

A pictorial introduction to differential geometry, leading to Maxwell's equations as three pictures.

Dr Jonathan Gratus

Physics Department, Lancaster University and the Cockcroft Institute of accelerator Science.

j.gratus@lancaster.ac.uk

<http://www.lancaster.ac.uk/physics/about-us/people/jonathan-gratus>

September 26, 2017

arXiv:1709.08492v1 [math.DG] 21 Sep 2017

Abstract

In this article we present pictorially the foundation of differential geometry which is a crucial tool for multiple areas of physics, notably general and special relativity, but also mechanics, thermodynamics and solving differential equations. As all the concepts are presented as pictures, there are no equations in this article. As such this article may be read by pre-university students who enjoy physics, mathematics and geometry. However it will also greatly aid the intuition of an undergraduate and masters students, learning general relativity and similar courses. It concentrates on the tools needed to understand Maxwell's equations thus leading to the goal of presenting Maxwell's equations as 3 pictures.

Prefix

When I was young, somewhere around 12, I was given a book on relativity, gravitation and cosmology. Being dyslexic I found reading the text torturous. However I really enjoyed the pictures. To me, even at that age, understanding spacetime diagrams was natural. It was obvious, once it was explained, that a rocket ship could not travel more in space than in time and hence more horizontal than 45° . Thus I could easily understand why, having entered a stationary, uncharged black hole it was impossible to leave and that you were doomed to reach the singularity. Likewise for charged black holes you could escape to another universe.

I hope that this document may give anyone enthusiastic enough to get a feel for differential geometry with only a minimal mathematical or physics education. However it helps having a good imagination, to picture things in 3 dimension (and possibly 4 dimension) and a good supply of pipe cleaners.

I teach a masters course in differential geometry to physicists and this document should help them to get some intuition before embarking on the heavy symbol bashing.

Contents

| | | |
|----------|-----------------------------------------------------------------------|-----------|
| 1 | Introduction | 3 |
| 2 | Manifolds, scalar and vector fields | 5 |
| 2.1 | Manifolds. | 5 |
| 2.2 | Submanifolds. | 6 |
| 2.3 | Boundaries of submanifold | 6 |
| 2.4 | Scalar fields. | 6 |
| 2.5 | Vector fields. | 6 |
| 3 | Orientation. | 8 |
| 3.1 | Orientation on Manifolds (internal) | 8 |
| 3.2 | Orientation on submanifolds: Internal and external | 9 |
| 3.3 | Internal and external, twisted and untwisted | 9 |
| 3.4 | Concatenation of orientations . . . | 10 |
| 3.5 | Inheritance of orientation by the boundary of a submanifold | 10 |
| 3.6 | Orientations of vectors | 11 |
| 3.7 | Twisting and untwisting | 11 |
| 4 | Closed forms. | 11 |

| | | | | | |
|------------------------|-----------------------------------------------------------------------|-----------|----|------------------------------------------------------------|----|
| 4.1 | Multiplication of vectors and forms by scalar fields. | 14 | 4 | A selection of submanifolds and their boundaries | 7 |
| 4.2 | Addition of forms. | 14 | 5 | Pathological examples which are not submanifolds. | 7 |
| 4.3 | The wedge product also known as the exterior product. | 14 | 6 | A representation of a scalar field. | 8 |
| 4.4 | Integration. | 15 | 7 | A scalar field as a function of x and y | 8 |
| 4.5 | Conservation Laws | 16 | 8 | A scalar field depicted as contours. | 8 |
| 4.6 | DeRham Cohomology. | 16 | 9 | A representation of a vector field. | 8 |
| 4.7 | Four dimensions. | 17 | 10 | Internal orientations. | 8 |
| 5 | Non-closed forms. | 18 | 11 | Equivalent orientations. | 9 |
| 5.1 | The exterior differential operator. | 19 | 12 | Creating a helix. | 10 |
| 5.2 | Integration and Stokes's theorem for non-closed forms. | 19 | 13 | Inheritance of internal orientation on a boundary. | 11 |
| 6 | Other operations with scalars, vectors and forms | 20 | 14 | Inheritance of external orientation on a boundary. | 11 |
| 6.1 | Combining a 1-form and a vector field to give a scalar field. | 20 | 15 | Representation of a top-form. | 12 |
| 6.2 | Vectors acting on a scalar field. | 20 | 16 | An untwisted 2-form on a 3-dim manifold. | 12 |
| 6.3 | Internal contractions. | 20 | 17 | An untwisted 2-form on a 3-dim manifold. | 12 |
| 7 | The Metric | 21 | 18 | An untwisted 1-form on a 3-dim manifold | 12 |
| 7.1 | Representing the metric: Unit ellipsoids and hyperboloids | 21 | 19 | Addition of 1-forms in 2-dim | 14 |
| 7.2 | Length of a vector | 22 | 20 | Wedge product of untwisted 1-form and 2-form. | 14 |
| 7.3 | Orthogonal subspace | 22 | 21 | Wedge product of two untwisted 2-form. | 14 |
| 7.4 | Angles and γ -factors | 23 | 22 | Wedge product of three untwisted 2-form. | 15 |
| 7.5 | Magnitude of a form. | 23 | 23 | Wedge product of twisted forms | 15 |
| 7.6 | Metric dual. | 23 | 24 | Wedge product of a twisted and an untwisted form. | 15 |
| 7.7 | Hodge dual. | 24 | 25 | Integration of a 2-form. | 15 |
| 7.8 | The measure from a metric | 24 | 26 | Integration of a 1-form. | 16 |
| 8 | Smooth maps | 24 | 27 | A conservation law. | 16 |
| 8.1 | Pullbacks of p -forms. | 25 | 28 | Conservation law for all space. | 17 |
| 8.2 | Pullbacks of metrics. | 27 | 29 | DeRham Cohomology on an annulus. | 17 |
| 8.3 | Pushforwards of vectors. | 27 | 30 | DeRham Cohomology on a Torus. | 17 |
| 9 | Electromagnetism and Electrodynamics | 27 | 31 | Failed non-zero integrals. | 18 |
| 9.1 | Electrostatics and Magnetostatics. | 29 | 32 | 2-form in 4-dimensions: plane picture. | 18 |
| 9.2 | Maxwell's equations as four space-time pictures | 30 | 33 | 2-form in 4-dimensions: moving line picture. | 18 |
| 9.3 | Lorentz force equation | 31 | 34 | Example of a non-closed 1-form (black) in 2-dim. | 19 |
| 9.4 | Constant space slice of Maxwell's equations. | 32 | 35 | A non-closed, non-integrable 1-form. | 19 |
| 10 | Conclusion | 32 | 36 | More non-closed forms. | 19 |
| List of Figures | | | 37 | Demonstration of Stokes's theorem. | 20 |
| 1 | Zooming in on a sphere. | 5 | 38 | Combining vectors and 1-forms | 20 |
| 2 | Zooming in on a corner. | 5 | 39 | Demonstration of internal contraction. | 20 |
| 3 | 0, 1, 2 and 3-dimensional manifolds. | 6 | | | |

| | | |
|----|-----------------------------------------------------------|----|
| 40 | Representations of the metric in 3-dim space. | 21 |
| 41 | The spacetime metric. | 21 |
| 42 | The metric ellipsoid and length. . . | 22 |
| 43 | The metric hyperboloid and length. . . | 22 |
| 44 | The metric ellipsoid and orthogonality. | 23 |
| 45 | The metric hyperboloid and orthogonality. | 23 |
| 46 | The metric ellipsoid and angles. . . | 23 |
| 47 | The metric hyperboloid and γ -factor. | 23 |
| 48 | The metric, magnitude of a 1-form | 24 |
| 49 | The metric dual. | 24 |
| 50 | The Hodge dual. | 25 |
| 51 | The measure from the metric. . . . | 25 |
| 52 | Projection 3-dim to 2-dim. | 25 |
| 53 | Projection 3-dim to 1-dim. | 26 |
| 54 | Pullback, 2-dim to 3-dim, 2-form . | 26 |
| 55 | Pullback, 2-dim to 3-dim, 1-form . | 26 |
| 56 | Pullback, 1-dim to 3-dim, 1-form . | 26 |
| 57 | Pullback, tangential | 26 |
| 58 | Pullback, 3-dim to 2-dim, 2-form . | 27 |
| 59 | Pullback, 3-dim to 2-dim, 1-form . | 27 |
| 60 | Pullback, 3-dim to 1-dim, 1-form . | 27 |
| 61 | The pullback of a metric onto an embedding. | 27 |
| 62 | The Push forward of the vector field. | 28 |
| 63 | The magnetic 2-form in space. . . | 29 |
| 64 | Gauss's Law in electrostatics. . . . | 29 |
| 65 | The B and H fields from a wire. . . | 30 |
| 66 | The B and H fields from a magnet. | 30 |
| 67 | Conservation of charge | 30 |
| 68 | Conservation of magnetic flux . . . | 31 |
| 69 | The excitation fields | 31 |
| 70 | The constitutive relations in the vacuum | 31 |
| 71 | The Lorentz force law. | 32 |
| 72 | Faraday's law of induction on the plane $z = 0$ | 32 |
| 73 | Maxwell Ampères law on the plane $z = 0$ | 33 |

List of Tables

| | | |
|---|------------------------------------------------------|----|
| 1 | External orientations. | 9 |
| 2 | Types of orientation for all objects. | 10 |
| 3 | Concatenation of orientations. . . . | 10 |
| 4 | Examples of twisting and untwisting. | 11 |
| 5 | The fields in electromagnetism . . . | 28 |
| 6 | The relativistic field in electromagnetism | 28 |

1 Introduction

Differential geometry is an incredibly useful tool in both physics and mathematics. Due to the heavy technical mathematics need to define all the objects precisely it is not usually introduced until the final years in undergraduate study. Physics undergraduates usually only see it in the context of general relativity, where spacetime is introduced. This is unfortunate as differential geometry can be applied to so many objects in physics. These include:

- The 3-dimensional space of Newtonian physics.
- The 4-dimensional spacetime of special and general relativity.
- 2-dimensional shapes, like spheres and torii.
- 6-dimensional phase space in Newtonian mechanics.
- 7-dimensional phase-time space in general relativity.
- The state space in thermodynamics.
- The configuration space of physical system.
- The solution space in differential equations.

There are many texts on differential geometry, many of which use diagrams to illustrate the concepts they are trying to portray, for example [1–8]. They probably date back to Schouten [8]. The novel approach adopted here is to present as much differential geometry as possible solely by using pictures. As such there are almost no equations in this article. Therefore this document may be used by first year undergraduates, or even keen school students to gain some intuition of differential geometry. Students formally studying differential geometry may use this text in conjunction with a lecture course or standard text book.

In geometry there is always a tension between drawing pictures and manipulating algebra. Whereas the former can give you intuition and some simple results in low dimensions, only by expressing geometry in terms of mathematical symbols and manipulating them can one derive deeper results. In addition, symbols do not care whether one is in 1,

3, 4 dimensions or even 101 dimensions (the one with the black and white spots), whereas pictures can only clearly represent objects in 1, 2, or 3 dimensions. This is unfortunate as we live in a 4-dimensional universe (space-time) and one of the goals of this document is to represent electrodynamics and Maxwell's equations.

As well as presenting differential geometry without equations, this document is novel because it concentrates on two aspects of differential geometry which students often find difficult: Exterior differential forms and orientations. This is done even before introducing the metric.

Exterior differential forms, which herein we will simply call **forms**, are important both in physics and mathematics. In electromagnetism they are the natural way to define the electromagnetic fields. In mathematics they are the principal objects that can be integrated. In this article we represent forms pictorially using curves and surfaces.

Likewise orientations are important for both integration and electromagnetism. There are two types of orientation, internal and external, also called twisted and untwisted, and these are most clearly demonstrated pictorially. Some physicists will be familiar with the statement that the magnetic field is a twisted vector field, also known as a pseudo-vectors or an axial vector. Here the twistedness is extended to forms and submanifolds.

The metric to late is one of the most important objects in differential geometry. In fact without it some authors say you are not in fact studying differential geometry but instead another subject called differential topology. Without a metric you can think of a manifold as made of a rubber, which is infinity deformable, whereas with a metric the manifold is made of concrete. The metric gives you the length of vectors and the angle between two vectors. It can also tell you the distance between two points. In higher dimensions the metric tells you the area, volume or n -volume of your manifold and submanifolds within it. It gives you a measure so that you can integrate scalar fields.

General relativity is principally the study

of 4-dimensional manifolds with a spacetime metric. It is the metric which defines curvature and therefore gravity. General relativity therefore is the study of how the metric affects particles and fields on the manifold and how these fields and particles effect the metric. There are many popular science books and programs which introduce general relativity by showing how light and other object move in a curved space, usually by showing balls moving on a curved surface. Therefore one of the things we do not cover in this document are the spacetime diagrams in special relativity and the diagrams associated with curvature, general relativity and gravity.

Since the metric is so important one may ask why one should study manifolds without metrics. There are a number of reasons for doing this. One is that many manifolds do not possess a metric. To be more precise, since on every manifold one may construct a metric, one should say it has no single preferred metric, i.e. one with physical significance. For example phase space does not possess a metric, neither do the natural manifolds for studying differential equations. Even in general relativity, one may consider varying the metric in order to derive Einstein's equations and the stress-energy-momentum tensor. Transformation optics, by contrast, may be considered as the study of spacetimes with two metrics, the real and the optical metric. As a result knowing about objects and operations which do not require a metric is very useful. The other reason for introducing differential forms before the metric is that, if one has a metric, then one can pass between 1-forms and vectors. Thus they both look very similar and it is difficult for the student to get an intuition about differential forms as distinct from vectors.

One of the goals of this article is to introduce the equations of electromagnetism and electrodynamics pictorially. Electromagnetism lends itself to this approach as the objects of electromagnetism are exterior differential forms, and most of electromagnetism does not need a metric. Indeed the four macroscopic Maxwell's equations can be written (and therefore drawn) without a metric. It is only when we wish to give the addi-

tional equations which prescribe vacuum or the medium that we need the metric. This has led to school of thought [9] which suggest that the metric may be a consequence of electromagnetism.

The challenge when depicting Maxwell's equations pictorially is that they are equations on four dimensional spacetime and we lack 4-dimensional paper. Unfortunately, were we blessed with such 4-dimensional paper, we would have to live in a universe with 5 spatial dimensions and therefore 6-dimensional spacetime. Whatever field the 5-dimensional equivalent of me would be attempting to represent in pictures he/she/it would presumably be bemoaning the lack of 6-dimensional paper.

This article is organised as follows. We start in section 2 with basic differential geometry, discussing manifolds, submanifolds, scalar and vector fields. In section 3 we introduce both untwisted and twisted orientation, dealing first with orientations on submanifolds, since this is the easiest to transfer to other objects like forms and vectors. We then introduce closed exterior differential forms, section 4, showing how one can add them and take the wedge product. We then talk about integration and hence conservation laws. This leads to a discussion about manifolds with "holes" in them. Finally in this section we discuss the challenge of representing forms in four dimensions. In section 5 we discuss non-closed forms and how to visualise and integrate them. In section 6 we summarise a number of additional operation one can do with vectors and forms. The metric is introduced in section 7 where we show how we can use it to measure the length of a vector, angles between vectors as well the the metric and Hodge dual. Section 8 is about smooth maps which shows how forms can be pulled from one manifold to another. Finally we reach our goal in section 9 which shows how to picture Maxwell's equations and the Lorentz force equation. We start with static electromagnetic fields as these are the easiest to visualise and then attempt to draw the full four dimensional pictures. We then return to three dimensional pictures but this time one

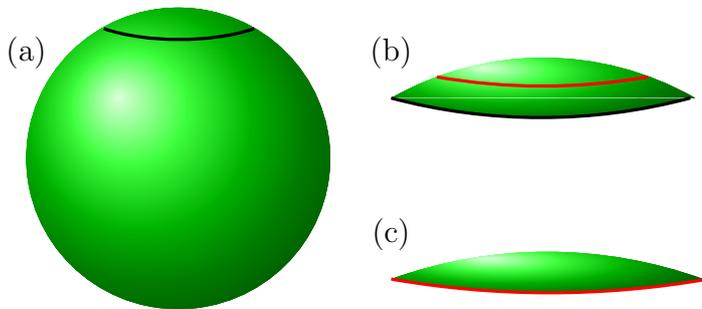


Figure 1: Zooming in on a sphere. As we go from (a) to (b) to (c), the patch of the sphere becomes flatter.

