THE SLICE METHOD FOR G-TORSORS

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Abstract. The notion of a (G, N)-slice of a G-variety was introduced by P.I. Katsylo in the early 80's for an algebraically closed base field of characteristic 0. Slices (also known under the name of relative sections) have ever since provided a fundamental tool in invariant theory, allowing reduction of rational or regular invariants of an algebraic group G to invariants of a "simpler" group. We refine this notion for a G-scheme over an arbitrary field, and use it to get reduction of structure group results for G-torsors. Namely we show that any (G, N)-slice of a *versal* G-scheme gives surjective maps $H^1(L, N) \to H^1(L, G)$ in fppf-cohomology for infinite fields L containing F. We show that every stabilizer in general position H for a geometrically irreducible G-variety V gives rise to a $(G, N_G(H))$-slice in our sense. The combination of these two results is applied in particular to obtain a striking new upper bound on the essential dimension of the simply connected split algebraic group of type E_7.

1. Introduction

In [Ka83] P.I. Katsylo introduced the notion of a slice (or section) for a regular action of a connected algebraic group G on a variety X over the field $\mathbb{C}$ of complex numbers as follows: A (G, N)-slice of X, for a subgroup N of G, is a (locally closed) subvariety S of X satisfying the following two conditions:

(1) $\overline{G \cdot S} = X$
(2) If $s \in S$ and $g \in G$ then $gs \in S$ if and only if $g \in N$.

This definition was inspired by Seshadri's work [Se62] from the early sixties. Katsylo observed that any (G, N)-slice induces an isomorphism $\mathbb{C}(X)^G \simeq \mathbb{C}(S)^N$ through restriction of rational functions. Applying this observation to the case $G = \mathrm{SL}_2$ and $X = V_d$ an irreducible G-representation, he showed that the fields $\mathbb{C}(V_d)^{\mathrm{SL}_2}$ are projectively rational, i.e., generated by algebraically independent homomogeneous rational functions.

A closely related notion is the one of a *Chevalley relative section*. This is a subvariety S of X such that restriction of regular functions induces an isomorphism $\mathbb{C}[X]^G \simeq \mathbb{C}[S]^N$, where $N = \{g \in G \mid gS = S\}$ is the normalizer of S. Existence of Chevalley relative sections and (G, N)-slices as defined by Katsylo simplify the calculation of regular resp. rational invariants of an algebraic group G.

Notions of slices and sections are widespread in the literature, not only forming an important technical tool in invariant theory, but also studied in their own right. In disguised form these concepts were already present in the 19th century, for instance in the work of Weierstrass and Hesse on normal forms of plane cubic curves. We refer to [Po94] for a systematic treatment and survey of this topic, all over an algebraically closed base field of characteristic 0.

Our interest in slices and sections has slightly different origins. It comes from the desire to classify torsors of an algebraic group G over an arbitrary field, or equivalently, to describe the sets $H^1_{fppf}(L, G)$ for an algebraic group G over F and field extension L of F. Torsors of algebraic groups often describe interesting algebraic objects like central simple algebras (for

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projective linear groups), quadratic forms (for orthogonal groups) or Cayley algebras (for the exceptional simple group G_2).

By getting information on the structure of torsors we also wish to improve the known upper bounds on the essential dimension of various algebraic groups. The essential dimension of an algebraic group G is a measure of how many algebraically independent parameters are needed to describe any of its torsors up to isomorphism. For example, isomorphism classes of $\mathbf{O}_n$-torsors correspond bijectively to isometry classes of non-degenerate quadratic forms of dimension n. It is known that such quadratic forms can always be diagonalized (provided char $F \neq 2$), which translates to the fact that $H^1(L, \mu_2^n) \to H^1(L, \mathbf{O}_n)$ is surjective for all field extensions L/F. This implies $\mathrm{ed}(\mathbf{O}_n) \leq \mathrm{ed}(\mu_2^n) = n$. Indeed, diagonalization should mean that a quadratic form "is described by at most n parameters".

Let N be a subgroup of an algebraic group over a field F. We will define a (G, N)-slice for a G-scheme X as an N-stable (locally closed) subscheme S of X such that the induced morphism $(G \times S)/N \to X$ of algebraic spaces is an open embedding. Here N acts on $G \times S$ through the formula $n \cdot (g, s) = (gn^{-1}, ns)$. This action is free. The fppf-quotient sheaf $(G \times S)/N$ is known to be an algebraic space.

We prefer to use the term "slice" rather than "section", following [CT88]. This is motivated by the analogy with the étale slices from Luna's theorem [Lu73] and the analogy with the notion of a slice from differential geometry, see e.g. [Pa60, Definition 2.11]. Our definition of a (G, N)-slice can be seen as a version of Katsylo's definition, which works well for arbitrary base fields F. See Corollary 3.5 and Proposition 5.1, which highlight the close connection between the two different definitions.

In Theorem 3.7 we show that existence of a (G, N)-slice of a versal G-scheme implies surjectivity of the maps $H^1_{fppf}(L, N) \to H^1_{fppf}(L, G)$ for infinite fields L containing F. Examples of versal G-schemes include linear representations and their associated projective spaces, as well as connected reductive groups on which G acts by group automorphisms; see Example 2.7. Every such reduction of structure group result implies the inequality $\mathrm{ed}(G) \leq \mathrm{ed}(N)$. The hope is that this will provide improved upper bounds for the essential dimension of G. Indeed in the case of $E_7^{s.c.}$, we obtain a much better upper bound on the essential dimension than was previously known.

Our principal method to construct slices of (geometrically irreducible) G-varieties comes from stabilizers in general position (SGP's). Namely Theorem 4.1 shows that for an SGP H of a geometrically irreducible G-variety V, there exists a $(G, N_G(H))$-slice, which is open in the fixed scheme V^H. The concept of an SGP has its origin in invariant theory over an algebraically closed field of characteristic 0. It has only recently become a tool used for arbitrary base fields.

We will give another construction of slices for smooth algebraic groups in section 5. Proposition 5.1 shows in particular, that in characteristic 0 every (G, N)-slice in Katsylo's sense gives a (G, N)-slice with our definition.

Theorem 3.7 could be considered a generalization of two existing results. Firstly, it generalizes to fields of characteristic other than zero an early result of Reichstein, which used (G, N)-sections (in the sense of [Re00, Definition 2.9]) of linear representations to give the inequality $\mathrm{ed}(G) \leq \mathrm{ed}(N)$; see Remark 3.9. Secondly, in [Ga09] Garibaldi, inspired by Rost's unpublished preprint [Ro99], considered the situation when there is a zero-dimensional slice of $\mathbb{P}(V)$ (consisting of a single point), where V is a linear representation of G, and proved the surjectivity of Theorem 3.7 when G is smooth. The surjectivities proved in Garibaldi's paper were fruitfully applied to describe cohomological invariants of various exceptional groups and Spin groups. So our Theorem might also be considered as a generalization of this result to a much wider class of slices.

The existence of a slice of the adjoint representation was shown during Grothendieck's proof that any smooth algebraic group G over an arbitrary field F contains a maximal torus [SGA3, XIV Theorem 1.1]. In particular, he showed that the regular elements contained in a given Cartan subalgebra form a slice of the adjoint representation, see [SGA3, XIII, Theorem 6.1(d)] (cf. Example 5.3). If $T \subset G$ is a split maximal torus in G, and $\mathrm{Aut}(G) \cong G$, then the existence of a maximal torus in $\bar{F}/F$-forms of G is equivalent to the surjectivity of the map $H^1(F, N_G(T)) \to H^1(F, G)$; in this situation, that surjectivity will now follow from Theorem 3.7.

Our results are also related to work of Chernousov, Gille and Reichstein [CGR08], who showed that every reductive algebraic group G has a finite subgroup S (contained in the normalizer of a maximal torus) for which the map $H^1(L, S) \to H^1(L, G)$ is surjective for every field extension L/F. However in practice these finite subgroups are usually not ideal for proving interesting new upper bounds on $\mathrm{ed}(G)$.

The rest of this paper is structured as follows: In section 2 we will fix conventions, recall some basics on algebraic spaces, torsors, twists, versal G-schemes, stabilizers in general position and essential dimension and prove some useful lemmas used later on. In section 3 we discuss (G, N)-slices of a G-scheme for an algebraic group G and subgroup N. Theorem 3.7 shows that every (G, N)-slice of a versal G-scheme (for instance a linear representation) gives rise to surjections $H^1(L, N) \to H^1(L, G)$ in fppf-cohomology for infinite fields L containing the base field. We proceed to construct slices of geometrically irreducible G-varieties out of stabilizers in general position in section 4 and give another construction for smooth algebraic groups in section 5. A simple application of the former result is given in section 6, which provides a new proof of Risman's Theorem in the theory of central simple algebras. Finally in section 7 we take the SGP on two copies of the 56-dimensional representation of $E_7^{s.c}$ to prove the upper bound $\mathrm{ed}(E_7^{s.c}) \leq 11$.

2. Preliminaries

2.1. **Definitions and conventions.** Unless otherwise specified, F will denote an arbitrary field, and $\bar{F}$ an algebraic closure. An *algebraic group* G will be a (not necessarily smooth) affine group scheme of finite type over a field (usually F). For an algebraic group G, the symbols $N_G(H)$ or $\mathrm{Norm}_G(Y)$ will denote the scheme-theoretic normalizer of G which preserves a subgroup H or closed subscheme Y respectively [Ja03]; the normalizer subgroup need not be smooth, even if G is smooth (for example, if $G = \mathbb{G}_m$ acts on $\mathbb{A}^1$ by $x.v := x^p v$, and Y is a non-zero closed point, then the normalizer is μ_p). A *variety* will be a reduced scheme which is separated and of finite type over a field.

We will frequently use the notion of algebraic spaces, as defined in [Stacks, 025Y]. Algebraic spaces are a generalization of schemes, that behave better under descent and appear naturally when working with quotients of free algebraic group actions on schemes. The category of schemes over F will be furnished with the big fppf (finitely presented faithfully flat) topology. Thus an *algebraic space* over F is a sheaf $X\colon (Sch/F)^{\mathrm{op}}_{fppf} \to \mathrm{Sets}$ such that

(1) For all schemes U, V over F and sheaf maps $U \to X$, $V \to X$ the fiber product of sheaves $U \times_X V$ is representable by a scheme.
(2) There exists a scheme U over F and a surjective étale sheaf map $U \to X$ (this means that for every scheme V over F and sheaf map $V \to X$ the morphism $U \times_X V \to V$ of schemes is surjective and étale).

A morphism of algebraic spaces is a natural transformation of functors. Every scheme gives rise to an algebraic space and the association $X \mapsto X$ defines a fully faithful embedding of the category (Sch/F) to the category of algebraic spaces over F. Thus we will identify every scheme over F with the algebraic space over F which it defines. If G is an algebraic group, a G-scheme will be a scheme with an action $G \times X \to X$ of G.

Definition 2.1. Let G be an algebraic group over F and $\pi\colon X \to Y$ a G-invariant morphism between algebraic spaces over F (i.e. π is G-equivariant, and G acts trivially on Y). The morphism π is said to be a *pseudo G-torsor* if the morphism $G \times X \to X \times_Y X$ given by

$(g, x) \mapsto (gx, x)$ is an isomorphism. A pseudo G-torsor π is called an *(fppf) G-torsor* if the morphism $\pi\colon X \to Y$ is fppf. Equivalently (cf. [Mi80, III, Proposition 4.1]) there exists an fppf covering $\{Y_i \to Y\}_{i\in I}$ such that for every i the pull-back $X \times_Y Y_i \to Y_i$ is G-equivariantly isomorphic to $G \times Y_i \to Y_i$, where G acts trivially on Y_i and by left multiplication on G. If $X \to \operatorname{Spec} F$ is a G-torsor, where F is a field, then we will also call X a G-torsor over F.

If an algebraic group G acts on an algebraic space X the fppf quotient sheaf X/G is defined as the sheafification of the presheaf $T \mapsto X(T)/G(T)$ on $(Sch/F)_{fppf}$ in the sense of [Stacks, 00WG]. By the universal property of sheafification every G-invariant morphism $X \to Y$ of algebraic spaces factors through the quotient map $X \to X/G$. Moreover, if $X \to Y$ is a G-torsor, then the induced morphism $X/G \to Y$ is an isomorphism by [Stacks, 044M].

2.2. **Free actions, twists and versal G-schemes.** Recall that an action of an algebraic group G on an algebraic space X is called *free*, if for every scheme T over F the action of the group $G(T)$ on the set $X(T)$ is free.

