Accepted Manuscript

The pro-p group of upper unitriangular matrices

Nadia Mazza

PII: S0022-4049(17)30034-8
DOI: http://dx.doi.org/10.1016/j.jpaa.2017.02.009
Reference: JPAA 5627

To appear in: *Journal of Pure and Applied Algebra*

Received date: 1 August 2016
Revised date: 11 January 2017

Please cite this article in press as: N. Mazza, The pro-p group of upper unitriangular matrices, *J. Pure Appl. Algebra* (2017), http://dx.doi.org/10.1016/j.jpaa.2017.02.009

THE PRO-p GROUP OF UPPER UNITRIANGULAR MATRICES

NADIA MAZZA

Abstract. We study the pro-p group G whose finite quotients give the prototypical Sylow p-subgroup of the general linear groups over a finite field of prime characteristic p. In this article, we extend the known results on the subgroup structure of G. In particular, we give an explicit embedding of the Nottingham group as a subgroup and show that it is selfnormalising. Holubowski ([13, 14, 15]) studies a free product $C_p * C_p$ as a (discrete) subgroup of G and we prove that its closure is selfnormalising of infinite index in the subgroup of 2-periodic elements of G. We also discuss change of rings: field extensions and a variant for the p-adic integers, this latter linking G with some well known p-adic analytic groups. Finally, we calculate the Hausdorff dimensions of some closed subgroups of G and show that the Hausdorff spectrum of G is the whole interval $[0, 1]$ which is obtained by considering partition subgroups only.

MSC: *Primary 20E18; Secondary 20H25*

Keywords: *pro-p group; infinite unitriangular matrix group; Nottingham group; Hausdorff dimension*

1. Introduction

In this paper we investigate the pro-p group whose finite quotients give the prototypical Sylow p-subgroup of the general linear groups over a finite field of prime characteristic p. For convenience, we will consider an odd prime p throughout the paper.

Sylow p-subgroups of finite general linear groups $\mathrm{GL}_n(q)$ for q a power of p have been minutely analysed by Weir in the 50s ([23]). His findings have subsequently been exploited by many; in particular Bier ([4, 5]) who extended some of Weir's results to the pro-p group $G(q)$ of upper unitriangular matrices with coefficients in the field $\mathbb{F}_q$ with q elements. By *upper unitriangular* matrix, we mean an upper triangular matrix with all diagonal coefficients equal to 1. In the first part of this article, we will elaborate on Weir, Bier's and Holubowski's results ([13, 14, 15]), and we will focus on the subgroup structure of $G(q)$, revisiting the notion of *partition subgroups* considered by Weir. We will also discuss the embeddings of a free product of the form $C_p * C_p$ as a discrete subgroup of $G(p)$ and of the Nottingham group $\mathcal{N}(q)$ ([6, 7, 8]) as a closed pro-p subgroup of $G(q)$. Then we will discuss how we can relate $G(q)$ and $G(p)$ for a field extension $\mathbb{F}_q/\mathbb{F}_p$. In Section 7 we present a p-adic version of the group $G(q)$ and briefly relate this group to some well-known p-adic analytic groups ([9]). In the last section of the paper we calculate the Hausdorff dimensions ([1, 2, 3, 10]) of the closed subgroups presented in the preceding sections. A short appendix includes background about the automorphism groups of the Sylow p-subgroups of the general linear groups $\mathrm{GL}_n(q)$ and about the Hausdorff dimension for profinite groups.

Definition 1.1. Let $q = p^f$ be a power of an odd prime number p, with $f \geq 1$. For each $n \in \mathbb{N}$, let $G_n(q)$ be the Sylow p-subgroup of $\mathrm{GL}_n(q)$ formed by the upper triangular matrices with diagonal coefficients all equal to 1. Let $V_n(q) = \mathbb{F}_q^n$ be the set of column vectors of size n with coefficients in $\mathbb{F}_q$.

Date: February 20, 2017.

Note that for each $n > 1$, we have

$$G_n(q) = \left(\begin{array}{c|c} G_{n-1}(q) & V_{n-1}(q) \\ \hline 0_{1\times(n-1)} & 1 \end{array} \right) \cong V_{n-1}(q) \rtimes G_{n-1}(q),$$

where $G_{n-1}(q)$ acts on $V_{n-1}(q)$ by left multiplication in the obvious way. Thus, the natural projections

$$(1) \qquad \ldots \longrightarrow G_{n+1}(q) \xrightarrow{\pi_{n+1}} G_n(q) \xrightarrow{\pi_n} \ldots \longrightarrow G_2(q) \xrightarrow{\pi_2} G_1(q) = 1$$

form an inverse system.

Definition 1.2. Let $G(q) = \varprojlim_n G_n(q)$ be the inverse limit of (1). The group $G(q)$ is a pro-p group, which we will call the *pro-p group of upper unitriangular matrices over* $\mathbb{F}_q$. If the prime power q is clear from the context, we write simply G and G_n instead of $G(q)$ and $G_n(q)$ respectively.

For each $n \in \mathbb{N}$, let $\theta_n \ : \ G \to G_n$ be the universal map, i.e. such that

$$\theta_m = \pi_{m+1} \cdots \pi_n \theta_n : G \to G_m \quad \text{for all } 1 \leq m < n.$$

Let also $N_n = \ker(\theta_n)$. Thus N_n is the normal subgroup of G formed by all the matrices whose upper left $n \times n$ diagonal block is the identity matrix.

From [24, Section 1.2], a filter base for the topology on G is the set of open normal subgroups

$$\mathcal{B} = \{\theta_n^{-1}(X) = XN_n \mid X \trianglelefteq G_n \ , \ n \in \mathbb{N}\}$$

while the set $\mathcal{U} = \{\theta_n^{-1}(X) = XN_n \mid X \subseteq G_n \ , \ n \in \mathbb{N}\}$ forms a fundamental system of open neighbourhoods of the identity in G ([12, p. 26-27]). In particular, G is countably based (cf. [24, Proposition 4.1.3]).

For any $i, j \in \mathbb{N}$ let e_{ij} denote the infinite elementary square matrix whose unique nonzero coefficient is (i, j) and is equal to 1, i.e. $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ for all $i, j, k, l \in \mathbb{N}$. So, if $a_1, \ldots, a_f \in \mathbb{F}_q^\times$ generate $\mathbb{F}_q$ as $\mathbb{F}_p$-vector space, then $G = \langle 1 + a_c e_{ij} \mid 1 \leq i < j \ , \ 1 \leq c \leq f\rangle$, where we write 1 for the identity element of G and $\mathbb{F}_q^\times$ for the multiplicative group of nonzero elements of $\mathbb{F}_q$. The set

$$\mathcal{E} = \{1 + e_{i,i+1} \mid i \in \mathbb{N}\}$$

generates G topologically and converges to 1. Indeed, it generates a dense subgroup of G, because $G_n = \langle 1 + e_{i,i+1} \mid 1 \leq i < n\rangle$ for all $n \in \mathbb{N}$; moreover, any open subgroup of G contains all but a finite number of such elements ([21, Section 2.4]).

The metric on G is defined as follows. Let $x, y \in G$ and fix a number $\epsilon \in (0, 1)$, for instance $\epsilon = p^{-1}$. Then

$$(2) \qquad d(x, y) = \epsilon^k \quad \text{where} \quad k = \max\{n \mid y^{-1}x \in N_n\}.$$

Then $d(\ , \ )$ is an ultrametric, i.e. subject to the axioms

- $d(x, y) \geq 0$ with equality if and only if $x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in G$ (the *ultrametric* axiom).

For $x \in G$ and $k \geq 0$, the open ball of centre x and radius ϵ^k is the set $B(x, \epsilon^k) = \{y \in G \mid d(x, y) < \epsilon^k\}$ of all the elements y of G such that $y^{-1}x \in N_l$ for some integer $l > k$. It differs from its closure only if k is an integer, in which case $\overline{B(x, \epsilon^k)} = \{y \in G \mid y^{-1}x \in N_k\} = B(x, \rho) = B(x, \epsilon^{k-1})$, for any $\epsilon^{k-1} \geq \rho > \epsilon^k$. In particular,

$$N_n = \{y \in G \mid y \in N_n\} = \overline{B(1, \epsilon^n)} = B(1, \epsilon^{n-1}) \quad \text{for all } n \in \mathbb{N}.$$

2. Partition subgroups of G

Let $\gamma_n(G)$ denote the n-th term in the lower central series of G, starting with $\gamma_1(G) = G$ and $\gamma_2(G) = [G, G]$, the derived subgroup of G. The notation $[A, B]$ for subsets A, B of a group denotes the subgroup spanned by the commutators $[a, b] = a^{-1}a^b$ with $a \in A$ and $b \in B$, where we write

$$x^y = y^{-1}xy \quad \text{and} \quad {}^y x = yxy^{-1}.$$

At the basis of each computation, lays the ubiquitous commutator relation

$$[1 + ae_{ij}, 1 + be_{kl}] = 1 + \delta_{jk}abe_{il} - \delta_{il}abe_{kj} \quad \text{for all } i, j, k, l \in \mathbb{N} \text{ and all } a, b \in \mathbb{F}_q. \tag{3}$$

Extending work of Weir, Bier investigated a sub-family of the subgroups that Weir called *partition subgroups* of G. The results she proves are only for these partition subgroups, but it is easy to see that they extend to all partition subgroups. For convenience, we introduce the following definition.

Definition 2.1. A *partition diagram* is a subset

$$\mu = \{(r_i, c_i) \in \mathbb{N}^2 \mid r_i < c_i\ ,\ i \in \mathbb{N}\}$$

such that

$$\text{whenever} \quad (i, j) \in \mu \quad \text{and} \quad (j, k) \in \mu\ , \quad \text{then} \quad (i, k) \in \mu\ . \tag{4}$$

That is, a partition diagram is a collection of pairs of distinct positive integers, which should be regarded as the coordinates of the nondiagonal *squares* (or coefficients) in the matrices of G:

$$(r, c) \in \mu \Longleftrightarrow 1 + e_{rc} \in G.$$

The corresponding *partition subgroup of* G is the subgroup

$$P_\mu = \{x \in G \mid x_{ij} = 0\ ,\ \forall\, (i, j) \notin \mu\}.$$

So, the constraint (4) on the elements of μ reflects the multiplication of the corresponding matrices (what Weir called "completing the rectangle") in P_μ, namely

$$(1 + ae_{ij})(1 + be_{jk}) = 1 + ae_{ij} + be_{jk} + abe_{ik}.$$

If μ is such that for each $j \geq 2$, if $(i, j) \in \mu$, then all the pairs $(k, j) \in \mu$ for all $k \leq i$, then we call μ a *partition* and write it as

$$\mu = (\mu_2, \mu_3, \dots) \quad \text{where} \quad \mu_j = \max\{l \mid (l, j) \in \mu\} \quad \text{for all } j \geq 2.$$

Then

$$P_\mu = \{x \in G \mid x_{ij} = 0\ ,\ \forall\, \mu_j < i < j\}$$

is formed by all the elements of G whose jth column has $(j - 1 - \mu_j)$ zeroes above the diagonal. A partition of the form $\mu = (0^{c-1}, c, c + 1, c + 2, \dots)$ defines the partition subgroup N_c, for any $c \geq 1$ (and if $c = 1$, then $N_1 = G$). An exponent "λ^s" in a partition μ means λ repeated s times.

Given a partition diagram μ, we denote $|\mu|$ its *shape*, i.e. the set of all squares on an infinite chessboard $\mathbb{N}^2$ which consist of the possible nonzero squares (i, j) in P_μ.

A square (i, j) *covers* (k, l) if $(i, j) \neq (k, l)$ and if $k \leq i$ and $l \geq j$. We say that (i, j) *avoids* $|\mu|$ if (i, j) covers a square outside of $|\mu|$.