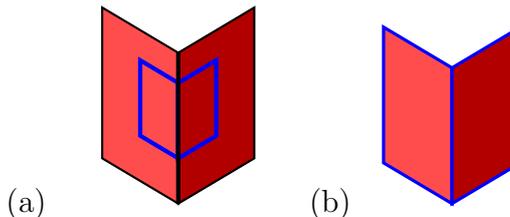


Figure 2: Zooming in on a corner. As we go from (a) to (b), the patch does not get any flatter. This is not a manifold.

time and two spatial dimensions. We then conclude in section 10 with a discussion of further pictures one may contemplate.

All figures are available in colour online.

2 Manifolds, scalar and vector fields

2.1 Manifolds.

Manifolds are the fundamental object in differential geometry on which all other object exist. These are particularly difficult to define rigorously and it usually take the best part of an undergraduate mathematics lecture course to define. This means that the useful tools discussed here are relegated to the last few lectures.

Intuitively we should think of a manifold as a shape, like a sphere or torus, which is smooth. That is, it does not have any corners. If we take a small patch of a sphere and enlarge it, then the result looks flat, like a small patch of the plane, figure 1. By contrast if we take a small patch of a corner then, no mater how large we made it, it would still never be flat, figure 2.

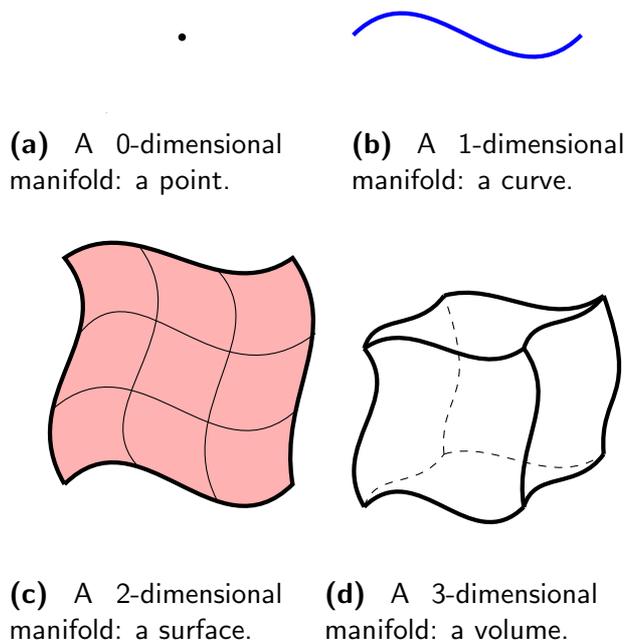


Figure 3: 0, 1, 2 and 3-dimensional manifolds.

In figure 3 are sketched manifolds with 0, 1, 2 and 3 dimensions. 0-dimensional manifolds are simply points, 1-dimensional manifolds are curves, 2-dimensional manifolds are surfaces and 3-dimensional manifolds are volumes.

Topologically, there are only two types of 1 dimensional manifolds, those which are lines and those which are loops. Higher dimensional manifolds are more interesting.

2.2 Submanifolds.

A submanifold is a manifold contained inside a larger manifold, called the **embedding manifold**. The submanifold can be either the same dimension or a lower dimension than the embedding manifold. Examples of embedded submanifolds are given in figure 4

There are all kinds of bizarre pathologies which can take place when one tries to embed one manifold into another, such as self intersection, figure 5a, or the a manifold approaching itself figure 5b. However we will avoid these cases.

Submanifolds may either be compact or non compact. A compact submanifold is bounded. By contrast a non compact submanifold extends all the way to “infinity” at

least in some direction. In figure 4 we see that examples 4a, 4c, 4f and 4g are all compact, whereas the others are not compact. A compact submanifold may either have a boundary, as in section 2.3 below, or close in on itself like a sphere or torus.

2.3 Boundaries of submanifold

Unlike the embedding manifold, it is also useful to consider submanifolds with boundaries. See figure 4. Both compact and non compact submanifolds can have boundaries. The importance of the boundary is when doing integration, in particular the use of Stokes’ theorem. The boundary of a submanifold is an intuitive concept. For example the boundary of a 3-dimensional ball is its surface, the 2-dimensional sphere, figure 4f. We assume that the boundary of an n -dimensional submanifold is an $(n - 1)$ -dimensional submanifold. We observe that the boundary of the boundary vanishes.

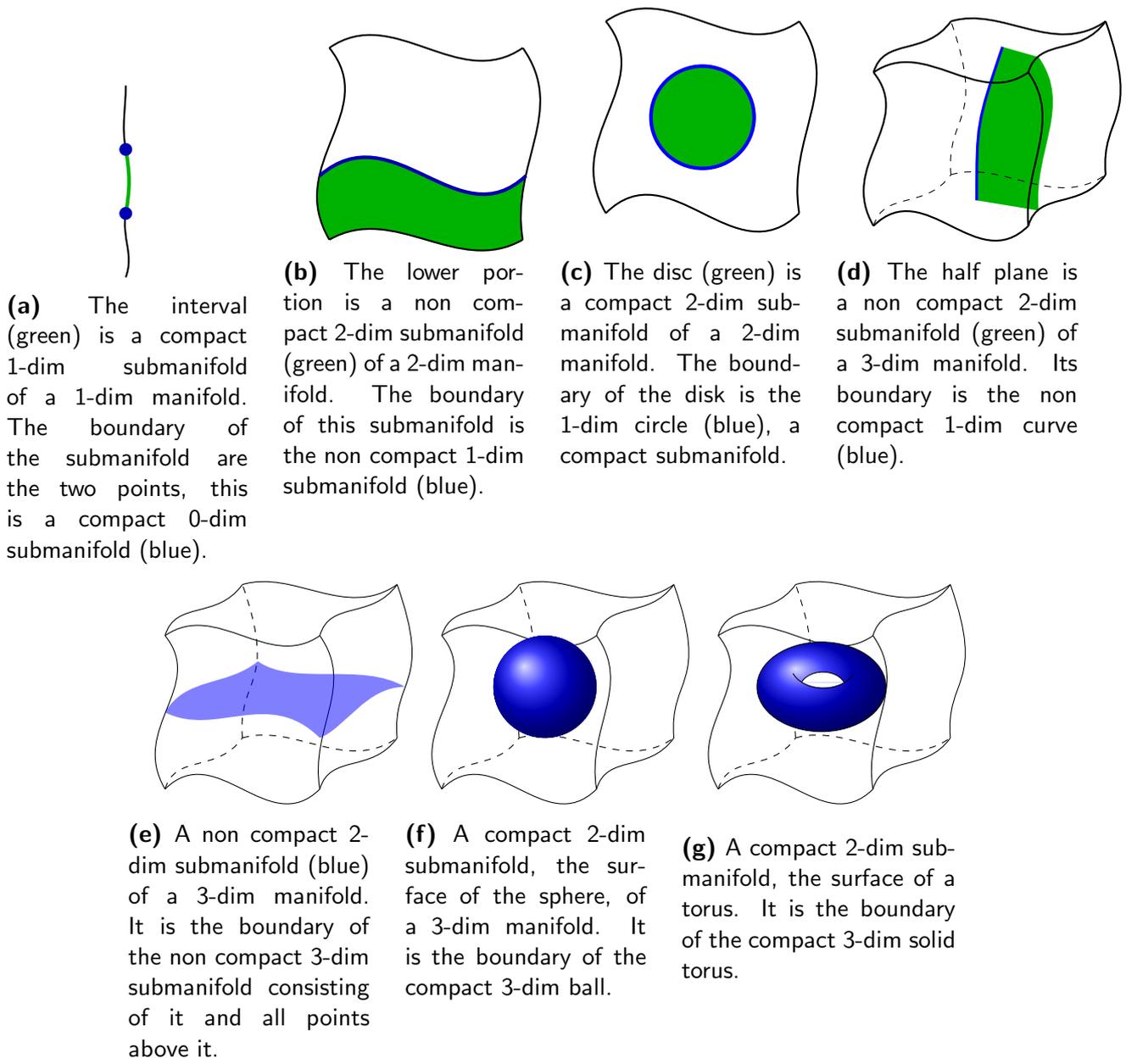
2.4 Scalar fields.

Having defined manifolds, the simplest object that one may define on a manifold is a scalar field. This is simply a real number associated with every point. See figure 6. The smoothness corresponds to nearby points having close values, also that these values can be differentiated infinitely many times. A scalar field on a 2-dimensional manifold may be pictured as function as in figure 7. Alternatively one can depict a scalar field in terms of its contours, figure 8. We will see below, section 4, that these contours actually depict the 1-form which is the the exterior derivative of the scalar field.

2.5 Vector fields.

A vector field is a concept most familiar to people. A vector field consists of an arrow, with a base point (bottom of the stalk) at each point on the manifold, see figure 9.

Although we have only drawn a finite number of vectors, there is a vector at each point. The smoothness of the vector field corresponds to nearby vectors not differing too



(a) The interval (green) is a compact 1-dim submanifold of a 1-dim manifold. The boundary of the submanifold are the two points, this is a compact 0-dim submanifold (blue).

(b) The lower portion is a non compact 2-dim submanifold (green) of a 2-dim manifold. The boundary of this submanifold is the non compact 1-dim submanifold (blue).

(c) The disc (green) is a compact 2-dim submanifold of a 2-dim manifold. The boundary of the disc is the 1-dim circle (blue), a compact submanifold.

(d) The half plane is a non compact 2-dim submanifold (green) of a 3-dim manifold. Its boundary is the non compact 1-dim curve (blue).

(e) A non compact 2-dim submanifold (blue) of a 3-dim manifold. It is the boundary of the non compact 3-dim submanifold consisting of it and all points above it.

(f) A compact 2-dim submanifold, the surface of the sphere, of a 3-dim manifold. It is the boundary of the compact 3-dim ball.

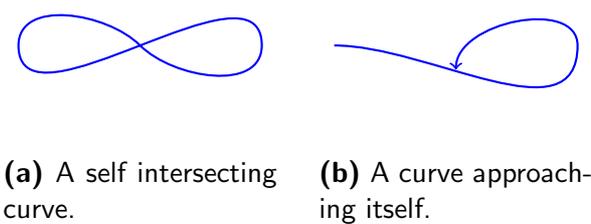
(g) A compact 2-dim submanifold, the surface of a torus. It is the boundary of the compact 3-dim solid torus.

Figure 4: A selection of submanifolds and their boundaries. Observe that the boundaries is itself a submanifold, but it does not have a boundary. This implies at the boundary must extend to infinity or close in on itself.

much.

To multiply a vector field by a scalar field, simply scale the length of each vector by the value of the scalar field at the base point of that vector. Of course if the scalar field is negative we reverse the direction of the arrow. We are all familiar with the addition of vectors from school, via the use of a parallelogram.

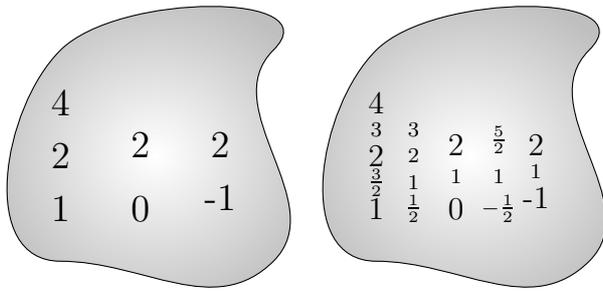
The vector presented in figure 9 are untwisted vectors. We will see below in section 3.6 that there also exist twisted vectors.



(a) A self intersecting curve.

(b) A curve approaching itself.

Figure 5: Pathological examples which are not submanifolds.



(a) Some of the values of the scalar field are shown. (b) Intermediate values of the scalar field are also shown.

Figure 6: A representation of a scalar field.

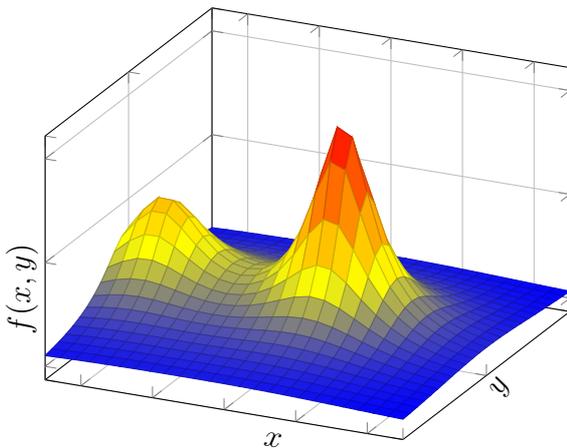


Figure 7: A scalar field as a function of x and y .

3 Orientation.

We now discuss the tricky role that orientation has to play in differential geometry, especially the difference between twisted and untwisted orientations. However the distinction is worth emphasising since the objects in electromagnetism are necessarily twisted and others are necessarily untwisted.

The take home message is:

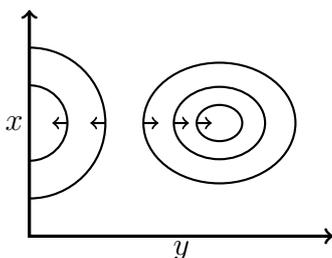


Figure 8: A scalar field depicted as contours. The arrows point up.

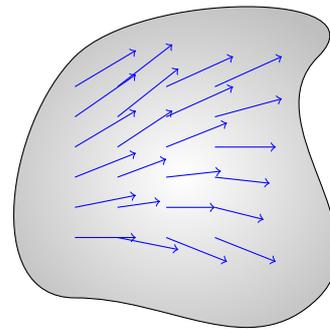


Figure 9: A representation of a vector field.

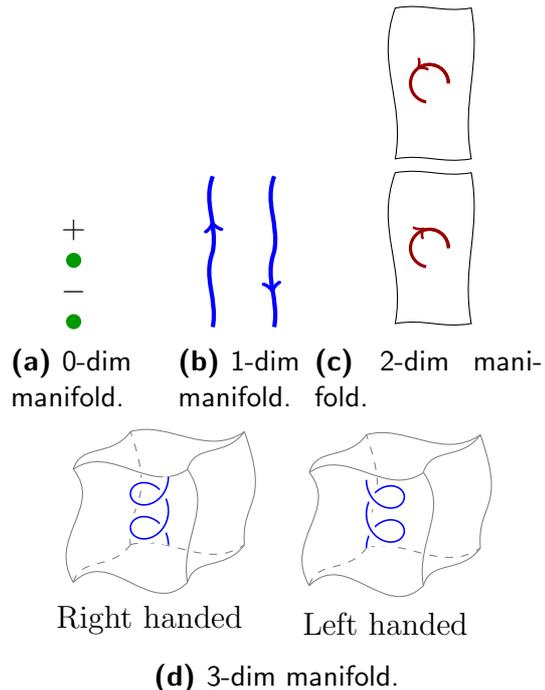


Figure 10: Possible (internal) orientations on 0, 1, 2 and 3 dimensional manifold.

- Orientation is important for integration and Stokes's theorem.
- The orientation usually only affects the overall sign. However it is important when adding forms.
- The pictures are pretty.

