AN INFINITE C*-ALGEBRA WITH A DENSE, STABLY FINITE *-SUBALGEBRA

NIELS JAKOB LAUSTSEN AND JARED T. WHITE

ABSTRACT. We construct a unital pre-C*-algebra A_0 which is stably finite, in the sense that every left invertible square matrix over A_0 is right invertible, while the C*-completion of A_0 contains a non-unitary isometry, and so it is infinite.

To appear in *Proceedings of the American Mathematical Society.*

1. INTRODUCTION

Let A be a unital algebra. We say that A is *finite* (also called directly finite or Dedekind finite) if every left invertible element of A is right invertible, and we say that A is *infinite* otherwise. This notion originates in the seminal studies of projections in von Neumann algebras carried out by Murray and von Neumann in the 1930s. At the 22nd International Conference on Banach Algebras and Applications, held at the Fields Institute in Toronto in 2015, Yemon Choi raised the following questions:

(1) Let A be a unital, finite normed algebra. Must its completion be finite?
(2) Let A be a unital, finite pre-C*-algebra. Must its completion be finite?

Choi also stated Question (1) in [7, Section 6].

A unital algebra A is said to be *stably finite* if the matrix algebra $M_n(A)$ is finite for each $n \in \mathbb{N}$. This stronger form of finiteness is particularly useful in the context of K-theory, and so it has become a household item in the Elliott classification programme for C*-algebras. The notions of finiteness and stable finiteness differ even for C*-algebras, as was shown independently by Clarke [8] and Blackadar [4] (or see [5, Exercise 6.10.1]). A much deeper result is due to Rørdam [9, Corollary 7.2], who constructed a unital, simple C*-algebra which is finite (and separable and nuclear), but not stably finite.

We shall answer Question (2), and hence Question (1), in the negative by proving the following result:

Theorem 1.1. *There exists a unital, infinite C*-algebra which contains a dense, unital, stably finite *-subalgebra.*

Let A be a unital *-algebra. Then there is a natural variant of finiteness in this setting, namely we say that A is **-finite* if whenever we have $u \in A$ satisfying $u^*u = 1$, then $uu^* = 1$. However, it is known (see, *e.g.*, [10, Lemma 5.1.2]) that a C*-algebra is finite if and only if it is *-finite, so in this article we shall not need to refer to *-finiteness again.

2010 *Mathematics Subject Classification.* 46L05 (primary); 20M25, 46L09 (secondary).
Key words and phrases. C*-algebra, stably finite, infinite, completion, free product.

2. Preliminaries

Our approach is based on semigroup algebras. Let S be a monoid, that is, a semigroup with an identity, which we shall usually denote by e. By an *involution* on S we mean a map from S to S, always denoted by $s \mapsto s^*$, satisfying $(st)^* = t^*s^*$ and $s^{**} = s$ $(s, t \in S)$. By a **-monoid* we shall mean a pair $(S, *)$, where S is a monoid, and $*$ is an involution on S. Given a *-monoid S, the semigroup algebra $\mathbb{C}S$ becomes a unital *-algebra simply by defining $\delta_s^* = \delta_{s^*}$ $(s \in S)$, and extending conjugate-linearly.

Next we shall recall some basic facts about free products of *-monoids, unital *-algebras, and their C*-representations.

Let S and T be monoids, and let A and B be unital algebras. Then we denote the free product (*i.e.* the coproduct) of S and T in the category of monoids by $S*T$, and similarly we denote the free product of the unital algebras A and B by $A*B$. It follows from the universal property satisfied by free products that, for monoids S and T, we have $\mathbb{C}(S*T) \cong (\mathbb{C}S)*(\mathbb{C}T)$.

Given *-monoids S and T, we can define an involution on $S * T$ by

$$(s_1 t_1 \cdots s_n t_n)^* = t_n^* s_n^* \cdots t_1^* s_1^*$$

for $n \in \mathbb{N}, s_1 \in S, s_2, \ldots, s_n \in S \setminus \{e\}, t_1, \ldots, t_{n-1} \in T \setminus \{e\}$, and $t_n \in T$. The resulting *-monoid, which we continue to denote by $S*T$, is the free product in the category of *-monoids. We can analogously define an involution on the free product of two unital *-algebras, and again the result is the free product in the category of unital *-algebras. We then find that $\mathbb{C}(S * T) \cong (\mathbb{C}S) * (\mathbb{C}T)$ as unital *-algebras.

