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Pre-Decision Side-Bet Sequences

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Abstract

Risk-averse Expected Utility (EU) decision makers with wealth-dependent utility functions may find themselves indifferent between accepting and rejecting an indivisible risky prospect. Bell (1988) showed that under these circumstances it is EU-enhancing for the decision maker to engage in a pre-decision side bet, accepting the indivisible risky prospect conditional upon winning the side bet. The side bet places the decision maker on the convex hull between the initial-wealth utility function and the utility function with risky-prospect-augmented wealth. We show that decision makers restricted to actuarially unfair side bets engage in a *sequence* of individually EU-enhancing side bets. This occurs because optimal stake size is modest for actuarially unfair side bets, whereby wealth remains within the interval of interim convexity upon losing the side bet. As optimal stake size falls strongly with each successive side-bet round, wealth remains within the interval of interim convexity. The EU enhancement conferred by each successive round is also strongly diminishing. Hence the side-bet sequence is eventually truncated when no further EU enhancement is available.

Keywords: Expected Utility, risk aversion, side bets, rationality, indivisibility, discreteness, actuarial fairness

JEL classification: D81

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1 Introduction

Early in the development of modern Expected Utility (EU) theory, the descriptive value and behavioral implications of utility-function convexity was acutely appreciated (Friedman and Savage, 1948; Markowitz, 1952). However once the normative approach took hold in modern EU theory, globally concave utility functions became the primary focus of investigation. It took a further forty years before Bell (1988) showed that local convexity can emerge on an interim basis for globally risk-averse decision makers who are in possession of a standing offer to acquire a discrete, indivisible risky prospect. Global risk aversion notwithstanding, then, a normatively rational EU decision maker can benefit by *increasing* her risk exposure through a pre-decision side bet when within such a region of interim local convexity (Bell, 1988).[1] This seemingly counterintuitive result applies in the neighborhood of any wealth level at which a risky prospect switches from being EU-diminishing to being EU-augmenting. Thus it holds for all EU decision makers possessing such switch points – i.e. all but those with linear or exponential utility functions.

We show that when a decision maker has access to actuarially unfair side bets, the optimal side bet's stake is smaller than in the actuarially fair case. When the decision maker loses an optimal actuarially fair side bet, she is ejected to the lower bound of the interval of interim convexity. However in the event of losing an actuarially *un*fair side bet, the decision-maker's wealth is diminished, but not by enough to fall out of the interval of interim convexity. Hence, a second optimally EU-enhancing side bet may be constructed, the stake of which is smaller than that of the first side bet, keeping the decision maker within the interval of interim convexity. When the decision maker is indifferent between her initial certain wealth and acquiring the indivisible risky prospect, not only will it be possible to construct a single actuarially unfair optimal side bet, but in general a sequence of individually optimal actuarially unfair side bets may be constructed, given continual availability of the side bets at ever-longer odds.

[1] Delquié (2008) presents two refinements: that the side bet's value will be maximal when designed to leave the decision maker indifferent between initial alternatives as increased by the side bet, and that it is not necessary for the side bet to be independent of the alternatives or of background wealth.

2 Single side bet

We follow Bell (1988) in illustrating the singe-side-bet case with a logarithmic-utility example in which the decision maker's initial wealth is $w_0 = \$10,000$. The decision maker is in possession of a standing offer to acquire the risky prospect $g_0 = (\$15,000, \frac{1}{2};\ \$0, \frac{1}{2})$ for the stub price of $\$5,000$. We write the prospect in net-final-wealth terms as $g(g_0, w_0) = (\$20,000, \frac{1}{2};\ \$5,000, \frac{1}{2})$. Define this to be the round-zero lottery $L_0 = g(g_0, w_0)$. Since

$$E[u(L_0)] = \tfrac{1}{2}\ln(\$20,000) + \tfrac{1}{2}\ln(\$5,000) \quad = \quad \ln(\$10,000) = u(w_0) \tag{1}$$

the decision maker is indifferent between acquiring the risky prospect and sticking with her initial (certain) wealth.