3.1 Orientation on Manifolds (internal)

We start by showing the possible orientations on manifolds. Not all manifolds can have an orientation. If a manifold can have an orientation it is called orientable. Examples of orientable manifolds include the sphere and torus. Non-orientable manifolds include the Möbius strip and the Kline bottle. Even if a

| | | Embedding dimension | | | |
|-----------------------|---|---------------------|--------------------------------|---------------------------------------------|---------------------------------------------|
| | | 0 | 1 | 2 | 3 |
| Submanifold dimension | 0 | $+$ $-$ ● ● | \uparrow \downarrow ● ● | \curvearrowright \curvearrowleft ● ● | \curvearrowright \curvearrowleft ● ● |
| | 1 | | \rightarrow \leftarrow | \rightarrow \leftarrow | \curvearrowright \curvearrowleft |
| | 2 | | | \curvearrowright \curvearrowleft ● ● | \rightarrow \leftarrow ● ● |
| | 3 | | | | $+$ $-$ ● ● |

Table 1: Possible external orientations on a 0, 1, 2 and 3-dim submanifold of a 0, 1, 2 and 3-dim embedding manifold.

manifold is orientable, one may choose not to put an orientation on the it. For manifolds of 0, 1, 2 or 3 dimensions the two possible (internal) orientations are shown in figure 10. These are as follows:

- An orientation for a 0-dimensional submanifold consists of either plus or minus, figure 10a. However convention usually dictates plus.
- On a 1-dimensional manifold it is an arrow pointing one way or another, figure 10b.
- On a 2-dimensional manifold it is an arc pointing clockwise or anticlockwise, figure 10c.
- On a three dimensional manifold one considers a helix pointing one way or another, figure 10d. Interestingly, observe that unlike in the 1-dim and 2-dim cases, there is no need for an arrow on the helix.

3.2 Orientation on submanifolds: Internal and external

Again submanifolds may be orientable or non-orientable. However with orientable manifold one now has a choice as to which type of orientations to chose: internal or external. With a manifold there is no concept of an external orientation because there is no embedding manifold in which to put the ex-

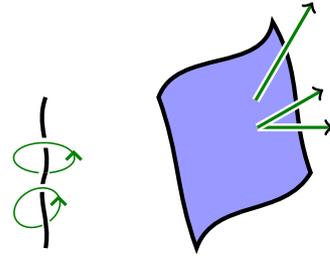


Figure 11: Equivalent external orientations of 1 and 2 dimensional submanifold in 3-dim. The actual length of the arc or arrow leaving is irrelevant. What matters is simply what the circulation around 1-dim submanifold and side of the 2-dim submanifold it points.

ternal orientation.

The internal orientation of submanifolds are exactly the same as those for manifolds, as shown in figure 10, with the type of the orientation: sign, arrow, arc or helix, depending on the dimension of the submanifold.

By contrast external orientations depend not just on the dimension of the submanifold but also on the dimension of the embedding manifold. They are similar, plus or minus sign, arrow, arc or helix, as the internal orientations. However in this case it takes place in the embedding external to the submanifold. In table 1 we see examples of external orientations for 0, 1, 2, 3 dimensional submanifolds embedded into 0, 1, 2, 3 dimensional manifolds. As we see, the type of the external orientation depends on the difference in the dimension of the embedding manifold and submanifold. Thus the external orientation is a plus or minus sign if the two manifolds have the same dimension. It is an arrow if they differ by 1 dimension, an arc if they differ by 2 dimensions and a helix if they differ by 3 dimensions.

We emphasise that when specifying an orientation, we can point the respective arrow or arcs in a variety of directions without changing the orientation. For example, in figure 11 we see how the same orientation can be represented by several arcs or arrows.

3.3 Internal and external, twisted and untwisted

For manifolds and submanifold it is useful to label internal orientations as untwisted and

| | Always definable & Essential | Internal | External |
|-------------|------------------------------|-----------|-----------|
| Manifold | No | Untwisted | Undefined |
| Submanifold | No | Untwisted | Twisted |
| Vectors | Yes | Untwisted | Twisted |
| Forms | Yes | Twisted | Untwisted |

Table 2: Objects which can have orientations and the correspondence between untwisted versus twisted and internal versus external.

| | | | | | | |
|--|-------------|--|---------|--|----------|--|
| | followed by | | becomes | | which is | |
| | followed by | | becomes | | which is | |
| | followed by | | becomes | | which is | |
| | followed by | | becomes | | which is | |
| | followed by | | becomes | | which is | |
| | followed by | | becomes | | which is | |

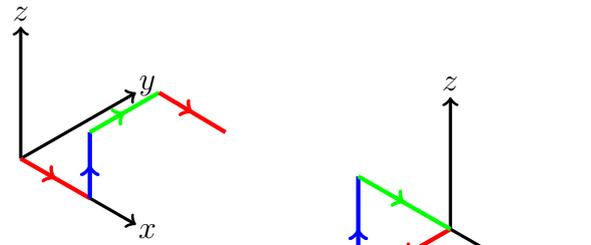
Table 3: Concatenation of orientations.

external orientations as twisted. As we have stated, as well as manifolds and submanifolds, vectors and forms also have orientations. These may also be twisted or untwisted and may also be internal or external. The pictures of the orientations depend on whether they are internal or external, but the assignment to which are twisted or untwisted depend on the object, according to table 2.

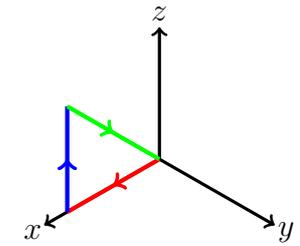
3.4 Concatenation of orientations

We can combine two orientations together. Simply take the two orientations and join them together, the second onto the end of the first. See table 3.

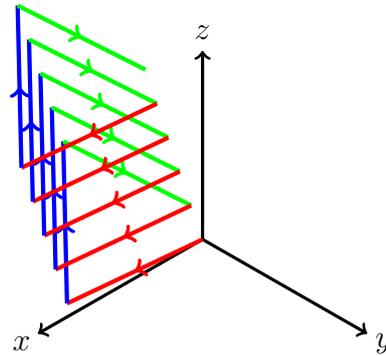
You can see that the untwisted orientations for two dimensions, figure 10c, i.e. clockwise or anticlockwise arise from the concatenation



(a) A curve created by going a short distance in x (red), followed by a short distance in z (blue) followed by going a short distance in y (green) and then repeating.



(b) The same diagram as in figure 12a but now with the axes all pointing towards the reader.



(c) The same diagram as in figure 12b but with a offset each loop to show the helix.

Figure 12: Demonstration that going a short distance in x then z then y and repeating creates a helix. The reader is encouraged to experiment with some pipe cleaners.

of two orientations, as in rows three and four of table 3. Likewise the helices, figure 10d arise by concatenating an arc with a perpendicular line as in rows five and six of table 3.

Using pipe cleaners may help you to see that that an arc followed by a perpendicular line is equivalent to a helix. Likewise a helix arises in 3 dimensions by tracing out a path, going in one direction, then a second, then the third and repeating as in figure 12.

3.5 Inheritance of orientation by the boundary of a submanifold

If a submanifold has an internal orientation and a boundary then the boundary inherits an internal orientation, see figure 13. Like-

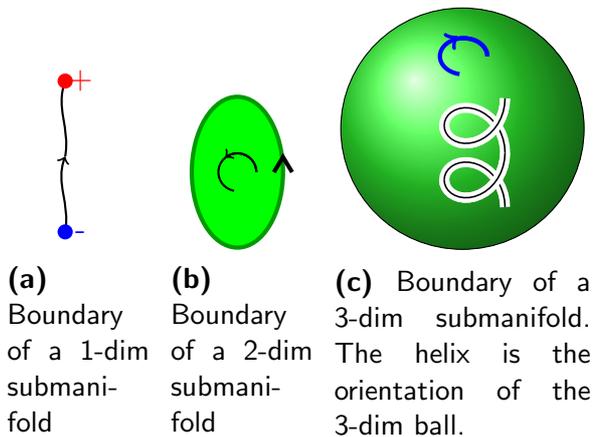


Figure 13: Inheritance of an internal orientation of a submanifold by its boundary for 1,2,3-dimensional submanifold embedded into a 3-dimensional manifold.

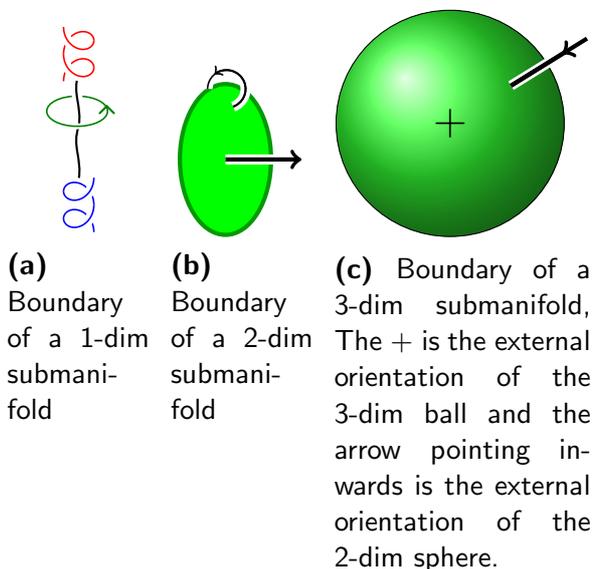


Figure 14: Inheritance of an external orientation of a submanifold by its boundary for 1,2,3-dimensional submanifold embedded into a 3-dimensional manifold.

wise if a submanifold has an external orientation then the boundary inherits the external orientation, figure 14.

3.6 Orientations of vectors

In section 2.5 and figure 9 we saw the usual type of vectors. These are untwisted vectors, since they have an untwisted orientations which is the internal orientations. There exists another type of vector called a twisted vector. These are also called axial or pseudo-vectors. An example of such a vector is the magnetic field \mathbf{B} . Twisted vectors have exter-

| Embedding dimension | Embedding orientation | Submanifold external orientation | Submanifold internal orientation |
|---------------------|-----------------------|----------------------------------|----------------------------------|
| 2 | | | |
| 3 | | | |

Table 4: Examples of twisting and untwisting.

nal orientations. They look like the external orientations of 1-dimensional submanifolds as depicted in table 1. In the case of vectors in 3-dimension one can see why they are called axial.

3.7 Twisting and untwisting

If a manifold has an orientation then we can use it to change the type of orientation of a submanifold. Thus we can convert an external orientation into an internal orientation. We can also convert an internal orientation into an external orientation. The convention we use here is to say that if we concatenate the external orientation (first) followed by the internal orientation (second) one must arrive at the orientation of the manifold. See examples in table 4.

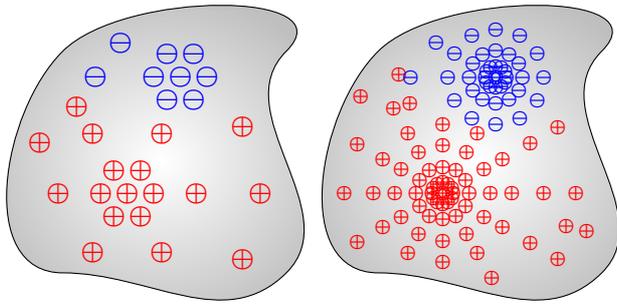
For example if the orientation of a 2-dim manifold is clockwise and the untwisted orientation is up then the twisted orientation is right. See fourth line of table 3.

Starting with an untwisted form, then twisting and then untwisting it does not change the orientation.

4 Closed forms.

In this section we introduce exterior differential forms, which we call simply forms. On an n -dimensional manifold a form has a degree, which is an integer between 0 and n . We refer for a form with degree p as a p -form. A closed form of degree p can be thought of as a collection of submanifolds¹ of dimension $n - p$.

¹Some authors [2, 8, 9] use a double sheet to indicate 1-forms and likewise they replace 1-dim form-



(a) Low density. (b) Increased density, while reducing the value of each dot.

Figure 15: Representation of a twisted top-form, i.e. an n -form on an n -dimensional manifold.

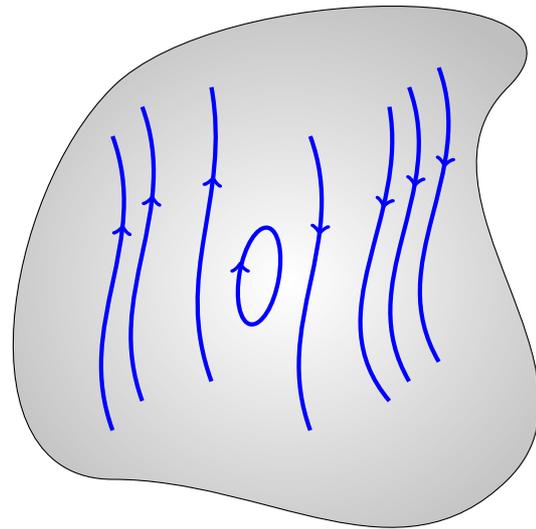


Figure 17: An untwisted 2-form on a 3-dim manifold.

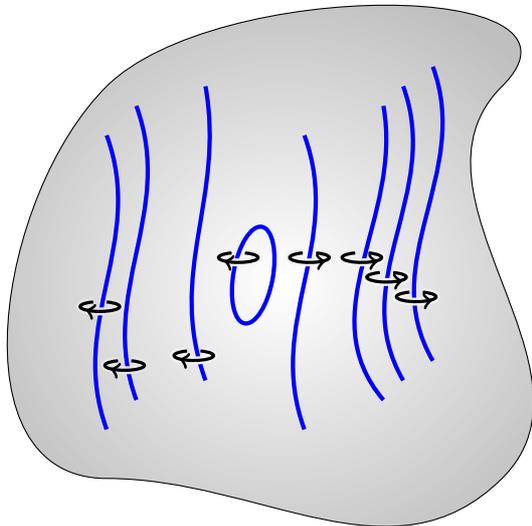


Figure 16: An untwisted 2-form on a 3-dim manifold.

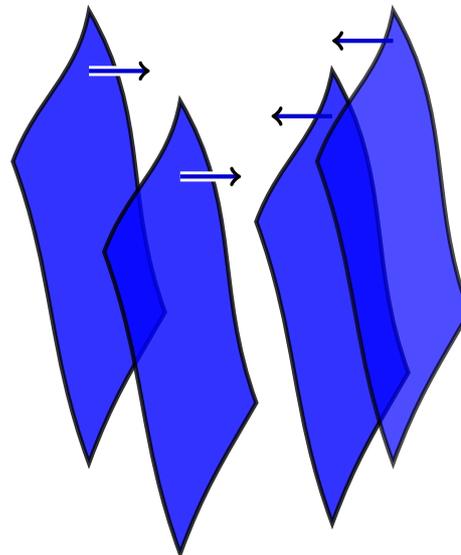


Figure 18: An untwisted 1-form on a 3-dim manifold. Note that the 1-form must vanish in the region between the two opposing orientations. A diagram of a twisted 1-form in 3 dimensions is given in figure 26.

submanifolds with cylinders to indicate $(n-1)$ -forms. It is true that for a single 1-form at a point it is necessary to have two $(n-1)$ -dim form-elements-submanifold in order indicate the magnitude of the form. Consider in figure 38 one needs at least 2-lines to indicate the combination of a 1-form with a vector where both are defined only at one point. However in this document we only consider smooth form-fields. I.e. a form defined at each point. Thus there is a form-submanifold passing through each point. Furthermore since the form-field is smooth form submanifolds for nearby points are nearly parallel. Therefore there is sufficient information to indicate the magnitude of the form via the density of the form-submanifolds. Looking at figure 38 again we see there is sufficient information to calculate the action of the vector on the form.

We call these **form-submanifolds**. Similar to a vector field, there is a submanifold of the correct dimension passing through each point in the manifold. They must not have boundaries, therefore they either continue off to “infinity” or they close in on themselves, like spheres. In section 5 we see that the form-submanifolds with boundaries correspond to non-closed forms.

Although the points, lines, surfaces corresponding to p -forms, for $p \geq 1$, do not have

values, they do have an orientation and as before these can be untwisted or twisted. As stated in table 2 these are the opposite way round than for submanifolds. Thus an untwisted form has an external orientation and a twisted form has an internal orientation. The untwisted forms are mathematically simpler and much easier to take wedge products of. See section 4.3 below. Indeed the standard method of prescribing the internal orientation of an n -dim manifold is in terms of an untwisted n -form.