Remark 2.2. In [4, 5], Bier only considers partitions. Moreover, she takes the "complementary" definition of a partition than the one we take here. That is, the parts in a partition denote the number of zeroes above the diagonal. Instead, we have chosen to use the same convention as Weir in [23], in order to include the more general partition subgroups defined by partition diagrams.

If $\mu \subseteq \mathbb{N}^2$ is a partition diagram, we call a *subpartition (diagram)* of μ a subset of μ which is a partition (diagram) on its own. So a partition diagram is a lattice. That is ([11, Section 8.2]), given any two subpartitions diagrams μ_1 and μ_2 of μ, their union and intersection are also subpartition diagrams of μ. The *union* of two partition diagrams is the smallest partition diagram which contains them (i.e. obtained by "completing the rectangles" in Weir's terminology), whilst their *intersection* is their set intersection. In particular, if $\mu_1 = (u_2, u_3, \dots)$ and $\mu_2 = (v_2, v_3, \dots)$ are subpartitions of μ, then

$$\begin{aligned} \mu_1 \cup \mu_2 &= (m_2, m_3, \dots) \quad \text{where} \quad m_j = \max\{u_j, v_j\} \quad \text{and} \\ \mu_1 \cap \mu_2 &= (n_2, n_3, \dots) \quad \text{where} \quad n_j = \min\{u_j, v_j\}. \end{aligned}$$

It follows that each partition diagram has a unique maximal subpartition

$$\mu_{\max} = \bigcup \mathfrak{P}_\mu \quad \text{where} \quad \mathfrak{P}_\mu = \{\mu' \subseteq \mu \mid \mu' \text{ is a subpartition of } \mu\ \}.$$

We say that a partition diagram μ *converges* to a partition if there exists $n \geq 2$ such that for any $(r, c) \in \mu$ with $c \geq n$, then $(i, c) \in \mu$ for all $1 \leq i \leq r$. That is, μ becomes a partition for n large enough. The *trivial partition* is the partition $(0^{\aleph_0})$, where $\aleph_0$ is the cardinality of $\mathbb{N}$.

[23, Theorem 2] describes the partition diagrams μ which define normal subgroups P_μ: namely μ is a partition and the boundary of $|\mu|$ should move monotonically downward to the right. The point is that if $P_\mu \trianglelefteq G$ and $(r, c) \in \mu$, then conjugation by any $1 + e_{ir}$ and $1 + e_{cj}$ implies that (i, c) and (r, j) must also be in μ for all $i \leq r$ and all $j \geq c$, i.e. μ contains all the squares covered by (r, c). So μ must be a partition, and its "boundary", determined by all the squares (i_c, c) with $i_c = \max\{r \mid (r, c) \in \mu\}$, must give an increasing sequence $i_2 \leq i_3 \leq i_4 \dots$.

A *rectangular* partition subgroup P_μ is a normal subgroup of G for μ of the form $\mu = (0^c, d^{\aleph_0})$, where $0 < d \leq c$ for some $c \in \mathbb{N}$ (we could extend to $d = 0$ by admitting the trivial subgroup of G as a rectangular partition subgroup). The shape of such $|\mu|$ explains the terminology. If $p > 2$, then the maximal abelian (and characteristic) subgroups of G have this form, with $d = c$ ([23, Theorem 6]). That is,

$$\mu = (0^c, c^{\aleph_0}) \quad \text{and} \quad P_\mu = \left(\begin{array}{r|c} I_{c+1} & * \\ \hline 0 & I_\infty \end{array} \right)$$

where the coefficients in the block $(^*)$ can take any value in $\mathbb{F}_q$.

Extending Pavlov ([20]) and Weir's ([23]) results, Bier proves that the automorphism group of G is generated by three types of continuous automorphisms: inner, diagonal (i.e. conjugation by an infinite diagonal matrix), and those induced by field automorphisms. Furthermore, shifts are surjective group homomorphism, where for $d \in \mathbb{N}$, the dth shift of $x \in G$ is the matrix $x[d]$ obtained by deleting the first d rows and columns of x.

Definition 2.3. We call a matrix $x \in G$ *periodic* (of period d) if there exists $d \in \mathbb{N}$ such that $x = x[d]$. A subgroup $H \leq G$ is periodic (of period d) if every element of H is periodic (of period d).

Here is a summary of Bier and Weir's results as they apply to $G = G(q)$.

Proposition 2.4.

(1) *Partition subgroups are closed.*
(2) *A partition subgroup P_μ is open if and only if the partition diagram μ is such that there exists $N \in \mathbb{N}$ for which $(i, j) \in \mu$ for all $1 \leq i < j$ and for all $j \geq N$.*
(3) *Let H be a closed subgroup of G. The following are equivalent.*
  (a) *H is a normal subgroup of G.*
  (b) *H is a normal partition subgroup of G.*
  (c) *H is a characteristic subgroup of G.*

(d) *H is a partition subgroup defined by an increasing partition μ, i.e. such that $\mu_j \leq \mu_{j+1}$.*

If H satisfies these equivalent conditions, we call H a normal partition *subgroup.*

(4) *Given a partition diagram μ, the normal core $\cap_g {}^g P_\mu$ of P_μ is the partition subgroup*

$$\cap_g {}^g P_\mu = P_{\mu'} \quad \text{where} \;\; \mu' = (\mu'_2, \mu'_3, \dots) \text{ with } \; \mu'_j = \min\{i \mid (i,k) \in \mu \;,\; \forall\, k \geq j\}$$

for all $j \geq 2$. In particular, if the maximal subpartition of μ converges to the trivial partition, then $P_{\mu'} = \{1\}$.

(5) *The normal closure $\langle (P_\mu)^G \rangle$ of a partition subgroup P_μ of G is the partition subgroup $P_{\mu''}$, where $\mu'' = (\mu''_2, \mu''_3, \dots)$ is the partition with $\mu''_j = \max\{i \mid (i,k) \in \mu \;,\; \forall\, k \leq j\}$.*

It is clear that partition subgroups are closed, since any sequence of elements in a partition subgroup P_μ which converges in G must converge in P_μ. The other statements are routine.

Weir obtained specific results for normal partition subgroups of the finite quotients $G_n(q)$, for all n, q, and these also apply to G.

Proposition 2.5. [23, Theorem 3] *Given a normal partition subgroup P_μ, then*

(1) *$[P_\mu, G] = P_{\mu'}$, where $|\mu'|$ are all the squares covered by $|\mu|$.*
(2) *Let P^* be the preimage of $Z(G/P_\mu)$ in G, then $P^* = P_{\hat{\mu}}$ is the normal partition subgroup where $|\hat{\mu}|$ are all the squares which do not avoid μ.*

Thus to get the commutator subgroup $[P_\mu, G]$ we "delete" the squares at the corners of $|\mu|$, i.e. if (i,j) and $(i+1, j+1)$ are both outside $|\mu|$ but $(i, j+1)$ is in $|\mu|$, then we delete $(i, j+1)$ in $|\mu'|$. On the other hand, if (i,j) and $(i+1, j+1)$ are both in $|\mu|$ but $(i+1, j)$ is not in $|\mu|$, then we add $(i+1, j)$ to $|\mu|$ to get $|\hat{\mu}|$.

As an example for Proposition 2.5, we get the subgroups $\gamma_d(G)$ in the lower central series of G by deleting successive *super diagonals*, where the dth super diagonal is the set of all squares $(i, i+d) \in \mathbb{N}^2$. Thus

$$\gamma_d(G) = P_{(0^{d-1},1,2,3,\dots)} \quad \text{for all } d \in \mathbb{N}, \text{ starting with} \quad \gamma_1(G) = G.$$

Similar considerations allow us to calculate the derived subgroups $G^{(d)} = [G^{(d-1)}, G^{(d-1)}]$ of G, starting with

$$G^{(2)} = [G, G] = \gamma_2(G) = \{1 + \sum_{j \geq i+2} a_{ij} e_{ij} \in G\}.$$

Thus elementary commutators calculations (Equation (3)), with $d \geq 2$, give

$$[1 + \sum_{j \geq i+2^{d-1}} a_{ij} e_{ij}, 1 + \sum_{j \geq i+2^{d-1}} b_{ij} e_{ij}] = 1 + \sum_{k \geq i+2^d} c_{ik} e_{ik}.$$

Hence, as partition subgroup,

$$G^{(d)} = P_{(0^{(-1+2^{d-1})},1,2,3,\dots)} \quad \text{for all } d \geq 2.$$

In particular, G is not soluble, because its derived series does not converge.

Partition subgroups can also be used to show that G is not hereditarily just infinite. A profinite group G is *hereditarily just infinite* if every every open subgroup is *just infinite* ([17, Definition I.3]). That is, every nontrivial closed normal subgroup of any open subgroup of G has finite index. By the above discussion, the open subgroups of G have infinitely many closed normal subgroups of infinite index (e.g. the subgroups in the lower central series).

From the basic commutator formula (3), we obtain the structure of the centralisers of partition subgroups of G.

Definition 2.6. Let μ be a partition diagram. The *orthogonal partition diagram* of μ is the partition diagram

$$\mu^\perp = \{(k,l) \in \mathbb{N}^2 \mid k < l \text{ and } \forall\, (i,j) \in \mu \text{ then } k \neq j \text{ and } l \neq i\}. \tag{5}$$

The *centre of* μ is the subpartition diagram

$$\zeta_\mu = \mu \cap \mu^\perp \quad \text{of } \mu. \tag{6}$$

For instance, if $\mu = \{(3,4)\}$, then

$$\mu^\perp = \{(k,l) \in \mathbb{N}^2 \mid k \neq 4\ ,\ l \neq 3\} \quad \text{and} \quad \zeta_\mu = \mu\ ,\ \text{i.e.,}$$

$$P_{\mu^\perp} = \begin{pmatrix} 1 & * & 0 & * & * & * & * \\ & 1 & 0 & * & * & * & * \\ & & 1 & * & * & * & * \\ & & & 1 & 0 & 0 & 0 \\ & & & & 1 & * & \cdots * \\ & & & & & \ddots & \dots \end{pmatrix} \quad \text{and} \quad P_{\zeta_\mu} = P_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & * & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & & & & \ddots & \ddots & \dots \end{pmatrix}.$$

By definition $C_G(P_\mu) = \bigcap_{\substack{(i,j)\in\mu \\ a_{ij}\in\mathbb{F}_q}} C_G(1 + a_{ij}e_{ij})$ and each $C_G(1 + a_{ij}e_{ij}) = C_G(1 + e_{ij})$ is a partition subgroup of G. Now $1 + ae_{kl} \in C_G(1 + e_{ij})$ for each $(i,j) \in \mu$ and $a \in \mathbb{F}_q$ if and only if

$$1 = [1 + e_{kl}, 1 + e_{ij}] = 1 + \delta_{il}e_{kj} - \delta_{jk}e_{il}.$$

So $l \neq i$ and $k \neq j$ for each $(i,j) \in \mu$. In other words,

$$1 + e_{kl} \in C_G(P_\mu) \Longleftrightarrow (k,l) \in \mu^\perp.$$

Which leads to the following conclusion.

Proposition 2.7. *Let μ be a partition diagram. Then*

$$C_G(P_\mu) = P_{\mu^\perp} \quad \textit{and} \quad P_{\zeta_\mu} \quad \textit{is the centre of } P_\mu.$$

In particular, $C_G(P_\mu) = \{1\}$ for any open partition subgroup.

3. Examples of torsion subgroups

The direct limit $\varinjlim_n G_n$ of the G_n's is a discrete torsion group, and so not a subgroup of G. Here $\varinjlim_n G_n$ is the group formed by all the square matrices x such that there exists $m \in \mathbb{N}$ for which $x \in G_m$. Now, each element of $\varinjlim_n G_n$ can be regarded as a torsion element in G in the obvious way, by taking each $x \in \varinjlim_n G_n$ to $\big(xN_n/N_n\big)_{n\in\mathbb{N}} \in G$. This mapping, let's call it ρ, is an injective homomorphism of abstract groups, which takes $\varinjlim_{n\in\mathbb{N}} G_n$ into G and with the property that its image is dense in G, i.e. $\bigcap_{n\in\mathbb{N}} \mathrm{im}(\rho)N_n/N_n = G$. Note that G contains "many" torsion elements which are not in $\mathrm{im}(\rho)$ (take for instance $1 + \sum_{1<j} e_{1j}$).

