivalent to a finite coproduct of dglas of the form $\mathcal{L}(M)$, for some minimal algebra M.*

Proof. Since a homologically disconnected cdga, A, is a finite product of homologically connected ones up to isomorphism, A is weakly equivalent to a finite product of minimal algebras. Hence $\mathcal{L}(A)$ is weakly equivalent to a finite coproduct of dglas of the form $\mathcal{L}(M)$. □

With this proposition in mind the following definitions are made.

Definition 1.77. *The category* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}^{c}_*$ *is the full subcategory of* $\mathrm{ho}(\hat{\mathscr{L}}_*)$ *with objects consisting of curved Lie algebras isomorphic to ones of the form $\mathcal{L}(M)$, where M is a minimal cdga of finite type.*

The category $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}^{dc}_*$ *is the full subcategory of* $\mathrm{ho}(\hat{\mathscr{L}}_*)$ *with objects consisting of curved Lie algebras isomorphic to finite coproducts of objects in* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}^{dc}_*$*.*

The next result is now clear.

Theorem 1.78. *The category* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}^{dc}_*$ *is equivalent to the categories* $\mathrm{fN}\mathbb{Q}{-}\mathrm{ho}\mathscr{S}^{dc}$*,* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}_{\geq 0}$*, and* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}$*.* □

To describe the equivalence in a more specific manner, first recall the definition of the Maurer-Cartan simplicial set.

Definition 1.79. *The Maurer-Cartan simplicial set is given by the functor*

$$\mathrm{MC}_\bullet \colon \hat{\mathscr{L}}_* \to \mathscr{S}$$

that maps a pseudo-compact curved Lie algebra $\mathfrak{g}$ to the simplicial space of MC elements in $\mathfrak{g} \otimes \Omega(\Delta^\bullet)$, where $\Omega(\Delta^n)$ is the Sullivan-de Rham algebra of polynomial forms on the standard topological cosimplicial simplex considered as a homologically graded cdga.

For more details on the Maurer-Cartan simplicial set see [15, 17, 24], for example. Note that the Sullivan-de Rham algebra of polynomial forms on the standard topological cosimplicial simplex must be considered as a homologically graded cdga so that the resulting object when tensored with a homologically curved Lie algebra is again a homologically curved Lie algebra.

Proposition 1.80. *The functors* $\mathrm{MC}_\bullet\colon \hat{\mathscr{L}}_* \to \mathscr{S}$ *and* $\mathcal{L}\Omega\colon \mathscr{S} \to \hat{\mathscr{L}}_*$ *form an adjoint pair.*

Proof. By definition, there exists an isomorphism of simplicial sets $\mathrm{MC}_\bullet(\mathfrak{g}) \cong F\mathcal{C}(\mathfrak{g})$. Therefore, the functors are compositions in the following diagram:

$$\hat{\mathscr{L}}_* \underset{\mathcal{L}}{\overset{\mathcal{C}}{\rightleftarrows}} \mathscr{A} \underset{\Omega}{\overset{F}{\rightleftarrows}} \mathscr{S} \quad .$$

The contravariant functors $\mathcal{L}$ and $\mathcal{C}$ are adjoint by Theorem 1.65, and the functors F and Ω are adjoint by [4], therefore the result follows. $\square$

Further, it follows (from Proposition 1.37 and [4]) that the composite functors $\mathrm{MC}_\bullet$ and $\mathcal{L}\Omega$ induce adjoint functors upon the homotopy categories of $\hat{\mathscr{L}}_*$ and $\mathscr{S}$. Restricting to the categories $\mathrm{fN}\mathbb{Q}{-}\mathrm{ho}\mathscr{S}^{dc}$ and $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{dc}$ these functors induce mutually inverse equivalences, as the following results show.

Proposition 1.81. *Given any connected non-negatively graded cdga, A, of finite type, the Lie algebra $\mathcal{L}(A)$ is weakly equivalent to an object belonging to the category* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{c}$.

Proof. Since A is connected there exists some unique minimal model, M_A, that is also of finite type. Therefore, since A is quasi-isomorphic to M_A the Lie algebras $\mathcal{L}(A)$ and $\mathcal{L}(M_A)$ are weakly equivalent. $\square$

Proposition 1.82. *Given any curved Lie algebra $\mathfrak{g} \in \mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{c}$ the cdga $\mathcal{C}(\mathfrak{g})$ is quasi-isomorphic to a connected non-negatively graded cdga of finite type.*

Proof. The Lie algebra will be weakly equivalent to one of the form $\mathcal{L}(M)$ for some minimal algebra M of finite type, thus $\mathcal{C}\mathcal{L}(M)$ is quasi-isomorphic to M and thus of the right form. $\square$

Theorem 1.83. *The functors* $\mathrm{MC}_\bullet$ *and* $\mathcal{L}\Omega$ *determine mutually inverse equivalences between the categories* $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{dc}$ *and* $\mathrm{fN}\mathbb{Q}{-}\mathrm{ho}\mathscr{S}^{dc}$.

Proof. First, note that the objects of the category $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{dc}$ are, up to equivalence, coproducts of objects of the category $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{c}$. Likewise the objects of the category $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}$ are, up to equivalence, products of objects of the category $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{c}$. Hence to show there is an equivalence of $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{dc}$ and $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}$ it is sufficient to show there is an equivalence of $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{c}$ and $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{c}$. By Propositions 1.81 and 1.82 the contravariant functors $\mathcal{C}$ and $\mathcal{L}$ restrict to an adjunction

$$\mathcal{C}\colon \mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{c} \leftrightarrows \mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{c}\colon \mathcal{L}.$$

Since Theorem 1.65 shows that the contravariant functors $\mathcal{C}$ and $\mathcal{L}$ form a Quillen pair it is evident that their restrictions form mutually inverse equivalences of the categories $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^c$ and $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^c$. Whence the categories $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{dc}$ and $\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc}$ are equivalent. Combining this with [24, Theorem C] there exists equivalences of the categories

$$\mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{dc} \sim \mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}^{dc} \sim \mathrm{f}\mathbb{Q}{-}\mathrm{ho}\mathscr{A}_{\geq 0}^{dc} \sim \mathrm{fN}\mathbb{Q}{-}\mathrm{ho}\mathscr{S}^{dc},$$

and the proof is complete. $\square$

Since equivalences of categories preserve colimits and limits, an analogue of [24, Theorem 1.7] and its corollary can be explained here immediate consequences of Theorem 1.83.

Corollary 1.84. *Given $\coprod_{i\in I}\mathfrak{g}_i \in \mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{dc}$, the simplicial set $\mathrm{MC}_\bullet(\coprod_{i\in I}\mathfrak{g}_i)$ is weakly equivalent to the disjoint union $\bigcup_{i\in I}\mathrm{MC}_\bullet(\mathfrak{g}_i)$.*

Let $\mathcal{MC}(-) := \pi_0(\mathrm{MC}_\bullet(-))$ denote the Maurer-Cartan moduli set, i.e. the set of Maurer-Cartan elements up to homotopy. This construction can be found for example in [15,17,24]. An alternate construction in the general case of pronilpotent dglas given in [28] is that the Maurer-Cartan moduli set can be described as the Maurer-Cartan set up to gauge equivalence. For more details regarding gauge equivalence and a proof of this statement, see [10, 31].

Corollary 1.85. *Given $\coprod_{i\in I}\mathfrak{g}_i \in \mathrm{f}\mathbb{Q}{-}\mathrm{ho}\hat{\mathscr{L}}_*^{dc}$, there exists an isomorphism of sets*

$$\mathcal{MC}\left(\coprod_{i\in I}\mathfrak{g}_i\right) \cong \bigcup_{i\in I}\mathcal{MC}(\mathfrak{g}_i).$$

Theorem 1.83 also has an application to mapping spaces. Recall that given a pseudo-compact curved Lie algebra, $(\mathfrak{g}, d_\mathfrak{g}, \omega_\mathfrak{g})$, and a unital cdga, A, (both homologically graded) their completed tensor product possesses a well defined pseudo-compact curved Lie algebra structure, see Proposition 1.7. Given such a tensor product, the MC elements can be studied in the usual manner as solutions to the MC equation

$$\omega_\mathfrak{g}\hat{\otimes}1 + d\xi + \frac{1}{2}[\xi,\xi].$$

The MC elements of such a tensor product correspond to morphisms of cdgas, as the following shows. Let $\mathrm{MC}(\mathfrak{g}, A) := \mathrm{MC}(\mathfrak{g}\hat{\otimes}A)$ be considered as a bifunctor.

Proposition 1.86. *Given a pseudo-compact curved Lie algebra, $(\mathfrak{g}, d_\mathfrak{g}, \omega_\mathfrak{g})$, and a cdga, A, the two bifunctors $\mathrm{MC}(\mathfrak{g}, A)$ and $\mathrm{Hom}_\mathscr{A}(\mathcal{C}(\mathfrak{g}), A)$ are naturally isomorphic.*

Proof. Given some degree minus one element of $\mathfrak{g}\hat{\otimes}A$, it corresponds precisely to a continuous linear morphism $k \to (\Sigma^{-1}\mathfrak{g})\hat{\otimes}A$ of degree zero. This continuous linear morphism, in turn, defines (and is defined by) a linear morphism $(\Sigma^{-1}\mathfrak{g})^* \to A$ which extends uniquely to a morphism of graded commutative algebras $\mathcal{C}(\mathfrak{g}) \to A$. The condition that this morphism is in fact one of cdga is precisely the one that the original element is a MC element. $\square$

This result extends to the level of homotopy. It is necessary to first, however, to recall the definition of a homotopy of MC elements.

Definition 1.87. *Let $k[z, dz]$ be the free unital cdga on generators z and dz of degrees 0 and 1 respectively, subject to the condition $d(z) = dz$.*

Given some unital cdga A, let $A[z, dz]$ denote the unital cdga given by the tensor $A \otimes k[z, dz]$. Further, denote the quotient morphisms by setting $z = 0, 1$ by

$$|_0, |_1 \colon A[z, dz] \to A.$$

Definition 1.88. *Given a pseudo-compact curved Lie algebra, $\mathfrak{g}$, and a unital cdga, A, two elements $\xi, \eta \in \mathrm{MC}(\mathfrak{g}, A)$ are said to be homotopic if there exists*

$$h \in \mathrm{MC}(\mathfrak{g}, A[z, dz])$$

such that $h|_0 = \xi$ and $h|_1 = \eta$.

As Proposition 1.86 shows, a homotopy of MC elements is nothing more than a Sullivan homotopy, i.e. a right homotopy with path object $A[z, dz]$. Therefore, two elements of $\mathcal{MC}(\mathfrak{g}, A)$ belong to the same class if, and only if, the two corresponding morphisms $\mathcal{C}(\mathfrak{g}) \to A$ belong to the same homotopy class.

Corollary 1.89. *Given $X, Y \in \mathrm{fN}\mathbb{Q}\mathrm{-ho}\mathscr{S}^{dc}$, then*

$$\mathrm{Hom}_{\mathrm{fN}\mathbb{Q}-\mathrm{ho}\mathscr{S}^{dc}}(X, Y) \cong \mathcal{MC}(\mathcal{L}\Omega(Y) \otimes \Omega(X)).$$

Proof. There is clearly an isomorphism

$$\mathrm{Hom}_{\mathrm{fN}\mathbb{Q}-\mathrm{ho}\mathscr{S}^{dc}}(X, Y) \cong \mathrm{Hom}_{\mathrm{f}\mathbb{Q}-\mathrm{ho}\mathscr{A}^{dc}}(\Omega(Y), \Omega(X)).$$

It therefore suffices to show that there exists an isomorphism of $\mathcal{MC}(\mathcal{L}\Omega(Y) \otimes \Omega(X))$ and homotopy classes of morphisms $\Omega(Y) \to \Omega(X)$ which is contained within the discussion above. □

Remark 1.90. In [23, Theorem 8.1] an explicit model for every connected component of the mapping space between two connected rational, nilpotent CW complexes of finite type, X and Y, is given; it is further assumed that either X is a finite CW complex or Y has a finite Postnikov tower, because this ensures the spaces of maps between X and Y are homotopically equivalent to finite type complexes. Whereas the result Corollary 1.89 gives a model for the whole mapping space. Therefore, in the case when the two spaces, X and Y, are both sufficiently nice (i.e. are both composed of finitely many connected components each rational, nilpotent, and of finite type with either X being a finite CW complex or Y having a finite Postnikov tower and such that the space of maps has finitely many connected components), the results [23, Theorem 8.1] and Corollary 1.89 could be combined to construct a model for the mapping space as a coproduct of finitely many MC moduli spaces. The material developed in this paper, however, does not allow the extension to the case where at least one of the spaces, X or Y, fails to meet the aforementioned constraints, or to the case when the space of maps has infinitely many connected components, and this can happen in some seemingly straightforward cases; the space of maps between 2-dimensional spheres, for example, has infinitely many connected components.

Section 1
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Section 2
Koszul duality and homotopy theory of curved Lie algebras

Abstract

This paper introduces the category of marked curved Lie algebras with curved morphisms, equipping it with a model structure. This model structure is—when working over an algebraically closed field of characteristic zero—Quillen equivalent to a model category of pseudo-compact unital commutative differential graded algebras, extending known results regarding the Koszul duality of unital commutative differential graded algebras and differential graded Lie algebras. As an application of the theory developed within this paper, algebraic deformation theory is extended to functors over pseudo-compact, not necessarily local, commutative differential graded algebras. Further, these deformation functors are shown to be representable.

Introduction

Pseudo-compact unital commutative differential graded algebras are dual to cocommutative counital differential graded coalgebras, and hence arise naturally as cochain complexes of topological spaces, at least in the simply connected case. Further, they play a notable role in rational homotopy theory [19, 24] and serve as representing objects in formal deformation problems [11, 18, 23]. Therefore, when working over a field of characteristic zero, it is natural to attempt to place them in the framework of a closed model category. Quillen [24] produced the first result of this kind—albeit under a strong assumption of connectedness. Later the connectedness assumption was removed by Hinich [11]. The crucial difference between the approaches of Quillen and Hinich was that Hinich had chosen a finer notion of weak equivalence in his model structure: he worked with filtered quasi-isomorphisms, whereas Quillen worked with quasi-isomorphisms.

Additionally, the constructions of Quillen and Hinich have a notable attribute: the model categories are Quillen equivalent to certain model structures on the category of differential graded Lie algebras. This type of equivalence is so-called *Koszul duality*. Whilst the construction of Hinich generalises the construction of Quillen, it does not completely generalise to the category of all cocommutative counital differential graded coalgebras; Hinich worked with conilpotent coalgebras in loc. cit. When the ground field is algebraically closed, the construction was completely generalised to all coalgebras by Chuang, Lazarev, and Mannan [5]. The authors therein chose to work in the dual setting of pseudo-compact unital commutative differential graded algebras. Note, although the construction was completely generalised to all coalgebras, in the dual setting the assumption that the algebras be pseudo-compact is made. One especially pleasant and useful property shown in op. cit. is, when working over an algebraically closed field, any pseudo-compact commutative differential graded algebra can be decomposed into a product of local pseudo-compact algebras. In fact, there exist only two types of local algebras—Hinich algebras where the unique maximal ideal is closed under the differential and acyclic algebras where any closed element is also a boundary. Hinich algebras are precisely those studied by Hinich in [11], hence the name.

The Koszul duality is also extended in [5]. The Koszul dual therein is the category of formal coproducts of curved Lie algebras. This construction, however, is somewhat asymmetric: because it describes a duality between the natural category of pseudo-compact unital cdgas and the somewhat unnatural category of formal coproducts of curved Lie algebras. Herein this category is replaced with a more natural one—namely the category of marked curved Lie algebras with curved morphisms—providing a more intuitive and symmetric description for a Koszul dual to the category of cocommutative counital differential graded coalgebras, and one that is easier to work with. A key difference between the tools of this paper and that of [5] is the use of curved morphisms.

Numerous papers discuss the homotopy theory of differential graded coalgebras over different operads; for example Positselski [22] constructed a homotopy theory for coassociative differential graded coalgebras. However, the coalgebras were assumed to be conilpotent and this construction is not known in the completely general case. Further, Positselski worked with curved objects suggesting that in more general cases when discussing a Koszul duality one side of the Quillen equivalence should be a category consisting of curved objects; a hypothesis that is strengthened by results of [5] and this paper.

The paper is organised as follows. Sections 2.1 and 2.2 recall (without proof) the necessary facts concerning the category of formal products of a given category, the extended Hinich category, and the category of pseudo-compact unital commutative differential graded algebras. For more details see the original paper [5].

In Section 2.3 the category of marked curved Lie algebras and curved morphisms is introduced. This category is similar to the category of curved Lie algebras with strict morphisms discussed in [5]. On the other hand, the morphisms are quite different and as such the category itself is quite different. For instance, in the category of curved Lie algebras with strict morphisms an object with non-zero curvature cannot be isomorphic to one with zero curvature, but using curved morphisms there is an abundance of such isomorphisms: any Lie algebra twisting by a Maurer-Cartan element gives rise a curved isomorphism (see Remark 2.20). The category of marked curved Lie algebras is equipped with a model structure (Definition 2.37), using the existing model structure of curved Lie algebras with strict morphisms (see [5]) together with *rectification by a marked point* (Definition 2.28). Rectification by a marked point provides a procedure in which a strict morphism can be obtained from a curved one. It should be noted, however, this construction is not functorial, because one has no canonical choice of marked point. Nonetheless, the morphisms obtained by two different choices of marked point are related by pre and post composition with isomorphisms.

Section 2.4 formulates the main result of the paper: a Quillen equivalence between the model category of marked curved Lie algebras and the model category of pseudo-compact unital commutative differential graded algebras (Theorem 2.60). This equivalence uses a pair of adjoint functors that have their origins in the Harrison and Chevalley-Eilenberg complexes of homological algebra, found (for example) in [1, 10] and [30] respectively. As alluded to above, a benefit of the Koszul duality of this paper over [5] is the symmetry of the construction.

Section 2.5 applies the material developed in the rest of the paper to introduce a certain class of deformation functors acting over pseudo-compact unital commutative differential graded algebras. Theorem 2.70 shows these deformation functors are representable in the homotopy category of pseudo-compact unital commutative differential graded algebras. Two definitions (2.69 and 2.72) are given for these deformation functors; one being slightly more general than the other. The less general Definition 2.72 enjoys the benefit of not requiring the knowledge of a decomposition of a pseudo-compact commutative differential graded algebra into the product of local pseudo-compact commutative differential graded algebras; this is possible via a functor of [5]. Here deformation theory is considered via the differential graded Lie algebra approach; see [9, 14, 18, 25, 26], for example.

The notion of a *Sullivan homotopy* was introduced by Sullivan [27] in his work on rational homotopy theory, and Appendix 2.A defines analogues in the categories of curved Lie algebras with strict morphisms and local pseudo-compact commutative differential graded algebras. These analogues serve the constructions of Section 2.5 by showing (when the objects considered are suitably nice) that the equivalence classes of Sullivan homotopic morphisms are in bijective correspondence with the classes of morphisms in the homotopy category (Theorem 2.79). An explicit construction of a path object in the category of curved Lie algebras with strict morphisms and similar—but more subtle—ideas in the category of local pseudo-compact commutative differential graded algebras led to the proof.

Notation and conventions

Throughout the paper it is assumed that all commutative and Lie algebras are over a fixed algebraically closed field, k, of characteristic zero. Algebraic closure is necessary for some key results, but some of the statements hold in a more general setting. Unmarked tensor products are assumed to be over k and algebras are assumed to be unital, unless stated otherwise. The graded vector space over k spanned by the vectors $a, b, c, \ldots$ (with specified degree) is denoted by $\langle a, b, c, \ldots \rangle$. Similarly, the free graded Lie algebra over k on generators $a, b, c, \ldots$ (with specified degrees) is denoted by $\mathrm{L}\langle a, b, c, \ldots \rangle$.

The following abbreviations are commonly used throughout the paper: 'dgla' for differential graded Lie algebra; 'cdga' for commutative differential graded algebra; 'dg' for differential graded; 'MC' for Maurer-Cartan; 'CMC' for closed model category in the sense of [24] (for a review of this material see [7]); 'LLP' for left lifting property; and 'RLP' for right lifting property.

Graded objects are assumed to be $\mathbb{Z}$-graded, unless otherwise stated. For both commutative and Lie algebras this grading is in the homological sense with lower indices. Although some Lie algebras in this paper are not necessarily complexes, they possess an odd derivation often referred to as the (pre-)differential and hence resemble complexes. In the homological grading, these differentials have degree -1.

Given any homogeneous element, x, of some given algebra, its degree is denoted by $|x|$. Therefore, in the homological setting a MC element is of degree -1 and the curvature element of a curved Lie algebra is of degree -2. The suspension, ΣV, of a homologically graded space is defined by $(\Sigma V)_i = V_{i-1}$. Applying the functor of linear discrete or topological duality takes homologically graded spaces to cohomologically graded ones, and vice versa, i.e. $(V^*)^i = (V_i)^*$. A homologically graded space can therefore be considered equivalently as a cohomological one by setting $V_i = V^{-i}$ for each $i \in \mathbb{Z}$. Additionally, ΣV^* is written for $\Sigma(V^*)$, and with this convention there is an isomorphism $(\Sigma V)^* \cong \Sigma^{-1}V^*$.

