ADDITIVE UNITS OF PRODUCT SYSTEMS

B.V. RAJARAMA BHAT, J. MARTIN LINDSAY, AND MITHUN MUKHERJEE

Abstract. We introduce the notion of additive units, or 'addits', of a pointed Arveson system. By the latter we mean a spatial Arveson system with a fixed normalised reference unit. We show that the addits form a Hilbert space whose codimension-one subspace of 'roots' is isomorphic to the index space of the Arveson system, and that the addits generate the type I part of the Arveson system. Consequently the isomorphism class of the Hilbert space of addits is independent of the reference unit. The addits of a pointed inclusion system are shown to be in natural correspondence with the addits of the generated pointed product system. The theory of amalgamated products is developed using addits and roots, and an explicit formula for the amalgamation of pointed Arveson systems is given, providing a new proof of its independence of the particular reference units. (This independence justifies the terminology 'spatial product' of spatial Arveson systems). Finally a cluster construction for inclusion subsystems of an Arveson system is introduced and we demonstrate its correspondence with the Cantor–Bendixson derivative in the context of the random closed set approach to product systems due to Tsirelson and Liebscher.

Introduction

A basic goal of the study of quantum dynamics is the classification of E_0-semigroups, that is suitably continuous one-parameter semigroups of unital *-endomorphisms of $B(\mathsf{H})$, the algebra of bounded operators on a separable Hilbert space H ([3]). Each E_0-semigroup is associated to an Arveson system, that is a suitably measurable one-parameter family of separable Hilbert spaces $\mathcal{E} = (\mathcal{E}_t)_{t>0}$ enjoying associative identifications $\mathcal{E}_{s+t} \simeq \mathcal{E}_s \otimes \mathcal{E}_t$ via unitary operators, and conversely, to each such Arveson system there is an associated E_0-semigroup. If cocycle conjugate E_0-semigroups are identified, and isomorphic Arveson systems are too, then these associations are rendered mutually inverse ([1],[2]; see also [14], and [27]).

A unit of an Arveson system is a nonzero measurable section $(u_s)_{s>0}$, which has the continuous factorisation property: $u_{s+t} = u_s \otimes u_t$, and Arveson systems are classified into type I, type II and (nonspatial or) type III, according to whether their set of units respectively, generates the system, is nonempty but fails to generate the system, or is empty. Spatial Arveson systems have an associated index space, a separable Hilbert space constructed from the set of units, whose dimension is called the index of the system. The index is an isomorphism invariant, and is additive under the tensor product operation on Arveson systems.

For type I Arveson systems the index is a complete invariant and, for each separable Hilbert space k there is a paradigm type I system with index equal to $\dim \mathsf{k}$, namely the Fock Arveson system $\mathcal{F}^{\mathsf{k}}$ ([1]); this is described in the appendix. The isomorphism classes of type II and type III systems are both known to be uncountable ([21],[22],[35],[34]). There is currently a lack of good invariants to distinguish these, and their classification is far from complete. Tsirelson has shown measure types of random sets, and generalised Gaussian processes, to be fertile sources of type II systems ([33],[32]); Liebscher has made a systematic study of Tsirelson's examples. To every product subsystem of an Arveson system $\mathcal{E}$ there corresponds a commuting family of orthogonal projections satisfying evolution and adaptedness relations, and the von Neumann algebra generated by them uniquely determines a (probability) measure type of random closed subsets of the unit interval. The measure types are stationary and factorising over disjoint intervals, and provide an isomorphism invariant for the Arveson system ([14]).

Completely positive contraction semigroups on operator algebras are called quantum dynamical semigroups. For a separable Hilbert space H, every unital quantum dynamical semigroup on $B(\mathsf{H})$

2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 46C05, 46L53.

Key words and phrases. Arveson systems, inclusion systems, quantum dynamics, completely positive semigroups, Cantor–Bendixson derivative, cluster construction.

dilates to an E_0-semigroup, and the minimal dilation is unique up to cocycle conjugacy; this provides an approach to the understanding of quantum dynamics ([4]). For E_0-semigroups on C^*- and W^*-algebras, one may associate product systems of Hilbert modules ([18],[24],[25]). Much of the theory of Arveson systems and E_0-semigroups on $B(\mathsf{H})$ carries over to product systems of Hilbert modules and E_0-semigroups on $B^{\mathrm{a}}(E)$, the algebra of adjointable operators on a Hilbert module E. However there is no tensor product operation for product systems of Hilbert modules. For pointed product systems of Hilbert modules, that is systems with a fixed normalised reference unit, Skeide overcame this by introducing a notion of spatial product ([28]). In the spatial product, units are identified and the index is again additive.