In [14], Holubowski studies string subgroups, which form a large class of torsion discrete subgroups of G. In particular string subgroups cannot contain any open subgroup of G.

Definition 3.1. A matrix $a \in G$ is a *string* if a is in the image of some injective group homomorphism

$$\prod_{n_i>1} G_{n_i} \hookrightarrow G \quad \text{of pairwise diagonal commuting block matrices of size greater than 1.}$$

Thus a has finite order and a^{-1} is a string with the same block structure as a. A *string subgroup* of G is a subgroup of G formed by strings.

So a string subgroup Q of G is isomorphic to a subgroup of a partition subgroup of G of the form $\prod_{n_i>1} G_{n_i}$ for some non-negative integers n_i, for $i \in \mathbb{N}$.

The equality

$$\prod_{i\in\mathbb{N}} G_{n_i} \cong G/P_\mu \quad \text{where} \quad \mu = (0^{n_1}, n_1^{n_2}, (n_1+n_2)^{n_3}, \dots, (\sum_{1\le i\le j} n_i)^{n_{j+1}}, \dots)$$

shows that, regarded as abstract groups (as in [14]), string subgroups are the complements of normal partition subgroups. That is,

$$G = P_\mu \cdot \prod_{i\in\mathbb{N}} G_{n_i} \quad , \quad P_\mu \cap \left(\prod_{i\in\mathbb{N}} G_{n_i}\right) = \{1\} \quad \text{and} \quad P_\mu \trianglelefteq G.$$

4. Free subgroups of G

In this section, we let $q = p$, and $G = \varprojlim_{n\in\mathbb{N}} G_n(p)$.

We discuss a particular discrete subgroup of G investigated by Holubowski (cf. [14]), and we also look at its closure in G. This subgroup of G is the product of two string subgroups, but is not a string subgroup itself.

Definition 4.1. Let

$$s = 1 + \sum_{n\in\mathbb{N}} e_{2n-1,2n} = \begin{pmatrix} 1 & 1 & & & \\ & 1 & & & \\ & & 1 & 1 & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \quad \text{and}$$

$$t = 1 + \sum_{n\in\mathbb{N}} e_{2n,2n+1} = \begin{pmatrix} 1 & & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix}$$

and regard s and t as elements of G.

Let $F = \langle x \rangle * \langle y \rangle$ be the free product of two groups of order p.

Holubowski ([14, Theorem 1]) defines a function $\varphi : F \to G$, by

$$\varphi(x) = s = 1 + \sum_{n\in\mathbb{N}} e_{2n-1,2n} \quad \text{and} \quad \varphi(y) = t = 1 + \sum_{n\in\mathbb{N}} e_{2n,2n+1}$$

and proves that φ is an injective group homomorphism. [14, Theorem 1] shows that the image is contained in the intersection of the subgroup of so-called *banded matrices* with the subgroup of periodic matrices of G. A *banded matrix* is a matrix $(a_{ij}) \in G$ for which there exists $d \in \mathbb{N}$

such that $a_{ij} = 0$ whenever $j > i+d$. In particular, $\mathrm{im}(\varphi)$ is not a closed subgroup of G because $\mathrm{im}(\varphi) < \bigcap_n \mathrm{im}(\varphi)N_n/N_n$. We let $Q = \overline{\varphi(F)}$ be the closure of $\varphi(F)$ in G.

A word $w(x,y) = x^{a_1}y^{b_1}\cdots x^{a_l}y^{b_l}$ is mapped to

$$\begin{aligned}\varphi(w(x,y)) = w(s,t) &= s^{a_1}t^{b_1}\cdots s^{a_l}t^{b_l}\\ &= 1 + \sum_{n\in\mathbb{N}}\left(\sum_{0\le j<2l} A_j e_{2n-1,2n+j} + \sum_{1\le j<2l} B_j e_{2n,2n+j}\right)\end{aligned}$$

where the coefficients A_j, B_j are monomials in $a_1,\dots,a_l,b_1,\dots,b_l$ of degrees at most $j+1$ and j respectively. For instance, for any $a,b,c,d \in \mathbb{F}_p$,

$$\begin{aligned}&s^at^bs^ct^d = \\ &1 + \sum_{n\in\mathbb{N}}\Big((a+c)e_{2n-1,2n} + (ab+cd+ad)e_{2n-1,2n+1} + \\ &\quad +abce_{2n-1,2n+2} + abcde_{2n-1,2n+3} + \\ &\quad +(b+d)e_{2n,2n+1} + bce_{2n,2n+2} + bcde_{2n,2n+3}\Big) \\ &= \begin{pmatrix} 1 & a+c & ab+cd+ad & abc & abcd & 0 & \dots \\ 0 & 1 & b+d & bc & bcd & 0 & \dots \\ 0 & 0 & 1 & a+c & ab+cd+ad & abc & \dots \\ & & & \ddots & \ddots & & \end{pmatrix}.\end{aligned} \tag{7}$$

Recall from Definition 2.3 that $X[2]$ is the matrix obtained from $X \in G$ by deleting the first 2 rows and columns. From Equation (7) and elaborating by induction on it, we record the following.

Proposition 4.2. *For any $w(x,y) \in F$, we have*

$$\varphi(w(x,y)) = w(s,t) = w(s,t)[2].$$

Moreover, the length of $w(x,y)$ can be read from the last nonzero squares in the first two rows of $w(s,t) = x^{a_1}y^{b_1}\cdots x^{a_l}y^{b_l}$. Namely,

(i) *if $a_1b_l \neq 0$, i.e. $a_j, b_j \neq 0$ for all $1 \le j \le l$, then the last nonzero squares in the first two rows of $w(s,t)$ are $(1, 2l+1)$ and $(2, 2l+1)$ respectively;*
(ii) *if $a_1 = 0 \neq b_l$, (i.e. a_1 is the only zero exponent) then the last nonzero squares in the first two rows of $w(s,t)$ are $(1, 2l-1)$ and $(2, 2l+1)$;*
(iii) *if $a_1 = b_l = 0$ and no other exponent is zero, then the last nonzero squares in the first two rows of $w(s,t)$ are $(1, 2(l-1))$ and $(2, 2l)$.*

In particular, we obtain 2-periodic elements of G whose last nonzero squares in any two successive rows $(i,j), (i+1,k)$ are such that $k - j \in \{0,2\}$. Therefore

$$Q < \{X \in G \mid X = X[2]\}$$

is a closed subgroup of infinite index in the subgroup of 2-periodic elements of G. Furthermore,

$$\varphi^{-1}(\gamma_d(G)) = \gamma_d(F) \quad \textit{for all } d \in \mathbb{N}.$$

A counting exercise gives the indices $|PN_n : N_n| = p^{2n-3}$ and $|QN_n : N_n| = p^{\xi_n}$, where P is the subgroup of 2-periodic elements of G and $\xi_n = n - 2 + \lfloor\frac{n+1}{2}\rfloor$, this latter obtained inductively on n.

Remark 4.3.

(1) It is important to emphasise that Holubowski regards $\varphi(F)$ (as most of the subgroups he investigates in [13, 14, 15]) as a discrete group, and this can be seen from the fact that $\mathrm{im}(\varphi)$ is not closed in G. Recall that F is a pro-p group for the topology defined by taking the set $\mathcal{F}$ of all the subgroups of F of finite p-power index (cf. [12, Example (iii), p. 29]). Letting R run through all the normal subgroups of F of finite p-power index, we obtain all the 2-generated finite p-groups as the quotients F/R. For instance $\gamma_2(F) \in \mathcal{F}$ and $F/\gamma_2(F) \cong C_p \times C_p$.

(2) Let us also mention the description of F from [22, p. 28], i.e. that of the fundamental group of a tree with fundamental domain a segment $\langle x\rangle \xrightarrow{\quad 1 \quad} \langle y\rangle$.

5. Nottingham group

As pointed out in the concluding remark of [13], the Nottingham group can be seen as a subgroup of G. We make this embedding of topological groups explicit in this section. There are equivalent definitions of the Nottingham group. We follow [7].

Definition 5.1. The Nottingham group $\mathcal{N} = \mathcal{N}_q$ is the group of algebra automorphisms of $\mathbb{F}_q[[t]]$ of the form

$$t \mapsto t + \sum_{j\geq 2} a_j t^j \quad a_j \in \mathbb{F}_q.$$

R. Camina investigated the subgroups of $\mathcal{N}$ and proved that $\mathcal{N}$ contains every countably based pro-p group as a closed subgroup.

Now the tantalising fact that $G = \varprojlim_n G_n$ is countably based as pro-p group, implies by Camina's result that G embeds into $\mathcal{N}$ as a closed subgroup. On the other hand, by linear algebra, the elements of $\mathcal{N}$ can be expressed as infinite unitriangular matrices, i.e. elements of G, and therefore $\mathcal{N}$ is a subset of G; but it certainly cannot be the whole of G, because $\mathcal{N}$ only consists of algebra automorphisms.

The convention is that $\mathcal{N}$ acts on the right of $\mathbb{F}_q[[t]]$. So, we can identify the nonconstant elements of $\mathbb{F}_q[[t]]$ as infinite row vectors

$$\sum_{j\geq 1} a_j t^j \in \mathbb{F}_q[[t]] \longleftrightarrow (a_1, a_2, a_3, \dots) \in \mathbb{F}_q^{\aleph_0}.$$

Matrix multiplication induces a linear transformation of $\mathbb{F}_q^{\aleph_0}$,

$$(a_1, a_2, a_3, \dots) \mapsto (a_1, a_2, a_3, \dots)x = (a_1, a_2 + x_{12}a_1, \dots, a_j + \sum_{1\leq i<j} x_{ij}a_i, \dots)\ ,$$

for all $(x_{ij}) \in G$. which translates as function on $\mathbb{F}_q[[t]]$ as follows:

$$\sum_{j\geq 1} a_j t^j \mapsto a_1 t + (a_2 + x_{12}a_1)t^2 + \cdots + \left(a_j + \sum_{1\leq i<j} x_{ij}a_i\right) t^j + \dots$$

Given that $\mathcal{N} = \langle e_1[\alpha_c], e_2[\alpha_c] \mid 1 \leq c \leq f\rangle$, where $\alpha_1, \dots, \alpha_f \in \mathbb{F}_q$ generate $\mathbb{F}_q$ as $\mathbb{F}_p$-vector space, and

$$e_r[\alpha_c]\ :\ t \mapsto t + \alpha_c t^{r+1} \in \mathcal{N}\ , \quad \text{we have}$$

$$(\sum_j a_j t^j) e_r[\alpha_c] = \sum_j a_j (t + \alpha_c t^{r+1})^j = \sum_j a_j \Big(\sum_{0\leq i\leq j} \binom{j}{i} \alpha_c^i t^{ri+j} \Big).$$

This suggests the following mapping $\sigma : \mathcal{N} \to G$, defined on the generators of $\mathcal{N}$ by

$$\sigma(e_r[\alpha_c]) = g_r[\alpha_c] \stackrel{\text{def}}{=} \sum_{1\leq i\leq j} \binom{i}{\frac{j-i}{r}} \alpha_c^{\frac{j-i}{r}}\ e_{ij} = 1 + \sum_{1\leq i<j} \binom{i}{\frac{j-i}{r}} \alpha_c^{\frac{j-i}{r}}\ e_{ij}\ , \tag{8}$$

where the sums run over all the positive integers $i \leq j$, resp. $i < j$, such that

$$\frac{j-i}{r} \in \mathbb{Z} \quad \text{and} \quad j \leq 2i$$

and all the other coefficients are zero. Note that in the first sum $g_r[\alpha]_{ii} = 1$ for all $i \in \mathbb{N}$.

