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Robust Comparative Statics in Contests

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Abstract

We drive several robust comparative statics results in a contest under minimal restrictions on the primitives. Some of our findings generalize existing results, while others clarify the relevance of structure commonly imposed in the literature. Contrasting prior results, we show, via an example, that equilibrium payoffs may be (strictly) decreasing in the value of the prize. We also obtain a condition under which equilibrium aggregate activity decreases in the number of players. Finally, we shed light on equilibrium existence and uniqueness. Differentiating this study from past work is our reliance on lattice-theoretic techniques, which allows for a more general approach.

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JEL Classifications: C61, C72, D72

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1 Introduction

Contests are games in which players exert effort to increase their chances of winning a prize. Corchón (2007) and Konrad (2009) provide overviews of the many applications these games have throughout economics. In this paper, we study the properties of a symmetric contest under the commonly used ratio form contest success function (CSF).[1] We explore how equilibrium behavior depends on the parameters of the model, and examine existence/uniqueness of equilibrium. Differentiating our analysis from previous work, we invoke tools from lattice theory, which allows for a more general approach. To the best of our knowledge, this is the first study to apply lattice-theoretic techniques in contests.

Under the ratio-form CSF, player i's probability of winning is given by $p_i = \frac{\phi(e_i)}{R+\sum_j \phi(e_j)}$, where e_k is the effort of player k, $R \geq 0$, is the "discount rate", and the function, ϕ, is the "production function over lotteries". We refer to $\phi(e_i)$ as the "force" allocated by player i. Prior work in this model has relied on first order conditions and the Implicit Function Theorem to conduct comparative statics. A shortcoming of that approach is that it requires stringent assumptions on the returns to scale of the contest technology.[2] These assumptions are not just of technical convenience, but they impose economically relevant structure on the model, which may or may not be justified, or necessary for the results in question. This creates ambiguity in identifying the underlying economic mechanisms at play; our analysis helps to clarify this ambiguity.

As pointed out by Acemoglu and Jensen (2013), contests *cannot* be supermodular or submodular games. Nevertheless, we show that a player's payoff function is strictly submodular in own force and rivals' total force over a subset of the strategy space, and is strictly supermodular over a disjoint subset. This

[1] Also called the logit-form CSF.

[2] By "technology", we mean, jointly, the cost function, C, and the production function, ϕ. In the contest literature, the term "decreasing returns to scale" is typically used to mean that either ϕ is concave (with C linear) or C is convex (with ϕ linear). We use the term decreasing (increasing) returns to scale to indicate that $\kappa = C \circ \phi^{-1}$ is convex (concave).

allows us to establish a regularity condition on the shape of the best response correspondence, which holds generically under the ratio-form CSF, and has several implications. Our approach suggests a means for how lattice-theoretic techniques can be applied in contests, which should help guide future research in developing more robust conclusions.

We show that for arbitrary technologies, and a strictly positive discount rate, there exists at most one symmetric pure-strategy equilibrium. When the discount rate is zero, multiple symmetric equilibria may exist – at most two – but uniqueness is restored if payoff functions are differentiable and/or at least three players compete in the contest. Although the existence of a pure-strategy equilibrium cannot be guaranteed for arbitrary technologies, we generate a number of comparative statics results that hold in any symmetric pure-strategy equilibrium, should one exist. Specifically, we show that individual effort is decreasing in the number of players, while the total force of one's rivals is increasing in the number of players. Furthermore, we show that individual effort is decreasing in the discount rate, and increasing in the value of the prize. Finally, per-player payoffs are shown to be decreasing in the number of players and the discount rate.

The comparative statics results mentioned above extend findings in Nti (1997), which assumes decreasing returns to scale, and invokes the Implicit Function Theorem. Contrasting Nti's results, we show, via an example, that equilibrium payoffs may be strictly decreasing in the value of the prize when the contest technology exhibits increasing returns to scale. The logic is clear: An increase in the value of the prize leads to an increase in the efforts of one's rivals, which reduces a player's payoff, ceteris paribus. The impact of an increase in the prize then depends on whether the direct positive effect outweighs the increased competition effect, or vice-versa. With decreasing returns, the competition effect is dampened by the fact that higher levels of force are "produced" at greater marginal cost. With increasing returns, a contrasting logic applies, and the competition effect is amplified.

We then provide some additional structure on the contest that ensures existence of a pure-strategy equilibrium, and explore how aggregate activity varies with the number of players. For these results, we draw on an equivalence between the contest and a particular case of the Cournot oligopoly model; this equivalence is also noted by Szidarovszky and Okuguchi (1997). We exploit this fact in order to apply results from Amir (1996) and Amir and Lambson (2000; henceforth AL). When the contest technology exhibits decreasing or mildly increasing returns to scale, we show that there exists a unique pure-strategy equilibrium, and no asymmetric equilbria. Under this same condition, equilibrium total force is increasing in the number of players. These results are in line with typical findings in the literature (e.g. Nti, 1997; Jensen, 2016).

When the contest technology exhibits more pronounced increasing returns to scale, and the discount rate is strictly positive, the structure of equilbria may be radically different. For a contest with n players, there always exists an equilibrium in which $n-1$ players are inactive, and one player chooses the optimal single-player effort. Moreover, for any $m < n$, if a symmetric equilibrium exists for the m-player contest, then there exists an equilibrium in the n-player contest in which any m players behave according to the symmetric equilibrium, and the other $n-m$ players are inactive. If players' payoff functions are strictly quasiconcave in own force, then a unique symmetric equilibrium exists, and in this equilibrium, total force is decreasing in the number of players.

The fact that total force may be decreasing in the number of players contrasts much of the literature in contests. Baye et al. (1993) obtain a related result, which shows that in an asymmetric all-pay auction, total effort can be increased by removing the player with the highest value. In contrast to Baye et al.'s result, our finding applies in a symmetric contest. Also related, Amegashie (1999) shows that total effort may be decreasing in the number of players in a rent-seeking game in which a player's prize depends on her effort.

Existence/uniqueness of equilibria in contests under the ratio-form CSF is also explored by Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamakazi (2008), and Jensen (2016). With the exception of Cornes and Hartley, all of these studies consider asymmetric contests, and focus exclusively on the case of decreasing returns to scale.[3] Here, we consider a symmetric contest, but relax the assumption of decreasing returns to scale. The existence result we provide under decreasing/mildly increasing returns nests these other results, for the case where all players are symmetric.

The existence result we provide with more pronounced increasing returns to scale has not been discussed in the contest literature, as far as we are aware. Perez-Castrillo and Verdier (1992) and Cornes and Hartley allow for increasing returns under the "Tullock" CSF (Tullock, 1980), where $\phi(x) = ax^r$ with $r > 1$, and a discount rate equal to zero. Similar to our finding, these authors show that multiple asymmetric equilibria may exist in which a subset of players are inactive. Contrasting our results, the authors find that equilibrium total effort is increasing in the number of active participants.

The lattice-theoretic techniques we apply were developed by Topkis (1978) in the operations research literature, and introduced in economics by works such as Vives (1990), Milgrom and Shannon (1994) and Milgrom and Roberts (1994). Amir (2005) provides a thorough survey of supermodularity and its application to economics; for a more general approach, see Topkis (1998).

2 Model

We consider a contest in which n symmetric players compete for a single prize of common value, V. Each player, $i \in \{1, ..., n\}$, chooses an effort, $e_i \in \mathbb{R}_+$, which increases her chances of winning the contest. The cost of effort is given by the function, $C : \mathbb{R}_+ \to \mathbb{R}_+$. If player i chooses effort e_i, and the other $n-1$ players choose efforts according to vector $\mathbf{e}_{-i} \in \mathbb{R}_+^{n-1}$, the probability that i wins the contest is given by the ratio-form CSF:

[3] Jensen (2016) introduces a condition that depends on the curvature of both C and ϕ. It will be seen that when all players are symmetric, his condition amounts to convexity of $\kappa = C \circ \phi^{-1}$, which corresponds to our interpretation of decreasing returns to scale (see footnote 2).

$$P(e_i, \mathbf{e}_{-i}) = \frac{\phi(e_i)}{R + \sum_{j=1}^{n} \phi(e_j)},$$

where $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. In settings such as patent races or inducement contests it may happen that no player wins the contest; the discount rate, $R \geq 0$, captures this possibility (see, e.g. Loury, 1979). In the baseline model, we assume $R > 0$, but we will address the case where $R = 0$ in Section 3.3. The ratio form is frequently used in the contest literature (see Skaperdas, 1996, for an overview). When $\phi(x) = x^r$, and $R = 0$, the CSF corresponds to the popular Tullock CSF; if $r = 1$ then the function corresponds to the "lottery" CSF. The expected payoff to player i is,

$$\pi_i(e_i, \mathbf{e}_{-i}) = \frac{\phi(e_i)}{R + \sum_{j=1}^{n} \phi(e_j)} V - C(e_i).$$

Player i chooses e_i to maximize π_i, taking $\mathbf{e}_{-i}$ as given. We make the following assumptions:

Assumption 1.