We have the following:

- An untwisted 0-form (internal orientation) is simply a scalar field and is best visualised as above in figures 6, 7 and 8. Note that a closed 0-form is a constant and not very interesting. From the prescription above, a 0-form on an n -dim manifold is an n -dim form-submanifold at each point.

A twisted 0-form (external orientation) requires an external orientation for a 0-dim submanifold (point) as in table 1.

- On an n -dimensional manifold, the form-submanifolds for an n -form consists of a collection of dots. For the rest of this report, we shall refer to n -form as **top-forms**.

For twisted top forms (internal orientation) these dots carry a plus or minus sign. See figure 15. The density of the dots corresponds loosely to the magnitude of the top-form. The smoothness of the field requires that between a high density of dots and a low density dots, there should be a region of intermediate density.

A non-vanishing twisted top-form with only plus dots is called a **measure**. We see below that these are very useful for integrating scalar fields (section 4.4) and for converting between vector fields and $(n - 1)$ -forms (section 6.3). We see in section 7.8 that a metric automatically gives rise to a measure.

Although, as with vector fields, there is really a “dot at each point”, therefore the density of dots is actually infinite. However we can think of a limiting process. Start with a set of discrete dots, then (approximately) double the density of dots, while at the same time halving the value associated

with each points. As such the number of dots multiplied by their sign inside a particular n -volume is approximately constant during the limiting process. Compare this to the scalar field, where as we stated, the value of each point remains constant during the limiting process.

For untwisted top-forms each point carries an external orientation, as in 0 dimensional submanifolds in table 1.

- On an n -dimensional manifold the closed $(n - 1)$ -forms correspond to 1-dimensional form-submanifolds, i.e. curves as depicted in figures 16 and 17. The curves do not have start points or end points. Therefore they must either close on themselves to form circles or they must go from and to “infinity”. They also do not intersect. Again there is really a curve going through each point. The smoothness again requires that between a high density of curves and a low density there should be an intermediate density, and that between curves of opposite sign there should be a gap.

Figure 16 shows an untwisted 2-form (external orientation) in 3 dimensions.

Twisted $(n - 1)$ -forms are lines with direction. (Internal orientation). One may consider these as flow lines of a fluid or similar. An example is shown in figure 17.

You can see from figures 9 and 17 that untwisted vectors and twisted $(n - 1)$ -forms look similar, and indeed they are. With just a measure we can convert a vector into a 1-form via the internal contraction. See section 6.3.

- On an n -dimensional manifold the closed 1-forms correspond to $(n - 1)$ -dimensional form-submanifolds.

Untwisted 1-forms are the submanifold corresponding to the contours or level surfaces of a scalar field, as in figure 8. The external orientation point in the direction of increasing value of the scalar field. The faster the rate of increase of the scalar field, the closer the level surfaces.

One usually takes the gradient of a scalar field which is the vector corresponding to the direction of steepest ascent. This gradient is perpendicular to the level surfaces.

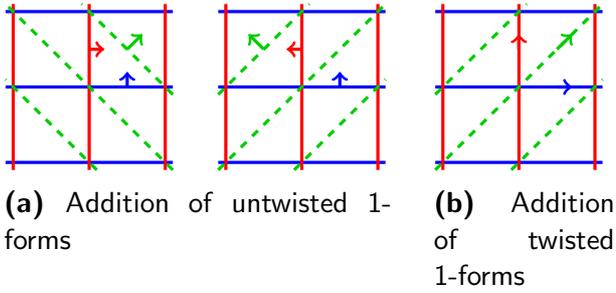


Figure 19: Addition of untwisted (left two) and twisted (right) 1-forms in 2-dim. The solid 1-form-submanifold (blue and red) are added to produce the dashed 1-form-submanifold (green).

However to do this one needs a concept of orthogonality which comes with the metric defined below in section 7.

An example of an untwisted 1-form in 3-dimensions is given in figure 18.

4.1 Multiplication of vectors and forms by scalar fields.

To multiply a p -form field by a scalar field simply increase the density of dots, curves, surfaces of the p -form by the value of the scalar field.

The action of multiplying a p -form by a negative scalar field changes the orientation of the p -form.

It is worth noting that if you multiply a closed p -form with $p < n$ by a non constant scalar the result will not be a closed form.

4.2 Addition of forms.

It is only possible to add forms of the same degree and twistedness. Thus two untwisted p -forms sum to an untwisted p -form and two twisted p -forms sum to a twisted p -form. In contrast to multiplying by a scalar, adding two closed forms gives rise to a closed form.

Although algebraically adding forms is trivial, the visualisation of the addition of two p -forms is surprisingly complicated. Figure 19 show how to add 1-forms in 2-dimension.

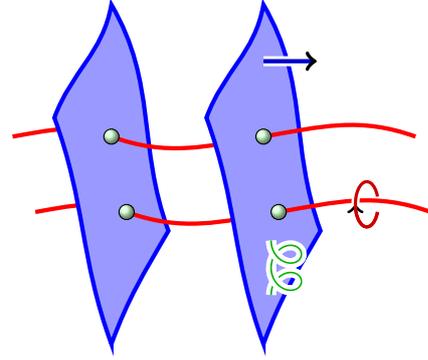


Figure 20: In 3-dimensions, the wedge product of an untwisted 1-form (blue) with a twisted 2-form (red) to give a twisted 3-form (green).

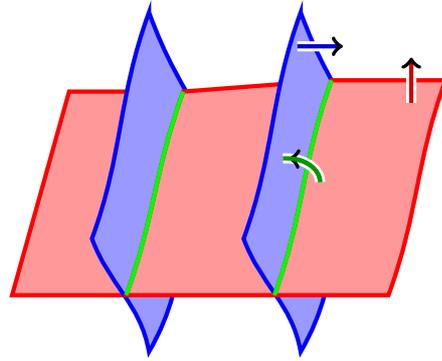


Figure 21: In 3-dimensions, the wedge product of an untwisted 1-form (vertical, blue) with an untwisted 1-form (horizontal, red) to give an untwisted 2-form (green).

4.3 The wedge product also known as the exterior product.

In order to take the wedge product of two forms simply requires taking their intersection, figures 20-24. In places where the two form-submanifolds are tangential then the resulting wedge product vanishes.

The wedge product of two untwisted forms is an untwisted form. Simply concatenate the two orientations. See figures 20 and 21. The wedge product for three untwisted 1-forms is given by the intersection of three $(n - 1)$ -dim form-submanifolds, figure 22.

The wedge product of two twisted forms is an untwisted form (figure 23) and the wedge product of a twisted form and an untwisted form is a twisted form (figure 24). However the rules of this is more complicated than simply concatenation. The only guaranteed method is to choose an orientation for the

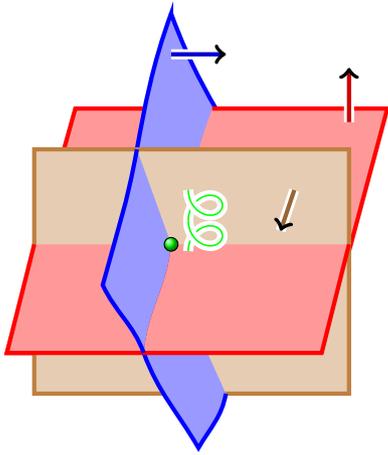


Figure 22: In 3-dimensions, the wedge product of three untwisted 1-forms. The first vertical (blue) followed by the second facing reader (brown) then the third horizontal (red). This gives an untwisted 3-form (green).

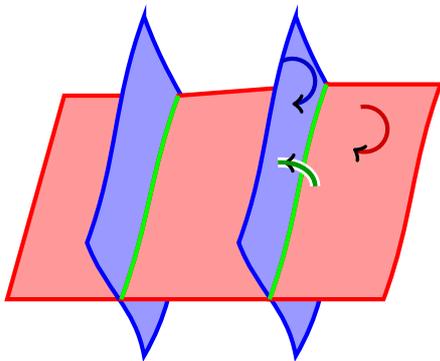


Figure 23: In 3-dimensions, the wedge product of the twisted 1-form (vertical, blue) with the twisted 1-form (horizontal, red) to give the green untwisted 2-form.

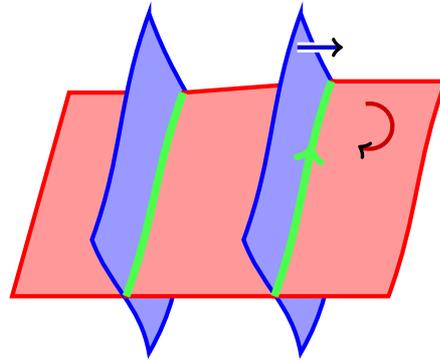


Figure 24: In 3-dimensions, the wedge product of the untwisted 1-form (blue) with the twisted 1-form (red) to give the green twisted 2-form.

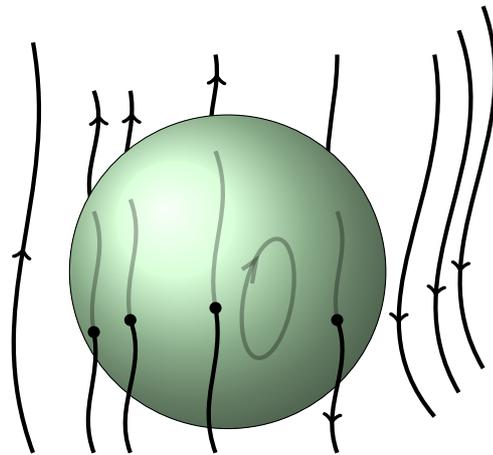


Figure 25: Integration of a twisted 2-form on a 2-sphere embedded in a 3-dimensional manifold. Observe that every curve that enters the sphere also leaves.

manifold, then convert the twisted forms into untwisted forms, take the wedge product and then apply a twisting if necessary. One can check that the result is invariant under the choice of orientation of the manifold. However it may depend on the twisting convention described in section 3.7.

4.4 Integration.

We can only integrate twisted top-forms, i.e. n -forms on an n -dimensional manifold, figure 15. To do this we simply add the number of positive dots and subtract the number of negative dots. Then take the limit the the density tends to infinity and the value tends to zero. Let figure 15 refer to a 2-dimensional manifold. Thus in figure 15a, assuming each dot value has 1, then its integral equals 9. In

figure 15b each dot must have value $\frac{1}{4}$ and the integral is $6\frac{1}{2}$. As the density of dots increases and their value tends to zero, this sum will converge. Contrast this to the case of the scalar field, figure 6, which does not converge as the density increases. In order to integrate a scalar field it is necessary to multiply it by a measure.

To integrate an untwisted top-form it is necessary to use the orientation of the manifold to convert the untwisted top-form into a twisted top-form. If the manifold is non orientable, such as the Möbius strip then one can only integrate twisted top-form.

What about integrating forms of lower dimension? To integrate a p -form we must integrate it over a p -dimensional submanifold. In addition untwisted forms are integrated over untwisted submanifolds and likewise for

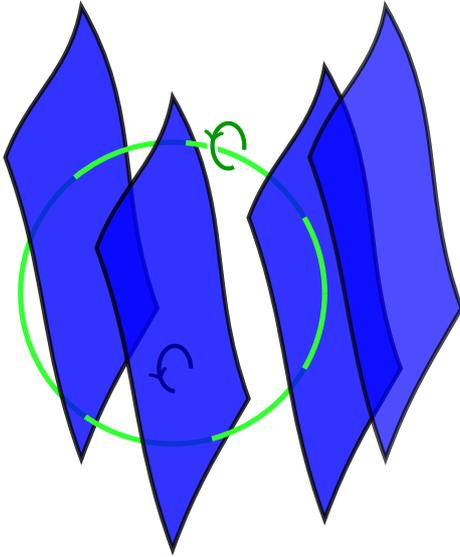


Figure 26: Integration of a twisted 1-form on a twisted circle embedded in a 3-dimensional manifold. Observe that every surface that crosses the circle must cross again in the opposite direction.

twisted forms and twisted submanifolds; with plus if the orientations agree and minus otherwise. We will see a generalisation of this in section 8.1 on pullbacks.

4.5 Conservation Laws

We can see from figures 25 and 26 that every curve or surface that intersects the submanifold also leaves. Therefore the integral over the sphere in figure 25 and over the circle in figure 26 are both zero. This gives rise to conservation laws which are very important in physics. For example the conservation of charge or the conservation of energy. In figure 27 we see that the closed twisted 3-form is integrated over the surface of a 3 dimensional cylinder. In this context we refer to the closed 3-form as a current.

However there are three conditions on the submanifold which together imply that the integral of a closed form is zero.

- The submanifold (of dimension p) must be the boundary of another submanifold (of dimension $p + 1$).
- The $(p + 1)$ -dimensional manifold must not go off to infinity.
- The larger manifold in which all the submanifolds are embedded must not have any “holes”.

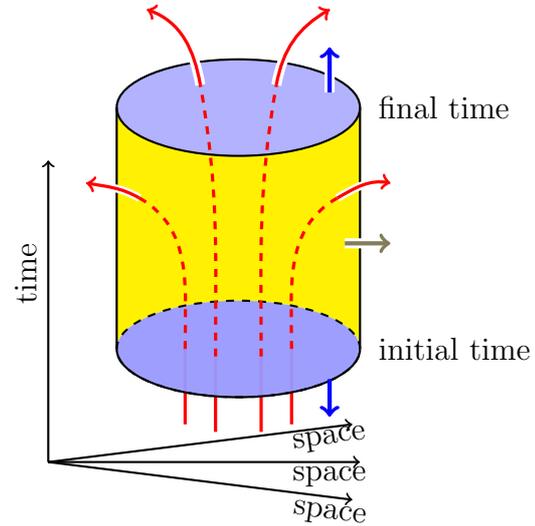


Figure 27: The closed twisted 3-form (red) is integrated over the 3-dim surface of a cylinder with net result zero. Any current which enters at the initial time (lower blue disc) must either leave at the final time (upper blue disc) or through the curved surface of the cylinder (yellow). Note that the orientation of the submanifold is out (given by the yellow and blue arrows). This is because the sum of the integrals is zero. However the contribution from initial time is negative.

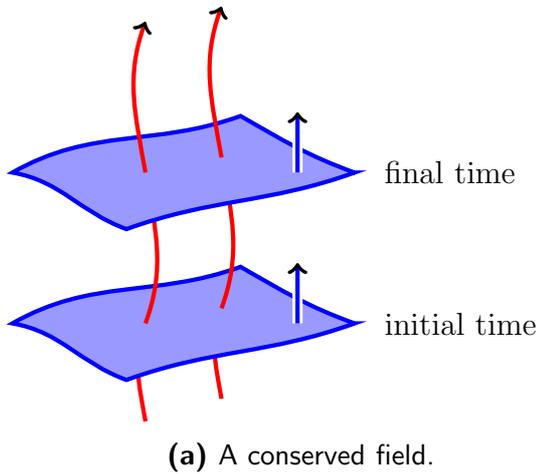
See section 4.6 below for a comment about holes.

In order to convert the integral of a closed current into a conservation over all time it is necessary to expand the curved surface of the cylinder out to infinity. Thus the initial and final discs become all of space. One must therefore guarantee that no current “escapes to infinity”. See figure 28.

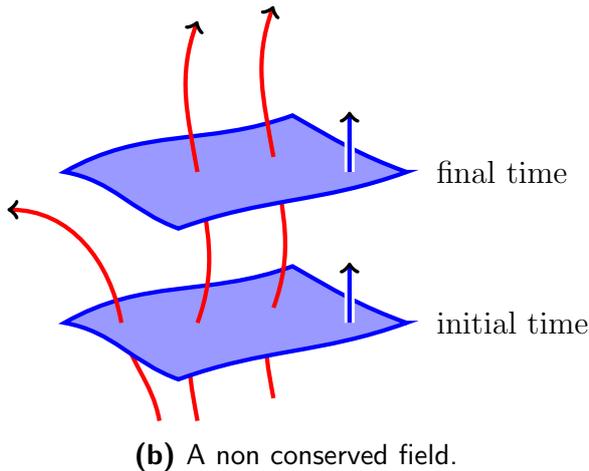
4.6 DeRham Cohomology.