Many cdgas considered in this paper are pseudo-compact. A cdga is said to be pseudo-compact if it is an inverse limit of finite dimensional commutative graded algebras with continuous differential. Taking the inverse limit induces a topology and the operations of the algebra are assumed to be continuous with respect to this topology. More details on pseudo-compact objects can be found in [8, 12, 28].

Consider a curved Lie algebra, $(\mathfrak{g}, d_{\mathfrak{g}}, \omega)$, and a pseudo-compact cdga, $A = \varprojlim A_i$. The *completed tensor product*, denoted $\hat{\otimes}$, is given by

$$\mathfrak{g}\hat{\otimes}A = \varprojlim_{i} \mathfrak{g} \otimes A_i,$$

where the tensor on the right hand side is given by the tensor product in the category of graded vector spaces. Note the adjective 'completed' is dropped almost everywhere. This tensor product possesses the structure of a curved Lie algebra: the curvature is given by $\omega\hat{\otimes}1$, the differential is given on elementary tensors by $d(x\hat{\otimes}a) = (d_{\mathfrak{g}}x)\hat{\otimes}a + (-1)^{|x|}x\hat{\otimes}(d_A a)$, and the bracket is given on elementary tensors by $[x\hat{\otimes}a, y\hat{\otimes}b] = [x,y]\hat{\otimes}(-1)^{|a||y|}ab$. This construction is useful in Section 2.5 when defining deformation functors in Definitions 2.69 and 2.72.

2.1 The category of formal products

The category of formal products was defined as a means to describe the category of pseudo-compact cdgas. Here the definition and some facts are recalled; for greater details and the proofs see the original paper [5]. Let $\mathcal{C}$ be a CMC.

Definition 2.1. *The category of formal products in $\mathcal{C}$, denoted* $\mathrm{Prod}(\mathcal{C})$*, is the category with objects given by morphisms from indexing sets to the set of objects of the category $\mathcal{C}$. An object is denoted by $\prod_{i\in I} A_i$, where I is some indexing set and for each $i \in I$ the morphism sends $i \mapsto A_i \in \mathcal{C}$. A morphism in* $\mathrm{Prod}(\mathcal{C})$*,*

$$f\colon \prod_{i\in I} A_i \to \prod_{j\in J} B_j,$$

is given by a morphism of sets $J \to I$ that sends $j \mapsto i_j$, and a morphism $f_j\colon A_{i_j} \to B_j$ of $\mathcal{C}$ for all $j \in J$. The morphism f_j is called the jth component of the morphism f.

Remark 2.2. The indexing set of an object in $\mathrm{Prod}(\mathcal{C})$ could be empty and in this case one has the terminal object for $\mathrm{Prod}(\mathcal{C})$.

Definition 2.3. *Let $\prod_{i\in I} A_i, \prod_{j\in J} B_j \in \mathrm{Prod}(\mathcal{C})$, their product is given by*

$$\prod_{i\in(I\coprod J)} A_i$$

and their coproduct is given by

$$\prod_{(i,j)\in I\times J} \left(A_i \coprod B_j\right).$$

Both constructions easily extend from the binary case.

Proposition 2.4. *Given a morphism $f\colon \prod_{i\in I} A_i \to \prod_{j\in J} B_j$, for each $i \in I$, let B^i denote the product in $\mathcal{C}$ of the B_j satisfying $i_j = i$. The morphisms $f_j\colon A_i \to B_j$ factor uniquely through a morphism $f^i\colon A_i \to B^i$.* □

Definition 2.5. *A morphism $f\colon \prod_{i\in I} A_i \to \prod_{j\in J} B_j$ of $\mathrm{Prod}(\mathcal{C})$ is said to be a*

- *weak equivalence if, and only if, the morphism $J \to I$ induced by f is a bijection and for every $j \in J$ the morphism f_j (or equivalently for every $i \in I$ the morphism f^i) is a weak equivalence in $\mathcal{C}$;*
- *cofibration if, and only if, for each $j \in J$ the morphism f_j is a cofibration in $\mathcal{C}$;*
- *fibration if, and only if, for each $i \in I$ the morphism f^i is a fibration in $\mathcal{C}$.*

The classes of morphisms in Definition 2.5 provide $\mathrm{Prod}(\mathcal{C})$ with the structure of a CMC; cf. [5, Theorem 4.8].

Example 2.6. To gain some intuition, an example of [5] is recalled. Let $\mathcal{C}$ be the category of connected topological spaces, then the category $\mathrm{coProd}(\mathcal{C}) := \mathrm{Prod}(\mathcal{C}^{op})$ is the category of all topological spaces that can be written as the disjoint union of connected spaces. In fact, in [16], the categories of finite (co)products were used to construct a disconnected rational homotopy theory.

2.2 The extended Hinich category and pseudo-compact cdgas

Recall from [5, Definition 2.2] that a local pseudocompact cdga is a pseudocompact cdga with a unique maximal graded ideal, that is possibly not closed under the differential. As shown in [5, Theorem 2.9, Lemma 4.2], the category of pseudo-compact cdgas (denoted herein $\mathcal{A}$) is equivalent to the category of formal products of local pseudo-compact cdgas. The category of local pseudo-compact cdgas is referred to as the *extended Hinich category* and denoted by $\mathcal{E}$. For greater details and the proofs see the original paper [5].

Proposition 2.7. *Any pseudo-compact cdga is isomorphic to a direct product of local pseudo-compact cdgas.* □

Definition 2.8. *A local pseudo-compact cdga is said to be*

- *a Hinich algebra if the maximal ideal is closed under the differential;*
- *an acyclic algebra if every cycle is a boundary.*

Every local pseudo-compact cdga is one of the above two types, cf. [5]. The category of Hinich algebras is referred to as the *Hinich category.*

Definition 2.9. *Given $A \in \mathcal{E}$ with maximal ideal M, the* full Hinich subalgebra *is the local pseudo-compact cdga given as $A^H := \{a \in A : da \in M\}$.*

Clearly, A^H is a Hinich algebra and if A is a Hinich algebra then $A^H = A$. If A is acyclic then A^H has codimension one in A.

Proposition 2.10. *Given a morphism $f\colon A \to B$ of local pseudo-compact cdgas, it restricts to a morphism $f^H\colon A^H \to B^H$ of Hinich algebras. Conversely, given $x \in A$ such that $x \notin A^H$, then $f(x) \notin B^H$.* □

It follows from Proposition 2.10 there cannot exist a morphism from an acyclic algebra to a Hinich algebra and any morphism from a Hinich algebra to an acyclic algebra must factor through the full Hinich subalgebra of the codomain.

Remark 2.11. A functor from the extended Hinich category to the Hinich category, is defined by $A \mapsto A^H$ and $(f\colon A \to B) \mapsto (f^H\colon A^H \to B^H)$. This functor forms the right adjoint in a Quillen adjunction (the left adjoint being the inclusion), cf. [5, Proposition 3.19].

Proposition 2.12. *The acyclic algebra $\Lambda = k[x]/\langle x^2\rangle$ with differential given by $dx = 1$, where $|x| = 1$, is the terminal object of $\mathcal{E}$.* □

Recall, from [11], that a morphism $A \to B$ in the Hinich category is a weak equivalence if, and only if, the induced morphism of differential graded Lie algebras $\mathrm{L}\,\Sigma^{-1}B^* \to \mathrm{L}\,\Sigma^{-1}A^*$ (with differentials induced by the differentials and multiplications) is a quasi-isomorphism.

Definition 2.13. *A morphism, $f\colon A \to B$, of $\mathcal{E}$ is said to be a*

- *weak equivalence if, and only if, it is a weak equivalence in the Hinich category or any morphism of acyclic algebras;*
- *fibration if, and only if, B^H is contained within its image;*
- *cofibration if, and only if, it is a retract of a morphism in the class consisting of the tensor products of cofibrations in the Hinich category with:*
  - *the identity $k \to k$;*
  - *the identity $\Lambda \to \Lambda$;*
  - *the natural inclusion $k \hookrightarrow \Lambda$.*

Remark 2.14. Definition 2.13 provides an extension of the model structure given by Hinich [11] for the Hinich category. It is in fact the unique one with this choice of weak equivalence, having all surjective morphisms being fibrations, and Λ being cofibrant; cf. [5].

Definition 2.13 provides a model structure for $\mathcal{E}$ making it a CMC (cf. [5, Theorem 3.17]) and hence provides a model structure for $\mathcal{A}$ via Definition 2.5 making it a CMC, since $\mathcal{A}$ is equivalent to $\mathrm{Prod}(\mathcal{E})$.

In the closing of this section the following important observation is made. This observation is useful in the proof of Theorem 2.70.

Proposition 2.15. *There exists an isomorphism of sets*

$$\mathrm{Hom}_{\mathcal{A}}\left(\prod_{i\in I} A_i, \prod_{j\in J} B_j\right) \cong \prod_{j\in J}\coprod_{i\in I} \mathrm{Hom}_{\mathcal{E}}(A_i, B_j).$$

$\square$

2.3 The category of marked Lie algebras

Here the category of marked Lie algebras is introduced and some basic properties discussed, including defining a model structure (Definition 2.37). In Section 2.4, the theory developed within this Section is used to prove the category of marked curved Lie algebras is Quillen equivalent to $\mathrm{Prod}(\mathcal{E})$, or equivalently $\mathcal{A}$, (Theorem 2.60).

2.3.1 Basic definitions

Similar to [5], this paper works with curved Lie algebras, but a key difference is the morphisms are curved: further the Lie algebras also possess a set of marked points. The set of marked points are introduced as an analogue of the indexing sets in the product of local pseudo-compact cdgas used to present pseudo-compact cdga. In fact, they will be used to index products of local pseudo-compact cdgas, cf. Definition 2.48. Despite the contrast with [5], a lot of the results proven therein prove useful in this setting.

Definition 2.16. *A curved Lie algebra is a triple* $(\mathfrak{g}, d, \omega)$ *where* $\mathfrak{g}$ *is a graded Lie algebra,* d *is a derivation with* $|d| = -1$*, and* $\omega \in \mathfrak{g}_{-2}$ *such that* $d\omega = 0$ *and* $d^2x = [\omega, x]$ *for all* $x \in \mathfrak{g}$*. The element* ω *is known as the curvature.*

Remark 2.17. The derivation of a curved Lie algebra is often referred to as the differential, but this is an abuse of notation since it need not square to zero (unless the curvature is zero). Despite this, the term 'differential' is used within this paper.

Definition 2.18. *A curved morphism of curved Lie algebras is a pair*

$$(f, \alpha)\colon (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}) \to (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$$

where $f\colon \mathfrak{g} \to \mathfrak{h}$ *is a graded Lie algebra morphism and* $\alpha \in \mathfrak{h}_{-1}$ *such that:*

- *$d_{\mathfrak{h}} f(x) = f(d_{\mathfrak{g}} x) - [\alpha, f(x)]$ for all $x \in \mathfrak{g}$, and*
- *$\omega_{\mathfrak{h}} = f(\omega_{\mathfrak{g}}) - d_{\mathfrak{h}}\alpha - \frac{1}{2}[\alpha, \alpha]$.*

The image of an element $x \in \mathfrak{g}$ of the curved morphism (f, α) is given by $f(x)+\alpha \in \mathfrak{h}$. The composition of two curved morphisms, when it exists, is defined as $(f, \alpha) \circ (g, \beta) = (f \circ g, \alpha + f(\beta))$. A morphism with $\alpha = 0$ is said to be strict.

A strict morphism is given by a dgla morphism such that $f(\omega_{\mathfrak{g}}) = \omega_{\mathfrak{h}}$; these are exactly the morphisms considered in [5]. Therefore, the α part of a curved morphism can be seen to deform how the differential and curvature commute with the graded Lie algebra morphism f. Strict morphisms are particularly useful since there exists a process to obtain a strict morphism from a curved one, cf. Definition 2.28.

Definition 2.19. *A curved isomorphism is a curved morphism*

$$(f, \alpha)\colon (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}) \to (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$$

with an inverse curved morphism

$$(f, \alpha)^{-1}\colon (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}}) \to (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$$

such that $(f, \alpha) \circ (f, \alpha)^{-1} = (id_{\mathfrak{h}}, 0)$ and $(f, \alpha)^{-1} \circ (f, \alpha) = (id_{\mathfrak{g}}, 0)$.

Remark 2.20. Observe that a curved Lie algebra with non-zero curvature may be isomorphic to one with zero curvature (i.e. a dgla). Take the curved isomorphism

$$(id, -\xi)\colon (\mathfrak{g}, d, \omega) \to \left(\mathfrak{g}, d + \mathrm{ad}_{\xi}, \omega + d\xi + \frac{1}{2}[\xi, \xi]\right)$$

which has inverse (id, ξ). The codomain has zero curvature if, and only if, ξ is a MC element of $(\mathfrak{g}, d, \omega)$. Some curved Lie algebras, however, do not possess any MC elements, unlike dglas where 0 is always a MC element. These morphisms correspond to twisting by ξ, written as $\mathfrak{g}^{\xi}$. More details concerning twisting can be found in [2], and how it is generalised to L_{∞}-algebras can be found in [4]. It should be noted that neither of the cited sources use the notion of a curved morphism.

Proposition 2.21. *Given a curved morphism $(f, \alpha)\colon (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}) \to (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$ and a MC element x of $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$, then $(f, \alpha)(x) = f(x) + \alpha$ is a MC element of $(\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$.*

Proof. It is a simple check to see $f(x) + \alpha$ satisfies the MC equation. □

Definition 2.22. *A curved Lie algebra, $(\mathfrak{g}, d, \omega)$, is said to be marked when it is equipped with a set of possibly non-distinct elements of $\mathfrak{g}_{-1}$ indexed by a non-empty set I, i.e. a set $\{x_i\}_{i\in I}$, with $|x_i| = -1$ for all $i \in I$.*

Remark 2.23. One could view a marked curved Lie algebra as a family of curved Lie algebras parametrised by the set of marked points. This point of view is not used here.

This set of marked points is supposed to mimic the indexing set in the product decomposition of a pseudo-compact cdga as local pseudo-compact cdgas. In fact, these sets are reflected by the adjoint functors in the Quillen equivalence constructed in Section 2.4. The marked points are of degree -1 and thus could be MC elements of the curved Lie algebra. Hence twisting by such an element results in the Lie algebra having zero curvature as shown in Remark 2.20. Although, one should be conscious of the fact that there may be no solutions to the MC equation in a curved Lie algebra, contrary to the case of a dgla where 0 is always a solution to the MC equation.

For the sake of brevity, a marked curved Lie algebra is here onwards denoted by its underlying graded Lie algebra and set of marked points, omitting the differential and curvature.

Definition 2.24. *A morphism of marked curved Lie algebras is a curved Lie algebra morphism $(f,\alpha)\colon(\mathfrak{g},\{x_i\}_{i\in I})\to(\mathfrak{h},\{y_j\}_{j\in J})$ along with a map of sets $\phi\colon I\to J$ such that $f(x_i)+\alpha=y_{\phi(i)}$. The morphism $x_i\mapsto y_{\phi(i)}$ is called the induced map of sets. A morphism (f,α) is an isomorphism of marked curved Lie algebras if (f,α) is an isomorphism of curved Lie algebras and the induced map of sets is a bijection.*

It is important to notice that, since repetitions of marked points are allowed, two of marked curved Lie algebras must have

Definition 2.25. *The category of marked curved Lie algebras, denoted $\mathscr{L}$, is the category whose objects are marked curved Lie algebras and morphisms are those of Definition 2.24.*

Definition 2.26. *A marked curved Lie algebra $((\mathfrak{h},d,\omega),\{x_i\}_{i\in J})$ is a marked curved Lie subalgebra of $((\mathfrak{g},d,\omega),\{x_i\}_{i\in I})$ if $\mathfrak{h}\subseteq\mathfrak{g}$ as graded Lie algebras and $J\subseteq I$ as sets.*

Proposition 2.27. *Given a curved Lie algebra morphism*

$$(f,\alpha)\colon(\mathfrak{g},d_{\mathfrak{g}},\omega_{\mathfrak{g}})\to(\mathfrak{h},d_{\mathfrak{h}},\omega_{\mathfrak{h}}),$$

taking $x\in\mathfrak{g}_{-1}$ and denoting $(f,\alpha)(x)=y$, a strict morphism is given by

$$\begin{aligned}&(id_{\mathfrak{h}},-y)\circ(f,\alpha)\circ(id_{\mathfrak{g}},x)=(f,0)\colon\\ &\qquad\left(\mathfrak{g},d_{\mathfrak{g}}+\mathrm{ad}_x,\omega_{\mathfrak{g}}+dx+\frac{1}{2}[x,x]\right)\to\left(\mathfrak{h},d_{\mathfrak{h}}+\mathrm{ad}_y,\omega_{\mathfrak{h}}+dy+\frac{1}{2}[y,y]\right).\end{aligned}$$

Proof. $(id_{\mathfrak{h}},-y)\circ(f,\alpha)\circ(id_{\mathfrak{g}},x)=(f,-y+\alpha+f(x))=(f,0)$. ☐

Specialising Proposition 2.27 to marked curved Lie algebras, recall that given

$$(f,\alpha)\colon(\mathfrak{g},\{x_i\}_{i\in I})\to(\mathfrak{h},\{y_j\}_{j\in J}),$$

for all $i\in I$ there exists $j_i\in J$ such that $(f,\alpha)(x_i)=y_{j_i}$.

Definition 2.28. *Given a morphism* $(f,\alpha)\colon (\mathfrak{g},\{x_i\}_{i\in I}) \to (\mathfrak{h},\{y_j\}_{j\in J})$ *and fixing some* $k \in I$*, the rectification of* (f,α) *by the marked point* x_k *is the morphism*

$$\begin{aligned}(id_{\mathfrak{h}}, -(f(x_k)+\alpha))&\circ(f,\alpha)\circ(id_{\mathfrak{g}},x_k)\\ &= (f,0)\colon (\mathfrak{g}^{x_k},\{x_i - x_k\}_{i\in I}) \to (\mathfrak{h}^{f(x_k)-\alpha},\{y_j - (f(x_k)+\alpha)\}_{j\in J}),\end{aligned}$$

obtained analogously to Proposition 2.27.

Proposition 2.29. *Any commutative diagram of marked curved Lie algebras with an initial vertex (one such that there exists no morphisms into it, and there exists a unique morphism, up to commutativity, to every other vertex) is isomorphic to a commutative diagram with strict morphisms.*

Proof. Choosing a marked point in the initial vertex induces a choice of marked point at every other vertex by images. Rectifying each morphism by these marked points completes the proof. □

Proposition 2.29 implies that when considering decompositions of morphisms, commutative squares, and lifting problems it is sufficient to consider strict morphisms.

2.3.2 Small limits and colimits

The category $\mathscr{L}$ does not possess an initial object, and, therefore, an initial object is formally added to $\mathscr{L}$. The resulting category is denoted by $\mathscr{L}_*$. It is assumed this initial object has no marked points to echo the terminal object of $\mathrm{Prod}(\mathcal{E})$ having no components, see Remark 2.2. The zero curved Lie algebra $((0,0,0),\{0\})$ is the terminal object for $\mathscr{L}_*$.

Proposition 2.30. *The product in* $\mathscr{L}_*$ *is given by the Cartesian product of the underlying graded Lie algebras, the differential is given by specialising to each component, the curvature is given by the Cartesian product of the curvature of each component, and the Cartesian product of the sets of marked points. The projection morphisms are those morphisms projecting onto each component by the identity.*

Proof. It is a straightforward check. □

Proposition 2.31. *The equaliser of two curved morphisms* $(f,\alpha),(g,\beta)\colon (\mathfrak{g},\{x_i\}_{i\in I}) \to (\mathfrak{h},\{y_j\}_{j\in J})$ *is given by the initial object if* $\alpha \neq \beta$ *and by* $\{x \in \mathfrak{g}\colon f(x) = g(x)\}$ *if* $\alpha = \beta$.

Proof. If $\alpha \neq \beta$ then 0 is not in the equaliser, because $f(0)-\alpha \neq g(0)-\beta$. Therefore, the initial object is the only object satisfying the conditions of the equaliser.