Lemma 2.2. *Let G be an algebraic group over F acting freely on an algebraic space X over F. Let $Y := X/G$ denote the fppf quotient sheaf. Then the sheaf Y is an algebraic space and the canonical morphism $\pi\colon X \to Y$ is an fppf G-torsor. Moreover for any algebraically closed field L/F we have $Y(L) = X(L)/G(L)$.*

Proof. By [Stacks, 06PH] Y is an algebraic space and $\pi\colon X \to Y$ is fppf. It remains to show that the map $G \times X \to X \times_Y X$, $(g, x) \mapsto (gx, x)$ is an isomorphism. Since the action of G on X is free, for schemes T over F, the maps

$$G(T) \times X(T) \to X(T) \times_{X(T)/G(T)} X(T), \quad (g, x) \mapsto (gx, x)$$

are bijective. Thus it suffices to show that the maps $X(T)/G(T) \to Y(T) = (X/G)(T)$ are injective. Since G acts freely on X, the morphism $G \times X \to X \times X, \quad (g, x) \mapsto (gx, x)$ is a monomorphism (by [Stacks, 07S2]), hence an equivalence relation. Therefore [Sk09, Proposition 1.13] (cf. [Ja03, §5.5]) implies that the presheaf quotient $(X/G)_{pre}$ is separated. Thus [Stacks, 00WB] implies the claim. For the last claim see [DG70, III, §1, 1.15]. □

Remark 2.3. Under some assumptions on X and G the quotient X/G in Lemma 2.2 will be a scheme. This is for example the case when G is finite and X is quasi-projective [Stacks, 07S7]. However for us it will be of no relevance, if X/G is a scheme, or only an algebraic space.

Lemma 2.4. *Suppose G acts freely on an algebraic space X and $f\colon X \to Z$ is a G-invariant morphism. Let $z\colon \operatorname{Spec}(R) \to Z$ be another morphism of algebraic spaces over F, where R is a commutative F-algebra. Then G_R acts freely on the algebraic space $X_z := X \times_Z \operatorname{Spec}(R)$ over R and induces an isomorphism $X_z/G_R \simeq (X/G)_z$ of algebraic spaces over R.*

Proof. It is clear that G_R acts on X_z and that this action is free. Moreover we have a G_R-equivariant isomorphism $X_z \simeq X \times_{X/G} (X/G)_z$. Since the projection $X \times_{X/G} (X/G)_z \to (X/G)_z$ is a G_R-torsor the claim follows. □

An important example of quotients by free actions is given by the twist construction as follows:

Definition 2.5. Let X be a G-scheme and E a G-torsor (both defined over F). Endow $E \times X$ with the (free) diagonal G-action. Then the quotient ${}^E X := (E \times X)/G$ is called the *twist* of X by E.

Note that for a G-torsor E any G-equivariant morphism of schemes $X \to Y$ gives rise to a morphism ${}^E X \to {}^E Y$ of algebraic spaces. Moreover if $E \simeq G$ is the trivial torsor the action morphism $G \times X \to X$ induces an isomorphism ${}^E X \simeq X$.

For smooth algebraic groups acting on quasi-projective varieties, the notions of versal and p-versal were introduced in [DR15]. We generalize these definitions below. For a prime p, a field is called *p-special* if all of its finite extensions have degree a power of p. Every p-special field is infinite. An algebraic group G over F is said to be *special*, if it only has the trivial torsor over fields containing F. It is said to be *p-special*, if it only has the trivial torsor over p-special fields containing F.

Definition 2.6. Let G be an algebraic group and V a G-scheme over F. Let p be a prime. Then V is called versal (resp. p-versal), if for every field extension L/F with L infinite (resp. p-special), every G-torsor E over L and every non-empty G-stable open subscheme U of V, the twist ${}^{E}U$ contains an L-rational point.

Clearly any versal G-scheme is p-versal for any prime p. The following examples of versal and p-versal G-varieties are listed in [DR15]:

Example 2.7.
(1) For V a linear G-representation both V and $\mathbb{P}(V)$ are versal.
(2) If V is a connected reductive group and G acts on V by group automorphisms, then V is versal.
(3) Let $V = A/B$ be a homogeneous space with A reductive and B a subgroup. Let G be a subgroup of A acting on V by left-translations. Suppose that V is geometrically irreducible and the image of $H^1_{fppf}(L,G) \to H^1_{fppf}(L,A)$ is contained in the image of $H^1_{fppf}(L,B) \to H^1_{fppf}(L,A)$ for every infinite field L/F. Then V is versal.
(4) Let X be a geometrically irreducible quasi-projective G-variety with a smooth F-point x having finite G-orbit $G \cdot x$ whose associated 0-cycle $[G \cdot x]$ is of degree prime to p. Then X is p-versal.

In [DR15] G is assumed to be smooth, but we will show that examples (1), (2), and (3) remain valid for non-smooth groups G as well. Later we will need that every linear representation V and its associated projective space $\mathbb{P}(V)$ are versal for general G. For this purpose we prove the following Lemma:

Lemma 2.8. *Let $G \to G'$ and $H \to G'$ be homomorphisms of algebraic groups over F and X be a G'-scheme over F. We view X as a G-scheme and H-scheme through the given homomorphisms to G'. Let p be a prime.*
(1) If E is a G-torsor over some field extension L/F and $E' = (E \times G')/G$ is the induced G'-torsor over L, then ${}^{E}X \simeq {}^{E'}X$ as algebraic spaces over L.
(2) Suppose that for every infinite (resp. p-special) field extension L/F the image of the map $H^1_{fppf}(L,G) \to H^1_{fppf}(L,G')$ is contained in the image of $H^1_{fppf}(L,H) \to H^1_{fppf}(L,G')$ and for every H-torsor T over L the twist ${}^{T}X$ has a dense set of L-rational points. Then X considered as a G-scheme is versal (resp. p-versal).
(3) Suppose that G' is special (resp. p-special) and X is a unirational variety, then X is versal (resp. p-versal) as a G-scheme.

Proof.
(1) This is well known and follows e.g. from [Gi71, Proposition 1.3.5].
(2) Let E be a G-torsor over some infinite (resp. some p-special) field extension L/F. Let E' denote the induced G'-torsor. By assumption there exists an H-torsor T which induces the same G'-torsor as E. Hence ${}^{E}X \simeq {}^{E'}X \simeq {}^{T}X$, which by assumption has a dense set of L-rational points. Hence ${}^{E}U$ contains an L-rational point for every non-empty G-stable open subscheme U of X.
(3) Take for H the trivial group. Since X is unirational X_L has a dense set of L-rational points for every infinite field L/F, so (2) applies.

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Remark 2.9. It follows from [DR15, Theorem 1.1 and Lemma 8.2] that for H smooth and X an irreducible quasi-projective variety the condition of Lemma 2.8(2) on the twists ${}^{T}X$ having dense sets of rational points holds if and only if X is versal (resp. p-versal) as an H-scheme. We do not know if this is true in our more general situation as well.

Corollary 2.10. *No smoothness assumption is needed in Example 2.7 (1), (2) and (3).*

Proof. In (1) we take $G' = \mathrm{GL}(V)$, which is special.

In (2) we take $G = G' = H$ and note that for every G-torsor E over a field extension L/F the twist ${}^E V$ is a connected reductive group over L, hence by Chevalley's Theorem (see [Bo91, Theorem 18.2(ii)]) contains a dense set of rational points.

In (3) if E' is an A-torsor induced from a B-torsor over L, then ${}^{E'}V = {}^{E'}(A/B)$ has a rational point, by [DG70, p. 373, Prop. III.4.4.6b]. Hence it is dominated by the reductive group $\mathrm{Aut}(E')$ over L. Therefore ${}^{E'}V$ contains a dense set of rational points. This is simply a rephrasing of the argument in [DR15]. □

Remark 2.11. We do not know if Example 2.7 (4) holds for non-smooth groups, but we will not need this kind of example in this paper.

In Example 2.7 (2) and (3), if the algebraic group V resp. A is not necessarily reductive, but only connected and smooth, then the corresponding variety V will still be versal (resp. p-versal) if $\mathrm{char}(F) = 0$ (resp. $\mathrm{char}(F) \neq p$). This comes from the fact that Chevalley's unirationality Theorem holds for smooth connected algebraic groups whenever the base field is perfect. This was also observed in [DR15].

2.3. **SGP's and generic freeness.** A *stabilizer in general position* (SGP) for an action of an algebraic group G on a geometrically irreducible variety X (defined over F) is a subgroup H of G such that for some dense open subscheme U of X every $u \in U(\bar{F})$ has (scheme-theoretic) stabilizer conjugate to $H_{\bar{F}}$. The G-action on X is said to be *generically free*, if the trivial subgroup of G is an SGP for that action. For a subgroup H of G the condition that H is an SGP for the G-action on a geometrically irreducible variety X can always be checked over an algebraic closure. Moreover we have the following Lemma

Lemma 2.12. *Suppose G acts on a geometrically irreducible F-variety X, for which F-rational points are dense. Assume the $G_{\bar{F}}$-action on $X_{\bar{F}}$ has an SGP $\tilde{H}$. Then the G-action on X has an SGP, say H, as well, and $H_{\bar{F}}$ is conjugate to $\tilde{H}$.*

Proof. Let $\tilde{U} \subset X_{\bar{F}}$ be a dense open subscheme such that every rational point of $\tilde{U}$ has stabilizer conjugate to $\tilde{H}$. First note that $\tilde{U}$ has only finitely many Galois conjugates. Indeed to prove this claim, we may assume without loss of generality that X is affine, so that $\tilde{U}$ is the complement of the zero set of finitely many polynomials. Since every of these polynomials has only finitely many non-zero coefficients and every coefficient has only finitely many Galois-conjugates, the claim follows. Now the intersection of the Galois conjugates of $\tilde{U}$ is dense open in $\tilde{U}$ and, by [Sp98, Proposition 11.2.8], descends to a dense open subscheme U of X. Taking for H the stabilizer of an F-rational point in U we get the desired result. □

By a Theorem of Richardson [Ric72] an SGP exists for any reductive algebraic group action on an irreducible smooth affine variety over an algebraically closed field in characteristic 0. Hence by Lemma 2.12 it also exists, in characteristic 0, for reductive group actions on geometrically irreducible smooth affine varieties with a dense set of rational points. This includes the case of linear reductive group actions. Moreover for linear actions of connected semisimple algebraic groups in characteristic 0 a lot of information about their SGP's is available in the literature, see e.g. [PV89, Section 7].

We will repeatedly make use of the following Lemma, which is due to Popov.

Lemma 2.13 ([Po89, Prop. 8])**.** *Let G act on geometrically irreducible varieties X and Y. Let $H_2 \subset H_1 \subset G$ be subgroups such that H_1 is an SGP for the G-action on X and H_2 is an SGP for the H_1-action on Y. Then H_2 is an SGP for the G-action on $X \times Y$.*

Proof. The reference [Po89] and the proof given therein assume that the base field is algebraically closed of characteristic 0. See [Lö15, Lemma 2.3] for a version of this proof which works for arbitrary base fields. This also corrects the proof of [Mac13, Prop.1.2], which is mistaken. □

Let G act on a scheme X over F. For any commutative F-algebra R and $x \in X(R)$ the functor $R' \mapsto \{g \in G(R') \mid gx_{R'} = x_{R'}\}$ from the category of commutative R-algebras to the category of groups is representable by a subgroup scheme of G_R, see [SGA3, I, 2.3.3] or [DG70, III, §2.2], which we denote by $(G_R)_x$ and call the *stabilizer R-group scheme* of x. We have $(G_R)_{gx} = g(G_R)_x g^{-1}$ for any $g \in G(R)$, $x \in X(R)$. When X is separated we will denote by X^G the closed subscheme of X which is fixed by G.

Lemma 2.14. *Suppose H is an SGP for the G-action on a geometrically irreducible variety X and U is a dense open subscheme as in the definition of an SGP. Then*

(1) The subscheme U contains a dense open subscheme U' which is G-stable.
(2) Suppose U is G-stable. Then $S := U \cap X^H$ is a non-empty $\mathrm{Norm}_G(X^H)$-stable open subscheme of X^H such that for every commutative F-algebra R and every $s \in S(R) \subset X(R)$ we have $(G_R)_s = H_R$.

Proof. (1) This is well known. See e.g. [Lö15, §2, before Lemma 2.3].

