Infinite dimensional representations of finite dimensional algebras and amenability. *

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Abstract

We present a novel quantitative approach to the representation theory of finite dimensional algebras motivated by the emerging theory of graph limits.

We introduce the rank spectrum of a finite dimensional algebra R over a finite field. The elements of the rank spectrum are representations of the algebra into von Neumann regular rank algebras, and two representations are considered to be equivalent if they induce the same Sylvester rank functions on R-matrices.

Based on this approach, we can divide the finite dimensional algebras into three types: finite, amenable and non-amenable representation types. We prove that string algebras are of amenable representation type, but the wild Kronecker algebras are not. Here, the amenability of the rank algebras associated to the limit points in the rank spectrum plays a very important part.

We also show that the limit points of finite dimensional representations of algebras of amenable representation type can always be viewed as representations of the algebra in the continuous ring invented by John von Neumann in the 1930's.

As an application in algorithm theory, we introduce and study the notion of parameter testing of modules over finite dimensional algebras, that is analogous to the testing of bounded degree graphs introduced by Goldreich and Ron. We shall see that for string algebras all the reasonable (stable) parameters are testable.

Keywords: representations of finite dimensional algebras, amenable algebras, soficity, Ziegler spectrum, skew fields, parameter testing, string algebras

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1 Introduction

By Maschke's theorem, the complex group algebra $\mathbb{C}G$ is semisimple if G is a finite group. Hence all the infinite dimensional representations of G can be decomposed into finite dimensional subrepresentations. The situation becomes much more interesting if the characteristic of the coefficient field divides the order of the group.

Example 1. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein group and K be the field of two elements. Let $K(t)$ be the transcendent extension of K by the element t. Then*

$$\phi(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \phi(b) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

defines an indecomposable representation $\phi : G \to GL(2, K(t))$, where $G = \{1, a\} \times \{1, b\}$. That is, the representation of G on the infinite dimensional space $K(t) \times K(t)$ cannot be decomposed into finite (or even infinite) dimensional subrepresentations.

Infinite dimensional representations of finite dimensional algebras has been studied for decades (see e.g. [24], [11]). The aim of our paper is to develop a theory for such representations guided by the convergence and limit theory of finite graphs (see the monograph of László Lovász [21]). Our philosophy is to view infinite dimensional representations of a given algebra (when it is possible) as a sort of limit of its finite dimensional representations.

The rank spectrum. Let K be a finite field and R be a finite dimensional K-algebra. Let R-Mod denote the set of finitely generated (left) R-modules (up to isomorphism). If $M \in$ R-Mod, then $\phi^M : R \to \mathrm{End}_K(M) \cong \mathrm{Mat}_{\dim(M)\times\dim(M)}(K)$ is the corresponding representation, where $\phi^M(r) = rm$ and $\dim(M)$ is the K-dimension of the module M. For any $k, l \geq 1$ the map ϕ^M extends naturally to the homomorphism

$$\phi^M_{k,l} : \mathrm{Mat}_{k\times l}(R) \to \mathrm{Mat}_{k\dim(M)\times l\dim(M)}(K)\,.$$

If $A \in \mathrm{Mat}_{k\times l}(R)$, then let

$$rk_M(A) := \frac{\mathrm{rank}(\phi^M_{k,l}(A))}{\dim(M)}\,.$$

Then rk_M is a Sylvester rank function, that is

- $rk_M(I) = 1$, where I is the unit element of R considered as a 1×1-matrix.
- $rk_M \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = rk_M(A) + rk_M(B)\,,$
- $rk_M \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \geq rk_M(A) + rk_M(B)\,,$

- $rk_M(AB) \leq \min(rk_M(A), rk_M(B))$,

provided that the R-matrices A, B, C have appropriate sizes. Observe that

$$rk_M = rk_{M \oplus M \oplus \cdots \oplus M}\,.$$

We will show that if for two modules $rk_M = rk_N$, then there exist $k, l \geq 1$ such that $M^k \cong N^l$. We say that a sequence $\{M_n\}_{n=1}^\infty \subseteq$ R-Mod is **convergent** if for all pairs $k, l \geq 1$ and $A \in \mathrm{Mat}_{k \times l}(R)$, $\lim_{n\to\infty} rk_{M_n}(A)$ exists. The notion of convergence is based on the Benjamini-Schramm convergence notion (see the Introduction of [15]). Since the set of all finite matrices over R is countable, from each sequence $\{M_n\}_{n=1}^\infty$ one can pick a convergent subsequence. Let us consider the locally convex, topological vector space $C = \mathbb{R}^{\mathrm{Mat}\,(R)}$, where $\mathrm{Mat}\,(R)$ is the countable set of all finite matrices over R. The set of all Sylvester rank functions on $\mathrm{Mat}\,(R)$, $\mathrm{Syl}(R)$ forms a convex, compact subset of C. We will also consider the subset of Sylvester rank functions corresponding to finite dimensional representations

$$\mathrm{Fin}(R) := \{rk_M \mid M \in \text{R-Mod}\}\,.$$

Let K be a finite field as above. A K-algebra S is called a **rank algebra** (see [17] for an introduction of rank functions on von Neumann regular rings) if

- S is a von Neumann regular ring,
- $\mathrm{Mat}\,(S)$ is equipped with a Sylvester rank function rank_S.
- $\mathrm{rank}_S(A) = 0$ implies $A = 0$.

The most important examples of rank algebras are skew fields, matrix rings over skew fields and more generally, semisimple Artinian rings. If we have a representation $\rho : R \to S$ into a rank algebra we still have an associated Sylvester rank function

$$rk_\rho(A) := \mathrm{rank}_S(\tilde{\rho}(A))\,,$$

where $\tilde{\rho}$ is the appropriate extension for matrices. We say that two such representations $\rho_1 : R \to S_1$ and $\rho_2 : R \to S_2$ are equivalent if $rk_{\rho_1} = rk_{\rho_2}$. The **rank spectrum** of R, $\mathrm{Rank}(R)$ is the set of Sylvester rank functions that are in the form rk_ρ for some representation $\rho : R \to S$. We say that the infinite dimensional representation $\rho : R \to S$ is the **limit** of the convergent sequence of finite dimensional representations $\{\rho_n : R \to \mathrm{Mat}_{m_n \times m_n}(K)\}_{n=1}^\infty$ if for any $A \in \mathrm{Mat}\,(R)$,

$$\lim_{n\to\infty} rk_{\rho_n}(A) = rk_\rho(A)$$

that is, if rk_ρ is the limit point of the sequence $\{rk_{\rho_n}\}_{n=1}^\infty$ in $\mathrm{Syl}(R)$.

Example 2. *Let $\{K(t_n) = E_n\}_{n=1}^\infty$ be a sequence of finite dimensional field extensions of K with generators $\{t_n\}_{n=1}^\infty$. Suppose that $\dim_K(E_n) \to \infty$. Let $G =$*

$\mathbb{Z}_2 \times \mathbb{Z}_2$ as in Example 1 and define the representation $\phi_n : KG \to Mat_{2\times 2}(E_n)$ by

$$\phi_n(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \phi_n(b) = \begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix}.$$

Then, the representations $\{\phi_n\}_{n=1}^{\infty}$ converge to the representation $\phi : KG \to Mat_{2\times 2}(K(t))$ described in Example 1.

Proposition 1.1. *For any finite dimensional algebra R, the set Rank(R) is a convex, compact subspace of Syl(R).*

The following question is the somewhat analogous to the famous Connes Embedding Problem [7].

Question 1. *(see* **Theorem 3***) Is the closure of Fin(R) in Syl(R) equals to Rank(R)?*

Note that if $rk \in \mathrm{Syl}(R)$ is a Sylvester rank function such that the values of rk are in the set $\mathbb{Z}/d$ for some $d \in \mathbb{N}$, then by Schofield's Theorem (Theorem 7.12 [25]), there exists a skew field D and a representation $\rho : R \to \mathrm{Mat}_{d\times d}(D)$ such that $rk = rk_\rho$

Hyperfiniteness. Let R be a finite dimensional algebra as above and $\{M_n\}_{n=1}^{\infty} \subset$ R-Mod be a set of modules. We say that $\{M_n\}_{n=1}^{\infty}$ is **hyperfinite**, if for any $\epsilon > 0$, there exists a finite family $\{N_1, N_2, \dots, N_t\} \subset$ R-Mod so that for any $n \geq 1$, there exists a submodule $P_n \subset M_n, \dim_K(P_n) \geq (1-\epsilon)\dim_K(M_n)$ and non-negative integers $\alpha_1, \alpha_2, \dots, \alpha_t$ such that $P_n \cong \bigoplus_{i=1}^{t} N_i^{\alpha_i}$. An algebra R of infinite representation type (that is the set of indecomposable modules Ind(R) in R-Mod is infinite) is called of **amenable representation type**, if R-Mod itself is a hyperfinite family. In the course of the paper, we shall see (Proposition 10.1) that string algebras are of amenable representation type and the wild Kronecker algebras (**Theorem 6**) are not. These results suggest a relation between the tame/wild and the amenable/non-amenable dichotomies.

Let d_R be a metric on Syl(R) defining the compact topology. Two modules $M, N \in$ R-Mod are ϵ-close to each other if there exist submodules $P \subset M, Q \subset N, P \cong Q$ such that $\dim_K(P) \geq (1-\epsilon)\dim_K(M)$, $\dim_K(Q) \geq (1-\epsilon)\dim_K(N)$. Motivated by graph theoretical results (Theorem 5. [14]), we make the following conjecture.

Conjecture 1. *Let R be an finite dimensional algebra of amenable representation type. Then, for any $\epsilon > 0$ we have a $\delta > 0$ such that for two elements $M, N \subset$ R-Mod, $1 - \delta \leq \frac{\dim_K(M)}{\dim_K(N)} \leq 1 + \delta$, $d_R(rk_M, rk_N) < \delta$, then M and N are ϵ-close to each other.*

The meaning of the conjecture is that if we have a finite dimensional algebra of amenable representation type, then the almost isomorphism of two R-modules of the same dimension can be decided by checking the ranks of finitely many matrices. This observation leads to the representational theoretical analogue of the Constant-Time Property Testing Algorithms developed by Goldreich and Ron for bounded degree graphs. (see Section 14). The main result of our paper is the following.

Theorem 1. *Conjecture 1 holds if R is a string algebra.*

Amenability of infinite dimensional algebras. Amenability and hyperfiniteness are intimately related notions in group theory. The notion of amenability for algebras were introduced by Misha Gromov [18] and plays an important part in the study of the rank spectrum.

Definition 1.1. *An element rk_ρ in the rank spectrum of R is amenable, if there exists a hyperfinite sequence $\{M_n\}_{n=1}^\infty \subset R$-Mod such that $\lim_{n\to\infty} rk_{M_n} = rk_\rho$.*

The following conjecture is motivated by the theorem of Oded Schramm on hyperfinite graph sequences [26].

Conjecture 2. *If $\rho : R \to S$ defines rk_ρ, where S is an amenable rank algebra, then rk_ρ is hyperfinite.*

We confirm this conjecture in the case, where $S = \mathrm{Mat}_{l\times l}(D)$ and D is an amenable skew field (**Theorem 5**). For the converse, one can hope that the limits of hyperfinite sequences can be represented by homomorphisms $\rho : R \to S$, where S is an amenable rank algebra.

The continuous algebra of John von Neumann. It is well-known that any countable amenable group algebra has a trace-preserving embedding into the unique separable hyperfinite II_1-factor. In the world of ranks the role of the hyperfinite II_1-factor will be played by an algebra constructed by John von Neumann in the 1930's [27]. The construction goes as follows. We consider the following sequence of diagonal embeddings.

$$K \to \mathrm{Mat}_{2\times 2}(K) \to \mathrm{Mat}_{4\times 4}(K) \to \mathrm{Mat}_{8\times 8}(K) \to \dots ,$$

where K is a finite field. Then all the embeddings are preserving the normalized ranks. Hence the direct limit $\varinjlim \mathrm{Mat}_{2^k\times 2^k}(\mathbb{C})$ is a rank algebra. The addition, multiplication and the rank function extends to the metric completion M_K of the direct limit algebra. The resulting algebra M_K is a simple continuous rank algebra (see also [17]).

Conjecture 3. *All the countable dimensional amenable rank algebras embeds into M_K. Conversely, all the countable dimensional rank subalgebra of M_K is amenable.*

Theorem 2. *Let $\mathcal{M} = \{M_n\}_{n=1}^\infty \subset R$-Mod be a hyperfinite convergent sequence and $rk_{\mathcal{M}} = \lim_{n\to\infty} rk_{M_n}$. Then there exists a representation $\rho : R \to M_K$ such that $rk_\rho = rk_{\mathcal{M}}$.*

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2 The continuous ultraproduct of rank algebras

First, let us prove Proposition 1.1. Let $\{S_n\}_{n=1}^\infty$ be rank algebras and $\{\rho_n : R \to S_n\}_{n=1}^\infty$ be a convergent sequence in the rank spectrum Rank(R), that is, for any matrix $A \in \mathrm{Mat}\,(R)$, $\lim_{n\to\infty} rk_{\rho_n}(A)$ exists. Let ω be a non-principal ultrafilter on the natural numbers and $\lim_\omega$ be the associated ultralimit. We define the **continuous ultraproduct** $\prod_\omega S_n$, a quotient of the classical ultraproduct, the following way. Let $I \subset \prod_{n=1}^\infty S_n$ be the set of elements $\{a_n\}_{n=1}^\infty$ such that

$$\lim_\omega \mathrm{rank}_{S_n}(a_n) = 0\,.$$

Clearly, I is a two-sided ideal in $\prod_{n=1}^\infty S_n$. Then $\Sigma = \prod_\omega S_n := \prod_{n=1}^\infty S_n/I$ is a von Neumann regular ring. Indeed, if $[\{a_n\}_{n=1}^\infty] \in \prod_\omega S_n$ and $x_n \in S_n, a_n x_n a_n = a_n$, then

$$[\{a_n\}_{n=1}^\infty][\{x_n\}_{n=1}^\infty][\{a_n\}_{n=1}^\infty] = [\{a_n\}_{n=1}^\infty]\,.$$

That is each element of $\prod_\omega S_n$ has a pseudoinverse. Now, if $A \in \mathrm{Mat}_{\,k\times l}(\Sigma) = [\{A_n\}_{n=1}^\infty]$, then let $\mathrm{rank}_\Sigma(A) = \lim_\omega \mathrm{rank}_{S_n}(A_n)$. Then rank_Σ is a Sylvester rank on Σ and $\mathrm{rank}_\Sigma(a) \neq 0$ if $a \in \Sigma$.
The representation $\rho_\Sigma : R \to \Sigma$ is defined by $\rho_\Sigma(r) := [\{\rho_n(r)\}_{n=1}^\infty]$. Then, $rk_\Sigma(A) = \lim_{n\to\infty} rk_{\rho_n}(A)$ holds for any matrix $A \in \mathrm{Mat}\,(R)$. Hence, Rank(R) is a closed subset of Syl(R).
If the elements rk_1 resp. rk_2 are represented by homomorphisms $\rho_1 : R \to S$ and $\rho_2 : R \to T$, where S, T are rank algebras, then $\lambda \cdot rk_1 + \mu \cdot rk_2$, $(\lambda, \mu \geq 0, \lambda + \mu = 1)$, is represented by $\rho : R \to S \oplus T$, where

$$\mathrm{rank}_{S\oplus T} = \lambda \cdot \mathrm{rank}_S + \mu \cdot \mathrm{rank}_T\,.$$

Hence, Rank(R) is a convex subset of Syl(R). $\square$

The following lemma is implicite in [15], nevertheless we include the proof for the sake of completeness.