(i) ϕ *is continuous and strictly increasing.*

(ii) C *is lower semi continuous and increasing.*

(iii) For all $\mathbf{e}_{-i} \in \mathbb{R}_+^{n-1}$, $\lim_{e \rightarrow \infty} \pi(e, \mathbf{e}_{-i}) < 0$

Assumptions 1(i)-(ii) imply that greater effort strictly increases a player's likelihood of winning the contest, but at a greater cost to the player; these assumptions are standard in models that utilize the ratio form CSF. We do not require continuity of C, which has relevant economic content, as lower semi continuity leaves open the possibility of (avoidable) fixed costs. Note, moreover, that we do not require that $\phi(0) = 0$ or $C(0) = 0$, which allows for the possibility of past sunk investments in effort. Assumption 1(iii) implies that we may, without further loss of generality, restrict attention to effort choices $e \in [0, \overline{e}]$ for some arbitrarily large $\overline{e} > 0$.

A Recasting

Let $x = \phi(e)$, $y = \sum_{j \neq i} \phi(e_j)$, and $z = x + y$ denote, respectively, the force allocated to the contest by some player i, the total force of all players other than i, and the total force. Let $\underline{x} = \phi(0)$, $\overline{x} = \phi(\overline{e})$, $\underline{y} = (n-1)\underline{x}$, and $\overline{y} = (n-1)\overline{x}$. We can then think of the players as choosing forces directly, rather than efforts. Since ϕ is strictly increasing and continuous, the inverse function, $\phi^{-1} : [\underline{x}, \overline{x}] \to [\underline{e}, \overline{e}]$, is well-defined, strictly increasing, and continuous. We let $\kappa = C \circ \phi^{-1}$, and re-write player i's payoff as follows:

$$\pi(x, y) = \frac{x}{R + x + y} V - \kappa(x). \tag{1}$$

Since the game is symmetric, we avoid the use of subscripts for ease of notation. For any $y \geq 0$ we let

$$X^*(y) = \arg\max \left\{ \pi(x, y) | x \in [\underline{x}, \overline{x}] \right\},$$

and let $r : \mathbb{R}_+ \to [\underline{x}, \overline{x}]$ denote an arbitrary single-valued selection from X^*. In our definition of X^* we allow y to take on any positive value, including those outside the feasible range of $[\underline{y}, \overline{y}]$. We can thus think of X^* as an extension of the best-response correspondence; but in a slight abuse of terminology, we refer to it simply as the best-response. Our assumptions on ϕ and C ensure that, for any fixed $y \geq 0$, $\pi(\cdot, y)$ is upper semi continuous; since the choice set is compact, $X^*(y)$ is non empty (and possibly set-valued).

3 Results

We divide our results into two sets. For our first set of results – in Section 3.1 – we impose no additional structure on the model. Although we cannot ensure existence of a pure-strategy equilibrium, we generate robust comparative statics results with regards to behavior in any symmetric equilibrium, should one exist. We follow this approach in order to avoid imposing structure on the model that is superfluous for the comparative statics results we provide. For our second set of results – in Section 3.2 – we provide conditions that ensure existence of a pure-strategy equilibrium, and study how aggregate behavior depends on the number of players. So that our results are consistent with many other models in the contest literature, we address the case where $R = 0$ in Section 3.3. Although many of our results continue to hold, the analysis changes slightly, and for ease of exposition, we address this case separately. All proofs are contained in the Appendix.

We focus on pure-strategy Nash equilibria; for brevity we simply write "equilibrium". We refer to ϕ and C jointly as the contest "technology". We say that the technology exhibits decreasing (increasing) returns to scale if κ is convex (concave). Throughout this paper, when we write "increasing" or "decreasing" we mean in the weak sense; otherwise we shall write "strictly increasing" or "strictly decreasing".

3.1 Robust Comparative Statics

In this section we explore how symmetric equilibrium behavior varies with the parameters of the model; namely, n, V, and R. A symmetric equilibrium satisfies $x^* \in X^*(y^*)$ and $x^* = \frac{y^*}{n-1}$. Note that since $X^*(y)$ is defined for all $y \in \mathbb{R}_+$, we need to ensure that the candidate equilibrium value of y corresponds to a feasible strategy profile for the other players. But since $x^* \in X^*(y^*)$ implies $x^* \in [\underline{x}, \overline{x}]$, clearly, $y^* = (n-1)x^*$ is feasible. We begin by establishing a general property of players' best-response correspondences. The following lemma is useful in doing so.

Lemma 1. *Let $x'' > x' \geq \underline{x}$ and $y'' > y' \geq 0$. If $x''x' < [>](y'' + R)(y' + R)$. Then,*

$$\pi(x'', y'') - \pi(x'', y') < [>]\pi(x', y'') - \pi(x', y'). \qquad (2)$$

It is well-known that no CSF can be supermodular or submodular in own and rivals' efforts over its entire domain. To illustrate, suppose $n = 2$ and let P denote the CSF. If the contest is decisive (i.e., one of the two players will win the contest),[4] then for all e_1 and e_2 it holds:

[4]Under the ratio form CSF, when $R > 0$, the contest is not decisive. Nevertheless, a

$$P(e_1, e_2) + P(e_2, e_1) \equiv 1$$

Suppose P is twice differentiable, then,

$$P_{12}(e_1, e_2) = -P_{12}(e_2, e_1).$$

It is easy to see that this implies whenever player 1's objective function is supermodular in (e_1, e_2), then player 2's objective function is submodular, and vice versa. Yet, Lemma 1 implies that, under the ratio form CSF, π is strictly submodular/supermodular over a subset of the strategy space. Specifically, π is strictly submodular in (x, y) on

$$\Phi_1 = \{(x, y) \in [\underline{x}, \overline{x}] \times \mathbb{R}_+ | x < y + R\}, \tag{3}$$

and π is strictly supermodular in (x, y) on

$$\Phi_2 = \{(x, y) \in [\underline{x}, \overline{x}] \times \mathbb{R}_+ | x > y + R\} \tag{4}$$

Using this fact, we establish the following regularity property of the best response correspondence.

Lemma 2. *Suppose Assumption 1 is satisfied. Let $y'' > y' \geq 0$, and let $x'' \in X^*(y'')$ and $x' \in X^*(y')$. If $x''x' < (y'' + R)(y' + R)$ then $x'' \leq x'$. If $x''x' > (y'' + R)(y' + R)$ then $x'' \geq x'$.*

Lemma 2 follows almost immediately by Topkis' Theorem (see, e.g., Theorem A.1 in AL), and it provides the key to our results in this section. It implies that every single-valued selection from X^* is decreasing (increasing) when it is fully contained in Φ_1 (Φ_2). As our next result shows, one implication of this property is that any symmetric equilibrium must be unique.

Proposition 1. *Suppose Assumption 1 is satisfied and $R > 0$. For a fixed $n \geq 2$, if a symmetric equilibrium exists, it must be unique.*

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similar logic applies.

Figure 1 illustrates a typical best-response correspondence. Lemma 2 implies that every single-valued selection from X^* must be decreasing (increasing) in the the region below (above) the line $x = y + R$. This does not preclude jumps in the best response back and forth between the two regions, but since a symmetric equilibrium satisfies $r(y^*) = \frac{y^*}{n-1} < y^* + R$, any symmetric equilibrium occurs in the region, Φ_1. Since the best response functions are all decreasing in this region, and the line $x = \frac{y}{n-1}$ is strictly increasing (in y), it is clear that there is, at most, one point of intersection between the two.