If $\alpha = \beta$, then it is a straightforward exercise to show that the space $\{x \in \mathfrak{g}\colon f(x) = g(x)\}$ respects the differential and bracket inherited from $\mathfrak{g}$. □

Proposition 2.32. *The coproduct in the category of marked curved Lie algebras is easiest to describe in the binary case: let* $(\mathfrak{g},\{x_i\}_{i\in I})$ *and* $(\mathfrak{h},\{y_j\}_{j\in J})$ *be marked curved Lie algebras, the coproduct* $(\mathfrak{g},\{x_i\}_{i\in I})\coprod(\mathfrak{h},\{y_j\}_{j\in J})$ *has as its underlying graded*

graded Lie algebra the free product of graded Lie algebras on $\mathfrak{g} * \mathfrak{h} * \mathrm{L}\langle z\rangle$*, where* z *is a formal element of degree* -1*. The differential is given by the rules:* $d|_{\mathfrak{g}} = d_{\mathfrak{g}}$*,* $d|_{\mathfrak{h}} = d_{\mathfrak{h}} - ad_z$ *and* $dz = \omega_{\mathfrak{g}} - \omega_{\mathfrak{h}} + \frac{1}{2}[z, z]$*. The set of marked points is given by the union of sets* $\{x_i\}_{i\in I} \cup \{y_j + z\}_{j\in J}$*. The resulting space has curvature equal to that of* $\mathfrak{g}$*. The two inclusion morphisms are given by*

$$(id_{\mathfrak{g}}, 0)\colon (\mathfrak{g}, \{x_i\}_{i\in I}) \hookrightarrow (\mathfrak{g}, \{x_i\}_{i\in I}) \coprod (\mathfrak{h}, \{y_j\}_{j\in J})$$

and

$$(id_{\mathfrak{h}}, z)\colon (\mathfrak{h}, \{y_j\}_{j\in J}) \hookrightarrow (\mathfrak{g}, \{x_i\}_{i\in I}) \coprod (\mathfrak{h}, \{y_j\}_{j\in J}).$$

Proof. It is clear $(\mathfrak{g}, \{x_i\}_{i\in I}) \coprod (\mathfrak{h}, \{y_j\}_{j\in J})$ is a well defined marked curved Lie algebra. Given a marked curved Lie algebra $(\mathfrak{a}, \{z_k\}_{k\in K})$ and morphisms

$$(f_{\mathfrak{g}}, \alpha) : (\mathfrak{g}, \{x_i\}_{i\in I}) \to (\mathfrak{a}, \{z_k\}_{k\in K})$$

and

$$(f_{\mathfrak{h}}, \beta) : (\mathfrak{h}, \{y_j\}_{j\in J}) \to (\mathfrak{a}, \{z_k\}_{k\in K}),$$

define $(f, \alpha) : (\mathfrak{g}, \{x_i\}_{i\in I}) \coprod (\mathfrak{h}, \{y_j\}_{j\in J}) \to (\mathfrak{a}, \{z_k\}_{k\in K})$ by $f|_{\mathfrak{g}} = f_{\mathfrak{g}}$, $f|_{\mathfrak{h}} = f_{\mathfrak{h}}$, and $f(z) = \beta - \alpha$. Clearly (f, α) is a well defined morphism and the diagram

$$\begin{tikzcd}
 & (\mathfrak{a}, \{z_k\}_{k\in K}) & \\
(\mathfrak{g}, \{x_i\}_{i\in I}) \arrow[ur, "(f_{\mathfrak{g}}, \alpha)"] \arrow[r] & (\mathfrak{g}, \{x_i\}_{i\in I}) \coprod (\mathfrak{h}, \{y_j\}_{j\in J}) \arrow[u, "(f, \alpha)"'] & (\mathfrak{h}, \{y_j\}_{j\in J}) \arrow[l] \arrow[ul, "(f_{\mathfrak{h}}, \beta)"']
\end{tikzcd}$$

commutes. Uniqueness of this construction is a quick check. □

The coproduct given in Proposition 2.32 is similar to the disjoint product of [16]: it can be informally thought of as taking the disjoint union of the two marked curved Lie algebras, adding a formal MC element, and then twisting the copy of $\mathfrak{h}$ with this formal element to flatten its curvature.

Proposition 2.33. *The coequaliser of two parallel morphisms*

$$(f, \alpha), (g, \beta)\colon (\mathfrak{g}, \{x_i\}_{i\in I}) \to (\mathfrak{h}, \{y_j\}_{j\in J})$$

is the quotient of $\mathfrak{h}$ *by the ideal generated by* $f(x) - g(x)$ *and* $\alpha - \beta$*, for all* $x \in \mathfrak{g}$ *with the set of marked points being the quotient in a similar manner.*

Proof. A painless check. □

Proposition 2.34. *The category* $\mathscr{L}_*$ *has all small limits and colimits.*

Proof. $\mathscr{L}_*$ contains an initial object, terminal object, products, equalisers, coproducts, and coequalisers, this is sufficient cf. [17]. □

2.3.3 Model structure

The category of curved Lie algebras with strict morphisms, denoted $\mathcal{G}$, plays an important role in defining a model structure for the category $\mathscr{L}_*$, therefore the model structure for $\mathcal{G}$ given in [5] is recalled. Note that all morphisms of $\mathcal{G}$ are strict (by definition).

Definition 2.35. *A morphism $f\colon \mathfrak{g} \to \mathfrak{h}$ of $\mathcal{G}$ is said to be*

- *a weak equivalence if, and only if, f is either a quasi-isomorphism of dglas or any morphism between curved Lie algebras that have non-zero curvature.*
- *a fibration if, and only if, it is surjective;*
- *a cofibration if, and only if, it has the LLP with respect to acyclic fibrations.*

Remark 2.36. Since all the morphisms of $\mathcal{G}$ are strict, there are no morphisms from a curved Lie algebra with zero curvature to one with non-zero curvature.

Recall Definition 2.28: given a morphism $(f,\alpha)\colon (\mathfrak{g}, \{x_i\}_{i\in I}) \to (\mathfrak{h}, \{y_j\}_{j\in J})$ such that $(f,\alpha)(x_k) = y_{j_k}$ for some $k \in I$, the rectification by the marked point x_k gives a strict morphism $(f,0)\colon \mathfrak{g}^{x_k} \to \mathfrak{h}^{y_{j_k}}$. This latter morphism can clearly be considered as a morphism in the category $\mathcal{G}$ by forgetting marked points.

Definition 2.37. *A morphism $(f,\alpha)\colon (\mathfrak{g}, \{x_i\}_{i\in I}) \to (\mathfrak{h}, \{y_j\}_{j\in J})$ of $\mathscr{L}_*$ is said to be*

- *a weak equivalence if, and only if, it induces a bijection of the marked points and the rectification by each marked point is a weak equivalence of $\mathcal{G}$;*
- *a fibration if, and only if, the graded Lie algebra morphism f is surjective;*
- *a cofibration if, and only if, it has the LLP with respect to all acyclic fibrations.*

It is clear that each of the classes of morphism given in Definition 2.37 is closed under taking retracts, and they are also closed under rectification.

Proposition 2.38. *A morphism of $\mathscr{L}_*$ belongs to any one of the classes of Definition 2.37 if, and only if, the rectification by every marked point (still being considered as a morphism in $\mathscr{L}_*$) belongs to the same class.*

Proof. It is straightforward to show the statement holds for weak equivalences and fibrations. Any cofibration has the LLP with respect to any acyclic fibration, and since a rectification can be viewed as a retract of the original morphism it also has the LLP with respect to any acyclic fibration. $\square$

Proposition 2.39. *A morphism of $\mathscr{L}_*$ is a weak equivalence if, and only if, it induces a bijection of marked points and the rectification by each marked point of the domain considered as a morphism of $\mathcal{G}$ is a weak equivalence. A morphism of $\mathscr{L}_*$ is a cofibration (resp. fibration) in Definition 2.37 if, and only if, the rectification by each marked point of the domain considered as a morphism of $\mathcal{G}$ is a cofibration (resp. fibration) in Definition 2.35.*

Proof. This is vacuously true for weak equivalences and almost as easily seen for fibrations. Taking the rectification of a cofibration and viewing it as a morphism of $\mathcal{G}$, it must have the LLP with respect to all acyclic fibrations of $\mathcal{G}$ since it has the LLP with respect to all acyclic fibrations of $\mathscr{L}_*$, and in particular those strict morphisms where the sets of marked points are assumed to be the singleton set $\{0\}$, i.e. where the morphism is already of the form of one in $\mathcal{G}$. The converse statement is almost analogous. $\square$

Proposition 2.40. *Given two composable weak equivalences, (f, α) and (g, β), if any two of (f, α), (g, β) and $(g \circ f, \beta + g(\alpha))$ are weak equivalences then so is the third.*

Proof. It is clear to see that if any two of the compositions induce bijections upon the sets of marked points then so must the third. Further, if any two of them are weak equivalences of $\mathcal{G}$ after rectification then again so must the third since the two of three property holds for $\mathcal{G}$. $\square$

By definition, any cofibration has the LLP with respect to all acyclic fibrations, accordingly it remains to show that any acyclic cofibration has the LLP with respect to all fibrations.

Proposition 2.41. *The acyclic cofibrations are precisely the morphisms that have the LLP with respect to the fibrations.*

Proof. This follows from Proposition 2.38, Proposition 2.39, and that the statement holds in the CMC $\mathcal{G}$. $\square$

In order to show a morphism of $\mathscr{L}_*$ can always be factorised as the composition of an acyclic cofibration followed by a fibration it is necessary to introduce the disk of a marked curved Lie algebra. This can informally be thought of as attaching cells to the curved Lie algebra to make it acyclic, despite curved Lie algebras not necessarily being complexes.

Definition 2.42. *Let $\mathfrak{g}_-$ denote the homogeneous elements of the curved Lie algebra $\mathfrak{g}$. Denote by $((D_\mathfrak{g}, \bar{d}, \bar{\omega}), \{u_{x_i}\}_{i\in I})$ the curved Lie algebra given by $D_\mathfrak{g} = \mathrm{L}\langle \bar{\omega}, u_g, v_g : g \in \mathfrak{g}_-\rangle$ subject only to the relations $\bar{d}u_g = v_g$ and $\bar{d}v_g = [\bar{\omega}, u_g]$, where $|u_g| = |g|$ for all $g \in \mathfrak{g}_-$. In particular, $\bar{\omega}$ is the curvature of $D_\mathfrak{g}$.*

Even in the case of a marked curved Lie algebra with zero curvature (i.e. a dgla with a set of marked points) the curved Lie algebra $D_\mathfrak{g}$ is not an acyclic complex, because a non-zero curvature element always exists as a generator.

Remark 2.43. The canonical strict morphism $(D_\mathfrak{g}, \{u_{x_i}\}_{i\in I}) \to (\mathfrak{g}, \{x_i\}_{i\in I})$ given by $u_g \mapsto g$, $v_g \mapsto d_\mathfrak{g} g$, and $\bar{\omega} \mapsto \omega_\mathfrak{g}$ for all $g \in \mathfrak{g}_-$ is a fibration.

Proposition 2.44. *An acyclic cofibration is given by the canonical strict morphism*

$$(\mathrm{L}\langle \bar{\omega}, u_{x_i}, v_{x_i}\rangle, \{u_{x_i}\}_{i\in I}) \to (D_\mathfrak{g}, \{x_i\}_{i\in I}).$$

Proof. Let $(g, 0) : (\mathfrak{m}, \{m_j\}_{j\in J}) \to (\mathfrak{n}, \{n_k\}_{k\in K})$ be a fibration and

$$\begin{tikzcd}
(\mathrm{L}\langle\bar\omega, u_{x_i}, v_{x_i} : i\in I\rangle, \{u_{x_i}\}_{i\in I}) \arrow[r, "{(f,0)}"] \arrow[d] & (\mathfrak{m}, \{m_j\}_{j\in J}) \arrow[d, "{(g,0)}"] \\
(D_{\mathfrak{g}}, \{u_{x_i}\}_{i\in I}) \arrow[r] & (\mathfrak{n}, \{n_k\}_{k\in K})
\end{tikzcd}$$

be a commutative diagram. The diagram has been assumed to be strict by Proposition 2.29. A lift $(h,0): (D_{\mathfrak{g}}, \{x_i\}_{i\in I}) \to (\mathfrak{m}, \{m_j\}_{j\in J})$ is defined as follows. For all $\bar\omega, u_{x_i}, v_{x_i} \in D_{\mathfrak{g}}$, $(h,0)$ has the same action as $(f,0)$. For all $u_x \in D_{\mathfrak{g}}$ that are not in $(\mathrm{L}\langle\bar\omega, u_{x_i}, x_{g_i}\rangle, \{u_{x_i}\}_{i\in I})$ there exists some $n\in\mathfrak{n}$ such that $u_x \mapsto n$ and since $(g,0)$ is surjective one can choose $m\in\mathfrak{m}$ such that $m\mapsto n$. Thus letting $h(u_x)=m$ completely defines a lift. ☐

Proposition 2.45. *Given a morphism $(\mathfrak{g}, \{x_i\}_{i\in I}) \to (\mathfrak{h}, \{y_j\}_{j\in J})$ of $\mathscr{L}_*$ it can be factorised as the composition of an acyclic cofibration followed by a fibration.*

Proof. Consider the pushout

$$\begin{tikzcd}
(\mathrm{L}\langle\bar\omega, u_{y_j}, v_{y_j} : j\in J\rangle, \{g_j\}_{j\in J}) \arrow[r] \arrow[d] & (\mathfrak{g}, \{x_i\}_{i\in I}) \arrow[d] \\
(D_{\mathfrak{h}}, \{y_j\}_{j\in J}) \arrow[r] & \left(D_{\mathfrak{h}} \coprod_{\mathrm{L}\langle\bar\omega, u_{y_j}, v_{y_j}: j\in J\rangle} \mathfrak{g}, \{z_i\}_{i\in I}\right).
\end{tikzcd}$$

Since it is the pushout of an acyclic cofibration the right hand morphism is also an acyclic cofibration. Furthermore, the universal property of a pushout provides a morphism

$$\left(D_{\mathfrak{h}} \coprod_{\mathrm{L}\langle\bar\omega, u_{y_j}, v_{y_j}: j\in J\rangle} \mathfrak{g}, \{z_i\}_{i\in I}\right) \to (\mathfrak{h}, \{y_j\}_{j\in J})$$

that is surjective and such that the composition

$$(\mathfrak{g}, \{x_i\}_{i\in}) \to \left(D_{\mathfrak{h}} \coprod\nolimits_{\mathrm{L}\langle\bar\omega, u_{y_j}, v_{y_j}: j\in J\rangle} \mathfrak{g}, \{z_i\}_{i\in I}\right) \to (\mathfrak{h}, \{y_j\}_{j\in J})$$

is equal to the original morphism. ☐

Proposition 2.46. *Any morphism in $\mathscr{L}_*$ can be factorised into a cofibration followed by an acyclic fibration.*

Proof. Let $(f,\alpha)\colon (\mathfrak{g}, \{x_i\}_{i\in I}) \to (\mathfrak{h}, \{y_j\}_{j\in J})$ be any morphism in $\mathscr{L}_*$. For some $k\in I$, let $(f,\alpha)(x_k) = y_{j_k}$ and consider the strict morphism

$$(f,0)\colon (\mathfrak{g}^{x_k}, \{x_i - x_k\}_{i\in I}) \to (\mathfrak{h}^{y_{j_k}}, \{y_j - y_{j_k}\}_{j\in J})$$

(obtained by rectification cf. Definition 2.28) as a morphism in $\mathcal{G}$ by forgetting the marked points. Working now in the category $\mathcal{G}$, there exists the factorisation

$$\begin{tikzcd}
& \mathfrak{m}_0 \arrow[dr, "\rho_0"] & \\
\mathfrak{g}^{x_k} \arrow[ur, "\iota_0"] \arrow[rr, "f"'] & & \mathfrak{h}^{y_{j_k}},
\end{tikzcd}$$

where ι_0 and ρ_0 are a cofibration and an acyclic fibration of $\mathcal{G}$ resp. In many cases, this factorisation along with a smart choice of marked points for $\mathfrak{m}$ and untwisting leads to the required factorisation of (f, α) in the category $\mathscr{L}_*$. However, in the general case one can 'glue' the images of the marked points of $\mathfrak{g}^{x_k}$ under ι_0 together as follows. Taking two marked points x_0 and x_1 in $\mathfrak{g}$, one takes the pushout (in $\mathcal{G}$)

$$\begin{tikzcd}
\mathrm{L}\langle x\rangle \arrow[r] \arrow[d] & \mathfrak{m}_0 \arrow[d, "\iota_1"] \arrow[ddr, bend left, "\rho_0"] & \\
\mathfrak{m}_0 \arrow[r] \arrow[drr, bend right, "\rho_0"'] & \mathfrak{m}_1 \arrow[dr, dotted, "\rho_1"] & \\
& & \mathfrak{h}^{y_{j_k}},
\end{tikzcd}$$

where the two morphisms $\mathrm{L}\langle x\rangle \to \mathfrak{m}_0$ are the inclusions of the two marked points x_0 and x_1 (which are cofibrations of $\mathcal{G}$, c.f. [5]). Being the pushout of a cofibration ι_1 is itself a cofibration.

With these constructions, the morphism $f\colon \mathfrak{g} \to \mathfrak{h}$ can be factorised as $\rho_1 \circ \tilde{\iota}_1$, where $\tilde{\iota}_1 := \iota_1 \circ \iota_0$ is a cofibration and ρ_1 is a fibration. Moreover, the marked points x_1 and x_2 have the same image under $\iota_1 \circ \iota_0$.

Assuming now that one has chosen marked points $x_0, \dots x_{i-1}$ and factorisation $f = \rho_{i-1} \circ \tilde{\iota}_{i-1}$, where $\tilde{\iota}_{i-1} = \iota_{i-1} \circ \cdots \circ \iota_0$ is a composition of cofibrations and ρ_{i-1} is (not necessarily acyclic) fibration. For a different marked point x_i, take the pushout (in $\mathcal{G}$)

$$\begin{tikzcd}
\mathrm{L}\langle x\rangle \arrow[r] \arrow[d] & \mathfrak{m}_{i-1} \arrow[d, "\iota_i"] \arrow[ddr, bend left, "\rho_{i-1}"] & \\
\mathfrak{m}_{i-1} \arrow[r] \arrow[drr, bend right, "\rho_{i-1}"'] & \mathfrak{m}_i \arrow[dr, dotted, "\rho_i"] & \\
& & \mathfrak{h}^{y_{j_k}},
\end{tikzcd}$$

where the vertical $\mathrm{L}\langle x\rangle \to \mathfrak{m}_{i-1}$ is the composition of the inclusion of x_0 into $\mathfrak{m}$ and ι_{i-1}, and the horizontal $\mathrm{L}\langle x\rangle \to \mathfrak{m}_{i-1}$ is the composition of the inclusion of x_1 into $\mathfrak{m}$ and ι_{i-1}. Again, because ι_i is a pushout of a cofibration, it is itself a cofibration.

The morphism $f\colon \mathfrak{g} \to \mathfrak{h}$ can now be factorised as $\rho_i \circ (\iota_i \circ \cdots \circ \tilde{\iota}_i)$, where $\tilde{\iota}_i := \iota_i \circ \cdots \circ \iota_1 \circ \iota_0$ is a cofibration and ρ_i is a fibration. Moreover, the marked points $x_1, x_2, \dots x_i$ have the same image under $\iota_i \circ \cdots \circ \iota_1 \circ \iota$.

Taking the colimit of the $\mathfrak{m}_i$, one arrives at an object $\mathfrak{m}_\infty$. Moreover, one has a cofibration $\tilde{\iota}_\infty \colon \mathfrak{g} \to \mathfrak{m}_\infty$ and a fibration $\rho_\infty \colon \mathfrak{m}_\infty \to \mathfrak{h}$, such that $f = \rho_\infty \circ \tilde{\iota}_\infty$.

To show that ρ_∞ is an acyclic fibration, one must show that there exists a lift in the following solid diagram:

$$\begin{tikzcd}
\mathfrak{a} \arrow[rr] \arrow[dd] \arrow[dr, dotted] & & \mathfrak{m}_\infty \arrow[dd, "\rho_\infty"] \\
& \mathfrak{m} \arrow[ur, dotted] \arrow[dr, dotted, "\rho"] & \\
\mathfrak{b} \arrow[rr] \arrow[ur, dotted] & & \mathfrak{h},
\end{tikzcd}$$

where the morphism $\mathfrak{a} \to \mathfrak{b}$ is a cofibration. The morphism $\mathfrak{a} \to \mathfrak{m}_\infty$ factorises through $\mathfrak{m}$ constructing the upper two dotted morphisms in the above diagram and creating the lift. Since p is an acyclic fibration, the final dotted morphism $\mathfrak{b} \to \mathfrak{m}$ is constructed, and hence the lift $\mathfrak{b} \to \mathfrak{m}_\infty$ is constructed.

The marked points of $\mathfrak{g}$ all have the same image under $\tilde{\iota}_\infty$ and so one can choose preimages of the marked points of $\mathfrak{h}$ under the surjective morphism ρ_∞ to construct morphisms of $\mathscr{L}_*$ with the required properties. Then, after untwisting, one has factorised the original morphism. □

Proposition 2.47. *$\mathscr{L}_*$ is a CMC with the model structure of Definition 2.37.*

Proof. It follows from the preceding results. □

2.4 Quillen equivalence

This section first recalls the Quillen equivalence of [5, Section 5] between $\mathcal{E}$ and $\mathcal{G}$. A nice feature of this construction is the functors interchange curved Lie algebras with non-zero curvature and acyclic algebras, as well as uncurved Lie algebras (or dglas) and Hinich algebras. Using the Quillen equivalence of $\mathcal{E}$ and $\mathcal{G}$ as a foundation, a Quillen equivalence between $\mathscr{L}_*$ and $\mathcal{A}$ (via $\mathrm{Prod}(\mathcal{E})$) is constructed.