(2) Clearly S is non-empty and $\mathrm{Norm}_G(X^H)$-stable. For any commutative F-algebra R and $s \in S(R)$ we want to show that $(G_R)_s = H_R$. Note that the inclusion $(G_R)_s \supset H_R$ is obvious. If $R = \bar{F}$, by the definition of an SGP, $(G_{\bar{F}})_s$ is conjugate to $H_{\bar{F}}$ and contains $H_{\bar{F}}$. The endomorphism $(G_{\bar{F}})_s \simeq H_{\bar{F}} \hookrightarrow (G_{\bar{F}})_s$ is a monomorphism of schemes. It is in fact an automorphism by [EGAIV, Proposition 17.9.6] since $(G_{\bar{F}})_s$ is of finite type over $\bar{F}$. This implies $(G_{\bar{F}})_s = H_{\bar{F}}$ as claimed.

To prove this equality for arbitrary commutative F-algebras R we consider the inertia scheme $X^T = (G \times X) \times_{X \times X} X$ for the G-action on X. Here the morphism $X \to X \times X$ is the diagonal and $G \times X \to X \times X$ is given by $(g, x) \mapsto (gx, x)$. By [DG70, III, §2.2] the fiber of the morphism $X^T \to X$ over any $x \in X(R)$ is isomorphic to the stabilizer R-group scheme $(G_R)_x$.

Now let $S^T = X^T \times_X S = (G \times X) \times_{X \times X} S$. The H-action on $G \times X \times S$ through $h \cdot (g, x, s) = (gh^{-1}, hx, s)$ induces a free H-action on S^T. The morphism $S^T \to S$ is H-invariant. By Lemma 2.4 the fiber $(S^T/H)_s$ of the induced morphism $f\colon S^T/H \to S$ over $s \in S(R)$ is isomorphic to $(S^T)_s/H_R \simeq (G_R)_s/H_R$. In particular for every $s \in S(\bar{F})$ the morphism $(S^T/H)_s \to \mathrm{Spec}(\bar{F})$ is an isomorphism.

Note that S^T/H is in fact a scheme of finite type over F, since $S^T \simeq (G \times S) \times_{X \times S} S$, hence $S^T/H \simeq (G/H \times S) \times_{X \times S} S$ and G/H is a finite type scheme over F by [DG70, III, §3, 5.4]. Thus by [DG70, III §2.2, Lemme] we conclude that f is a monomorphism. Since f has a section, it follows that f is an isomorphism. Hence for $s \in S(R)$ the morphism $(G_R)_s/H_R \to \mathrm{Spec}(R)$ is an isomorphism too. So $(G_R)_s \to \mathrm{Spec}(R)$ is an H_R-torsor. The existence of the identity section shows that $(G_R)_s = H_R$ as claimed. □

2.4. **Essential dimension.** For the definition of essential dimension and essential p-dimension of an algebraic group G, see [Me09]; it is the essential dimension (resp. p-dimension) of the fppf cohomology functor $L \mapsto H^1_{fppf}(L, G)$. The set $H^1_{fppf}(L, G)$ classifies fppf G-torsors over L (i.e. over $\mathrm{Spec}(L)$).

When G is smooth every fppf G-torsor is locally trivial in the étale topology and these sets can be identified with the Galois cohomology set:

$$H^1_{fppf}(L, G) = H^1_{\text{ét}}(L, G) = H^1(\mathrm{Gal}(L_{\mathrm{sep}}/L), G(L_{\mathrm{sep}})).$$

So when G is smooth, we may simply write $H^1(L, G)$ for any of these sets.

Lemma 2.15. *Let $H \to G$ be a homomorphism of algebraic groups (e.g. the inclusion of a subgroup) and p a prime. If the map $H^1_{fppf}(L, H) \to H^1_{fppf}(L, G)$ is surjective for every infinite (resp. p-special) field L/F, then $\mathrm{ed}(G) \leq \mathrm{ed}(H)$ (resp. $\mathrm{ed}_p(G) \leq \mathrm{ed}_p(H)$).*

Proof. For (absolute) essential dimension this is shown as in the proof of [Me09, Proposition 1.3]. Note that elements of $H^1_{fppf}(L,G)$ with L finite do not contribute to the essential dimension of G, since all such elements have essential dimension zero over F. For essential p-dimension, see [LMMR13, Proposition 2.4]. $\square$

Lemma 2.16. [Me09, Corollary 4.2] *If an algebraic group G acts on an F-vector space V linearly and generically freely then* $\mathrm{ed}(G) \le \dim(V) - \dim(G)$.

3. Reduction of structure group from slices of versal G-schemes

Given a G-scheme V and a (locally closed) subscheme $S_0 \subset V$ stable under the action of a subgroup N of G we have a free action of N on $G \times S_0$ via the formula $n \cdot (g, s_0) = (gn^{-1}, ns_0)$. The action morphism

$$G \times S_0 \to V, \quad (g, s_0) \mapsto gs_0$$

is N-invariant. Hence it gives rise to a morphism $(G \times S_0)/N \to V$ of algebraic spaces, which is G-equivariant. Here we equip $(G \times S_0)/N$ with the G-action induced from $g \cdot (g', s_0) = (gg', s_0)$.

Definition 3.1. Let N be a subgroup of an algebraic group G. For us a (G,N)-*slice* of a G-scheme V will be a non-empty N-stable subscheme $S_0 \subset V$ such that the induced G-equivariant morphism $(G \times S_0)/N \to V$ of algebraic spaces is an open immersion.

Example 3.2. Suppose a G-scheme V contains an F-rational point x with an open orbit in V. Then $S_0 = \{x\}$ is a (G,N)-slice of V, where N is the stabilizer of x in G. Indeed, in this case the morphism $(G \times S_0)/N \to V$ is given by the inclusion of the open orbit in V.

Lemma 3.3. *Let N be a subgroup of G. The following conditions on a non-empty N-stable subscheme S_0 of a G-scheme V are equivalent:*

(1) S_0 is a (G,N)-slice.
(2) The morphism $m\colon G \times S_0 \to V$ is open and for $V_0 = m(G \times S_0) = G \cdot S_0$ the induced morphism $G \times S_0 \to V_0$ is an N-torsor.
(3) The morphism $m\colon G \times S_0 \to V$ is flat and locally of finite presentation, and for every commutative F-algebra R, every $g \in G(R)$ with $gS_0(R) \cap S_0(R) \neq \emptyset$ is contained in $N(R)$.

Proof. If (1) holds, then both $G \times S_0 \to (G \times S_0)/N$ and $(G \times S_0)/N \to V$ are flat and locally of finite presentation, hence so is $m\colon G \times S_0 \to V$. Moreover for $g \in G(R)$, $s_0, s_0' \in S_0(R)$ such that $gs_0 = s_0'$ the elements (g, s_0) and (e, s_0') of $(G \times S_0)(R)$ have the same image in $V(R)$, hence also in $((G \times S_0)/N)(R)$. Therefore there exists $n \in N(R)$ with $(e, s_0') = (gn^{-1}, ns_0)$. Hence $g = n \in N(R)$, and (3) follows.

If (3) holds, then $m\colon G \times S_0 \to V$ is open (by [Stacks, 01UA]). The induced morphism $G \times S_0 \to V_0$ is fppf, so it suffices to show that for every commutative F-algebra R and $g, g' \in G(R)$, $s_0, s_0' \in S_0(R)$ with $gs_0 = g's_0'$ there exists a unique $n \in N(R)$ such that $(g', s_0') = (gn^{-1}, ns_0)$. Uniqueness is clear and existence follows from $(g')^{-1}gS_0(R) \cap S_0(R) \neq \emptyset$. So (2) follows.

If (2) holds then $(G \times S_0)/N \simeq V_0$ and the morphism $(G \times S_0)/N \to V$ is given by the inclusion of V_0 in V. $\square$

Remark 3.4. Let S_0 be a (G,N)-slice of a G-scheme V over F.

(1) If V and N are smooth then G and S_0 are smooth as well, since the composite

$$G \times S_0 \to (G \times S_0)/N \to V \to \operatorname{Spec} F$$

of smooth morphisms is smooth.

(2) Let $S := \overline{S_0}$ be the scheme-theoretic image of S_0 in V. If V is of finite type over F and $\mathrm{Norm}_G(S)$ is smooth then $N = \mathrm{Norm}_G(S)$. To show this, notice N normalizes S_0, hence also S (since the image of $N \times S_0$ under the action morphism $N \times V \to V$ lies

in $\overline{S_0}$ and $\overline{N \times S_0} = N \times \overline{S_0}$ in $N \times V$), from which we get $N \subset \mathrm{Norm}_G(S)$. Now let $g \in \mathrm{Norm}_G(S)(\bar{F})$. Since $gS(\bar{F}) = S(\bar{F})$ and $S_0(\bar{F})$ is dense and open in $S(\bar{F})$ we have $gS_0(\bar{F}) \cap S_0(\bar{F}) \neq \emptyset$. Hence $g \in N(\bar{F})$ by Lemma 3.3. Since $\mathrm{Norm}_G(S)$ is smooth we get the desired result by [KMRT98, 22.3].

Corollary 3.5. *Let N be a subgroup of an algebraic group G and V be a reduced G-scheme of finite type over F. Let S be a non-empty N-stable subscheme of V. Suppose that the morphism $m\colon G \times S \to V,\ (g,s) \to gs$ is dominant over $\bar{F}$, and that for every commutative F-algebra R, every $g \in G(R)$ with $gS(R) \cap S(R) \neq \emptyset$ lies in $N(R)$. Then the flat locus of m is of the form $G \times S_0$ for some N-stable non-empty open subscheme S_0 of S and S_0 is a (G,N)-slice of V.*

Proof. By [Stacks, 052B and 0399] the flat locus of $m\colon G \times S \to V$ is non-empty and open in $G \times S$. Note that m is equivariant for the $G \times N$-action on $G \times S$ and V through $(g,n)\cdot(g',s) = (gg'n^{-1}, ns)$ and $(g,n) \cdot v = gv$, respectively. Since the formation of the flat locus commutes with flat base change [Stacks, 047C], the flat locus of m is of the form $G \times S_0$ for some N-stable non-empty open subscheme S_0 of S. Moreover $G \times S_0 \to V$ is flat and locally of finite presentation. Applying Lemma 3.3 we get that S_0 is a (G,N)-slice of V. □

The following observation will be useful for non-faithful G-actions:

Lemma 3.6. *Let $\pi\colon G \to G'$ be a surjective homomorphism of algebraic groups and N' be a subgroup of G'. Set $N = \pi^{-1}(N')$. Let V be a G'-scheme and $S_0 \subset V$ a N'-stable subscheme. Then S_0 is a (G',N')-slice of V if and only if it is a (G,N)-slice of V.*

Proof. The composition $G \times S_0 \to G' \times S_0 \to (G' \times S_0)/N'$ is N-invariant, hence it induces a morphism $(G \times S_0)/N \to (G' \times S_0)/N'$. We claim it is an isomorphism. Since $\ker(\pi)$ acts trivially on S_0 the morphism $(G \times S_0) \to (G \times S_0)/N$ factors through $G' \times S_0$. We get an N'-invariant morphism $G' \times S_0 \to (G \times S_0)/N$, which is easily seen to induce an inverse of $(G \times S_0)/N \to (G' \times S_0)/N'$.

Now we conclude the proof by observing that the morphism $(G \times S_0)/N \to V$ is the composition of the above isomorphism with the canonical morphism $(G' \times S_0)/N' \to V$. □

The definition of a (G,N)-slice was designed in order to make the following result work. Its usefulness will be seen in the following sections.

Theorem 3.7. *Let S_0 be a (G,N)-slice of a G-scheme V and let p be a prime. Then if V is versal (resp. p-versal), then the map $H^1_{fppf}(L,N) \to H^1_{fppf}(L,G)$ is surjective for every infinite (resp. p-special) field L/F. In particular,* $\mathrm{ed}(G) \leq \mathrm{ed}(N)$ *(resp.* $\mathrm{ed}_p(G) \leq \mathrm{ed}_p(N)$*).*

Proof. Let L be an infinite (resp. p-special) field containing F and E a G-torsor over L. By [DG70, p. 373, Prop. III.4.4.6b] in order to show that the class of E in $H^1_{fppf}(L,G)$ comes from $H^1_{fppf}(L,N)$ it suffices to show that the twist ${}^E(G/N)$ has an L-rational point. By definition of a slice the fppf quotient $(G \times S_0)/N$ is representable by a G-stable non-empty open subscheme of V, which we call U. The morphism $G \times S_0 \xrightarrow{\pi_1} G \to G/N$ induces a G-equivariant morphism $U \to G/N$. We get an induced morphism ${}^EU \to {}^E(G/N)$. Since V is versal (resp. p-versal), ${}^EU$ contains an L-rational point. Its image under ${}^EU \to {}^E(G/N)$ is an L-rational point of ${}^E(G/N)$. □

Remark 3.8. Saying that the map $H^1_{fppf}(L,N) \to H^1_{fppf}(L,G)$ is surjective for every p-special field L/F is equivalent to saying that for every field K/F and G-torsor E over K there exists a finite extension K'/K of degree prime to p such that the class of $E_{K'}$ in $H^1(K',G)$ lies in the image of the map $H^1_{fppf}(K',N) \to H^1_{fppf}(K',G)$. This follows from the continuity of the functors $H^1_{fppf}(-,N)$, $H^1_{fppf}(-,G)$ and the existence of a p-special closure of K, see [LMMR13, §2].