Lemma 2 is also useful for performing monotone comparative statics in our model. One can see from Figure 1 that an increase in the number of players rotates the line $x = \frac{y}{n-1}$ downwards. As a consequence, it is apparent that the individual force in any symmetric equilibrium must decrease, while the joint force of the other players must increase, following an increase in n. Our next result formalizes this result, and also shows how payoffs change in response to a change in n. In what follows, we let x_t denote the symmetric equilibrium force when the parameter of interest is equal to t. Similarly, we let y_t, z_t, and π_t denote equilibrium others' force, equilibrium total force, and the equilibrium per-player payoff, respectively.

Proposition 2. *Suppose Assumption 1 is satisfied. For a fixed $n \geq 2$, in a symmetric equilibrium, for $n'' > n'$,*

(i) The individual force (and effort) is decreasing in the number of players: $x_{n''} \leq x_{n'}$.

(ii) The other players' joint force is increasing in the number of players: $y_{n''} \geq y_{n'}$.

(iii) The expected per-player payoff is decreasing in the number of players: $\pi_{n''} \leq \pi_{n'}$.

Consistent with other results that invoke lattice-theoretic techniques, our findings only ensure weak monotonicity of individual forces/efforts and payoffs in n. The following example shows that strict monotonicity cannot be guaranteed in general.

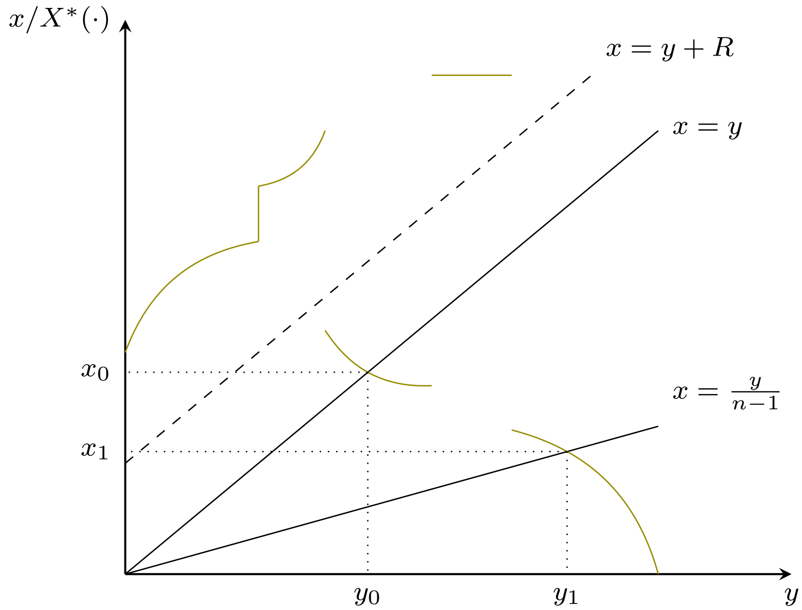

Figure 1: An example of a typical best-response correspondence (in green). Each selection from X^* must be increasing in the region above the line $x = y + R$, and decreasing in the region below this line.

Example 1. *Let $\phi(e) = e$, $V = 10$, $R = 0$,*[5] *and*

$$C(e) = \begin{cases} \sqrt{e} & e \leq 1 \\ (e+1)^2 - 3 & e \geq 1 \end{cases}$$

For each $n = 2, \ldots, 10$ there exists a unique symmetric equilibrium in which all players choose $e^* = x^* = 1$.

We next study how symmetric equilibrium behavior depends on the other parameters of the model, namely, the value of the prize, V, and the discount rate, R. Before doing so, we introduce a new parameter that may be of interest in some applications. Suppose that each player's cost function depends on a parameter, $\theta \in \mathbb{R}$. Assume that C is strictly submodular in (e, θ). That is, for all $e'' > e'$ and $\theta'' > \theta'$ assume:

[5] The main idea in this example would also hold for R strictly positive but sufficiently small.

$$C(e'', \theta'') - C(e', \theta'') < C(e'', \theta') - C(e', \theta')$$

If C is twice differentiable in e and θ, then the condition above is implied by $C_{12} < 0$ (equivalently, $\kappa_{12} < 0$). That is, an increase in θ strictly decreases the marginal cost of effort.

Proposition 3. *Fix $n \geq 2$ and suppose Assumption 1 is satisfied. Then in a symmetric equilibrium,*

(i) Individual and total forces/efforts are increasing in the value of the prize: For $V'' > V'$, $x_{V''} \geq x_{V'}$ and $z_{V''} \geq z_{V'}$.

(ii) Individual and total forces/efforts are increasing in the cost parameter: For $\theta'' > \theta'$, $x_{\theta''} \geq x_{\theta'}$ and $z_{\theta''} \geq z_{\theta'}$.

(iii) Individual and total forces/efforts are decreasing in the discount rate: For $R'' > R'$, $x_{R''} \leq x_{R'}$ and $z_{R''} \leq z_{R'}$.

Parts (i) and (ii) of Proposition 3 follow from the fact that π is strictly supermodular in (x, V) and (x, θ). By Topkis' Theorem, an increase in either of these parameters shifts a player's best response upward, and leads to an increase in individual force/effort in any symmetric equilibrium. Figure 2 illustrates: Following an increase in V or θ, the player's best response shifts up from the green to the blue curve, and the symmetric equilibrium increases from the point E_0 to the point E_1.

To understand part (iii) of Proposition 3, first note that π depends on R and y insofar as it depends on the sum, $y + R$. Adapting Lemmas 1 and 2, it is straightforward to show that, for any fixed $y \geq 0$, π is strictly submodular in (x, R) on Φ_1, and that any selection, $r(y)$, is decreasing in R whenever it is contained in Φ_1. So, suppose R increases from R' to R''. When a player's best response correspondence is contained in Φ_1 (for both parameter values), this portion of the best response must shift down, following the increase in the parameter. Figure 3 illustrates; the green curve is the best response when $R = R'$, and the blue curve is the best response when $R = R'' > R'$. In

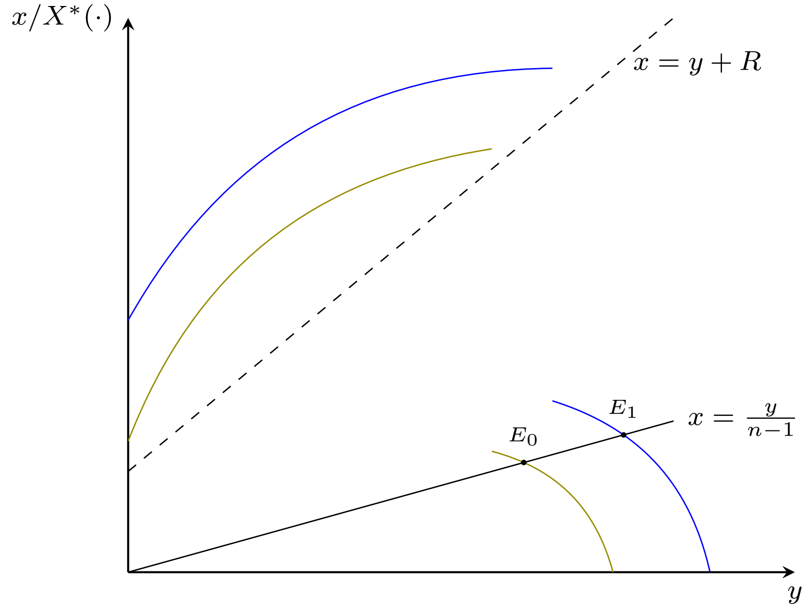

Figure 2: An illustration of the impact of an increase in V or θ on the best response correspondence. The green curve represents the best response before the parameter increase; the blue curve represents the best response after the parameter increase. Following the parameter change, the symmetric equilibrium increases from E_0 to E_1.

the region between the lines, $x = y + R'$ and $x = y + R''$, the relationship between the two best-response functions is difficult to ascertain in general. In the region below the line $x = y + R'$, there is a clear ordering between the two, with the best-response shifting down following the increase in the parameter. As any symmetric equilibrium occurs below this line, we can conclude that the symmetric equilibrium force decreases.

We now explore the relationship between the parameters of the model, and equilibrium payoffs. When the contest technology exhibits decreasing returns to scale, Nti shows that equilibrium payoffs increase in V. The next example shows that this result does not hold in general. Indeed, equilibrium per-player payoffs may be strictly decreasing in the prize or the cost parameter, θ.