Let A be a *-algebra. If there exists an injective *-homomorphism from A into some C*-algebra, then we say that A admits a *faithful C*-representation.* In this case, A admits a norm such that the completion of A in this norm is a C*-algebra, and we say that A admits a *C*-completion.* Our construction will be based on C*-completions of *-algebras of the form $\mathbb{C}S$, for S a *-monoid.

We shall denote by S_∞ the free *-monoid on countably many generators; that is, as a monoid S_∞ is free on some countably-infinite generating set $\{t_n, s_n : n \in \mathbb{N}\}$, and the involution is determined by $t_n^* = s_n$ $(n \in \mathbb{N})$. For the rest of the text we shall simply write t_n^* in place of s_n. We define BC to be the bicyclic monoid $\langle p, q : pq = e\rangle$. This becomes a *-monoid when an involution is defined by $p^* = q$, and the corresponding $*$-algebra $\mathbb{C}BC$ is infinite because $\delta_p\delta_q = \delta_e$, but $\delta_q\delta_p = \delta_{qp} \neq \delta_e$.

Lemma 2.1. *The following unital *-algebras admit faithful C*-representations:*

(i) $\mathbb{C}(BC)$,
(ii) $\mathbb{C}(S_\infty)$.

Proof. (i) Since BC is an inverse semigroup, this follows from [2, Theorem 2.3].

(ii) By [3, Theorem 3.4] $\mathbb{C}S_2$ admits a faithful C*-representation, where S_2 denotes the free monoid on two generators $S_2 = \langle a, b\rangle$, endowed with the involution determined by $a^* = b$. There is a *-monomorphism $S_\infty \hookrightarrow S_2$ defined by $t_n \mapsto a(a^*)^n a$ $(n \in \mathbb{N})$ and this induces a *-monomorphism $\mathbb{C}S_\infty \hookrightarrow \mathbb{C}S_2$. The result follows. ☐

By a *state* on a unital *-algebra A we mean a linear functional $\mu\colon A \to \mathbb{C}$ satisfying $\langle a^*a, \mu\rangle \geq 0$ $(a \in A)$ and $\langle 1, \mu\rangle = 1$. We say that a state μ is *faithful* if $\langle a^*a, \mu\rangle > 0$ $(a \in A \setminus \{0\})$. A unital *-algebra with a faithful state admits a faithful C*-representation via the GNS representation associated with the state.

The following theorem appears to be folklore in the theory of free products of C^*-algebras; it can be traced back at least to the seminal work of Avitzour [1, Proposition 2.3] (see also [6, Section 4] for a more general result).

Theorem 2.2. *Let A and B be unital $*$-algebras which admit faithful states. Then their free product $A * B$ also admits a faithful state, and hence it has a faithful C^*-representation.*

We make use of this result in our next lemma.

Lemma 2.3. *The unital *-algebra $\mathbb{C}(BC * S_\infty)$ admits a faithful C^*-representation.*

Proof. We first remark that a separable C*-algebra A always admits a faithful state. To see this, note that the unit ball of A^* with the weak*-topology is a compact metric space, and hence also separable. It follows that the set of states $S(A)$ is weak*-separable. Taking $\{\rho_n : n \in \mathbb{N}\}$ to be a dense subset of $S(A)$, we then define $\rho = \sum_{n=1}^{\infty} 2^{-n}\rho_n$, which is easily seen to be a faithful state on A.