Lemma 2.1. *Let K be a finite field, R be a finite dimensional K-algebra and $\rho : R \to S$ be a representation of R into a rank algebra. Then there exists a countable dimensional subalgebra $S' \subset S$ that contains the image of ρ. That is, all the elements of the rank spectrum can be represented by countable dimensional rank algebras.*

Proof. Let X be a finite dimensional linear subspace of S containing the unit. Denote by $P(X)$ the finite dimensional linear subspace of S spanned by elements in the form

$$\{x_1 x_2 \mid x_1, x_2 \in X\}\,.$$

Also, let $R(X)$ be an arbitrary finite set such that for any $x \in X$, there exists $y \in R(X)$ such that $xyx = x$. Let $X_1 = \mathrm{Im}\,\rho(R)$ and $X_{n+1} = R(P(X_n))$. Then $\cup_{n=1}^\infty X_n$ is a countable dimensional von Neumann regular ring containing the image of R. $\square$

3 The Ziegler spectrum

The goal of this section is to recall the notion of the Ziegler spectrum [28] (see also [23]) and show that finitely generated indecomposable modules are isolated points of the closure of $\mathrm{Fin}(R)$.
Let R be a finite dimensional algebra over a finite field K. A pp-formula ϕ of type t is given by a matrix

$$A = \{a_{ij}\}_{1\leq i\leq m, 1\leq j\leq n} \in \mathrm{Mat}_{m\times n}(R)\,.$$

Let M be a module over R. We say that $\underline{v} = \{v_1, v_2, \ldots, v_t\} \in M^t$ satisfies ϕ if there exists $\{y_{t+1}, y_{t+2}, \ldots, y_n\} \in M^{n-t}$ such that for any $1 \leq i \leq m$

$$\sum_{j=1}^{t} a_{ij}v_j + \sum_{j=t+1}^{n} a_{ij}y_j = 0\,.$$

The elements $\underline{v}$ satisfying ϕ form the *pp*-subspace $M(\phi) \subset M^t$. We say that two formulas (of the same type) are equivalent, $\phi \cong \psi$, if for any M, $M(\phi) = M(\psi)$. Also, $\phi \geq \psi$ if for any M, $M(\phi) \supset M(\psi)$. For any $t \geq 1$, the equivalence classes of pp-formulas of type t form the modular lattice $pp^t(R)$. The Ziegler spectrum of R, $Zg(R)$, consists of the pure-injective indecomposable modules over R. The basic open sets of $Zg(R)$ are given by pp-pairs $\langle\phi, \psi\rangle$, where $\phi \geq \psi$.

$$U_{\phi,\psi} = \{M \mid M(\phi) \supsetneq M(\psi)\}\,.$$

Then, the set of isolated points in $Zg(R)$ are exactly the finitely generated indecomposable modules and they form a dense subset in the quasi-compact (not necessarily Hausdorff) space $Zg(R)$ (Corollary 5.3.36. and Corollary 5.3.37. [23]). Let ϕ be a *pp*-formula of type t and let M be a finitely generated module. Set

$$D_M([\phi]) := \frac{\dim_K(M(\phi))}{\dim_K(M)}\,.$$

Then, D_M is a dimension funcion on $pp^t(R)$, that is

- $D_M(0) = 0$
- $D_M(1) = 1$
- $D_M(a \wedge b) + D_M(a \vee b) = D_M(a) + D_M(b)$
- $D_M(a) \leq D_M(b)$ if $a \leq b$

(Note that we do not require that $a \lneq b => D_M(a) \lneq D_M(b)$.) Now let $\rho : R \to S$ be a representation (possibly infinite dimensional) into a rank algebra. Recall [17] that the finitely generated right submodules of S, Mod-S, are projective, all exact sequences of such modules split and the rank function defines a dimension function $\dim_S$ on the modular lattice Mod-S. Let $A : S^m \to S^n$ be a module endomorphism (we nonchalantly consider A as an $m \times n$-matrix in $\mathrm{Mat}\,(S)$). Then

$$\mathrm{rank}_S(A) = \dim_S(\mathrm{Im}\,(A)) = n - \dim_S(\mathrm{Ker}\,(A))\,.$$

Let us consider S as a left R-module.

Lemma 3.1. *For any $\phi \in pp^t(R)$, $S(\phi) \in$ Mod-S and its dimension $\dim_S(S(\phi))$ can be computed by knowing only the element rk_ρ in the rank spectrum.*

Proof. Recall that $S(\phi)$ is the set of elements $\{s_1, s_2, \dots, s_t\} \in S^t$ such that there exists $\{y_{t+1}, y_{t+2}, \dots, y_n\} \in S^{n-t}$ so that for any $1 \leq i \leq m$

$$\sum_{j=1}^{t} a_{ij} s_j + \sum_{j=t+1}^{n} a_{ij} y_j = 0\,,$$

where the matrix A defines ϕ. Therefore, $S(\phi)$ is a right S-module. Let Q be the set of elements $\{z_1, z_2, \dots, z_n\} \in S^n$ such that

$$\sum_{j=1}^{n} a_{ij} z_j = 0\,,$$

and $z_i = 0$, if $1 \leq i \leq t$. Then

$$0 \to Q \to \mathrm{Ker}\,(A) \to S(\phi) \to 0$$

is an exact sequence, where Q is the kernel of a matrix $B \in \mathrm{Mat}\,(R) \subset \mathrm{Mat}\,(S)$. That is Q and $\mathrm{Ker}\,(A)$ are both finitely generated right S-modules and $S(\phi)$ is also finitely generated since it is a quotient of $\mathrm{Ker}\,(A)$. We have that

$$D_S(\phi) = \dim_S(S(\phi)) = \dim_S(Q) - \dim_S(\mathrm{Ker}\,(A)) = t + rk_\rho(B) - rk_\rho(A) \qquad \square$$

Corollary 3.1. *If $\{M_n\}_{n=1}^\infty \in R$-Mod is a convergent sequence, then for any pp-formula ϕ, $\{D_{M_n}(\phi)\}_{n=1}^\infty$ converges.*

So, one can identify the elements of the rank spectrum with certain dimension functions on $pp^t(R)$. Note that if $\{M_n\}_{n=1}^\infty$ is a convergence sequence of modules, $\langle \phi, \psi \rangle$ is a pp-pair, for any $n \geq 1$ $M_n(\phi) \neq M_n(\psi)$ but

$$\lim_{n\to\infty} (D_{M_n}(\phi) - D_{M_n}(\psi)) = 0\,.$$

then if $\rho : R \to S$ represents the limit of $\{M_n\}_{n=1}^\infty$ in the rank spectrum, we have $S(\phi) = S(\psi)$. This phenomenon signifies an important difference between the topology of the Ziegler-spectrum and the rank spectrum.

4 The geometry of the rank spectrum

The goal of this section is to understand the geometry of the closure of $\mathrm{Fin}(R)$ (the ranks associated to finitely generated modules) in the rank spectrum. Let $M \in$ R-Mod. By the Krull-Schmidt theorem, M can be written in a unique way as $M \cong \oplus_{i=1}^{s} Q_i^{n_i}$, where the Q_i's are indecomposable modules. The **weight** of an indecomposable module Q_i in M is defined as

$$w_{Q_i}(M) := \frac{n_i \dim_K(Q_i)}{\dim_K(M)} \,.$$

That is,

$$M \cong \bigoplus_{Q \in \mathrm{Ind}(R)} Q^{\frac{w_Q(M)\dim_K(M)}{\dim_K(Q)}}$$

and $\sum_{Q \in \mathrm{Ind}(R)} w_Q(M) = 1$.

Proposition 4.1. *If $M, N \in R$-Mod, then $rk_M = rk_N$ if and only if $w_Q(M) = w_Q(N)$ for all $Q \in Ind(R)$. In particular, $rk_Q \neq rk_P$, whenever $P \neq Q \in Ind(R)$.*

Proof. First, suppose that $M \cong \oplus_{i=1}^{s} Q_i^{n_i}$.

Lemma 4.1. $rk_M = \sum_i^s w_{Q_i}(M) rk_{Q_i}$

Proof. Let $A \in \mathrm{Mat}_{\,m\times n}(R)$ be a matrix. Then, using the definition in the Introduction,

$$\mathrm{rank}(\phi^M_{m,n}(A)) = \sum_{i=1}^{s} w_{Q_i}(M) \frac{\dim_K(M)}{\dim_K(Q_i)} \mathrm{rank}(\phi^{Q_i}_{m,n}(A)) \,.$$

That is,

$$rk_M(A) = \frac{\mathrm{rank}(\phi^M_{m,n}(A))}{\dim_K(M)} = \sum_{i=1}^{s} w_{Q_i}(M) rk_{Q_i}(A) \,.$$

So, by the lemma above, if for all $Q \in \mathrm{Ind}(R), w_Q(M) = w_Q(N)$ holds, then $rk_M = rk_N$. Note that it means that for certain integers k and l, $M^k \cong N^l$. $\square$

Now, suppose that for some Q, $w_Q(M) \neq w_Q(N)$. Let $\langle \phi, \psi \rangle$ be a pp-pair, that isolates Q, that is

- $Q(\phi) \neq Q(\psi)$
- $P(\phi) = P(\psi)$ if $P \ncong Q, P \in \mathrm{Ind}(R)$.

By, Lemma 1.2.3. [23], $M(\phi) = \bigoplus_{Q \in \mathrm{Ind}(R)} Q^{\frac{w_Q(M)\dim_K(M)}{\dim_K(Q)}}(\phi)$. Hence,

$$D_M(\phi) - D_M(\psi) = w_Q(M)(D_Q(\phi) - D_Q(\psi)) \tag{1}$$

That is,

$$D_M(\phi) - D_M(\psi) = w_Q(M)(D_Q(\phi) - D_Q(\psi)) \neq D_N(\phi) - D_N(\psi)\,.$$

Hence by Lemma 3.1, $rk_M \neq rk_N$. $\square$

Note that (1), has the following corollary.

Corollary 4.1. *If $\{M_m\}_{n=1}^\infty \subset$ R-Mod is a convergent sequence, then for all $Q \in$ Ind(R) the sequence $\{w_Q(M_n)\}_{n=1}^\infty$ is convergent as well.*

Now, let $Q_1, Q_2, \dots$ be an enumeration of the indecomposable modules in R-Mod. Let $\{a_n\}_{n=1}^\infty$ be non-negative real numbers, such that $\sum_{n=1}^\infty a_n = 1$. Then $\underline{a} = \sum_{n=1}^\infty a_n rk_{Q_n}$ is a well-defined element of Rank(R). We denote these set of elements by Conv(R).

Proposition 4.2. *Let $\{\underline{a^i} = \sum_{n=1}^\infty a_n^i rk_{Q_n}\}_{i=1}^\infty$ be a convergent sequence such that $\lim_{i\to\infty} \underline{a^i} = \underline{b} = \sum_{n=1}^\infty b_n rk_{Q_n} \in$ Conv(R). Then for any $n \geq 1$, $\lim_{i\to\infty} a_n^i = b_n$.*

Proof. Let suppose that for some $n \geq 1$ and subsequence $\{i_k\}_{k=1}^\infty$

$$\lim_{k\to\infty} a_n^{i_k} = c_n \neq b_n\,.$$

Let $\langle \phi, \psi \rangle$ be a pp-pair that isolates Q_n. Then,

$$\lim_{k\to\infty} (D_{\underline{a}^{i_k}}(\phi) - D_{\underline{a}^{i_k}}(\psi) = c_n(D_{Q_n}(\phi) - D_{Q_n}(\psi)) \neq$$

$$\neq b_n(D_{Q_n}(\phi) - D_{Q_n}(\psi)) = D_{\underline{b}}(\phi) - D_{\underline{b}}(\psi)\,,$$

leading to a contradiction. $\square$

Corollary 4.2. *The limit points of Ind(R) in the rank spectrum form a non-empty closed set that is disjoint from Conv(R).*

5 Sofic algebras

Let R be finite dimensional algebra over the finite field K as in the previous sections and $\rho : R \to S$ be a homomorphism to a rank algebra. In this section we will get a sufficient condition under which rk_ρ is an element of the closure of Fin(R) (see Question 1). Let $\mathcal{A}$ be a countable dimensional algebra over the finite field K with basis $\{1 = e_1, e_2, \dots\}$. Following Arzhantseva and Paunescu [1], we call $\mathcal{A}$ a **sofic algebra**

- if there exists a non-negative function $j : \mathcal{A} \to \mathbb{R}$ such that $j(a) > 0$ if $a \neq 0$, and a sequence of real numbers $k_1 > k_2 > \dots$ tending to zero,

- for any $n \geq 1$, there exists a unital, linear map $\phi_n : \mathcal{A} \to \mathrm{Mat}_{m_n \times m_n}(K)$ so that
$$\frac{\mathrm{Rank}(\phi_n(ab) - \phi_n(a)\phi_n(b))}{m_n} < k_n \,,$$
whenever $a, b \in \mathrm{Span}(e_1, e_2, \dots, e_n)$
- $\frac{\mathrm{Rank}(\phi_n(a))}{m_n} > \frac{j(a)}{2}$, if n is large enough.

We call such a sequence $\{\phi_n\}_{n=1}^\infty$ a sofic representation sequence of $\mathcal{A}$. It is not hard to see (Proposition 5.1 [15]) that $\mathcal{A}$ is sofic if and only if there exists an injective, unital homomorphism $\phi : \mathcal{A} \to \mathrm{Mat}\,^{\mathcal{M}}_{\omega}$, where $\mathrm{Mat}\,^{\mathcal{M}}_{\omega}$ is the continuous ultraproduct of the matrix algebras $\mathcal{M} := \{\mathrm{Mat}_{m_n \times m_n}(K)\}_{n=1}^\infty$. Recall from Section 2 that the rank $rk_{\mathcal{M}}$ on $\mathrm{Mat}\,^{\mathcal{M}}_{\omega}$ is given by $rk_{\mathcal{M}}([\{a_n\}] = \lim_\omega rk_{m_n}(a_n)$, where rk_{m_n} is the normalized rank function on the matrix algebra $\mathrm{Mat}_{m_n \times m_n}(K)$. By the proof of Proposition 1.1, it follows that any element of the closure of $\mathrm{Fin}(R)$ is associated to a unital homomorphism $\rho : R \to \mathrm{Mat}\,^{\mathcal{M}}_{\omega}$. Now we prove the converse.

Theorem 3. *Let $\mathcal{A}$ be a sofic algebra over our finite base field K with a rank $rk_{\mathcal{A}}$ derived from the injective homomorphism $\phi : \mathcal{A} \to Mat^{\mathcal{M}}_{\omega}$. Let R be a finite dimensional algebra over a finite field K, and $\rho : R \to \mathcal{A}$ be the corresponding element of the rank spectrum Rank(R). Then rk_ρ is in the closure of Fin(R).*

Proof. By Proposition 5.1 [15], we have a sequence of unital maps $\{\phi_n : \mathcal{A} \to \mathrm{Mat}_{m_n \times m_n}(K)\}_{n=1}^\infty$ such that for any $\epsilon > 0$ and for any $a, b \in \mathcal{A}$
$$\{n \mid rk_{m_n}(\phi_n(ab) - \phi_n(a)\phi_n(b)) < \epsilon\} \in \omega$$
and for any $k, l \geq 1$ and $A \in \mathrm{Mat}_{k,l}(\mathcal{A})$
$$\{n \mid \frac{\dim_K(\phi_n^{k,l}(A))}{m_n} - rk_\rho(A)| < \epsilon\} \in \omega \,,$$
where $\phi_n^{k,l}$ is the extension of ϕ_n onto $\mathrm{Mat}_{k,l}(\mathcal{A})$. Therefore, by taking a subsequence we can assure that for any $a, b \in \mathcal{A}$
$$\lim_{n\to\infty} rk_{m_n}(\phi_n(ab) - \phi_n(a)\phi_n(b)) = 0 \tag{2}$$
and for any $k, l \geq 1$ and $A \in \mathrm{Mat}_{k,l}(\mathcal{A})$
$$\lim_{n\to\infty} \frac{\dim_K(\phi_n^{k,l}(A))}{m_n} = rk_\rho(A). \tag{3}$$

If the sofic representation sequence $\{\phi_n\}_{n=1}^\infty$ satisfies (3), then we call it a **convergent sofic representation sequence**. It is enough to construct for all $n \geq 1$ a subspace $V_n \subset K^{m_n}$ such that

- $\lim_{n\to\infty} \frac{\dim_K(V_n)}{m_n} = 1$.

- For any $a \in R$, $\phi'_n(a)(V_n) \subset V_n$,

where $\phi'_n = \phi_n \circ \rho$. Indeed, for such sequence of subspaces $\{V_n\}_{n=1}^\infty$ the maps ϕ'_n define R-module structures on V_n and for any matrix $A \in \mathrm{Mat}_{k,l}(\mathcal{A})$,

$$\lim_{n\to\infty} \left(\frac{\dim_K(\phi_n^{k,l}(A))}{m_n} - \frac{\dim_K(\psi_n^{k,l}(A))}{m_n} \right) = 0,$$

where $\psi_n^{k,l}$ is the restriction of $\phi_n^{k,l}$ onto V_n^l. By (2), for any pair $a, b \in R$,

$$\lim_{n\to\infty} \frac{\dim_K(\mathrm{Ker}\,(\phi'_n(ab) - \phi'_n(a)\phi'_n(b))}{m_n} = 0\,.$$

Hence, for $V_n = \cap_{a,b\in R}\mathrm{Ker}\,(\phi'_n(ab) - \phi'_n(a)\phi'_n(b))$, $\lim_{n\to\infty} \frac{\dim_K(V_n)}{m_n} = 1$. We need to show that if $v \in V_n$, then $\phi'_n(c)(v) \in V_n$, that is for any $a, b \in R$

$$\phi'_n(ab)\phi'_n(c)(v) = \phi'_n(a)\phi'_n(b)\phi'_n(c)(v)\,. \tag{4}$$

Since $\phi'_n(ab)\phi'_n(c)(v) = \phi'_n(abc)(v)$ and $\phi'_n(a)\phi'_n(b)\phi'_n(c)(v) = \phi'_n(a)\phi'_n(bc)(v) = \phi'_n(abc)(v)$, (4) follows. $\square$

Let $\rho : R \to \mathrm{Mat}_{n\times n}(D)$ be an infinite dimensional representation, where D is a skew field over the base field K. Clearly, there exists a countable dimensional subskew field $E \subset D$ such that $\mathrm{Im}\,(\rho) \subset \mathrm{Mat}_{n\times n}(E)$. Since $\mathrm{Mat}_{n\times n}(E)$ is sofic if and only if E is sofic and $\mathrm{Mat}_{n\times n}(E)$ has a unique rank, by Proposition 3, we have that if E is sofic, then rk_ρ is in the closure of $\mathrm{Fin}(R)$. It is an open question, whether there are non-sofic skew fields or not. In [15] it was proved that all the amenable skew fields (we discuss them in Section 7) and the free skew field (we will prove that it is a non-amenable skew field in Section 7) are sofic. Hence for these skew fields and homomorphisms $\rho : R \to \mathrm{Mat}_{n\times n}(E)$, rk_ρ is always in the closure of $\mathrm{Fin}(R)$.