Now let the decision maker consider a pre-decision side bet $g_1 = (\$1,000, \frac{1}{2}; -\$1,000, \frac{1}{2})$. Although g_1 and g_0 are stochastically independent, the decision maker considers a compound, conditional policy of only acquiring the round-zero lottery L_0 if the side bet proves successful. This compound lottery, which presumes the Reduction of Compound Lotteries (ROCL) axiom, may be written as follows.

$$L_1(g(g_0, w_0), g_1) = \left(\$21,000, \tfrac{1}{4};\ \$6,000, \tfrac{1}{4};\ \$9,000, \tfrac{1}{2}\right) \tag{2}$$

Although the side bet g_1 is EU-diminishing in isolation due to risk aversion, when g_0 is implemented conditional upon successful resolution of g_1, the resulting compound lottery is EU-augmenting.

$$E[u(L_1)] = \tfrac{1}{4}\ln(\$21,000) + \tfrac{1}{4}\ln(\$6,000) + \tfrac{1}{2}\ln(\$9,000) \quad > \quad \ln(\$10,000) = u(w_0) \tag{3}$$

The Certainty Equivalent (CE) of this compound lottery is \$10,002.[2] This is a marginal (\$2) improvement, reflecting the arbitrary nature of the side bet g_1. By appropriate choice of the side bet, it will in principle be possible to improve upon this CE. Two classes of solutions are of interest: unconstrained optimal side bets g_1^*, and counterparty-constrained optimal side bets $g_1^*|_{\overline{s},\overline{\mu}}$. In the former, the side-bet counterparty limits neither the maximum stake $\overline{s}$ nor the

[2] $E[u(L_1)] = u(\$10,002)$

expected return on a unit stake, i.e. the maximum mean return $\overline{\mu}$. In the latter, the side-bet counterparty may impose a ceiling on the maximum permissible stake, and/or on the maximum mean return $\overline{\mu} < 0$, which captures the actuarially unfair odds associated with 'house edge'. If the counterparty is a casino, table limits impose a ceiling on the maximum stake, and the mean return is limited to $\overline{\mu} = -\frac{1}{37}$ (European roulette) or $\overline{\mu} = -\frac{1}{19}$ (American roulette).

An EU-augmenting side bet can be constructed only if the decision maker's wealth falls within the interval of interim convexity. The bounds of this interval also determine the payoffs and probabilities of the optimal side bet. Figure 1 illustrates the decision-maker's utility function $u(\cdot)$ without the risky prospect and $v(\cdot)$ with the risky prospect. The common tangent to $u(\cdot)$ and $v(\cdot)$ identifies the lower and upper bounds of the interval of interim convexity (w_0', w_0''). The solution method for identifying these bounds analytically is presented in Appendix A. This yields the numerical values $(w_0', w_0'') = (9037.16, 10774.6)$. Given the decision-maker's assumed wealth in this example, an EU-augmenting side bet is possible since $10,000 \in (9037.16, 10774.6)$, and the unconstrained optimal side bet is given by $g_1^* = (w_0'' - w_0, \frac{w_0 - w_0'}{w_0'' - w_0'};\ w_0' - w_0, \frac{w_0'' - w_0}{w_0'' - w_0'}) =$ $(\$774.6, 0.554;\ \ -\$962.84, 0.446)$. With this optimal pre-decision side bet g_1^*, the compound lottery becomes

$$L_1(g(g_0, w_0), g_1^*) = (\$20774.6, 0.277;\ \ \$5774.6, 0.277;\ \ \$9037.16, 0.446) \tag{4}$$