By Lemma 2.1, both $\mathbb{C}(BC)$ and $\mathbb{C}(S_\infty)$ admit C*-completions. Since both of these algebras have countable dimension, their C*-completions are separable, and, as such, each admits a faithful state, which we may then restrict to obtain faithful states on $\mathbb{C}BC$ and $\mathbb{C}S_\infty$. By Theorem 2.2, $(\mathbb{C}BC) * (\mathbb{C}S_\infty) \cong \mathbb{C}(BC * S_\infty)$ admits a faithful C*-representation. □

3. Proof of Theorem 1.1

The main idea of the proof is to embed $\mathbb{C}S_\infty$, which is finite, as a dense *-subalgebra of some C*-completion of $\mathbb{C}(BC * S_\infty)$, which will necessarily be infinite. In fact we have the following:

Lemma 3.1. *The *-algebra $\mathbb{C}S_\infty$ is stably finite.*

Proof. As we remarked in the proof of Lemma 2.1, $\mathbb{C}S_\infty$ embeds into $\mathbb{C}S_2$. It is also clear that, as an algebra, $\mathbb{C}S_2$ embeds into $\mathbb{C}F_2$, where F_2 denotes the free group on two generators. Hence $\mathbb{C}S_\infty$ embeds into $\mathrm{vN}(F_2)$, the group von Neumann algebra of F_2, which is stably finite since it is a C*-algebra with a faithful tracial state. It follows that $\mathbb{C}S_\infty$ is stably finite as well. □

We shall next define a notion of length for elements of $BC * S_\infty$. Indeed, each $u \in (BC * S_\infty) \setminus \{e\}$ has a unique expression of the form $w_1 w_2 \cdots w_n$, for some $n \in \mathbb{N}$ and some $w_1, \ldots, w_n \in (BC \setminus \{e\}) \cup \{t_j, t_j^* : j \in \mathbb{N}\}$, satisfying $w_{i+1} \in \{t_j, t_j^* : j \in \mathbb{N}\}$ whenever $w_i \in BC \setminus \{e\}$ $(i = 1, \ldots, n-1)$. We then define $\operatorname{len} u = n$ for this value of n, and set $\operatorname{len} e = 0$. This also gives a definition of length for elements of S_∞ by considering S_∞ as a submonoid of $BC * S_\infty$ in the natural way. For $m \in \mathbb{N}_0$ we set

$$L_m(BC * S_\infty) = \{u \in BC * S_\infty : \operatorname{len} u \leq m\}, \quad L_m(S_\infty) = \{u \in S_\infty : \operatorname{len} u \leq m\}.$$

We now describe our embedding of $\mathbb{C}S_\infty$ into $\mathbb{C}(BC * S_\infty)$. By Lemma 2.3, $\mathbb{C}(BC * S_\infty)$ has a C*-completion $(A, \|\cdot\|)$. Let $\gamma_n = (n\|\delta_{t_n}\|)^{-1}$ $(n \in \mathbb{N})$ and define elements a_n in $\mathbb{C}(BC * S_\infty)$ by $a_n = \delta_p + \gamma_n \delta_{t_n}$ $(n \in \mathbb{N})$, so that $a_n \to \delta_p$ as $n \to \infty$. Using the universal property of S_∞ we may define a unital *-homomorphism $\varphi \colon \mathbb{C}S_\infty \to \mathbb{C}(BC * S_\infty)$ by setting $\varphi(\delta_{t_n}) = a_n$ $(n \in \mathbb{N})$ and extending to $\mathbb{C}S_\infty$. In what follows, given a monoid S and $s \in S$, δ'_s will denote the linear functional on $\mathbb{C}S$ defined by $\langle \delta_t, \delta'_s \rangle = \mathbb{1}_{s,t}$ $\quad (t \in S)$, where $\mathbb{1}_{s,t} = 1$ if $s = t$ and $\mathbb{1}_{s,t} = 0$ otherwise.

Lemma 3.2. *Let $w \in S_\infty$ with $\operatorname{len} w = m$. Then*

(i) $\varphi(\delta_w) \in \operatorname{span}\{\delta_u : u \in L_m(BC * S_\infty)\}$;

(ii) *for each $y \in L_m(S_\infty)$ we have*

$$\langle \varphi(\delta_y), \delta'_w \rangle \neq 0 \Leftrightarrow y = w.$$

Proof. We proceed by induction on m. When $m = 0$, w is forced to be e and hence, as φ is unital, $\varphi(\delta_e) = \delta_e$, so that (i) is satisfied. In (ii), y is also equal to e, so that (ii) is trivially satisfied as well.