Example 2. *Suppose* $n = 2$, $\phi(e) = e$, $R = 0$,[6] *and*

[6]The main idea in this example would also hold for R strictly positive but sufficiently small.

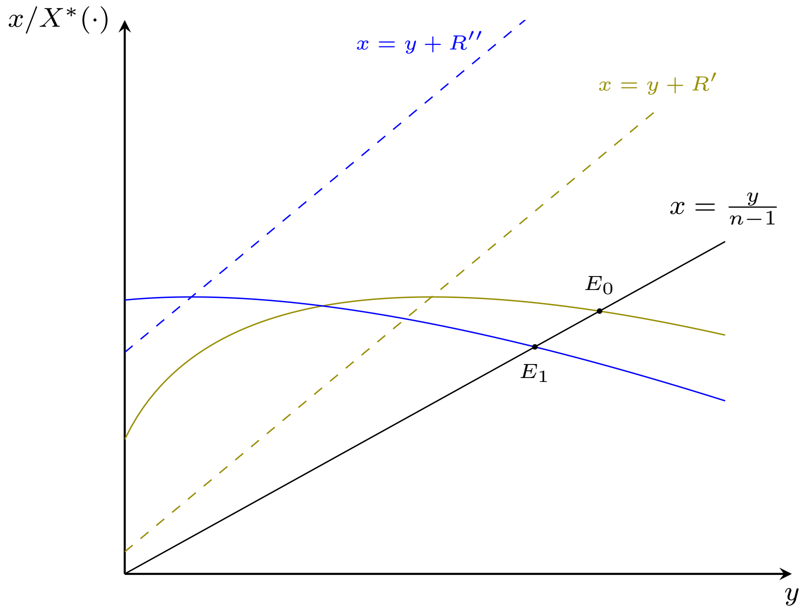

Figure 3: An illustration of the impact of a change in R on the best response. The green curve is the best response when $R = R'$; the blue curve is the best response when $R = R'' > R'$. Following an increase in R, the equilibrium decreases from E_0 to E_1.

$$C(e, \theta) = \begin{cases} \frac{e}{\theta} & e \leq 1 \\ \frac{e^\alpha}{\alpha\theta} + \frac{\alpha-1}{\alpha\theta} & e \geq 1 \end{cases}$$

If $1 \leq \frac{V\theta}{4} \leq \frac{1-\alpha}{1-2\alpha}$, *the symmetric equilibrium per-player effort/force is*

$$e^* = x^* = \left(\frac{V\theta}{4}\right)^{\frac{1}{\alpha}}.$$

The equilibrium per-player payoff is

$$\pi^* = \frac{1-\alpha}{\alpha\theta} - V\left(\frac{1-2\alpha}{4\alpha}\right).$$

For $0 < \alpha < \frac{1}{2}$, *it is clear that* π^* *is is strictly decreasing in* V *for fixed* θ, *and strictly decreasing in* θ *for fixed* V. *For instance, if* $\alpha = \frac{1}{3}$ *and* $\theta = 1$ *then for* $4 \leq V \leq 8$, $x^* = \left(\frac{V}{4}\right)^3$, *and* $\pi^* = 2 - \frac{V}{4}$. *If* $\alpha = \frac{1}{3}$ *and* $V = 1$ *then for* $4 \leq \theta \leq 8$, $e^* = x^* = \left(\frac{\theta}{4}\right)^3$ *and* $\pi^* = \frac{2}{\theta} - \frac{1}{4}$.

There are two competing forces acting on a player's equilibrium payoff following an increase in the prize. First, is a direct positive effect, since the value of winning the contest increases. Second, there is an indirect negative effect, resulting from an increase in the efforts of one's rivals. A priori, it is unclear which effect dominates. Nti shows that the direct positive effect outweighs the indirect negative effect when the contest technology exhibits decreasing returns to scale. For such technologies, the indirect effect is muted since higher levels of force are produced at greater marginal cost. For contest technologies with increasing returns to scale, a contrasting logic applies: Players' optimal efforts are more sensitive to changes in the prize, and the indirect effect is exacerbated. This effect is evident in Example 2, as $\frac{\partial^2 x^*}{\partial \alpha \partial V} < 0$. That is, a decrease in α, which leads to more pronounced increasing returns to scale, implies that x^* is more sensitive to changes in the prize. As the example shows, with increasing returns it may well be that the indirect effect dominates the direct effect. A similar intuition holds for changes in θ; as a result, the relationship between equilibrium per-player payoffs and V/θ is ambiguous in general. But as our next result shows, there is a clear relationship between the discount rate, R, and equilibrium payoffs.

Proposition 4. *Suppose Assumption 1 is satisfied. In a symmetric equilibrium, the per-player expected payoff is decreasing in the discount rate: For* $R'' > R'$, $\pi_{R'} \geq \pi_{R''}$.

As with an increase in V or θ, there are two competing forces acting on equilibrium payoffs following a decrease in R. There is a direct positive effect, since a decrease in R increases the likelihood of a player winning the prize. But there is an indirect negative effect caused by the increase in the efforts of one's rivals (see Proposition 3(iii)). As it happens, the positive effect always outweighs the negative effect. Indeed, the proof of Proposition 4 shows that, following a decrease in R, although the efforts of one's rivals increase, the sum $y + R$ decreases.

To understand why this result must hold in general, recall from the discussion following Proposition 3(iii) that X^* depends on y and R insofar as it depends on $y + R$. Following a decrease in R from R'' to R', a player's best-response shifts outward by a horizontal distance of $R'' - R'$. Figure 4 illustrates: The green curve is (a portion of) the best response when $R = R''$, and the blue curve is the best response when $R = R' < R''$. When the parameter is R'', we see that equilibrium others' total force is y''. At the point $\hat{y} > y''$, it holds: $\hat{y} + R' = y'' + R''$. Since x'' is a best response to y'' when $R = R''$, then x'' must also be a best response to $\hat{y}$ when $R = R'$. Since any selection from the best response (when $R = R'$) is decreasing in y below the blue dashed line, any point of intersection between the blue best response curve and the line $x = \frac{y}{n-1}$ must occur at a value $y' \in [y'', \hat{y}]$. This means $y' + R' \leq y'' + R''$. Using this fact, it is straightforward to show that a player's equilibrium payoff must increase following a decrease in R.[7]

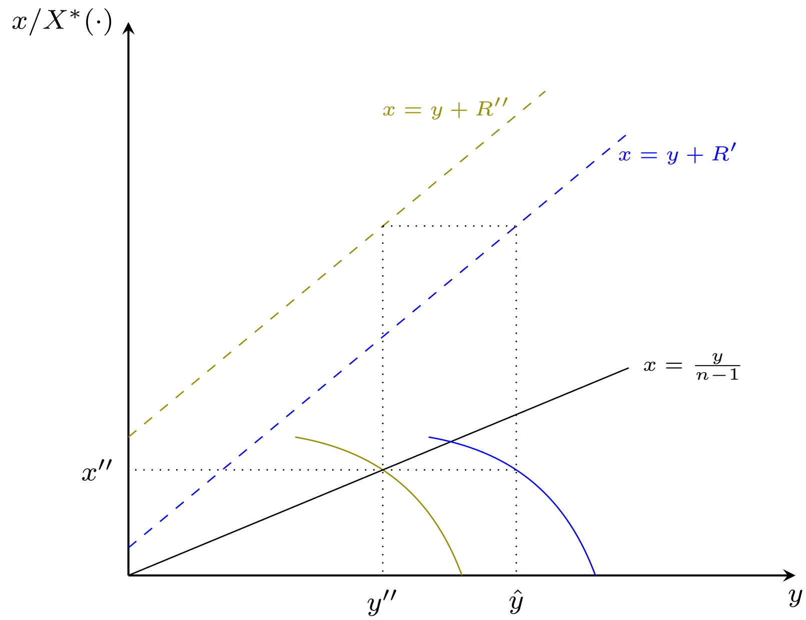

Figure 4: An illustration of the impact of a change in R on (a portion of) the best response correspondence. The green curve is the best-response when $R = R''$; the blue curve is the best response when $R = R' < R''$.