and the decision maker increases her utility

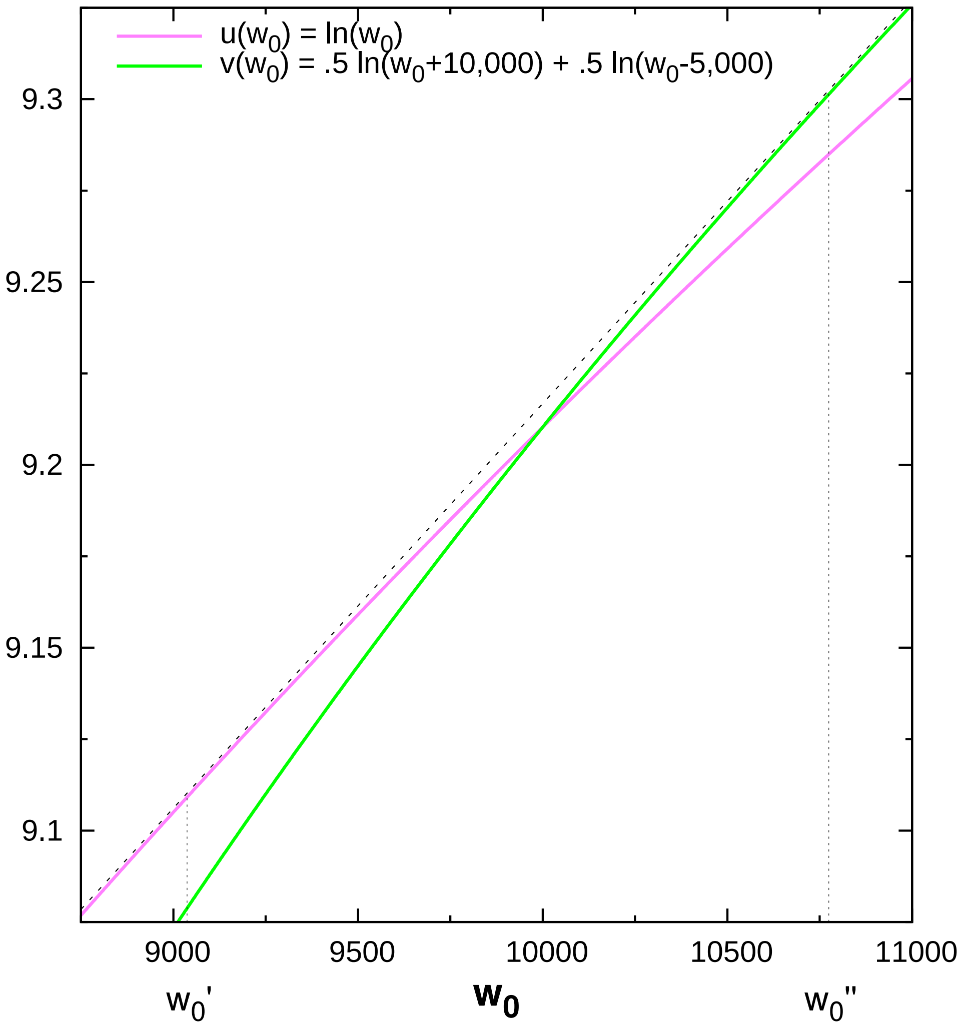
$$E[u(L_1(g(g_0, w_0), g_1^*))] \quad > \quad E[u(L_1(g(g_0, w_0), g_1))] \quad > \quad u(w_0) \tag{5}$$

to the CE wealth level of $\$10,052.8$, which is clearly greater than the $\$10,002$ of the arbitrary side bet g_0.

We may also develop a counterparty-constrained optimal-side-bet example which recognizes that large-scale providers of side-bet services – gaming industry firms – typically do so on an actuarially unfair basis, or with limits (e.g. table limits) that protect consumers from themselves and the house against doubling strategies. Here we focus on the restriction to actuarially unfair side bets, specifically in the form of the $\mu = -\frac{1}{37}$ unit-bet expected return in European roulette.[3]

[3] A European roulette wheel has 36 numbered pockets, half red and half black, and one green pocket, which is excluded from payout odds: a single-number bet pays out only 35:1 plus the original stake, even though the odds

Figure 1: Common tangent to $u(\cdot)$ and $v(\cdot)$ which determines the bounds of the interval of interim convexity (w_0', w_0'') for the round-one side bet.