In particular, for $\alpha_1 = 1$, if we write $e_r = e_r[1]$ and $g_r = g_r[1]$, then the ith row of g_r contains the ith row of Pascal's triangle starting from the diagonal 1, and spaced by $(r-1)$ zeroes between each coefficient in a row. Note that $g_r[\alpha_c] \in \gamma_r(G)$ for all $r \geq 1$.

For example,

$$g_1 = \begin{pmatrix} 1 & 1 & 0 & \dots & & & & \\ & 1 & 2 & 1 & 0 & \dots & & \\ & & 1 & 3 & 3 & 1 & 0 & \dots \\ & & & \ddots & \ddots & \ddots & & \end{pmatrix} \quad \text{and}$$

$$g_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & \dots & & & & & & \\ & 1 & 0 & 2 & 0 & 1 & 0 & \dots & & & \\ & & 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \dots \\ & & & & & & \ddots & \ddots & \ddots & & \end{pmatrix}$$

Matrix multiplication yields

$$(a_1, a_2, \dots)g_1 = (a_1, a_1 + a_2, 2a_2 + a_3, a_2 + 3a_3 + a_4, \dots)$$

which corresponds to

$$a_1 t + (a_1 + a_2)t^2 + (2a_2 + a_3)t^3 + (a_2 + 3a_3 + a_4)t^4 + \dots \in \mathbb{F}_q[[t]]$$

and so gives in particular $(1, 0, 0, \dots)g_1 = (1, 1, 0, \dots)$, i.e. $te_1 = t + t^2$. Accordingly, for any "canonical" vector $t^i \in \mathbb{F}_q[[t]]$ the corresponding "canonical" row vector $v_i \in \mathbb{F}_q^{\mathbb{N}_0}$ has a unique nonzero coefficient equal to 1 in the ith coordinate, so that $v_i g_r[\alpha_c]$ is the ith row of $g_r[\alpha_c]$,

$$(0^{i-1}, 1, 0^{r-1}, i\alpha_c, 0^{r-1}, \binom{i}{2}\alpha_c^2, 0^{r-1}, \dots, 0^{r-1}, \alpha_c^i, 0^{r-1}, \dots)$$

which corresponds to the element

$$t^i + i\alpha_c t^{i+r} + \alpha_c^2 \binom{i}{2} t^{i+2r} + \dots + \alpha_c^i t^{i(r+1)} \in \mathbb{F}_q[[t]].$$

Routine computations give

$$(g_1^2)_{ij} = \sum_{k \geq 1} (g_1)_{ik}(g_1)_{kj} = \sum_{i \leq k \leq 2i} \binom{i}{k-i}\binom{k}{j-k}$$

for $i \leq j \leq 4i$ and $(g_1^2)_{ij} = 0$ otherwise. That is, a "row-palindrome" matrix

$$\begin{pmatrix} 1 & 2 & 2 & 1 & \dots & & & \\ & 1 & 4 & 8 & 10 & 8 & 4 & 1 \dots \\ & & \dots & & & & & \end{pmatrix}.$$

Since each row of $g_r[\alpha_c]$ has finitely many nonzero entries, a recursive algorithm (or a more elaborate procedure) gives us the inverse; for instance

$$(g_1)^{-1} = \begin{pmatrix} 1 & -1 & 2 & -5 & 14 & \dots \\ & 1 & -2 & 5 & 14 & \dots \\ & & 1 & -3 & 9 & \dots \\ & & & 1 & -4 & \dots \\ & & \ddots & & \ddots & \end{pmatrix} \quad \text{and}$$

$$(g_2)^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 & 3 & 0 & -12\dots \\ & 1 & 0 & -2 & 0 & 7 & 0\dots \\ & & 1 & 0 & -3 & 0 & 12\dots \\ & & & 1 & 0 & -4 & 0\dots \\ & & \ddots & & \ddots & & \end{pmatrix} .$$

In particular, each row of $g_r[\alpha_c]^{-1}$ has infinitely many nonzero terms, and $g_r[\alpha_c]^{-1} \in \gamma_r(G)$ for all $r \geq 1$.

The key point is that the elements in $\mathrm{im}(\sigma)$ are entirely determined by their first row, where σ is defined by Equation (8). That is, if $x \in \mathcal{N}$ is given by $tx = t + \sum_j a_j t^j$, then the equation $(t^i)x = (t + \sum_j a_j t^j)^i$ defines the coefficients in the ith row of $\sigma(x)$.

It is routine to check that the matrices $g_r[\alpha_c]$ correspond to the image under σ of the algebra automorphisms $e_r[\alpha_c] \in \mathcal{N}$, and that they are subject to the same relations.

Proposition 5.2. *$\sigma(\mathcal{N})$ is a closed subgroup of G of infinite index in G. In particular, $\sigma(\mathcal{N})$ does not contain any open subset of G.*

Proof. We have seen above that σ is a homomorphism of abstract groups, and it is clearly injective. For any $g \in \sigma(\mathcal{N})$ and for any $\delta \in (0, 1)$, the open ball $B(g, \delta)$ is not contained in $\sigma(\mathcal{N})$, since it contains infinitely many linear transformations of $\mathbb{F}_q^{\aleph_0}$ which are not algebra automorphisms. Therefore, $\sigma(\mathcal{N})$ does not contain any open subset of G and has infinite index. To prove that σ is continuous, and so that $\sigma(\mathcal{N})$ is a closed subgroup of G, we show that the preimage by σ of any neighbourhood $B(\sigma(u), \epsilon^{n-1}) = \sigma(u)N_n$ is a neighbourhood of u for any $u \in \mathcal{N}$ and $n \in \mathbb{N}$. Note that $\sigma^{-1}(N_n) = \langle e_m[\alpha_c] \mid 1 \leq c \leq f \ , \ m \geq n \rangle$ is an open normal subgroup of $\mathcal{N}$ ([7]). So we have

$$\sigma^{-1}\big(\sigma(u)N_n\big) = u \ \langle e_m[\alpha_c] \mid 1 \leq c \leq f \ , \ m \geq n \rangle$$

which is an open set of $\mathcal{N}$, as required. □

Next, we turn to the normaliser $N_G(\sigma(\mathcal{N}))$ of $\sigma(\mathcal{N})$ in G. Klopsch proved in [18] that every automorphism of $\mathcal{N}$ is standard, provided $p \geq 5$. That is, $\mathrm{Aut}(\mathcal{N}) \cong \mathcal{A}(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$, where $\mathcal{A}(q)$ is the group of all the algebra automorphisms $\{t \mapsto \sum_{n \geq 1} \lambda_n t^n \mid \lambda_n \in \mathbb{F}_q \ , \ \lambda_1 \neq 0\}$ of $\mathbb{F}_q[[t]]$.

Proposition 5.3. *Suppose $p \geq 5$. Then $N_G(\sigma(\mathcal{N})) = \sigma(\mathcal{N})$, i.e. $\sigma(\mathcal{N})$ is selfnormalising in G.*

Proof. Consider the inclusion $N_G(\sigma(\mathcal{N}))/C_G(\sigma(\mathcal{N})) \hookrightarrow \mathrm{Aut}(\sigma(\mathcal{N})) \cong \mathrm{Aut}(\mathcal{N})$ given by mapping $gC_G(\sigma(\mathcal{N}))$ to conjugation by g in $\sigma(\mathcal{N})$. From the elementary commutator relation (3), we observe that $C_G(\sigma(\mathcal{N})) = \cap_{r,c} C_G(g_r[\alpha_c]) = \{1\}$, where $r \in \mathbb{N}$ and $1 \leq c < f$. Moreover, G is a pro-p group and so $N_G(\sigma(\mathcal{N})) \hookrightarrow S_p$, where S_p is a Sylow p-subgroup of $\mathrm{Aut}(\mathcal{N}) \cong \mathcal{A}(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$. Now, $\mathrm{Aut}(\mathbb{F}_q) = \langle \Phi \ : \ \alpha \mapsto \alpha^p \rangle$ is the cyclic group spanned by the Frobenius homomorphism Φ, which has order $f = \log_p q$ ([19, VII.5 Theorem 12]). In particular, for $\alpha \in \mathbb{F}_q - \mathbb{F}_p$, we have $\Phi(g_1[\alpha]) \not\equiv g_1[\Phi(\alpha)] \pmod{\gamma_2(G)}$, implying that this mapping cannot be given by conjugation by an element of G. It follows that $N_G(\sigma(\mathcal{N}))$ is isomorphic to a p-subgroup of $\mathcal{A}(q)$. Since $\mathcal{N}$ is the unique Sylow p-subgroup of $\mathcal{A}(q) \cong \mathcal{N} \rtimes \mathbb{F}_q^\times$, the result follows. □

6. Field extensions

Given $q = p^f$ for $f \in \mathbb{N}$ and p an odd prime, let us regard $\mathbb{F}_q$ as an f-dimensional $\mathbb{F}_p$-vector space. Left multiplication in $\mathbb{F}_q$ induces an injective ring homomorphism between endomorphism rings of vector spaces

$$\widehat{\alpha}_f \ : \ \operatorname{End}\mathbb{F}_q \to \operatorname{End}(\mathbb{F}_p^f) \cong \operatorname{Mat}_f(\mathbb{F}_p).$$

Choosing such $\widehat{\alpha}_f$ induces an injective group homomorphism (cf. Definition 1.2)

$$\alpha_f \ : \ G(q) \to G(p).$$

Note that $\widehat{\alpha}_f(\lambda) \neq 0 \in \operatorname{Mat}_f(\mathbb{F}_p)$ if and only if $\lambda \neq 0 \in \mathbb{F}_q$, in which case $\widehat{\alpha}_f(\lambda) \in \mathrm{GL}_f(p)$. Also, $\widehat{\alpha}_f(\lambda) = \lambda I_f$ if and only if λ is in the subfield $\mathbb{F}_p$ of $\mathbb{F}_q$.

Lemma 6.1. *The map α_f is continuous. So $G(q)$ is isomorphic to a closed subgroup of $G(p)$.*

Proof. Write $\alpha = \alpha_f$, and $N_k(p)$ and $N_k(q)$ for the normal open subgroups of $G(p)$ and $G(q)$ respectively. Let $\alpha(x) \in G(p)$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
\alpha^{-1}\big(B(\alpha(x), \epsilon^n)\big) &= \alpha^{-1}\big(\{z \in G(p) \mid z^{-1}\alpha(x) \in N_{n+1}(p)\}\big) \\
&\overset{(\dagger)}{=} \alpha^{-1}\big(\{\alpha(y) \mid y \in G(q)\ ,\ \alpha(y)^{-1}\alpha(x) \in N_{n+1}(p)\}\big) \\
&= \{y \in G(q) \mid \alpha(y^{-1}x) \in N_{n+1}(p)\} \\
&= \{y \in G(q) \mid y^{-1}x \in N_k(q)\} \\
&= B(x, \epsilon^k)
\end{aligned}$$

where $k \in \mathbb{N}$ is defined by the inequalities $kf \leq n+1 < (k+1)f$ and $(\dagger)$ follows from taking only those $z \in G(p)$ which have nonempty preimage by α. Therefore, the preimage under α of an open neighbourhood of $\alpha(x)$ is an open neighbourhood of x for all $x \in G(q)$, which proves that α is continuous ([16, Ch. 3, Theorem 1]). $\square$

In particular, for any $f, k \in \mathbb{N}$, we have $\alpha_f^{-1}(N_{kf}(p)) = N_k(q)$.