[7] Figure 4 illustrates a scenario in which $y + R$ strictly decreases in equilibrium following the decrease in R. But this need not be the case: Lemma 2 does not preclude the possibility that $\hat{x} = \frac{\hat{y}}{n-1}$ is also a best response to $\hat{y}$ when $R = R'$ (although this would not be possible in the way we've drawn the figure).

Summarizing

The comparative statics results provided in Propositions 2 - 4 generalize results in Nti (1997), which assumes decreasing returns to scale, and utilizes first-order conditions. We have shown that neither decreasing returns to scale nor differentiability are essential drivers of these results. Rather, it is the structure implied by the ratio form CSF itself that drives the generic properties of the best response correspondence, which we established in Lemma 2. At the same time, Example 2 shows that the assumption of decreasing returns to scale *is* an important driver of the typical finding that equilibrium payoffs increase in the value of the prize.

3.2 Equilibrium Existence and Effects of Entry

We next turn to the question of existence, and study how aggregate behavior varies with the number of players. Our results in this section make use of the fact that the contest we study is equivalent to a symmetric Cournot oligopoly with inverse demand, $\frac{V}{R+Q}$, and cost function κ. Noting this equivalence, the next results are consequences of our Proposition 1, and results in AL. We strengthen Assumption 1 as follows:

Assumption 2.

(i) For all $e > 0$, $\phi(e)$ is twice continuously differentiable with $\phi'(e) > 0$. Moreover, $\phi(0) = 0$.

(ii) For all $e \geq 0$, $C(e)$ is twice continuously differentiable with $C'(e) \geq 0$.

We do not wish to rule out the widely-used Tullock CSF, which is not differentiable at zero when $r < 1$. For this reason, we do not require $\phi(0)$ to be differentiable; Assumption 2 is otherwise consistent with assumptions in AL.

Noting that $z = x + y$, a player's payoff can be expressed as,

$$\tilde{\pi}(z, y) = \frac{z - y}{R + z}V - \kappa(z - y). \tag{5}$$

We can then think of some player i as choosing the total force, taking the total force of the other players, as given. That is, player i solves,

$$\max\{\tilde{\pi}(z,y)|\overline{x}+y \geq z \geq y\} \tag{6}$$

We let $Z^*(y)$ denote the argmax in (6), and let r_z denote an arbitrary single-valued selection from Z^*. Note that Assumption 2 ensures that $\kappa(x)$ is twice continuously differentiable for all $x > 0$. For all $z > y$, let $\Delta(z, y)$ denote the cross partial of $\tilde{\pi}$ with respect to z and y:

$$\Delta(z,y) = \frac{V}{(R+z)^2} + \kappa''(z-y).$$

Note that Δ is defined on the lattice,

$$\Phi = \{(z,y) \in \mathbb{R}_+ |(n-1)\overline{x} \geq y \geq 0, y + \overline{x} \geq z > y\}.$$

As in AL, the sign of Δ on Φ plays a critical role in our analysis. When $\Delta > 0$ on Φ, $\tilde{\pi}$ is strictly supermodular in (z, y), which implies that any selection from Z^* is increasing in y. Conversely, $\Delta < 0$ implies that $\tilde{\pi}$ is strictly submodular in (z, y), which implies that any selection from Z^* is decreasing in y. First we consider the case where $\Delta > 0$ on Φ.

Proposition 5. *In addition to Assumptions 1(iii) and 2, suppose $\Delta > 0$ on Φ. Then,*

(i) For each $n \in \mathbb{N}$, there exists a unique symmetric equilibrium, and no asymmetric equilibria.

(ii) The equilibrium total force is increasing in n: For $n'' > n'$, $z_{n''} \geq z_{n'}$.

Proposition 5 provides conditions under which the existence of a unique symmetric equilibrium can be guaranteed, and also shows that equilibrium total force is increasing in the number of players. The existence result follows from AL, while uniqueness follows by Proposition 1. We point out that Proposition 5(ii) does not, in general, imply that equilibrium total effort is increasing in the number of players.[8] Yet in some situations, comparisons in total force are more meaningful than comparisons in total effort. Consider, for example, a contest designer interested in encouraging the development of a new technology. What is relevant is the probability with which at least one player succeeds. This probability, $\frac{z}{R+z}$, depends only on the equilibrium total force.

The key for Proposition 5 is the condition, $\Delta > 0$, which limits the returns to scale of the contest technology, but also depends on the magnitudes of the discount rate and the prize. Jensen (2016) provides a result similar to Proposition 5 in an asymmetric contest under a different sufficient condition. In the special case where all players are symmetric, Jensen's Assumption 3 amounts to,

$$\frac{C''}{C'} \geq \frac{\phi''}{\phi'},$$

which is equivalent to convexity of κ. The condition, $\Delta > 0$, generalizes Jensen's result, for the case of a symmetric contest. To demonstrate this generalization, note that $\Delta > 0$ if κ is convex, but as our next example shows, Δ may be strictly positive, even if κ is strictly concave everywhere.

Example 3. *Suppose $V = 1$, $R = .01$, $\phi(e) = e$, and*

$$\kappa(x) = C(x) = x - \frac{1}{2(x+2)} + \frac{1}{4}$$

Note that $\kappa'' < 0$ and,

$$\Delta(z, y) = \frac{1}{(z + .01)^2} - \frac{1}{(z - y + 2)^3}$$

It may be verified that for all y, $\tilde{\pi}(z, y) < 0$ for $z \geq 1$. So we may, without loss of generality, restrict attention to $z \leq 1$. For all $y \leq z \leq 1$, it may be verified that $\Delta > 0$, and thus Proposition 5 applies immediately.

We now consider the case where $\Delta < 0$. Our next two results are consequences of AL's Theorems 2.5 and 2.6, respectively.

[8] See Table 1 in Nti (1997).

Proposition 6. *In addition to Assumptions 1 and 2, suppose* $\Delta < 0$ *on* Φ. *Then, for any* $n \in \mathbb{N}$,

(i) For any $m < n$, *if a symmetric equilibrium exists in the* m*-player contest (with individual force* x_m, *say) then the following configuration constitutes an equilibrium for the* n *player contest: Each of any* m *players chooses force* x_m *while the remaining* $n - m$ *players exert zero effort. In particular, an* n*-player equilibrium always exists in which one player chooses the optimal single-player effort and the other* $n-1$ *players choose zero effort.*

(ii) A unique symmetric equilibrium exists if, for each $y \in [0, \overline{y}]$, $\pi(\cdot, y)$ *is strictly quasiconcave.*

(iii) No other equilibrium other than those described in parts (i) and (ii) can exist.

When $\Delta < 0$ over its domain, several asymmetric equilibria may exist; moreover, there always exists an equilibrium in which $n-1$ players are inactive. An equilibrium with a single active player is not standard in contests. Note that the condition, $\Delta < 0$ requires that $R > 0$. It should be clear that an equilibrium with only one active player could never exist if $R = 0$. Note moreover, that $\Delta < 0$ requires that the contest technology exhibits sufficiently strong increasing returns to scale. The structure of equilibria is reminiscent of the structure found by Perez-Castrillo and Verdier (1992) and Cornes and Hartley (2005) under the Tullock CSF with increasing returns, and a discount rate equal to zero.[9] Although both of our results are driven by increasing returns, neither is a special case of the other. This is most clearly illustrated by noting that our result requires $R > 0$, while these other findings assume $R = 0$.

Our next result shows how total equilibrium forces vary with the number of players when $\Delta < 0$.

[9] Chowdhury and Sheremeta (2011) also show the potential for multiple asymmetric equilibria under the lottery CSF when a player's prize depends on her own effort, and the effort of her rival. Here, we include no such spillovers.

Proposition 7. *In addition to Assumptions 1 and 2, suppose $\Delta(z, y) < 0$ on Φ. Then,*

(i) Under the hypothesis of Proposition 6(i), all the asymmetric equilibria for all $m < n$ are invariant in the number of players n.

(ii) Under the hypothesis of Proposition 6(ii), the total equilibrium force, z_n, is decreasing in n.

Proposition 7(i) clearly holds since, in the asymmetric equilibria with $m < n$, an additional player exerts no effort. Proposition 7(ii) contrasts much of the work in contests under the ratio form CSF.[10] Before discussing, we provide an example illustrating this result.