As remarked earlier, when passing from $\mathcal{E}$ to $\mathcal{A}$ (via $\mathrm{Prod}(\mathcal{E})$) it is possible to extend the Quillen equivalence by passing from $\mathcal{G}$ to its category of formal coproducts: cf. [5]. However, this result is unnatural due to its asymmetry. Here—by replacing the category of formal coproducts with the category of curved Lie algebras with curved morphisms—a more symmetric result is obtained.

Definition 2.48. *The functor $\mathrm{CE} \colon \mathcal{G} \to \mathcal{E}$ is given by sending a curved Lie algebra $(\mathfrak{g}, d, \omega)$ to the local pseudo-compact cdga $\hat{S}\Sigma^{-1}\mathfrak{g}^*$ with differential induced via the Leibniz rule and continuity by its restriction to $\Sigma^{-1}\mathfrak{g}^*$. The restriction is given by the sum of three components:*

- *$\Sigma^{-1}\mathfrak{g}^* \to k$ given by the evaluation of the curvature ω of $\mathfrak{g}$,*
- *$\Sigma^{-1}\mathfrak{g}^* \to \Sigma^{-1}\mathfrak{g}^*$ given by pre-composition of the differential d, and*

- $\Sigma^{-1}\mathfrak{g}^* \to S^2\Sigma^{-1}\mathfrak{g}^*$ *given by the pre-composition of the bracket on* $\mathfrak{g}$.

Remark 2.49. If the curvature of a curved Lie algebra is 0, then the first part of the differential in Definition 2.48 disappears recovering the construction of Hinich [11].

Proposition 2.50. *For any given curved Lie algebra* $(\mathfrak{g}, d, \omega) \in \mathcal{G}$*, the local pseudo-compact cdga* $\mathrm{CE}(\mathfrak{g}, d, \omega)$ *is cofibrant in* $\mathcal{E}$.

Proof. Every object of $\mathcal{G}$ is fibrant and CE maps fibrations to cofibrations. □

Definition 2.51. *The functor* $\mathcal{L}\colon \mathcal{E} \to \mathcal{G}$ *is given by sending a local pseudo-compact cdga* A *to the free graded Lie algebra* $\mathrm{L}\,\Sigma^{-1}A^*$ *with the differential induced via the Leibniz rule and its restriction to* $\Sigma^{-1}A^*$*, given by the sum of*

- *the pre-composition of the differential of* A*, and*
- *the unique derivation of* $\mathrm{L}\,\Sigma^{-1}A^*$ *whose restriction to* $\Sigma^{-1}A^*$ *is given by*

$$m^* - 1 \otimes \mathrm{id}_A + \mathrm{id}_A \otimes 1 : A^* \to A^* \otimes A^*,$$

  where m *is the multiplication on* A.

The curvature of $\mathcal{L}(A)$ *is given by the morphism* $k \to \Sigma^{-1}A^*$ *induced by the composition of the augmentation of* A *with the differential of* A.

Remark 2.52. If A is a Hinich algebra the image of the differential on A is its maximal ideal m_A which is precisely the kernel of the augmentation of A and so $\mathcal{L}(A)$ has zero curvature.

Proposition 2.53. *For any given local pseudo-compact cdga,* A*, the curved Lie algebra* $\mathcal{L}(A)$ *is a cofibrant object in* $\mathcal{G}$.

Proof. Every object of $\mathcal{E}$ is fibrant and $\mathcal{L}$ maps fibrations to cofibrations. □

Moving from the categories $\mathcal{G}$ and $\mathcal{E}$ to the categories $\mathscr{L}_*$ and $\mathrm{Prod}(\mathcal{E})$, the functors $\mathcal{L}$ and CE are extended as follows.

Definition 2.54. *A functor* $\widetilde{\mathrm{CE}}\colon \mathscr{L}_* \to \mathrm{Prod}(\mathcal{E})$ *is given by*

$$\widetilde{\mathrm{CE}}((\mathfrak{g}, d, \omega), \{x_i\}_{i\in I}) := \prod_{i\in I} \mathrm{CE}(\mathfrak{g}, d + ad_{x_i}, \omega + dx_i + \frac{1}{2}[x_i, x_i]).$$

Given a morphism of marked curved Lie algebras, $(f, \alpha)\colon (\mathfrak{g}, \{x_i\}_{i\in I}) \to (\mathfrak{h}, \{y_j\}_{j\in J})$*, let the* k*th component of the morphism* $\widetilde{\mathrm{CE}}(\mathfrak{h}, \{y_j\}_{j\in J}) \to \widetilde{\mathrm{CE}}(\mathfrak{g}, \{x_i\}_{i\in I})$ *be given by applying* CE *to the rectification* $\mathfrak{g}^{x_k} \to \mathfrak{h}^{y_{j_k}}$ *considered as a morphism in* $\mathcal{G}$.

Remark 2.55. The functor $\widetilde{\mathrm{CE}}$ takes a marked Lie algebra to a formal product of Hinich algebras and acyclic algebras depending upon whether the marked point is a MC element or not.

Definition 2.56. *A functor* $\widetilde{\mathcal{L}}\colon \operatorname{Prod}(\mathcal{E}) \to \mathscr{L}_*$ *is given by*

$$\widetilde{\mathcal{L}}\left(\prod_{i\in I} A_i\right) := \coprod_{i\in I}(\mathcal{L}(A_i), \{0\}),$$

where $\coprod$ *is the coproduct of the category* $\mathscr{L}_*$*, cf. Proposition 2.32. Given any morphism* $f\colon \prod_{i\in I} A_i \to \prod_{j\in J} B_j$ *of* $\operatorname{Prod}(\mathcal{E})$*, let the morphism*

$$\coprod_{j\in J}(\mathcal{L}(B_j), \{0\}) \to \coprod_{i\in I}(\mathcal{L}(A_i), \{0\})$$

be obtained by combining each of the components

$$(\mathcal{L}(f_j), 0)\colon (\mathcal{L}(B_j), \{0\}) \to (\mathcal{L}(A_{i_j}), \{0\}).$$

Proposition 2.57. *The functor* $\widetilde{\mathcal{L}}$ *is left adjoint to the functor* $\widetilde{\mathrm{CE}}$.

Proof. The statement of the proposition is equivalent to the statement there exists an isomorphism of the sets

$$\operatorname{Hom}_{\operatorname{Prod}(\mathcal{E})}\left(\widetilde{\mathrm{CE}}(\mathfrak{g}, \{x_i\}_{i\in I}), \prod_{j\in J} A_j\right) \cong \operatorname{Hom}_{\mathscr{L}_*}\left(\widetilde{\mathcal{L}}\left(\prod_{j\in J} A_j\right), (\mathfrak{g}, \{x_i\}_{i\in I})\right).$$

Given a morphism $f\colon \prod_{i\in I}\mathrm{CE}(\mathfrak{g}^{x_i}) \to \prod_{j\in J} A_j$ of $\operatorname{Prod}(\mathcal{E})$, applying the functor $\mathcal{L}$ to the components, f_j, the morphisms $\mathcal{L}(f_j)\colon \mathcal{L}(A_j) \to \mathcal{L}(\mathrm{CE}(\mathfrak{g}^{x_{i_j}}))$ are obtained. Since the functors $\mathcal{L}$ and CE are adjoint, these morphisms are equivalent to the morphisms $\mathcal{L}(f_j)\colon \mathcal{L}(A_j) \to \mathfrak{g}^{x_{i_j}}$. Therefore, by the universal property of coproducts, for each $j \in J$ there exists the commutative diagram

$$\begin{tikzcd}
 & \mathfrak{g} \\
\mathcal{L}(A_j) \arrow[r] \arrow[ur, "(id, x_{i_j}) \circ \mathcal{L}(f_j)"] & \coprod_{j\in J}\mathcal{L}(A_j). \arrow[u]
\end{tikzcd}$$

Clearly, the morphisms are marked and therefore it has been shown that $\widetilde{\mathcal{L}}(f)$ is a morphism in $\operatorname{Hom}_{\mathscr{L}_*}(\widetilde{\mathcal{L}}(\prod_{j\in J} A_j), (\mathfrak{g}, \{x_i\}_{i\in I}))$.

Conversely, given a morphism $f\colon \widetilde{\mathcal{L}}(\prod_{j\in J} A_j) \to (\mathfrak{g}, \{x_i\}_{i\in I})$ one can easily use the fact that CE and $\mathcal{L}$ are an adjoint pair to show—in a similar manner to the preceding—that after applying the functor $\widetilde{\mathrm{CE}}$, a morphism equivalent to

$$\prod_{i\in I}\mathrm{CE}(\mathfrak{g}^{x_i}) \to \prod_{j\in J} A_j$$

is obtained. ∎

Proposition 2.58. *The functors* $\widetilde{\mathcal{L}}$ *and* $\widetilde{\mathrm{CE}}$ *both map fibrations to cofibrations.*

Proof. Given a fibration $f\colon \prod_{i\in I} A_i \to \prod_{j\in J} B_j$ of $\mathrm{Prod}(\mathcal{E})$, for each $i \in I$ the morphism $f^i\colon A_i \to \prod_{\{j:j\mapsto i\}} B_j$ is a fibration of $\mathcal{E}$. Hence for each $i \in I$ the morphism $\mathcal{L}(f^i)$ is a cofibration. Let

$$\begin{tikzcd}
\left(\mathcal{L}\left(\prod_{\{j:j\mapsto i\}} B_j\right), \{0\}\right) \arrow[r] \arrow[d] & \coprod_{j\in J}(\mathcal{L}(B_j), \{0\}) \arrow[r] \arrow[d] & (X, \{\xi_k\}_{k\in K}) \arrow[d] \\
(\mathcal{L}(A_i), \{0\}) \arrow[r] & \coprod_{i\in I}(\mathcal{L}(A_i), \{0\}) \arrow[r] & (Y, \{\nu_k\}_{k\in K}),
\end{tikzcd}$$

be a commutative diagram in $\mathscr{L}_*$ and $(X, \{\xi_k\}_{k\in K)}) \to (Y, \{\nu_k\}_{k\in K})$ be an acyclic fibration. Forgetting the marked points, the left hand morphism is a cofibration of $\mathcal{G}$ and so (by remembering the marked points again and) after rectification of the diagram (cf. Proposition 2.29) there exists some lift $\mathcal{L}(A_i) \to X$ in $\mathcal{G}$, for each $i \in I$. Using the universal property of coproducts, there exists a morphism $\coprod_{i\in I}\mathcal{L}(A_i) \to X$ making everything commute. Hence, by undoing the rectifications and taking care with the marked points, the morphism $\widetilde{\mathcal{L}}(f)$ is a cofibration. The proof for $\widetilde{\mathrm{CE}}$ is similar. ☐

Proposition 2.59. *The functors* $\widetilde{\mathrm{CE}}$ *and* $\widetilde{\mathcal{L}}$ *preserve weak equivalences.*

Proof. Given a weak equivalence $(f, \alpha)\colon (\mathfrak{g}, \{x_i\}_{i\in I}) \to (\mathfrak{h}, \{y_i\}_{i\in I})$ of $\mathscr{L}_*$, it is evident by definition that for all $i \in I$ the morphism $\mathrm{CE}(\mathfrak{h}^{y_i}) \to \mathrm{CE}(\mathfrak{g}^{x_i})$ is a weak equivalence of $\mathcal{E}$, whence $\widetilde{\mathrm{CE}}(f, \alpha)$ is a weak equivalence.

Suppose $f\colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is a weak equivalence of $\mathrm{Prod}(\mathcal{E})$. Clearly $\widetilde{\mathcal{L}}(f)$ induces a bijection upon marked points. Now, f has components $f_i\colon A_i \to B_i$ which are weak equivalences in $\mathcal{E}$, hence $(\mathcal{L}(f_i), 0)$ is a weak equivalence for each i. Since $\widetilde{\mathcal{L}}(A)$ is cofibrant for all A (by Proposition 2.58) one can conclude the coproducts $\coprod_i \widetilde{\mathcal{L}}(A_i)$ and $\coprod_i \widetilde{\mathcal{L}}(B_i)$ descend to the homotopy category, where each morphism $(\mathcal{L}(f_i), 0)$ is an isomorphism, which implies $\widetilde{\mathcal{L}}(f)$ is an isomorphism and so $\widetilde{\mathcal{L}}(f)$ is a weak equivalence. ☐

Theorem 2.60. *The categories* $\mathrm{Prod}(\mathcal{E})$ *and* $\mathscr{L}_*$ *are Quillen equivalent.*

Proof. One applies the preceding results with [7, Theorem 9.7]. ☐

2.5 Deformation functors over pseudo-compact cdgas

An application of the above constructions to algebraic deformation theory is contained within this section; more precisely, the theory is extended to deformation functors over (not necessarily local) pseudo-compact cdgas. The main result, Theorem 2.70, shows these deformation functors are representable in $\mathrm{Ho}(\mathcal{A})$. The story of the approach to deformation theory via dglas has its roots with Nijenhuis [20, 21], Drinfeld [6], and Kontsevich [13] among others, who noticed that deformation theories in characteristic zero are governed by the MC elements of dglas.

Here the category of sets is denoted by $\mathscr{S}$ and the notation (A, m_A) refers to a local pseudo-compact cdga A with maximal ideal given by m_A.

2.5.1 MC elements and local pseudo-compact cdga morphisms

Fixing a curved Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$ and a local pseudo-compact cdga, (A, m_A), recall that the tensor product $(\mathfrak{g}\hat{\otimes}A, d, \omega_{\mathfrak{g}}\hat{\otimes}1)$ is a well defined curved Lie algebra. The MC elements of the tensor product $\mathfrak{g}\hat{\otimes}A$ can be considered, in the usual sense, as those elements, $x \in \mathfrak{g}\hat{\otimes}A$, solving the MC equation: $\omega_{\mathfrak{g}}\hat{\otimes}1 + dx + \frac{1}{2}[x,x] = 0$. However, for the construction of the deformation functor given later in this section, only the subset of those MC elements belonging to $\mathfrak{g}\hat{\otimes}m_A$ are examined.

Definition 2.61. *Let $\widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A)$ denote the set of MC elements belonging to the subset $\mathfrak{g}\hat{\otimes}m_A$.*

In fact $\widetilde{\mathrm{MC}}$ can be viewed as a bifunctorial construction from the product category of curved Lie algebras with local pseudo-compact cdga into the category of sets.

Proposition 2.62. *The functors $(\mathfrak{g}, A) \mapsto \mathrm{Hom}(\mathrm{CE}(\mathfrak{g}), A))$ and $(\mathfrak{g}, A) \mapsto \widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A)$ are naturally isomorphic.*

Proof. A degree -1 element in $\mathfrak{g}\hat{\otimes}m_A$ is a degree 0 element in $\Sigma^{-1}\mathfrak{g}\hat{\otimes}m_A$, further this element determines (and is determined by) a continuous linear morphism $(\Sigma^{-1}\mathfrak{g})^* \to m_A$. In turn, this continuous linear morphism determines (and is determined) by a morphism of local pseudo-compact commutative graded algebras: $\mathrm{CE}(\mathfrak{g}) \to A$. The condition that this morphism commutes with the differential is precisely the condition that the element belongs to $\widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A)$. □

The equivalence of functors in Proposition 2.62 motivates the notion of a homotopy between elements of the set $\widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A)$.

Definition 2.63.

- *Let $k[z, dz]$ denote the free unital cdga on the generators z and dz with degrees 0 and -1 respectively, with the differential given by $d(z) = dz$.*
- *Given $A \in \mathcal{E}$, let $A[z, dz]$ be the cdga given by $A\hat{\otimes}k[z, dz]$, and $|_0, |_1 \colon A[z, dz] \to A$ be the quotient morphisms given by setting z to 0 or 1, respectively.*

Remark 2.64. $k[z, dz]$ is the familiar de Rham algebra of forms on the unit interval.

Definition 2.65. *Two elements $\xi, \eta \in \widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A)$ are said to be homotopic if there exists an element $h \in \widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A[z, dz])$ such that $h|_0 = \xi$ and $h|_1 = \eta$.*

Using the isomorphism of functors in Proposition 2.62, $h \in \widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A[z, dz])$ corresponds to a morphism of local pseudo-compact cdgas $h\colon \mathrm{CE}(\mathfrak{g}) \to A[z, dz]$ which when specialising to $z = 0$ and $z = 1$ restricts to the morphisms corresponding to $\xi, \eta \in \widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A)$, respectively. Thus, a homotopy of MC elements is nothing more than a Sullivan homotopy between the corresponding morphisms of local pseudo-compact cdgas, cf. Appendix 2.A.

Definition 2.66. *Let $\widetilde{\mathcal{MC}}(\mathfrak{g}\hat{\otimes}A)$ denote the set of equivalence classes of $\widetilde{\mathrm{MC}}(\mathfrak{g}\hat{\otimes}A)$ modulo the homotopy relation. This set is called the Maurer-Cartan moduli set of the curved Lie algebra $\mathfrak{g}$ with coefficients in the local pseudo-compact cdga A.*

Remark 2.67. The MC moduli set can be obtained in several ways in slightly specialised cases. For example, it can be seen as the set of connected components of the MC simplicial set; see [16], it should be noted, though, that pseudo-compact dglas are considered in op. cit. Additionally, a result of Schlessinger and Stasheff [26] states that for a pronilpotent dgla, two MC elements are homotopic if, and only if, they are gauge equivalent: see [3] or [29] for a discussion and a proof.

The homotopy relation of MC elements and the Sullivan homotopy of morphisms of pseudo-compact local cdgas are so closely related that Proposition 2.62 also holds on the level of homotopy.

Proposition 2.68. *Given a curved Lie algebra $\mathfrak{g}$ and a local pseudo-compact cdga (A, m_A), the functors*

$$(\mathfrak{g}, A) \mapsto \mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(\mathrm{CE}(\mathfrak{g}), A)) \text{ and } (\mathfrak{g}, A) \mapsto \widetilde{\mathcal{MC}}(\mathfrak{g}\hat{\otimes}A)$$

are naturally isomorphic.

Proof. The two definitions of homotopy are equivalent since $\mathrm{CE}(\mathfrak{g})$ is cofibrant (see Theorem 2.79), and so the result follows. □

2.5.2 Deformations over pseudo-compact cdgas

For a dgla $\mathfrak{g}$ and a local Artin algebra (A, m_A), recall the tensor $\mathfrak{g}\hat{\otimes}m_A$ possesses a well-defined dgla structure. Shadowing the inspiration of Drinfeld, Kontsevich, et al., where the deformation functor associated to $\mathfrak{g}$ is defined as the functor mapping (A, m_A), to the set of MC elements of $\mathfrak{g}\hat{\otimes}m_A$ modulo gauge equivalence, the following definition is made.

Definition 2.69. *Fixing a marked curved Lie algebra $(\mathfrak{g}, \{x_i\}_{i\in I}) \in \mathscr{L}_*$, a deformation functor $\mathrm{Def}_{(\mathfrak{g},\{x\}_{i\in I})} \colon \mathscr{A} \to \mathscr{S}$ is given by*

$$A = \prod_{j\in J} A_j \mapsto \prod_{j\in J}\coprod_{i\in I} \widetilde{\mathcal{MC}}\left(\mathfrak{g}^{x_i}\hat{\otimes}A_j\right).$$

Using Proposition 2.15 and Proposition 2.68 one arrives at the following theorem.

Theorem 2.70. *The deformation functor $\mathrm{Def}_{(\mathfrak{g},\{x_i\}_{i\in I})} \colon \mathscr{A} \to \mathscr{S}$ is representable in the homotopy category of $\mathcal{A}$ by $\widetilde{\mathrm{CE}}(\mathfrak{g}, \{x_i\}_{i\in I})$.*

Proof. Let $A \in \mathcal{A}$,

$$\begin{aligned}\mathrm{Def}_{(\mathfrak{g},\{x_i\}_{i\in I})}(A) &= \prod_{j\in J}\coprod_{i\in I} \widetilde{\mathcal{MC}}\left(\mathfrak{g}^{x_i}\hat{\otimes}A_j\right)\\ &\cong \prod_{j\in J}\coprod_{i\in I} \mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}\left(\mathrm{CE}(\mathfrak{g}^{x_i}), A_j\right)\\ &\cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{A})}\left(\widetilde{\mathrm{CE}}(\mathfrak{g}, \{x_i\}_{i\in I}), A\right).\end{aligned}$$

□

Corollary 2.71. *The functor* $\mathrm{Def}_{(\mathfrak{g},\{x_i\}_{i\in I})}$ *is homotopy invariant in both the marked Lie algebra and pseudo-compact cdga variables.*

Proof. $\widetilde{\mathrm{CE}}$ is homotopy invariant and $\mathrm{Def}_{(\mathfrak{g},\{x_i\}_{i\in I})}$ is representable in the homotopy category of pseudo-compact cdgas, thus the result follows. □

The functor given in Definition 2.69 relies upon the decomposition of a pseudo-compact cdga into the product of local ones. By recalling a pair of Quillen adjoint functors originally given in [5] a more satisfying equivalent definition that does not rely upon a decomposition can be given in the case where the curved Lie algebra possesses one marked point. Let the functor $F\colon \mathcal{E} \to \mathrm{Prod}(\mathcal{E})$ be given by the inclusion of a local pseudo-compact cdga to the formal product over a singleton set, and let $G\colon \mathrm{Prod}(\mathcal{E}) \to \mathcal{E}$ be given by taking the formal product to the actual product of $\mathcal{E}$. It is proven in [5] these functors are Quillen adjoint. As defined here, G uses the decomposition of a pseudo-compact cdga into the product of local ones. However, it is possible to define G without this luxury, cf. [5].