As mentioned in section 4.5 in order to guarantee that the integral of a closed p -form over a closed p -dimensional submanifold is zero we need that the larger manifold in which all the submanifolds are embedded has no “holes”. As an example of a manifold with holes consider the annulus, which is disc with a hole, given in figure 29. When integrating around a circle which does not encircle the hole then the integral is zero. However when integrating around the hole then the integral is non-zero.

The use of integration to identify holes in a



(a) A conserved field.



(b) A non conserved field.

Figure 28: In 4-dimensions, integration of the twisted 3-form (red), showing it is conserved as long as nothing “escapes to infinity”. The 3-dimensional submanifolds (blue) correspond to “equal time slice”. Note that, comparing with figure 27 we have let the curved surface (yellow) pass to infinity. We have also swapped the orientation of the initial time-slice so that it has the same orientation as the final time slice.

manifold is known as DeRham Cohomology.

Note that the “hole” need not be removed from the manifold like a hole is removed from a disc to create an annulus. In figure 30, one integrates over a circle which encircles the “hole” of a torus.

If the manifold does not have any holes, then as stated the integral is zero. It is interesting to see attempts to create a non zero integral as in figure 31. In figure 31a the form-submanifolds terminate, whereas in figure 31b the form-submanifolds all intersect at a point. In both cases these correspond to non closed forms as we will see below in section 5. The difference is that in 31a the 1-

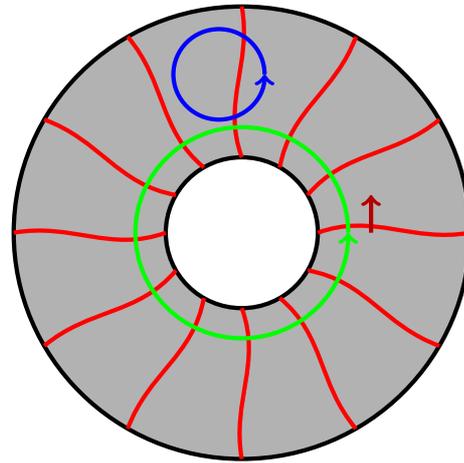


Figure 29: A annulus which is a 2-dim manifold with a hole. When the untwisted 1-form (red) is integrated in a circle (blue) which does not enclose the hole the result is zero. By contrast when one integrates over circle (green) which does enclose the hole the result is non zero.

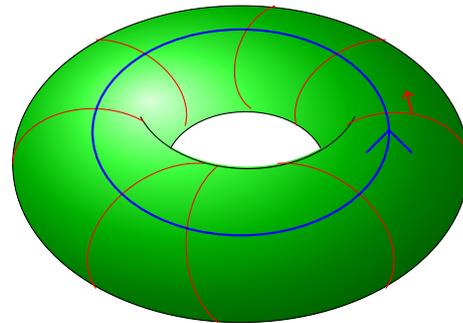
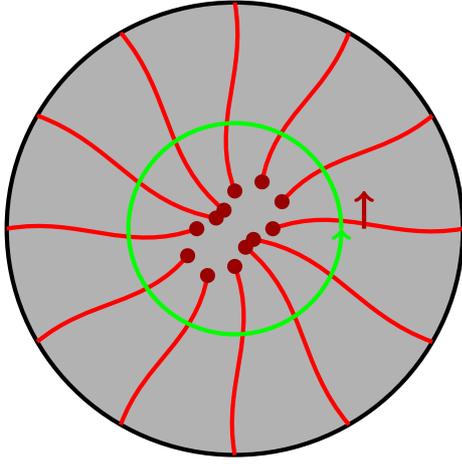


Figure 30: On a 2-dim torus, integrating the 1-form (red) over the circle (blue) is not zero, despite the fact that the 1-form is closed.

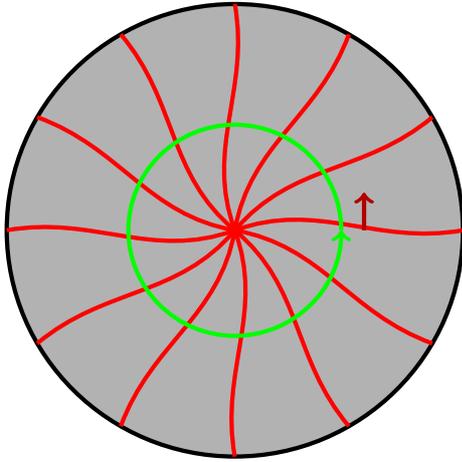
form is continuous, in 31b it is discontinuous. We will see physical examples of such discontinuous forms in electrostatics and magnetostatics, section 9.1 which correspond to the fields generated by point and line sources such as electrons and wires.

4.7 Four dimensions.

There are significant yet obvious problems when trying to visualise forms in 4-dimensions. However there are important reasons to try. As we will see below, we of course live in a 4-dimensional universe and the pictorial representation of electrodynamics would be particularly enlightening. Secondly there are phenomena which only exist in four or more dimensions. In lower dimen-



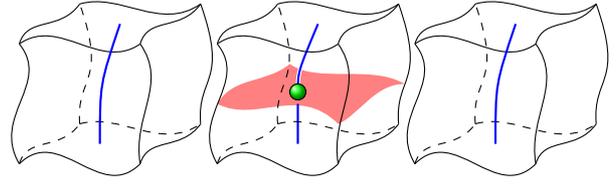
(a) In this case the form-submanifolds terminate. This corresponds to the 1-form (red) not being closed.



(b) In this case the form-submanifolds all intersect. This corresponds to the 1-form (red) not being continuous at this point.

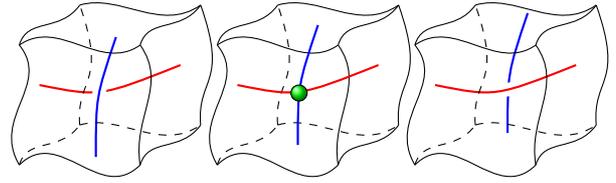
Figure 31: Two attempts to create a non zero integral as in figure 29, but on a manifold which does not have a hole. On the 2-dimensional disc, the 1-form (red) is integrated over circle (green).

sions any p -form, for $p \geq 0$ wedged with itself is automatically zero. This is because every form-submanifold is tangential to itself. This does not apply to scalar fields, viewed as 0-forms, which can be squared. However there are 2-forms in 4-dimensions which are non zero when wedged with themselves. In particular, this includes the electromagnetic 2-form which we write F . Thirdly, all closed form-submanifolds up to now have a single smooth submanifold passing through each point. However 2-forms in 4-dimensions, which are non-zero when wedged with themselves, correspond, at each point, to two 2-



(a) $t < 0$ (b) $t = 0$ (c) $t > 0$

Figure 32: The plane representation of a 2-form-submanifold in 4-dimensions. The 2-form-submanifold is represented by both the line (blue), for all time, and the momentary plane (red) at the time $t = 0$. The self wedge is represented the dot (green).



(a) $t < 0$ (b) $t = 0$ (c) $t > 0$

Figure 33: The moving line representation of a 2-form-submanifold in 4-dimensions. The 2-form-submanifold is represented by two lines (red and blue). The self wedge is represented by the dot (green).

dimensional form-submanifolds which intersect at that point.

One way to attempt to visualise a 2-form-submanifold in 4-dimensions, is to use 3-dimensional space and 1-dimension of time. This is particularly relevant for visualising spacetime. There are two possibilities. Let's assume that the two 2-dimensional form-submanifolds intersect at time $t = 0$. Then one possibility, shown in figure 32 is that for all $t \neq 0$ the form-submanifold is a line, whereas at $t = 0$ it is that line and an intersecting plane. The alternative representation is given by two moving lines as seen in figure 33. The result of wedging the 2-form with itself gives a 4-form. This top-form is represented by a dot, which is the intersection of the 2-form-submanifold with itself.

5 Non-closed forms.

Non closed p -forms consist of a small submanifold of dimension $(n - p)$ or surface at

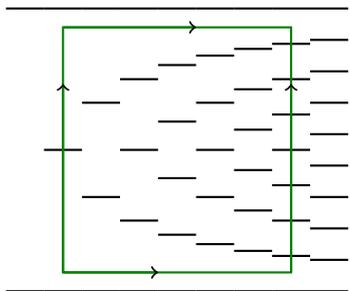


Figure 34: Example of a non-closed 1-form (black) in 2-dim. If we integrate from bottom left to top right on the green square, we see we get a different value depending on the path taken.

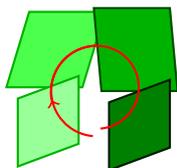


Figure 35: An Attempt to show a non-closed, non-integrable 1-form in 3-dim (green). If we try and close the elements up in to surface we see that we fail. Starting the red arc on the lightest green and going round anticlockwise we end up on the darkest green, we have helixed down.

each point. We call these **form-elements**. The difference is that in non-closed forms, the form-elements do not connect up to form a submanifold. Examples are given in figures 34 and 25. We can see from figure 34 that the integral now depends on path taken. Thus non-closed forms correspond to non conservative fields.

In some (but not all) cases a non-closed form can be thought of as a collection of closed form-submanifolds, but each of these has a scalar field on it, see figure 36. Such forms are known as **integrable**. For example the 1-form (in 2-dim) in figure 34 is integrable whereas the 1-form (in 3-dim) in figure 35 is not.

5.1 The exterior differential operator.

The exterior differential operator takes a p -form and gives a $(p+1)$ -form. One may think of this as taking the boundary of the form-submanifolds as given by figures 13 and 14.

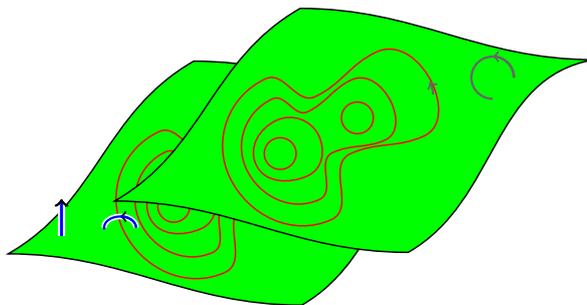


Figure 36: An integrable non-closed 1-form in 3-dim: Two 1-form-submanifolds (green) each with a scalar field, represented by contour lines (red). The exterior differential of this 1-form are the closed red contour lines. We assume that the scalar field is increasing towards the inner contours. Both possible twistedness orientations are shown on the same diagram. For untwisted orientations (lower sheet, blue) concatenate the inward arrow of the contours with the up arrow of the submanifold sheets. The twisted orientation (upper sheet, grey) simply inherits the orientation of the sheet.

The exterior derivative of a scalar field is simply the 1-form-submanifolds corresponding to the contours, also known as level $(n - 1)$ -surfaces, of the scalar field. For untwisted scalar fields the orientation points in the direction the scalar field increases, as in figure 8.

The exterior differential of an integrable non-closed form are the contours of the scalar field on each form-submanifold. See figure 36.

Clearly if the p -form is closed then it has no boundary and the result is zero. In addition, since a boundary has no boundary, the action of the exterior derivatives operator twice vanishes.

5.2 Integration and Stokes's theorem for non-closed forms.

The integration of non-closed forms is no longer necessarily zero, even if there are no holes in the manifold. Recall to integrate a p -form we must integrate it over a p -dimensional submanifold. In addition untwisted forms are integrated over untwisted submanifolds and likewise for twisted forms

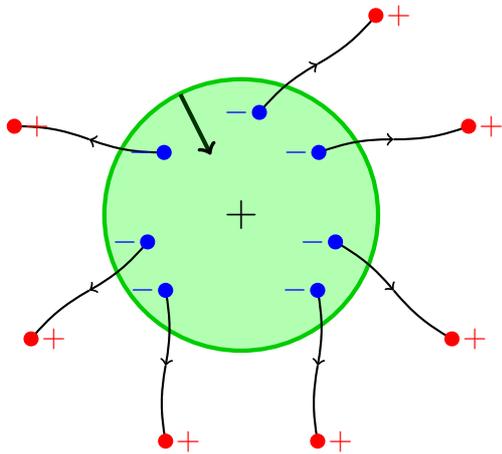


Figure 37: Demonstration of Stokes's theorem in 2-dimensions. The orientation of the twisted disk is $+$ and the twisted circle is inwards. The twisted 1-form is not closed and its exterior derivative is the positive and negative dots. The integration of these dots over the disk is -7 . Likewise the integration of the 1-form over the circle is also -7 .

and twisted submanifolds; with plus if the orientations agree and minus otherwise.

In figure 37 we see a demonstration of Stokes's theorem. That is, the integral of a $(p-1)$ -form over a $(p-1)$ -dimensional boundary of a submanifold equals the integral of the exterior derivative of the $(p-1)$ -form over the p -dimensional submanifold. Again we have to assume the embedding manifold has no holes.

With a bit of work defining the divergence and curl in terms of exterior differential operator, one can see that our version of Stokes's theorem is a generalisation of the divergence and Stokes's theorem in 3-dimensional vector calculus.

6 Other operations with scalars, vectors and forms

6.1 Combining a 1-form and a vector field to give a scalar field.

Given a vector field and a 1-form field, one can combine them together to create a scalar field. The scalar is given by the number form-

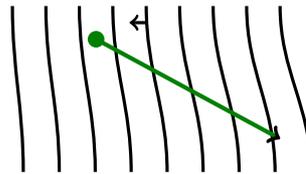


Figure 38: Combining an untwisted 1-form (black) and an untwisted vector field (green) to give a scalar field, in 2-dim. The scalar at the base point of the vector would have value -5 .

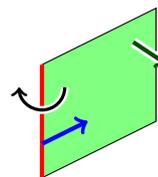


Figure 39: In a 3-dim manifold. The internal contraction of the untwisted 2-form (red) with the untwisted vector (blue) to produce the untwisted 1-form (green). The orientation of the vector followed by the orientation of the 2-form gives the orientation of the 1-form.

submanifolds the vector crosses. See figure 38. From this we see that if a 1-form contracted with a vector is zero, then the vector must lie within the 1-form-submanifold.

6.2 Vectors acting on a scalar field.

A vector acting on a scalar field gives a new scalar field. This is given by contracting the exterior derivative of the scalar field with the vector, as defined above. Thus it corresponds to the number of times a vector crosses the contour submanifolds of the scalar field.

6.3 Internal contractions.

Internal contractions is an operation that takes a vector field and a p -form and gives a $(p-1)$ -form. Pictorially it corresponds to the $(p-1)$ -form-submanifold created by extending the p -form-submanifold in the direction of the vector. Thus increases the dimension of the form-submanifold by 1. See figure 39. If the vector lies in the p -form-submanifold the internal contractions vanishes.

This is a generalisation of the contracting of

a vector and a 1-form given in subsection 6.1. Since the vector lies in the resulting $(p - 1)$ -form-submanifold, internally contracting by the same vector twice is zero.

7 The Metric

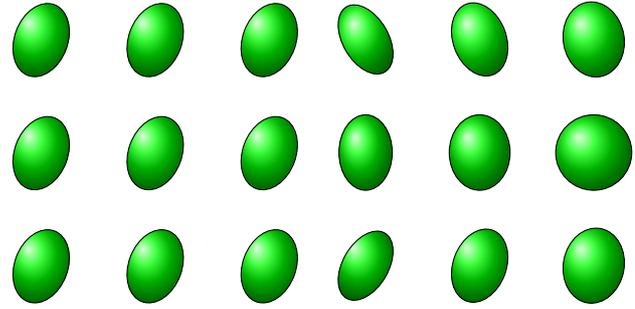
The metric gives you the length of vectors (figures 42 and 43) and the angle between two vectors (figures 44 and 45). It can also tell you how long a curve is between two points and given two points it can tell you which is the shortest such curve, which is called a geodesic. Thus it can tell you the distance between two points. This is the length along the geodesic between them.