Example 4. *Let $\phi(x) = x$, $R = 3$, $V = 10$, and*

$$C(x) = \kappa(x) = 5\ln(x+2) - \frac{5}{x+3} - 5\ln(2) + \frac{5}{3}.$$

Note that

$$\Delta(z, y) = \frac{10}{(z+3)^2} - \frac{5}{(z-y+2)^2} - \frac{10}{(z-y+3)^3}$$

It may be verified that for all y, $\tilde{\pi}(z, y) < 0$ for any $z \geq 1.4$. Then, without loss of generality, we can focus on the sign of Δ when $z \leq 1.4$. It may be verified that for all $y \leq z \leq 1.4$, $\Delta < 0$. There is a symmetric equilibrium with:

- $n = 1$: $x_1 = z_1 \approx .618$
- $n = 2$: $x_2 \approx .2068$, $z_2 \approx .4137$
- $n = 3$: $x_3 \approx .1188$, $z_3 \approx .3565$
- $n = 4$: $x_4 \approx .083$, $z_4 \approx .332$

[10] One exception is Amegashie (1999). Under the lottery CSF, Amegashie shows that total effort may be decreasing in n, when a player's prize includes a fixed component, and a variable component, which increases linearly in a player's effort.

- *$n = 5$: $x_5 \approx .0637$, $z_5 \approx .3186$*

In addition to the symmetric equilibrium, there are multiple asymmetric equilibria, where, for any n, one player chooses x_1, and the others exert no effort, another where 2 players choose x_2 and the others choose zero, etc.

When $\Delta < 0$, it can be shown that a player's best response, X^*, is fully contained in Φ_1 for all y. That is, R is sufficiently large, relative to the prize, that for any $y \geq 0$, $x \in X^*(y)$ implies $x < y + R$. By Lemma 2 this means that each selection $r(\cdot)$ from X^* is decreasing for all y. In fact, it can be shown that the best response functions are all *strictly* decreasing (when interior), and that an increase in y leads a player to reduce her force by so much that the total force decreases. To understand why, recall that $\Delta < 0$ implies that any selection, r_z from Z^* is decreasing. For any y it holds $r_z(y) = r(y) + y$, where r is some selection from X^*. Since r_z is decreasing, this implies that the slope of r is bounded from above by -1. Then, in a symmetric equilibrium, we know by Proposition 2 that an increase in the number of players leads to an increase in the total force of one's rivals. Since total force is decreasing in y, it follows that an increase in n leads to a decrease in symmetric equilibrium total force.

Proposition 7 implies that, when $\Delta < 0$, a contest designer could maximize total force by excluding all but one player from the contest. It is worth relating this finding to the well-known Exclusion Principal of Baye et al. (1993). Baye et al. show that in an asymmetric contest under the all-pay auction CSF,[11] a contest designer can increase total effort by excluding the player with the highest value. Intuitively, there is a "discouragement effect" whereby the presence of a high-value player discourages other players from exerting effort. Removing the high-value player "levels the playing field", and encourages greater effort among the remaining players (enough to offset the effort of the removed player). The choice of CSF plays a critical role in driving the Exclusion Principle. For instance, when marginal cost is constant, it is well-known that the result does not hold under the Tullock CSF.[12] Intuitively, the Tullock

[11] Under the all-pay auction CSF, a player wins the contest with certainty if her effort is greater than every other player's effort.

[12] See, e.g., Fang (2002); Matros (2006) and Menicucci (2006).

CSF induces a softer form of competition than the all-pay auction CSF, and the discouragement effect is less pronounced.[13] In contrast to the Exclusion Principal, our result applies in a symmetric contest. Rather than being driven by a leveling-the-playing-field idea, our result is driven by increasing returns to scale in the production of effort. Reminiscent of a natural monopoly in industrial organization, restricting the contest to a single player allows this player to take full advantage of her increasing returns to scale.

3.3 The case, R = 0

Since much of the work in the contest literature utilizing the ratio form CSF assumes $R = 0$, in this section we explicitly address this case. Note that if $R = 0$ and $\phi(0) = 0$ then the ratio-form CSF is not well defined when $e_i = 0$ for all i. We follow much of the literature and assume that if all players exert zero effort, then each player wins with probability $\frac{1}{n}$. Although most of our results carry over to the case where $R = 0$, the uniqueness of a symmetric equilibrium cannot be guaranteed. The following example illustrates:

Example 5. *Let $\phi(e) = e$, $n = 2$, $V = 5$, $R = 0$, and*

$$C(e) = \begin{cases} e & e \leq \frac{1}{2} \\ 2 + \frac{4(e-2)^3}{9} & \frac{1}{2} \leq e \leq 2 \\ e & e \geq 2 \end{cases}$$

There are two symmetric equilibria; one equilibrium in which $e^* = x^* = \frac{1}{2}$, and another in which $e^* = x^* = 2$.

Example 5 demonstrates that when $R = 0$, multiple symmetric equilibria may exist. Figure 5 illustrates the best-response correspondence for a player in this example. There are a couple features worth mentioning. First, is the fact that there are exactly two players. When $n = 2$, the lines $x = y + R$

[13] Matros and Rietzke (2017) show that an exclusion result along the lines of Baye et al. can be obtained under the Tullock CSF and constant marginal cost if one considers a richer structure of interactions.

and $x = \frac{y}{n-1}$ coincide. The best response functions are therefore increasing in the region above the line, $x = \frac{y}{n-1} = y$, and decreasing below this line. Thus, $R = 0$ and $n = 2$ represents a knife-edge case in which the best-response functions are not decreasing in the relevant region where symmetric equilibria occur, as is the case when $R > 0$ or $n > 2$. The second critical feature in this example is the fact that κ is not everywhere differentiable. Our next result highlights the relevance of these features.

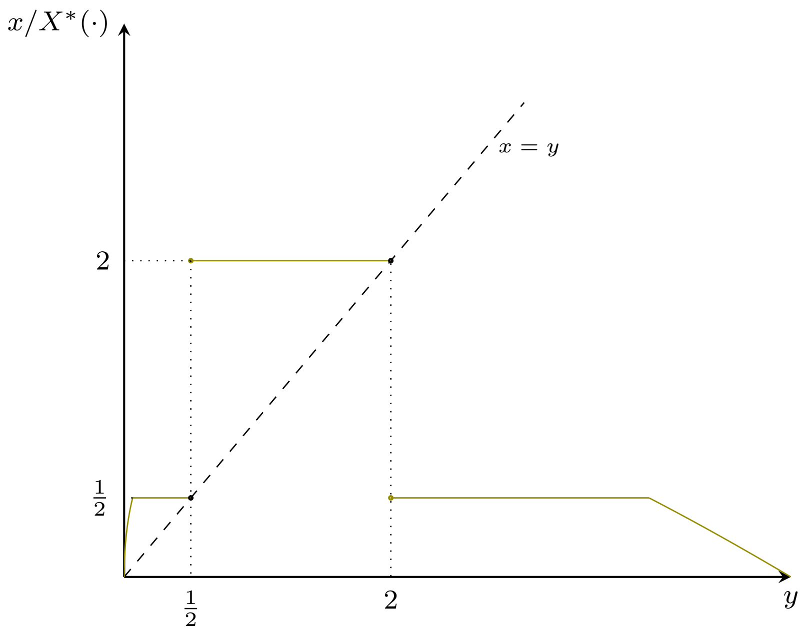

Figure 5: The best-response correspondence (in green) for a player in Example 5.

Proposition 8. *Suppose Assumption 1 is satisfied and $R = 0$. Then there exist at most two symmetric equilibria. Moreover, if $n \geq 3$ or κ is differentiable, then there exists at most one symmetric equilibrium.*

Proposition 8 shows that when $R = 0$, the contest possesses as most two distinct symmetric equilibria, but if $n > 2$ or κ is differentiable, then any symmetric equilibrium is unique. To understand this result, first note that for $n > 2$, the line $x = \frac{y}{n-1}$ lies in the region Φ_1. In this case, selections from the best response are decreasing in the relevant region where any symmetric equilibrium occurs, and uniqueness of symmetric equilibrium is restored. When κ

is differentiable, it can be shown that every selection from the best response is strictly decreasing when it is contained in Φ_1 and strictly increasing when contained in Φ_2. Clearly, this rules out a best response along the lines of Figure 5. More generally, it can be shown that, whenever multiple symmetric equilibria exist when $n = 2$, the best response *must* be constant over some interval in Φ_1; differentiability rules out this possibility.