6 Amenable elements in the rank spectrum

As we have seen in the previous section, one can construct convergent sequences of finite dimensional representations of a finite dimensional algebra R by mapping R into a sofic algebra $\mathcal{A}$. Now we investigate that using this construction, how can we obtain amenable elements (see Definition 1.1)in the rank spectrum of R. So, let $\mathcal{A}$ be a countable dimensional algebra over the finite field K, with basis $\{1 = e_1, e_2, \dots\}$ and let $\{\phi_n : \mathcal{A} \to \mathrm{Mat}_{m_n\times m_n}(K)\}_{n=1}^\infty$ be unital linear maps forming a convergent, sofic representation system. We call the family $\{\phi_n\}_{n=1}^\infty$ hyperfinite (see [15]) if for any $\epsilon > 0$, there exists $L_\epsilon > 0$ such that for any $n \geq 1$, we have independent K-linear subspaces $N_n^1, N_n^2, \dots, N_n^{k(n)} \subset K^{m_n}$ satisfying the following three conditions:

- For any $1 \leq i \leq k(n)$, $\dim_K(N_n^i) \leq L_\epsilon$

- $\frac{\sum_{i=1}^{k(n)} \dim_K(N_n^i)}{m_n} \geq 1-\epsilon$
- The subspaces $\bigvee_{a\in \mathrm{Span}\{e_1,e_2,\dots,e_s\}} \phi_n(a)(N_n^i)$ are independent and for any $1 \leq i \leq k(n)$, $\frac{\dim_K\left(\bigvee_{a\in \mathrm{Span}\{e_1,e_2,\dots,e_s\}} \phi_n(a)(N_n^i)\right)}{\dim_K(N_n^i)} \leq 1+\epsilon$, where s is the integer part of $1/\epsilon$.

So, as opposed to the case of finite dimensional algebras, the pieces N_n^i are not exactly $\mathcal{A}$-modules, only in an approximate sense. First, we prove a technical lemma that will make the proof of some of our results easier. It states that part of the third condition is not necessary to check in order to prove that a certain sequence is hyperfinite.

Lemma 6.1. *Let $\{\phi_n : \mathcal{A} \to Mat_{m_n\times m_n}(K)\}_{n=1}^\infty$ be a convergent sofic representation sequence. Suppose that for any $\epsilon > 0$, there exists $L_\epsilon > 0$ such that for any $n \geq 1$, we have independent K-linear subspaces $Z_n^1, Z_n^2, \dots, Z_n^{k(n)} \subset K^{m_n}$ satisfying the following three conditions:*

- *For any $1 \leq i \leq k(n)$, $\dim_K(Z_n^i) \leq L_\epsilon$.*
- $\frac{\sum_{i=1}^{k(n)} \dim_K(Z_n^i)}{m_n} \geq 1-\epsilon$.
- *For any $1 \leq i \leq k(n)$, $\frac{\dim_K\left(\bigvee_{a\in Span\{e_1,e_2,\dots,e_s\}} \phi_n(a)(Z_n^i)\right)}{\dim_K(Z_n^i)} \leq 1+\epsilon$, where s is the integer part of $1/\epsilon$.*

Then $\{\phi_n\}$ is hyperfinite.

Proof. We start with a simple linear algebra lemma.

Lemma 6.2. *Let $V \subset Z$ be finite dimensional K-spaces and $T, S \subset End_K(Z)$. Suppose that $\dim_K(T(V)+V) \leq (1+\epsilon_T)\dim_K(V)$, $\dim_K(S(V)+V) \leq (1+\epsilon_S)\dim_K(V)$, then*

- *There exists a linear subspace $W \subset V$ such that $T(W) \subset V$ and $\dim_K(W) \geq (1-e_T)\dim_K(V)$.*
- $\dim_K(TS(V)+V) \leq (1+e_T+e_S)\dim_K(V)$.

Proof. Let $m_T : V \to \frac{T(V)+V}{V}$ be the natural quotient map. Since $\dim_K(\mathrm{Im}\,(m_T)) \leq \epsilon_T \dim_K(V)$, we have that

$$\dim_K(\mathrm{Ker}\,(m_T)) \geq (1-\epsilon_T)\dim_K(V)\,.$$

On the other hand, $T(\mathrm{Ker}\,(m_T)) \subset V$. For the second part, let $S(V)+V = Q\oplus V$ for some complementing space Q, then $TS(V)+V \subset T(Q)+T(V)+V$, hence

$$\dim_K(TS(V)+V) \leq (1+\epsilon_T+\epsilon_S)\dim_K(V) \qquad \square.$$

Lemma 6.3. *For any $n \geq 1$ and $1 \leq i \leq k(n)$, there exists a subspace $Q_n^i \subset Z_n^i$ so that*

- $$\dim_K(Q_n^i) \geq (1-\sqrt{\epsilon}) \dim_K(Z_n^i)\,.$$
- *For any $a \in Span\{e_1, e_2, \ldots, e_{s(\epsilon)}\}$*
$$\phi_n(a)(Q_n^i) \subset Z_n^i\,,$$
where $s(\epsilon)$ is given in such a way that $|Span\{e_1, e_2, \ldots, e_{s(\epsilon)}\}| \leq \frac{1}{\sqrt{\epsilon}}$.

Proof. For $a \in \mathrm{Span}\{e_1, e_2, \ldots, e_{s(\epsilon)}\}$, let $W_a \subset Z_n^i$ (Lemma 6.2) such that

- $\dim_K(W_a) \geq (1-\epsilon) \dim_K(Z_n^i)\,.$
- $\phi_n(a)(W_a) \subset Z_n^i\,.$

Let
$$Q_n^i = \cap_{a \in \mathrm{Span}\{e_1, e_2, \ldots, e_{s(\epsilon)}\}} W_a\,.$$
Then,
$$\dim_K(Q_n^i) \geq (1-\sqrt{\epsilon}) \dim_K(Z_n^i)$$
and for any $a \in \mathrm{Span}\{e_1, e_2, \ldots, e_{s(\epsilon)}\}$
$$\phi_n(a)(Q_n^i) \subset Z_n^i \qquad \square$$

The existence of the systems $\{Q_n^i\}_{i=1}^{k(n)}$ clearly implies the statement in Lemma 6.1 $\square$

If $\{\phi_n : \mathcal{A} \to \mathrm{Mat}_{\,m_n \times m_n}(K)\}_{n=1}^{\infty}$ is a hyperfinite sofic representation system, then the associated sofic representation system

$$\{\hat{\phi}_n : \mathrm{Mat}_{\,l \times l}(\mathcal{A}) \to \mathrm{Mat}_{\,m_n l \times m_n l}(K)\}$$

is still hyperfinite. Indeed, the spaces $M_n^i := (N_n^i)^l \subset K^{m_n l}$ will be the independent subspaces satisfying the approximate module condition. Now, let $\rho : R \to \mathrm{Mat}_{\,l \times l}(\mathcal{A})$ be a unital homomorphism from a finite dimensional algebra R. By Theorem 3, we have a sequence of subspaces $V_n \subset K^{m_n l}$ so that

- $\hat{\phi}_n \circ \rho(V_n) \subset V_n$ (that is V_n is an R-module).
- $\lim_{n \to \infty} \frac{\dim_K(V_n)}{m_n l} = 1\,,$

representing the element rk_ρ in the rank spectrum.

Proposition 6.1. *rk_ρ is an amenable element of the rank spectrum, that is, $\{V_n\}_{n=1}^{\infty}$ is a hyperfinite sequence of R-modules.*

Proof. The first step is to show that the sequence $\{V_n\}_{n=1}^{\infty}$ is hyperfinite in the weaker sense, described in Lemma 6.1. As opposed to the graph theoretical case, it is not a priori obvious. The reason is that although the subspaces $\{V_n\}_{n=1}^{\infty}$ are almost as big as the total spaces $K^{m_n l}$ it is not true that they contain all (or even one single copy) of the pieces M_n^i. The following lemma resolves this problem. First, fix a basis $\{f_i\}_{i=1}^{\infty}$ for the algebra $\mathrm{Mat}_{l\times l}(A)$. By taking a subsequence, we can suppose that the subspaces $\bigvee_{a\in\mathrm{Span}\{f_1,f_2,\dots,f_s\}}\hat{\phi}_n(a)(M_n^i)$ are independent, where s is the integer part of $1/\epsilon$ and for any $1\le i\le k(n)$,

$$\frac{\dim_K\left(\bigvee_{a\in\mathrm{Span}\{f_1,f_2,\dots,f_s\}}\hat{\phi}_n(a)(M_n^i)\right)}{\dim_K(M_n^i)}\le 1+\epsilon\,,$$

Lemma 6.4. *For any $\epsilon>0$ and large enough $n\ge 1$, we have independent subspaces $Q_n^1,Q_n^2,\dots Q_n^{l(n)}$ in V_n such that*

- *For any $1\le i\le l(n)$, $\dim_K(Q_n^i)\le L_\epsilon$.*
- $\frac{\sum_{i=1}^{l(n)}\dim_K(Q_n^i)}{m_n}\ge 1-2\epsilon\,.$
- *For any $1\le i\le l(n)$, $\frac{\dim_K\left(\bigvee_{a\in Span\{f_1,f_2,\dots,f_s\}}\hat{\phi}_n(a)(Q_n^i)\right)}{\dim_K(Q_n^i)}\le 1+\epsilon$, where s is the integer part of $1/\epsilon$.*

Proof. We call the subspaces M_n^i and M_n^j equivalent if there exists a linear bijection $\alpha_{ij}:K^{m_n l}\to K^{m_n l}$ so that

- α_{ij} bijectively maps M_n^i into M_n^j.
- For any $a\in\mathrm{Span}\{f_1,f_2,\dots,f_s\}$ and $v\in M_n^i$
$$\alpha_{ij}(\hat{\phi}_n(a)(v))=\hat{\phi}_n(a)(\alpha_{ij}(v))\,.$$

Notice that by our finiteness conditions, there exists a constant $C>0$ such that for any $n\ge 1$, the number of equivalence classes is at most C. Now, let $\mu=\{M_n^{i_1},M_n^{i_2},\dots,M_n^{i_{\mu(t)}}\}$ be an equivalence class and let $\alpha_j:K^{m_n l}\to K^{m_n l}$ be the corresponding bijections mapping $M_n^{i_1}$ into $M_n^{i_j}$. For $\hat{\lambda}=\{\lambda_1,\lambda_2,\dots,\lambda_{i_{\mu(t)}}\}\in K^{\mu(t)}$ define

$$M_n^{\hat{\lambda}}:=(\sum_{j=1}^{\mu(t)}\lambda_j\alpha_j)(M_n^{i_1})\,.$$

Then

$$\dim_K\left(\bigvee_{a\in\mathrm{Span}\{f_1,f_2,\dots,f_s\}}\hat{\phi}_n(a)(M_n^{\hat{\lambda}})\right)\le(1+\epsilon)\dim_K(M_n^{\hat{\lambda}})\,.$$

Also, if $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_q$ are independent vectors in $K^{\mu(t)}$, then the corresponding subspaces $M_n^{\hat{\lambda}_1}, M_n^{\hat{\lambda}_2}, \dots, M_n^{\hat{\lambda}_q}$ are independent as well. We call an equivalence class μ *large* if

$$\mu(t)\dim_K(M_n^\mu) \geq \frac{(1-\epsilon)\epsilon}{100C} m_n l$$

holds, where M_n^μ is representing the class μ. It is not hard to check that

$$\dim_K\Big(\bigoplus_{\mu \text{ is large}} \oplus_{i=1}^{\mu(t)} M_n^{i,\mu}\Big) \geq (1-\frac{3}{2}\epsilon) m_n l\,, \tag{5}$$

holds, where $\{M_n^{1,\mu}, M_n^{2,\mu}, \dots, M_n^{\mu(t),\mu}\}$ are the subspaces in the class μ. Now let μ_n be a large class for some $n \geq 1$ and $M_n^{i_1}$ be its representative. For $v \in M_n^{i_1}$, let $H_v^{\mu_n} \subset K^{\mu_n(t)}$ be the set of vectors $\hat{\lambda}$ such that $\sum_{j=1}^{\mu_n(t)} \lambda_j \alpha_j(v) \in V_n$. Since $\lim_{n\to\infty} \frac{\dim_K(V_n)}{m_n l} = 1$, we get

$$\lim_{n\to\infty} \frac{\dim_K(\cap_v H_v^{\mu_n})}{\mu_n(t)} = 1\,. \tag{6}$$

For each large class μ_n, when n is large enough we choose a basis $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_p$ in $\cap_v H_v^{\mu_n}$ and set $Q_{i,\mu_n}^n := M_n^{\hat{\lambda}_i}$. Then, by (5) and (6) the subspaces Q_{i,μ_n}^n will satisfy the conditions of our proposition, provided that n is large enough. $\square$

So, we have a convergent sequence of representations

$$\{\psi_n : R \to \mathrm{End}_K(V_n)\}_{n=1}^\infty$$

that are hyperfinite in the weaker sense, that is for any $\epsilon > 0$ we have $L_\epsilon > 0$ and for large enough n independent subspaces $Q_n^1, Q_n^2, \dots, Q^{l(n)} \subset V_n$ such that

- For any $1 \leq i \leq l(n)$, $\dim_K(Q_n^i) \leq L_\epsilon$.
- $\sum_{i=1}^{l(n)} \dim_K(Q_n^i) \geq (1-2\epsilon)\dim_K(V_n)\,.$
- For any $1 \leq i \leq l(n)$,

$$\dim_K\left(\bigvee_{r\in R} \psi_n(r)(Q_n^i)\right) \leq (1+\epsilon)\dim_K(Q_i^n)\,.$$

Note that we used the fact that if s is large then $\mathrm{Span}\{f_1, f_2, \dots, f_s\}$ contains $\rho(R)$. Our proposition immediately follows from the following lemma.

Lemma 6.5. *Let $\tau : R \to End_K(V)$ be a unital representation and $L \subset V$ be a subspace such that*

$$\dim_K\left(\bigvee_{r\in R} \tau(r)(L)\right) \leq (1+\epsilon)\dim_K(L)\,.$$

Then there exists a subspace $L' \subset L$, $\frac{\dim_K(L')}{\dim_K(L)} \geq 1 - |R|\epsilon$ such that L' is an R-module (that is for any $r \in R$, $\tau(r)L' \subset L'$).

Proof. By Lemma 6.2, for any $r \in R$, there exists a subspace $L_r \subset L$ such that

$$\dim_K(L_r) \geq (1-\epsilon)\dim_K(L)$$

and $\tau(r)L_r \subset L$. Then $\dim_K(\cap_{r\in R} L_r) \geq (1-|R|\epsilon)\dim_K(L)$ and for any $r \in R$, $\tau(r)(\cap_{r\in R} L_r) \subset L$. Hence the subspace $L' = \bigvee_{r\in R}(\tau(r)(\cap_{r\in R} L_r))$ satisfies the condition of our lemma. □

Proposition 6.2. *Let $K[X]$ be the polynomial algebra over K. Then the family of all the finite dimensional representations K[X]-Mod are hyperfinite.*

Corollary 6.1. *If $\phi : R \to Mat_{l\times l}(K[X])$ is a homomorphism, then ϕ^*(K[X]-Mod) is a hyperfinite family in R-Mod.*

Proof. (of Proposition 6.2) It is enough to show that the class of indecomposable finite dimensional $K[X]$-modules is hyperfinite. By Jordan's theorem indecomposable elements in K[X]-Mod are in the form of $K[X]/f^nK[X]$, where $n \geq 1$ and $f \in K[X]$ is an irreducible, monic polynomial. The module structure of $M_{f^n} = K[X]/f^nK[X]$ is given the following way,

- $M_{f^n} = \mathrm{Span}\{1, x, x^2, \dots, x^{\deg(f)n-1}\}$
- $X \cdot t^i = t^{i+1}$, if $1 \leq i \leq \deg(f)n - 2$
- $X \cdot t^{\deg(f)n} - 1 = t^{\deg(f)n} - f^n(t)$.