By a pointed Arveson system we mean a spatial Arveson system together with a fixed normalised reference unit. For pointed Arveson systems $(\mathcal{E}, u)$ and $(\mathcal{F}, v)$, Skeide's spatial product may be identified with $\mathcal{E} \otimes v \bigvee u \otimes \mathcal{F}$, the product subsystem of the tensor product Arveson system $\mathcal{E} \otimes \mathcal{F}$ generated by $\mathcal{E} \otimes v$ and $u \otimes \mathcal{F}$. This raises the natural question, is this necessarily all of $\mathcal{E} \otimes \mathcal{F}$? Powers answered this in the negative, by solving the corresponding equivalent problem for E_0-semigroups using his 'sum construction' ([23]); see also [26], [7] and [29]). Motivated by this question, the amalgamated product via a contractive morphism of Arveson systems (which are not necessarily spatial) was introduced in [8] (see Section 5). This generalises the spatial product of pointed Arveson systems since the latter may be viewed as the amalgamated product via the morphism defined through Dirac dyads of the normalized units. (It also answers Powers' problem for the Powers sum arising from not-necessarily-isometric intertwining semigroups.) A priori the spatial product may depend on the reference units. Since, as Tsirelson has shown, the automorphism group of an Arveson system may not act transitively on its set of units ([36]) the answer to this dependency question is not obvious. It was settled in the negative in [5], see also [15]. Our work yields another proof of this fact.

In this paper we introduce and systematically exploit the notions of addit and root, for pointed Arveson systems. We also introduce a cluster construction for product subsystems of an Arveson system; on the one hand the construction provides a new way of obtaining the type I part of a spatial Arveson system, on the other hand it reflects the extraction of the derived sets of random closed subsets of the unit interval in the above correspondence. Whereas Liebscher's work heavily relies on direct integral constructions and the measure theory of random sets, by contrast our cluster construction is done explicitly by elementary Hilbert space means, via an inclusion subsystem.

The structure of the paper is as follows. In Sections 1 and 2, we give a brief overview of the basic theory of product systems, Arveson systems and inclusion systems, and set out the notations and terminology used in the paper. An appendix describes the paradigm case of Fock Arveson systems $\mathcal{F}^{\mathsf{k}}$, for a separable Hilbert space k, and introduces the 'Guichardet picture' for these. In Section 3 addits and roots are defined. These are additive counterparts to units; roots are addits which are orthogonal to the reference unit. Addits comprise a Hilbert space with roots occupying a codimension-one subspace. The roots of the pointed Arveson systems $(\mathcal{F}^{\mathsf{k}}, \Omega^{\mathsf{k}})$, in which Ω^{k} is the vacuum unit, are shown to be indexed by the elements of k, via an isometric isomorphism. From this we show that, for any normalised unit, the type I part of a spatial Arveson system is generated by the unit together with its roots, and the dimension of the Hilbert space of roots equals the index of the Arveson system. Thus the isomorphism class of the Hilbert space of addits of a pointed Arveson system $(\mathcal{E}, u)$ is independent of the choice of unit u of the spatial Arveson system $\mathcal{E}$. In Section 4, we extend the notions of addit and root to pointed inclusion systems (E, u), and establish a natural bijection between the addits of such a system and the addits of $(\mathcal{E}, \widehat{u})$ where $\mathcal{E}$ is the generated (algebraic) product system and $\widehat{u}$ is the normalised unit obtained from u by 'lifting'. The behaviour of roots under amalgamated products of both spatial and pointed Arveson systems is studied in Section 5. In that section we give an explicit formula for the amalgamated product of pointed Arveson systems which provides another proof of its independence of the reference units, and thus also of the fact that, up to cocycle conjugacy, the Powers sum of E_0-semigroups is independent of the choice of intertwining isometries. In Section 6 we describe our cluster construction for subsystems $\mathcal{F}$ of an Arveson system $\mathcal{E}$. When $\mathcal{E}$ is spatial, and $\mathcal{F}$ is generated by a normalised unit, the cluster is shown to be the type I part of $\mathcal{E}$. Finally, extending part of Proposition 3.33 of [14], we show that the measure type corresponding to a subsystem and the measure type of its cluster are precisely related via the Cantor–Bendixson derivative.