Our final result shows that, although the uniqueness of a symmetric equilibrium cannot be guaranteed when $R = 0$, most of our other comparative statics results continue to hold.

Proposition 9. *In the model with $R = 0$, the statements of Lemmas 1-2, and Proposition 5 hold. Moreover, the statements of Propositions 2-4 hold, when one replaces "in a symmetric equilibrium" with "in the smallest and largest symmetric equilibria".*

As discussed in Section 3.2, $R > 0$ is a necessary condition for $\Delta < 0$ on Φ; Propositions 6-7 are therefore excluded from the statement of Proposition 9. Aside from these results, our other comparative statics findings carry over to the case where $R = 0$.

Tullock CSF

It is worth relating our findings in Section 3.2 to the commonly used Tullock CSF, where $\phi(e) = e^r$, and $R = 0$. For this CSF, it holds $\kappa(x) = x^{\frac{1}{r}}$, and Δ becomes:

$$\Delta(z, y) = \frac{V}{z^2} + \frac{1}{r}\left(\frac{1}{r} - 1\right)(z - y)^{\frac{1}{r}-2}$$

It can be shown that $\Delta > 0$ on its domain if and only if $r \leq 1$. For $r \leq 1$, Propositions 5/9 then imply that there exists a unique symmetric equilibrium, and no asymmetric equilibria, for *any* n. Perez-Castrillo and Verdier (1992) show that for any $r > 1$, asymmetric equilibria exist when n is sufficiently large (specifically, $n > \frac{r}{r-1}$). Thus, although $\Delta > 0$ is, in general, only a sufficient condition for the existence/non-existence result in Proposition 5, for the case of the Tullock CSF, it is also necessary.

4 Conclusion

In this paper, we established a number of robust comparative statics results in a contest under the ratio-form CSF with minimal restrictions on the primitives. We furthermore shed new light on the issues of equilibrium existence and uniqueness. Our results help to clarify the relevance of the structure typically imposed in this model. The main innovation of this paper is the application of lattice-theoretic techniques, which have not previously been applied in contests. Utilizing these tools, we established a strong regularity condition on the shape of a player's best response correspondence, out of which our comparative statics results follow. Our approach sheds light on how lattice-theoretic techniques can be applied in contests, which should help guide future research in developing more robust conclusions.

5 Appendix

Proof of Lemma 1

Let $x'' > x' \geq \underline{x}$ and $y'' > y' \geq 0$. Let $\tilde{y}'' = y'' + R$ and $\tilde{y}' = y' + R$. The reader can easily verify that expression (2) is equivalent to

$$\frac{x''}{x'' + \tilde{y}''} - \frac{x''}{x'' + \tilde{y}'} < [>] \frac{x'}{x' + \tilde{y}''} - \frac{x'}{x' + \tilde{y}'},$$

which holds if and only if $x''x' < [>] \tilde{y}''\tilde{y}'$. $\square$

Proof of Lemma 2

We will show the first statement in the lemma; the proof of the second statement is analogous. Let $y'' > y' \geq 0$, $x'' \in X^*(y'')$ and $x' \in X^*(y')$. Suppose $x''x' < (y'' + R)(y' + R)$. Proceed by contradiction: Suppose, contrary to the lemma, $x'' > x'$. Then by Lemma 1,

$$0 \leq \pi(x'', y'') - \pi(x', y'') < \pi(x'', y') - \pi(x', y') \leq 0,$$

where the l.h.s. inequality follows since $x'' \in X^*(y'')$, and the r.h.s. inequality follows since $x' \in X^*(y')$.[14] We have a contradiction; hence it must be that $x'' \le x'$. $\square$

Proof of Proposition 1

Fix $n \ge 2$. We show that if a symmetric equilibrium exists, it must be unique. Let x_1 and x_2 be two symmetric equilibrium levels of force, and let let $y_i = (n-1)x_i$, $i = 1, 2$. Proceed by contradiction, and suppose that these two equilibria are distinct; in particular, suppose $x_1 > x_2$; equivalently, $y_1 > y_2$. Note that for $i = 1, 2$, x_i and y_i satisfy: $x_i \in X^*(y_i)$ and $x_i = \frac{y_i}{n-1}$. But since $\frac{y_i}{n-1} \le y_i$, it must hold that $x_i < y_i + R$ for $i = 1, 2$; hence, $x_1 x_2 < (y_1 + R)(y_2 + R)$. Since $y_1 > y_2$ (by assumption), Lemma 2 implies $x_1 \le x_2$, which yields a contradiction. Therefore, the symmetric equilibrium must be unique. $\square$

Proof of Proposition 2

Parts (i)-(ii)

Fix $n'' > n'$, and suppose that a symmetric pure-strategy equilibrium exists in the n'' and n' player contests. From the arguments in the first part of this proof, we know that the symmetric equilibrium must be unique. Let x'' (x') denote the individual equilibrium individual force when the contest has n'' (respectively, n') players; let $y'' = (n'' - 1)x''$ and $y' = (n' - 1)x'$.

We first show part (ii). First note that if $y' \le (n'' - 1)\underline{x}$, then as feasibility requires $y'' \ge (n'' - 1)\underline{x}$, it follows that $y'' \ge y'$. Then suppose $y' > (n'' - 1)\underline{x}$. We will show that there cannot be a symmetric equilibrium with $y < y'$ when $n = n''$. Fix $y_0 \in [(n'' - 1)\underline{x}, y')$, and let $x_0 \in X^*(y_0)$. Clearly if $x_0 \ge y_0 + R$ then $x_0 > \frac{y_0}{n''-1}$. If $x_0 < y_0 + R$, then Lemma 2 implies $x_0 \ge x'$. But since $n'' > n'$ and $y' > y_0$ it holds, $x' = \frac{y'}{n'-1} > \frac{y'}{n''-1} > \frac{y_0}{n''-1}$; thus, $x_0 > \frac{y_0}{n''-1}$. We have now established that for all $y \in [(n'' - 1)\underline{x}, y')$, $x \in X^*(y)$ implies $x > \frac{y}{n''-1}$; thus, there cannot be a symmetric equilibrium with $y < y'$ when $n = n''$. It follows that $y'' \ge y'$. This establishes part (ii).

[14] The choice set, $[\underline{x}, \overline{x}]$, is independent of y, so x' (x'') is certainly feasible when others' joint force is y'' (y').

Next, we show $x'' \leq x'$. We have already shown that $y'' \geq y'$. If $y'' = y'$ then $n'' > n'$, implies $x'' = \frac{y''}{n''-1} < \frac{y'}{n'-1} = x'$. If $y'' > y'$ then since $x'' < y'' + R$ and $x' < y' + R$, Lemma 2 implies $x'' \leq x'$. Since individual force decreases following the increase in n, clearly individual effort decreases as well. This establishes part (i).

Part (iii)

Let π'' (π') denote the equilibrium individual expected payoff in the contest with n'' (respectively, n') players where $n'' > n'$. We have the following string of inequalities:

$$\begin{aligned}
\pi' &= \frac{x'}{R + x' + y'} V - \kappa(x') \\
&\geq \frac{x''}{R + x' + y'} V - \kappa(x'') \\
&\geq \frac{x''}{R + x'' + y''} V - \kappa(x'') \\
&= \pi''
\end{aligned}$$

The first inequality holds by definition of x'. The second inequality follows since, as we showed in part (ii), $y'' \geq y'$. This establishes part (iii) and the proposition. $\square$

Proof of Proposition 3

We prove parts (i) and (ii) jointly. Fix n, and let t denote either the parameter V or θ. It is easily verified that π is strictly supermodular in t and x. By Topkis' Theorem (see, e.g. Topkis, 1978, or Theorem A.1 in AL), it follows that any selection, $r(\cdot)$, from X^* is increasing in t for each y. It is clear that this implies that any intersection between a player's reaction curve and the line $x = \frac{y}{n-1}$ must lie further from the origin following an increase in t. Since individual force increases, then individual effort and total force/effort must increase, as n is fixed. This proves parts (i) and (iii).