Denote the (actuarially unfair) house payout odds by a. For a single-number bet in European roulette, the payout odds are 35:1, that is $a = \frac{35}{1}$. The associated win probability is $p = \frac{1+\mu}{1+a}$, which for the single-number bet is $p = (1 + \frac{-1}{37})/(1 + \frac{35}{1}) = \frac{1}{37}$. Meanwhile, the interim-convexity bounds remain as above.[4] To determine the optimal round-one counterparty-constrained optimal side bet, we maximize

$$p\left[\frac{1}{2}\ln(w_0 + 10,000 + sa) + \frac{1}{2}\ln(w_0 - 5,000 + sa)\right] + (1-p)\ln(w_0 - s) \tag{6}$$

by appropriate choice of stake s and payout odds a. As the win probability p is a function

against the number turning up are 36:1 $\Leftrightarrow p = \frac{1}{37}$.

[4] The round-one side bet's bounds depend on round-zero parameters alone.

of a and the fixed parameter μ, there are no other undetermined parameters. This is solved by setting $s = 573.33$ and $sa = 732.77$ – i.e. smaller than in the unconstrained, actuarially fair problem – from which follows that $a = 732.77/573.33 = 1.27809$ and consequently $p = (1-(1/37))/(1+1.27809) = 0.42710$. Due to the house advantage $\mu = \frac{-1}{37}$, the expected value of this side bet is $|\mu|s = \$15.50$ less than in the unconstrained problem, i.e. $E[w_0+psa-(1-p)s] = \$10,000 - \$15.50 = \$9,984.50$.

$$L_1(g(g_0, w_0), g_1^*|_{\overline{\mu}}) = (\$20732.77, 0.21355;\ \$5732.77, 0.21355;\ \$9,426.67, 0.5729) \tag{7}$$

With the optimal counterparty-constrained pre-decision side bet $g_1^*|_{\overline{\mu}}$, the CE of this compound lottery (7) becomes \$10,030.69, in which the \$30.69 increase over the pre-side-bet CE is 58% of the \$52.80 increase achieved with the actuarially fair side bet g_1^*. Still, with the counterparty-constrained optimal side bet $g_1^*|_{\overline{\mu}}$ the compound lottery $L_1(g(g_0, w_0), g_1^*|_{\overline{\mu}})$ nevertheless proves to be EU-augmenting.

$$E[u(L_1(g(g_0, w_0), g_1^*))] \;>\; E[u(L_1(g(g_0, w_0), g_1^*|_{\overline{\mu}}))] \;>\; E[u(L_1(g(g_0, w_0), g_1))] \;>\; u(w_0) \tag{8}$$

3 Sequences of individually optimal side bets

3.1 Counterparty-unconstrained, actuarially fair case

After losing the side bet g_1^* in the round-one compound lottery $L_1(g(g_0, w_0), g_1^*)$ and therefore the \$962.84 stake, the decision-maker's wealth is reduced to $w_1 = w_0' = 9037.16$. Although it is possible to solve for the optimal round-two counterparty-unconstrained side bet,[5] this side bet is EU diminishing. Hence for the counterparty-unconstrained case, the side-bet sequence is degenerate, being truncated at g_1^* as in Bell (1988). This degeneracy does not carry over to the counterparty-constrained case, however.