Let $\epsilon > 0$ and $k \geq 1/\epsilon$. We will show that for any module M_{f^n}, we have independent subspaces $N^1_{f^n}, N^2_{f^n}, \dots, N^{k(f^n)}_{f^n} \subset M_{f^n}$ such that

- For any $1 \leq i \leq k(f^n)$, $\dim_K(N^i_{f^n}) \leq 2k^2$
- For any $1 \leq i \leq k(f^n)$,
$$\frac{\dim_K(X(N^i_{f^n}) + N^i_{f^n})}{\dim_K(N^i_{f^n})} \leq 1 + \epsilon$$
- $\sum_{i=1}^{k(f^n)} \dim_K(N^i_{f^n}) \geq (1-\epsilon)\dim_K(M_{f^n})$.

If $\deg(f)n \leq 2k^2$, let $k(f^n) = 1$ and $N^1_{f^n} = M_{f^n}$. If $\deg(f)n > 2k^2$, let C be the integer such that

$$Ck - 1 < \deg(f)n \leq (C+1)k - 1$$

and for $1 \leq i \leq C+1$ let

$$N^i_{f^n} := \mathrm{Span}\{t^{(i-1)k}, t^{(i-1)k} + 1, \dots, t^{ik-1}\}.$$

Then $\dim_K(X(N^i_{f^n}) + N^i_{f^n}) = \dim_K(N^i_{f^n}) + 1$ and all the three inequalities above are satisfied. □

7 Amenable and non-amenable skew fields

In this section we recall some basic definitions and results about amenable and non-amenable skew fields. We will confirm Conjecture 2 for amenable algebras $\mathrm{Mat}_{l\times l}(D)$, when D is an amenable skew field. We also prove the non-amenability of the free skew field D^r_K. This result is hopefully interesting on its own right, nevertheless it will be important for us in the proof of Theorem 6.
Let $\mathcal{A}$ be a countable dimensional algebra over the base field K. We say that the algebra is (left) amenable ([18], [12]) if there exists a sequence of finite dimensional K-linear subspaces $\{W_n\}_{n=1}^\infty \subset \mathcal{A}$ such that for each $a \in \mathcal{A}$

$$\lim_{n\to\infty} \frac{\dim_K(W_n + aW_n)}{\dim_K(W_n)} = 1\,.$$

Let us recall some basic facts on amenable algebras.

- If $\mathcal{A}$ is an amenable domain, then it has the Ore property and its classical skew field of quotients is amenable as well (Proposition 2.1, [12]).
- If $\mathcal{A} = K\Gamma$ is a group algebra, then $\mathcal{A}$ is amenable if and only if Γ is an amenable group [12], [2].
- If $E \subset D$ are skew fields over the base field K, and E is non-amenable, then so is D (Theorem 1. [12])
- If D and E are amenable skew fields and $D\otimes_K E$ is a domain, then $D\otimes_K E$ is amenable (Proposition 2.4 [12]).

Now let KF_r be the free algebra $K\langle x_1, x_2, \dots, x_t\rangle$ on r indeterminates. Let KF_r-Mod be the set of all finite dimensional modules over KF_r. So, any element $M \in KF_r$-Mod defines a Sylvester rank function on KF_r and the convergence of such representations is well-defined (see [15]) exactly the same way as for finite dimensional algebras. Note that a convergent sequence of representations can always be viewed as a sofic approximation of the algebra KF_r/I, where I is the ideal of elements $a \in KF_r$ for which $\lim_{n\to\infty} rk_{M_n}(a) = 0$.
Let $\mathcal{M} = \{M_n\}_{n=1}^\infty \subset KF_r$-Mod be a convergent sequence of modules and $\phi_\mathcal{M} : KF_r \to \mathrm{Mat}_\omega^\mathcal{M}$ be the ultraproduct representation as in the previous sections (we suppose that K is a finite field). Then the division closure of the image of KF_r in $\mathrm{Mat}\,\mathcal{M}_\omega$ is a skew field if and only if the rank function rk_ω is integer-valued (Theorem 2. [15]).
Furthermore, D is an amenable skew field if and only if the convergent sequence $\mathcal{M}$ is hyperfinite (Theorem 3. [15]). According to Proposition 7.1 [15] the free skew field D^r_K on r generators over the base field K is sofic (for the free skew field see e.g.[8].) That is, there exists a convergent sequence $\{M_n\}_{n=1}^\infty \subset KF_r$-Mod , for which the division closure in $\mathrm{Mat}_\omega^\mathcal{M}$ is the free skew field. In [12] using operator algebraic methods we proved that the free field over the complex numbers is non-amenable. Now we prove the following theorem.

Theorem 4. *The free skew field on $r > 1$ generators over an arbitrary base field K is non-amenable.*

Proof. First, observe that for any base field K, the free algebra KF_r is isomorphic to its opposition algebra KF_r^{op}, hence the free skew field $D = D_K^r$ is isomorphic to its opposite algebra D^{op}. Recall that $D \otimes_K D$ is a domain (Theorem 3.1, [10]). Hence, it is enough to prove that there exists an embedding of $D \otimes D^{op}$-modules

$$\pi : (D \otimes D^{op})^2 \to D \otimes D^{op} . \tag{7}$$

Indeed, if D is amenable, then by the properties listed above $D \otimes D^{op}$ is an amenable domain and therefore $D \otimes D^{op}$ has the Ore property. Thus two non-trivial left ideals of $D \otimes D^{op}$ have non-zero intersection, in contradiction with the existence of the embedding (7). In order to construct the embedding, we follow the lines of Lemma 4.21 in [5] (I am indebted to Andrey Lazarev for calling my attention to this paper). First, consider the two-terms free resolution of KF_r as a bimodule over itself

$$0 \to KF_r \otimes_K [x_1, x_2] \otimes_K KF_r \to KF_r \otimes_K KF_r \to KF_r \to 0$$

as in [5]. Here $[x_1, x_2]$ denotes the 2-dimensional vector space spanned by x_1, x_2. If we tensor the resolution above by D on the right, we obtain an exact sequence of $KF_r - D$-bimodules

$$0 \to KF_r \otimes_K [x_1, x_2] \otimes_K D \to KF_r \otimes_K D \to D \to 0 \,,$$

since $\mathrm{Tor}_1^{KF_r}(KF_r, D) = 0$ by the definition of the functor Tor. Now let us tensor this exact sequence by D on the left. Notice that $\mathrm{Tor}_1^{KF_r}(D, D) = 0$ ([25] Theorems 4.7,4.8). Hence, we obtain an embedding of $D - D$-bimodules

$$i : D \otimes_K [x_1, x_2] \otimes_K D \to D \otimes D \,.$$

That is we have the embedding of left $D \otimes D^{op}$-modules

$$j : (D \otimes D^{op})^2 \to D \otimes D^{op}$$

we sought for. This finishes the proof of our theorem. □

Theorem 5. *Let $\rho : R \to Mat_{l \times l}(D)$ be an infinite dimensional representation of a finite dimensional algebra R over the finite field K, where D is a countable dimensional amenable skew field. Then the associated element $rk_\rho \in Rank(R)$ is amenable.*

Proof. Let $\{d_1, d_2, \dots, d_r\}$ be the set of all entries in the matrices $\{\rho(s)\}_{s \in R}$. Let $\pi : KF_r \to D$ be the unital homomorphism mapping x_i to d_i. Let $E \subset D$ be the sub skew field generated by $\{d_1, d_2, \dots, d_r\}$ (that is the division closure of $\mathrm{Im}\,(\pi)$). From now on, we can suppose that ρ maps R into $\mathrm{Mat}_{l \times l}(E)$ and π maps KF_r into E. Let $S := KF_r / \mathrm{Ker}\,(\pi)$, and $\theta : KF_r \to S$ and $\hat{\pi} : S \to E$ be the natural quotient maps, so $\pi = \hat{\pi} \circ \theta$. Then, the homomorphism $\zeta : R \to$

$\mathrm{Mat}_{l\times l}(S)$ can be defined in a unique way to satisfy $\hat{\pi}\circ\zeta=\rho$ (note that $\hat{\pi}$ is injective). Finally, pick a lifting $\mu: S\to KF_r$. that is μ is an injective, unital, linear map such that $\theta\circ\mu=\mathrm{Id}_S$. Then, $\pi\circ\mu=\hat{\pi}$.
Since E is amenable, it is sofic. Hence, we have a sofic representation sequence $\{\phi_n: E\to \mathrm{Mat}_{m_n\times m_n}(K)\}_{n=1}^\infty$. Therefore, we have the sofic representation sequence

$$\{\phi_n\circ\hat{\pi}: S\to \mathrm{Mat}_{m_n\times m_n}(K)\}_{n=1}^\infty$$

that factors through the sofic representation system

$$\{\phi_n\circ\pi: KF_r\to \mathrm{Mat}_{m_n\times m_n}(K)\}_{n=1}^\infty .$$

By Proposition 11.1 [15], $\{\phi_n\circ\pi\}_{n=1}^\infty$ is hyperfinite hence $\{\phi_n\circ\hat{\pi}\}_{n=1}^\infty$ is a hyperfinite sofic representation. Therefore by Proposition 6.1, $rk_\zeta=rk_\rho$ is an amenable element of the rank spectrum. $\square$

8 The wild Kronecker algebras are of non-amenable representation types

First recall the notion of Kronecker quiver algebras and the wildness phenomenon (see e.g. [3]). Let $r\geq 3$ and Q_r be the finite dimensional algebra with basis $p_1,p_2,\{e_i\}_{i=1}^r$, where

- $p_1^2=p_1, p_2^2=p_2, p_1+p_2=1, p_1p_2=p_2p_1=0.$
- For any $1\leq i,j\leq r$, $e_ip_1=e_i$, $p_1e_i=0$, $e_ip_2=0$, $p_2e_i=e_i$, $e_ie_j=0$.

The algebra Q_r defined the way above is the Kronecker quiver algebra of index r. The wildness of Q_r means that the classification of finitely generated indecomposable modules over Q_r is as complicated as the classification of finitely generated indecomposable modules over noncommutative free algebras. To make it precise, there exists a representation $\pi_r: Q_r\to \mathrm{Mat}_{2\times 2}(KF_{r-1})$ such that

- The associated functor $\pi_r^*: KF_{r-1}$-Mod $\to Q_r$-Mod is injective.
- If $M\in KF_{r-1}$-Mod is indecomposable, then $\pi_r^*(M)$ is indecomposable as well.

The following representation satisfies the conditions above: $\pi_r(p_1)=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $\pi_r(p_2)=\begin{bmatrix}0&0\\0&1\end{bmatrix}$, $\pi_r(e_1)=\begin{bmatrix}0&0\\1&0\end{bmatrix}$, $\pi_r(e_j)=\begin{bmatrix}0&0\\x_{j-1}&0\end{bmatrix}$ if $j>1$.

Proposition 8.1. *Let $\pi_r: Q_r\to Mat_{2\times 2}(KF_{r-1})$ the representation as above. Then the family of modules $\{\pi_r^*(M_n)\}_{n=1}^\infty\subset Q_r$-Mod is hyperfinite if and only if $\{M_n\}_{n=1}^\infty\subset KF_{r-1}$-Mod is hyperfinite.*

Proof. The "if" part follows from Proposition 6.1. For the converse, let us suppose that $\{\pi_r^*(M_n)\}_{n=1}^\infty$ is a hyperfinite family. That is, for any $\epsilon > 0$, we have $L_\epsilon > 0$ and for any $n \geq 1$ we have independent Q_r-submodules $N_n^1, N_n^2, \dots, N_n^{k(n)} \subset \pi_r^*(M_n)$ such that

- $\dim_K(N_n^i) \leq L_\epsilon$
- $\sum_{i=1}^{k(n)} \dim_K(N_n^i) \geq (1-\epsilon)2\dim_K(M_n)$. Note that as a vector space $\pi_r^*(M_n)$ is isomorphic to $M_n \oplus M_n$ and p_i acts on $\pi_r^*(M_n)$ as the projection onto the i-th coordinate.

Let us consider the subspaces $p_1 N_n^i \subset M_n \oplus 0$, $p_2 N_n^i \subset 0 \oplus M_n$. We use the notation $[p_1 N_n^i]$ resp. $[p_2 N_n^i]$ for the subspaces in M_n that are the projections of the spaces above onto the first resp. second component of $M_n \oplus M_n$. Then, for any $n \geq 1$, $1 \leq i \leq k(n)$

- $[p_1 N_n^i] \subset [p_2 N_n^i]$
- $x_j [p_1 N_n^i] \subset [p_2 N_n^i]$ if $1 \leq j \leq r-1$.

Also, the subspaces $\{[p_1 N_n^i]\}_{i=1}^{k(n)}$ are independent. Since

$$\sum_{i=1}^{k(n)} (\dim_K[p_1 N_n^i] + \dim_K[p_2 N_n^i]) \geq (1-\epsilon)2\dim_K(M_n)\,,$$

we have the inequality

$$\sum_{i=1}^{k(n)} \dim_K[p_1 N_n^i] \geq (1-2\epsilon)\dim_K(M_n)\,.$$

Lemma 8.1. *Let $C > 0$, $\{a_i\}_{i=1}^{k(n)}$, $\{b_i\}_{i=1}^{k(n)}$, be positive real numbers satisfying the following inequalities.*

- $\sum_{i=1}^{k(n)} a_i \geq (1-2\epsilon)C$
- $\sum_{i=1}^{k(n)} b_i \leq C$
- $a_i \leq b_i$.

Let $S = \{i \mid \frac{b_i}{a_i} < 1 + \sqrt{\epsilon}\}$. Then $\sum_{i \neq S} a_i \leq 2\sqrt{\epsilon}C$.

Proof. We have

$$C \geq \sum_{i=1}^{k(n)} b_i \geq \sum_{i \notin S} (1+\sqrt{\epsilon})a_i + \sum_{i \in S} a_i \geq$$
$$\geq (1-2\epsilon)C + (\sum_{i \notin S} a_i)\sqrt{\epsilon}\,.$$

Therefore $\sum_{i \neq S} a_i \leq 2\sqrt{\epsilon}C$. $\square$

Let us apply the lemma above, for $C = \dim_K(M_n)$, $a_i = [p_1 N_i^n]$, $b_i = [p_2 N_i^n]$. We get that

- $\sum_{i \notin S} \dim_K([p_1 N_i^n]) \leq 2\sqrt{\epsilon} \dim_K(M_n)$.
- If $i \in S$, then $\dim_K \left([p_1 N_i^n] \oplus \bigoplus_{j=1}^{r-1} x_j [p_1 N_i^n] \right) \leq (1 + \sqrt{\epsilon}) \dim_K([p_1 N_i^n])$.
- $\dim_K([p_1 N_i^n]) \leq L_\epsilon$.

Hence by definition, $\{M_n\}_{n=1}^\infty$ is a hyperfinite family of KF_{r-1}-modules. $\square$

Theorem 6. *The wild Kronecker quiver algebras are of non-amenable representation type.*

Proof. Let $r \geq 2$ and $\mathcal{M} = \{M_n\}_{n=1}^\infty \subset KF_r$-Mod be a convergent sequence of KF_r-modules such that for all KF_r-matrices A

$$\lim_{n \to \infty} rk_{M_n}(A) \in \mathbb{Z} .$$

Let $\phi_\omega : KF_r \to \mathrm{Mat}_\omega^\mathcal{M}$ be the limit representation of the sequence $\{M_n\}_{n=1}^\infty$ Then by Proposition 8.1 [15], the division closure of $\phi_\omega(KF_r)$ in the ring $\mathrm{Mat}_\omega^\mathcal{M}$ is a skew field. We also know that

- The division closure above is an amenable skew field if and only if $\{M_n\}_{n=1}^\infty$ is hyperfinite (Proposition 11.1 and 12.1 [15]).
- There exists a convergent sequence of modules $\{M_n\}_{n=1}^\infty$ such that the division closure is the free skew field on r-generators with base field K (Proposition 7.1 [15]).

Hence by Proposition 8.1 and Theorem 4, there exist non-hyperfinite families of modules for the algebra Q_r, $r \geq 3$. $\square$

9 Hyperfinite families and the continuous algebra of von Neumann

The goal of this section is to prove Theorem 2. In fact, we will prove a stronger interpolation theorem. Let K be a finite field, R be a finite dimensional algebra over K and let $\mathcal{M} = \{M_n\}_{n=1}^\infty \subset$ R-Mod be a hyperfinite convergent sequence of modules and ω be a non-principal ultrafilter.