Definition 2.72. *Fixing a marked curved Lie algebra* $(\mathfrak{g},\{x\}) \in \mathscr{L}_*$*, a deformation functor* $\widehat{\mathrm{Def}}_{(\mathfrak{g},\{x\})}\colon \mathscr{A} \to \mathscr{S}$ *is given by* $A \mapsto \widetilde{\mathcal{MC}}\left(\mathfrak{g}^x \hat{\otimes} G(A)\right)$.

Proposition 2.73. *Fix a marked curved Lie algebra* $(\mathfrak{g},\{x\}) \in \mathscr{L}_*$*. The functors* $\widehat{\mathrm{Def}}_{(\mathfrak{g},\{x\})}$ *and* $\mathrm{Def}_{(\mathfrak{g},\{x\})}$ *are naturally isomorphic.*

Proof. It is immediate from the definitions. □

Theorem 2.74. *The functor* $\widehat{\mathrm{Def}}_{(\mathfrak{g},\{x\})}$ *is representable in the homotopy category of pseudo-compact cdgas by* $\widetilde{\mathrm{CE}}(\mathfrak{g},\{x\})$.

Proof. It follows from Proposition 2.73. □

2.A Sullivan homotopy and path objects

In Section 2.5 the notion of a Sullivan homotopy is used to relate homotopy classes of MC elements in certain curved Lie algebras with the homotopy classes of morphisms in $\mathcal{E}$. A Sullivan homotopy is reminiscent of a right homotopy in a CMC, but in $\mathcal{E}$ this is not quite the case: the candidate for a so-called path object is not pseudo-compact. One could attempt to fix this failure by extending $\mathcal{E}$ suitably. This approach, however, is not taken here as it is not necessary.

Within this section interest is restricted to the categories $\mathcal{E}$ and $\mathcal{G}$; most importantly, all morphisms between curved Lie algebras are strict.

Recall, from [7] (for example), the definition of a (very good) path object.

Definition 2.75. *A path object for an object* X *in a CMC* $\mathcal{C}$ *is an object* X^I *of* $\mathcal{C}$ *together with morphisms* $i\colon X \to X^I$ *and* $p\colon X^I \to X \prod X$ *such that* i *is a weak equivalence and* $p \circ i$ *is the diagonal map* $(\mathrm{id}_X, \mathrm{id}_X)\colon X \to X \prod X$.

A path object X^I *is said to be*

- *a good path object, if the morphism* $X^I \to X \prod X$ *is a fibration, and*

- *a very good path object, if it is a good path object and the morphism $X \to X^I$ is a (necessarily acyclic) cofibration.*

Definition 2.76. *Given $\mathfrak{g} \in \mathcal{G}$, let $\mathfrak{g}[z, dz] \in \mathcal{G}$ be given by $\mathfrak{g} \otimes k[z, dz]$, and the quotient morphisms given by setting z to 0 and 1 are denoted $|_0, |_1 \colon \mathfrak{g}[z, dz] \to \mathfrak{g}$, resp.*

Recall Definition 2.63. As already remarked, the objects $A[z, dz]$ and $\mathfrak{g}[z, dz]$ resemble path objects for A and $\mathfrak{g}$, respectively. However, $k[z, dz]$ is not pseudo-compact, and $A[z, dz] \notin \mathcal{E}$. Therefore, $A[z, dz]$ cannot be a path object for A in $\mathcal{E}$.

Proposition 2.77. *Given $\mathfrak{g} \in \mathcal{G}$, $\mathfrak{g}[z, dz]$ is a very good path object for $\mathfrak{g}$ in $\mathcal{G}$.*

Proof. When $\mathfrak{g}$ has zero curvature (i.e. a dgla) the statement is already known. Assuming $\mathfrak{g}$ has non-zero curvature, the canonical morphism $\mathfrak{g} \to \mathfrak{g}[z, dz]$ is between two curved Lie algebras with non-zero curvature and, therefore, a weak equivalence. Moreover, the diagonal morphism $(id_{\mathfrak{g}}, id_{\mathfrak{g}}) \colon \mathfrak{g} \to \mathfrak{g} \prod \mathfrak{g}$ can be factorised as $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}[z, dz] \to \mathfrak{g} \prod \mathfrak{g}$ with the morphism $\mathfrak{g}[z, dz] \to \mathfrak{g} \prod \mathfrak{g}$ clearly surjective and hence a fibration. Further, the morphism $\mathfrak{g} \to \mathfrak{g}[z, dz]$ can easily be shown to be a cofibration by showing it has the LLP with respect to all acyclic fibrations. □

Definition 2.78.

- *Let $A, B \in \mathcal{E}$. Two parallel morphisms of $\mathcal{E}$, $f, g \colon A \to B$, are said to be Sullivan homotopic if there exists a continuous local cdga morphism $h \colon A \to B[z, dz]$ such that $h|_0 = f$ and $h|_1 = g$.*

- *Let $\mathfrak{g}, \mathfrak{h} \in \mathcal{G}$. Two parallel morphisms of $\mathcal{G}$, $f, g \colon \mathfrak{g} \to \mathfrak{h}$, are said to be Sullivan homotopic if there exists a curved Lie algebra morphism $h \colon \mathfrak{g} \to \mathfrak{h}[z, dz]$ such that $h|_0 = f$ and $h|_1 = g$.*

Since all objects of $\mathcal{G}$ are fibrant, the right homotopy with the path object given above is an equivalence relation, cf. [7, Lemma 4.16]. Therefore a Sullivan homotopy in $\mathcal{G}$ is simply a right homotopy. When the source object is cofibrant this notion of homotopy coincides with that given by the weak equivalences of the category $\mathcal{G}$, whence the notion of a Sullivan homotopy coincides with that of the model structure.

It was proven in [15, Theorem 3.6] that for a cofibrant Hinich algebra, A, and an arbitrary Hinich algebra, B, the set of equivalence classes of Sullivan homotopic morphisms $A \to B$ is in bijection with the set of morphisms $A \to B$ in the homotopy category of the Hinich category. This theorem extends to $\mathcal{E}$.

Theorem 2.79 (Lazarev)**.** *Given a cofibrant $A \in \mathcal{E}$ and any arbitrary $B \in \mathcal{E}$, two parallel morphisms $f, g \colon A \to B$ are Sullivan homotopic if, and only if, they represent the same morphism in $\mathrm{Ho}(\mathcal{E})$. More precisely, there is a bijective correspondence between equivalence classes of Sullivan homotopic morphisms $A \to B$ in $\mathcal{E}$ and the set of morphisms $A \to B$ in $\mathrm{Ho}(\mathcal{E})$.*

Proof. The proof follows, mutatis mutandis, from [15, Theorem 3.6]. □

It should be mentioned, albeit in the conclusion of this appendix, Pridham [23] constructed a path object functor for Hinich algebras. This construction is briefly recalled here. Given a finite-dimensional nilpotent local cdga whose maximal ideal respects the differential, A, consider the pullback

$$\begin{tikzcd} A[z,dz] \times_{k[z,dz]} k \arrow[r] \arrow[d] & k \arrow[d, hook] \\ A[z,dz] \arrow[r, "\varepsilon"'] & k[z,dz], \end{tikzcd}$$

where ε is the induced augmentation. This pullback is isomorphic to the algebra $m_A[z,dz] \oplus k$. The path object of A is given by the completion of $m_A[z,dz] \oplus k$ with respect to ideal $m_A[z,dz]$. This construction is extended to the category of Hinich algebras by using the exactness of the completion functor for finitely generated algebras. For more details on this path object construction see Pridham's paper [23] where, using this path object functor, a cylinder object functor is also constructed.

Section 2
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Section 3
Minimal models of quantum homotopy Lie algebras via the BV-formalism

Abstract

Using the BV-formalism of mathematical physics an explicit construction for the minimal model of a quantum L_∞-algebra is given as a formal super integral. The approach taken herein to these formal integrals is axiomatic, and they can be approached using perturbation theory to obtain combinatorial formulae as shown in the appendix. Additionally, there exists a canonical differential graded Lie algebra morphism mapping formal functions on homology to formal functions on the whole space. An inverse L_∞-algebra morphism to this differential graded Lie algebra morphism is constructed as a formal super integral.

Introduction

The Batalin-Vilkovisky (BV-)formalism was originally introduced in physics as a tool to quantise gauge theories and is named after the creators Igor Batalin and Grigori Vilkovisky [4, 5]. One of the strengths of the BV-formalism is it describes how to deal with certain super path integrals, understood as formal power series using perturbation theory. The BV-formalism has also found success in other fields, leading to many results including: deformation quantisation [9], an alternative description of the graph complex [26], an alternative proof of the Kontsevich theorem [53], and manifold invariants [10]. All told, the BV-formalism provides a framework in which odd symplectic geometry, homological algebra, and path integrals interact successfully. The geometric formulation of the BV-formalism was pioneered by Khudaverdian, cf. [31, 33–35]. A modern formulation of the BV-formalism was given by Schwarz [57], but it should be noted that there are many papers where the BV-geometry is considered from various standpoints, see [20, 31, 32, 38, 58] for example. BV-algebras themselves have also been generalised to BV_∞-algebras: c.f. [3, 6, 19, 39].

The formal geometry of the BV-formalism is well suited to the study of (quantum) L_∞-algebras which can be studied in the same language. Accordingly, this viewpoint is taken herein to construct *minimal models* of quantum L_∞-algebras, see Definition 3.39.

Quantum L_∞-algebras arose in work of Zwiebach [62] on closed string field theory. They have appeared in work of Markl [47] under the name 'loop homotopy algebras', and have appeared in work of the first author joint with Lazarev [7]. Quantum L_∞-algebras are a 'higher genus' version of a cyclic L_∞-algebra: the definition is recalled in Section 3.3. One particularly amenable viewpoint of a (quantum) L_∞-algebra structure on a super vector space is a solution to the Maurer-Cartan equation in an appropriate differential graded Lie algebra. The Maurer-Cartan equation is known in physics—more specifically in quantum field theory—as the (quantum or classical) master equation and its use is central to this paper. Indeed, in this language, a quantum L_∞-algebra is a solution to the quantum master equation just as a cyclic

L_∞-algebra is a solution to the classical master equation. Maurer-Cartan elements have many uses in mathematics besides defining L_∞-algebra structures: they govern deformation functors [8, 46, 50, 52, 56], in certain cases correspond to morphisms of certain commutative differential graded algebras, and model rational topological spaces [27, 42, 43, 49, 54]

It is known that minimal models exist for many sorts of homotopy algebras [28,36]. To prove existence and uniqueness of minimal models is usually fairly straightforward. Indeed, in this paper, this is the content of Proposition 3.36, which we obtain as a consequence of standard facts concerning Maurer-Cartan moduli sets of differential graded Lie algebras. However, this argument is not at all constructive—one is often concerned not just with the existence of minimal models but also wishes to have explicit formulae to hand.

Explicit formulae for the structure maps of minimal models for A_∞-algebras were given in [37, 48, 51] as sums over trees. A more general approach was taken in [12] where an explicit formula for minimal models, in terms of sums over 'stable graphs', for an algebra over the cobar-construction of a differential graded modular operad was constructed.

We take a different approach to deducing formulae for minimal models in the present paper. Indeed, the formulae in terms of stable graphs are reminiscent of those given by perturbative expansions of path integrals using Feynman diagrams and this is the perspective we pursue here. We will show that the minimal model of a quantum L_∞-algebra can be given by a simple explicit integral formula coming from the BV-formalism (Theorem 3.42). These sorts of integrals have already been studied in the context of quantum field theory [10, 15]. The advantage of this approach is that we obtain a simpler and more conceptual proof of the minimal model formulae.

In fact, the combinatorial formulae in terms of stable graphs that one obtains using the results of [12] can be deduced from the integral formula given in this paper by standard methods of expansions of path integrals in terms of Feynman diagrams: this is done in Appendix 3.A.

Recently, minimal models have found applications in theoretical physics. More specifically, minimal models have been found to have applications in string field theory and quiver gauge theory, [1, 2, 29, 44, 59].

The paper is organised as follows. Section 3.1 recalls results of linear formal odd symplectic geometry. In particular, the master equation (or Maurer-Cartan equation) and the notion of a strong deformation retract from one odd symplectic vector space to another are both recalled. Section 3.2 introduces the theory of integration used in this setting (Definition 3.13), provides a proof of an analogue of Stokes Theorem (Proposition 3.18), and discusses the relevant parts of the BV-formalism. Section 3.3 recalls details surrounding the theory of (quantum) L_∞-algebras and Proposition 3.22 introduces an important and useful filtration. The main result of the paper (Theorem 3.42) is contained within Section 3.4. That is, the explicit construction of the minimal model for a given quantum L_∞-algebra via a formal super integral. As a straightforward corollary of Theorem 3.42, a minimal model for a harmonic odd cyclic L_∞-algebra can be given via a formal super integral. Section 3.4 closes by providing an inverse L_∞-algebra morphism to the differential graded Lie algebra morphism embedding the functions on homology into the space of all functions: this is the content of Theorem 3.50. The combinatorial approach to formal super integration is briefly discussed in Appendix 3.A. More precisely, within this appendix those formal super integrals considered throughout the paper are shown to admit a presentation as formal sums over stable graphs: Theorems 3.57 and 3.58. The same result is given in [15], but proven by different means. Appendix 3.A closely follows the argument of [16]. Indeed, the ordinary notion of a graph is a special case of a stable graph and, by restricting our formula to usual graphs, one can recover that of [16]. A very similar result to Theorem 3.57 is given in [17, Example 3.10]. Further, in loc. cit. the authors explore the relationship between Feynman diagram expansions and graphical calculus of Reshetikhin-Turaev.

Notations and conventions

Fix the real numbers, $\mathbb{R}$, as the base field. For technical reasons, the base field is extended to the field of formal Laurent series $\mathbb{R}((\hbar))$ at some points of the paper. All unmarked tensors are assumed to be over the appropriate base field, unless otherwise stated. We will be concerned with the category of differential $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces ('super vector spaces'). Some of the definitions and results contained within this paper could also be made sense of in the $\mathbb{Z}$-graded context once suitable adaptations are made.

The degree (or parity) of a homogeneous element v in a super vector space is denoted $|v|$. Following well established notation those elements of homogeneous degree 0 are referred to as even and those of homogeneous degree 1 are referred to as odd. Accordingly, the dimension of a super vector space is given as $(m|n)$, where m is the dimension (in the non-graded sense) of the subspace of even elements and n is the dimension of the subspace of odd elements. The total dimension of a super vector space of dimension $(m|n)$ is given by $m + n$. A super vector space will be finite-dimensional if, and only if, it is of finite total dimension. The tensor product $V \otimes W$ of super vector spaces V and W has differential defined as: $d_{V\otimes W}(v \otimes w) = (d_V v)\otimes w+(-1)^{|v|}v\otimes(d_W w)$ and thus the category of super vector spaces is symmetric monoidal with symmetry isomorphism given by $s(v \otimes w) = (-1)^{|v||w|}w \otimes v$.

Denote by $\Pi\mathbb{R}$ the super vector space of total dimension one (over $\mathbb{R}$) concentrated in odd degree. The functor given by taking a super vector space V to the tensor $V \otimes \Pi\mathbb{R}$ is denoted by Π (the super vector space ΠV is called the *parity reversion* of V). Likewise for super vector spaces over $\mathbb{R}((\hbar))$. The space $\mathrm{Hom}(V, W)$ denotes the super vector space with even part the space of morphisms $V \to W$ (i.e. those linear maps preserving the grading) and odd part the space of morphisms $V \to \Pi W$ (i.e. those linear maps which reverse the grading). This can be equipped with the differential $df = d_W \circ f - (-1)^{|f|} f \circ d_V$, making it into an internal Hom functor, and hence the category of super vector spaces is a closed symmetric monoidal category.

In particular, an associative, a commutative, or a Lie algebra is always the appropriate notion in the category of super vector spaces. The expressions 'differential (super)graded', 'commutative differential graded algebra' and 'differential graded Lie algebra' are abbreviated to 'dg', 'cdga' and 'dgla', respectively.

Given a dgla $\mathfrak{g}$ and a cdga A, recall the tensor product $\mathfrak{g}\otimes A$ possesses a well defined structure of a dgla: the bracket is given on elementary tensors by $[x \otimes a, y \otimes b] = [x, y] \otimes (-1)^{|a||y|}ab$.

The notion of a pseudo-compact super vector space is used extensively within this text. A pseudo-compact super vector space is one given by an inverse limit of super vector spaces of finite total dimension. As such, a pseudo-compact super vector space is equipped with a topology induced by the inverse limit, and thus all linear maps of pseudo-compact super vector spaces are assumed to be continuous. The dual of a pseudo-compact super vector space is, therefore, the topological dual. This has the luxury of always having $(V^*)^* \cong V$ without any finiteness conditions. Similarly, it will always be the case that $(V \otimes V)^* \cong V^* \otimes V^*$ since the tensor product of two pseudo-compact super vector spaces $A = \varprojlim_i A_i$ and $B = \varprojlim_j B_j$ will always be the completed tensor product, in other words $A \otimes B$ is the pseudo-compact super vector space given by $\varprojlim_{i,j} A_i \otimes B_j$. Similarly, if V is a discrete super vector space and $A = \varprojlim_i A_i$ is a pseudo-compact super vector space, the tensor product $A \otimes V$ is always assumed to mean $\varprojlim_i A_i \otimes V$. More details on pseudo-compact objects can be found in the literature [18, 30, 60]. In particular, it should be noted that the functor $V \mapsto V^*$ gives a symmetric monoidal equivalence between the category of pseudo-compact super vector spaces and the opposite category of super vector spaces. Thus, for example, a pseudo-compact dg algebra is equivalently a dg coalgebra.

The completed symmetric algebra of a finite-dimensional super vector space V is an example of a pseudo-compact cdga (it is the dual of the cofree dg cocommutative coalgebra on V) and appears regularly throughout this paper. Explicitly, the completed symmetric algebra is the pseudo-compact algebra $\hat{S}(V) := \prod_{i=0}^{\infty} S^i(V)$, that is as the direct product of symmetric tensor powers (over either $\mathbb{R}$ or $\mathbb{R}((\hbar))$) of the super vector space V. The symmetric algebra $S(V)$ is a subalgebra of $\hat{S}(V)$.

We will often refer to a pronilpotent dgla (or cdga), meaning an inverse limit of nilpotent dglas (or cdgas). Nilpotent here will mean 'global' nilpotence: the descending central series stabilises at zero.

Statement of results

For convenience and motivation, we give some definitions and state the main theorems of the paper here.

Let V be a super vector space with an odd symmetric, non-degenerate bilinear form. A quantum L_∞-algebra structure on V is a MC element in a certain dgla $\mathfrak{h}[\Pi V] \subset \hat{S}\Pi V^*[\hbar]]$, see Definition 3.20 and Definition 3.33. Moreover, there exists a filtered quasi-isomorphism of dglas, ι, that induces a bijection of quantum L_∞-algebra structures on $\mathrm{H}(V)$ to V, see Proposition 3.36.

Definition 3.39 *Given a quantum L_∞-algebra structure (V, m), the minimal model of (V, m) is a quantum L_∞-algebra $(\mathrm{H}(V), m')$ such that $\iota(m')$ is homotopic to m (as a MC element in $\mathfrak{h}[\Pi V]$).*

There exists a canonical isotropic subspace $\mathcal{L}_s \subset \Pi V$ (see Section 3.1.3) which can be endowed with a non-degenerate quadratic function σ, see Remark 3.15 and the beginning discussion of Section 3.2.2.