In higher dimensions the metric tells you the area, volume or n -volume of your manifold and submanifolds within it. It gives you a measure so that you can integrate scalar fields, figure 51. It enables you to convert 1-forms into vectors and vectors into 1-forms (called the metric dual). It also enables you to convert p -forms into $(n - p)$ -forms (called the Hodge dual). It doesn't, however, give you an orientation.

Metrics possess a signature. A list of n positive or negative signs, where n is the dimension of the manifold. There are a number of important dimensions and signatures:

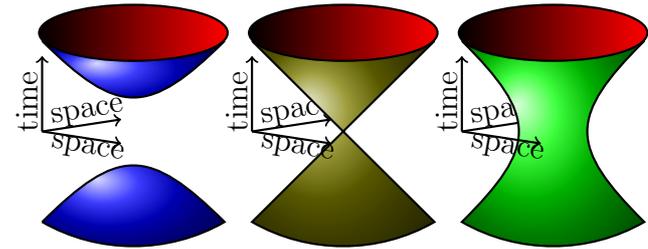
- One dimensional Riemannian manifolds.
- Two dimensional Riemannian manifolds, i.e. signature $(+, +)$. These are surfaces, which may be closed like spheres and torii, with a number of "holes". They may also be open like planes. They also included non orientable surfaces such as the Möbius strip.
- Three dimensional Riemannian manifolds, i.e. signature $(+, +, +)$. These include the three dimensional space that we live in.
- Four dimensional spacetime, i.e. signature $(-, +, +, +)$. Here the minus sign refers to the time direction and the three pluses to space.

However for this document we will also be looking at three dimensional spacetime, sig-



(a) The unit spheres in flat Riemannian space. (b) The unit spheres in Riemannian space.

Figure 40: Representations of the metric in 3-dim space.



(a) The two-sheet (upper and lower) hyperboloid. (b) The double cone. (c) The one-sheet hyperboloid.

Figure 41: The spacetime metric (in 3 dimensions). The two-sheet hyperboloid (a) is used for measuring timelike vectors. The upper hyperboloid is for future pointing vectors, and the lower for past pointing vectors. The one-sheet hyperboloid (c) is used for measuring the length of spacelike vectors. Between the two hyperboloids is the double cone, which all lightlike vectors lie on.

nature $(-, +, +)$, and two dimensional spacetime, signature $(-, +)$. This is because they are easier to draw.

7.1 Representing the metric: Unit ellipsoids and hyperboloids

Here we represent the metric is by the unit ellipsoid or hyperboloid. See also [6]. Around a point consider all the vectors which have unit length. For a Riemannian manifold, these vectors form an ellipse in 2 dimensions and an ellipsoid in 3 dimensions. In figure 40 we show the ellipsoid for a Riemannian manifold. If the ellipsoids are the same at each point then

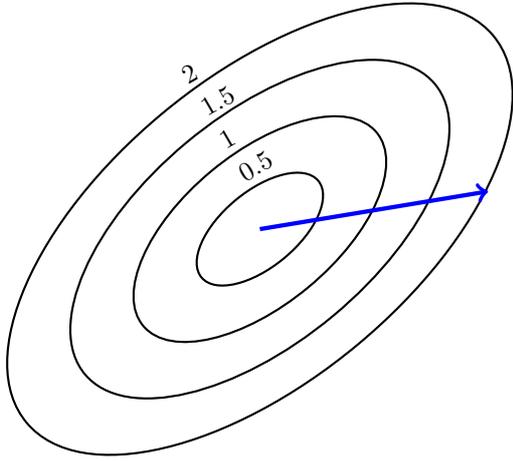


Figure 42: Using the metric ellipsoid to calculate the length of a vector. The unit ellipsoid is scaled to various sizes. The vector has length 2.

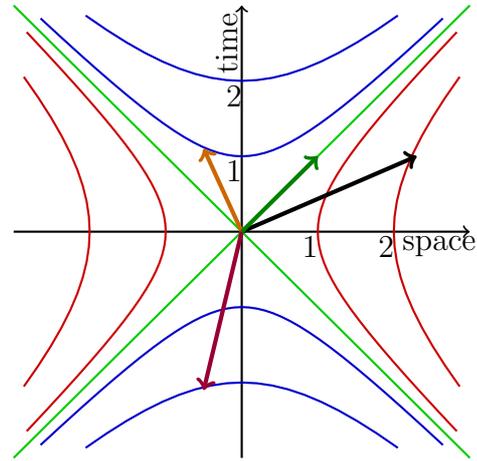


Figure 43: Using the metric hyperboloid to measure the length of vectors.

- The diagonal (green) lines are the path of light rays. Thus the diagonal vector (dark green) is a lightlike vector.
- The left and right (red) curves are the hyperboloid for measuring the length of spatial vectors. (In two dimensions these two curves are disjoint. However in three or more dimensions these two lines connect to create the one-sheet hyperboloid.) The right-pointing (black) vector is spatial and has length 2.
- The upper and lower (blue) lines are for measuring the length of timelike vectors. The upward (orange) line is timelike and has length 1 in the forward direction. The downward (purple) line is timelike and has length 2 in the backward direction.

the metric is flat as in figure 40a. By contrast for a curved metric the ellipsoids must be different at each point as in figure 40b.

For spacetime metrics the ellipsoids is replaced by a hyperboloid, as in figure 41. Since there is a minus in the signature, vectors are divided into three types:

- Timelike vectors, these represent the 4-velocity of massive particles. These vectors point to one of the components of the two-sheet hyperboloid, as seen in figure 41a. As the name suggest, the two-sheet hyperboloid has two components, the upper and lower. This distinguishes between future point and past pointing vectors.
- Lightlike vectors, the 4-velocity of light. These vector lie on the double cone. See figure 41b. Again one can distinguish between past and future pointing lightlike vectors.
- Spacelike vectors, These are not the 4-velocity of anything. These vectors point to the one-sheet hyperboloid. One cannot distinguish between past and future spacelike vectors.

7.2 Length of a vector

Using the metric ellipsoid or hyperboloid to calculate the length of a vector is easy. For a Riemannian metric the ellipsoid represent all vectors of unit length. Therefore simply compare length of the vector with the radius of ellipsoid in the direction of the vector as in

figure 42.

We see in figure 43, we use the two-sheet (upper and lower) hyperboloid to measure the length of forward and backward pointing timelike vectors. We also use the one-sheet hyperboloid (left and right curves) to measure the spacelike vectors.

7.3 Orthogonal subspace

Given a vector then the metric can be used to specify all the directions which are orthogonal to that vector. Orthogonal vectors are also known as perpendicular vectors or vectors at right angles to each other. This is given by the line tangent to the ellipsoid (figure 44) or hyperboloid (figure 45) at the point where the vector crosses it. In 3 dimensions the set of orthogonal directions form a plane and in 4 dimensions they would form a 3-volume.

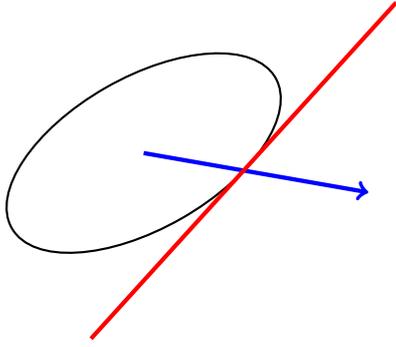


Figure 44: Using the metric ellipsoid to prescribe the orthogonal space (red) to a vector (blue). The orthogonal space is tangential to the ellipsoid.

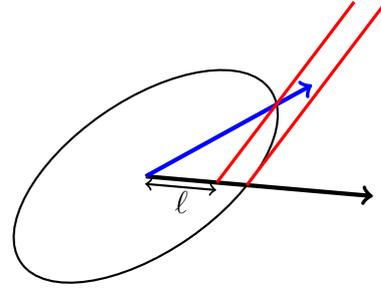


Figure 46: Using the metric ellipsoid to measure the angle between two vectors. The (red) lines are orthogonal to one of the vectors (black). The length ℓ is the cosine of the angle between the two vectors.

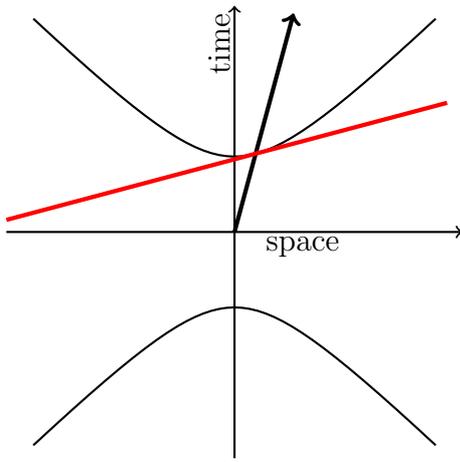


Figure 45: Using the metric hyperboloid to prescribe the orthogonal space (red) to a vector (black). The orthogonal space is tangential to the hyperboloid.

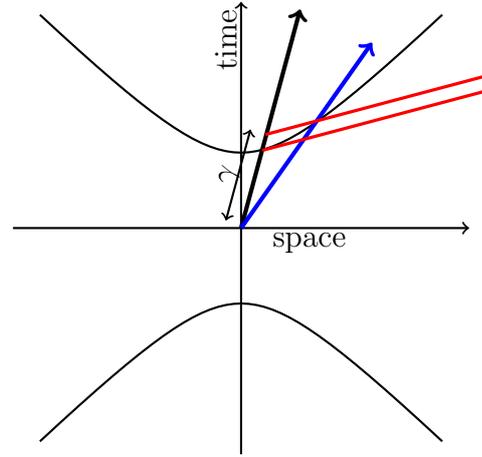


Figure 47: Using the metric hyperboloid to measure the angle between two vectors. The (red) lines are orthogonal to one of the vectors (black). The length γ is the γ -factor required to boost one of the vectors to the other.

7.4 Angles and γ -factors

Given two vectors then a Riemannian metric can be used to find the cosine of the angle between them as shown in figure 46. This cosine will always be a value between -1 and 1 .

In the case of a spacetime metric, then in figure 47 we get a value which is greater than 1 . We call this value the γ -factor. It is one of the key values in special relativity. Assuming that the two vectors are the 4-velocity of two particles, then the γ -factor is a function of the relative velocity between the two particles.

7.5 Magnitude of a form.

One can use the metric to measure the magnitude of forms. On an n -dimensional Riemannian manifold the magnitude to a 1-form

is given by counting the $(n - 1)$ -dimensional form-submanifolds inside the ellipsoid and dividing by 2 . In figure 48 we see how to use the metric ellipse to measure the magnitude to a 1-form in 2 dimensions. Measuring the magnitude of a 2-form, requires counting the number of $(n - 2)$ -dimensional form-submanifolds inside the ellipsoid and dividing by π . Similar formula exist for higher degrees and for spacetime manifolds.

7.6 Metric dual.

The metric can be used to convert a vector into a 1-form and a 1-form into a vector. Given a vector then the 1-form is the $(n - 1)$ -form-submanifold which is perpendicular to the vector, as given in figures 44 and 45. See also figure 49.

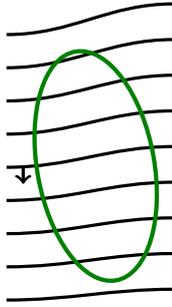


Figure 48: Using the metric to measure the magnitude of a 1-form in 2 dimensions. Simply count the number of form-submanifold inside the ellipse and divide by 2. In this case the 1-form has magnitude 3.5

The twistedness of the orientation is unchanged. For spacelike vectors the orientation is preserved, whereas for timelike vectors the orientation is reversed. For lightlike vectors, the vector is orthogonal to itself. Therefore it lies in its own orthogonal complement. For these the orientation points in the direction of the nearby spacelike vectors.

7.7 Hodge dual.

The Hodge dual takes a p -form and gives the unique $(n - p)$ -form which is orthogonal to it.

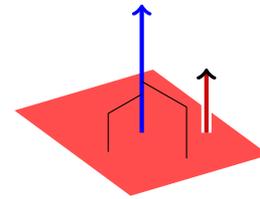
It changes the twistedness of the form. In Riemannian geometry, if one starts with an untwisted p -form it gives a twisted $(n - p)$ -form with the same orientation. If one starts with a twisted p -form it gives the untwisted form with either the same or the opposite orientation. This is given by double twisting as follows:

- If p is even or $(n - p)$ is even then the orientation is unchanged.
- If both p and $(n - p)$ are odd then the orientation is reversed.

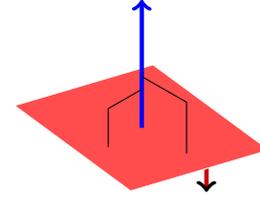
For non Riemannian geometry the formula for the resulting orientation is more complicated.

7.8 The measure from a metric

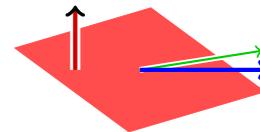
A metric naturally gives rise to a measure. That is a twisted top-form. This is actually the Hodge dual of the untwisted constant scalar field 1. It is given by packing together the metric ellipsoids. See figure 51. We can integrate this measure to give the size of any manifold even of it is not orientable.



(a) Spacelike vector (blue). The orientation (red) of the 1-form is unchanged.



(b) Timelike vector (blue). The orientation (red) of the 1-form is reversed.



(c) Lightlike vector (blue). The orientation (red) of the 1-form points in the direction of the nearby spacelike vectors (green).

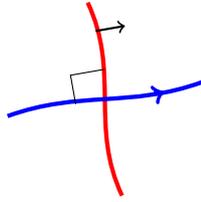
Figure 49: The metric dual converts a vector into a 1-form and visa versa. The untwisted vector (blue) is converted into the untwisted 1-form (red).

8 Smooth maps

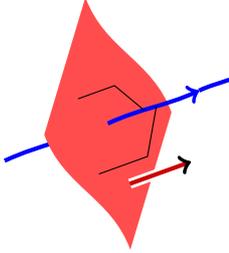
A smooth map is a function that maps one manifold into another.

The simplest case of a smooth map is a diffeomorphism where there is a one to one correspondence between points in the source manifold and points in the target manifold. As a result the source and target manifolds must have the same dimension and in some respects they look the same, that is all the concepts in sections 2-6. By contrast when a metric is included the two manifolds can look very different. One example is in transformation optics where one maps between one manifold where light rays travel in straight lines and another manifold where light rays travel around the hidden object.

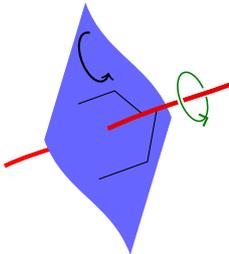
One class of smooth maps that we have encountered is that of the embedding of submanifolds as seen in figure 4. In this case we map a lower dimensional (embedded) manifold into a higher dimensional (embedding) manifold. In this case every point in the em-



(a) In 2-dim. Hodge dual of the untwisted 1-form (red), becomes the twisted 1-form (blue).



(b) In 3-dim. Hodge dual of the untwisted 1-form (red), becomes the twisted 2-form (blue).



(c) In 3-dim. Hodge dual of the untwisted 2-form (red), becomes the twisted 1-form (blue).

Figure 50: Hodge dual of untwisted forms (red) to twisted forms (blue) in 2 and 3-dimensional Riemannian geometry. For space-time geometry the resulting orientation may be reversed.

bedded submanifold corresponds to a unique point in the embedding manifold. However some points in the embedding manifold do not correspond to a point in the embedded manifold.

Another class are regular projections. These map a higher dimensional manifold into a lower dimensional manifold. See figures 52 and 53. In this case every point in the image manifold corresponds to a submanifold of points in the domain.

Other maps however are none of the above and may include self intersections (figure 5), folding and a variety of other more complicated situations.