Theorem 7. *Let $\rho_\omega : R \to Mat_\omega^\mathcal{M}$ be the ultraproduct representation corresponding to the sequence of modules above. Then there exists a subalgebra $Im(\rho_\omega) \subset S \subset Mat_\omega^\mathcal{M}$ such that $S \cong M_K$.*

According to the theorem, when we pick a countable dimensional subalgebra from the ultraproduct, we can always land inside M_K.

Proof. The proof will be given in a series of lemmas and propositions.

Lemma 9.1. *Let $a_1, a_2, \dots, a_r$ be positive integers, $\epsilon > 0$ and $U \in \omega$. Let us have an integer m_i for each $i \in U$ (we suppose that m_i tends to infinity as i tends to infinity). Suppose that for any $i \in U$ and $1 \leq j \leq r$ we have a non-negative integer c_{ij} such that*

$$(1-\epsilon)m_i \leq \sum_{j=1}^{r} c_{ij}a_j \leq m_i\,.$$

Then we have a subset $V \subset U$, $V \in \omega$, and non-negative integers $\{p_j\}_{j=1}^r$, $\{l_k\}_{k=1}^\infty$ such that for any $i \in V$

$$(1-2\epsilon)m_i \leq l_i(\sum_{j=1}^{r} p_j a_j) \leq m_i\,.$$

Proof. For large enough $i \in U$, let

$$\frac{\epsilon m_i}{2r(\max_{1\leq j\leq r} a_j)} < l_i < \frac{\epsilon m_i}{r(\max_{1\leq j\leq r} a_j)}$$

be an integer. Then, for any $1 \leq j \leq r$, we have $c_{ij} = l_i d_{ij} + g_{ij}$, where $0 \leq g_{ij} < l_i$ and

$$0 \leq d_{ij} \leq \frac{2r\max_{1\leq j\leq r} a_j}{\epsilon}\,.$$

Since

$$\sum_{j=1}^{r} l_i a_j \leq \frac{\epsilon m_i}{r(\max_{1\leq j\leq r} a_j)} \sum_{j=1}^{r} a_j \leq \epsilon m_i\,,$$

we have that $\sum_{j=1}^r g_{ij}a_j \leq \epsilon m_i$ that is

$$(1-2\epsilon)m_i \leq l_i \sum_{j=1}^{r} d_{ij}a_j \leq m_i\,.$$

Since the set $\{d_{ij}\}_{i\in U, 1\leq j\leq r}$ is bounded, there are non-negative integers $p_1, p_2, \dots, p_r$ such that

$$V = \{i \mid\ d_{ij} = p_j, \text{ for any } 1 \leq j \leq r\} \in \omega\,.$$

Therefore if $i \in V$, we have

$$(1-2\epsilon)m_i \leq l_i(\sum_{j=1}^{r} p_j a_j) \leq m_i\,. \qquad \square$$

Definition 9.1. *The module $A \in R$-Mod ϵ-tiles the module $B \in R$-Mod, if there exists $k \geq 1$ and $A^k \cong C \subset B$ such that $\dim_K(C) \geq (1-\epsilon)\dim_K(B)$.*

Lemma 9.2. *Let $U \in \omega$ be a subset of the naturals. Suppose that $\{M_i\}_{i \in U}$ is a hyperfinite family. Then for any ϵ, there exists $A \in R$-Mod and $V \subset U$, $V \in \omega$ such that for each $i \in V$ the module A ϵ-tiles M_i.*

Proof. By hyperfiniteness, we have modules $A_1, A_2, \dots, A_r \in$ R-Mod such that for each $n \geq 1$ there exist constants $\{c_{nj}\}_{1 \leq j \leq r}$ so that

$$\oplus_{j=1}^{r} A_j^{c_{nj}} \cong B_n \subset M_n$$

satisfying the inequality

$$(1 - \frac{\epsilon}{2}) \dim_K(M_n) \leq \dim_K(B_n)\,.$$

By Lemma 9.1, there exist constants $\{p_j\}_{j=1}^{r}$, a module $A = \oplus_{j=1}^{r} A_j^{p_j}$ and $V \subset U$, $V \in \omega$ so that A ϵ-tiles M_i if $i \in V$. ∎

Lemma 9.3. *Suppose that the module $A \in R$-Mod ϵ-tiles all elements of the sequence $\{B_n\}_{n=1}^{\infty} \subset R$-Mod, where $\lim_{n\to\infty} \dim_K(B_n) = \infty$. Then for any $l \geq 1$, A^l 2ϵ-tiles B_n, if n is large enough.*

(the proof is straightforward)

Lemma 9.4. *Let $0 < \epsilon < 1$ and $A^k \cong C \subset B$ finitely generated R-modules such that*

$$(1-\epsilon)\dim_K(B) \leq \dim_K(C)\,.$$

Let $\delta \leq \frac{\epsilon}{\dim_K^2(A)}$ and $B' \subset B$ is a submodule such that

$$(1-\delta)\dim_K(B) \leq \dim_K(B')\,.$$

Then, there exists $k' \geq 0$ and $A^{k'} \cong C' \subset B'$ such that

$$(1-2\epsilon)\dim_K(B') \leq \dim_K(C')\,.$$

Proof. Let $b = \dim_K(B)$, that is,

$$(1-\epsilon)b \leq k \dim_K(A) \leq b\,.$$

Let us write A^k into the form of $A \otimes_K K^k$ and for each $a \in A$ define the vector space

$$V_a := \{v \mid a \otimes v \in B'\}\,.$$

Since

$$\dim_K(a \otimes K^k) + \dim_K(B') = \dim_K\left((a \otimes K^k) \cap B'\right) + \dim_K\left((a \otimes K^k) \vee B'\right)$$

we have that

$$k + (1-\delta)b \leq k + \dim_K(B') \leq \dim_K(V_a) + b\,.$$

That is, $k - \delta b \leq \dim_K(V_a)$. Hence, $k - \dim_K(A)\delta b \leq \dim_K(\cap_{a\in A} V_a)$. Observe that

$$C' = A \otimes_K (\cap_{a\in A} V_a) \subset B'$$

and

$$(1-2\epsilon)b \leq k \dim_K(A) - \delta b \dim_k^2(A) \leq \dim_K(C').$$

This finishes the proof of our lemma. $\square$

Now we put together the previous lemmas to have a single technical proposition.

Proposition 9.1. *Let $\{M_n\}_{n=1}^\infty \subset R$-Mod be a convergent, hyperfinite sequence. Then there exist modules $\{N_i\}_{i=1}^\infty \subset R$-Mod, subsets $\mathbb{N} \supseteq V_1 \supseteq V_2 \supseteq \dots, V_i \in \omega$ and a monotonically decreasing sequence of real numbers $\{\alpha_n\}_{n=1}^\infty$, $\alpha_n \to 0$ such that*

- *For any i, N_i α_i-tiles M_j, whenever $j \in V_i$.*
- *For any i, N_i α_i-tiles N_{i+1}.*
- *For any i, $\dim_K(N_{i+1})$ is divisible by $\dim_K(N_i)$.*
- *For any $k, l \geq 1$ and matrix $A \in Mat_{k\times l}(R)$.*

$$\lim_{n\to\infty} rk_{M_n}(A) = \lim_{i\to\infty} rk_{N_i}(A).$$

Proof. We proceed by induction. Our inductional hypothesis goes as follows. Suppose that $N_1, N_2, \dots N_l$ have already been constructed together with the sets $V_1 \supseteq V_2 \supseteq \dots \supseteq V_l$ and $\alpha_1 > \alpha_2 > \dots > \alpha_l$, $\alpha_i < \frac{1}{i}$, satisfying the following conditions.

- For any $1 \leq i \leq l-1$, N_i α_i-tiles M_j if $j \in V_i$.
- For any $1 \leq i \leq l-1$, N_i α_i-tiles N_{i+1}.
- N_l $\alpha_l/2$-tiles M_j if $j \in V_l$.
- For any $1 \leq i \leq l-1$, $\dim_K(N_{i+1})$ is divisible by $\dim_K(N_i)$.

By Lemma 9.2, we have a set $V'_{l+1} \subset V_l$, $V'_{l+1} \in \omega$ and $N'_{l+1} \in$ R-Mod such that the module N'_{l+1} $\alpha_{l+1}/2$-tiles M_j if $j \in V'_{l+1}$, where $\alpha_{l+1} \leq \frac{\alpha_l}{2\dim_K(N_l)}$. Then by Lemma 9.4, there exists some $m \geq 1$ such that N_l α_l-tiles $(N'_{l+1})^m$. Hence, N_l α_l-tiles $(N'_{l+1})^{m\dim_K(N_l)}$ as well. Let $N_{l+1} = (N'_{l+1})^{m\dim_K(N_l)}$. By Lemma 9.3, there exits $n_j > 0$ such that N_{l+1} α_{l+1}-tiles M_j provided that $j \in V'_{l+1}$ and $j > n_j$. So, let

$$V_{l+1} = V'_{l+1} \cap \{n \mid n > n_j\}.$$

Clearly, $V_{l+1} \in \omega$. Then, we satisfy the inductional hypothesis with $V_1 \supseteq V_2 \supseteq \dots \supseteq V_{l+1}$ and the modules $N_1, N_2, \dots, N_{l+1}$. In order to finish the proof of our proposition, we need to show that for any $k, l \geq 1$ and matrix $A \in \mathrm{Mat}_{k\times l}(R)$

$$\lim_{n\to\infty} rk_{M_n}(A) = \lim_{i\to\infty} rk_{N_i}(A).$$

Therefore, it is enough to prove the following approximation lemma.

Lemma 9.5. *Let $N \subset M$ be finitely generated R-modules such that $\dim_K(N) \geq \dim_K(M)(1-\epsilon)$ for some $0 < \epsilon < 1$. Let $k, l \geq 1$ and $A \in Mat_{k \times l}(R)$. Then*

$$|rk_M(A) - rk_N(A)| \leq 2\epsilon l$$

.

Proof. By definition,

$$rk_N(A) = \frac{\mathrm{rank}_{N^l}(A)}{\dim_K(N)}, \quad rk_M(A) = \frac{\mathrm{rank}_{M^l}(A)}{\dim_K(M)},$$

where A is viewed as a linear map from M^l to M^k (and from N^l to N^k) and $\mathrm{rank}_{M^l}(A) := \dim_K(\mathrm{Im}\ |_{M^l}\ (A))$. Then,

$$|rk_M(A) - rk_N(A)| = \left| \frac{\mathrm{rank}_{M^l}(A)}{\dim_K(M)} - \frac{\mathrm{rank}_{N^l}(A)}{\dim_K(N)} \right| \leq$$

$$\leq \left| \frac{\mathrm{rank}_{M^l}(A) - \mathrm{rank}_{N^l}(A)}{\dim_K(M)} \right| + \left| \frac{\mathrm{rank}_{N^l}(A)}{\dim_K(M)} - \frac{\mathrm{rank}_{N^l}(A)}{\dim_K(N)} \right| \leq 2\epsilon l$$

Hence our lemma and the proposition follows. ∎

Now we turn to the proof of Theorem 7. We will use the spaces and constants of Proposition 9.1. Set $n_i = \dim_K(N_i)$, $m_j = \dim_K(M_j)$. We need three maps for the proof of our theorem, $\rho_\omega : R \to \prod_\omega \mathrm{End}_K(M_j) = \mathrm{Mat}_\omega^{\mathcal{M}}$ is already constructed in Section 2.

<u>The map $\rho : R \to M_K$:</u>

Let V, W be finite dimensional K-spaces such that $\dim_K(W) = l \dim_K(V)$ and let $W = W_1 \oplus W_2 \oplus \cdots \oplus W_l$ be a decomposition together with isomorphisms $s_j : V \to W_j$, $1 \leq j \leq l$. Then, we have the corresponding diagonal homomorphism $D : \mathrm{End}_K(V) \to \mathrm{End}_K(W)$ defined by

$$D(A)(w_1 \oplus w_2 \oplus \cdots \oplus w_l) =$$

$$= \left(s_1(A(s_1^{-1}(w_1))) \oplus s_2(A(s_2^{-1}(w_2))) \oplus \cdots \oplus s_l(A(s_l^{-1}(w_l)))\right).$$

Let $N_{i+1} = N_i^{l_i} \oplus V_{i,1} \oplus \cdots \oplus V_{i,b_i}$, where the first component is the submodule defined in Proposition 9.1 and $\{V_{i,j}\}_{j=1}^{b_i}$ are arbitrary subspaces satisfying $\dim_K(V_{i,j}) = n_i$. Note that the subspaces $V_{i.j}$ can be constructed by the divisibility condition and that $l_i n_i / n_{i+1} \geq 1 - \alpha_i$. Let $\tau_i : \mathrm{End}_K(N_i) \to \mathrm{End}_K(N_{i+1})$ be the corresponding diagonal map. So, we have a sequence of injective maps

$$\mathrm{End}_K(N_1) \xrightarrow{\tau_1} \mathrm{End}_K(N_2) \xrightarrow{\tau_2} \ldots$$

Hence we have a direct limit of matrix algebras

$$T = \varinjlim \mathrm{End}_K(N_i).$$

Let $\overline{T}$ be the closure of the algebra with respect to the unique rank metric, where $d(A,B) = \text{Rank}(A-B)$. Then by [19], $\overline{T} \cong M_K$. Note that this result is originally due to von Neumann. Let $\rho_i : R \to \text{End}_K(N_i)$ be the map given by the R-module structure. Observe that

$$rk_{N_{i+1}}(\rho_{i+1}(r) - \tau_i \circ \rho_i(r)) \leq \alpha_i \,.$$

Hence, the sequence $\{\rho_i(r)\}_{r=1}^{\infty}$ is Cauchy in $\overline{T}$. Thus, $\lim_{i\to\infty} \rho_i(r) = \rho(r)$ defines a homomorphism $\rho : R \to \overline{T} \cong M_K$.

The map $\Phi : M_K \to \prod_\omega \text{End}_K(M_j)$:

Let $j \in V_i$ and $M_j \cong N_i^{t_{i,j}} \oplus Z_{i,j}$ a decomposition into linear subspaces, where the first component $N_i^{t_{i,j}}$ is the submodule obtained by the α_i-tiling of M_j by the module N_i and $Z_{i,j}$ is an arbitrary complementing subspace. So, we have

$$\dim_K(Z_{i,j}) \leq \alpha_i \dim_K(M_j) \,. \tag{8}$$

Using the decomposition above, we define a non-unital injective homomorphism $\Phi_{i,j} : \text{End}_K(N_i) \to \text{End}_K(M_j)$ by

$$\Phi_{i,j}(A) = A \oplus A \oplus \cdots \oplus A \oplus 0 \,,$$

where there are $t_{i,j}$ copies of A in $\Phi_{i,j}(A)$. For $j \in \mathbb{N}$, let

- $q(j) = j$ if $j \in V_j$.
- $q(j) = \max\{i \mid j \in V_i\}$, if $j \notin V_j$.

Let $A \in T$ such that $A \in \text{End}_K(N_i)$ but $A \notin \tau_{i-1}(\text{End}_K(N_{i-1}))$ Then, let

- $\Psi_j(A) = 0$ if $j \notin V_i$.
- $\Psi_j(A) = \Phi_{q(j),j}(\tau_{q(j)-1} \circ \tau_{q(j)-2} \circ \cdots \circ \tau_i(A))$ if $j \in V_i$, that is $q(j) \geq i$.

Finally, let

$$\Phi'(A) = [\{\Psi_j(A)\}_{j=1}^{\infty}] \in \prod_\omega \text{End}_K(M_j) \,.$$

Lemma 9.6. *Φ' defines a unital, rank preserving homomorphism from T to $\prod_\omega End_K(M_j)$.*

Proof. Let $A \in \text{End}_K(N_s)$, $A \notin \tau_{s-1}(\text{End}_K(N_{s-1}))$ and $B \in \text{End}_K(N_t)$, $B \notin \tau_{t-1}(\text{End}_K(N_{t-1}))$. Then, by Proposition 9.1,

$$\{j \in\mid \Psi_j(AB) = \Psi_j(A)\Psi_j(B)\} \in \omega$$

and

$$\{j \in\mid \Psi_j(A+B) = \Psi_j(A) + \Psi_j(B)\} \in \omega \,.$$

Since the direct limit ring has a unique rank, the only thing remains to be proved is that Φ' is unital. By (8),

$$rk_{M_j}(\Psi_j(1) - \mathrm{Id}_{\mathrm{End}_K(M_j)}) \leq \alpha_{q(j)}\,.$$

Since for any $i \geq 1$,

$$\{j \in \mathbb{N} \mid q(j) \geq i\} \in \omega$$

we have that

$$\lim_{\omega}(rk_{M_j}(\Phi_j(1) - \mathrm{Id}_{\mathrm{End}_K(M_j)}) = 0\,.$$

Thus, Φ' is indeed unital. □

Now let $\Phi : M_K \to \prod_\omega \mathrm{End}_K(M_j)$ be the closure of Φ' that is

$$\Phi(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \Phi'(x_n)\,.$$

Since Φ' is rank preserving, Φ is a well-defined homomorphism. There, we have three homomorpisms.