[5] $g_2^* = (\$1,733.17, 7.3765 \times 10^{-4}; -\$4.89, 0.9993)$

3.2 Actuarially unfair case

When side bets are constrained to be actuarially unfair, the round-one side-bet stake is smaller than the difference between initial wealth w_0 and the lower bound of the interval of convexity w_0'. Hence, after losing the side bet $g_1^*|_{\overline{\overline{\mu}}}$ in the round-one compound lottery $L_1(g(g_0, w_0), g_1^*|_{\overline{\overline{\mu}}})$ and therefore the associated stake $s_{1\overline{\overline{\mu}}}^* = \$573.33$, the decision-maker's wealth is reduced to $w_1 = \$9,426.67 \in (w_0', w_0'')$, which is within the interval of convexity. Consequently the decision-maker can benefit from a further side-bet round. In turn the stake of the optimal round-two side bet $s_{2\overline{\overline{\mu}}}^* = \$135.25$ is also smaller than $w_1 - w_0'$. Upon losing the side bet of the round-two compound lottery $L_2(g(g_0, w_1), g_2^*|_{\overline{\mu}})$, the decision-maker's wealth is reduced to $w_2 = \$9,291.43 \in (w_0', w_0'')$, which is within the interval of convexity. Again the decision maker can benefit from a further side-bet round. Table 1a presents this sequence of individually EU-augmenting optimal side bets against a European-roulette counterparty. These side bets have been computed under the assumption that the minimum stake-size increment is 0.01 (one penny). The round-four side bet's $\text{CE}_{4\overline{\overline{\mu}}}^*$ is larger than w_3 in the fourth decimal digit. Similarly $\text{CE}_{5\overline{\mu}}^* > w_4$ in the sixth decimal digit, and $\text{CE}_{6\overline{\mu}}^* > w_5$ in the eighth decimal digit. Notice that availability of ever-longer-odds side bets is necessary for extending the sequence, which nevertheless is truncated to six rounds due to the discrete, one-penny stake-size increment.

Table 1a also presents the operator's sum of expected payoffs $|\overline{\overline{\mu}}| s_{i\overline{\overline{\mu}}}^* \prod_{j=0}^{i-1} q_j$, where $q_i = (1 - p_i) = 1 - ((1 + \overline{\overline{\mu}})/(1 + a_{i\overline{\overline{\mu}}}^*))$ is the probability that the decision maker loses her i^{th} side bet to the operator.[6] q_{i-1} is the probability that the decision maker loses her $i-1$ side bet, and consequently finds herself undertaking the EU-enhancing side bet in round i. The column total \$17.89 in Table 1a is the gaming operator's expected gross revenue from the decision maker's sequence of individually rational side bets.

Table 1b presents the sequence of individually EU-augmenting optimal side bets against an American-roulette counterparty. Due to the more disadvantagous mean return ($\overline{\mu} = \frac{-1}{19}$), the round-one stake is considerably smaller than when the counterparty operates a European roulette wheel (\$358.83 rather than \$573.33). After this initial stake is lost, wealth remains closer to the original point of indifference (i.e. $w_0 = \$10,000$). Hence, subsequent-round stakes are larger than in the $\overline{\overline{\mu}} = \frac{-1}{37}$ case. However, $\overline{\mu} = \frac{-1}{19}$ supports fewer individually utility-enhancing rounds

[6] We assume that the decision maker cannot fail to make it to the round-1 side-bet stage, and hence $q_0 = 1$.

Table 1: Parameters of optimal EU-augmenting round-i side bets.

(a) European roulette ($\overline{\overline{\mu}} = \frac{-1}{37}$).

i	w_{i-1}	$s^*_{i\overline{\overline{\mu}}}$	$a^*_{i\overline{\overline{\mu}}}$	$\lvert\overline{\overline{\mu}}\rvert s^*_{i\overline{\overline{\mu}}}$	$\mathrm{CE}^*_{i\overline{\overline{\mu}}}$	$\lvert\overline{\overline{\mu}}\rvert s^*_{i\overline{\overline{\mu}}} \prod_{j=0}^{i-1} q_j$
1	10,000.00	573.33	1.28	15.49	10,030.69	15.49
2	9,426.67	135.25	9.85	3.66	9,428.03	2.10
3	9,291.42	19.41	75.58	0.52	9,291.45	0.27
4	9,272.01	1.80	827.27	0.05	>9,272.01	0.03
5	9,270.21	0.23	6482.14	0.01	>9,270.21	0.00
6	9,269.98	0.03	49702.21	0.00	>9,269.98	0.00
$\sum$		730.05		19.73		17.89

(b) American roulette ($\overline{\mu} = \frac{-1}{19}$).

i	w_{i-1}	$s^*_{i\overline{\mu}}$	$a^*_{i\overline{\mu}}$	$\lvert\overline{\mu}\rvert s^*_{i\overline{\mu}}$	$\mathrm{CE}^*_{i\overline{\mu}}$	$\lvert\overline{\mu}\rvert s^*_{i\overline{\mu}} \prod_{j=0}^{i-1} q_j$
1	10,000.00	358.83	1.82	18.86	10,017.53	18.86
2	9,641.17	131.89	8.10	6.94	9,642.96	4.61
3	9,509.28	30.43	39.16	1.60	9,509.39	0.95
4	9,478.85	9.76	127.33	0.51	>9,478.85	0.30
$\sum$		530.91		27.94		24.72