If G is connected then for any finite field L/F the map $H^1_{fppf}(L,N) \to H^1_{fppf}(L,G)$ is also surjective, since in that case $H^1(L,G)$ is trivial by Lang's Theorem.

Remark 3.9. If $\operatorname{char} F = 0$, and there is a linear subspace S_0 of a linear representation V, which is a (G, N)-section, in the terminology of [Re00], then one can also show the existence of a (G, N)-slice of V, and so by Theorem 3.7 we have $\operatorname{ed}(N) \leq \operatorname{ed}(G)$. But in this case, this inequality on essential dimensions already follows from [Re00, Lemma 4.1] (notice that for characteristic zero, Reichstein's original definition of essential dimension is equivalent to the one above [BF03, §6]). To see this, if W is a generically free G-representation, then $S_0 \times W$ is a (G, N)-section of $V \times W$, so the Lemma applies. Therefore, Theorem 3.7 is a generalization of this result to fields of arbitrary characteristic, non-smooth groups, and non-linear actions.

4. Producing slices from SGP's

In this section we will describe a technique to produce a slice of a geometrically irreducible G-variety V admitting an SGP. Moreover we apply Theorem 3.7 to versal G-varieties V to obtain reduction of structure for G-torsors. This will in particular apply to linear representations of G. The following Theorem is a variant of [Po94, 1.7.8], where relative sections are replaced by slices in our sense.

Theorem 4.1. *Let V be a geometrically irreducible G-variety. Suppose that H is an SGP for the G-action on V (in particular an SGP exists). Let $N := N_G(H)$ be its normalizer. Then there exists an open N-stable subscheme S_0 of V^H, which is a (G, N)-slice of V.*

Proof. By Lemma 2.14 there is a G-stable non-empty open subscheme U of V such that $S := U \cap V^H$ is a N-stable non-empty open subscheme in V^H and for every commutative F-algebra R and $s \in S(R)$ the stabilizer R-group scheme $(G_R)_s$ of s is equal to H_R. In particular for any $g \in G(R)$ and $s \in S(R)$ with $gS(R) \cap S(R) \neq \emptyset$, comparision of stabilizer R-group schemes yields $gH_R g^{-1} = H_R$, hence $g \in N(R)$. Therefore the claim follows from Corollary 3.5. □

Corollary 4.2. *Let V be a geometrically irreducible G-variety which admits an SGP H. Let $N := N_G(H)$ and p be a prime. If V is versal (resp. p-versal), then the map $H^1_{fppf}(L, N) \to H^1_{fppf}(L, G)$ is surjective for every infinite (resp. p-special) field L/F. In particular,* $\operatorname{ed}(G) \leq \operatorname{ed}(N)$ *(resp.* $\operatorname{ed}_p(G) \leq \operatorname{ed}_p(N)$*).*

Proof. This follows immediately from Theorems 4.1 and 3.7. □

Example 4.3. Let G be a connected reductive algebraic group over F and let $V = G$, which we view as a G-scheme through the conjugation action. By Example 2.7, V is versal. Let T be a maximal torus of G. We claim that T is an SGP for the G-action on V. So Corollary 4.2 shows the well known surjection $H^1(L, N_G(T)) \to H^1(L, G)$ for any field extension L/F.

In order to prove the claim we may assume without loss of generality that $F = \bar{F}$. First note that there is a dense open subscheme T_0 of T, such that the centralizer of every $t \in T_0(F)$ is equal to T. In fact, this can be seen by embedding G in some GL_n, where T is diagonalized: let $\chi_1, \ldots, \chi_n$ denote the standard characters of the diagonal torus of GL_n and take for T_0 the intersection of the open subschemes $D_{(\chi_i/\chi_j)|_T}$, where (i, j) runs over all pairs with $(\chi_i)|_T \neq (\chi_j)|_T$. The morphism $G \times T_0 \to G,\ (g, t) \mapsto gtg^{-1}$ is dominant, hence contains an open subscheme U in its image. The stabilizer of every $u \in U(F)$ is conjugate to T. This shows the claim.

Remark 4.4. The result of Example 4.3 can also be obtained by considering the adjoint representation of G. We will show this in more generality in section 5.

Corollary 4.5. *Let V be a linear representation of G which admits an SGP H. Write $N := N_G(H)$. Then the map $H^1_{fppf}(L, N) \to H^1_{fppf}(L, G)$ is surjective for every infinite field L/F. Moreover* $\operatorname{ed}(G) \leq \operatorname{ed}(N)$ *and* $\operatorname{ed}_p(G) \leq \operatorname{ed}_p(N)$ *for every prime p.*

Proof. Since V is versal (hence also p-versal for every prime p), this is only a special case of Corollary 4.2. □

5. Slices for smooth algebraic groups G

In section 4 we saw how to construct a slice of a geometrically irreducible G-variety, when an SGP exists. This may not always be the case, especially for actions of non-reductive groups. So we give another construction similar to the one in Corollary 3.5, but with the flat locus replaced by the smooth locus. This will allows us to check the condition $gS(R) \cap S(R) \neq \emptyset \Rightarrow g \in N(R)$ only for algebraically closed field extensions R of F instead of arbitrary commutative F-algebras. Our goal consists of proving the following result:

Proposition 5.1. *Let N be a subgroup of G. Let V be a G-scheme over F and S be a non-empty N-stable subscheme of V. Suppose that for every algebraically closed field extension L/F and $g \in G(L)$ with $gS(L) \cap S(L) \neq \emptyset$ we have $g \in N(L)$.*

(1) If $m\colon G \times S \to V, \quad (g, s) \mapsto gs$ is smooth, then S is a (G, N)-slice of V.
(2) Suppose G is smooth, S and V are geometrically integral and of finite type over F, and $m_{\bar{F}}$ is dominant and separable (in the sense of [Sp98, §8]*). Then the smooth locus of m is of the form $G \times S_0$ for a (non-empty) N-stable open subscheme S_0 of S and S_0 is a (G, N)-slice of V.*

In order to prove Proposition 5.1 we will need the following Lemma:

Lemma 5.2. *Let $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ be morphism of algebraic spaces over F such that*

(1) Z is a scheme,
(2) $\psi \circ \phi\colon X \to Z$ is smooth,
(3) $\phi\colon X \to Y$ is fppf,
(4) $Y(L) \to Z(L)$ is injective for every algebraically closed field L containing F.

Then $\psi\colon Y \to Z$ is an open embedding.

Proof. First ψ is universally injective by the proof of [Stacks, 040X]. Since every universally injective étale morphism of algebraic spaces is an open immersion (by [Stacks, 05W5]), it remains to show ψ is étale, i.e., for every étale morphism $\alpha\colon U \to Y$ with U a scheme, the composition $\psi \circ \alpha\colon U \to Z$ is étale. By [Stacks, 0AHE] ψ is smooth, and in particular locally of finite type [Stacks, 06MH]. Hence $\psi \circ \alpha$ is smooth as well, so by [Stacks, 0397] it suffices to show that $\psi \circ \alpha$ is locally quasi-finite. Now both ψ (universally injective) and α (étale) are locally quasi-finite by [Stacks, 06RW], hence so is their composition. □

Proof of Proposition 5.1. (1) Let $\phi\colon G \times S \to (G \times S)/N$ and $\psi\colon (G \times S)/N \to V$ be the canonical morphisms, so that $\psi \circ \phi = m\colon G \times S \to V$. For L/F an algebraically closed field $((G \times S)/N)(L) = (G(L) \times S(L))/N(L)$ and the injectivity of ψ on L-points translates to the condition $gS(L) \cap S(L) \neq \emptyset \Rightarrow g \in N(L)$, which we have assumed. Therefore Lemma 5.2 implies that ψ is an open immersion, hence S is a (G, N)-slice of V.

(2) First of all by Corollary 3.5, the flat locus of m is of the form $G \times S'$ for a non-empty (hence dense) N-stable open subscheme S' of S. It contains the smooth locus of m. Thus after replacing S by S' we may assume that m is flat. By [Sp98, Theorem 17.3] and [EGAIV, (4) 17.11.1] the smooth locus of $m_{\bar{F}}$ is non-empty open. Hence by [Stacks, 047C] the smooth locus of m is non-empty open as well and (by $G \times N$-equivariance of m as in the proof of Corollary 3.5) of the form $G \times S_0$ for a (non-empty) N-stable subscheme S_0 of S. Now the claim follows from part (1).

□

Example 5.3. Let G be a smooth (not necessarily reductive) algebraic group over F, acting on $\mathfrak{g} = \mathrm{Lie}(G)$ through the adjoint representation. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$. Let S denote the open subscheme of $\mathfrak{c}$ formed by the regular elements. By [SGA3, XIII, Corollaire 5.4] the morphism $G \times S \to \mathfrak{g}$, $(g, s) \mapsto \mathrm{Ad}(g)(s)$ is smooth. Let $N := \mathrm{Norm}_G(\mathfrak{c})$. For L/F algebraically closed, $g \in G(L)$ and $x, x' \in S(L)$ with $\mathrm{Ad}(g)(x) = x'$, the element x' is contained in both $\mathfrak{c}_L$ and $\mathrm{Ad}(g)\mathfrak{c}_L$. Since every regular element is only contained in one Cartan-subalgebra [SGA3, XIII, Proposition 4.6], we get $g \in N(L)$. Hence by Proposition 5.1, S is a (G, N)-slice of $\mathfrak{g}$. This is also directly proven in [SGA3, XIII, Theorem 6.1].

Example 5.4. Let G be a connected reductive algebraic group over a field F. Let ζ be a root of unity in F and let m denote its order. Suppose θ is an automorphism of G of order m. Then $G(0) := (G^\theta)^0$ is connected reductive and $G(0)$ preserves $\mathfrak{g}(1) := \{x \in \mathfrak{g} \mid d\theta(x) = \zeta x\} \subset \mathfrak{g}$ under the adjoint action, where $\mathfrak{g} = \mathrm{Lie}(G)$. This is the setting of Vinberg's θ-groups, introduced by E.B. Vinberg in the 70's [Vi76].

From now on (in this example) assume that F is perfect and either $\mathrm{char}(F) = 0$ or $p = \mathrm{char}(F) > 2$ and p is good for G (this is in particular the case for any $p \geq 7$). Let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{g}(1)$, i.e., a maximal commutative subspace of $\mathfrak{g}(1)$ consisting of semisimple elements. We assume that $\mathfrak{c}$ remains maximal over $\bar{F}$. Let

$$\mathfrak{z}_{\mathfrak{g}(1)}(\mathfrak{c}) = \{z \in \mathfrak{g}(1) \mid [z, c] = 0 \ \forall\ c \in \mathfrak{c}\}.$$

By [Le09, Lemma 1.12] $\mathfrak{z}_{\mathfrak{g}(1)}(\mathfrak{c}) = \mathfrak{c} \oplus \mathfrak{u}$ for a subspace $\mathfrak{u}$ consisting of nilpotent elements. In particular $\mathrm{Norm}_{G(0)}(\mathfrak{c}) = \mathrm{Norm}_{G(0)}(\mathfrak{z}_{\mathfrak{g}(1)}(\mathfrak{c}))$, which we will denote by N. We claim that there is a $(G(0), N)$-slice S_0 of $\mathfrak{g}(1)$, which is an open subscheme of $\mathfrak{z}_{\mathfrak{g}(1)}(\mathfrak{c})$.