- $\rho : R \to M_K$
- $\rho_\omega : R \to \prod_\omega \mathrm{End}_K(M_j)\,.$
- $\Phi : M_K \to \prod_\omega \mathrm{End}_K(M_j)\,.$

The following lemma implies that $\mathrm{Im}\,(\phi)$ contains $\mathrm{Im}\,(\rho_\omega)$, hence it completes the proof of Theorem 7.

Lemma 9.7. $\Phi \circ \rho = \rho_\omega\,.$

Proof. Let $\kappa_j : R \to \mathrm{End}_K(M_j)$ be the homomorphism given by the module structure. By (8), if $j \in V_i$ and $r \in R$,

$$rk_{M_j}(\Phi_{i,j}(\rho_i(r)) - \kappa_j(r)) \leq \alpha_i\,.$$

Therefore, for any $i \geq 1$

$$rk_\omega(\Phi \circ \rho_i(r) - \rho_\omega(r)) \leq \alpha_i\,.$$

Since, $\lim_{i\to\infty} \rho_i(r) = \rho(r)$ and Φ is rank preserving map, $\Phi \circ \rho(r) = \rho_\omega(r)\,.$ □

10 String algebras are of amenable representation types

The significance of string algebras is due to the fact that the system of f.g. indecomposable modules over such algebras can explicitely be described (see e.g. [23] and the references therein). First, let us recall the notion of a string algebra. Let Q be a finite quiver with vertex set Q_0 and arrow set Q_1. Let KQ be the associated path algebra and $I \triangleleft KQ$ be an ideal, generated by monomials in the path algebras, satisfying the following four conditions.

1. For any vertex $a \in Q_0$, there are at most two arrows with source a, and at most two arrows with target a.

2. For any arrow $\alpha \in Q_1$, there exists at most one $\beta \in Q_1$ such that $s(\beta) = t(\alpha)$ and $\beta\alpha \notin I$.

3. For any arrow $\alpha \in Q_1$, there exists at most one $\beta \in Q_1$ such that $t(\beta) = s(\alpha)$ and $\alpha\beta \notin I$.

4. There exists $q \geq 1$ such that any Q-path of length at least q is inside I.

Example 3. *$KQ = K\langle x, y\rangle, I = \langle x^m, y^n, xy, yx\}$ is a string algebra. The path-algebra of the 2-Kronecker quiver is itself a string algebra.*

Let $R = KQ/I$ be a string algebra. First we describe the so-called string modules over R, For every arrow $\alpha \in Q_1, \alpha^{-1}$ will denote its formal inverse such that $s(\alpha) = t(\alpha^{-1})$ $t(\alpha) = s(\alpha^{-1})$. We denote by Q_1^{-1} the set of all inverse arrows. The elements of $Q_1 \cup Q_1'$ are called *letters*. A *string* of length $n \geq 1$ is a sequence $C_1C_2 \dots C_n$ such that

1. $t(C_{i+1}) = s(C_i)$ if $1 \leq i \leq n-1$.

2. $C_i \neq C_{i+1}^{-1}$ if $1 \leq i \leq n-1$.

3. No substring $C_jC_{j+1}, \dots C_k$ or its inverse $C_k^{-1} \dots C_{j+1}^{-1}C_j^{-1}$ lies in the ideal I.

Additionally, any vertex $a \in Q_0$ is considered to be a string of length 0. Note that for any string $C_1C_2 \dots C_n$, $(C_1C_2 \dots C_n)^{-1} = C_n^{-1}C_{n-1}^{-1} \dots C_1^{-1}$ is also a string. The set of strings is denoted by W. For each string $C_1C_2 \dots C_n = S \in W$ we can associate an R-module $M(S)$ the following way. A K-basis for $M(S)$ is $z_0, z_1, \dots, z_n$ For a vertex $a \in Q_0$ and e_a (path of length zero starting and ending at a):

$$e_a z_i = \begin{cases} z_i \text{ if } i > 0 \text{ and } s(C_i) = a \\ z_0 \text{ if } i = 0 \text{ and } t(C_1) = a \\ 0 \text{ otherwise} \end{cases}$$

For an arrow $\alpha \in Q_1$ we define

$$\alpha z_i = \begin{cases} z_{i-1} \text{ if } C_{i-1} = \alpha \\ z_{i+1} \text{ if } C_i = \alpha^{-1} \\ 0 \text{ otherwise} \end{cases}$$

Observe that $M(S) \cong M(S^{-1})$. On the other hand, if $S_1 \neq S_2$, $M(S_1) \ncong M(S_2)$ whenever $S_1 \neq S_2^{-1}$.

Now we describe the band modules over the string algebra R. A string $S = S_{(0)} = C_1C_2 \dots C_n$ is called *cyclic* if $S_{(1)} = C_2C_3 \dots C_1$,$S_{(2)} = C_3C_4 \dots C_2$, $\dots$,$S_{(n-1)} = C_nC_1 \dots C_{n-1}$ are also strings. Let $S = C_1C_2 \dots C_n$ be a cyclic string and V be a finite dimensional vector space over K. Fix an indecomposable transformation ϕ on V by identifying V with $K[x]/f(x)^n$, where f is an irreducible monic polynomial, $d = n\deg(f)$, and ϕ is the multiplication by x. The band module $M(S,\phi)$ is defined the following way. $M(S,\phi) = \oplus_{i=1}^n V_i$, where $V_i \cong V$, that is for each $v \in V$ and $1 \leq i \leq n$ we fix $v_i \in V_i$. For any $\alpha \in Q_1$ and $i \neq 1,2$:

$$\alpha v_i = \begin{cases} v_{i-1} \text{ if } C_i = \alpha \\ v_{i+1} \text{ if } C_{i+1} = \alpha^{-1} \\ 0 \text{ otherwise} \end{cases}$$

Also,

$$\alpha v_1 = \begin{cases} v_n \text{ if } C_1 = \alpha \\ (\phi^{-1}(v))_2 \text{ if } C_2 = \alpha^{-1} \\ 0 \text{ otherwise} \end{cases}$$

$$\alpha v_2 = \begin{cases} (\phi(v))_1 \text{ if } C_2 = \alpha \\ v_3 \text{ if } C_3 = \alpha^{-1} \\ 0 \text{ otherwise} \end{cases}$$

By the Butler-Ringel classification of indecomposable modules [6], $M(S,\phi) \cong M(T,\phi')$ if and only if $\phi = \phi'$ and $S_{(i)} = T$ for some $i \geq 1$. Also, a band module and a string module is never isomorphic and any finitely generated indecomposable module over a string algebra is either a string module or a band module.

Proposition 10.1. *Any string algebra R is of hyperfinite type.*

The proof will be given in two lemmas.

Lemma 10.1. *The family of string modules over a string algebra R is hyperfinite.*

Proof. By 6.1, it is enough to show that there exists some $m_\epsilon > 0$ such that if $n \geq m_\epsilon$ and $w = C_1C_2\dots C_n$ is a string, then there exist independent subspaces $W_0, W_1, \dots, W_t \subset M(w)$ such that

- $$\dim_K(W_i \oplus \bigoplus_{\alpha \in Q_1} \alpha W_i) \leq (1+\epsilon)\dim_K(W_i)$$

  (we only need to check the elements $\alpha_\in Q_1$ not the whole path algebra by Lemma 6.2)

- $$\dim_K(\oplus_{i=0}^t W_i) \geq (1-\epsilon)\dim_K(M(w))\,.$$

Let m be an integer such that $\frac{m-2}{m} < 1+\epsilon$ and let $m_\epsilon \geq \frac{m}{\epsilon}$. Finally, let $t \geq 1$ such that $tm \leq n < (t+1)m$.

Let W_i be the subspace spanned by the vectors $\{z_{im}, z_{im+1}, \dots, z_{(i+1)m-1}\}$. Then,

$$\dim_K(W_i \oplus \bigoplus_{\alpha \in Q_1} \alpha W_i) \leq m+2 \leq (1+\epsilon)\dim_K(W_i)\,.$$

Also,

$$\dim_K(M(w)) - \dim_K(\oplus_{i=0}^t W_i) \leq m \leq \epsilon \dim_K(M(w)) . \qquad \square$$

Lemma 10.2. *The family of band modules over a string algebra R is hyperfinite.*

Proof. The proof will be very similar to the previous one. Fix $\epsilon > 0$. It is enough to prove that there exists $L_\epsilon > 0$, such that for any band module $M(S,\phi)$ there exist independent subspaces $\{V_\rho\}_{\rho \in G}$ such that

- For all $\rho \in G$, $\dim_K(V_\rho) \leq L_\epsilon$.
- $\dim_K(\sum_{\alpha \in Q_1} \alpha V_\rho + V_\rho) \leq (1+\epsilon)\dim_K(V_\rho)$
- $\sum_{\rho \in G} \dim_K(V_\rho) \leq (1-\epsilon)\dim_K M(S,\phi)$

Let $m \in \mathbb{N}$ such that $\frac{m+2}{m} \leq 1+\epsilon$ and $m_\epsilon > \frac{2m}{\epsilon}$. Let $S = C_1C_2\dots C_n$. First, suppose that $n > m_\epsilon$. Consider the basis $\{1, x, x^2, \dots, x^{d-1}\}$ for V and let $t \in \mathbb{N}$ such that $tm + 2 \leq n < (t+1)m + 1$. Then, for any pair $0 \leq i \leq t-1$ and $0 \leq j \leq d-1$ let W_i^j be the subspace of $M(S,\phi)$ spanned by the set $\{x^j_{im+2}, x^j_{im+3}, \dots, x^j_{(i+1)m+1}\}$. Then, it is easy to see that for any pair i,j

$$\dim_K(\sum_{\alpha \in Q_1} \alpha W_j^i + W_j^i) \leq (1+\epsilon)\dim_K(W_i^j) .$$

and

$$\sum_{0 \leq i \leq t-1} \sum_{0 \leq j \leq d-1} \dim_K(W_i^j) \leq (1-\epsilon)\dim_K M(S,\phi) .$$

Now, suppose that for the length of the cyclic string, we have $n \leq m_\epsilon$ and for the dimension of V we have $d > m_\epsilon$. Let t be as above, and for $0 \leq q \leq t-1$ let $Z_q \in M(S,\phi)$ be the subspace spanned by the set

$$\{\cup_{i=1}^n \cup_{j=qm}^{(q+1)m-1} x_i^j\} .$$

By the definition of the band module structure,

$$\dim_K(\sum_{\alpha \in Q_1} \alpha Z_q + Z_q) \leq (1+\epsilon)\dim_K(Z_q) .$$

and

$$\sum_{q=0}^{t-1} \dim_K Z_q \geq (1-\epsilon)\dim_K M(S,\phi) .$$

Also, for any q, $\dim_K(Z_q) \leq m m_\epsilon$. Observe that there are only finitely many band modules $M(S,\phi)$ for which $n, d \leq m_\epsilon$. Let M be the maximal K-dimension of these modules. So, in order to finish the proof of the lemma we need to set $L_\epsilon := \max(m m_\epsilon, M)$. $\square$

11 Benjamini-Schramm convergence implies the convergence of string modules

For the next three sections we fix a string algebra R over the finite field K. The sole purpose of the next two sections is to establish a relation between the convergence of string modules and the Benjamini-Schramm convergence of the associated edge-colored graphs. This will be the preparation for the proof of Theorem 1. Let $M(S)$ be the string module corresponding to the string $S = (C_1 C_2 \dots C_n)$. We can associate a graph G_S to S in a very natural way.

- The vertex set $V(G_S)$ has n elements $\{x_0, x_1, \dots, x_n\}$.
- For any $1 \leq i \leq n$, we have a directed edge between x_{i-1} and x_i. If $C_i = \alpha \in Q_1$, then the edge is directed from x_{i-1} to x_i and colored by α. If $C_i = \alpha^{-1} \in Q_1$, then the edge is directed from x_i to x_{i-1} and colored by α. That is, all the edge-colors are from Q_1.

Note that the graphs G_S and G_T are isomorphic as edge-colored, directed graphs if $S = T$ or $S = T^{-1}$. By definition, if x is a vertex of G_S and e, f are edges both pointing into resp. pointing out of x, then the edge-colors of e and f are different.

Definition 11.1. *A directed Q_1-edge colored graph is called a string graph if all of its components are in the form of G_S for some string S. So, for each finitely generated R-module M that is a sum of string modules, we have a unique string graph G_M and two R-modules are isomorphic if and only if the corresponding string graphs are isomorphic.*

Now we recall the Benjamini-Schramm graph convergence (see [4] and [14]). A rooted string graph is a connected string graph with a distinguished vertex x (the root). The radius of a rooted string graph is the shortest path distance between a vertex of H and the root. The finite set of rooted string graphs of radius less or equal than r is denoted by U^r, that is $U^{r-1} \subset U^r$. Let $H \in U^r$ and G be a string graph and let $P(H, G)$ denote the set of vertices y in G such that the rooted r-neighborhood of y is isomorphic to H (as rooted, edge-colored, directed graphs). Let $p(H, G) := \frac{|P(H,G)|}{|V(G)|}$. That is, $p(H, G)$ is the probability that a randomly chosen vertex of G has rooted r-neighborhood isomorphic to H. The following definition was originally given for simple graphs [4].

Definition 11.2. *A sequence of string graphs $\{G_n\}_{n=1}^{\infty}$ is convergent (in the sense of Benjamini and Schramm) if for any $r \geq 1$ and $H \in U^r$, $\lim_{n\to\infty} p(H, G_n)$ exists.*

Note that for any H, G and $k \geq 1$, $p(H, G) = p(H, G^k)$, where G^k is the disjoint union of k copies of G and it is easy to see that $p(H, G_1) = p(H, G_2)$ if and only if $G_1^m \cong G_2^n$ for some $m, n \geq 1$. So, by Proposition 4.1, we have the following lemma.

Lemma 11.1. *Let M, N be finite direct sums of string modules. They represent the same element in the rank spectrum if and only if $p(H, G_M) = p(H, G_N)$ holds for all $r \geq 1$ and $H \in U^r$.*

Now we can state our main proposition in the section.

Proposition 11.1. *Let $\{M_n\}_{n=1}^{\infty} \subset R$-Mod be a sequence of modules such that for any $n \geq 1$, M_n is the sum of string modules. Suppose that the graphs $\{G_{M_n}\}_{n=1}^{\infty}$ converges in the sense of Benjamini and Schramm. Then $\{M_n\}_{n=1}^{\infty}$ is a convergent sequence of R-modules.*

Proof. The proof will be given in a series of lemmas. First we need the definition of ϵ-isomorphism for string graphs.

Definition 11.3. *The string graphs G_1 and G_2 are ϵ-isomorphic, if they contain subgraphs $J_1 \subset G_1$ and $J_2 \subset G_2$ such that J_1 and J_2 are isomorphic string graphs and $|V(J_1)| \geq (1-\epsilon)|V(G_1)|, |V(J_2)| \geq (1-\epsilon)|V(G_2)|$.*

Lemma 11.2. *For any $\epsilon > 0$, there exists $\delta > 0$ and $n \geq 1$ such that if the string graphs G_1 and G_2 has the same amount of vertices and $|p(H, G_1) - p(H, G_2)| < \delta$ for any $H \in U^r$, $1 \leq r \leq n$, then G_1 and G_2 are ϵ-isomorphic.*

Proof. (of Lemma 11.2) Our lemma holds for simple planar graphs with a vertex degree bound by the main result of [22] (see also Theorem 5. in [15]). Our strategy is to reduce our lemma to the Newman-Sohler result, by enconding our Q_1-edge colored, directed graph with a simple planar graph. Let G be a string graph. The encoding simple graph $\hat{G}$ is constructed as follows. The vertex set of $\hat{G}$ consists of the vertex set of G plus two vertices a_{xy}, b_{xy} per each edge of G. If (x, y) is an edge of of G, then (x, a_{xy}), $(a_{x,y}, b_{x,y})$, $(b_{x,y}, y)$ will be edges of $\hat{G}$. That is, we substitute the original edge with a path of length three. For each element of $\alpha \in Q_1$, we choose a planar graph (the edge-color coding graph) T_α with a distinguished vertex t_α, such that each T_α have the same amount of vertices and all the vertex degrees of T_α are at least three. For each edge (x, y) of G we stick a copy of T_α to a_{xy} (by identifying a_{xy} and the distinguished vertex t_α) if the edge is directed towards x and its color is α. On the other hand, we stick a copy of T_α to b_{xy} if the edge is directed towards y and its color is α. Clearly,

- We can reconstruct G from $\hat{G}$.
- If $\{G_n\}_{n=1}^{\infty}$ is convergent then $\{\hat{G_n}\}_{n=1}^{\infty}$ is a convergent sequence of simple planar graphs.