(4 versus 6). Despite the sum of stakes being lower (\$530.91 versus \$730.05), the operator's sum of expected gross payoffs $\left(\sum_i |\mu| s^*_{i\mu} \prod_{j=0}^{i-1} q_j\right)$ is greater under the $\overline{\mu} = \frac{-1}{19}$ restriction than under the $\overline{\overline{\mu}} = \frac{-1}{37}$ restriction (\$24.72 versus \$17.89).

3.3 Maximally unfair case

In Section 3.2, the decision maker was allowed to choose her EU-maximizing side bet from a degenerate menu restricted to the least-disadvantageous class of betting offered by the gaming industry: single-number wagers on a European roulette wheel. Table 1b shows that a sequence of individually optimal EU-enhancing side bets can be constructed when the menu is restricted to the more disadvantageous American roulette. Here we ask, what is the most *disadvantageous* per-unit-bet expected return $\underline{\mu}$ that the decision maker will still find EU enhancing? The gaming operator maximizes his per-unit-bet expected return by restricting the decision maker's menu of available wagers to the $\mu = \underline{\mu}$ singleton subset, such that at least one wager $g|_{\underline{\mu}}$ is

Table 2: Parameters of individually optimal EU-augmenting round-i side bets for maximally disadvantageous, but still EU-augmenting mean return ($\underline{\mu} = -0.19$).

i	w_{i-1}	$s^*_{i\underline{\mu}}$	$a^*_{i\underline{\mu}}$	$\lvert\underline{\mu}\rvert s^*_{i\underline{\mu}}$	$\mathrm{CE}^*_{i\underline{\mu}}$	$\lvert\underline{\mu}\rvert s^*_{i\underline{\mu}} \prod_{j=0}^{i-1} q_j$
1	10,000.00	1.11	45.338	0.21	>10,000.00	0.21
2	9,998.89	0.90	75.998	0.17	>9,998.89	0.17
3	9,997.99	0.16	564.422	0.03	>9,997.99	0.03
4	9,997.83	0.07	1146.683	0.01	>9,997.83	0.01
$\sum$		2.24		0.42		0.42

EU-enhancing $E[u(L_1(g(g_0, w_0), g|_{\underline{\mu}}))] - u(w_0) = \epsilon > 0$.

Two of the parameters determining $\underline{\mu}$ may be treated as exogenous for present purposes. The first captures the decision maker's 'just-noticeable difference', operationalized as the number of significant digits to which EU and ϵ are measured. The second captures the house minimum-stake-size increment (here assumed to be 1 penny). Operationally we maximize $|\underline{\mu}|$ subject to (i) the discrete 1-penny minimum-stake-size increment and (ii) the requirement that the round-i side bet be EU-enhancing ($w_{i-1} - \mathrm{CE}^*_{i\underline{\mu}} > 0$) at any arbitrarily fine-grained just-noticeable difference. Table 2 presents the sequence of individually EU-enhancing side bets which follows the first-round stake and mean-return combination $(s^*_{1\underline{\mu}}, \underline{\mu}) = (1.11, -0.19)$. Even though the house mean return is greater than in single-number bets in American or European roulette ($|\underline{\mu}| > |\overline{\mu}| > |\overline{\overline{\mu}}|$), the operator's sum of expected gross payoffs over the four rounds amounts to only 42 cents. In the next section, we show that the gambling operator can increase its overall sum of expected payoffs by setting the per-unit-bet expected return to maximize the sum of ex-ante expected house payoffs *given* the repeated-side-bet structure.