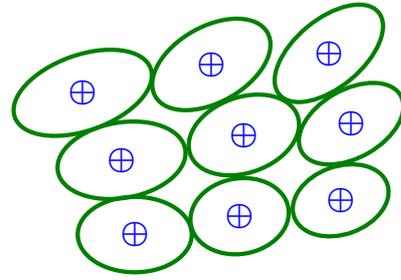


Figure 51: The measure from the metric. In 2 dim, the twisted top form is given by packing together the metric ellipses.

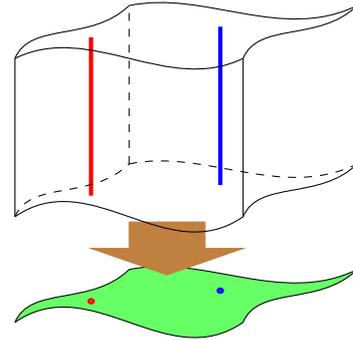


Figure 52: A projection mapping a 3-dim manifold onto a 2-dim manifold (green). All the points on each line (red and blue) are projected onto the respective dots.

8.1 Pullbacks of p -forms.

No matter how complicated the map is it is always possible to pullback the form. We will only consider here the pullback with respect to diffeomorphisms, embeddings and projections. Pullbacks preserve the degree of the form and also the twistedness of the forms.

The pullback with respect to diffeomorphisms simply deforms the shape of the form-submanifolds. However unless there is a metric the question is: deforms with respect to what? This corresponds to our statement that without a metric one cannot distinguish between manifolds which are diffeomorphic and therefore as we said in the introduction, one is dealing with rubber sheet geometry.

For the pullback with respect to an embedding, one simply takes the intersection of submanifold with the form-submanifolds as seen in figures 54, 55 and 56. This preserves the degree of the form. If we pullback a p -form from an m -dimensional manifold to an n -dimensional manifold then the p -form-

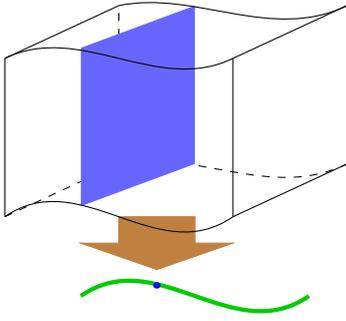


Figure 53: A projection mapping a 3-dim manifold onto a 1-dim manifold (green). All the points on the plane (blue) are projected onto the dot.

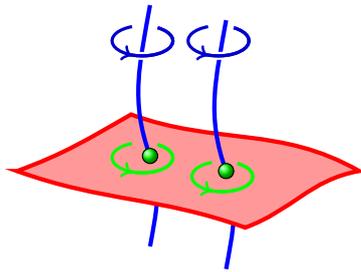


Figure 54: The pullback with respect to a 2-dim submanifold (red plane) embedded in a 3-dim manifold. The pullback of an untwisted 2-form (blue curves) gives an untwisted 2-form (green dot).

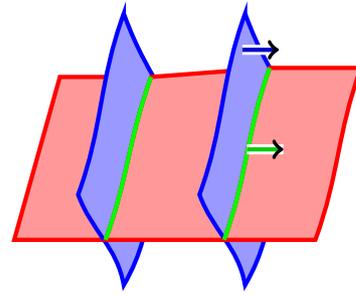


Figure 55: The pullback with respect to a 2-dim submanifold (red plane) embedded in a 3-dim manifold. The pullback of an untwisted 1-form (blue planes) gives an untwisted 1-form (green lines).

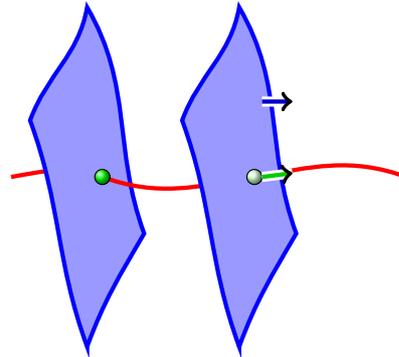


Figure 56: The pullback with respect to a 1-dim submanifold (red curve) embedded in a 3-dim manifold. The pullback of an untwisted 1-form (blue plane) gives an untwisted 1-form (green dots).

submanifolds are of dimension $(m - p)$, hence the intersection of the $(m - p)$ -dim form-submanifolds with the n -dimensional manifolds will have dimension $(n - p)$ corresponding correctly to a p -form. This assumes the form-submanifolds and the submanifold intersects transversely. However if the submanifold and the form-submanifold intersect tangentially then the result is zero. See figure 57.

For untwisted forms there is a natural pullback of the orientation. For twisted forms it is necessary to pullback onto a twisted submanifold. To be guaranteed to get the orientation correct it is easiest to choose an orientation for the embedding manifold, untwist both the form and the submanifold and then twist the pullbacked form. This gives the result that: The orientation of the pullbacked form concatenated with the orientation of the submanifold equals the orientation of the original form. In particular the pullback of a twisted p -form onto a twisted submanifold is plus if

the orientations agree and minus otherwise, as was discussed in section 5.2.

For the pullback with respect to projections the resulting form-submanifold is simply all the points which mapped onto the original form-submanifold. See figures 58 and 59. For untwisted forms the orientation is obvious. For twisted forms one can use orientation for both the source and target manifolds, untwist the form and convert it back. Alternatively one can decide an orientation of the projection and use that. We will not describe this

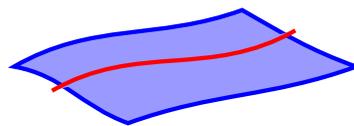


Figure 57: The pullback with respect to a 1-dim submanifold (red curve) embedded in a 3-dim manifold. The 1-form (blue plane) is tangential to 1-dim submanifold. The result is zero.

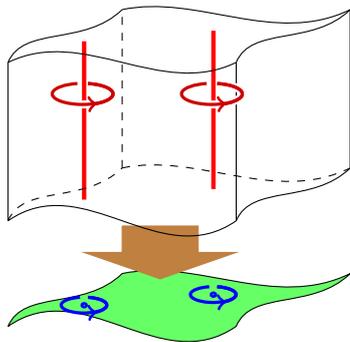


Figure 58: The pullback with respect to a projection from a 3-dim manifold to a 2-dim manifold (green plane). The pullback of the untwisted 2-form (blue dots) gives the 2-form (red lines).

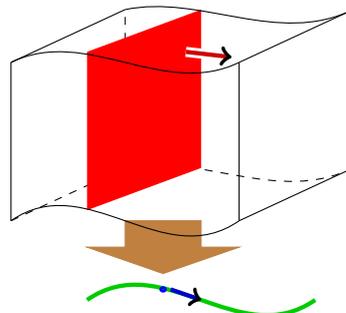


Figure 60: The pullback with respect to a projection from a 3-dim manifold to a 1-dim manifold (green curve). The pullback of the untwisted 1-form (blue dot) gives the 1-form (red plane).

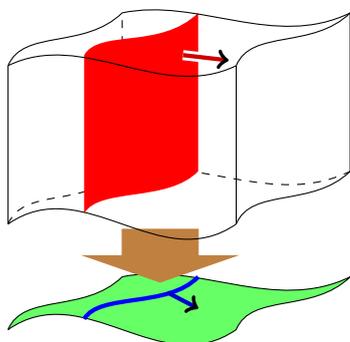


Figure 59: The pullback with respect to a projection from a 3-dim manifold to a 2-dim manifold (green plane). The pullback of the untwisted 1-form (blue curve) gives the 1-form (red plane).

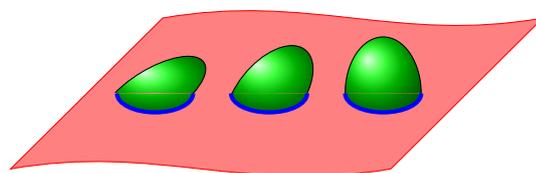


Figure 61: The pullback of a Riemannian metric onto an embedding as represented by unit spheres. The unit spheres in 3-dim (green) are pulled back into unit circles (blue) on the 2-dim submanifold (red).

here.

8.2 Pullbacks of metrics.

We can only pullback a metric with respect to diffeomorphisms and embedding. With respect to diffeomorphisms it makes the two manifolds have the same geometry.

The pullback of a metric with respect to an embedding is called the induced metric. This is the metric we are familiar with on the sphere or torus which is embedded into flat 3-space. Figure 61 shows the result of the induced metric.

We cannot pullback a metric with respect to a projection as we don't have enough information. In particular we do not know the length of vectors in the direction of the pro-

jection.

8.3 Pushforwards of vectors.

We can always pushforward a vector field with respect to a diffeomorphism. However, in general the pushforward of a vector field with respect to any other map will not result in a vector field. As can be seen in figure 62 some points in the target manifold have no vector associated with them and others have more than one.

However the pushforward of vectors are very useful. For example the velocity of a single particle is the tangent to its worldline. This is only defined on its worldline.

I will not discuss these further here.

9 Electromagnetism and Electrodynamics

Maxwell's equations, when written in standard Gibbs-Heaviside notation are four dif-

| Vector notation | names | Form notation | Relation between vector and form |
|-------------------------------|----------------------------------------------------------------------------------------------------|--------------------------|----------------------------------|
| \mathbf{E} untwisted vector | Electric field. | E untwisted 1-form | Metric dual |
| \mathbf{D} untwisted vector | Displacement field. | D twisted 2-form | Metric dual and Hodge dual |
| \mathbf{B} twisted vector | “B-field”, Magnetic field or Magnetic flux density. | B untwisted 2-form | Metric dual and Hodge dual |
| \mathbf{H} twisted vector | “H-field”, Magnetic field intensity, Magnetic field strength, Magnetic field or Magnetising field. | H untwisted 1-form | Metric dual |
| ρ untwisted scalar | charge density. | ϱ twisted 3-form | Hodge dual |
| \mathbf{J} untwisted vector | current density. | J twisted 2-form | Metric dual and Hodge dual |

Table 5: The fields in electromagnetism, in terms of the familiar vector notation, the corresponding form notation and the operations needed to pass between the two.

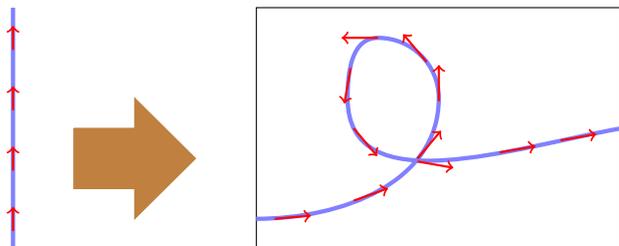


Figure 62: Push forward of the vector field (red) of the 1-dim manifold (blue) embedding into the 2-dim manifold (white) produces vectors (red). However these vectors do not form a vector field as they are not defined in some places and in others have multiple values.

| Spacetime form | Pull back onto space | Remaining component |
|------------------------------|--------------------------|----------------------|
| F untwisted 2-form | B untwisted 2-form | E untwisted 1-form |
| \mathcal{H} twisted 2-form | D twisted 2-form | H twisted 1-form |
| \mathcal{J} twisted 3-form | ϱ twisted 3-form | J twisted 2-form |

Table 6: Combining the fields in electromagnetism into the relativistic quantities.

ferential equations of four vector fields \mathbf{E} , \mathbf{B} , \mathbf{D} and \mathbf{H} . The source for the electromagnetic fields is the charge density ρ and the current \mathbf{J} . In our notation it is clearer to replace these with forms. In table 5 the relationship between the vector notation and the form notation is given.

As can be seen in table 5, both the fields \mathbf{B} and \mathbf{H} may be referred to as the magnetic field. Here we will simply refer to them as the “B-field” and “H-field”.

As we said, Maxwell’s equations are differential equations in \mathbf{E} , \mathbf{B} , \mathbf{D} and \mathbf{H} . However there is insufficient information to be able to solve these equations. They need to be related by additional information which relate the four quantities \mathbf{E} , \mathbf{B} , \mathbf{D} and \mathbf{H} . These are called the constitutive relations. We have

just stated the vacuum constitutive relations, namely that $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$. However in a medium, such as glass, water, a plasma, a magnet, a human body or whatever else you may wish to study, these constitutive relations are more complicated. The simplest constitutive relations consider a constant permittivity ϵ and permeability μ . In this medium there is no dispersion, i.e. radiation of different frequency travel at the same speed. This would mean for example that rain drops would not give rise to rainbows and may therefore be described as “antediluvian”! They are however easy to work with. A more physically reasonable constitutive relations is to allow the permittivity or permeability to depend on frequency. This is a good model for water, wax or the human body. If the electric or magnet fields are very strong then the permittivity and permeability

depend on the electric or magnet fields. These are non linear materials. Exotic and meta-materials can have more complicated constitutive relations. The degree of complexity of these relations is limited only by the imagination of the scientists studying the material and their ability to build them. Indeed physicists are currently looking at models of the vacuum where the simple relations $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ break down in the presents of intense fields, either due to quantum effects or because Maxwell's equations themselves breakdown.

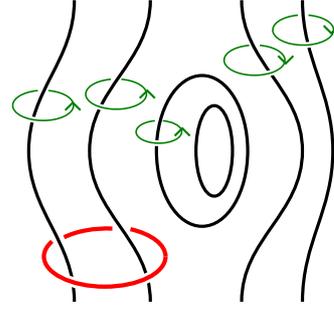


Figure 63: The magnetic 2-form B in space as unbroken curves. Since the curves are unbroken this corresponds to the non divergence of B . The external orientation is given (green). The circle (red) encloses some of the B -lines.

Just as in relativistic mechanics where we combine energy and momentum into a single relativistic vector the 4-momentum, so we combine the electric and magnetic B-field into a single spacetime untwisted 2-form F . Likewise we combine the displacement current and H-field into a single twisted 2-form \mathcal{H} , and we combine ρ and J in a single twisted 3-form \mathcal{J} , as summarised in table 6. As a result relativity mixes the \mathbf{E} and \mathbf{B} fields and separately mixes the \mathbf{D} and \mathbf{H} fields. The constitutive relations then relate F and \mathcal{H} . The simplest case is the vacuum where the relationship is simply given by the Hodge dual.

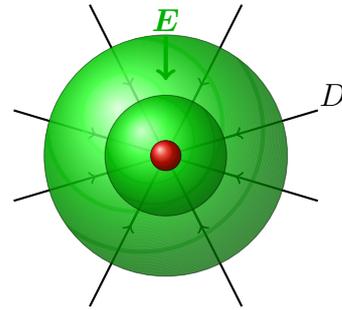


Figure 64: Gauss's Law in electrostatics. The positive charge, is given by the twisted 3-form ρ is the dot (red) in the centre. This is the exterior derivative of the twisted 2-form D (black) lines. In the vacuum or similar constitutive relations, then D is the Hodge dual of the untwisted 1-form E (green spheres).

As stated, Maxwell's equations give the electric and magnetic fields for a given current. It is also necessary to know how the charges, that produce the current, respond to the electromagnetic fields. This is given by the Lorentz force equation.

In this section we start, section 9.1, with electrostatics and magnetostatics as these are in three dimensions and therefore easier to depict pictorially. We then, in section 9.2 to draw Maxwell's equations as four spacetime pictures. In section 9.3, we depict the Lorentz force equation and a spacetime picture. In section 9.4 we look at Maxwell's equations in 3-dimensional spacetime. This is the pullback of the 4-dimensional spacetime equations onto a fixed plane. This contrasts the the static cases when we pullback onto a fixed timeslice.

9.1 Electrostatics and Magnetostatics.

In this subsection we deal with electrostatics and magnetostatics as 3-dimensional pictures are easier to draw. One has that \mathbf{B} is divergence-free, Gauss's law which states that the divergence of \mathbf{D} equals the charge, Faraday's law which states that \mathbf{E} is curl-free and Ampere's law which states that the curl of \mathbf{H} is the current. In the language of forms this corresponds to the fact that B and E are closed, whereas the exterior derivative of D and H are ρ and J respectively.