Conversely, the ring inclusion $\mathbb{F}_p \hookrightarrow \mathbb{F}_q$ induces a continuous injective group homomorphism

$$\beta_f \ : \ G(p) \to G(q) \quad \text{and we have}$$

$$\alpha_f\beta_f(x) = x \otimes I_f = \left(\begin{array}{c|c|c|c} I_f & x_{12}I_f & x_{13}I_f & \dots \\ \hline & I_f & x_{23}I_f & \dots \\ \hline & & I_f & \dots \\ \hline & & & \ddots \end{array}\right) \in G(p) \quad \text{for all } x \in G(p).$$

For short, fix $f \in \mathbb{N}$ and let $\alpha = \alpha_f$, $\widehat{\alpha} = \widehat{\alpha}_f$ and $\beta = \beta_f$. Let $H = \operatorname{im}(\alpha) \leq_c G(p)$ and $K = \operatorname{im}(\beta) \leq_c G(q)$. Clearly neither H nor K are open subgroups because they have infinite index in $G(p)$ and $G(q)$ respectively. To find the index of H in $G(p)$, we regard the elements of $G(p)$ in $f \times f$ block form, where each block is of the form

$$\begin{array}{ccc} \big((f-1)i+1, (f-1)j+1\big) & \cdots\cdots & \big((f-1)i+1, fj\big) \\ \vdots & & \vdots \\ \big(fi, (f-1)j+1\big) & \cdots\cdots & \big(fi, fj\big) \end{array} \quad \text{with } i \leq j. \tag{9}$$

In each of these blocks with $i < j$, the image of $\widehat{\alpha}$ is isomorphic to a copy of $\mathbb{F}_q$, so that in each block, the index is equal to $|\operatorname{Mat}_f(\mathbb{F}_p)\ :\ \widehat{\alpha}(\mathbb{F}_q)| = p^{f^2-f}$. There are countably infinitely many such blocks, giving $|G(p) : H| = (p^{f^2-f})^{\aleph_0}$.

Similarly, to calculate the index of $G(p)$ in $G(q)$ we see that for each coefficient (i, j) with $i < j$, we have an index $|\mathbb{F}_q : \mathbb{F}_p| = p^{f-1}$, so that $|G(q) : K| = (p^{f-1})^{\aleph_0}$.

From Proposition 2.4, we gather that $H \ntrianglelefteq G(p)$ and that $K \ntrianglelefteq G(q)$ since they are not partition subgroups.

In order to determine the normalisers $N_{G(p)}(H)$ and $N_{G(q)}(K)$, we use a result of Weir on the automorphisms of the finite groups $G_n(q)$. For convenience, we have put Weir's theorem and some technical considerations of conjugation in Appendix A.

Proposition 6.2. *Suppose the above notation. The following hold.*

(i) $N_{G(q)}(K) = K$, *where* $K = \beta_f(G(p))$.
(ii) $N_{G(p)}(H) = H$, *where* $H = \alpha_f(G(q))$.

Proof. We prove the second part: $N_{G(p)}(H) = H$. Let $\alpha = \alpha_f$ and $H_m = HN_m(p)/N_m(p)$ for any $m \in \mathbb{N}$. We first show that in the finite quotients $G_{nf}(p)$ we have

$$N_{G_{nf}(p)}(H_{nf}) = H_{nf}C_{G_{nf}(p)}(H_{nf})$$

where $C_{G_{nf}(p)}(H_{nf}) = \langle 1 + e_{ij} \mid 1 \le i \le f \ ,\ n - f < j \le n\rangle$ is the subgroup of all elements of $G_{nf}(p)$ whose only nonzero nondiagonal squares are in the upper right $f \times f$ corner, by Lemma A.2 below.

Using Theorem A.1, it remains to show that no other automorphism of H_{nf} of p-power order can be expressed as a conjugation by an element of $G_{nf}(p)$. Thus we need to consider $\mathcal{P}$ and possibly $\mathcal{L}$ in case p divides $f = \log_p q$. Now, a field automorphism of $G_n(q)$ becomes an automorphism of H_{nf} which fixes all the elements $\alpha(1+e_{r,r+1})$ and therefore cannot be given by a conjugation by a matrix in $G_{nf}(p)$ because the elements which centralise $\alpha(K)N_{nf}(p)/N_{nf}(p)$ also centralise H_{nf}. So Lemma A.2 proves that a field automorphism on H_{nf} cannot be given by an inner automorphism of $G_{nf}(p)$.

Finally, neither central, nor extremal automorphisms of H_{nf} can be given by inner automorphisms of $G_{nf}(p)$ by a similar argument to that used in the proof of [23, Theorem 8]. Indeed, in Equation (13) in the proof of Lemma A.2 below, if a morphism $\alpha_f(1 + e_{r,r+1}) \mapsto \alpha_f(1+e_{r,r+1}+ce_{1,n})$, for $r > 1$ and $c \in \mathbb{F}_q$, were given by an inner automorphism of $G_{nf}(p)$, say conjugation by $1 + \sum_{i<j} a_{ij}e_{ij}$, then Equation (13) has no solution; that is, on the one hand we would need $a_{ij} = 0$ for all $f < i < j \le n$, and on the other, conjugating $\alpha_f(1 + e_{r,r+1})$ by such element is not of the form $\alpha_f(1 + e_{r,r+1} + ce_{1,n})$.

It follows that $N_{G_{nf}(p)}(H_{nf}) = H_{nf}C_{G_{nf}(p)}(H_{nf})$ as asserted. Now, to obtain the normaliser $N_{G(p)}(H)$, we let $n \to \infty$, and the claim follows.

The proof that $N_{G(q)}(K) = K$ can be handled in a similar way and follows easily from the above. We leave this to the reader. □

7. p-ADIC VARIATION

In this section, we consider a variant on the pro-p groups $G(q) = \varprojlim_{n\in\mathbb{N}} G_n(q)$ of Definition 1.2. Namely, let p be an odd prime and for any $n, k \in \mathbb{N}$, write

$$G_n(\mathbb{Z}/p^k) = \{x \in \mathrm{GL}_n(\mathbb{Z}/p^k) \mid x_{ij} = 0 \ \forall\ i > j \quad \text{and} \quad x_{ii} = 1\} \quad \text{for all } (n, k) \in I,$$

where $I = \{(n, k) \in \mathbb{N}^2\}$ is a poset for the order relation $(n, k) \le (m, l)$ if and only if $n \le m$ and $k \le l$. Thus I is a directed system.

For any $(n, k) \le (m, l)$ in I, there are obvious surjections:

- $G_m(\mathbb{Z}/p^l) \to G_n(\mathbb{Z}/p^l)$ analogous to the surjective group homomorphisms from Section 1, using that $G_{n+1}(\mathbb{Z}/p^l) \cong (\mathbb{Z}/p^l)^n \rtimes \mathrm{GL}_n(\mathbb{Z}/p^l)$, where $(\mathbb{Z}/p^l)^n$ is the natural module for $\mathrm{GL}_n(\mathbb{Z}/p^l)$, and
- $G_m(\mathbb{Z}/p^l) \to G_m(\mathbb{Z}/p^k)$ induced by the reduction $\mathbb{Z}/p^l \to \mathbb{Z}/p^k$ of the coefficients.

Hence we get an inverse system in which all the maps are surjective

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & G_{n+1}(\mathbb{Z}/p^{k+1}) & \longrightarrow & G_n(\mathbb{Z}/p^{k+1}) & \longrightarrow & G_{n-1}(\mathbb{Z}/p^{k+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & G_{n+1}(\mathbb{Z}/p^{k}) & \longrightarrow & G_n(\mathbb{Z}/p^{k}) & \longrightarrow & G_{n-1}(\mathbb{Z}/p^{k}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & G_{n+1}(\mathbb{Z}/p^{k-1}) & \longrightarrow & G_n(\mathbb{Z}/p^{k-1}) & \longrightarrow & G_{n-1}(\mathbb{Z}/p^{k-1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The inverse limit of this system is the group

$$G(\mathbb{Z}_p) = \varprojlim_{(n,k)\in I} G_n(\mathbb{Z}/p^k) \cong \varprojlim_{n} G_n(\mathbb{Z}_p) \cong \varprojlim_{k} G(\mathbb{Z}/p^k),$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers. By definition, $G(\mathbb{Z}_p)$ is a pro-p-group ([21, Proposition 2.2.1] or [24, Theorem 1.2.5 (b)]), because the class of pro-p groups is closed under taking closed subgroups and arbitrary direct products. It follows from the product and subgroup topologies that the open sets of G must be the cosets of the factor groups $G/U_n(k)$ where

$$\begin{aligned}
U_n(k) &= \{x \in G(\mathbb{Z}_p) \mid x_{ij} \in p^k\mathbb{Z}_p \ \forall\ 1 \le i < j \le n\} \\
&= \ker\big(\ G(\mathbb{Z}_p) \twoheadrightarrow G_n(\mathbb{Z}/p^k)\ \big) \\
&= \left\{ \left(\begin{array}{c|c} G_n(p^k\mathbb{Z}_p) & * \\ \hline 0 & * \end{array} \right) \;\middle|\; * \in \mathbb{Z}_p \right\}
\end{aligned}$$

for all $n, k \in \mathbb{N}$. So $U_n(k) \triangleleft G$ and $G/U_n(k) \cong G_n(\mathbb{Z}/p^k)$.

For later use, we want to extract from this set a *filtration*, i.e. a totally ordered set of open normal subgroups of G. For each $n \in \mathbb{N}$, let

$$V_n = \left\{ \left(\begin{array}{c|c} G_n(p^n\mathbb{Z}_p) & * \\ \hline 0 & * \end{array} \right) \;\middle|\; * \in \mathbb{Z}_p \right\} \tag{10}$$

so that $V_n \trianglelefteq_o G$, with $V_1 \ge V_2 \ge V_3 \ge \dots$ and $G/V_n \cong G_n(\mathbb{Z}/p^n)$ for all $n \in \mathbb{N}$.

To prove that this is a base of open neighbourhood of 1 in G, it suffices to show that each $U_n(k)$ is a union of cosets of the V_m's. By inspection, we obtain

$$U_n(k) = \bigcup_{g \in G_m(p^k\mathbb{Z}_p/p^m\mathbb{Z}_p)} \left(\begin{array}{c|c} g & * \\ \hline 0 & * \end{array} \right) \quad \text{where} \quad m = \max\{n,k\}, \quad \text{and}$$

the coefficients of $g \in G_m(p^k\mathbb{Z}_p/p^m\mathbb{Z}_p)$ are the coefficients of g in $p^m\mathbb{Z}_p$ running through a set of representatives of $p^k\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}/p^{(m-k)}$. That is,

$$U_n(n) = V_n \quad , \quad U_n(k) = \bigcup_{g \in G_n(p^k\mathbb{Z}_p/p^n\mathbb{Z}_p)} \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \mathrm{I}_\infty \end{array} \right) V_n \quad \text{if } k < n, \text{ and}$$

$$U_n(k) = \bigcup_{g \in G_k(p^n\mathbb{Z}_p/p^k\mathbb{Z}_p)} \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \mathrm{I}_\infty \end{array}\right) V_k \quad \text{if } k > n.$$

Although $G(\mathbb{Z}_p)$ is defined over the p-adic integers, it is clear from the definition of a p-adic analytic group ([9, Section 9]) that $G(\mathbb{Z}_p)$ is not p-adic analytic because one cannot find homeomorphisms between the open subsets of $G(\mathbb{Z}_p)$ and $\mathbb{Z}_p$-modules of finite rank. However, for each $n \in \mathbb{N}$, the group $G_n(\mathbb{Z}_p)$ is the prototype of a compact p-adic analytic group (cf. [9, § 5.1]). From the fact that $G(\mathbb{Z}_p) \cong \varprojlim_n G_n(\mathbb{Z}_p)$, we observe that the inverse limit of compact p-adic analytic groups need not be p-adic analytic.

Obviously, there are some similarities between subgroup structures of $G(\mathbb{Z}_p)$ and $G(\mathbb{F}_q)$ for any q. In particular, we have partition subgroups and free products (of the form $\mathbb{Z}_p * \mathbb{Z}_p$). Using the ideals $p^k\mathbb{Z}_p$ in $\mathbb{Z}_p$ we note that $G(\mathbb{Z}_p)$ has in fact a plethora of closed subgroups. We flag some "ideal" partition subgroups.