So, by the Newmark-Sohler Theorem, for any $\epsilon' > 0$, there exists $n \geq 1$ and $\delta > 0$ such that if $|p(H, G_1) - p(H, G_2)| < \delta$ holds for any $H \in U^r$, $1 \leq r \leq n$ and G_1 and G_2 have the same amount of vertices, than $\hat{G_1}$ and $\hat{G_2}$ are ϵ'-isomorphic. Also, it is easy to see that for any $\epsilon > 0$ there exists $\epsilon' > 0$ such that if $\hat{G_1}$ and $\hat{G_2}$ are ϵ'-isomorphic, than G_1 and G_2 are ϵ-isomorphic. Hence our lemma follows. $\square$

Lemma 11.3. *For any $\epsilon > 0$, there exists $n \geq 1$ such that if for the string graphs G_1 and G_2*

$$1 - \delta \leq \frac{|V(G_1)|}{|V(G_2)|} \leq 1 + \delta$$

and $|p(H, G_1) - p(H, G_2)| < \delta$ for any $H \in U^r$, $1 \leq r \leq n$, then G_1 and G_2 are ϵ-isomorphic.

Proof. First, pick a δ and n to the constant $\epsilon/2$ as in Lemma 11.2. Let $0 < \delta' \leq \min(\delta/2, \epsilon/10)$ be a constant such that for any $H \in U^r$, $1 \leq r \leq n$,

$$|p(H, G_2) - p(H, G_3)| < \frac{\delta}{2}$$

provided that $G_2 \subset G_3$ and $|V(G_3)| \leq (1 + \delta')|V(G_2)|$. The existence of such δ' can easily be seen. Now we show that δ' and n satisfy the condition of our lemma. We can suppose that $|V(G_2)| \leq |V(G_1)|$. By adding strings of one single vertex, we can get $G_2 \subset G_3$ such that $|V(G_3)| = |V(G_1)|$. Then $|V(G_3)| \leq (1 + \delta')|V(G_2)|$. Hence for any $H \in U^r$, $1 \leq r \leq n$,

$$|p(H, G_1) - p(H, G_3)| < \delta\,.$$

Therefore, by Lemma 11.2, G_1 and G_2 are $\epsilon/2$-isomorphic. Since $\delta' < \epsilon/10$, G_2 and G_1 are ϵ-isomorphic. ☐

Lemma 11.4. *For any $\epsilon > 0$, there exists $\delta > 0$ such that if G_M and G_N are δ-isomorphic string graphs, then the modules M and N are ϵ-isomorphic.*

Proof. We need some notations. If $x \in G_M$, then let δ_x be the corresponding element in M. If $x \xrightarrow{\alpha} y$ is an edge, then $\alpha\delta_x = \delta_y$. Also, if there is no edge pointing out of x colored by α, then $\alpha\delta_x = 0$. By the definition of the string algebra, there exists $q > 0$ such that any path of the quiver Q of length at least q is in I, where $R = KQ/I$. Therefore, if the q-neighborhood of a vertex $x \in G_M$ is in a subgraph L, then

$$R\delta_x \subseteq \mathrm{Span}\{\delta_y \mid y \in V(L)\}\,. \tag{9}$$

Suppose that $L_1 \subset G_M$, $L_2 \subset G_N$ are isomorphic string graphs such that $|V(L_1)| \geq (1-\delta)|V(G_M)|, |V(L_2)| \geq (1-\delta)|V(G_N)|$. We call $x \in V(L_1)$ an inside point if its q-neighborhood is contained in L_1. Obviously, if δ is small enough, then $|I_1| \geq (1-\epsilon)|V(G_M)|, |I_2| \geq (1-\epsilon)|V(G_N)|$, where I_1 resp. I_2 are the sets of inside points in L_1 resp. in L_2. Since L_1 and L_2 are isomorphic, the modules generated by $\{\delta_x \mid x \in I_1\}$ and by $\{\delta_x \mid x \in I_2\}$ are also isomorphic. Thus our lemma follows. ☐

Therefore, from the previous lemmas we have the following corollary.

Corollary 11.1. *For any $\epsilon > 0$, there exists $\delta > 0$ and $n \geq 1$ such that if M and N are sums of string modules and for any $H \in U^r$, $1 \leq r \leq n$,*

$$|p(H, G_M) - p(H, G_N)| < \delta$$

and also, $1 - \delta \leq \frac{\dim_K(M)}{\dim_K(N)} \leq 1 + \delta$, then M and N are ϵ-isomorphic.

Now, by Lemma 9.5, our proposition follows. □

12 Convergence of string modules implies Benjamini-Schramm convergence

The goal of this section is to prove the following converse of Proposition 11.1.

Proposition 12.1. *Let $\{M_n\}_{n=1}^\infty \subset$ R-Mod be a convergent sequence of modules such that each module M_n is a sum of string modules. Then $\{G_{M_n}\}_{n=1}^\infty$ is convergent in the sense of Benjamini and Schramm.*

Proof. Let G be a string graph and $S = C_1 C_2 \dots C_k$ be a string. Let $L \cong G_S$ be a subgraph in G. We have $V(L) = \{l_0, l_1, \dots, l_k\}$, where

- The edge (l_{i-1}, l_i) is directed towards l_{i-1} and colored by C_i, if $C_i \in Q_1$.
- The edge (l_{i-1}, l_i) is directed towards l_i and colored by C_i^{-1}, if $C_i \in Q_1^{-1}$.

We call l_k the right resp. l_0 the left endvertex of L. By the definition of the strings, if L_1, L_2 are subgraphs in G isomorphic to G_S and their right resp. left endvertices coincide, then L_1 and L_2. For the string S and the string graph G, let $R(S, G)$ be the set of vertices x in G such that x is the right vertex of a substring L of G isomorphic to G_S. Let

$$r(S, G) := \frac{|R(S, G)|}{|V(G)|}\,.$$

We say that $\{G_n\}_{n=1}^\infty$ is **stringconvergent** if for any S, $\lim_{n\to\infty} r(S, G_n)$ exists. The following combinatorial lemma is straightforward to prove.

Lemma 12.1. *If $\{G_n\}_{n=1}^\infty$ is stringconvergent, then it is convergent in the sense of Benjamini and Schramm.*

Let $N \in$ R-Mod be a sum of string modules and G_N be its string graph. That is, for any $x \in V(G_N)$, we have a base element $\delta_x \in N$ and if $x \xrightarrow{\alpha} y$ is an edge of G_N, then $\alpha(\delta_x) = \delta_y$

Lemma 12.2 (The String Counting Lemma)**.** *For any string $S = C_1 C_2 \dots C_k$, there exists a pp-pair $\langle \phi_S, \psi_S \rangle$ such that*

$$\dim_K(N(\phi_S)) - \dim_K(N(\psi_S)) = R(S, G_N)$$

holds for any module N that is a sum of string modules.

Proof. For $1 \leq i \leq k$ let E_i be the equation

- $C_i n_i - n_{i-1} = 0$, if $C_i \in Q_1$.
- $C_i n_{i-1} - n_i = 0$, if $C_i \in Q_1^{-1}$.

Also, let E_0 be the equation $n_o = 0$. Let

$$N(\phi_S) = \{n_k \in N \mid \exists\, n_0, n_1, \dots, n_{k-1} \subset N \text{ such that } E_1, E_2, \dots, E_k \text{ holds}\}.$$

$$N(\psi_S) = \{n_k \in N \mid \exists\, n_0, n_1, \dots, n_{k-1} \subset N \text{ such that } E_0, E_1, \dots, E_k \text{ holds}\}.$$

Then clearly, $\phi_S \geq \psi_S$. Let $\pi : N(\phi_S) \to \mathrm{Span}\{\delta_x : x \in R(S, G_N)\}$ be the natural restriction map.

Lemma 12.3. *The map π is surjective and $Ker(\pi) = N(\psi_S)$.*

Proof. Let $x \in R(S, G_N)$ and $L = [x_0, x_1, \dots, x_k], x_k = x$ be the subgraph of G_N isomorphic to G_S. Then for $n_i = \delta_{x_i}$, $0 \leq i \leq k$ the equations $E_1, E_2, \dots, E_k$ hold. Hence π is surjective. If $z \in N(\phi_S)$ and the δ_x-coordinate of z is λ, then the δ_{x_0}-coordinate of z is λ as well. Hence, if $\pi(z) = 0$, then $z \in N(\psi_S)$. $\square$

So we have

$$r(S, M_n) = \frac{\dim_K(M_n(\phi_S))}{\dim_K(M_n)} - \frac{\dim_K(M_n(\psi_S))}{\dim_K(M_n)}.$$

By Corollary 3.1, the sequence $\{r(S, M_n)\}_{n=1}^\infty$ converges whenever the modules $\{M_n\}_{n=1}^\infty$ converge. Hence our proposition follows from Lemma 12.1. $\square$

13 The proof of Theorem 1

Let d_R be a metric on $\mathrm{Syl}(R)$ defining the compact topology. Then, for any $\epsilon > 0$ there exists $\delta > 0$ and matrices $A_1, A_2, \dots, A_n \in \mathrm{Mat}\,(R)$ so that if for any $1 \leq i \leq n$, $|rk_1(A_i) - rk_2(A_i)| < \delta$, then $d_R(rk_1, rk_2) < \epsilon$. Conversely, for any set of matrices $A_1, A_2, \dots, A_n \in \mathrm{Mat}\,(R)$ and $\delta > 0$, there exists $\epsilon > 0$ so that if $d_R(rk_1, rk_2) < \epsilon$, then for all $1 \leq i \leq n$, $|rk_1(A_i) - rk_2(A_i)| < \delta$. Particularly, by Lemma 9.5, for any $\epsilon > 0$ there exists $\delta > 0$ such that if M, N are δ-isomorphic, then $d_R(rk_M, rk_N) < \epsilon$. First, we prove a weak version of Theorem 1.

Proposition 13.1. *For any $\epsilon > 0$, there exists $\delta > 0$ such that if $M, N \in R$-Mod are sums of string modules, $1 - \delta \leq \frac{\dim_K(M)}{\dim_K(N)} \leq 1 + \delta$ and $d_R(rk_M, rk_N) < \delta$, then M and N are ϵ-isomorphic.*

Proof. We proceed by contradiction. Suppose that $\{M_n\}_{n=1}^\infty$, $\{N_n\}_{n=1}^\infty$ are sums of string modules such that $\lim_{n\to\infty} \frac{\dim_K(M_n)}{\dim_K(N_n)} = 1$ and none of the pairs M_n, N_n are ϵ-isomorphic. We can also assume that $\{M_n\}_{n=1}^\infty$, $\{N_n\}_{n=1}^\infty$ are convergent sequences. By Proposition 12.1, $\{G_{M_n}\}_{n=1}^\infty$ and $\{G_{N_n}\}_{n=1}^\infty$ are convergent in the sense of Benjamini and Schramm. Hence, by Lemma 11.3, for any $\delta > 0$ there exists n_δ such that for any $n \geq n_\delta$, G_{M_n} and G_{N_n} are δ-isomorphic. So, by

Lemma 11.4, M_n and N_n are always ϵ-isomorphic, if n is large enough, leading to a contradiction. $\square$

Now suppose that Theorem 1 does not hold. Again, we have two convergent sequence of modules $\{M_n\}_{n=1}^{\infty}$ and $\{N_n\}_{n=1}^{\infty}$ such that

- $\lim_{n\to\infty} \frac{\dim_K(M_n)}{\dim_K(N_n)} = 1$.
- $\lim_{n\to\infty} rk_{M_n} = \lim_{n\to\infty} rk_{N_n}$.
- For each $n \geq 1$, M_n, N_n are not ϵ-isomorphic.

In Lemma 10.2 we observed that for any $\kappa > 0$, there exists $T_\kappa \geq 0$ such that if for a band module B, $\dim_K(B) \geq T_\kappa$ holds, then there is a module L_B such that

- L_B is a sum of string modules.
- B and L_B are κ-isomorphic.

Now, we fix some constants. Let $\kappa_1 > 0$ such that if L_1, L_2 are sums of string modules and $d_R(L_1, L_2) < \kappa_1$, then L_1 and L_2 are $\epsilon/100$-isomorphic. Let $\kappa_2 > 0$ be a constant such that $\kappa_2 < \epsilon/100$ and if $M, N \in$ R-Mod are κ_2-isomorphic, then $d_R(M, N) \leq \kappa_1/3$. For each $n \geq 1$, we consider the decompositions,

$$M_n = M_n^1 \oplus M_n^2 \quad N_n = N_n^1 \oplus N_n^2 \,,$$

where M_n^1, N_n^1 are sums of band modules of dimension less than T_{κ_2} and M_n^2, N_n^2 are sums of string modules and band modules of dimension greater or equal than T_{κ_2}.

Lemma 13.1. *The sequences $\{M_n^1\}_{n=1}^{\infty}, \{M_n^2\}_{n=1}^{\infty}, \{N_n^1\}_{n=1}^{\infty}$ and $\{N_n^2\}_{n=1}^{\infty}$ are all convergent. Also, the limits $\lim_{n\to\infty} \frac{\dim_K(M_n^1)}{\dim_K(M_n)}, \lim_{n\to\infty} \frac{\dim_K(M_n^2)}{\dim_K(M_n)}, \lim_{n\to\infty} \frac{\dim_K(N_n^1)}{\dim_K(N_n)}$ and $\lim_{n\to\infty} \frac{\dim_K(N_n^2)}{\dim_K(N_n)}$ exist.*

Le $B_1, B_2, \dots, B_s$ be the set of band modules in R-Mod that have dimension less than T_{κ_2}. By Corollary 4.1, for any $1 \leq i \leq s$, the limit $\lim_{n\to\infty} w_{B_i}(M_n)$ exists. By the definition of the rank (see also Lemma 4.1) if $P_1, P_2, \dots, P_l \in$ R-Mod, then for any $A \in \mathrm{Mat}\,(R)$,

$$rk_{\oplus_{i=1}^l P_i}(A) = \sum_{i=1}^{l} \frac{\dim_K(P_i)}{\dim_K(\oplus_{i=1}^l P_i)} rk_{P_i}(A)\,. \tag{10}$$

Recall that

$$M_n^1 = \oplus_{i=1}^s B_i^{\frac{w_{B_i}(M_n)\dim_K(M_n)}{\dim_K(B_i)}}\,. \tag{11}$$

Therefore, $\frac{\dim_K(M_n^1)}{\dim_K(M_n)} = \sum_{i=1}^s w_{B_i}(M_n)$. Then by (11), $\{M_n^1\}_{n=1}^{\infty}$ and $\{N_n^1\}_{n=1}^{\infty}$ are convergent, hence by (10), $\{M_n^2\}_{n=1}^{\infty}$ and $\{N_n^2\}_{n=1}^{\infty}$ are convergent as well. $\square$

Also, for large n, $B_i^{\frac{w_{B_i}(M_n)\dim_K(M_n)}{\dim_K(B_i)}}$ and $B_i^{\frac{w_{B_i}(N_n)\dim_K(N_n)}{\dim_K(B_i)}}$ are ϵ-isomorphic, hence for large n, M_n^1 and N_n^1 are ϵ-isomorphic. By our assumptions, we have modules $L_n \subset M_n^2$, $O_n \subset M_n^2$ that are sums of string modules such that for any $n \geq 1$, L_n and M_n^2 resp. O_n and N_n^2 are κ_2-isomorphic. So, by the definition of κ_2, we have that $d_R(L_n, M_n^2) \leq \kappa_1/3$, $d_R(O_n, N_n^2) \leq \kappa_1/3$. By the previous lemma, if n is large enough, then $d_R(M_n^2, N_n^2) \leq \kappa_1/3$. Therefore if n is large enough, then $d_R(L_n, O_n) \leq \kappa_1$. Thus by the definition of κ_1, L_n and O_n are $\epsilon/100$-isomorphic. Since $\kappa_2 < \epsilon/100$, we can see that M_n^2 and N_n^2 are ϵ-isomorphic. Since for large n, M_n^1, N_n^1 are ϵ-isomorphic, we get that for large n, M_n and N_n are ϵ-isomorphic as well, leading to a contradiction. Hence our theorem follows.

14 Parameter testing for modules

A module parameter p is a bounded real function on R-Mod, where R is a finite dimensional algebra over a finite field K. So, if $M \cong N$ then $p(M) = p(N)$ (see [13] for parameters of bounded degree graphs).