Let $\mathfrak{c}_{reg}$ denote the open subscheme of regular elements in $\mathfrak{c}$ and by $R(\mathfrak{c}) = \mathfrak{c}_{reg} \oplus \mathfrak{u}$ its preimage in $\mathfrak{z}_{\mathfrak{g}(1)}(\mathfrak{c})$. It is N-stable as well. Let $m\colon G(0) \times R(\mathfrak{c}) \to \mathfrak{g}(1),\ (g, x) \mapsto \mathrm{Ad}(g)(x)$. By [Le09, Corollary 2.4] the morphism $m_{\bar{F}}$ is dominant and separable. Moreover for L/F algebraically closed, $g \in G(0)(L)$ and $x, x' \in R(\mathfrak{c})(L)$ with $\mathrm{Ad}(g)(x) = x'$, and x_s, x'_s their semisimple parts, we have

$$\mathfrak{z}_{\mathfrak{g}_L}(x_s) = \mathfrak{z}_{\mathfrak{g}_L}(\mathfrak{c}_L) = \mathfrak{z}_{\mathfrak{g}_L}(x'_s) = \mathfrak{z}_{\mathfrak{g}_L}(\mathrm{Ad}(g)x_s) = \mathrm{Ad}(g)\mathfrak{z}_{\mathfrak{g}_L}(x_s).$$

Intersecting both sides with $\mathfrak{g}(1)_L$ yields $g \in N(L)$. Therefore Proposition 5.1 applies and we get a $(G(0), N)$-slice S_0 of $\mathfrak{g}(1)$, which is open in $R(\mathfrak{c})$, as claimed.

Note that this slice is related to, but different from, the Kostant-Weierstrauss slice of $\mathfrak{g}(1)$ constructed by Levy.

Corollary 5.5. *Let $(H, \mathfrak{c}, V)$ be a triple consisting either of a smooth algebraic group H over F and a Cartan subalgebra $\mathfrak{c}$ of $V = \mathrm{Lie}(H)$, or of a θ-group $H = G(0)$ over F as in Example 5.4 (with the same assumptions on the reductive group G and the base field F) and a Cartan subspace $\mathfrak{c}$ of $V = \mathfrak{g}(1)$. Let K be a normal subgroup of H contained in the kernel of the H-action on V. Then the map $H^1_{fppf}(L, \mathrm{Norm}_{H/K}(\mathfrak{c})) \to H^1_{fppf}(L, H/K)$ is surjective for every infinite field L containing F.*

Proof. In view of Theorem 3.7 and Lemma 3.6 this follows straight from the existence of $(H, \mathrm{Norm}_H(\mathfrak{c}))$-slices in V as shown in Examples 5.3 and 5.4. $\square$

6. Common maximal étale subalgebras

In this section we give an application of Corollary 4.5 to prove that central simple algebras of the same degree, whose tensor product is of index ≤ 2, have a common maximal étale subalgebra; see Corollary 6.3.

We start with the group $G = \mathrm{GL}_n \times \mathrm{GL}_n$ acting on two copies of $M_n(F)$ via the formula $(b, c) \cdot a = bac^T$. The following easy result was shown in [Lö15, Proposition 2.4]. For convenience of the reader we include a proof.

Lemma 6.1. *The image H of the homomorphism $\mathbb{G}_m^n \to G,\ t \mapsto (t, t^{-1})$ is an SGP for the G-action on $M_n \oplus M_n$ with Lie algebra $\mathfrak{h} = \{(d, -d) \in M_n \times M_n \mid d \in D\}$, where D are the diagonal matrices.*

Proof. Since G has an open orbit on M_n given by the invertible matrices, the image of the homomorphism $\mathrm{GL}_n \to G,\ b \mapsto (b, (b^{-1})^T)$ is an SGP for the G-action on M_n. Moreover this group acts on M_n through $b \cdot a = bab^{-1}$. Hence for any semisimple regular $a \in V_{\bar{F}}$ the stabilizer is a maximal torus of $(\mathrm{GL}_n)_{\bar{F}}$. Since these are all conjugate the claim follows by Lemma 2.13. □

We now consider the following twisted version of the above representation. Let A_1 and A_2 be central simple algebras of the same degree such that $A_1 \otimes_F A_2$ has index ≤ 2. Set $G = \mathrm{GL}_1(A_1) \times \mathrm{GL}_1(A_2)$ and let V be a left ideal of $A_1 \otimes_F A_2$ of reduced dimension 2. The representation

$$G \to \mathrm{GL}_1(A_1 \otimes_F A_2) \hookrightarrow \mathrm{GL}(V),$$

where the first map takes (a_1, a_2) to $a_1 \otimes a_2$, decomposes into two irreducible copies over any splitting field of A_1 and A_2 as previously. Write $\mathfrak{g} = \mathrm{Lie}(G) = A_1 \times A_2$.

Corollary 6.2. *The G-action on V has an SGP H. Let $\mathfrak{h} = \mathrm{Lie}(H)$. The compositions*

$$f_i \colon \mathfrak{h} \hookrightarrow \mathfrak{g} = A_1 \times A_2 \to A_i$$

are injective and their images are étale subalgebras $E_i \subset A_i$ with $E_1 \simeq E_2$ as F-algebras.

Proof. When F is infinite, the F-rational points are dense in V, hence the existence of H follows from Lemmas 2.12 and 6.1. Otherwise, if F is finite, then A_1 and A_2 are split, and the existence follows directly from Lemma 6.1. The statement on Lie algebras obviously holds over $\bar{F}$, hence it follows over F by descent. □

Corollary 6.3 (Risman's Theorem)**.** *(1) Let A_1 and A_2 be central simple algebras of the same degree such that* $\mathrm{ind}(A_1 \otimes_F A_2) \leq 2$. *Then A_1 and A_2 contain a common maximal étale subalgebra.*
(2) Let D be a division algebra and Q a quaternion algebra over F such that $D \otimes_F Q$ is not a division algebra. Then D contains a separable subfield which splits Q.

Proof. (1) This follows immediately from Corollary 6.2
(2) The assumption implies that there exists a central simple algebra B of degree $\deg(D)$ Brauer-equivalent to $(D \otimes_F Q)^{\mathrm{op}}$. Then $\mathrm{ind}(D \otimes_F B) = \mathrm{ind}(Q) \leq 2$. So in view of part (1) D and B share a common maximal étale subalgebra L. Since D is a division algebra, L is a field. Moreover L splits Q.

□

Remark 6.4. Corollary 6.3 is the well known common slot theorem in case A_1 and A_2 (resp. D) are quaternion algebras. We refer to [Kn93] for an elegant proof of that theorem and references to a variety of other proofs (all of a different flavor than ours). For arbitrary 2-primary degree, the following weaker result has been proven by D. Krashen in [Kr10, Corollary 4.4]: There exists an étale F-algebra of degree $2^n m$ with $m \geq 1$ odd, which splits both A_1 and A_2.

Part (2) of Corollary 6.3 is Risman's Theorem [Ris75, Theorem 1] for finite dimensional division algebras. In fact the two parts are equivalent. The link between part (1) and part (2) was brought to our attention by Z. Reichstein. The proof of the reverse implication goes as follows: We may assume that $\mathrm{ind}(A_2) \leq \mathrm{ind}(A_1)$. Let Q be a quaternion algebra Brauer equivalent to $A_1 \otimes_F A_2$ and let D a division algebra Brauer equivalent to A_1^{op}. Then $D \otimes_F Q$ is Brauer equivalent to A_2 which has index $\leq \mathrm{ind}(A_1) = \deg(D)$. Hence $D \otimes_F Q$ is not a division algebra. Now by part (2) D contains a splitting field L of Q. We let L' be a maximal separable subfield of D containing L and let $E = (L')^r \subset A_1 = M_r(D)$. Since E splits D and Q, it does also split A_2. As $\dim(E) = \deg(A_1) = \deg(A_2)$ it follows that A_2 contains E as well.

We proceed to describe the normalizer N of the SGP H on V and interpret the resulting reduction of structure result (applied to the image of N in $\mathrm{GL}(V)$) from an algebraic point of view. This in fact gives another proof of Risman's Theorem (Corollary 6.3).

Proposition 6.5. *Let H be the SGP for the G-action on V from Corollary 6.2. It is the image of the homomorphism*

$$\mathrm{GL}_1(E) \to \mathrm{GL}_1(E) \times \mathrm{GL}_1(E) \hookrightarrow \mathrm{GL}_1(A_1) \times \mathrm{GL}_1(A_2) = G,$$

for some common maximal étale subalgebra E of A_1, A_2, where the first map is given by $e \mapsto (e, e^{-1})$. We have

$$N_G(H) = N_{\mathrm{GL}_1(A_1)}(\mathrm{GL}_1(E)) \times_{\mathrm{Aut}(E)} N_{\mathrm{GL}_1(A_2)}(\mathrm{GL}_1(E)).$$

Proof. The inclusion $N_G(H) \supset N_{\mathrm{GL}_1(A_1)}(\mathrm{GL}_1(E)) \times_{\mathrm{Aut}(E)} N_{\mathrm{GL}_1(A_2)}(\mathrm{GL}_1(E))$ is obvious, so it remains to show the reverse containment. For $i = 1, 2$ the image of $N_G(H)$ in $\mathrm{GL}_1(A_i)$ under the projection homomorphism normalizes the image of H in $\mathrm{GL}_1(A_i)$, which is $\mathrm{GL}_1(E)$. Therefore $N_G(H) \subset N_{\mathrm{GL}_1(A_1)}(\mathrm{GL}_1(E)) \times N_{\mathrm{GL}_1(A_2)}(\mathrm{GL}_1(E))$. Moreover let R be a commutative R-algebra and $(a_1, a_2) \in N_G(H)(R)$. Then for all $e \in E_R$ we have $a_1 e a_1^{-1} = (a_2 e^{-1} a_2^{-1})^{-1} = a_2 e a_2^{-1}$, hence a_1 and a_2 have the same image in $\mathrm{Aut}(E)(R)$. This shows the claim. □

Corollary 6.6. *Let $n \in \mathbb{N}$ and let A_1 and A_2 be central simple algebras of degree n containing a common maximal étale subalgebra E, such that $\mathrm{ind}(A_1 \otimes_F A_2) \le 2$. Let*

$$G = \mathrm{GL}_1(A_1) \times \mathrm{GL}_1(A_2) \twoheadrightarrow \bar{G} := N_{\mathrm{GL}_1(A_1 \otimes_F A_2)}(\mathrm{GL}_1(A_1)) \subset \mathrm{GL}_1(A_1 \otimes_F A_2)$$

and let $\overline{N}$ be the image of $N_{\mathrm{GL}_1(A_1)}(\mathrm{GL}_1(E)) \times_{\mathrm{Aut}(E)} N_{\mathrm{GL}_1(A_2)}(\mathrm{GL}_1(E))$ in $\overline{G}$. Then for every field extension L/F the natural map $H^1(L, \overline{N}) \to H^1(L, \overline{G})$ is surjective.

Moreover we have natural correspondences

L-isomorphism classes of quintuples $(A_1', A_2', E_1', E_2', \phi)$ with A_1', A_2' central simple of degree n, $E_1' \subset A_1'$, $E_2' \subset A_2'$ maximal étale and $\phi\colon E_1' \xrightarrow{\sim} E_2'$ such that $A_1' \otimes_L A_2' \simeq (A_1 \otimes_F A_2)_L$	$\longleftrightarrow$	$H^1(L, \overline{N})$
L-isomorphism classes of pairs of central simple L-algebras (A_1', A_2') of degree n such that $A_1' \otimes_L A_2' \simeq (A_1 \otimes_F A_2)_L$	$\longleftrightarrow$	$H^1(L, \overline{G})$.

Under these identifications the map $H^1(L, \overline{N}) \to H^1(L, \overline{G})$ takes the isomorphism class of $(A_1', A_2', E_1', E_2', \phi')$ to the isomorphism class of (A_1', A_2').

Proof. First of all the surjectivity of the maps $H^1(L, \overline{N}) \to H^1(L, \overline{G})$ follows from Proposition 6.5, Corollary 4.5 and Lemma 3.6.

Let $\overline{\overline{G}} = \mathrm{Aut}_F(A_1) \times \mathrm{Aut}_F(A_2)$ and $\overline{\overline{N}} = \mathrm{Aut}_F(A_1, E) \times_{\mathrm{Aut}(E)} \mathrm{Aut}_F(A_2, E)$ the image of $\overline{N}$ in $\overline{\overline{G}}$. Note that $\overline{\overline{N}}$ is the automorphism group scheme of the quintuple $(A_1, A_2, E, E, \mathrm{id}_E)$ and $\overline{\overline{G}}$ is the automorphism group scheme of the pair (A_1, A_2). Therefore we have natural correspondences:

L-isomorphism classes of quintuples $(A_1', A_2', E_1', E_2', \phi)$ with A_1', A_2' central simple of degree n, $E_1' \subset A_1'$, $E_2' \subset A_2'$ maximal étale and $\phi\colon E_1' \xrightarrow{\sim} E_2'$	$\longleftrightarrow$	$H^1(L, \overline{\overline{N}})$
L-isomorphism classes of pairs of central simple L-algebras (A_1', A_2') of degree n	$\longleftrightarrow$	$H^1(L, \overline{\overline{G}})$.