Proposition 7.1. *Let $I = p^k\mathbb{Z}_p$ be an ideal of $\mathbb{Z}_p$ and μ a partition diagram. Let*

$$P_\mu(I) = \langle 1 + p^k e_{ij} \mid (i,j) \in \mu \rangle \leq G(\mathbb{Z}_p).$$

The following hold.

(i) *$P_\mu(I)$ is closed.*
(ii) *$P_\mu(I)$ is open if and only if $I = \mathbb{Z}_p$ and μ converges to a partition such that there exists $N \in \mathbb{N}$ with $(i,j) \in \mu$ of all $1 \leq i < j$ and all $j \geq N$.*
(iii) *$P_\mu(I)$ is normal in $G(\mathbb{Z}_p)$ if and only if $P_\mu(I)$ is characteristic in $G(\mathbb{Z}_p)$ if and only if the partition subgroup P_μ is normal in $G(\mathbb{Z}_p)$.*

The proof is straightforward using Proposition 2.4.

8. Hausdorff dimension of closed subgroups of G

In this section, we apply [1, Proposition 2.6] to some closed subgroups of G and calculate their Hausdorff dimension. Note that in [1], the author refers to the *Billingsley dimension* instead of *Hausdorff*, which is defined over the set of real numbers. We adopt the terminology used in later papers ([2, 3]), which use Abercrombie's results too. We refer the reader to [10] for an in-depth background on fractal dimensions and measure theory. We limit ourselves to the essential facts as they apply to $G = G(q)$ from Definition 1.2, and include an appendix with some additional theory which may be useful to the reader. For convenience, we take as definition of Hausdorff dimension that given in Abercrombie's result.

Definition 8.1. [1, Proposition 2.6] Let $H = \varprojlim_{n \in \mathbb{N}} H_n$, be a closed subgroup of $G = \varprojlim_n G_n = \varprojlim_n G/N_n$, where $H_n = HN_n/N_n$, and where $N_1 \geq N_2 \geq N_3 \geq \dots$ is a filtration of G by open normal subgroups. The *Hausdorff dimension of H* is the real number

$$\begin{aligned} &\dim(H) = \lim_{n\to\infty} \frac{\log|H_n|}{\log|G_n|} \quad \text{whenever the limits exists.} \\ &\text{Otherwise} \quad \dim(H) \geq \liminf_{n\to\infty} \frac{\log|H_n|}{\log|G_n|} \end{aligned} \tag{11}$$

The *Hausdorff spectrum* of G is the subset

$$\mathrm{Spec}(G) = \{\dim(H) \mid H \leq_c G\}$$

of the Hausdorff dimensions of all the closed subgroups of G. Thus $\{0,1\} \subseteq \mathrm{Spec}(G) \subseteq [0,1]$.

Recall from the elementary law of logarithms

$$e^{\ln a} = a = b^{\log_b a} = e^{\ln b \log_b a} \quad \text{that} \quad \log_b a = \frac{\ln a}{\ln b},$$

so that we can take any base for the logarithm defining $\dim(H)$. We will take $\log = \log_q$ unless otherwise specified.

From the definition, if $|H| < \infty$, then $\dim(H) = 0$, and consequently, any subset $K \subseteq G$ of finite index has $\dim(K) = 1$. Therefore, the "interesting" dimensions may only be obtained by taking closed subsets of G of infinite index.

8.1. **Hausdorff dimension of partition subgroups of G.**

The partitions subgroups of G (of infinite index) are of the form P_μ for a partition diagram $\mu = \{(i,j) \in \mathbb{N}^2 \mid 1 \le i < j\} \subset \mathbb{N}^2$, subject to $(i,j),(j,k) \in \mu \Rightarrow (i,k) \in \mu$; or a partition $\mu = (\mu_2, \mu_3, \dots)$, where $\mu_j = \max\{i \mid (i,j) \in \mu\}$ for all $j \ge 2$ as defined in Section 2. Hence $P_\mu = \{x \in G \mid x_{ij} = 0 \ , \ \forall\, (i,j) \notin \mu\}$, which gives

$$\frac{\log |P_\mu N_n : N_n|}{\log |G_n|} = \frac{2|\mu|_n}{n(n-1)} \tag{12}$$

where $|\mu|_n$ denotes the cardinality of the subset of the squares of μ up to, and including, the nth column:

$$|\mu|_n = |\{(i,j) \in \mu|\ 1 \le i < j \le n\}| \quad \text{for any integer } n \ge 2.$$

With this notation, we can state and prove the main result in this section.

Theorem 8.2. *Let $\alpha \in [0,1]$. Then there exists a partition subgroup P_μ for which $\dim(P_\mu) = \alpha$. In particular $\mathrm{Spec}(G) = [0,1]$.*

Moreover, for all $d \ge 1$,

$$\dim(\gamma_d(G)) = \dim(G^{(d)}) = 1$$

for the subgroups of G in the lower central and derived series of G.

Proof. The existence of a partition subgroup for $\alpha \in \{0,1\}$ is clear. Suppose $\alpha \in (0,1)$ and let $(a_n)_{n\ge 2} \subset \mathbb{Q}$ be the sequence of rational numbers defined as follows:

$$a_n = \frac{2b_n}{n(n-1)} \quad \text{where} \quad b_n = \lfloor \frac{\alpha n(n-1)}{2} \rfloor \quad \text{for all } n \in \mathbb{N},$$

where $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ denotes the integer part of any $x \in \mathbb{R}$.

We claim that $(a_n)_{n\ge 2}$ converges to α. The inequalities $b_n \le \dfrac{\alpha n(n-1)}{2} < b_n + 1$ imply that

$$a_n = \frac{2b_n}{n(n-1)} \le \alpha < \frac{2(b_n+1)}{n(n-1)} = a_n + \frac{2}{n(n-1)}.$$

Let $\varepsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that $|\alpha - a_n| < \varepsilon$ for all $n \ge N$. Define N as being the least positive integer such that $\varepsilon \ge \dfrac{2}{N(N-1)}$. Then for any $n \ge N$, we have

$$|\alpha - a_n| < \frac{2}{n(n-1)} \le \varepsilon \quad \text{which proves that} \quad \lim_{n\to\infty} a_n = \alpha.$$

For $n \ge 3$, consider

$$\begin{aligned} b_n - b_{n-1} &= \lfloor \frac{\alpha n(n-1)}{2} \rfloor - \lfloor \frac{\alpha(n-1)(n-2)}{2} \rfloor \\ &= \lfloor \frac{\alpha(n-1)(n-2)}{2} + \alpha(n-1) \rfloor - \lfloor \frac{\alpha(n-1)(n-2)}{2} \rfloor \end{aligned}$$

where $0 < \alpha(n-1)$, so that the difference is nonnegative for all $n \geq 2$. More precisely, from the inclusion $b_n - b_{n-1} \in \big(\alpha(n-1) - 1\ ,\ \alpha(n-1) + 1\big)$, which contains a unique integer, we see that whenever the difference is positive, then $b_n - b_{n-1}$ is of the order of $\alpha(n-1)$.

Now, define the partition $\mu = (\mu_2, \mu_3, \dots) \subset \mathbb{N}$ as follows: $\mu_2 = b_2$ and then $\mu_n = b_n - b_{n-1}$ for all $n \geq 3$. Because the differences $b_n - b_{n-1}$ are nonnegative integers less than n, the sequence μ defines a partition subgroup P_μ. That is, P_μ is the subgroup of G whose nonzero squares in column n are the top $b_n - b_{n-1}$ ones (possibly none). For $n \geq 2$, we have

$$|\mu|_n = b_2 + \sum_{3 \leq j \leq n} b_j - b_{j-1} = b_n \quad \text{so that} \quad |P_\mu N_n / N_n| = q^{b_n}.$$

It follows that

$$\dim(P_\mu) = \lim_{n\to\infty} \frac{\log |P_\mu N_n : N_n|}{\log |G_n|} = \lim_{n\to\infty} \underbrace{\frac{2b_n}{n(n-1)}}_{=a_n} = \alpha$$

proving the first part of the theorem.

To prove the second part of the statement, recall that

$$\gamma_d(G) = P_{(0^{d-1},1,2,3,\dots)} \quad \text{and} \quad G^{(d)} = P_{(0^{(-1+2^{d-1})},1,2,3,\dots)} \quad \text{for all } d \geq 2.$$

So

$$|\gamma_d(G) N_n / N_n| = |G_{n+1-d}| = q^{\frac{(n+1-d)(n-d)}{2}} \quad \text{and}$$
$$|G^{(d)} N_n / N_n| = |G_{n+1-2^{d-1}}| = q^{\frac{(n+1-2^{d-1})(n-2^{d-1})}{2}}.$$

It follows that

$$\dim(\gamma_d(G)) = \lim_{n\to\infty} \frac{\log |\gamma_d(G) N_n : N_n|}{\log |G_n|} = \lim_{n\to\infty} \frac{(n+1-d)(n-d)}{n(n-1)} = 1\ ,$$

and similarly

$$\dim(G^{(d)}) = \lim_{n\to\infty} \frac{(n+1-2^{d-1})(n-2^{d-1})}{n(n-1)} = 1 \quad \text{for all } d \geq 2.$$

□

Remark 8.3. From the proof of the theorem, we see that one could try to modify the definition of μ in order to obtain a normal partition subgroup with prescribed Hausdorff dimension. This "tweaking" consists in shifting each "bulging" square arising whenever $b_n - b_{n-1} > b_{n+1} - b_n$ by a finite number of columns to the right until it reaches the next "landing". The next examples may shed some light on this.

Example 8.4. (1) Let $\alpha = \dfrac{1}{\pi} \approx 0.318309886$. The sequence $(a_n)_{n\in\mathbb{N}}$ obtained by the method of the proof of Theorem 8.2 gives the integers $b_n = \lfloor \dfrac{n(n-1)}{2\pi} \rfloor$ and $\mu_n = b_n - b_{n-1}$, with $\mu_2 = b_2$. We calculate

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
μ_n	0	0	1	2	1	2	2	3	3	3	4	3	4	5	5	5	5	6	6

That is, the subgroup

$$P_\mu = \begin{pmatrix} 1 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & \dots \\ & 1 & 0 & 0 & * & 0 & * & * & * & * & * & * & * & * & \dots \\ & & 1 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & \dots \\ & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & \dots \\ & & & & \ddots & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & & \dots \end{pmatrix}.$$

So the proportion of squares in P_μ (relative to the total number of squares in G) up to the fourth column is $\frac{1}{6}$, while up to the twentieth column we have $\frac{60}{190} = \frac{6}{19} \approx 0.315789$ (an error of 0.080% to 3 decimal places).

Expanding on Remark 8.3, we can tweak the partition μ to get a normal partition $\mu' = (0^2, 1^2, 2^3, 3^4, 4^2, 5^4, 6, 6, \dots)$, such that $P_{\mu'} \lhd G$ and $\dim(P_{\mu'}) = \dim(P_\mu) = \pi^{-1}$.

One can check that the same values for μ_n for $n \leq 20$ are obtained with $\alpha = \frac{7}{22}$ instead of π^{-1}.

(2) Let $\alpha = e^{-3} \approx 0.049787$. The corresponding partition μ is

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
μ_n	0	0	0	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	1

So the proportion of squares in P_μ up to the twentieth column is $\frac{9}{190} \approx 0.047368$ (an error of 4.858% to 3 d.p.). Here finding a normal partition subgroup with Hausdorff dimension e^{-3} seems more difficult.

Theorem 8.2 gives one amongst "many" partition diagrams which give partition subgroups of prescribed Hausdorff dimension. Closed subgroups of G with a rational Hausdorff dimension can easily be described as partition subgroups using (maybe more natural) partition diagrams.