Examples:

- Let $G(M)$ be the smallest generating system of the module M. Then $g(M) := \frac{|G(M)|}{\dim_K(M)}$ is a module parameter. This parameter is analogous to the covering number of finite graphs.

- Let $I(M)$ be the largest system $\{m_1, m_2, \ldots, m_{I(M)}\}$ of elements in the module M so that $\sum_{i=1}^{|I(M)|} r_i m_i = 0$ implies that $r_i = 0$ for any $1 \leq i \leq |I(M)|$. Then $i(M) := \frac{|I(M)|}{\dim_K(M)}$ is a module parameter, analogous to the independence number of a finite graph.

- Let $Q \in$ R-Mod. Then the weight function $w_Q(M)$ is a module parameter.

- Let $Q \in$ R-Mod. Then $L_Q(M) := \frac{\dim_K(\mathrm{Hom}_R(Q,M))}{\dim_K(M)}$ resp. $R_Q(M) := \frac{\dim_K(\mathrm{Hom}_R(M,Q))}{\dim_K(M)}$ left and right homomorphism numbers are module parameters analogous to the left and right homomorphism numbers of finite graphs.

Definition 14.1. *The module parameter p : R-Mod $\to \mathbb{R}$ is stable if it satisfies the following two conditions:*

1. *For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $N \subseteq M$ and $\dim_K(N) \geq (1-\delta)\dim_K(M)$ then $|p(M) - p(N)| < \epsilon$.*

2. *For any $M \in$ R-Mod the limit $\lim_{k\to\infty} p(M^k)$ exists.*

By Lemma 9.4, for any matrix $A \in \mathrm{Mat}\,(R)$, the matrix parameter $p_A(M) = rk_M(A)$ is stable.

Proposition 14.1. *The parameters g, i, w_Q, L_Q and the parameter R_Q provided that Q is an injective module are stable parameters.*

Proof. Clearly, $w_Q(M^k) = w_Q(M)$, $L_Q(M^k) = L_Q(M)$, $R_Q(M^k) = R_Q(M)$. Also, $|G(M \oplus N)| \leq |G(M)| + |G(N)|$ and $|I(M \oplus N)| \geq |I(M)| + |I(N)|$. So, by Fekete' Theorem on Subadditive and Superadditive Functions

$$\lim_{k\to\infty} \frac{|G(M^k)|}{k \dim_K(M)} \quad \text{and} \quad \lim_{k\to\infty} \frac{|I(M^k)|}{k \dim_K(M)}$$

exist. Note that similar observations were made by Cohn on projective module parameters in [9]. It remains to show that the first condition of stability holds for our parameters.

$g : \text{R-Mod} \to \mathbb{R}$. Let $W \subset M$ be a K-linear subspace such that $N \oplus W = M$. Clearly, if $\{t_1, t_2, \dots, t_s\}$ is a basis of the space W and $\{n_1, n_2, \dots, n_q\}$ is a generating system for N, then $\{t_1, t_2, \dots, t_s, n_1, n_2, \dots, n_q\}$ is a generating system for M. Therefore

$$G(M) \leq G(N) + \delta \dim_K(M) \tag{12}$$

if $\dim_K(N) \geq (1-\delta)\dim_K(M)$. Now let $\{m_1, m_2, \dots, m_l\}$ be a generating system for M and $\{w_i\}_{i=1}^l \subset W$ be elements such that $m_i + w_i \in N$, for any $1 \leq i \leq l$. Let $n \in N$ and $n = \sum_{i=1}^l r_i m_i$. Then

$$n = \sum_{i=1}^{l} r_i(m_i + w_i) - \sum_{i=1}^{l} r_i w_i \tag{13}$$

hence, $\sum_{i=1}^l r_i w_i \in N$. Let $[W]$ be the R-module generated by W, so $\dim_K([W]) \leq \dim_K(R)\dim_K(W)$. Let $\{u_1, u_2, \dots, u_s\}$ be a K-basis for $[W] \cap N$. Then by (13), $\{u_1, u_2, \dots, u_s\} \cup \{m_1 + w_1, m_2 + w_2, \dots, m_l + w_l\}$ is a generating system for N. That is,

$$G(N) \leq G(M) + \delta \dim_K(M) \dim_K(R)\,. \tag{14}$$

Now, (12) and (14) imply that the first condition of stability holds for the parameter g.

$i : \text{R-Mod} \to \mathbb{R}$. Clearly, $I(N) \subset I(M)$. Suppose that we have a sequence $\{N_k \subset M_k\}_{k=1}^\infty$ such that

- $\dim_K(M_k) \to \infty$.
- $\lim_{k\to\infty} \frac{\dim_K(N_k)}{\dim_K(M_k)} = 1$, such that $|i(N_k) - i(M_k)| \geq \epsilon$.

We can suppose that $I(M_k) \geq \epsilon \dim_K(M_k)$. Let $Z_1, Z_2, \dots, Z_t$ be the finite set of isomorphism classes of principal R-modules. For each k, we have $S_k \subset M_k$,

$S_k \cong \bigoplus_{i=1}^t Z_i^{q_i}$ such that $\sum_{i=1}^t q_i = I(M_k)$. By Lemma 9.4, we have $r_i \leq q_i$ and $T_k \subset N_k$, $T_k = \sum_{i=1}^t Z_i^{r_i}$, $\sum_{i=1}^t r_i \leq I(N_k)$ such that

$$\lim_{k\to\infty} \frac{\dim_K(Z_i)(q_i - r_i)}{\dim_K(M)} = 0\,.$$

Therefore, $\lim_{k\to\infty} |i(N_k) - i(M_k)| = 0$ in contradiction with our assumption. This implies that the first condition of stability holds for the parameter i.

$\underline{w_Q : \text{R-Mod} \to \mathbb{R}\,.}$ Again, suppose that we have a sequence $\{N_k \subset M_k\}_{k=1}^\infty$ such that

- $\dim_K(M_k) \to \infty\,.$
- $\lim_{k\to\infty} \frac{\dim_K(N_k)}{\dim_K(M_k)} = 1$ and for any $k \geq 1$

$$|w_Q(M_k) - w_Q(N_k)| \geq \epsilon. \tag{15}$$

Taking a subsequence, we can assume that $\{M_k\}_{k=1}^\infty$ is convergent. That is, by Lemma 9.5, for any matrix $A \in \text{Mat}\,(R)$,

$$\lim_{k\to\infty} |rk_{M_k}(A) - rk_{N_k}(A)| = 0\,.$$

By (1),

$$w_Q(M_k) = \frac{D_{M_k}(\phi) - D_{M_k}(\psi)}{D_Q(\phi) - D_Q(\psi)} \quad w_Q(N_k) = \frac{D_{N_k}(\phi) - D_{N_k}(\psi)}{D_Q(\phi) - D_Q(\psi)}$$

that is $\lim_{k\to\infty} |w_Q(M_k) - w_Q(N_k)| = 0$ in contradiction with (15). So, we established the first stability condition for the parameter w_Q.

$\underline{R_Q : \text{R-Mod} \to \mathbb{R}\,.}$ Since Q is injective, any homomorphism $\phi : N \to Q$ extends to a homomorphism $\phi' : M \to Q$. Therefore,

$$\dim_K(\text{Hom}(N, Q)) \leq \dim_K(\text{Hom}(M, Q))\,. \tag{16}$$

Let $\pi : \text{Hom}(M, Q) \to \text{Hom}(N, Q)$ be the restriction map. Clearly,

$$\dim_K(\text{Ker}\,(\pi)) \leq (\dim_K(M) - \dim_K(N)) \dim_K(Q)\,.$$

that is,

$$\dim_K(\text{Hom}(M, Q)) \leq \dim_K(\text{Hom}(N, Q)) + (\dim_K(M) - \dim_K(N)) \dim_K(Q)\,. \tag{17}$$

Now, (16) and (17) imply the first stability condition for the parameter R_Q, provided that Q is injective.

$\underline{L_Q : \text{R-Mod} \to \mathbb{R}\,.}$ Let us consider the following equations on M^Q.

- For any pair $q_1, q_2 \in Q$

$$m_{q_1} + m_{q_2} - m_{q_1+q_2} = 0\,.$$

- For any pair $r \in R$, $q \in Q$

$$m_{rq} - rm_q = 0\,.$$

The solution set of the equations above is isomorphic with $\mathrm{Hom}(Q, M)$. So, the first stability condition for the parameter L_Q immediately follows from the fact that all matrix parameters $p_A(M) = rk_M(A)$ are stable.

The theory of constant-time graph algorithms was developed in the last decade (see e.g. [16]). Let us briefly recall the main idea. Say, we want to estimate the value of a certain graph parameter p for an immensely large graph G of small vertex degrees. For certain parameters (such as the matching number) we can do the estimate in constant-time (that is, independently of the size of the graph). A parameter p is called **testable** for the class of graphs $\mathcal{G}$, if for any $\epsilon > 0$ there exists $\delta > 0$ and a family of test-graphs $F_1, F_2, \dots, F_t$ such that if we learn all the F_i-subgraph densities for $G \in \mathcal{G}$ up to an error of δ, we can compute $p(G)$ up to an error of ϵ. The point is that using a fixed amount of random samplings of G, we can estimate all the subgraph densities above up to an error of δ with very high probability, no matter how large our graph G is. We have an analogous definition for modules.

Definition 14.2. *A module parameter $p : R\text{-}Mod \to \mathbb{R}$ is testable if for any $\epsilon > 0$, there exists $\delta > 0$, test-matrices $A_1, A_2, \dots, A_t$ and $n \geq 1$, such that if for a modules $M \in R\text{-}Mod$, $\dim_K(M) \geq n$, and we know all the $rk_M(A_i)$'s up to an error of δ, then we can compute $p(M)$ up to an error of ϵ.*

The following theorem is motivated by the results of Newman-Sohler [22] and Hassidim-Kelner-Nguyen-Onak [20].

Theorem 8. *If Conjecture 1 holds for an algebra R (e.g. R is a string algebra), then every stable parameter $p : R\text{-}Mod \to \mathbb{R}$ is testable.*

Proof. Let p be a stable parameter for R. By our assumptions, we have matrices $A_1, A_2, \dots, A_t$ and $\kappa > 0$ such that if $|rk_M(A_i) - rk_N(A_i)| \leq \kappa$ holds for any $1 \leq i \leq t$ and

$$1 - \kappa < \frac{\dim_K(M)}{\dim_K(N)} \leq 1 + \kappa$$

then $|p(M) - p(N)| \leq \epsilon/3$. We can pick $\lambda > 0$ in such a way that

- $\lambda \leq \kappa$
- if a module Q λ-tiles M (see Section 9), then $|rk_Q(A_i) - rk_M(A_i)| \leq \kappa/2$ holds for any $1 \leq i \leq t$.

In the proof of Lemma 9.2 we saw that there exists a finite set of modules $\{Q_1, Q_2, \dots, Q_s\}$ such that for any $M \in$ R-Mod at least one of the Q_j's λ-tiles M. Pick n so large that if $1 \leq j \leq s$ and $\dim_K(Q_j^l) \geq n/2$, then $\lim_{k\to\infty} p(Q^k) - p(Q_j^l)| \leq \epsilon/3$. We also assume that $\dim_K(Q_j) \leq \kappa n$. Let $p_i = p(Q_j^n)$, then for each $1 \leq j \leq s$

$$|\lim_{k\to\infty} p(Q_j^k) - p_i| \leq \epsilon/3. \tag{18}$$

Our algorithm goes as follows. We set $\delta = \kappa/2$. If we learn all the values $rk_M(A_i)$'s up to an error of δ, we can find at least one $1 \leq j \leq s$ such that

$$|rk_{Q_j}(A_i) - rk_M(A_i)| \leq \kappa \tag{19}$$

holds for all $1 \leq i \leq t$. These comparisons are all the computations we do after learning the estimated values $rk_M(A_i)$. Hence, the computation time does not depend on the dimension of M. Notice however, the role of $n \geq 1$. In the graph parameter case, all the interesting parameters are additive with respect to disjoint union. Here we only have the stability property.

Note that we might find more than one Q_j's for which (19) holds and we cannot be sure that the Q_j we choose λ-tiles M. Nevertheless by our assumption, for some $c > 0$

$$1 - \kappa \leq \frac{\dim_K(M)}{\dim_K(Q_j^c)} \leq 1 + \kappa\,,$$

hence by the choice of κ, $|p(M) - p(Q_j^c)| \leq \epsilon/3$, whenever $\dim_K(M) \geq n$. Therefore by (18), $|p_j - p(M)| \leq \epsilon$. So, p_j estimates $p(M)$ up to an error of ϵ. $\square$

Remark: One can easily see that the parameters L_Q and w_Q are testable for any finite dimensional algebra R. Indeed, they can be computed from the rank of some matrices.

References

[1] G. Arzhantseva and L. Paunescu, Linear sofic groups and algebras. (preprint) http://arxiv.org/pdf/1212.6780v1.pdf

[2] L. Bartholdi, On amenability of group algebras. I. *Israel J. Math.* **168** (2008), 153–165.

[3] M. Barot, Introduction to the representation theory of algebras. *Springer* (2015).

[4] I. Benjamini and O. Schramm, Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6** (2001), no. 23, 13 pp. (electronic).

[5] C. Brown, J. Chuang and A. Lazarev, Derived localization of algebras and modules. (preprint) http://arxiv.org/pdf/1505.01146v2.pdf

[6] M. C. R. Butler and C. M. Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras. *Comm. Algebra* **15** (1987), no. 1-2, 145–179.

[7] V. Capraro and M. Lupini, Introduction to Sofic and Hyperlinear Groups and Connes' Embedding Conjecture, Lecture Notes in Mathematics **2136**, Springer Verlag (2015)

[8] P. M. Cohn, Free rings and their relations. *London Mathematical Society Monographs*, **19**. Academic Press, Inc., London, 1985.

[9] P. M. Cohn, Rank functions on projective modules over rings. *Contemp. Math.*,**131**, Part 2, Amer. Math. Soc., Providence, RI, 1992. 271–277.

[10] P.M. Cohn, The universal skew field of fractions of a tensor product of free rings.*Colloq. Math.* **72** (1997), no. 1, 1-8.

[11] W. Crawley-Boevey, Infinite dimensional modules in the representation theory of finite dimensional algebras. *Algebras and modules, I (Trondheim, 1996)*, 29-54, CMS Conf. Proc., 23, Amer. Math. Soc., Providence, RI, (1998).

[12] G. Elek, The amenability and non-amenability of skew fields. *Proc. Amer. Math. Soc.* **134** (2006), no. 3, 637–644.

[13] G. Elek, Parameter testing in bounded degree graphs of subexponential growth, *Random Structures and Algorithms*, **37** no. 2 (2010), 248–270.

[14] G. Elek, Finite graphs and amenability, *J. Funct. Anal.* **263** (2012), no. 9, 2593-2614.

[15] G. Elek, Convergence and limits of linear representations of finite groups,*Journal of Algebra*, **450** 588–615.

[16] O. Goldreich and D. Ron, Property testing in bounded degree graphs. *Algorithmica* **32** (2002), no. 2, 302–343.

[17] K. R. Goodearl, von Neumann regular rings. *Robert E. Krieger Publishing Co.*, Inc., Malabar, FL, (1991)

[18] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps. I, Math. Phys. Anal. Geom. **2** (1999), no. 4, 323-415.

[19] I. Halperin, von Neumann's manuscript on inductive limits of regular rings. *Canad. J. Math.***20** (1968) 477-483.

[20] A. Hassidim, J.A. Kelner, H.N. Nguyen and K. Onak, Local graph partitions for approximation and testing. *50th Annual IEEE Symposium on Foundations of Computer Science* (2009), 22–31.

[21] L. Lovász, Large networks and graph limits. *American Mathematical Society Colloquium Publications*, **60**. American Mathematical Society, Providence, RI, 2012.

[22] I. Newman and C. Sohler, Every property of hyperfinite graphs is testable. *SIAM J. Comput.* **42** (2013), no. 3, 1095–1112.

[23] M. Prest, Purity, spectra and localisation. *Encyclopedia of Mathematics and its Applications*, **121**. Cambridge University Press, Cambridge, (2009).

[24] C. M. Ringel, Infinite-dimensional representations of finite-dimensional hereditary algebras. *Symposia Mathematica, Vol. XXIII* (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977), pp. 321–412, Academic Press, London-New York, 1979.

[25] A. H. Schofield, Representations of rings over skew fields, London Mathematics Society, Lecture Notes **92** (1985)

[26] O. Schramm, Hyperfinite graph limits. *Electron. Res. Announc. Math. Sci.* **15** (2008), 17-23.

[27] J. von Neumann, Examples of continuous geometries, *Proc. Nat. Acad. Sci. U.S.A.*, **22** (1936), 101–108.

[28] M. Ziegler, Model theory of modules. *Ann. Pure Appl. Logic* **26** (1984), no. 2, 149-213.

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