Theorem 3.42 *Given a quantum* L_∞*-structure* (V, m)*, the integral formula*

$$m' = \hbar \log \int_{\mathcal{L}_s} e^{\frac{m}{\hbar}} e^{\frac{-\sigma}{2\hbar}}$$

defines a quantum L_∞*-algebra* $(\mathrm{H}(V), m')$ *and, moreover, it is the minimal model of* (V, m)*.*

Theorem 3.50 *Let* A *be a pseudo-compact cdga. The morphism of sets*

$$\mathrm{MC}(\mathfrak{h}[\Pi V], A) \to \mathrm{MC}(\mathfrak{h}[\Pi\, \mathrm{H}(V)], A)$$

given by mapping $\sum_{i \in I} f_i \otimes a_i$ *to the function given by*

$$\hbar \log \int_{\mathcal{L}_s} e^{\frac{1}{\hbar} \sum_{i \in I} f_i \otimes a_i} e^{\frac{-\sigma}{2\hbar}}.$$

provides the inverse L_∞*-morphism to the filtered quasi-isomorphism* ι*.*

3.1 Formal odd symplectic geometry

A brief account of formal linear odd symplectic geometry is contained within this section. Recall that an odd symplectic super vector space is a super vector space with an odd bilinear form that is also non-degenerate and skew-symmetric. An odd symplectic vector space is considered as a formal odd symplectic manifold. As such, some of the terminology used here reflects the geometric setting and many of the results stated in terms of super vector spaces have known analogues and generalisations. For a general treatment of the BV-formalism see, for example, [33–35, 57]. The notation V will generally be used to denote a super vector space endowed with an odd symmetric bilinear form and W will generally be used to denote an odd symplectic super vector space. As such, a first example of an odd symplectic vector space is the following: let ω be an odd symmetric form on V that is also non-degenerate, then an odd symplectic form ΠV can be given by the formula $\tau(\Pi x, \Pi y) = (-1)^{|x|}\omega(x, y)$ for all $x, y \in V$.

3.1.1 Preliminaries

The algebra of functions on an odd symplectic vector space, W, is given by $\hat{S}W^*$. If W has a differential, say d, then $\hat{S}W^*$ possesses both a canonical differential obtained from d (which herein is denoted, by an abuse of notation, also d) and an operator of order two corresponding to the odd symplectic form (the Laplacian, see Definition 3.3). Every odd symplectic vector space is of even total dimension, and there exists a canonical basis $\{x_i, \xi_j\}_{i,j \in \{1,\ldots,n\}}$ for W^* with x_i even and ξ_i odd such that the odd symplectic form is of canonical form: $\omega = \sum_i d_{DR} x_i d_{DR} \xi_i$, where d_{DR} is the de Rham differential.

Definition 3.1. *Given an odd symplectic vector space, a Lagrangian subspace is a maximal isotropic subspace, i.e. a subspace such that the restriction of the odd symplectic form vanishes and is of maximal total dimension with this property.*

Remark 3.2. A Lagrangian subspace must have total dimension half that of the whole space, and any isotropic subspace extends to a Lagrangian one.

Lagrangian subspaces serve as convenient subspaces for integration and are particularly important for this paper. More details are contained in Section 3.2.1.

Definition 3.3. *Let W be an odd symplectic vector space with basis $\{x_i, \xi_j\}_{i,j\in\{1,2,\ldots,n\}}$ for W^*. The Laplacian acts on formal functions by*

$$\Delta(g) = \sum_{i=1}^{n} \partial_{x_i}\partial_{\xi_i} g.$$

Remark 3.4. The definition of the Laplacian does not depend upon the choice of basis in W.

Definition 3.5. *A dg BV-algebra is a unital cdga A endowed with an odd differential operator, Δ, of order 2 such that:*

- $\Delta^2 = 0 = \Delta(1)$*; and*
- $d\Delta + \Delta d = 0$.

For more details on dg BV-algebras and their generalisation to BV_∞-algebras see [6]. The notation Δ for the Laplacian and for the odd differential operator in the above is deliberate as the next proposition shows.

Proposition 3.6. *The Laplacian defines a dg BV-algebra structure on $\hat{S}W^*$ with differential d and BV-operator Δ. Hence, $\hat{S}W^*$ has the structure of a dg odd Poisson algebra with differential $d + \Delta$ and odd bracket given by*

$$[x, y] = (-1)^{|x|}\Delta(xy) - (-1)^{|x|}\Delta(x)y - x\Delta(y).$$

Proof. It is a straightforward check to show that $\hat{S}W^*$ is a dg BV-algebra. Further, it is a known fact that given a (dg) BV-algebra, the bracket given in the proposition defines the structure of a (dg) odd Poisson algebra. $\square$

For more details regarding BV-algebras and odd Poisson (or Gerstenhaber) algebras see [55].

3.1.2 Master equations

To make sense of some constructions it is necessary to extend the base field from $\mathbb{R}$ to $\mathbb{R}((\hbar))$ and extend super vector spaces from W to $\overline{W} := W \otimes_{\mathbb{R}} \mathbb{R}((\hbar))$. The cdga $\hat{S}\overline{W}^*$ is a dg BV-algebra with differential d and BV-operator $\hbar\Delta$, where d and Δ have been extended $\hbar, \hbar^{-1}$-linearly. Hence, $\hat{S}\overline{W}^*$ is a dg odd Poisson algebra with differential $d + \hbar\Delta$. Note that the symmetric tensors here are taken over $\mathbb{R}((\hbar))$.

For all functions, f, in some pronilpotent ideal, we define the formal power series:

$$e^f = \sum_{n=0}^{\infty} \frac{f^n}{n!}.$$

Likewise, f such that $(f-1)$ is in some pronilpotent ideal, we define the formal power series:

$$\log(f) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(f-1)^n}{n}.$$

For any f in some pronilpotent ideal we have $\log(e^f) = f$, and for any g such that $(g-1)$ is in some pronilpotent ideal we have $e^{(\log g)} = g$.

Definition 3.7. *Let $m = \sum_{i=0}^{\infty} \hbar^i m_i \in \hat{S}\overline{W}^*$ be an even function with $m_i \in \hat{S}W^*$ for all $i \geq 0$ such that $\frac{m}{\hbar}$ belongs to a pronilpotent ideal of $\hat{S}\overline{W}^*$. The function m is said to satisfy the quantum master equation (QME) if $(d+\hbar\Delta)e^{\frac{m}{\hbar}} = 0$.*

Proposition 3.8. *A solution to the QME is equivalent to a solution to the Maurer-Cartan (MC) equation, i.e. $\forall m \in \hat{S}\overline{W}^*$ such that $\frac{m}{\hbar}$ lies in a pronilpotent ideal*

$$(d+\hbar\Delta)e^{\frac{m}{\hbar}} = 0 \Leftrightarrow e^{\frac{m}{\hbar}}\left((d+\hbar\Delta)m + \frac{1}{2}[m,m]\right) = 0.$$

Proof. The failure of Δ to be a derivation is measured by the odd Poisson bracket (as in Proposition 3.6) and so one determines the relationship

$$(d+\hbar\Delta)e^{\frac{m}{\hbar}} = \frac{1}{\hbar}e^{\frac{m}{\hbar}}\left((d+\hbar\Delta)m + \frac{1}{2}[m,m]\right).$$

The result now follows. □

The phrases ‘solution to the QME’ and ‘MC element’ will be used interchangeably. Writing out the MC equation in terms of the expansion $m_0 + \hbar m_1 + \hbar^2 m_2 + \ldots$ leads to an equivalent system of equations collecting powers of $\hbar$. The first equation $d(m_0) + \frac{1}{2}[m_0, m_0] = 0$ is the classical master equation (CME) for the function m_0 (hence m_0 defines a cyclic L_∞-algebra), and the second equation $d(m_1) + \Delta(m_0) + [m_0, m_1] = 0$ defines a *unimodular* L_∞-algebra (see [7, 24]). In [7] the problem of lifting a solution to the CME to a solution of the QME is addressed and unimodularity plays a key role therein.

3.1.3 Strong deformation retracts

Let W be a symplectic super vector space. Given a strong deformation retract (SDR) of $V = \Pi W$ onto some choice of representatives for the homology of V, which is, moreover, compatible with the bilinear form in an appropriate way (see below), one arrives at a canonical choice of isotropic/Lagrangian subspace for W. The Lagrangian subspace arrived at in this way is used heavily in Section 3.2.1. This ‘cyclic’ SDR from a space onto its homology is equivalent to that of a Hodge decomposition, c.f. [12, 13]. A Hodge decomposition always exists for a finite-dimensional (super) vector space.

Definition 3.9. *Let V and U be two super vector spaces, both equipped with odd symmetric bilinear forms. A cyclic SDR from V to U is a pair of even super vector space morphisms $i\colon U \hookrightarrow V$ and $p\colon V \twoheadrightarrow U$ and an odd linear morphism $s\colon V \to V$ such that:*

- $pi = id_U$;
- $ds + sd = id_V - ip$;
- $si = 0$; $ps = 0$; $s^2 = 0$;
- $\langle ix, iy\rangle = \langle x, y\rangle$; $\ker(p) \perp \mathrm{im}(i)$; *and* $\langle sx, y\rangle = (-1)^{|x|}\langle x, sy\rangle$.

Remark 3.10. Forgetting the last condition, concerning the bilinear forms, in Definition 3.9 one obtains the usual, well-established, notion of an SDR of super vector spaces. Since we will always assume an SDR is cyclic in this paper we will suppress the adjective cyclic from now on.

The conditions $si = 0$, $ps = 0$, and $s^2 = 0$ are called the side conditions and are not always included; the reason being that they can be imposed at no cost as the following simple proposition, taken from [7, Lemma B.6.], shows.

Proposition 3.11. *Let U, V be two super vector spaces equipped with odd symmetric bilinear forms and (i, p, s) be morphisms satisfying all the conditions of an SDR except the side conditions, then s can be replaced with a morphism s' in such a way that the triple (i, p, s') is an SDR.*

Proof. If s does not satisfy $si = 0$ and $ps = 0$, it can be replaced with $\tilde{s} = (ds + sd)s(ds + sd)$. By elementary, yet tedious calculations, the triple $(i, p, \tilde{s})$ now satisfies everything except (possibly) $\tilde{s}^2 = 0$. Replacing $\tilde{s}$ with $s' = \tilde{s}d\tilde{s}$ means the triple (i, p, s') is an SDR. □

The properties of Definition 3.9 ensure that given an SDR of V onto $\mathrm{H}(V)$, one has a decomposition $V = \mathrm{im}(i) \oplus \mathrm{im}(s) \oplus \mathrm{im}(d)$. Furthermore, one has the orthogonality relations:

$$\mathrm{im}(i)^\perp = \mathrm{im}(s) \oplus \mathrm{im}(d), \quad \mathrm{im}(d)^\perp = \mathrm{im}(i) \oplus \mathrm{im}(d), \quad \text{and} \quad \mathrm{im}(s)^\perp = \mathrm{im}(i) \oplus \mathrm{im}(s).$$

Let the bilinear form on V be non-degenerate. The decomposition $V = \mathrm{H}(V) \oplus \mathrm{im}(s) \oplus \mathrm{im}(d)$ gives rise to a decomposition of the odd symplectic vector space $W = \Pi V$ as $W = \Pi\,\mathrm{H}(V) \oplus \Pi\,\mathrm{im}(s) \oplus \Pi\,\mathrm{im}(d)$. Therefore, define the subspace $\mathcal{L}_s = \Pi\,\mathrm{im}(s)$ and notice it is a Lagrangian subspace of $\Pi\,\mathrm{im}(i)^\perp$.

Proposition 3.12. *If $\mathcal{L}_s$ is non-zero, then the total dimension of the subspace of odd elements is even.*

Proof. One can define an odd symmetric non-degenerate form $(x, y) = \langle x, dy\rangle$ on $\mathcal{L}_s$. Therefore, the form is skew symmetric when restricted to odd coordinates and—by forgetting the grading—the form defines an odd symplectic form on the super vector space of odd coordinates, hence there must be an even number of odd coordinates. □

3.2 The BV formalism

Here the necessary facts concerning the BV-formalism are recalled. For a more in depth exploration of the BV-formalism and BV-geometry see one of the many resources [4, 5, 15, 31, 31, 33–35, 53, 57, 58].

3.2.1 Integration and BV Stokes' Theorem

The integration over Lagrangian subspaces defined in this section provides an important tool for Section 3.4. This integration is often only formally defined, but in some cases restricts to the standard definition of (super) integration. The reader who is already comfortable with integrals over super vector spaces may wish to skip this section, or at least the early discussion. It should be noted that although we take an axiomatic approach to integration, the integrals considered here could, alternatively, be evaluated using perturbation theory, c.f. Appendix 3.A.

Let V be a super vector space of total dimension m concentrated entirely in even degree, i.e. a usual vector space. Let A be a non-degenerate, positive definite bilinear form on V. We begin with case of the integral

$$\int_V f e^{-A}\mu,$$

where f is some polynomial function on V. It is well known how to calculate such integrals: taking a change of coordinates, one diagonalises A and calculates

$$\int_{\mathbb{R}^m} f(x)e^{(-\sum_{i=1}^m x_i^2)}dx_1\ldots dx_m$$

using integration by parts and the Gaussian integral $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$. Next, we wish to extend to super vector spaces. For an odd coordinate ξ on some super vector space one has

$$\int_{\mathbb{R}} 1 d\xi = 0 \quad \text{and} \quad \int_{\mathbb{R}} \xi d\xi = 1.$$

This completely defines odd integration (since odd coordinates square to zero).

Let V now be a super vector space of dimension $(m|n)$ and A is a non-degenerate, positive definite bilinear form. The integral of a polynomial function, f, over V is given by

$$\int_V f e^{-A}\mu := \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} f e^{-A} dx_1 \ldots dx_m d\xi_1 \ldots d\xi_n.$$

The full machinery of integration over super manifolds is not necessary in this paper. However, for a more general approach to integration over super manifolds see [35].

We now wish to make one final extension to the integrals being presented here. Namely, we wish to no longer make the assumption that A is positive definite. For odd coordinates, no longer having this assumption makes no difference and we again concentrate our attention to the case where V is a super vector space of total dimension m concentrated entirely in even degree. Let A now be a non-degenerate bilinear form on V. After a change of variables, one has $A = \sum_{i=1}^k x_i^2 - \sum_{i=k+1}^m x_i^2$. For $1 \leq i \leq k$ the integral

$$\int_{\mathbb{R}} f(x)e^{-x_i^2}dx_i,$$

where f is some polynomial function, exists. It is, therefore, the case when $k+1 \leq i \leq m$ when we have an issue, i.e. the integral

$$\int_{\mathbb{R}} f(x)e^{x_i^2}dx_i,$$

does not exist using standard integration. To rectify this issue, introduce the function

$$g_f(t) := \int_{\mathbb{R}^m} f(x) e^{(\sum_{i=1}^k x_i^2 + \sum_{i=k+1}^m (tx_i)^2)} dx_1 \dots dx_m,$$

defined for all non-zero real numbers. Expanding $g_f(t)$ as a formal power series, one can use analytic continuation to define g_f for all non-zero complex numbers. The original integral is, therefore, equal to $g_f(i)$. For example $\int_{\mathbb{R}} e^{x^2} \mu_x = -i\sqrt{\pi}$. It is important to note these integrals only exist formally.

One can integrate $g_{x_i^{2n}}(t)$ by parts to establish a recursive relation, just like in the positive definite case. Using these two recursive relations and normalising the integrals by diving through by

$$\int_V e^{-A} \mu$$

we make the following elementary definition after setting the scene.

Let W be an exact odd symplectic vector space such that $W = L_1 \oplus L_2$, where L_1 and L_2 are Lagrangian subspaces of W and $d\colon L_1 \to L_2$ is an isomorphism. A suitable choice of coordinates $\{x_1, \dots, x_k\}$ on the even part of L_1 diagonalises the quadratic function $x \mapsto \langle x, dx \rangle = \sum_{i=1}^j x_i^2 - \sum_{i=j+1}^k x_i^2$. Let $\epsilon(x_i) = 1$ if x_i belongs to the first sum and $\epsilon(x_i) = -1$ if x_i belongs to the second sum. Similarly, there exists coordinates $\{\xi_{k+1}, \dots, \xi_n\}$ on the odd part of L_1 such that the quadratic function $\xi \mapsto \langle \xi, d\xi \rangle = -(\xi_{k+1}\xi_{k+2} + \xi_{k+3}\xi_{k+4} + \dots + \xi_{n-1}\xi_n)$, since L_1 has an even number of odd coordinates (c.f. Proposition 3.12). For $i \in \{1, 3, 5, \dots, n-1\}$, the coordinates ξ_i and ξ_{i+1} will be said to be a pairing of odd coordinates.

Definition 3.13. *For even $x_i \in L_1$,*

$$\int_{\mathbb{R}} e^{\frac{-1}{2\hbar}\langle x_i, dx_i \rangle} \mu_{x_i} = 1,$$

and, for all $n \in \mathbb{N}$, the recursive relation

$$\int_{\mathbb{R}} x_i^{2(n+1)} e^{\frac{-1}{2\hbar}\langle x_i, dx_i \rangle} \mu_{x_i} = \epsilon(x_i)(2n+1)\hbar \int_{\mathbb{R}} x_i^{2n} e^{\frac{-1}{2\hbar}\langle x_i, dx_i \rangle} \mu_{x_i}$$

will be called integration by parts (for even coordinates). For a pairing of odd coordinates, ξ_i and ξ_{i+1},

$$\int_{\mathbb{R}^2} e^{\frac{1}{2\hbar}\xi_i\xi_{i+1}} \mu_{\xi_i}\mu_{\xi_{i+1}} := 1, \quad \int_{\mathbb{R}^2} \xi_i e^{\frac{1}{2\hbar}\xi_i\xi_{i+1}} \mu_{\xi_i}\mu_{\xi_{i+1}} := 0,$$
$$\int_{\mathbb{R}^2} \xi_{i+1} e^{\frac{1}{2\hbar}\xi_i\xi_{i+1}} \mu_{\xi_i}\mu_{\xi_{i+1}} := 0, \quad \text{and} \quad \int_{\mathbb{R}^2} \xi_i\xi_{i+1} e^{\frac{1}{2\hbar}\xi_i\xi_{i+1}} \mu_{\xi_i}\mu_{\xi_{i+1}} := \hbar.$$

The integral is extended in the obvious manner to polynomial functions $f \in S\overline{W}^*$ over L_1 by extending $\hbar, \hbar^{-1}$-linearly and by setting

$$\int_{L_1} f e^{\frac{-\sigma}{2\hbar}} := \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left(f e^{\frac{-\sigma}{2\hbar}} \big|_{L_1} \right) \mu_{x_1} \dots \mu_{x_k} \mu_{\xi_{k+1}} \dots \mu_{\xi_n},$$

where σ is the quadratic function corresponding to $? \mapsto \langle ?, d? \rangle$. This integral is a Laurent polynomial in $\mathbb{R}[\hbar, \hbar^{-1}]$, because the integrand is restricted to L_1 and the integral is taken over all the coordinates of L_1.

Remark 3.14. In Definition 3.13 integration by parts is defined for even coordinates. There is no integration by parts defined for odd coordinates, because odd coordinates square to zero.

Remark 3.15. The above argument for even variables in the case where the bilinear form is not necessarily positive definite is known (in a different guise) as the Wick rotation. The non-degeneracy of $\langle x, dx\rangle$ on L_1 means the integration is against a 'Gaussian measure', dealing with convergence issues for even variables, although these integrals need only exist formally: see [26].

Proposition 3.16. *The integrals of Definition 3.13 do not depend upon the choice of coordinates.*

Proof. The (hidden) normalisation by

$$\int_{L_1} e^{\frac{-\sigma}{2\hbar}}$$

in Definition 3.13 causes the integrals to be the ratio of two integrals and, thus, they do not depend upon the choice of coordinates. $\square$

An analogue of Stokes' Theorem is now recalled after a couple of auxiliary results. The more general original result is found in [57]. For an alternative proof using the usual exterior calculus in the linear case see [26].

Proposition 3.17. *Let $\mathcal{L} \subset W$ be a Lagrangian subspace such that $\sigma(y) = \langle y, dy\rangle$ is non-degenerate. For x_i even*

$$\int_{\mathbb{R}} \partial_{x_i}(x_i^m e^{\frac{-1}{2\hbar}\sigma(x_i)})\mu_{x_i} = 0.$$

For ξ_i and ξ_{i+1} an odd pairing and $f \in SW^*$ any polynomial, for $j \in \{i, i+1\}$

$$\int_{\mathbb{R}^2} \partial_{\xi_j}(f e^{\frac{1}{2\hbar}\xi_i\xi_{i+1}})\mu_{\xi_i}\mu_{\xi_{i+1}} = 0.$$

Proof. In the even case, compute the partial derivative and when m is odd integrate by parts. The odd case is immediate from the calculation. $\square$

Proposition 3.18. *Let $\mathcal{L} \subset W$ be a Lagrangian subspace such that $\sigma(y) = \langle y, dy\rangle$ is non-degenerate and let $f \in SW^*$ be any polynomial, then*

$$\int_{\mathcal{L}} \Delta\left(f e^{\frac{-\sigma}{2\hbar}}\right) = 0.$$

Proof. It follows from computation and Proposition 3.17. $\square$

3.2.2 Integrating solutions to the QME

Section 3.2.1 assumes the vector space in question is acyclic, i.e. $\mathrm{H}(W) = 0$, leading to the existence of a Lagrangian subspace on which $\langle x, dx\rangle$ is non-degenerate, see Remark 3.15. In general, the odd symplectic vector space in question may have non-trivial homology and thus we must now consider this more general case. Recall the decomposition

$$W = \Pi\,\mathrm{im}(i) \oplus \Pi\,\mathrm{im}(i)^{\perp}$$

given in Section 3.1.3. Clearly, $\mathrm{H}(\Pi\,\mathrm{im}(i)^{\perp}) = 0$, so the preceding results can be applied to $\Pi\,\mathrm{im}(i)^{\perp}$. Moreover, recall that one has a canonical choice of Lagrangian subspace $\mathcal{L}_s \subset \Pi\,\mathrm{im}(i)^{\perp}$.