Next, we show part (iii). Let $R'' > R'$, and suppose a symmetric equilibrium exists for both parameter values. Let x'' and x' denote the symmetric equilibrium individual force when the parameter is R'', respectively R'. Let $y'' = (n-1)x''$ and $y' = (n-1)x'$. Since π depends on y and R insofar as it depends on the sum, $y+R$, X^* depends only on this sum. We write $X^*(y+R)$ to denote the set of best-replies to y when the parameter is R. Note that our result follows immediately if $x'' = \underline{x}$; so, assume $x'' > \underline{x}$; equivalently, $y'' > \underline{y}$.

Let $\hat{y} = R'' - R' + y'' > y''$. By construction, $\hat{y} + R' = y'' + R''$; therefore, $X^*(\hat{y}+R') = X^*(y''+R'')$. By definition of x'', this means $x'' \in X^*(\hat{y}+R')$. We now show that there cannot be a symmetric equilibrium in which $y < y''$ when $R = R'$. Fix $y_0 \in [\underline{y}, y'')$, and let $x_0 \in X^*(y_0 + R')$. Clearly, if $x_0 > y_0 + R'$ then $x_0 > \frac{y_0}{n-1}$. If $x_0 \leq y_0 + R'$, then since $x'' \leq y'' < \hat{y} + R'$, it holds $x_0 x'' < (y_0 + R')(\hat{y} + R')$. As, $x_0 \in X^*(y_0 + R')$, $x'' \in X^*(\hat{y} + R')$, and $\hat{y} > y'' > y_0$, Lemma 2 implies $x_0 \geq x'' = \frac{y''}{n-1} > \frac{y_0}{n-1}$.

We have now shown that for all $y \in [\underline{y}, y'')$, $x \in X^*(y+R')$ implies $x > \frac{y}{n-1}$; thus, there cannot exist a symmetric equilibrium in which $y < y''$ when $R = R'$. It therefore must be that $y' \geq y''$; equivalently, $x' \geq x''$. Since individual force increases following a decrease in R, then individual effort and total force/effort must increase, as n is fixed. This establishes part (iii) and the proposition. □

Proof of Proposition 4

Let $R'' > R'$ and suppose that a symmetric equilibrium exists for both parameter values. Let x'' (x') denote the equilibrium per-player force when the parameter is R'' (R'). Let $y'' = (n-1)x''$ and $y' = (n-1)x'$. Finally, let $X^*(y + R)$ denote the best response to y when the parameter is R.[15]

Let $\hat{y} = y'' + R'' - R' > y''$. As we showed in the proof of Proposition 3(ii), it must be that $x'' \in X^*(\hat{y} + R')$. Moreover, since $x'' = \frac{y''}{n-1} < \frac{\hat{y}}{n-1} < \hat{y} + R'$, Lemma 2 implies that for all $y > \hat{y}$, if $x \in X^*(y + R')$ and $x < y + R'$, then $x \leq x'' < \frac{y}{n-1}$. Thus, there cannot exist a symmetric equilibrium with $y > \hat{y}$ when $R = R'$. Therefore, $y' \leq \hat{y}$, which implies $y' + R' \leq \hat{y} + R' = y'' + R''$. Next, let π'' (π') denote the equilibrium payoff to a player when the parameter

[15] Recall from the proof of Proposition 3(ii) that a player's best response depends on y and R insofar as it depends on the sum, $y + R$.

is R'' (respectively, R'). It holds,

$$\begin{aligned}
\pi' &= \frac{x'}{x' + y' + R'} V - \kappa(x') \\
&\geq \frac{x''}{x'' + y' + R'} V - \kappa(x'') \\
&\geq \frac{x''}{x'' + y'' + R''} V - \kappa(x'') \\
&= \pi''
\end{aligned}$$

The first inequality follows by definition of x'; the second follows since $y'' + R'' \geq y' + R'$. This establishes the proposition. □

Proof of Propositions 5-7

The contest is equivalent to a symmetric Cournot oligopoly with inverse demand function $\tilde{P}(z) = \frac{V}{R+z}$, and cost function, $\tilde{C}(x) = \kappa(x)$. The existence of a symmetric equilibrium, and non-existence of any asymmetric equilibria established in Proposition 5 follows from Theorem 2.1 in AL. The uniqueness of this equilibrium follows from Proposition 1. Part (ii) of Proposition 5 follows immediately by Theorem 2.2(b) in AL. Propositions 6 and 7 follow by AL's Theorems 2.5 and 2.6, respectively. □

Proof of Proposition 8

We first show that if $n \geq 3$ or κ is differentiable, then any symmetric equilibrium must be unique. First note that Lemmas 1 and 2 immediately generalize to the case $R = 0$. If $n \geq 3$ then in any symmetric equilibrium it holds, $x^* = \frac{y^*}{n-1} < y^* + R = y^*$. Using this fact, the proof of Proposition 1 applies immediately to the case $R = 0$. Then suppose $n = 2$ and κ is differentiable. Proceed by contradiction, and suppose there exist at least two distinct symmetric equilibria with individual forces $x'' > x'$. Let y'' and y' denote the force of the other player in this equilibrium (although $y'' = x''$ and $y' = x'$, for clarity, we use different notation to denote the other player's action). Now, it may easily be verified that,

$$\pi(x'', y'') - \pi(x', y'') = \pi(x'', y') - \pi(x', y')$$

Since $x'' \in X^*(y'')$ and $x' \in X^*(y')$ it holds,

$$0 \leq \pi(x'', y'') - \pi(x', y'') = \pi(x'', y') - \pi(x', y') \leq 0.$$

Thus, the two inequalities must hold with equality. Therefore, $\pi(x'', y'') = \pi(x', y'')$ and $\pi(x'', y') = \pi(x', y')$, which means $x'' \in X^*(y')$ and $x' \in X^*(y'')$.

Now, since κ is differentiable, this implies π is differentiable in x for each $y > 0$. Since $y'' > y' \geq 0$, $\pi(\cdot, y'')$ is differentiable. Also note that $x'' > x' \geq \underline{x}$ means that x'' is interior. Since $x'' \in X^*(y')$ and $x'' \in X^*(y'')$ this implies that x'' must satisfy the following first order conditions:

$$\frac{y'}{(x'' + y')^2} V - \kappa'(x'') = \frac{y''}{(x'' + y'')^2} V - \kappa'(x'') = 0$$

which means,

$$\frac{y'}{(x'' + y')^2} = \frac{y''}{(x'' + y'')^2} \tag{7}$$

Let $\Gamma(y) = \frac{y}{(x''+y)^2}$. Note that for all $y \in [y', y'')$ it holds, $\Gamma'(y) = \frac{x''-y}{(x''+y)^3} > 0$. This implies $\Gamma(y'') > \Gamma(y')$, which contradicts (7). Thus, if κ is differentiable and a symmetric equilibrium exists, it must be unique.

We now complete the proof of the proposition by showing that there can exist at most two symmetric equilibria. We have already shown that if $n \geq 3$ then the contest possesses at most one symmetric equilibrium. So, the only relevant case to address is for $n = 2$. We proceed by contradiction: Suppose there exist at least three distinct symmetric equilibria. Choose any three of these equilibria; let x' denote the smallest individual force of these three, and let $x'' > x'$ denote the largest individual force of these three. Let $y'' > y'$ represent the corresponding forces of the other player.

Since our hypothesis is that we have three distinct symmetric equilibria, there must be a point, $y_0 \in (y', y'')$ such that $y_0 = x_0 \in X^*(y_0)$. Recall from the first part of this proof that $x'' \in X^*(y')$. Since $x'' > y'$ and $x_0 = y_0$, it holds, $x''x_0 > y'y_0$. As $y_0 > y'$, Lemma 2 implies $x_0 \geq x''$; equivalently $y_0 \geq y''$, which yields a contradiction. Our hypothesis that there exist at least three distinct symmetric equilibria leads to a contradiction, and thus there can exist at most two. $\square$

Proof of Proposition 9

The fact that Lemmas 1-2 extend in this case is immediate. The proofs for Propositions 2-4 are also nearly identical when $R = 0$, so we do not reproduce these here. The only issue in applying the result of AL for Proposition 5 is that π is discontinuous at the zero vector when $\underline{x} = R = 0$. However, Assumption 2 implies that π is continuous in x for any $y > 0$. In particular, this means that $X^*(y)$ is non-empty for any $y > 0$. Since, in any symmetric equilibrium, all players must exert strictly positive effort when $\underline{x} = R = 0$, the arguments made by AL can easily be adapted to deal with this point of discontinuity. We therefore do not reproduce the proof here. $\square$

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