Some notational conventions. For Hilbert space vectors $u \in \mathsf{H}$ and $x \in \mathsf{K}$, $|x\rangle\langle u|$ denotes the bounded operator $\mathsf{H} \to \mathsf{K}$, $v \mapsto \langle u, v\rangle x$. For a subset A of the domain of a vector-valued function g, g_A denotes the function which equals g except that it takes the value 0 outside A (*cf.* indicator function notation). We use $\mathcal{P}$ to denote power set, and $\subset\subset$ for subset of finite cardinality.

1. Product systems

In this section we introduce our notations and briefly recall the basic terminology of continuous product systems of Hilbert spaces. Key references are Arveson's monograph ([3]) and Liebscher's memoir ([14]).

Definition 1.1. An (*algebraic*) *product system* $\mathcal{E}$ consists of a family of Hilbert spaces $(\mathcal{E}_t)_{t>0}$ with associated unitary *structure maps*

$$B^{\mathcal{E}}_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \otimes \mathcal{E}_t \qquad (s, t > 0),$$

satisfying the natural consistency conditions

$$(I^{\mathcal{E}}_r \otimes B^{\mathcal{E}}_{s,t})B^{\mathcal{E}}_{r,s+t} = (B^{\mathcal{E}}_{r,s} \otimes I^{\mathcal{E}}_t)B^{\mathcal{E}}_{r+s,t} \qquad (r, s, t > 0)$$

where $I^{\mathcal{E}}_s := I_{\mathcal{E}_s}$ $(s > 0)$. It is called an *Arveson system* if each fibre $\mathcal{E}_t$ is separable and the system is endowed with measurable structure: the families $(\mathcal{E}_t)_{t>0}$ and $(B^{\mathcal{E}}_{s,t})_{s,t>0}$ are both 'measurable'.

Remarks. (i) In the literature, the structure maps are usually taken to be the adjoints $W^{\mathcal{E}}_{s,t} = (B^{\mathcal{E}}_{s,t})^* : \mathcal{E}_s \otimes \mathcal{E}_t \to \mathcal{E}_{s+t}$. Here we use the equivalent B's instead in order to maintain consistency with inclusion systems (defined below).

(ii) For the precise meaning of measurability meant here, we refer to [3] and the essentially equivalent formulation given in [14].

(iii) Frequently one supresses the structure maps and identifies $\mathcal{E}_{s+t}$ and $\mathcal{E}_s \otimes \mathcal{E}_t$, or writes $x \cdot y$ for the preimage in $\mathcal{E}_{s+t}$ of $x \otimes y$ when $x \in \mathcal{E}_s$ and $y \in \mathcal{E}_t$.

(iv) If $\dim \mathcal{E}_t = 1$ for each $t > 0$ then a choice of unit vector $u_t \in \mathcal{E}_t$ for each $t > 0$ reduces the consistency condition to the multiplier relation

$$m(s, t)m(r, s + t) = m(r, s)m(r + s, t)$$

for the map $m : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{T}$ given by $m(s, t)u_s \otimes u_t = B^{\mathcal{E}}_{s,t}u_{s+t}$.

Definition 1.2. Let $\mathcal{E}$ be a product system and let $T > 0$. The family of unitary operators $U^{\mathcal{E},T} = (U^{\mathcal{E},T}_t)_{t\in\mathbb{R}}$ on $\mathcal{E}_T$ defined by periodic extension of the prescription

$$U^{\mathcal{E},T}_t = \begin{cases} I^{\mathcal{E}}_T & \text{if } t = 0 \\ (B^{\mathcal{E}}_{t,T-t})^*\Pi^T_t B^{\mathcal{E}}_{T-t,t} & \text{if } 0 < t < T \end{cases},$$

in which Π^T_t denotes the tensor flip $\mathcal{E}_{T-t} \otimes \mathcal{E}_t \to \mathcal{E}_t \otimes \mathcal{E}_{T-t}$, is called the *unitary flip group on* $\mathcal{E}_T$.

It is easily verified that $U^{\mathcal{E},T} = (U^{\mathcal{E},T}_t)_{t\in\mathbb{R}}$ forms a one-parameter group.