3.4 Maximizing gaming operator's sum of expected payoffs

Within our interval of interest $|\mu| \in [\frac{1}{37}, 0.19]$, the optimal first-round side-bet stake is monotonically decreasing in the absolute value of the operator's mean return $\frac{\partial s^*_{1\mu}}{\partial|\mu|} < 0$, $\frac{\partial^2 s^*_{1\mu}}{\partial|\mu|^2} > 0$. But the product $|\mu| s^*_{1\mu}$ does not inherit this property. We therefore ask whether the operator's expected payoff on the first-round side bet $|\mu| s^*_{1\mu}$ reaches a maximum for some per-unit-bet return $|\mu|$ within the interval $[\frac{1}{37}, 0.19]$. Figure 2 shows that this is indeed is the case, and that the maximum is reached at $|\breve{\mu}| = 0.054$, which is approximately the average between $\frac{1}{19}$ and $\frac{1}{18}$. Under the assumptions and parameters of this working example, the per-unit-bet expected return $|\overline{\mu}| = \frac{1}{19} = 0.053$ is within one-thousandth (0.001) of the optimal expected return $|\breve{\mu}| = 0.054$.

Figure 2: Expected payoff to gaming operator from the first-round side bet for different values of the $|\mu|$ parameter between $|\overline{\overline{\mu}}| = \frac{1}{37} = 0.\overline{027}$ and $|\underline{\mu}| = 0.19$. Maximum occurs at $(0.054, 18.89)$.

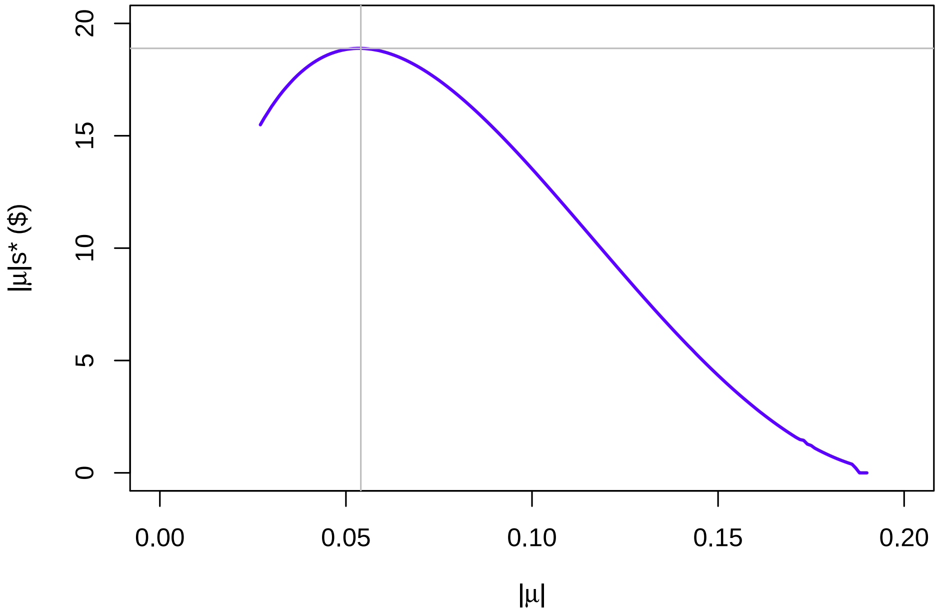


Table 3 shows that the operator's first-round expected payoff $|\breve{\mu}|s^*_{1\breve{\mu}} = \$18.89$ under the optimal expected return $|\breve{\mu}| = 0.054$ is a marginal improvement over the operator's first-round expected payoff $|\overline{\mu}|s^*_{1\overline{\mu}} = 18.86$ under American-roulette expected return $|\overline{\mu}| = \frac{1}{19}$. The operator's sum of expected gross payoffs over the four individually EU-enhancing side-bet rounds is $24.99, which compares favorably with $24.72 under $|\overline{\mu}| = \frac{1}{19}$. Thus also in terms of the gaming operator's expected gross revenue, the optimal-expected-return case $\breve{\mu} = -0.054$ dominates the American-roulette expected return case $\overline{\mu} = \frac{-1}{19}$, but only marginally. In the present setting, American roulette's $\overline{\mu} = \frac{-1}{19}$ is a good practical approximation to the optimal $\breve{\mu} = -0.054$.