The map $H^1(L, \overline{\overline{N}}) \to H^1(L, \overline{\overline{G}})$ is the natural forgetful map. Moreover we have a commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & \overline{N} \arrow[r] \arrow[d, hook] & \overline{\overline{N}} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \overline{G} \arrow[r] & \overline{\overline{G}} \arrow[r] & 1
\end{tikzcd}$$

with $\mathbb{G}_m$ central in $\overline{N}$ and $\overline{G}$. The induced connecting map $H^1(L, \overline{\overline{G}}) \to H^2(L, \mathbb{G}_m) = \mathrm{Br}(L)$ takes the pair (A_1', A_2') to the Brauer class $[A_1'] + [A_2'] - [(A_1)_L] - [(A_2)_L]$. Hence the claim follows from the exact sequences in cohomology. $\square$

7. Essential dimension of $E_7^{s.c.}$

In this section E_7 will denote the split simply connected algebraic group of type E_7. The best published bounds on the essential dimension of this group are $7 \leq \mathrm{ed}(E_7) \leq 29$, where the lower bound is valid for fields of characteristic not 2 [CS06], and the upper bound is valid for fields of characteristic not 2 or 3 [Mac13]. We will show that $\mathrm{ed}(E_7) \leq 11$ for all fields of characteristic not 2 or 3.

Let V_{56} denote the smallest non-trivial linear representation of $E_7 := E_7^{s.c.}$; it is unique and 56-dimensional, and we will give a construction below. The following three linear representations of E_7 are not generically free, and have no trivial subrepresentations. Over the complex numbers these are the only ones [El72a], [Po86], but for the rest of this section we will allow arbitrary base fields of characteristic not 2 or 3. Firstly, the adjoint representation has a 7-dimensional slice which gives a reduction of structure to the normalizer of a maximal torus. Secondly, V_{56} has SGP E_6, so by Theorem 4.1 there is a 1-dimensional slice which gives a reduction of structure to $\mathrm{Norm}_{E_7}(E_6)$; also the 0-dimensional slice of the versal $\mathbb{P}(V_{56})$ gives a reduction to $E_6 \rtimes \mu_4$, which was considered in [Ga09] and [Mac13]. Thirdly, we have $V := V_{56} \oplus V_{56}$, which we consider below.

We will use Theorem 4.1 to obtain a 16-dimensional slice of V, whose closure has a 37-dimensional normalizer N. Then we will show that N acts generically freely on a 48-dimensional subrepresentation of V_{56}, and hence $\mathrm{ed}(E_7) \leq \mathrm{ed}\, N \leq 48 - 37 = 11$.

7.1. Coordinates for the E_8 root system.

We choose to describe the E_8 root system as follows, because it will make the relevant E_7 action easier to understand, since the slice will take a particularly simple form in these coordinates. The following are elements of the lattice $\mathbb{Z}^8$, which we equip with the standard bilinear form scaled by a factor of 1/2.

$$\begin{aligned}
\alpha_1 &= (0,-1,1,0 \mid 0,-1,-1,0) & \alpha_5 &= (0,0,0,1 \mid 1,1,-1,0) \\
\alpha_2 &= (0,0,0,1 \mid 1,-1,1,0) & \alpha_6 &= (-1,1,0,0 \mid -1,-1,0,0) \\
\alpha_3 &= (0,0,0,1 \mid -1,1,1,0) & \alpha_7 &= (2,0,0,0 \mid 0,0,0,0) \\
\alpha_4 &= (0,0,0,-2 \mid 0,0,0,0) & \alpha_8 &= (-1,-1,-1,0 \mid 0,0,0,1)
\end{aligned}$$

This defines a system of simple roots of type E_8, using numbering as in Bourbaki, and we will denote the associated set of roots by Λ_{E_8}. So the free $\mathbb{Z}$-module generated by these simple roots, together with the set of roots Λ_{E_8}, defines a root datum. Given any field F we can construct an algebraic group E_8 over F together with a split maximal torus T_8 whose root datum is as above [Co14, 6]. Furthermore, below we describe subroot systems of types $(A_1)^3$, D_4, and E_7, which correspond to subgroups $(\mathrm{SL}_2)^3, \mathrm{Spin}_8$, and E_7 of E_8.

To describe these subroot systems, we find it convenient to consider the orbits of Λ_{E_8} under the group generated by the sign changes of each of the coordinates, together with the following

Representative	#	Inclusion	Representative	#	Inclusion
(0002 | 0000)	2	$D_4 \subset E_7$	(2000 | 0000)	6	$(A_1)^3 \subset E_7$
(0000 | 2000)	6	$D_4 \subset E_7$	(1110 | 0001)	16	$W \subset V$
(0001 | 1110)	16	$D_4 \subset E_7$	(1001 | 1001)	48	V
(0111 | 1000)	48	E_7	(1000 | 0111)	48	V
(0110 | 0110)	48	E_7	(0000 | 0002)	2	$\pm\rho$

Table 1. Orbits of the 240 E_8 roots

operation of order 3 on the roots:

$$(a_1, a_2, a_3, b \mid c_1, c_2, c_3, d) \mapsto (a_2, a_3, a_1, b \mid c_2, c_3, c_1, d).$$

In [Wi14] this operation is referred to as triality, but here we have rearranged the coordinates to make the symmetry more transparent. The generated group is $(\mathbb{Z}/2)^8 \rtimes \mathbb{Z}/3$, and its 10 orbits among the 240 E_8 roots are listed in Table 1, where # is the orbit size.

The 126 roots labelled in Table 1 as E_7 form a root system of type E_7 with system of simple roots given by $\alpha_1, \cdots, \alpha_7$; they are the roots whose 8th coordinate is zero. This is a modified version of the description of E_7 roots given in [Wi14]. The 24 roots labelled as D_4 form a root system of type D_4, with system of simple roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5$.

The 112 roots labelled in Table 1 as V correspond to weights of the E_7 representation $V = V_{56}^+ \oplus V_{56}^-$, described below, with each copy being distinguished by the sign of the 8th coordinate. The symbol ρ denotes the highest root of E_8 with respect to the given system of simple roots.

The Lie algebra $\mathfrak{e}_8$ decomposes into root spaces, with respect to the Cartan subalgebra $\mathfrak{t} := \mathrm{Lie}(T_8)$. For each simple root we will define the *α_i-height* as the coefficient of α_i when expressed as a sum of simple roots. We can define a $\mathbb{Z}$-grading on $\mathfrak{e}_8$ by defining $\mathfrak{e}_8(i)$ to be the sum over root spaces whose α_8-height is i, with the exception of $\mathfrak{e}_8(0)$, which will also contain $\mathfrak{t}$. This grading has dimensions $1, 56, 134, 56, 1$ in degrees $-2, -1, 0, 1, 2$ respectively. Now we can define $V_{56}^+ := \mathfrak{e}_8(1)$ and $V_{56}^- := \mathfrak{e}_8(-1)$. The algebraic group E_8 acts on $\mathfrak{e}_8$ via the adjoint representation. The subgroup E_7 preserves both V_{56}^+ and V_{56}^-, since this holds on the Lie algebra level. As E_7 representations, V_{56}^+ and V_{56}^- are isomorphic (we will sometimes also write $V_{56} := V_{56}^+$). The span of the weight spaces for the weights $(\pm1, \pm1, \pm1|0|0, 0, 0|d)$ is 8-dimensional when d is either $+1$ or -1, and we will denote these subspaces by $V_8^+ \subset V_{56}^+$ and $V_8^- \subset V_{56}^-$, respectively (we will also write $V_8 := V_8^+$). Below we will write $W := V_8^+ \oplus V_8^-$.

So our E_7 representation of interest will be $V := V_{56}^+ \oplus V_{56}^-$. Choose a Chevalley basis $\{X_\alpha, h_i\}$ of $\mathfrak{e}_8$, where α runs through the roots of E_8 and $h_1, \ldots, h_8$ are the elements of the Cartan subalgebra of $\mathfrak{e}_8$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_8$.

Proposition 7.1. *The subgroup* Spin_8 *of* E_7 *described above is an SGP for the* E_7*-action on* V*. Moreover* V^{Spin_8} *is equal to* $W = V_8^+ \oplus V_8^-$ *and* $\mathfrak{n}_W := \{x \in \mathfrak{e}_7 \mid [x, W] \subset W\} = \mathfrak{spin}_8 \oplus (\mathfrak{sl}_2)^3$.

Proof. Let Λ_{E_7} denote the roots of E_7, Λ_{D_4} the roots of Spin_8, Λ_V the weights of V and Λ_W the weights of W. For $\alpha \in \Lambda_{E_7}$ and $\mu \in \Lambda_V$ we have

$$[X_\alpha, X_\mu] = \begin{cases} \pm X_{\alpha+\mu}, & \text{if } \alpha + \mu \in \Lambda_V \\ 0, & \text{otherwise.} \end{cases}$$

Hence $V^{\mathrm{Spin}_8} = V^{\mathfrak{spin}_8}$ is the subspace of V generated by all X_μ, with $\mu \in \Lambda_V$, such that $\alpha + \mu \notin \Lambda_V$ for all $\alpha \in \Lambda_{D_4}$. Similarly $\mathfrak{n}_W$ is the subspace of $\mathfrak{e}_7$ generated by all X_α, with $\alpha \in \Lambda_{E_7}$, such that for all $\mu \in \Lambda_W$ either $\alpha + \mu \in \Lambda_W$ or $\alpha + \mu \notin \Lambda_V$. Hence the assertions $V^{\mathrm{Spin}_8} = W$ and $\mathfrak{n}_W = \mathfrak{spin}_8 \oplus (\mathfrak{sl}_2)^3$ are easily verified by inspecting Table 1.

The split simply connected group E_6 is an SGP of the action of E_7 on V_{56}, by [Ga01, Example 3.5], and as an E_6 representation, we have the decomposition $V_{56} = F \oplus F \oplus V_{26} \oplus V_{26}^*$. Also, Spin_8 is an SGP of E_6 acting on V_{56}, by [Mac13, Prop.1.3] together with Lemma 2.13. Therefore, Spin_8 is an SGP of the action of E_7 on V. This also follows from [El72a] when $F = \mathbb{C}$.

□

7.2. **Pinning of E_8.** For each E_8-root α, we fix an isomorphism $p_\alpha \colon \mathbb{G}_a \to U_\alpha$ to the unipotent root subgroup; this is known as a *pinning*. This choice defines constants $c_{\lambda,\alpha}$ for each $\lambda, \alpha \in \Lambda_{E_8}$ by the formula (the adjoint representation) $p_\alpha(t)X_\lambda = X_\lambda + c_{\lambda,\alpha} t X_{\lambda+\alpha}$. Since E_8 is simply-laced, if λ and $\lambda + \alpha$ are both weights, then $c_{\lambda,\alpha} = \pm 1$. But in what follows, and in particular to prove generic freeness, it is not enough to only know the structure constants up to sign. So we will choose our Chevalley basis in such a way that for every $\alpha = \pm\alpha_i$, plus or minus a simple root of E_7, and $\lambda, \lambda+\alpha \in \Lambda_V$ we have $c_{\lambda,\alpha} = 1$; this is possible by [Va00, §2].

We will use the notation $w_i := p_{\alpha_i}(1)p_{-\alpha_i}(-1)p_{\alpha_i}(1) \in E_7(F)$ for $i = 1, \ldots, 7$. These elements generate the extended Weyl group of E_7 and normalize the given split maximal torus of E_7 (and also of E_8). The element w_i has the effect of swapping the coordinates of weights that are connected by an edge labelled i in the associated weight diagram (see Figure 1 below; also [Va00, Fig.2]), and negating any coordinate that was moved rightwards; we follow the convention that positive roots act in a leftwards direction.

In the following Lemma, the subgroups Spin_8 and $(\mathrm{SL}_2)^3$ of E_7 are those generated by the root subgroups (images of p_α) for the roots in Table 1 which are labelled as D_4 and A_1^3 respectively. The subgroup S_3 is the subgroup generated by the following two elements:

$$w_{(12)} := w_6 w_5 w_4 w_3 w_2 w_4 w_5 w_6, \qquad w_{(23)} := w_1 w_3 w_4 w_5 w_2 w_4 w_3 w_1.$$

The choice of names for $w_{(12)}$ and $w_{(23)}$ are motivated by the fact that these two elements act on the vectors $(a_1, a_2, a_3, b \mid c_1, c_2, c_3, d)$ by simultaneous swapping of the $1, 2$ (resp. $2, 3$)-coordinates. Although we don't include the details, it is enough to check that this is how $w_{(12)}$ and $w_{(23)}$ act on the 8 simple roots α_i, which span the space. Therefore, together these elements generate the symmetric group on three letters.