Assume $m \geq 1$ and that (i) and (ii) hold for all elements of $L_{m-1}(S_\infty)$. We can write w as $w = vx$ for some $v \in S_\infty$ with $\operatorname{len} v = m - 1$ and some $x \in \{t_j, t_j^* : j \in \mathbb{N}\}$.

First consider (i). By the induction hypothesis, we can write $\varphi(\delta_v) = \sum_{u \in E} \alpha_u \delta_u$, for some finite set $E \subset L_{m-1}(BC * S_\infty)$ and some scalars $\alpha_u \in \mathbb{C}$ $(u \in E)$. Suppose that $x = t_j$ for some $j \in \mathbb{N}$. Then

$$\varphi(\delta_w) = \varphi(\delta_v)\varphi(\delta_{t_j}) = \left(\sum_{u \in E} \alpha_u \delta_u \right) (\delta_p + \gamma_j \delta_{t_j}) = \sum_{u \in E} \alpha_u \delta_{up} + \alpha_u \gamma_j \delta_{ut_j},$$

which belongs to $\operatorname{span} \{\delta_u : u \in L_m(BC * S_\infty)\}$ because

$$\operatorname{len}(up) \leq \operatorname{len}(u) + 1 \leq m \quad \text{and} \quad \operatorname{len}(ut_j) = \operatorname{len}(u) + 1 \leq m$$

for each $u \in L_{m-1}(BC * S_\infty)$. The case $x = t_j^*$ is established analogously.

Next consider (ii). Let $y \in L_m(S_\infty)$. If $\operatorname{len} y \leq m - 1$ then, by (i), we know that $\varphi(\delta_y) \in \operatorname{span} \{\delta_u : u \in L_{m-1}(BC * S_\infty)\} \subset \ker \delta'_w$. Hence in this case $y \neq w$ and $\langle \varphi(\delta_y), \delta'_w \rangle = 0$.

Now suppose instead that $\operatorname{len} y = m$, and write $y = uz$ for some $u \in L_{m-1}(S_\infty)$ and $z \in \{t_j, t_j^* : j \in \mathbb{N}\}$. By (i) we may write $\varphi(\delta_u) = \sum_{s \in F} \beta_s \delta_s$ for some finite subset $F \subset L_{m-1}(BC * S_\infty)$ and some scalars $\beta_s \in \mathbb{C}$ $(s \in F)$, and we may assume that $v \in F$ (possibly with $\beta_v = 0$). We prove the result in the case that $z = t_j$ for some $j \in \mathbb{N}$, with the argument for the case $z = t_j^*$ being almost identical. We have $\varphi(\delta_z) = \delta_p + \gamma_j \delta_{t_j}$ and it follows that

$$\varphi(\delta_y) = \varphi(\delta_u)\varphi(\delta_z) = \sum_{s \in F} \beta_s \delta_{sp} + \beta_s \gamma_j \delta_{st_j}.$$

Observe that $sp \neq w$ for each $s \in F$. This is because we either have $\operatorname{len}(sp) < m = \operatorname{len}(w)$, or else sp ends in p when considered as a word over the alphabet $\{p, p^*\} \cup \{t_j, t_j^* : j \in \mathbb{N}\}$, whereas $w \in S_\infty$. Moreover, given $s \in F$, $st_j = w = vx$ if and only if $s = v$ and $t_j = x$. Hence

$$\langle \varphi(\delta_y), \delta'_w \rangle = \beta_v \gamma_j \mathbb{1}_{t_j, x} = \langle \varphi(\delta_u), \delta'_v \rangle \gamma_j \mathbb{1}_{t_j, x}.$$