4 Conclusion

In this note we revisit Bell's (1988) finding that a pre-decision side bet can be a rational, EU-enhancing strategy for determining whether to take on a large, indivisible, risky prospect. Providers of side-bet services – gaming operators such as casinos and betting shops – do so on an actuarially unfair basis. When pre-decision side bets are constrained to be actuarially unfair, the side bet's optimal stake size is biased downward. Upon losing the side bet, the decision

Table 3: Parameters of individually optimal EU-augmenting round-i side bets which maximize the gaming operator's sum of expected payoffs ($\breve{\mu} = -0.054$).

i	w_{i-1}	$s^*_{i\breve{\mu}}$	$a^*_{i\breve{\mu}}$	$\lvert\breve{\mu}\rvert s^*_{i\breve{\mu}}$	$\mathbf{CE}^*_{i\breve{\mu}}$	$\lvert\breve{\mu}\rvert s^*_{i\breve{\mu}} \prod_{j=0}^{i-1} q_j$
1	10,000.00	349.80	1.855	18.89	10,016.98	18.89
2	9,650.20	130.48	8.088	7.05	9,651.98	4.71
3	9,519.72	33.57	35.652	1.81	9,519.83	1.09
4	9,486.15	9.59	128.326	0.52	>9,486.15	0.30
$\sum$		523.44		28.27		24.99

maker's wealth consequently remains within the interval of interim convexity, instead of being ejected to its lower boundary. Hence, the decision maker rationally engages in a further side-bet round. Under the assumptions of Bell's (1988) example, we find that the decision maker rationally engages in up to four side-bet rounds (in one case, six rounds). For each successive optimal side bet, the stake becomes smaller and the required odds become longer. This ever-longer-odds requirement may hinder implementation of extended side-bet sequences in practice. Nevertheless side-bet sequences are surprisingly general from a theoretical standpoint, as they arise in all families of non-linear, non-exponential risk-averse utility functions. Together with Bell's (1988) seminal result, the present finding expands the range of empirical phenomena that can be explained within the framework of normative rationality. However, it also introduces a further set of methodological considerations that must be confronted in the design of risk-aversion-elicitation procedures and in the empirical estimation of risk-aversion coefficients.

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A Side-bet bounds

The tangent line to $u(\cdot)$ at $(w_0', u(w_0'))$, where $u(w_0) = \ln(w_0)$, is

$$
\begin{aligned}
y &= u(w_0') + u'(w_0')(w_0 - w_0') && (9)\\
&= \ln(w_0') + \frac{w_0 - w_0'}{w_0'} && (10)
\end{aligned}
$$

while the tangent line to $v(\cdot)$ at $(w_0'', v(w_0''))$, where $v(w_0) = \frac{1}{2}\ln(w_0 + 10,000) + \frac{1}{2}\ln(w_0 - 5,000)$, is given by

$$
\begin{aligned}
y &= u(w_0'') + u'(w_0'')(w_0 - w_0'') && (11)\\
&= \frac{1}{2}\left(\ln(w_0'' + 10,000) + \ln(w_0'' - 5,000) + \frac{w_0 - w_0''}{w_0'' + 10,000} + \frac{w_0 - w_0''}{w_0'' - 5,000}\right) . && (12)
\end{aligned}
$$

In order for these lines to be the same (i.e. a common tangent to $u(\cdot)$ and $v(\cdot)$) then they must share the same slope

$$
\frac{1}{w_0'} = \frac{1}{2(w_0'' + 10,000)} + \frac{1}{2(w_0'' - 5,000)} \qquad (13)
$$

and vertical intercept

$$
\ln(w_0') - 1 = \frac{1}{2}\left(\ln(w_0'' + 10,000) + \ln(w_0'' - 5,000) - \frac{w_0''}{w_0'' + 10,000} - \frac{w_0''}{w_0'' - 5,000}\right) . \qquad (14)
$$

Solving these two equations computationally yields $(w_0', w_0'') = (9037.16, 10774.6)$.