Example 8.5. To get a partition subgroup with Hausdorff dimension $\alpha = \frac{1}{2}$, let

$$\mu = \{(i, i+2) \mid i \in \mathbb{N}\}\ . \quad \text{Then} \quad P_\mu = \begin{pmatrix} 1 & 0 & * & 0 & * & 0 & * & \dots \\ & 1 & 0 & * & 0 & * & 0 & \dots \\ & & 1 & 0 & * & 0 & * & \dots \\ & & & 1 & 0 & * & 0 & \dots \\ & & & & 1 & 0 & * & \dots \\ & & & & & 1 & 0 & \dots \\ & & & & & & 1 & \dots \\ & & & & & & & \ddots \end{pmatrix} .$$

Note that P_μ is not normal in G, and that P_μ is formed by a half of the super diagonals of G. Therefore

$$\dim(P_\mu) = \lim_{n\to\infty} \frac{\log |P_\mu N_n : N_n|}{|G_n|} = \frac{1}{2}\ .$$

8.2. **Hausdorff dimension of finitely determined closed subgroups.**

Let $H \leq_c G$, where $G = G(q)$ be determined by a finite total number of rows, columns and super diagonals. That is, there exists a finite number $R_{i_1}, \dots, R_{i_r}$ of rows and $D_{j_1}, \dots, D_{j_d}$ of super diagonals such that for any $x \in H$ any coefficient x_{ij} of x is given as a function of certain coefficients in the rows and diagonals R_{i_l}, D_{j_m} above. We call such a subgroup *finitely determined*. For instance, any d-periodic subgroup is finitely determined by its first d rows: if $H = H[d]$, then $x = (x_{ij})$ is subject to the constraints $x_{ij} = x_{\bar{i}\bar{j}}$, where $\bar{i} \equiv i \pmod d$ and $\bar{j} - \bar{i} = j - i$. Similarly, a string subgroup which embeds into a direct product of subgroups isormorphic to $\prod_{1<n_i<N} G_{n_i}$, for some upper bound N on the size of the diagonal blocks, is determined by its first N super diagonals.

In the examples seen above, we observed that the subgroup $\sigma(\mathcal{N})$ isomorphic to the Nottingham group is determined by its first row (cf. paragraph preceeding Proposition 5.2), while the free product $\varphi(F)$, as a subgroup of $G(p)$, is determined by its first 2 rows, by Proposition 4.2.

Recall from Section 4 that

$$\varphi(F) = \varphi(F)[2] = \langle s = 1 + \sum_{n\in\mathbb{N}} e_{2n-1,2n} \ , \ t = 1 + \sum_{n\in\mathbb{N}} e_{2n,2n+1}\rangle \cong C_p * C_p \ .$$

More generally, for a d-periodic subgroup H, the subgroup $H_n = HN_n/N_n$ of G_n is the subgroup whose coefficients in the first d rows can be chosen freely, and these uniquely determine the remaining ones in the bottom $(n-d)$ rows.

$$\left(\begin{array}{cccccc|cc}
1 & a_{12} & \dots & \dots & \dots & a_{1n} & a_{1,n+1} & \dots \\
 & \ddots & \dots & \dots & \dots & \vdots & \dots & \dots \\
 & & 1 & a_{d,d+1} & \dots & a_{d,n} & a_{d,n+1} & \dots \\
 & & & \ddots & \bullet \dots & \bullet & \bullet_{i,n+1} & \dots \\
 & & & & 1 & \bullet & \bullet_{n-1,n+1} & \dots \\
 & & & & & 1 & \bullet_{n,n+1} & \dots \\
\hline
 & & & & & & * & \\
 & & & & & & &
\end{array}\right)$$

where $\bullet$ denote the coefficients that are determined by the freely chosen chosen a_{ij}.

Although not finitely generated, nor finitely presented in general, finitely determined subgroups of G are "small" in G, in the following sense.

Lemma 8.6. *Suppose that $H \leq_c G$ is finitely determined. Then* $\dim(H) = 0$.

Note that a closed subgroup of G which is determined by a finite number of columns is also determined by a finite number of rows, so that the lemma applies to this class of subgroups too.

Proof. We prove the claim for $H \leq_c G$ determined by a finite number of rows $i_1 < \cdots < i_d$. Let $H_n = HN_n/N_n$ and K_d the subgroup of G formed by all the d-periodic elements. So K_d is determined by its first d-rows. For $n > d$ we calculate

$$\begin{aligned}\log|H_n| &\leq \log|K_dN_n/N_n| = \log|G_n| - \log|G_{n-d}| \\ &= \frac{n(n-1)}{2} - \frac{(n-d)(n-d-1)}{2} = \frac{d(2n-d-1)}{2} \ .\end{aligned}$$

It follows that

$$\dim(H) \leq \lim_{n\to\infty} \frac{d(2n-d-1)}{n(n-1)} = 0 \quad \text{and so} \quad \dim(H) = 0.$$

Similarly, if H is determined by d super diagonals, then for $n > d$, we have (counting the coefficients in the successive super diagonals up to the dth one)

$$\log|H_n| \leq (n-1) + (n-2) + \cdots + (n-d) = \frac{d(2n-d-1)}{2}$$

and we conclude as above. The lemma follows. □

Here are immediate consequences of this observation.

Corollary 8.7. *The following hold.*

(1) *The subgroup $\sigma(\mathcal{N})$ of G isomorphic to the Nottingham group has Hausdorff dimension* 0.
(2) *If $q = p$, the subgroup $\varphi(F) \cong C_p * C_p$ of G has Hausdorff dimension* 0.
(3) *If H is a d-periodic subgroup of G for some positive integer d, then* $\dim(H) = 0$.
(4) *If H is a string subgroup of G of the form $H \leq \prod_{1<n_i\leq N} G_{n_i}$ for some integer $N \geq 2$, then H has Hausdorff dimension* 0.

8.3. Hausdorff dimension and field extensions.

We use the same notation as in Section 6. Let $H = \alpha_f(G(q)) \leq_c G(p)$ and $q = p^f$ for some $f \in \mathbb{N}$. Given $n \in \mathbb{N}$ write $n = rf + s$ with $0 \leq s < f$ and $r \geq 0$. Write $\log = \log_p$. We calculate

$$\frac{\log|G_r(q)|}{\log|G_n(p)|} \leq \frac{\log|HN_n(p) : N_n(p)|}{\log|G_n(p)|} \leq \frac{\log|G_{r+1}(q)|}{\log|G_n(p)|}$$
$$\frac{\log|q^{\frac{r(r-1)}{2}}|}{\log|p^{\frac{n(n-1)}{2}}|} \leq \frac{\log|HN_n(p) : N_n(p)|}{\log|G_n(p)|} \leq \frac{\log|q^{\frac{r(r+1)}{2}}|}{\log|p^{\frac{n(n-1)}{2}}|}$$
$$\frac{fr(r-1)}{(rf+s)(rf+s-1)} \leq \frac{\log|HN_n(p) : N_n(p)|}{\log|G_n(p)|} \leq \frac{fr(r+1)}{(rf+s)(rf+s-1)} .$$

Left- and right hand side terms both converge to $\frac{1}{f}$ as $n \to \infty$, i.e. as $r \to \infty$. Therefore $\dim\big(\alpha_f(G(q))\big) = \frac{1}{f}$.

Let $K = \beta_f(G(p)) \leq_c G(q)$ and $q = p^f$. By definition of β_f, for each $n \geq 2$, the index of $KN_n(q)/N_n(q) \cong G_n(p)$ in $G_n(q)$ is equal to $\frac{p^{\frac{n(n-1)}{2}}}{(p^f)^{\frac{n(n-1)}{2}}}$, so that

$$\lim_{n\to\infty} \frac{\log|KN_n(q) : N_n(q)|}{\log|G_n(q)|} = \frac{1}{f} .$$

8.4. Hausdorff spectrum of $G(\mathbb{Z}_p)$.

To calculate the Haudorff dimension of closed subgroups of $G(\mathbb{Z}_p)$, we consider the filtration given in Equation (10):

$$V_n = \left\{ \left(\begin{array}{c|c} G_n(p^n\mathbb{Z}_p) & * \\ \hline 0 & * \end{array} \right) \mid * \in \mathbb{Z}_p \right\}$$

with factor groups $G/V_n \cong G_n(\mathbb{Z}/p^n)$ for all $n \in \mathbb{N}$.

Theorem 8.2 proves that the dimension spectrum of $G(\mathbb{Z}_p)$ is the whole interval $[0, 1]$, which can be attained using solely partition subgroups of $G(\mathbb{Z}_p)$.

Corollary 8.8. *Let $\alpha \in [0, 1]$. Then there exists a partition subgroup P_μ for which $\dim(P_\mu) = \alpha$. In particular $\mathrm{Spec}(G(\mathbb{Z}_p)) = [0, 1]$.*

For all proper ideals $I \subset \mathbb{Z}_p$ and any ideal partition subgroup $P_\mu(I)$ as in Proposition 7.1, we have $\dim(P_\mu(I)) = 0$.

Moreover, for all $d \geq 1$,

$$\dim(\gamma_d(G(\mathbb{Z}_p))) = \dim(G(\mathbb{Z}_p)^{(d)}) = 1$$

for the subgroups of $G(\mathbb{Z}_p)$ in the lower central and derived series of $G(\mathbb{Z}_p)$.

Appendix A. Automorphisms of the finite groups $G_n(q)$

Let $G = G_n(q)$ where $q = p^f$ for some $f, n \in \mathbb{N}$. So $G_n(q)$ is generated by all the matrices of the form $1 + a_i e_{rs}$, where $a_1, \ldots, a_f$ form a basis of $\mathbb{F}_q$ as $\mathbb{F}_p$-vector space and $1 \leq r < s \leq n$. A minimal set of generators is formed by all such elements of the form $1 + a_i e_{r,r+1}$. In [23], A. Weir determines the group of automorphisms of $G_n(q)$ and describes the maps by their action on the elements $1 + a_i e_{r,r+1}$ in a minimal set of generators.

Theorem A.1. [23, Theorem 8] *The group* $\mathrm{Aut}(G_n(q))$ *of automorphisms of* $G_n(q)$ *is generated by the subgroups* $\langle\tau\rangle$, $\mathcal{L}$, $\mathcal{D}$,$\mathcal{I}$ *and* $\mathcal{P}$, *where*

$$\begin{aligned}
\langle\tau\rangle(1+e_{ij}) &= 1+e_{n+1-j,n+1-i} \quad \text{for all} \quad 1\le i<j\le n \ ;\\
\mathcal{L} &= \langle \nu \ : \ 1+a_ie_{r,r+1}\mapsto 1+\nu(a_i)e_{r,r+1} \ a_i\in\mathbb{F}_q \ , \ 1\le r<n \ , \ \nu\in\mathrm{Aut}(\mathbb{F}_q)\rangle \ ;\\
\mathcal{D} &= \ \text{conjugation by non scalar diagonal matrices;}\\
\mathcal{I} &= \ \text{conjugation by elements of } G_n(q);\\
\mathcal{P} &= \mathcal{Z}\times\mathcal{U} \quad \text{where}\\
\mathcal{Z} &= \left\langle \begin{array}{cc} \tau_r^i \ : & 1+a_ie_{s,s+1}\mapsto 1+a_ie_{s,s+1}+\delta_{r,s}b_ie_{1,n} \ ,\\ & b_i\in\mathbb{F}_q \ , \ 1\le i\le f \ , \ 1\le s<n \ ,2\le r\le n-2 \end{array}\right\rangle \quad \text{and}\\
\mathcal{U} &= \langle 1+ae_{12}\mapsto 1+ae_{12}+abe_{2n} \ , \ 1+ae_{n-1,n}\mapsto 1+ae_{n-1,n}+abe_{1,n-1}\rangle \ .
\end{aligned}$$

In particular, $\langle\tau\rangle$ *has order* 2, $\mathcal{L}$ *is cyclic of order* f, $\mathcal{D}$ *has order* $(q-1)^{n-1}$, $\mathcal{I}$ *has order* $|G_n(q)/Z(G_n(q))| = q^{\frac{n^2-n-2}{2}}$, $\mathcal{P}$ *is elementary abelian of order* $q^{f(n-3)+2}$.