Any root of E_8 can be written as $\sum c_i \alpha_i$ for integers c_i, and we will arrange these coefficients as in the Dynkin diagram. For example $000^1 0000$ refers to the root α_2, and the highest root of E_8 is $\rho = 246^3 5432$.

7.3. **Reduction of structure group to the normalizer of Spin_8.**

Lemma 7.2. $N := \mathrm{Norm}_{E_7}(V_8) = N_{E_7}(\mathrm{Spin}_8) \cong ((\mathrm{SL}_2)^3 \times \mathrm{Spin}_8)/\mu_2^2) \rtimes S_3$.

Proof. We have an inclusion $N_{E_7}(\mathrm{Spin}_8) \subset \mathrm{Norm}_{E_7}(V_{56}^{\mathrm{Spin}_8}) = N$. Moreover all of the subgroups Spin_8, $(\mathrm{SL}_2)^3$, and S_3 normalize Spin_8, hence are contained in $N_{E_7}(\mathrm{Spin}_8)$. We know $(\mathrm{SL}_2)^3$ and Spin_8 commute. To describe the intersection of their centres, we will view the three copies of SL_2 as the root groups of $\pm\beta_i$, where

$$\beta_1 := (2,0,0,0|0,0,0,0), \quad \beta_2 := (0,2,0,0|0,0,0,0) \quad \text{and} \quad \beta_3 := (0,0,2,0|0,0,0,0).$$

As E_8 roots, these are $\beta_1 = 000^0 0010$, $\beta_2 = 012^1 2210$, and $\beta_3 = 234^2 3210$.

Now observe that $\beta_1^\vee(-1)\beta_2^\vee(-1) = \alpha_2^\vee(-1)\alpha_3^\vee(-1)$ and $\beta_1^\vee(-1)\beta_3^\vee(-1) = \alpha_3^\vee(-1)\alpha_5^\vee(-1)$ lie in the group of F-points of the centres of both $(\mathrm{SL}_2)^3$ and Spin_8. In fact it can be checked that these two elements generate the group of F-points of the intersection of the centres, and hence that the intersection is $\simeq (\mu_2)^2$.

The subgroup of E_7 generated by Spin_8 and $(\mathrm{SL}_2)^3$ is normalized by S_3 and intersects trivially with S_3, since the non-trivial elements of S_3 induce non-trivial automorphisms of based root systems $(D_4, \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\})$. So the subgroup of E_7 generated by Spin_8, $(\mathrm{SL}_2)^3$ and S_3 is isomorphic to $((\mathrm{SL}_2)^3 \times \mathrm{Spin}_8)/\mu_2^2) \rtimes S_3$, yielding an inclusion $((\mathrm{SL}_2)^3 \times \mathrm{Spin}_8)/\mu_2^2) \rtimes S_3 \subset N_{E_7}(\mathrm{Spin}_8) \subset N$. In order to show the reverse containment we may pass to an algebraic closure. So assume F is algebraically closed.

By [KMRT98, 21.5 (10)] $\mathrm{Lie}(N) = \mathfrak{n}_W$ and by Proposition 7.1 this is equal to $\mathfrak{spin}_8 \oplus \mathfrak{sl}_2^3$. Therefore the connected component N^0 is smooth, 37-dimensional, and generated by $(SL_2)^3$ and Spin_8.

Assume $g \in (N \backslash (N^0 \rtimes S_3))(F)$. Since all maximal tori of N^0 are conjugate over $F = \bar{F}$, by multiplying g with a suitable element of $N^0(F)$ we can assume g preserves the given split torus. Then we can consider the image $\bar{g} \in W(E_7)$ in the Weyl group. Since conjugation by g preserves N^0, $\bar{g}$ preserves the root system D_4. Since all automorphisms of the Dynkin diagram of D_4 are realized by elements of S_3, all possible automorphisms of the root system of D_4 are induced from a torus-preserving element of $N^0 \rtimes S_3$. Therefore we can assume our $\bar{g}$ fixes the D_4 root system. In other words, $\bar{g}$ fixes all but the first three coordinates of the E_8 roots. To see how it acts on the first three coordinates, consider:

$$\bar{g}(2000|0000) = \bar{g}(1000|0111) - \bar{g}(-1000|0111) = (\pm 1000|0111) - (\pm 1000|0111) = (\pm 2000|0000).$$

Here we have used that the only roots of the form $(***0|0111)$ of E_8 (as listed in Table 1) are the roots $(\pm 1000|0111)$. A similar argument applies to the 2nd and 3rd coordinates. But this means $\bar{g}$ acts in the same way as an element of $W(A_1^3)$. Since the Weyl group of E_7 acts faithfully on the set of weights of V_{56}, this contradicts the assumption that $g \notin (N^0 \rtimes S_3)(F)$. Therefore, $N \cong N^0 \rtimes S_3$, which concludes the proof. □

Theorem 7.3. *Let $N := N_{E_7}(\mathrm{Spin}_8)$. For every field L/F the map $H^1(L, N) \to H^1(L, E_7)$ is surjective.*

Proof. When L is infinite the result follows from Proposition 7.1 and Corollary 4.5. If L is finite, then Lang's Theorem asserts $H^1(L, E_7) = 1$ [La56]. □

7.4. Generic freeness of V_{48}.

Theorem 7.4. *The action of N on the 48-dimensional irreducible subrepresentation of $V_{56} = V_{48} \oplus V_8$ is generically free.*

Proof. Figure 1 is the weight diagram of V_{56}, the E_7-representation (see [PSV96, Fig.21]), which can be used as a visual aid. Each node corresponds to a weight (and hence a 1-dimensional weight space), and two nodes are joined by an edge labelled i if their difference is the simple root α_i. The diagram is presented in such a way that adding a positive simple root move weights from right to left, so that the highest weight of the representation $(\rho - \alpha_8)$ is the node in the top left corner. The node in the lower right is the weight α_8. Nodes connected by the simple roots of D_4 are solid, while the others are dotted.

By Figure 1, as a Spin_8 representation, V_{56} decomposes into an 8-dimensional trivial representation (the isolated nodes), and 6 irreducible representations, each of dimension 8. These 8 dimensional spaces pair up, and when decomposed as an $((\mathrm{SL}_2)^3 \times \mathrm{Spin}_8)/\mu_2^2$ representation, V_{48} decomposes into 3 representations each of dimension 16, and we will write $V_{48} = V_1 \oplus V_2 \oplus V_3$. Here V_i has highest weight $\lambda_1 := 134^2 3221$, $\lambda_2 := 135^3 4321$, and $\lambda_3 := 246^3 5321$ for $i = 1, 2, 3$ respectively. In the coordinates of 7.1 these are:

$$\lambda_1 = (1, 0, 0, 0|0, 1, 1, 1), \quad \lambda_2 = (0, 1, 0, 0|1, 0, 1, 1), \quad \lambda_3 = (0, 0, 1, 0|1, 1, 0, 1).$$

First we will show it is enough to check that the connected component of the identity N^0 acts generically freely. Namely we have a homogeneous N-invariant polynomial on V_{56} which is non-zero on the V_i, given by the E_7-invariant quartic q on V_{56}; see for example [He12, Lemma 6] for a proof that q is non-zero on each 2-dimensional space $V_\lambda \oplus V_{\rho-\lambda}$, where ρ is the highest root of E_8, and λ is any weight of V_{48} (each V_i is a direct sum of such subspaces). The morphism

$$V_{48} = V_1 \oplus V_2 \oplus V_3 \to \mathbb{A}^3, \quad (u_1, u_2, u_3) \mapsto (q(u_1), q(u_2), q(u_3))$$

is surjective on $\bar{F}$-points and N-equivariant if we endow $\mathbb{A}^3$ with the natural (faithful) permutation action of $N/N^0 = S^3$. So any $u = (u_1, u_2, u_3)$ which has trivial stabilizer in N^0 and is

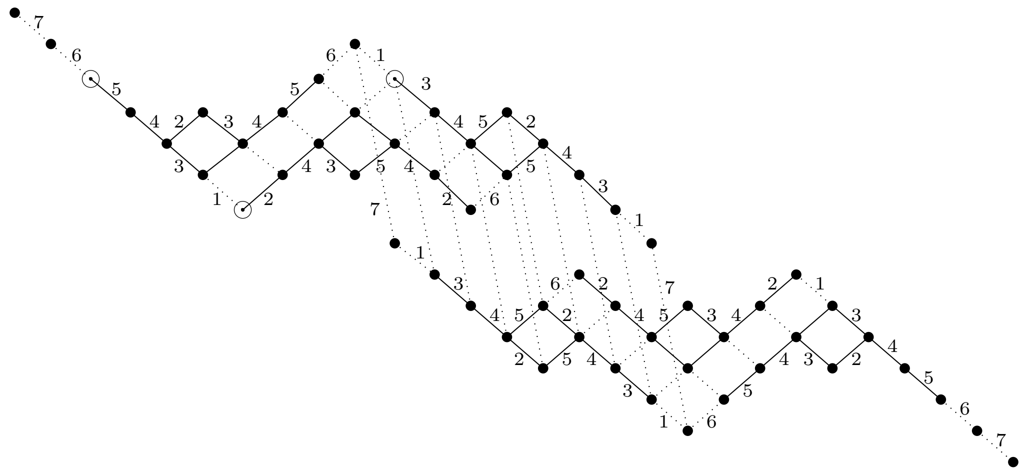

FIGURE 1. Weight diagram of the E_7 representation V_{56}. The vertices marked $\odot$ are, from left to right, the weights $\lambda_3, \lambda_2, \lambda_1$.

contained in the preimage of the open dense subset $\{(a_1, a_2, a_3) \mid a_1 \neq a_2 \neq a_3 \neq a_1\} \subset \mathbb{A}^3$, must have trivial stabilizer in N. This shows the claim.

We will label the three copies of SL_2 in N by L_i for $i = 1, 2, 3$, where L_i acts trivially on V_j for $i \neq j$. In other words L_i is generated by the root groups of $\pm\beta_i$, from the proof of Lemma 7.2.

To finish the proof of Theorem 7.4 we will need the following Lemmas. Recall that $\mathrm{Spin}_6 \cong \mathrm{SL}_4$ and $\mathrm{SO}_2 \cong \mathbb{G}_m$.

Lemma 7.5. *The* $\mathrm{Spin}_8 \times L_1$ *orbit of any* $v \in V_1$ *such that* $q(v) \neq 0$ *contains an element which is a non-zero scalar multiple of* $X_{\lambda_1} + X_{\rho-\lambda_1}$. *The stabilizer of* $X_{\lambda_1} + X_{\rho-\lambda_1}$ *in* $\mathrm{Spin}_8 \times L_1$ *is isomorphic to* $\mathrm{Spin}_6 \times \mathbb{G}_m$.

Proof. One can see from the weight diagram that V_1 is the tensor product of an 8-dimensional irreducible Spin_8-representation with the natural 2-dimensional vector L_1-representation. The action may be viewed as $(a, b) \in \mathrm{SO}_8 \times \mathrm{SL}_2$ acting on 8×2 matrices by $X \mapsto aXb^T$. Here we are viewing F-points of SO_8 as $a \in \mathrm{SL}_8$ such that $a^t a = I_8$. So by the argument in [SK77, p.109], this action has an SGP isomorphic to $\mathrm{SO}_6 \times \mathrm{SO}_2$, and hence the SGP in $\mathrm{Spin}_8 \times L_1$ is as required. In fact, [SK77, p.109-110] considers the action of $\mathrm{SO}_8 \times \mathrm{GL}_2$ instead, and they show that there is an open orbit in $F^8 \otimes F^2$ (due to a dimension count), and the function $f(X) = \det(X^T X)$ is a relative invariant. Since the SGP is reductive, the open orbit is affine, and so its complement is a hypersurface. Since f is the only irreducible relative invariant [SK77, p.110] (see also [BGL14, Example 6.5]), the hypersurface is irreducible and defined by f [SK77, p.60]. Although [SK77] assumes the field is $\mathbb{C}$, the arguments needed here work in characteristic not 2.