Theorem 1.3 ([14], Theorem 7.7)**.** *Let $\mathcal{E}$ be a product system and let $\tau > 0$. Then the following are equivalent*:

(i) *$\mathcal{E}$ is an Arveson system with respect to some measurable structure.*
(ii) *For all $t > 0$, $\mathcal{E}_t$ is separable, and for all $T \in]0, \tau[$, $U^{\mathcal{E},T}$ is strongly continuous.*

Let $\mathcal{E}$ be a product system and suppose that, for each $t > 0$, $\mathcal{F}_t$ is a closed subspace of $\mathcal{E}_t$ and that, for each $s, t > 0$, $B^{\mathcal{E}}_{s,t}(\mathcal{F}_{s+t}) = \mathcal{F}_s \otimes \mathcal{F}_t$. Then $\mathcal{F} = (\mathcal{F}_t)_{t>0}$ is a product system with structure maps $B^{\mathcal{F}}_{s,t} : \mathcal{F}_{s+t} \to \mathcal{F}_s \otimes \mathcal{F}_t$ $(s, t > 0)$ given by compression of the structure maps of $\mathcal{E}$. Such systems are called *product subsystems of* $\mathcal{E}$.

The following automatic measurability result is a significant consequence of Liebscher's approach to product systems. Note that in his approach the parameter set of an Arveson system $\mathcal{E}$ is extended to $\mathbb{R}_+$, with $\mathcal{E}_0 = \mathbb{C}$.

Theorem 1.4. *Let $\mathcal{F}$ be a product subsystem of an Arveson system $\mathcal{E}$. Then $\mathcal{F}$ is an Arveson subsystem, in other words the measurable structure of $\mathcal{E}$ induces measurable structure on $\mathcal{F}$.*

Proof. Let $(e^n)_{n\geqslant 1}$ be a family of sections of $\mathcal{E}$ determining its measurable structure. We must show that

(a) the sections $\big(P_t e^n_t\big)_{n\geqslant 1}$ are measurable, and

(b) the family of operators $\big(W^{\mathcal{F}}_{s,t} := V^*_{s+t} W^{\mathcal{E}}_{s,t}(V_s \otimes V_t)\big)_{s,t\geqslant 0}$ is measurable,

for the inclusion operators $V_t : \mathcal{F}_t \to \mathcal{E}_t$ and orthogonal projections $P_t := V_t V_t^* = P_{\mathcal{F}_t}$. Without loss of generality we may suppose that $(e^n_t)_{n\geqslant 1}$ is an orthonormal basis of $\mathcal{E}_t$ for all $t > 0$, moreover it suffices to prove these for $s, t \leqslant 1$ (see [14]).

Let $\big(P^{\mathcal{F}}_{s,t}\big)_{0\leqslant s,t\leqslant 1}$ be the strongly continuous family of orthogonal projections in $B(\mathcal{E}_1)$ defined in (5.1) below, and set $e := e^1$. By Parseval's identity, the measurability of $t \mapsto e^p_t \cdot e^q_{1-t}$ $(p, q \in \mathbb{N})$, and the strong continuity of $t \mapsto P^{\mathcal{F}}_{0,t}$ it follows that, for all $l, m \geqslant 1$ and $t \in [0, 1]$,

$$\begin{aligned}\langle e^l_t, P_t e^m_t\rangle &= \langle e^l_t \cdot e_{1-t}, P^{\mathcal{F}}_{0,t}(e^m_t \cdot e_{1-t})\rangle \\ &= \sum_{p,q\geqslant 1} \langle e^l_t \cdot e_{1-t}, e^p_1\rangle \, \langle P^{\mathcal{F}}_{0,t} e^p_1, e^q_1\rangle \, \langle e^q_1, e^m_t \cdot e_{1-t}\rangle\end{aligned}$$

which is now manifestly measurable in t. This proves (a). By another application of Parseval's identity, we see that

$$\begin{aligned}\langle V^*_{s+t} e^l_{s+t}, W^{\mathcal{F}}_{s,t}(V^*_s e^m_s \otimes V^*_t e^n_t)\rangle &= \langle e^l_{s+t}, W^{\mathcal{E}}_{s,t}(P_s e^m_s \otimes P_t e^n_t)\rangle \\ &= \sum_{p,q\geqslant 1} \langle e^l_{s+t}, W^{\mathcal{E}}_{s,t}(e^p_s \otimes e^q_t)\rangle \, \langle e^p_s, P_s e^m_s\rangle \, \langle e^q_t, P_t e^n_t\rangle\end{aligned}$$

for $l, m, n \geqslant 1$ and $s, t \in [0, 1]$, so (b) follows from (a). □

Given two product subsystems $\mathcal{E}^1$ and $\mathcal{E}^2$ of a product system $\mathcal{E}$, the smallest product system of $\mathcal{E}$ containing $\mathcal{E}^1$ and $\mathcal{E}^2$ is denoted $\mathcal{E}^1 \bigvee \mathcal{E}^2$. Thus if $\mathcal{E}$ is an Arveson system then $\mathcal{E}^1 \bigvee \mathcal{E}^2$ is an Arveson subsystem of $\mathcal{E}$.