The elements of $\mathcal{Z}$ are called *central* automorphisms, because they induce the identity on $G/Z(G)$, and those of $\mathcal{U}$ are called *extremal* automorphisms. The maps τ_1^i, τ_{n-1}^i are inner automorphisms, and so need not be added in $\mathcal{Z}$. The function τ corresponds to the symmetry of the Dynkin diagram of type A_{n-1}, flipping the squares about the antidiagonal.

We now prove the technical lemma which we used in the proof of Proposition 6.2. We use the same notation as in the proposition. In particular, $\alpha_f \ : \ G(q)\to G(p)$, induced by regarding $\mathbb{F}_q$ as an f-dimensional $\mathbb{F}_p$-vector space, has image H, and $\beta_f \ : \ G(p)\to G(q)$, induced by the inclusion of the coefficients $\mathbb{F}_p\subseteq\mathbb{F}_q$, has image K.

Lemma A.2. *Let* $H_m = HN_m(p)/N_m(p)$ *and* $L_m = \alpha_f(K)N_m(p)/N_m(p)$ *for any* $m\in\mathbb{N}$. *Then* $C_{G_{nf}(p)}(L_{nf}) = C_{G_{nf}(p)}(H_{nf}) = \langle 1+e_{ij} \mid 1\le i\le f \ , \ n-f<j\le n\rangle$, *is the subgroup formed by all the matrices whose only nonzero nondiagonal squares lie in the upper right* $f\times f$ *corner.*

Proof. Let $X = \langle 1+e_{ij} \mid 1\le i\le f \ , \ n-f<j\le n\rangle$. Clearly the elements of X commute with any element of H_{nf} because the first and last diagonal $f\times f$ blocks of any element in H_{nf} is the identity $f\times f$ matrix.

To show that these are exactly the elements which centralise H_n, it is enough to see that no other element centralises an element of L_{nf}. Let $y = \alpha_f(1+e_{u+1,v+1}) = 1+\sum_{1\le k\le f} e_{uf+k,vf+k}$ for $0\le u<v<n$ and suppose that $x = 1+\sum_{1\le i<j\le n} a_{ij}e_{ij}\in C_{G_{nf}(p)}(L_{nf})$. Put $x^{-1} = 1+\sum_{1\le i<j\le n} b_{ij}e_{ij}$. We calculate

$$ {}^{x}y = y + \underbrace{\sum_{1\le k\le f}\left(\sum_{vf+k<j}\left(b_{vf+k,j}e_{uf+k,j}+\sum_{i<uf+k} a_{i,uf+k}b_{vf+k,j}e_{ij}\right)\right)}_{(*)} \tag{13}$$

and solve the equation $(*) = 0$. Note that all the indices (i,j) appearing in $(*)$ are distinct. Therefore $b_{vf+k,j} = 0$ for all $j>vf+k$, all $1\le k\le f$ and all $1\le v<n$. By definition of these coefficients, we must then also have $a_{ij} = 0$ for all $f<i<j\le n$. Now, take $y = \alpha_f(1+e_{12}) = \begin{pmatrix} I_f & I_f & 0\\ & I_f & 0\\ & & I_{(n-2)f}\end{pmatrix}$ and suppose that $x = \begin{pmatrix} A & B & C\\ & I_f & 0\\ & & I_{(n-2)f}\end{pmatrix}$ centralises y.

We calculate ${}^x y = \begin{pmatrix} I_f & A & 0 \\ & I_f & 0 \\ & & I_{(n-2)f} \end{pmatrix}$, which gives $A = I_f$. Similarly, for $y = \alpha_f(1+e_{23})$ we obtain that B is the zero $f \times f$ matrix. Inductively on $\alpha_f(1+e_{r,r+1})$ each of the successive $f \times f$ blocks of the $f \times (n-2)f$ matrix C must be zero, except the last one in which the squares can take any value. This proves the lemma. □

Appendix B. Fractional dimension for profinite groups

We review the concept of fractional dimension for profinite groups, as introduced in [1], and refer to [10] for the concepts used.

First some background on measure theory. Let X be a set and write $\mathcal{P}(X)$ for the power set of X. A non-empty subset $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-field if $\mathcal{S}$ is closed under taking complements and countable unions. The *Borel sets* of X are the sets belonging to the σ-field generated by the closed subsets of X. Given a σ-field, one can show that

$$\liminf_{j\to\infty} E_j = \bigcup_{k\geq 1}\bigcap_{j\geq k} E_j \in \mathcal{S} \quad \text{and} \quad \limsup_{j\to\infty} E_j = \bigcap_{k\geq 1}\bigcup_{j\geq k} E_j \in \mathcal{S}\ .$$

The former set is formed by all elements which are in all but a finite number of E_j, while the latter set is formed by all elements which belong to infinitely many E_j.

A *measure* defined on a σ-field $\mathcal{S}$ is a function $\mu : \mathcal{S} \to [0,\infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\cup_j E_j) = \sum_j \mu(E_j)$ for any countable collection of disjoint sets E_j. We call μ an *outer measure* if $\mathcal{S} = \mathcal{P}(X)$, and we call it a *probability measure* if $\mu(X) = 1$.

Suppose that X is a profinite group $X = \varprojlim_{n\in\mathbb{N}} X_n$ with projection maps $\theta_n : X \to X_n$ onto the finite quotients and maps $\pi_{n,m} : X_n \to X_m$ such that $\theta_m = \pi_{n,m}\theta_n$ for all $m \leq n$ (with $\pi_{n,n} = \mathrm{Id}_{X_n}$). We consider the *standard basis*

$$\mathcal{B} = \{\theta_n^{-1}(x_n) \mid x_n \in X_n\ ,\ n \in \mathbb{N}\}$$

for the topology on X. For short, let $N_n = \ker(\theta_n)$ for all $n \in \mathbb{N}$. It is well-known in measure theory that the so-called Haar measure is the unique probability measure on X which is X-invariant. This measure satisfies

$$\mu(\theta_n^{-1}(x_n)) = \mu(N_n) = |X_n|^{-1} \quad \text{and} \quad \mu(\{x\}) = 0\ \forall\, x \in X.$$

This second property means that μ is *non-atomic.*

Now define $\Delta_\mu : \mathcal{P}(X) \to \mathbb{R}$ as follows. For all $\delta, \gamma \in \mathbb{R}$ with $\delta > 0$ define for $M \in \mathcal{P}(X)$,

$$l^\gamma_{\mu,\theta}(M) = \inf_{\mathcal{C}} \sum_{B\in\mathcal{C}} (\mu(B))^\gamma$$

where $\mathcal{C}$ is a cover of M by balls $B \in \mathcal{B}$ such that $\mu(B) < \theta$. Hence let

$$l^\gamma_\mu = \lim_{\theta\to 0} l^\gamma_{\mu,\theta}\ .$$

One can show that there exists a unique real number $\Delta_\mu(M)$ satisfying

$$l^\gamma_\mu(M) = \infty \quad \text{for all } \gamma < \Delta_\mu(M), \text{ and} \quad l^\gamma_\mu(M) = 0 \quad \text{for all } \gamma > \Delta_\mu(M)\ .$$

Thus $\Delta_\mu(M)$ is the *Hausdorff dimension of* M, or *Billingsley dimension of* M ([1]).

Abercrombie proves the following.

Proposition B.1. *Δ_μ is a positive increasing set function and for each $\gamma \in \mathbb{R}$, the function l^γ_μ is an outer measure on X. Furthermore,*

(i) *For all $(M_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)^{\mathbb{N}}$, then $\Delta_\mu(\cup_n M_n) = \sup_n\big(\Delta_\mu(M_n)\big)$.*
(ii) *If $M \in \mathcal{B}$ such that $\Delta_\mu(M) > 0$, then $\Delta_\mu(M) = 1$.*

(iii) *Let Y be a closed subset of X, i.e.* $Y = \bigcap_{n\in\mathbb{N}} YN_n/N_n$. *Then*

$$\Delta_\mu(Y) \geq \liminf_{n\to\infty} \frac{\log|YN_n : N_n|}{\log|X_n|} \quad \textit{with} \quad \Delta_\mu(Y) = \lim_{n\to\infty} \frac{\log|YN_n : N_n|}{\log|X_n|}$$

whenever the limit exists.

Acknowledgements. We wish to thank the referee for a careful proofreading, correcting a few errors, and for helpful suggestions.

References

[1] J.L. Abercrombie, *Subgroups and subrings of profinite rings*, Math. Proc. Cambr. Phil. Soc. **116** (1994), 209–222.
[2] Y. Barnea, B. Klopsch, *Index-subgroups of the Nottingham group*, Adv. Math. **180** (2003), no. 1, 187–221.
[3] Y. Barnea, A. Shalev, *Hausdorff dimension, pro-p groups, and Kac-Moody algebras*, Trans. Amer. Math. Soc. **349** (1997), no. 12, 5073–5091.
[4] A. Bier, *Commutators and powers of infinite unitriangular matrices*, Linear Algebra Appl. **457** (2014), 162–178.
[5] A. Bier, *On lattices of closed subgroups in the group of infinite triangular matrices over a field*, Linear Algebra Appl. **485** (2015), 132–152.
[6] R. Camina, *Some natural subgroups of the Nottingham group*, J. Algebra, **196** (1997), 101-113.
[7] R. Camina, *Subgroups of the Nottingham group*, J. Algebra **196** (1997), no. 1, 101-113.
[8] R. Camina, *The Nottingham group*, New horizons in pro-p groups, 205-221, Progr. Math., **184**, Birkhäuser, 2000.
[9] J. Dixon, M. du Sautoy, A. Mann and D. Segal *Analytic pro-p-groups*, London Mathematical Society Lecture Note Series, **157** Cambridge University Press, Cambridge, 1991.
[10] K. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics **85**, Cambridge University Press, Cambridge, 1986.
[11] M. Hall, *The theory of groups*, Macmillan, 1959.
[12] P. J. Higgins, *Introduction to topological groups*, London Mathematical Society Lecture Note Series. 15. Cambridge, 1974.
[13] W. Holubowski, *Most finitely generated subgroups of infinite unitriangular matrices are free*, Bull. Austral. Math. Soc. **66** (2002), 419–423.
[14] W. Holubowski, *Free subgroups of the group of infinite unitriangular matrices*, Internat. J. Algebra Comput. **13** No. 1 (2003), 81–86.
[15] W. Holubowski, *Subgroups of unitriangular matrices*, J. Math. Sci. **145** (2007), no. 1, 47734780.
[16] J. Kelley, *General topology*, Springer, 1975.
[17] G. Klaas, C. R. Leedham-Green, W. Plesken, *Linear pro-p groups of finite width*, Lecture Notes in Mathematics **1674**, Springer, 1997.
[18] B. Klopsch, *Automorphisms of the Nottingham group*, J. Algebra **223** (2000), 37–56.
[19] S. Lang, *Algebra*, Addison-Wesley, 1965.
[20] P. Pavlov, *Sylow p-subgroups of the full linear group over a simple field of characteristic p (Russian)*, Izv. Akad. Nauk SSSR, Ser. Mat. **16** (1952), 437–458.
[21] L. Ribes, P. Zalesskii, *Profinite groups*, Springer, 2000.
[22] J.-P. Serre, Trees, Springer, 2003.
[23] A. Weir, *Sylow p–subgroups of the general linear group over finite fields of characteristic p*, Proc. Amer. Math. Soc., **6** (1955), 454-464.
[24] J. Wilson, *Profinite groups*, London Math. Soc. Monographs New Series **19**, Oxford University Press, 1998.

Department of Mathematics and Statsitics, University of Lancaster, Lancaster, LA1 4YF, UK
E-mail address: n.mazza@lancaster.ac.uk