Definition 3.19. *Given an odd symplectic vector space $W = H(W) \oplus \Pi\,\mathrm{im}(i)^{\perp}$, the integral of a polynomial $f \in S\overline{W}^*$ over $\mathcal{L}_s$ is given by $(id_{H(W)} \otimes \int_{\mathcal{L}_s})(f) \in S(\overline{H(W)}^*)$.*

So far integration has been defined for arbitrary polynomials $f \in S\overline{W}^*$. We will need to integrate certain infinite series in $\hat{S}\overline{W}^*$, however this raises the issue of convergence. For instance, the integral $\int_{\mathbb{R}} \left(\sum_{k=1}^{\infty} \frac{x^{2k}}{h^k}\right) e^{\frac{-1}{2\hbar}x^2}\mu_x$ does not converge. It will be necessary to be able to integrate exponentials of the form $e^{\frac{f}{\hbar}}$ where f is a formal power series belonging to a certain subspace of $\hat{S}W^*[|\hbar|]$. It turns out that integrals converge when integrating functions of this exponential form, as is now explained.

Definition 3.20. *Introduce the weight grading on the cdga $\hat{S}W^*[|\hbar|]$ by requiring that for a monomial $f \in \hat{S}W^*$ of degree n, the element $f\hbar^g$ has weight $2g + n$. Let $\mathfrak{h}[W]$ be the subspace of $\hat{S}W^*[|\hbar|]$ containing those elements of weight grading > 2.*

Remark 3.21. Not only is $\mathfrak{h}[W]$ important because it defines a subspace of $\hat{S}W^*[|\hbar|]$ where we can make sense of integration, but it is important because it is where quantum L_∞-algebra structures on ΠW are defined: see Definition 3.33.

Proposition 3.22. *For all $i \geq 1$ let F_i be the subspace of $\mathfrak{h}[W]$ given by all vectors of weight grading $\geq i - 2$. The filtration $\{F_i\}_{i\geq 1}$ is Hausdorff and complete.* □

It is clear the cdga $\mathfrak{h}[W]$ inherits the structure of an odd Poisson algebra from $\hat{S}W^*[|\hbar|]$. Further, it is clear that $\mathfrak{h}[W]$ is pronilpotent, whereas $\hat{S}W^*[|\hbar|]$ is not.

The following is an elementary observation.

Proposition 3.23. *Integration over an even coordinate or a pairing of odd coordinates fixes the weight grading.* □

Proposition 3.24. *For $f \in \mathfrak{h}[W]$, the formal Laurent series f' given by*

$$f' = \hbar \log \left(\int_{\mathcal{L}_s} e^{\frac{f}{\hbar}} e^{\frac{-\sigma}{2\hbar}} \right)$$

converges and consists of non-negative powers of $\hbar$ only, i.e. it is a formal power series in $\hbar$. Moreover, $f' \in \mathfrak{h}[\mathrm{H}(W)]$.

Proof. The first statement follows immediately from Appendix 3.A and, in particular, Theorem 3.58. The second statement now follows from the first and Proposition 3.23. $\square$

Lemma 3.25. *Let $m \in \mathfrak{h}[W]$ be a solution to the QME, then $m' \in \mathfrak{h}[\mathrm{H}(W)]$ given by*

$$m' = \hbar \log \left(\int_{\mathcal{L}_s} e^{\frac{m}{\hbar}} e^{\frac{-\sigma}{2\hbar}} \right)$$

satisfies the QME. $\square$

The proof is suppressed here as it is a corollary of Proposition 3.30 given later. Lemma 3.25 is well known in various guises: see [10, 15, 40]. In fact, in loc. cit. the lemma includes an additional statement: if the Lagrangian subspace $\mathcal{L}_s$ is perturbed by a small amount a homotopic solution to the QME is achieved. This latter statement is a corollary of Theorem 3.42.

3.2.3 Homotopy of solutions to the QME

In this section, we show that integration respects the notion of homotopy between MC elements. The notion of a homotopy between MC elements in a fixed dgla is standard and is equivalent to gauge equivalence for pronilpotent dglas: see [7, 56, 61]. Since the dglas considered within this paper are pronilpotent, this section will make the assumption that all dglas are pronilpotent. Given a dgla $\mathfrak{g}$, let $\mathrm{MC}(\mathfrak{g})$ denote the solutions to the MC equation in $\mathfrak{g}$.

Definition 3.26.

- *Let $\mathbb{R}[t, dt]$ be the free cdga over $\mathbb{R}$ with generators t and dt (even and odd respectively) subject to the condition $d(t) = dt$. Clearly there exist two evaluation maps $|_0, |_1 \colon \mathbb{R}[t, dt] \to \mathbb{R}$ defined by setting t to 0 and 1, respectively.*

- *For some dgla $\mathfrak{g}$, let $\mathfrak{g}[t, dt]$ be the dgla given by $\mathfrak{g} \otimes \mathbb{R}[t, dt]$. Clearly there exist two evaluation maps $|_0, |_1 \colon \mathfrak{g}[t, dt] \to \mathfrak{g}$ defined by setting t to 0 and 1, respectively.*

Definition 3.27. *Let $\mathfrak{g}$ be a dgla. $\xi, \eta \in \mathrm{MC}(\mathfrak{g})$ are said to be homotopic if there exists $H(t) \in \mathrm{MC}(\mathfrak{g}[t, dt])$ such that $H(0) = \xi$ and $H(1) = \eta$.*

The Maurer-Cartan moduli set, denoted $\mathcal{MC}(\mathfrak{g})$, is the set of equivalence classes of $\mathrm{MC}(\mathfrak{g})$ under the homotopy relation.

For an odd symplectic vector space W, consider a homotopy $H(t) \in \mathfrak{h}[W][t, dt]$. Clearly, $H(t) = A(t) + B(t)dt$. Further, there exists the equivalent exponential form

$$e^{\frac{H(t)}{\hbar}} = e^{\frac{A(t)}{\hbar}} + \frac{1}{\hbar} e^{\frac{A(t)}{\hbar}} B(t) dt.$$

We can extend Definition 3.13 A-linearly:

Definition 3.28. *Let A be a cdga. Given an element $\sum_i f_i \otimes a_i \in SW^* \otimes A$, let*

$$\int_{\mathcal{L}_s} \left(\sum_i f_i \otimes a_i \right) e^{\frac{-\sigma}{2\hbar}} := \sum_i \left(\int_{\mathcal{L}_s} f_i e^{\frac{-\sigma}{2\hbar}} \right) \otimes a_i .$$

Proposition 3.29. *If $W = \Pi \operatorname{im}(i) \oplus \Pi \operatorname{im}(i)^\perp$ is an orthogonal decomposition, then the Laplacian splits $\Delta = \Delta_{\Pi \operatorname{im}(i)} + \Delta_{\Pi \operatorname{im}(i)^\perp}$. Moreover, given an element $\sum_i f_i \otimes a_i \in SW^* \otimes A$,*

$$\Delta_{\Pi \operatorname{im}(i)} \int_{\mathcal{L}_s} \left(\sum_i f_i \otimes a_i \right) e^{\frac{-\sigma}{2\hbar}} = \int_{\mathcal{L}_s} \Delta \left(\sum_i f_i \otimes a_i \right) e^{\frac{-\sigma}{2\hbar}}$$

Proof. The first statement is immediate. The second statement follows from three facts: the observation $\Delta_{\Pi \operatorname{im}(i)}$ commutes with integration and restriction (because it is composed of partial derivatives on $\Pi \operatorname{im}(i)$), the first statement, and Proposition 3.18. $\square$

Proposition 3.30. *Let $\sum_i f_i \otimes a_i \in \mathfrak{h}[W] \otimes A$ be a solution to the QME. The function*

$$\hbar \log \left(\int_{\mathcal{L}_s} e^{\frac{1}{\hbar} \sum_i f_i \otimes a_i} e^{\frac{-\sigma}{2\hbar}} \right)$$

is a solution to the QME in $\mathfrak{h}[\mathrm{H}(W)] \otimes A$.

Proof. The statement follows readily from Proposition 3.24 and Proposition 3.29, noticing that the latter continues to hold in the limit. $\square$

As a simple corollary of this proposition it can be seen that integrating a homotopy of MC elements in $\mathfrak{h}[W]$ leads to a homotopy of MC elements in $\mathfrak{h}[\mathrm{H}(W)]$, by setting $A = \mathbb{R}[t, dt]$.

Proposition 3.31. *Given a homotopy $H(t)$ of $\eta, \nu \in \mathrm{MC}(\mathfrak{h}[W])$, $H'(t)$ given by*

$$H'(t) = \hbar \log \left(\int_{\mathcal{L}_s} e^{\frac{H(t)}{\hbar}} e^{\frac{-\sigma}{2\hbar}} \right)$$

defines a homotopy of $\eta', \nu' \in \mathrm{MC}(\mathfrak{h}[\mathrm{H}(W)])$. $\square$

In fact, given a specific form of $H(t)$, a specific form of $H'(t)$ can be given.

Proposition 3.32. *Given a homotopy $H(t) = A(t) + B(t)dt$ of $\eta, \nu \in \mathrm{MC}(\mathfrak{h}[W])$, the induced homotopy $H'(t) = A'(t) + B'(t)dt$ of $\eta', \nu' \in \mathrm{MC}(\mathfrak{h}[\mathrm{H}(W)])$ is given by*

$$A'(t) = \hbar \log \int_{\mathcal{L}_s} e^{\frac{A(t)}{\hbar}} e^{\frac{-\sigma}{2\hbar}} \quad \text{and} \quad B'(t) = e^{\frac{-A'(t)}{\hbar}} \left(\int_{\mathcal{L}_s} e^{\frac{A(t)}{\hbar}} B(t) e^{\frac{-\sigma}{2\hbar}} \right).$$

Proof. A quick check proves the formulae are correct. $\square$

3.3 Prerequisites on homotopy Lie algebras

This section recalls those facts concerning (quantum) L_∞-algebras seen as relevant in the context of this paper, fixing both terminology and notation.

3.3.1 Quantum homotopy Lie algebras

Similar to L_∞-algebra and cyclic L_∞-algebra structures, a quantum L_∞-algebra structure is given by a MC element in a dgla.

Definition 3.33. *Let V be a super vector space equipped with an odd non-degenerate symmetric bilinear form. A quantum L_∞-algebra structure on V is an even element $m(\hbar) = m_0 + \hbar m_1 + \hbar^2 m_2 + \cdots \in \mathfrak{h}[\Pi V]$ that satisfies the QME. The pair (V, m) is referred to as a quantum L_∞-algebra.*

Remark 3.34. To elaborate on the weight grading a little, m_0 must be at least cubic, m_1 must be at least linear, and there are no restrictions on m_i for all $i \geq 2$.

The restriction on the weight grading in Definition 3.20 used to construct $\mathfrak{h}[\Pi V]$ is a reflection of the stability condition for modular operads, c.f. [21]. Thus one can define quantum L_∞-algebras as algebras over the Feynman transform of the modular closure of the cyclic operad governing commutative algebras. This view is not used here.

Remark 3.35. Quantum L_∞-algebras can be thought of as a 'higher genus' version of cyclic L_∞-algebras as follows. Given a quantum L_∞-algebra $m(\hbar) = m_0 + \hbar m_1 + \ldots$, the canonical derivation associated to m_0 defines an odd cyclic L_∞-algebra on V (see [45]) and, moreover, forgetting the cyclic structure one obtains just an L_∞-algebra. For greater details regarding L_∞-algebras one should consult the various literature: [7, 12, 25, 41]. Further, the derivation associated with m_0 paired with the function m_1 defines a unimodular L_∞-algebra structure on V: see [7, 24] for details.

3.3.2 Minimal models of quantum homotopy Lie algebras

Let (V, m) be a quantum L_∞-algebra. Recall there exist two even super vector space morphisms $i\colon \mathrm{H}(V) \to V$ and $p\colon V \to \mathrm{H}(V)$ compatible with the odd symmetric forms, coming from an SDR (see Section 3.1.3).

Proposition 3.36. *Given an SDR from V to $\mathrm{H}(V)$, the dgla morphism*

$$\iota\colon \mathfrak{h}[\Pi\,\mathrm{H}(V)] \to \mathfrak{h}[\Pi V]$$

defined by $f \mapsto ifp$ is a filtered quasi-isomorphism and hence induces a bijection between the MC moduli sets, i.e. the homotopy classes of quantum L_∞-algebra structures on $\mathfrak{h}[\Pi\,\mathrm{H}(V)]$ and $\mathfrak{h}[\Pi V]$ are in bijective correspondence.

Proof. Take the filtrations by weight grading, c.f. Proposition 3.22, then clearly one has quasi-isomorphisms

$$\frac{F_i\mathfrak{h}[\Pi\,\mathrm{H}(V)]}{F_{i+1}\mathfrak{h}[\Pi\,\mathrm{H}(V)]} \to \frac{F_i\mathfrak{h}[\Pi V]}{F_{i+1}\mathfrak{h}[\Pi V]}.$$

The fact that filtered quasi-isomorphisms induce isomorphisms of MC moduli sets is well-known (see [8, 22], or the Koszul duality of [43, 49]). □

Remark 3.37. The finer notion of a filtered quasi-isomorphism is required here in order to induce an isomorphism of MC moduli sets. Two dglas which are just quasi-isomorphic may not have isomorphic MC moduli sets as show in the example below.

Example 3.38. Let $\mathfrak{g} = \{a, [a,a] : |a| = -1, da = -\frac{1}{2}[a,a]\}$ be a dgla. Clearly $\mathfrak{g}$ is acyclic and quasi-isomorphic to the zero dgla, but $\mathcal{MC}(\mathfrak{g}) = \{0, a\} \neq \mathcal{MC}(0)$.

Definition 3.39. *Given a quantum* L_∞*-algebra* (V, m)*, the minimal model of* (V, m) *is a quantum* L_∞*-algebra* $(\mathrm{H}(V), m')$ *such that* $\iota(m')$ *is homotopic to* m *(as a MC element in* $\mathfrak{h}[\Pi V]$*).*

Remark 3.40. Usually the minimal model for, say, an L_∞-algebra V can be taken to be an L_∞-algebra on the homology $\mathrm{H}(V)$ which is L_∞*-quasi-isomorphic* to V. In the case of quantum L_∞-algebras this does not quite work due to the presence of the non-degenerate bilinear form: there is no longer an especially good notion of a quantum L_∞-map which is not an isomorphism. Therefore, given a quantum L_∞-algebra on the homology $\mathrm{H}(V)$, we need to say in what sense it is equivalent to the original quantum L_∞-algebra on V. In the definition above this is done by extending by zero the quantum L_∞-algebra on $\mathrm{H}(V)$ to all of V and requiring it to be homotopic (as a Maurer-Cartan element) to the original quantum L_∞-algebra.

It should be noted that if we took this definition in the context of usual L_∞-algebras it would be equivalent to the usual definition, stated at the beginning of this remark.

Proposition 3.36 shows the existence and uniqueness up to homotopy of minimal models for quantum L_∞-algebras. However, this does not give an explicit construction or formula for the minimal model. This will be addressed in the next section.

3.4 Integral formulae for minimal models

This section contains the main result of the paper (Theorem 3.42): the quantum L_∞-algebra $(\mathrm{H}(V), m')$ given by the integral formula in Lemma 3.25 is proven to provide the minimal model for the original quantum L_∞-structure on (V, m).

3.4.1 The construction of the minimal model

Recall from Section 3.2.1 it is possible to integrate a solution to the QME to obtain a solution to the QME on homology.

Viewing integration as a morphism of sets $\rho\colon \mathrm{MC}(\mathfrak{h}[\Pi V]) \to \mathrm{MC}(\mathfrak{h}[\Pi\,\mathrm{H}(V)])$ it is a one-sided inverse to $\iota\colon \mathrm{MC}(\mathfrak{h}[\Pi\,\mathrm{H}(V)]) \to \mathrm{MC}(\mathfrak{h}[\Pi V])$, where the restriction of ι is denoted the same by an abuse of notation.

Proposition 3.41. *The morphism* ρ *is a left inverse of* ι*. The morphism* ρ *descends to the level of MC moduli spaces, and therefore induces the inverse bijection on the level of Maurer-Cartan moduli sets.*

Proof. It is an easy check to see that $\rho \circ \iota = id_{\mathrm{MC}(\mathfrak{h}[\Pi\,\mathrm{H}(V)])}$. To prove the second statement, one recalls Proposition 3.31. □

Theorem 3.42. *Given a quantum* L_∞*-structure* (V, m)*, the integral formula*

$$m' = \rho(m) = \hbar \log \int_{\mathcal{L}_s} e^{\frac{m}{\hbar}} e^{\frac{-\sigma}{2\hbar}}$$

defines a quantum L_∞*-algebra* $(\mathrm{H}(V), m')$ *and, moreover, it is the minimal model of* (V, m)*.*

Proof. The first statement is a rephrasing of Lemma 3.25. Next, one simply applies the preceding result to see that $\iota(m')$ is homotopic to m. □

Remark 3.43. Using the results of Appendix 3.A, we can now deduce as a corollary of Theorem 3.42 the known combinatorial formulae for the minimal model in terms of stable graphs given in [12].

Ordinarily, when attempting to lift a solution m to the classical master equation to a solution to the quantum master equation, one is met by a series of obstructions: one would require certain cohomology classes in $\hat{S}\Pi V^*$ to vanish. The first obstruction is just $\Delta(m)$. If this first obstruction is zero not just in cohomology but on the chain level then it turns out, by a straightforward calculation, that all higher obstructions vanish. Such an m, with $\Delta(m) = 0$, is called harmonic. Summarising, we have the following proposition [7].

Proposition 3.44. *Let* (V, ξ) *be an odd cyclic* L_∞*-algebra and let* m *denote the Hamiltonian function associated to* ξ*. If* m *is harmonic* $(\Delta(m) = 0)$*, then* (V, m) *is a quantum* L_∞*-algebra lifting* ξ*.* □

Therefore, having a quantum lift allows one to apply Theorem 3.42, resulting in the following.

Corollary 3.45. *Let* (V, ξ) *be an odd cyclic* L_∞*-algebra and let* m *denote the Hamiltonian function associated to* ξ*. If* m *is harmonic, then the minimal model of* (V, ξ) *is given by* $(\mathrm{H}(V), X_{\rho(m)})$*, where* X_f *denotes the derivation associated to a Hamiltonian* f*.* □

3.4.2 An inverse L-infinity morphism

The morphism ι is a (filtered) quasi-isomorphism of dglas, and as such there exists an inverse L_∞-algebra morphism to ι, meaning an L_∞-algebra morphism which induces the inverse to ι on the level of homology. To construct this morphism a preliminary result is recalled.

Definition 3.46. *Let* V *and* W *be two dglas. An* L_∞*-morphism* $f\colon V \to W$ *is a cdga morphism* $\hat{S}\Pi V^* \to \hat{S}\Pi W^*$*.*

MC elements of dglas play a significant role in the theory of L_∞-algebras: as well as being used to define L_∞-algebra structures, they also correspond to morphisms of pseudo-compact cdgas as shown in the following well known result (which can be extended to include L_∞-algebras, c.f. [6] for example).

Proposition 3.47. *Let V be a dgla and A be a unital pseudo-compact cdga. The functor given by taking $A \mapsto \mathrm{MC}(V \otimes A)$ is represented by $\hat{S}\Pi V^*$.* □

Remark 3.48. An L_∞-morphism of dglas $V \to W$ gives rise, for any unital pseudo-compact cdga A, to a map of sets $\mathrm{MC}(V \otimes A) \to \mathrm{MC}(W \otimes A)$ functorial in A, i.e. a natural transformation. Moreover, any such natural transformation is, by Yoneda's Lemma, equivalent to having an L_∞-morphism $(V, m_V) \to (W, m_W)$. For greater details see [14].

Therefore, if the morphism ρ can be extended to include dg coefficients in a functorial manner, this is equivalent to the required L_∞-algebra morphism.