In figure 63 we see that the B lines are unbroken and therefore B is closed. In figure 64 we see Gauss's law, that a charge twisted 3-form ρ is the exterior derivative of the twisted 2-form D . For the vacuum or for any constitutive relation where the permittivity is con-

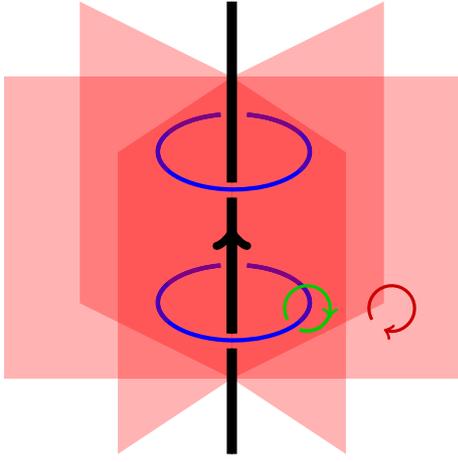


Figure 65: Ampere's law: The constant line current, the twisted 2-form J (Black), is the exterior derivative of the twisted 1-form H . In the vacuum or similar constitutive relations, then H is the Hodge dual of the untwisted 2-form B (blue circles).

stant or depends only on frequency, then the D is the permittivity times the Hodge dual of the electric 1-form E . In figure 65 we see Ampere's law. The current, a twisted 2-form

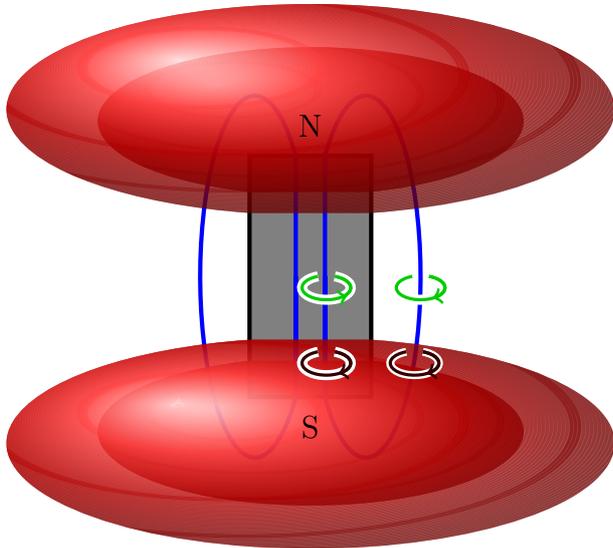


Figure 66: A permanent magnet. The untwisted magnetic 2-form B (blue curves) and twisted 1-form H (red ellipsoids) generated by a permanent magnet (grey rectangle). Observe that outside of the magnet H fields are perpendicular to the B fields. Thus the Hodge dual, figure 50, of H coincides with B . Inside the magnet however the orientations of H and B are in opposite directions. This is due to the non vacuum constitutive relations of permanent magnets.

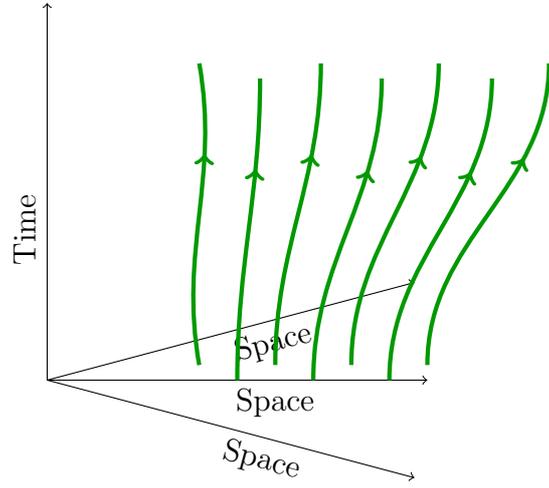


Figure 67: Conservation of charge: There exists a closed twisted 3-form \mathcal{J} called the electric current (green). The green line may be interpreted as charged particles, distributed over all of spacetime.

J in the wire, is the exterior derivative of the twisted 1-form H , and H is the Hodge dual of B .

In figure 66 we show the fields for a permanent magnet. This is an example of a slightly more complicated constitutive relations. Outside of the magnet, the twisted 1-form H is the Hodge dual of the untwisted 2-form B . However in the magnet, the orientations of H and B are in the opposite direction, due to the constitutive relations for a permanent magnet.

9.2 Maxwell's equations as four spacetime pictures

Figures 67 to 70 represent conservation of charge and Maxwell's equations in spacetime. Thus there are four axes, one of time and three of space.

We start with conservation of charge, figure 67, which is not one of Maxwell's equations but a simple consequence of them. In figure 67 we see the 1-dim form-submanifolds of \mathcal{J} do not have a boundary and therefore \mathcal{J} is closed. One may think of the form-submanifolds of \mathcal{J} as worldlines of the individual point charges such as electrons or ions.

Look again at figure 63, but now consider that the fields are not static. We may imagine that the magnetic field lines move in space.

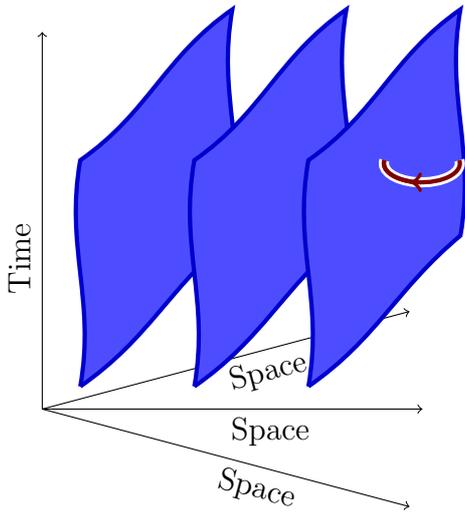


Figure 68: Conservation of magnetic flux: There exists a closed untwisted 2-form F called the electromagnetic field (blue). This encodes the two fields E and B . Since we are in 4-dimensions the orientation is a loop outside the 2-dim form-submanifolds. The closure of F is equivalent to the two macroscopic Maxwell equations which contain E and B .

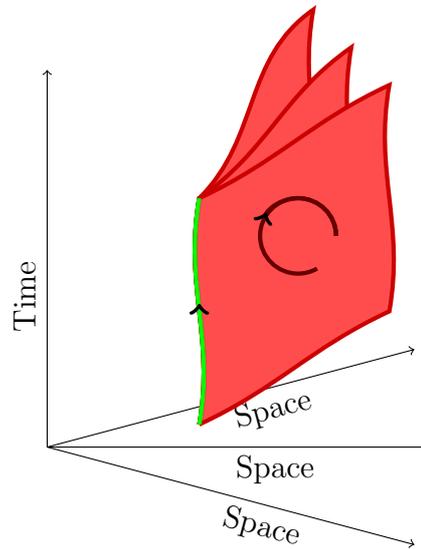


Figure 69: The excitation fields: There exists a twisted 2-form \mathcal{H} called the excitation field (red). It encodes the fields D and H . In general it is not closed and its exterior derivative is \mathcal{J} (green). This is equivalent to the two macroscopic Maxwell equations which contain D and H .

As a result some of the field lines may cross the red circle. This will generate an electromotive force as given by the integral form of Faraday's law of induction. In this case the form-submanifolds of B will map out a 2-dim form-submanifold in 4-dim spacetime. These form-submanifold will not have boundaries and therefore it corresponds to the untwisted closed 2-form F . In figure 68 we depict the closed 2-form F as 2-dim without boundary in 4-dim spacetime. However one should recall that 2-forms in 4-dimensions can self intersect as indicated in section 4.7.

Repeating the same procedure for figure 64. If the charges ρ are not static then they map out worldlines in spacetime. This is the closed 3-form \mathcal{J} as in figure 67. As a consequence the 1-dim form-submanifolds for the 2-form D map out the 2-dim form-submanifolds for the 2-form \mathcal{H} . This gives figure 69.

As stated before, Maxwell's equations need to be augmented using the constitutive relations. In the vacuum one has that D and H are the Hodge duals of E and B , as can be seen in figures 64 and 65. In figure 70 we see that the twisted 2-form \mathcal{H} is the Hodge dual of untwisted 2-form F . In other media the relationship between F and \mathcal{H} is more com-

plicated.

9.3 Lorentz force equation

In electrodynamics there is one remaining equation, the Lorentz force law. As depicted in figure 71. This states that in spacetime the

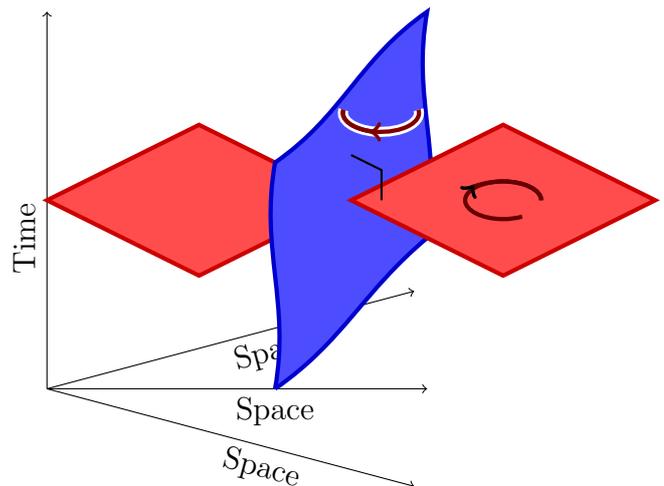


Figure 70: The constitutive relations in the vacuum: The twisted 2-form \mathcal{H} (red) is the Hodge dual of the untwisted 2-form F (blue). Here we attempt to show that \mathcal{H} and F are orthogonal 2-dim form-submanifolds in 4-dimensions. They have the same orientation.

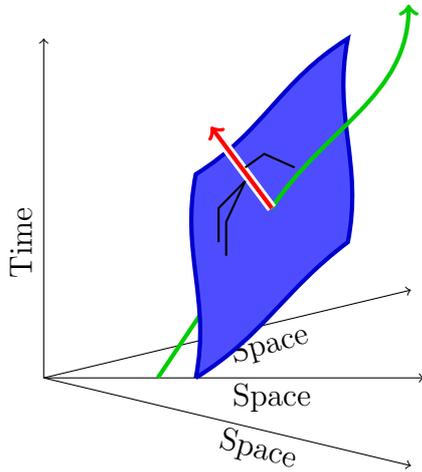


Figure 71: An attempt to draw the Lorentz force law on a particle. Here the 2-form F is the 2-dim form-submanifold (blue), the world-line of the particle is the curve (green) and the force is the straight arrow (red). The force is orthogonal to both the 2-form F and the 4-velocity of the particle.

force is orthogonal to both the 4-velocity of the particle and the electromagnetic 2-form F .

9.4 Constant space slice of Maxwell's equations.

We may consider the static figures 63 and 64 correspond to the pullback, as discussed in section 8, of the fields F , \mathcal{H} and \mathcal{J} on a 3-dimensional slice through spacetime.

An alternative slice of Maxwell's equations in spacetime is to choose a plane in space and consider what the electromagnetic fields look like when pulled back onto this plane for all time. We will call this plane the $z = 0$ plane. The result is a diagram with one time coordinate and two space coordinates.

The 2-form F is a closed untwisted 1-dimensional form-submanifold. However this time it contains information about both B and E . Figure 72 shows Faraday's law of induction for a loop in the plane $z = 0$. Since F is closed the difference between the field at the top and bottom, which is the difference in the B field, must correspond to field leaving the loop, which is the integral of the electromotive force. Similarly figure 73 is the Maxwell-Ampères law.

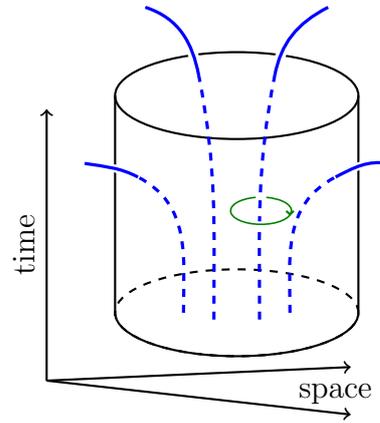


Figure 72: Faraday's law of induction on the plane $z = 0$, integrated over time and space. The blue lines are pullback of the electromagnetic 2-form F onto the plane $z = 0$. The blue lines are continuous, i.e. as F is closed and therefore the pullback of F onto the plane $z = 0$ is closed. Thus the difference between the number of lines entering the bottom disc and the number leaving the top disc equals the number of lines leaving the sides of the cylinder.

10 Conclusion

We have presented a large proportion of differential geometry, concentrating on differential forms, in order to arrive at figures 67-71 which are the diagrams that describe electrodynamics.

In terms of pictorially presenting differential geometry, there are still some open challenges. One is in the attempt to depict 2-forms in 4 dimensions, section 4.7. One question is whether for a given 2-form, figure 32 or 33 is correct and whether there are alternatives which give more or different information.

There are many phenomena in electromagnetism which may gain from a pictorial representation. For example, how is it best to draw the the electromagnetism potential, and can one shed light on the Aharonov-Bohm effect? How does one depict the electromagnetic fields as well as the flow of electrons and ions in a plasma. In this case, ideally one would have 7 dimensional paper.

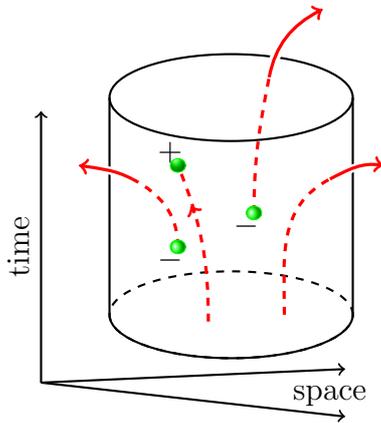


Figure 73: Maxwell Ampère's law on the plane $z = 0$, integrated over time and space. The green dots are the intersections of the positive and negative charges as they cross $z = 0$. The red lines are the pullback of the excitation 2-form \mathcal{H} onto the plane $z = 0$. These terminate at charges. Thus the difference between the number of lines entering the bottom disc and the number leaving the top and sides of the cylinder equals the charge inside the cylinder.

Acknowledgments

I am grateful grateful for the support provided by STFC (the Cockcroft Institute ST/G008248/1 and ST/P002056/1), EPSRC (the Alpha-X project EP/J018171/1 and EP/N028694/1). I am grateful to the masters students who took my course Advanced Electromagnetism and Gravity and 2017 and read the document, making suggestions and to Taylor Boyd and Richard Dadhley who pointed out some inconsistencies.

All figures are used with kind permission of Jonathan Gratus [10].

References

- [1] Karl F Warnick and Peter H Russer. Differential forms and electromagnetic field theory. *Progress In Electromagnetics Research*, 148:83–112, 2014.
- [2] William L Burke. *Applied differential geometry*. Cambridge University Press, 1985.
- [3] Bernard Jancewicz. *Multivectors and Clifford algebra in electrodynamics*. World Scientific, 1989.
- [4] Enzo Tonti. Finite formulation of the electromagnetic field. *Progress In Electromagnetics Research*, 32:1–44, 2001.
- [5] Karl F Warnick, Richard H Selfridge, and David V Arnold. Electromagnetics made easy: differential forms as a teaching tool. In *Frontiers in Education Conference, 1996. FIE'96. 26th Annual Conference., Proceedings of*, volume 3, pages 1508–1512. IEEE, 1996.
- [6] Charles W Misner and John A Wheeler. Classical physics as geometry. *Annals of physics*, 2(6):525–603, 1957.
- [7] Friedrich W Hehl and Yuri N Obukhov. A gentle introduction to the foundations of classical electrodynamics: The meaning of the excitations (d, h) and the field strengths (e, b). *arXiv preprint physics/0005084*, 2000.
- [8] Jan Arnoldus Schouten. *Tensor analysis for physicists*. Courier Corporation, 1954.
- [9] Friedrich W Hehl and Yuri N Obukhov. *Foundations of classical electrodynamics: Charge, flux, and metric*, volume 33. Springer Science & Business Media, 2012.
- [10] Jonathan Gratus. Lancaster university internal report. 2017.