As $\gamma_j > 0$, this implies that $\langle \varphi(\delta_y), \delta'_w \rangle \neq 0$ if and only if $\langle \varphi(\delta_u), \delta'_v \rangle \neq 0$ and $t_j = x$, which, by the induction hypothesis, occurs if and only if $u = v$ and $t_j = x$. This final statement is equivalent to $y = w$. □

Corollary 3.3. *The map φ is injective.*

Proof. Assume towards a contradiction that $\sum_{u \in F} \alpha_u \delta_u \in \ker \varphi$ for some non-empty finite set $F \subset S_\infty$ and $\alpha_u \in \mathbb{C} \setminus \{0\}$ $(u \in F)$. Take $w \in F$ of maximal length. Then

$$0 = \left\langle \varphi \left(\sum_{u \in F} \alpha_u \delta_u \right), \delta'_w \right\rangle = \sum_{u \in F} \alpha_u \langle \varphi(\delta_u), \delta'_w \rangle = \alpha_w \langle \varphi(\delta_w), \delta'_w \rangle,$$

where the final equality follows from Lemma 3.2(ii). That lemma also tells us that $\langle \varphi(\delta_w), \delta'_w \rangle \neq 0$, forcing $\alpha_w = 0$, a contradiction. □

We can now prove our main theorem.

Proof of Theorem 1.1. Recall that $(A, \|\cdot\|)$ denotes a C*-completion of $\mathbb{C}(BC * S_\infty)$, which exists by Lemma 2.3, and A is infinite since $\delta_p, \delta_q \in A$. Let $A_0 \subset A$ be the image of φ. Corollary 3.3 implies that $A_0 \cong \mathbb{C}S_\infty$, which is stably finite by Lemma 3.1. Moreover, $\varphi(\delta_{t_n}) = a_n \to \delta_p$ as $n \to \infty$, so that $\delta_p \in \overline{A_0}$, and we see also that $\delta_{t_n} = \frac{1}{\gamma_n}(a_n - \delta_p) \in \overline{A_0}$ $(n \in \mathbb{N})$. The elements δ_p and δ_{t_n} $(n \in \mathbb{N})$ generate A as a C*-algebra, and since $\overline{A_0}$ is a C*-subalgebra containing them, we must have $A = \overline{A_0}$, which completes the proof. □

Acknowledgements. We are grateful to Yemon Choi for several helpful discussions whilst working on this problem. We are also grateful to the referee for his/her useful comments, in particular those concerning the references for Theorem 2.2.

References

[1] D. Avitzour, Free products of C*-algebras, *Trans. Amer. Math. Soc.* **271** (1982), 423–435.

[2] B. A. Barnes, Representations of the ℓ^1-algebra of an inverse semigroup, *Trans. Amer. Math. Soc.* **218** (1976), 361–396.

[3] B. A. Barnes and J. Duncan, The Banach algebra $\ell^1(S)$, *J. Funct. Anal.* **18** (1975), 96–113.

[4] B. Blackadar, Notes on the structure of projections in simple C*-algebras (Semesterbericht Funktionalanalysis), Technical Report W82, Universität Tübingen, March 1983.

[5] B. Blackadar, *K-Theory for operator algebras.* Second edition. Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998.

[6] D. P. Blecher and V. I. Paulsen, Explicit construction of universal operator algebras and applications to polynomial factorization, *Proc. Amer. Math. Soc.* **112** (1991), 839–850.

[7] Y. Choi, Directly finite algebras of pseudofunctions on locally compact groups, *Glasgow Math. J.* **57** (2015), 693–707.

[8] N. P. Clarke, A finite but not stably finite C*-algebra, *Proc. Amer. Math. Soc.* **96** (1986), 85–88.

[9] M. Rørdam, A simple C*-algebra with a finite and an infinite projection, *Acta Math.* **191** (2003), 109–142.

[10] M. Rørdam, F. Larsen, and N. J. Laustsen, *An introduction to K-theory for C*-algebras*, London Mathematical Society Student Texts 49, Cambridge University Press, Cambridge, 2000.

Niels Jakob Laustsen/Jared T. White, Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, United Kingdom

E-mail address: n.laustsen@lancaster.ac.uk, j.white6@lancaster.ac.uk