Definition 3.49. *Let A be a pseudo-compact cdga. The morphism of sets*

$$\tilde{\rho}\colon \mathrm{MC}(\mathfrak{h}[\Pi V], A) \to \mathrm{MC}(\mathfrak{h}[\Pi\,\mathrm{H}(V)], A)$$

is given by mapping $\sum_{i\in I} f_i \otimes a_i$ to the function given by

$$\hbar \log \int_{\mathcal{L}_s} e^{\frac{1}{\hbar}\sum_{i\in I} f_i\otimes a_i} e^{\frac{-\sigma}{2\hbar}}.$$

The morphism $\tilde{\rho}$ is clearly functorial in both arguments, and by Proposition 3.30 it is well-defined. Further, $\tilde{\rho}$ is a one-sided inverse to ι on the level of MC sets. Let $\tilde{\iota}\colon \mathrm{MC}(\mathfrak{h}[\Pi\,\mathrm{H}(V)], A) \to \mathrm{MC}(\mathfrak{h}[\Pi V], A)$ be the A-linear morphism corresponding to ι, defined in the obvious way.

Theorem 3.50. *$\tilde{\rho}$ is a left inverse of $\tilde{\iota}$. The morphism $\tilde{\rho}$ provides the inverse L_∞-morphism to ι.*

Proof. The first statement is straightforward. Using Yoneda's Lemma, as explained in Remark 3.48, one arrives at the proof of the second statement. □

3.A Integrals as sums over graphs

Formal integrals like those used throughout the paper are commonly treated using the formalism of Feynman Diagrams. More precisely, these integrals can often be written as formal series with sums taken over certain graphs. Within this appendix, a presentation as a formal series summing over stable graphs will be given for the integrals considered in this paper. The same presentation is given in [15, Chapter 2, Section 3]. Our proof, however, is different and is a mild generalisation of the proof of the case of 'usual' graphs given in [16]. A very similar result is given in [17, Example 3.10]. It should be noted, however, that the obtained formulae in this section are precisely those given in the context of minimal models for algebras over modular operads in [12]. To give the combinatorial presentation, a brief discourse to introduce the relevant material is necessary.

3.A.1 Stable graphs

Stable graphs were introduced by Ginzburg and Kapranov [23] in the context of modular operads and later used in giving formulae for minimal models by Chuang and Lazarev [11, 12]. Here only the briefest of details will be recalled and for more details one should consult those papers cited.

Definition 3.51. *A graph G is given by the following data:*

- *A finite set of half edges,* $\mathrm{Half}(G)$*, and a finite set of vertices,* $\mathrm{Vert}(G)$*, with a morphism* $f\colon \mathrm{Half}(G) \to \mathrm{Vert}(G)$ *and an involution* $\sigma\colon \mathrm{Half}(G) \to \mathrm{Half}(G)$*.*
- *The set of edges,* $\mathrm{Edge}(G)$*, is the set of two-cycles of σ and the legs,* $\mathrm{Leg}(G)$*, are the fixed points of σ.*
- *For a vertex v the valence is the cardinality of $f^{-1}(v)$, i.e. the number of half edges attached to v.*

Definition 3.52. *A stable graph is a graph G such that every vertex v is decorated with a non-negative integer $g(v)$, called the genus of the vertex v, such that $2g(v)+n(v) \geq 3$. The homology* $\mathrm{H}_\bullet(-)$ *of a stable graph is given by the homology of corresponding one dimensional CW-complex. The genus of a stable graph (denoted $g(G)$) is given by* $\dim(\mathrm{H}_1(G)) + \sum_{v\in\mathrm{Vert}(G)} g(v)$*.*

Definition 3.53. *Given a stable graph G its Euler Characteristic $\chi(G)$ is given by the difference* $\dim(\mathrm{H}_0(G)) - g(G)$*.*

Example 3.54. For a connected stable graph G where every vertex has genus zero, one recovers the classical result for graphs $\chi(G) = |\mathrm{Vert}(G)| - |\mathrm{Edge}(G)|$.

3.A.2 Feynman Expansions

The ideas behind the arguments used in this section are largely standard and closely follow those of Etingof [16]. Indeed, the formulae of [16] can be extracted from our formulae by restricting to those stable graphs where every vertex has genus zero.

Throughout this section W is an odd symplectic vector space with a decomposition $W = \mathrm{H}(W) \oplus \mathcal{L}_s \oplus \Pi\,\mathrm{im}(d)$ given by an SDR, see 3.1.3. Recall the form $\sigma(?) = \langle ?, d?\rangle$ is non-degenerate on $\mathcal{L}_s$ and denote the inverse form on $\mathcal{L}_s^*$ by σ^{-1}. Integration over $\mathcal{L}_s$ is given by $id_{\mathrm{H}(W)} \otimes \int_{\mathcal{L}_s}$.

Definition 3.55. *Let $f \in \mathfrak{h}[W]$. Given a connected stable graph G, the Feynman amplitude* $F(G) \in (\mathrm{H}(W)^*)^{\otimes|\mathrm{Leg}(G)|}$ *is given as follows:*

- *place at every vertex with genus i, j half edges, and k legs the component in the symmetric tensor* $(\mathcal{L}_s^*)^{\otimes j} \otimes (\mathrm{H}(W)^*)^{\otimes k}$ *of the coefficient of h^i in f.*
- *take contraction of tensors along each edge using the form σ^{-1}.*

If G is disconnected, then $F(G)$ is given by the product of the amplitudes given by its connected components. The empty stable graph has amplitude 1.

As is the case in all Feynman expansions, the key is Wick's Theorem.

Theorem 3.56 (Wick's Theorem)**.** *Let $\phi_1, \ldots, \phi_m \in \mathcal{L}_s^*$. If $m = 2k$,*

$$\int_{\mathcal{L}_s} \phi_1 \ldots \phi_m e^{\frac{-\sigma}{2}} = \sum_{\text{pairings}} \sigma^{-1}(\phi_{i_1}, \phi_{i_2}) \ldots \sigma^{-1}(\phi_{i_{k-1}}, \phi_{i_k}).$$

If m is odd, the integral is zero.

Proof. For the case m is odd the result is immediate. The case m is even is almost as straightforward, but requires to be broken down further: it is necessary to consider the cases of even integration and odd integration independently. For even integration it suffices to prove the result for $\phi_1 = \cdots = \phi_m = x$ for a canonical basis element x, i.e. one can assume $\mathcal{L}_s$ is of total dimension one. The result now readily follows from the definition of integration. For odd integration it, again, reduced to a simple case: it suffices to prove the result for a pairing of canonical odd coordinates ξ_i and ξ_j. Since odd elements square to zero it must be the case that $m = 2$ and the result is then obvious. $\square$

Theorem 3.57. *Let $f \in \mathfrak{h}[W]$, then*

$$\int_{\mathcal{L}_s} e^{\frac{f}{\hbar}} e^{\frac{-\sigma}{2\hbar}} = \sum_G \hbar^{-\chi(G)} \frac{F(G)}{|\operatorname{Aut}(G)|},$$

where the sum is over all (possibly disconnected) stable graphs.

Proof. One can write the restriction of f to $\mathrm{H}(W)^* \oplus \mathcal{L}_s^*$ as $\sum_{i,j,k} \frac{1}{j!k!} \hbar^i f_{i,j,k}$, where j is the number of linear factors of $\mathcal{L}_s$ in $f_{i,j,k}$ and k is the number of linear factors of $\mathrm{H}(W)$. Making a substitution of $y\hbar^{\frac{1}{2}} = x$ in $\mathcal{L}_s$ and after expanding $e^{\frac{f}{\hbar}}$ in terms of its Taylor expansion, one can write

$$\int_{\mathcal{L}_s} e^{\frac{f}{\hbar}} e^{\frac{-\sigma}{2\hbar}} = \sum_N Z_N,$$

where the sum is over $N = (n_{i,j,k})$, where each $n_{i,j,k}$ is an integer and is zero if $2i + j + k < 3$, and

$$Z_N = \int_{\mathcal{L}_s} \left(\prod_i \prod_j \prod_k \frac{\hbar^{\left(i + \frac{j}{2} - 1\right) n_{i,j,k}}}{(j!k!)^{n_{i,j,k}} n_{i,j,k}!} f_{i,j,k}^{n_{i,j,k}} \right) e^{\frac{-\sigma}{2\hbar}}.$$

Clearly, every $f_{i,j,k}$ is a product of linear functions and therefore, using Wick's Theorem (3.56), the integral for each N is given combinatorially as follows: every $f_{i,j,k}$ gives a decorated flower, i.e. a vertex with genus i, j half-edges, and k legs. One can then choose a pairing, p, of half-edges of all flowers and contract using σ^{-1} to produce a function F_p. Thus

$$Z_N = \prod_i \prod_j \prod_k \frac{\hbar^{\left(i + \frac{j}{2} - 1\right) n_{i,j,k}}}{(j!k!)^{n_{i,j,k}} (n_{i,j,k}!)} \sum_p F_p.$$

A choice of pairing p of half-edges can be visualised as a glueing. Thus, a gluing creates a stable graph G and F_p is, in fact, precisely the Feynman amplitude $F(G)$. What's more, it is clear that any stable graph with $n_{i,j,k}$ vertices of genus i, valence j, and having k legs can be obtained from a pairing in this way. Since the goal is to sum over stable graphs, one must take care to deal with the redundancies arising from the fact that the same graph can be obtained in multiple ways from different pairings of half-edges. To this end, consider the permutations of half-edges that preserve decorated flowers. This group of permutations involves three parts: the permutations of flowers with a given genus, valence, and number of legs; permutations of the half-edges of a flower; and permutations of the legs of a flower. To be precise, the group is the semi-direct product

$$\prod_i \prod_j \prod_k S_{n_{i,j,k}} \ltimes (S_j^{n_{i,j,k}} \times S_k^{n_{i,j,k}}),$$

which has order $\prod_i \prod_j \prod_k (j!k!)^{n_{i,j,k}}(n_{i,j,k}!)$. It is clear to see that the group acts transitively on all pairings of half-edges that result in a particular stable graph and the stabiliser of such a pairing is the automorphism group of the resulting graph. Therefore, the number of pairings resulting in a stable graph G is given by $\frac{\prod_i \prod_j \prod_k (j!k!)^{n_{i,j,k}}(n_{i,j,k}!)}{|\operatorname{Aut}(G)|}$. Putting this all together, the result is obtained. $\square$

Notice the series on the right hand side of Theorem 3.57 involves arbitrary (possibly negative) powers of $\hbar$. This is to be expected since $e^{\frac{f}{\hbar}}$ has arbitrary powers of $\hbar$ and integration fixes the weight grading. These negative powers come from disjoint unions of graphs such as the one with two vertices, one edge, and four legs (two at each vertex). Taking the logarithm of the series in Theorem 3.57 has the initially surprising effect of reducing the sum to over connected stable graphs, and thus multiplying by $\hbar$ results in only non-negative powers of $\hbar$.

Theorem 3.58.

$$\hbar \log \int_{\mathcal{L}_s} e^{\frac{f}{\hbar}} e^{\frac{-\sigma}{2\hbar}} = \sum_{G \text{ connected}} \hbar^{g(G)} \frac{F(G)}{|\operatorname{Aut}(G)|},$$

where the sum is over all connected stable graphs.

Proof. Writing any disconnected graph as $G = G_1^{k_1} \dots G_l^{k_l}$, for non-isomorphic connected graphs G_i, it is clear that $F(G) = F(G_1)^{k_1} \dots F(G_l)^{k_l}$ and $\chi(G) = k_1\chi(G_1) + \cdots + k_l\chi(G_l)$. Further, $|\operatorname{Aut}(G)| = \prod_i \big(|\operatorname{Aut}(G_i))|^{k_i}(k_i!)\big)$. Thus, exponentiating the series

$$\frac{1}{\hbar} \sum_{G \text{ connected}} \hbar^{g(G)} \frac{F(G)}{|\operatorname{Aut}(G)|},$$

one arrives at the series of Theorem 3.57. $\square$

Section 3
Bibliography

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Conclusion

The Maurer-Cartan equation is, without doubt, an important and far-reaching equation. This thesis has displayed the flexibility of the Maurer-Cartan equation in only a few of its widespread applications in mathematics and mathematical physics. Here the results of this thesis will be briefly reviewed and the relation these results have to existing research literature will be discussed. Some possible directions for future research are also mentioned.

The Koszul dualities developed in Section 1 and Section 2 suggest that in order to generalise existing dualities one should have a category of curved objects on one side of the duality. This hypothesis is further strengthened by the work of Positselski [Pos11], as well as the work of Chuang, Lazarev, and Mannan [CLM14]. Loosely speaking, allowing the relaxation of the nilpotency condition for the derivation in a differential graded object, allows one to relax conditions on the dual object. In fact, having a square zero derivation should be viewed as a condition on a curved object: it is the condition that the algebra in question has zero curvature. Indeed, in dualities, the curvature of an algebra is a reflection the failure of its dual object to satisfy certain conditions. For example, in the duality between augmentated commutative differential graded algebras and differential graded Lie algebras, the differential graded Lie algebra analogue of an augmentation of a commutative differential graded algebra is a MC element. Since curved objects are not endowed with the canonical MC element (the 0 element) unless the curvature is zero, its dual commutative differential graded algebra is not necessarily augmented.

As a natural generalisation to differential graded objects, curved objects have found a growing number of applications in mathematics and mathematical physics since they were first introduced by Positselski [Pos93]. Curved objects have, for example, found applications in: derived categories of D-branes in Landau-Ginzburg models [Laz07, Orl09, Pre12, Swe75]; non-commutative geometry [Blo10, BD10, Sch03]; graph complexes [LS12]; differential geometry [CD01]; and quantum field theory [Cos11a].

The duality proven in Section 1 allows one to construct curved Lie algebra models for unbased rational topological spaces and is similar to (and in fact uses results from) the disconnected theory of Lazarev and Markl [LM15]. The authors in loc. cit. constructed Lie models for based rational topological spaces (as well as commutative algebra models in both the based and unbased case). One beneficial difference of using curved Lie algebras over differential graded Lie algebras in Section 1 is that the coproduct is categorical. That is, the analogue of the disjoint union of topological spaces for curved Lie algebras is a categorical coproduct. Having the categorical coproduct immediately allows one to know certain properties of the construction 'for free', whereas Lazarev and Markl had to work to show their disjoint product satisfied the properties necessary for their disconnected theory. It seems likely that one could describe a theory using curved Lie algebras equivalent to Lazarev and Markl's disconnected theory for Lie models of based spaces in a way that is almost analogous to Section 1.

Further, using the theory developed in Section 1, one is able to construct Lie models for mapping spaces (under certain restrictions) using rational models, derived functors, and the formalism of Maurer-Cartan moduli sets. This is building upon the previous results using rational homotopy theory [BPS89, BS97, Hae82] and using Lie models [Ber15, BFM09, BFM11]. These previous results, however, do not lend themselves to calculations. The homotopy groups of function spaces were computed in terms of Harrison-André-Quillen cohomology by Block and Lazarev [BL05], however their methods do not extend to constructions of rational homotopy models of function spaces. This gap was filled in further work of Lazarev [Laz13]. Additionally, as stated in Remark 1.90, under nice conditions one can combine the results [Laz13, Theorem 8.1] and Corollary 1.89 to construct a model for the mapping space as a coporduct of finitely many Maurer-Cartan moduli spaces.Outside of the nice setting required, however, one cannot make such constructions. Such failures can happen in some apparently straightforward cases: the space of maps between 2-dimensional spheres, for example, is one of the cases in which this construction cannot be made, as stated in the remark. It is unclear, to the author, how one should attempt to overcome these difficulties and achieve a generalised result.

The theory of marked curved Lie algebras developed in Section 2 can be seen as associating a curved Lie algebra to every marked point, i.e. a family of curved Lie algebras, thus giving a model for (curved) Lie algebroids in the setting considered. This perspective and the theory of Lie algebroids is, however, never used in this thesis and it has only recently come to the attention of the author. In particular, the author cannot, as yet, comment on whether or not such a perspective could prove to be beneficial.

Applying the theory of marked curved Lie algebras and the Quillen equivalence developed in Section 2 to algebraic deformation theory, i.e. the definition of deformation functors over (not necessarily local) pseudo-compact commutative differential graded algebras, is a straightforward construction. This algebraic deformation theory does not require the language of stacks or infinity categories unlike other definitions of deformation functors (still over local algebras), c.f. [Lur09, Man99, Pri10]. Further, the deformation functors defined in Section 2 are amenable to calculations using only the theory of MC moduli sets and derived functors. It is slightly disappointing, however, that the deformation functor given in Definition 2.69 is not given in a more aesthetically pleasing form. It would be preferable to present it in a form similar to that of Definition 2.72, i.e. as a set rather than a product of a coproduct of sets, similar to the case when the marked curved Lie algebra has one marked point. However, one could not hope to find such a form, because the coproduct does not commute with the Maurer-Cartan Moduli space functor. This is a problem that completely disappears when there is only one marked point.

The Batalin-Vilkovisky formalism has been found many applications in mathematics and mathematical physics since it was first introduced by Batalin and Vilkovisky [BV81, BV83]. Providing a framework in which odd symplectic geometry, homological algebra, and path integrals interact successfully, it is not surprising that the BV-formalism has found many applications outside of its original intended purpose. For example, the Batalin-Vilkovisky formalism has been used in the fields of deformation quantisation [CF01], an alternative description of the graph complex [HL09b], an alternative proof of the Kontsevich theorem [QZ11], link invariants [Iac08], and manifold invariants [CM10]. The fact that integration over certain isotropic spaces of super vector spaces with non-trivial homology can be used to construct solutions to the QME on the homology is a known fact, c.f. [Cos11b, CM10]. However, showing the relationship between the original function and the 'effective' function on homology obtained after integrating is not something available (as far as the author knows) in the literature. That being said, such integral methods were used in the work of Cattaneo and Mnev [CM10] to find 3-manifold invariants.

L_∞-algebra structures are common in mathematical physics, where they are often called strong homotopy Lie algebras. Moreover, the BV-formalism can be described in the language of (quantum) L_∞-algebras. Indeed, the first place quantum L_∞-algebras appeared was the work of Zwiebach (who had been following the BV-formalism) on closed string field theory [Zwi93], where the genus zero structures (i.e. $\hbar = 0$) were spotted to be L_∞-algebras. Markl [Mar01] further investigated this link and noted that the genus zero structure (or has Markl wrote 'tree level' specialisation) is a structure richer than just an L_∞-algebra—it is a cyclic L_∞-algebra. Braun and Lazarev [BL15] asked the reverse question: given a cyclic L_∞-algebra, under what conditions can we lift this structure to a quantum L_∞-algebra? They gave a complete answer in the case of 'odd doubles' [BL15, Theorem 6.7]. There is not much literature available explicitly concerning quantum L_∞-algebras, but such structures clearly have strong ties to field theory and L_∞-algebras. For instance, a question that readily arises is the following: do quantum L_∞-algebras govern deformation problems in a similar way to the way that L_∞-algebras and dglas do?

Using the language of L_∞-algebras to present the Batalin-Vilkovisky formalism as a machine to understand integrals is not something that has been studied extensively, at least to the knowledge of the author. Understanding the Batalin-Vilkovisky quantisation procedure from the perspective of L_∞-algebras and in particular the application to understanding integrals could lead to some interesting questions and research.

It was initially hoped—by the author and Christopher Braun—that it would not be necessary to include the Feynman diagram formalism in Section 3, but attempts to prove Proposition 3.24 without it proved difficult. In hindsight, including the Feynman diagram formalism on this occasion does lead to a more complete result. That being said, a conceptual proof of Proposition 3.24 may very well exist. In support of this idea, a simple conceptual proof can be given in the case when the homology is zero.

Lemma 3.59. *Given an exact odd symplectic vector space V and a quantum L_∞-algebra structure $m \in \mathfrak{h}[\Pi V]$, the formal Laurent Series*

$$m' = \hbar \log \int_{x \in \mathcal{L}} e^{\frac{m}{\hbar}(x)} e^{\frac{-1}{2\hbar}\langle x, dx\rangle}$$

consists of non-negative powers of $\hbar$ only, i.e. it is a formal power series in $\hbar$.

Proof. Recall, the integral respects the MC set. Observe that $\mathbb{R}((\hbar))$ is a dgla where every element is a MC element. Moreover each element is the unique representative of its homotopy class. Therefore, given the minimal model $\phi \in \mathbb{R}$ of m, we can conclude m' is homotopic to ϕ and must therefore be equal. □

Section 3 has, perhaps, the most scope for future research. The relationship between the Batalin-Vilkovisky formalism and L_∞-algebras appears to be strong and warrants further investigation, as already remarked. An immediate goal for the author is to extend the results of Section 3 from quantum L_∞-algebras to classical L_∞-algebras. Before completely generalising to classical L_∞-algebras, it will perhaps be beneficial to begin with the unimodular case. Jointly with Christopher Braun, some informal progress has been made in this direction and it is hypothesised that a conceptual argument will be achieved using the doubling constructions of [BL15]. Further, it ought to be possible to prove analogues to Theorem 3.42 over different operads. For example, one could construct minimal models for quantum A_∞-algebras. Moving from the Lie to associative case introduces several difficulties—one would need to define formal integration for associative algebras, for a start.

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