Now $q(X_{\lambda_1} + X_{\rho-\lambda_1}) \neq 0$, and q restricted to V_1 is a $\mathrm{Spin}_8 \times L_1$-invariant function of degree 4, so the invariant ring $F[V_1]^{\mathrm{Spin}_8 \times L_1} = F[q]$ by [BGL14, Example 6.5]. In particular, q must be a scalar multiple of f, and hence any non-zero $v \in V_1$ such that $q(v) \neq 0$ must be the in open $(\mathrm{SO}_8 \times \mathrm{GL}_2)$-orbit. The result follows. $\square$

Lemma 7.6. *Let* $H \simeq \mathrm{SL}_4$ *be the subgroup of* E_7 *generated by the root groups for* $\pm\alpha_2, \pm\alpha_4, \pm\alpha_5$, *and let* $T_1 \simeq \mathbb{G}_m$ *denote the image of the coroot* $\alpha_2^\vee + \alpha_5^\vee + 2(\alpha_4^\vee + \alpha_3^\vee - \alpha_7^\vee) \colon \mathbb{G}_m \to E_7$. *Then the subgroups* H *and* T_1 *commute and intersect in the image of* $\mu_2 \hookrightarrow E_7$, $t \mapsto \alpha_2^\vee(t)\alpha_5^\vee(t)$. *Moreover the subgroup generated by* H *and* T_1 *is the stabilizer of* $X_{\lambda_1} + X_{\rho-\lambda_1} \in V_1$ *in* $\mathrm{Spin}_8 \times L_1 \subset E_7$.

Proof. Looking at the bonds in the weight diagram connected to λ_1 and $\rho-\lambda_1$ it is clear that the stabilizer contains H. Write $\beta^\vee := \alpha_2^\vee+\alpha_5^\vee+2(\alpha_4^\vee+\alpha_3^\vee-\alpha_7^\vee)$. Using the bilinear pairing one checks that $\lambda_1\circ\beta^\vee=(\rho-\lambda_1)\circ\beta^\vee=0$ and $\alpha_i\circ\beta^\vee=0$ for $i=2,4,5$. Therefore T_1 (the image of $\beta^\vee$) lies in the stabilizer and commutes with H. Clearly the intersection $H\cap T_1$ contains the image of $\mu_2\hookrightarrow E_7,\ t\mapsto\alpha_2^\vee(t)\alpha_5^\vee(t)$. Since the image of the coroot $\alpha_7^\vee$ intersects Spin_8 trivially this containment cannot be proper. So we have a 16-dimensional subgroup $\mathrm{SL}_4\cdot\mathbb{G}_m$ contained in the stabilizer. By Lemma 7.5, this must equal the whole stabilizer. □

Remark 7.7. See also [Ga09, 12.2] and [El72b, Theorem 5], which consider a similar situation.

So the subgroups $H\simeq\mathrm{SL}_4$, $T_1\simeq\mathbb{G}_m$, $L_2\simeq\mathrm{SL}_2$, and $L_3\simeq\mathrm{SL}_2$ described above generate an SGP for the action of N^0 on V_1. To finish the proof of generic freeness, we will prove the following Lemma:

Lemma 7.8. *The group generated by H, T_1, L_2, and L_3 in E_7 acts generically freely on $V_2\oplus V_3$.*

Proof. As an SL_4-representation, $V_2\oplus V_3$ decomposes into 4 standard representations, and 4 copies of the dual (see Figure 2). By arranging the weight spaces into a pair of 4 by 4 matrices, it is easier to understand the action. For simplicity we will consider the group $L_2\times L_3\times H\times T_1$, which is the product of four commuting subgroups of E_7, rather than the image of its projection to E_7, which has kernel $\mu_2\times\mu_4$. We will identify T_1 with $\mathbb{G}_m$ through the isomorphism $\mathbb{G}_m\to T_1,\ \xi\mapsto\beta^\vee(\xi)$.

We arrange the 32 weight spaces as follows, written in the coordinates of the E_8 fundamental roots. So the columns correspond to the SL_4 representations, λ_2 is the upper left entry of the first matrix, and λ_3 is the 4th row and 3rd column of the second matrix. Also, β_2 is the difference between the weights in the first two columns (and the same rows) for each matrix, while the different between columns 3 and 4 is β_3.

$$\begin{bmatrix}135^34321 & 123^22111 & 235^34321 & 001^11111\\ 135^24321 & 123^12111 & 235^24321 & 001^01111\\ 134^24321 & 122^12111 & 234^24321 & 000^01111\\ 134^23321 & 122^11111 & 234^23321 & 000^00111\end{bmatrix}\begin{bmatrix}123^13321 & 111^01111 & 245^24321 & 011^01111\\ 123^23321 & 111^11111 & 245^34321 & 011^11111\\ 124^23321 & 112^11111 & 246^34321 & 012^11111\\ 124^24321 & 112^12111 & 246^35321 & 012^12111\end{bmatrix}$$

Choose a basis of the representation consisting of weight vectors X_λ. Then we can describe an element of $V_2\oplus V_3$ by its coefficients of this basis, which we will write $(x,y)\in M_4\oplus M_4$, as a pair of matrices. Let us identify the subgroup H of E_7 generated by the root groups of $\pm\alpha_2,\pm\alpha_4,\pm\alpha_5$ with SL_4 using the identifications

$$p_{\alpha_2}(t)=\begin{bmatrix}1&t&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\qquad p_{\alpha_4}(t)=\begin{bmatrix}1&0&0&0\\0&1&t&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\qquad p_{\alpha_5}(t)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&t\\0&0&0&1\end{bmatrix},$$

$$p_{-\alpha_2}(t)=\begin{bmatrix}1&0&0&0\\t&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\qquad p_{-\alpha_4}(t)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&t&1&0\\0&0&0&1\end{bmatrix},\qquad p_{-\alpha_5}(t)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&t&1\end{bmatrix}.$$

Similarly L_i (for $i=2,3$) is identified with SL_2 using $p_{\beta_i}(t)=\left[\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right]$, $p_{-\beta_i}(t)=\left[\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right]$.

Since we have assumed our structure constants are as in [Va00, Theorem 1], $g\in\mathrm{SL}_4$ acts as $(x,y)\mapsto(gx,D(g^{-1})^TDy)$, where $D=\mathrm{diag}(-1,1,-1,1)$. The reason for conjugating by D is to ensure the structure constants are $c_{\lambda,\alpha}=1$ for simple roots α. To determine how L_2 and L_3 act, we use the procedure described in [Va00, Theorem 2] applied to the roots β_2 and β_3. For example, the root strings for β_2 and β_3 are 654234567 and 13425431654234567 respectively, so to show that $c_{\lambda_2-\beta_2,\beta_2}=1$, where $\lambda_2-\beta_2$ is the $(1,2)$ entry of the left matrix, we calculate $n(\lambda_2-\beta_2,\beta_2)=4$, in the notation of [Va00, Theorem 2].

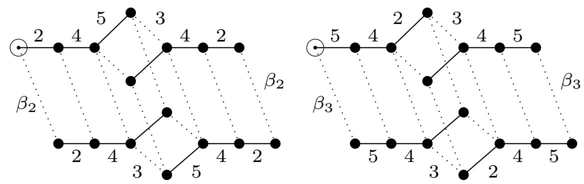

FIGURE 2. Weight diagram of the $H \times L_2 \times L_3$ representation $V_2 \oplus V_3$; we also label the connections for α_3, to clarify how it sits inside Figure 1. The vertices marked $\odot$ are, from left to right, the weights λ_2, λ_3.

In fact, for $i = 2$ or 3, whenever λ and $\lambda + \beta_i$ are both weights of $V_2 \oplus V_3$, we find $c_{\lambda,\beta_i} = 1$. This, together with the action of T_1, which is determined by the bilinear pairing between characters and cocharacters, shows that $L_2 \times L_3 \times H \times T_1$ acts through the formula

$$(a, b, g, \xi) \cdot (x, y) = \left(gx \begin{bmatrix} \xi a^T & 0 \\ 0 & \xi^{-1} b^T \end{bmatrix}, D(g^{-1})^T Dy \begin{bmatrix} \xi^{-1} a^T & 0 \\ 0 & \xi b^T \end{bmatrix} \right).$$

For example, $(E_{12}, 0) = X_{\lambda_2 - \beta_2}$, and we see $p_{\beta_2}(t)(X_{\lambda_2-\beta_2}) = (tE_{11} + E_{12}, 0) = X_{\lambda_2-\beta_2} + tX_{\lambda_2}$, as required. A generic $x \in M_2$ is invertible, and so its H-orbit contains a scalar matrix. The stabilizer of a non-zero scalar x is the subgroup defined by the equation $g = \left[\begin{smallmatrix} \xi^{-1}(a^{-1})^T & 0 \\ 0 & \xi(b^{-1})^T \end{smallmatrix}\right]$. Therefore the $L_2 \times L_3 \times H \times T_1$-action on the first copy of M_4 has an SGP, which we may identify with $L_2 \times L_3 \times T_1$. It acts on the second copy of M_4 through the formula

$$(a, b, \xi) \cdot y = D \begin{bmatrix} \xi a & 0 \\ 0 & \xi^{-1} b \end{bmatrix} Dy \begin{bmatrix} \xi^{-1} a^T & 0 \\ 0 & \xi b^T \end{bmatrix}.$$

Note that this representation decomposes into the 4 sub-representations given by the four 2 by 2 blocks in M_4. The representation of $L_2 \times L_3 \times T_1$ on the sum $M_2 \oplus M_2$ of the diagonal M_2's is given by

$$(a, b, \xi) \cdot (y_1, y_2) = (\bar{a} y_1 a^T, \bar{b} y_2 b^T),$$

where $\bar{a} = dad, \bar{b} = dbd$ with $d = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = d^{-1}$. As an L_i-representation, the only non-trivial component in $M_2 \oplus M_2$ is equivalent to the adjoint representation, which has SGP a maximal torus. So the stabilizer of a generic pair of matrices (y_1, y_2) is a maximal torus of $L_2 \times L_3 \times T_1$. Since all maximal tori are conjugate we have an SGP for this action given by $\mathbb{G}_m^3$ embedded in $L_2 \times L_3 \times T_1$ on the diagonals.

The induced action of $\mathbb{G}_m^3$ on the upper off-diagonal copy of M_2 has weights $(1, 1, 2)$, $(1, -1, 2)$, $(-1, 1, 2)$ and $(-1, -1, 2)$, hence it has an SGP $\simeq \mu_2 \times \mu_4$. Therefore we conclude that the $L_2 \times L_3 \times H \times T_1$ action on $M_4 \oplus M_4$ has SGP isomorphic to $\mu_2 \times \mu_4$, which is the kernel of the projection to E_7; using the above formula for the action of $L_2 \times L_3 \times H \times T$, we see the kernel is the image of the embedding $\mu_2 \times \mu_4 \to L_2 \times L_3 \times H \times T_1$ defined by $(b, g) \mapsto (g^2 bI_2, bI_2, gI_4, gb)$. Therefore we have shown that the subgroup of E_7 generated by L_2, L_3, H and T_1 acts generically freely on $V_2 \oplus V_3$. □

End of proof of Theorem 7.4: Putting the above Lemmas together, we see that the action of N on $V_{48} = V_1 \oplus V_2 \oplus V_3$ is generically free, as required. □

Theorem 7.9. $\operatorname{ed}(E_7) \leq 11$.

Proof. Theorem 7.3 implies $\operatorname{ed}(E_7) \leq \operatorname{ed}(N)$ and by Lemma 2.16 Theorem 7.4 yields $\operatorname{ed}(N) \leq 48 - \dim N$. By Lemma 7.2 $\dim(N) = 3 \cdot 3 + \frac{8 \cdot 7}{2} = 37$, so the claim follows. □

Remark 7.10. Consider a (not necessarily split) simply connected semisimple algebraic group G of type E_7 over F. Under what assumptions do our results on reduction of structure and the upper bound on $\operatorname{ed}(G)$ still hold?

(1) The upper bound $\mathrm{ed}(G) \leq 11$ still holds if G has trivial Tits algebras. Indeed such a group is of the form $\mathrm{Aut}(X)$ for some E_7-torsor X, so it has the same torsors as the split E_7.
(2) If G only has Tits algebras of index ≤ 2 then we still have a 112-dimensional representation V which decomposes as $V_{56} \oplus V_{56}$ over an algebraic closure. Assume that F is infinite. Then by Lemma 2.12 there exists an SGP H for the G-action on V, which becomes conjugate to Spin_8 over $\bar{F}$. Then applying Theorem 4.5 we still get surjectivity of the maps $H^1(L, N) \to H^1(L, G)$ for field extensions L/F and the inequality $\mathrm{ed}(G) \leq \mathrm{ed}(N)$, where $N = N_G(H)$. However in that case we have no good upper bound on $\mathrm{ed}(N)$ at our disposal.

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