Definition 1.5. Let $\mathcal{E}$ and $\mathcal{F}$ be product systems. A family of bounded operators $\phi = (\phi_t : \mathcal{E}_t \to \mathcal{F}_t)_{t>0}$ is a *morphism of product systems* if it satisfies

$$B^{\mathcal{F}}_{s,t}\, \phi_{s+t} = (\phi_s \otimes \phi_t) B^{\mathcal{E}}_{s,t} \qquad (s, t > 0)$$

and the quasicontractivity condition $e^{-kt}\|\phi_t\| \leqslant 1$ $(t > 0)$, for some $k \in \mathbb{R}$; it is an *isomorphism* if each ϕ_t is unitary. A *morphism of Arveson systems* is a morphism of the underlying product systems which consists of a measurable family of operators.

Theorem 1.6 ([14], Corollary 7.16)**.** *Let $\phi : \mathcal{E} \to \mathcal{F}$ be an isomorphism of product systems. Suppose that $\mathcal{E}$ and $\mathcal{F}$ are Arveson systems. Then ϕ and ϕ^{-1} are measurable, and thus ϕ is an isomorphism of Arveson systems.*

Definition 1.7. Let $\mathcal{E}$ be an Arveson system. A *unit* of $\mathcal{E}$ is a nonzero measurable section of $\mathcal{E}$ satisfying

$$u_{s+t} = u_s \cdot u_t \qquad (s, t > 0);$$

it is *normalised* if it satisfies $\|u_t\| = 1$ $(t > 0)$. The collection of units of $\mathcal{E}$, respectively normalised units of $\mathcal{E}$, is denoted $\mathcal{U}^{\mathcal{E}}$, respec. $\mathcal{U}^{\mathcal{E}}_1$, and $\mathcal{E}$ is called *spatial* if $\mathcal{U}^{\mathcal{E}} \neq \emptyset$.

The *type I part of* $\mathcal{E}$, denoted $\mathcal{E}^I$, is the smallest product subsystem of $\mathcal{E}$ containing all the units of $\mathcal{E}$, and $\mathcal{E}$ is said to be *of type I* if $\mathcal{E}^I$ is $\mathcal{E}$ itself. Thus, for a spatial Arveson system $\mathcal{E}$,

$$\mathcal{E}^I_T = \overline{\mathrm{Lin}}\{u^1_{t_1} \cdot \cdots \cdot u^n_{t_n} : n \in \mathbb{N}, u^1, \cdots, u^n \in \mathcal{U}^{\mathcal{E}}, \mathbf{t} \in J^{(n)}_T\} \qquad (T > 0)$$

where $J^{(n)}_T := \{\mathbf{t} \in (\mathbb{R}_{>0})^n : \sum t_i = T\}$.

Let $\mathcal{E}$ be a spatial Arveson system. For each $u, v \in \mathcal{U}^{\mathcal{E}}$, the function $t \mapsto \langle u_t, v_t\rangle$ is measurable and satisfies Cauchy's multiplicative functional equation $f(s + t) = f(s)f(t)$, and so there is $\gamma(u, v) \in \mathbb{C}$ such that $\langle u_t, v_t\rangle = e^{t\gamma(u,v)}$ $(t > 0)$. The resulting map $\gamma : \mathcal{U}^{\mathcal{E}} \times \mathcal{U}^{\mathcal{E}} \to \mathbb{C}$ is called the *covariance function* of $\mathcal{E}$. It is conditionally positive definite: $\sum \overline{\lambda_i}\lambda_j \gamma(u^i, u^j) \geqslant 0$ for $n \in \mathbb{N}$, $u^1, \cdots, u^n \in \mathcal{U}^{\mathcal{E}}$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ satisfying $\sum \lambda_i = 0$. It follows that the prescription

$$\langle f, g\rangle := \sum_{u,v\in\mathcal{U}^{\mathcal{E}}} \gamma(u, v)\overline{f(u)}f(v)$$

defines a nonnegative sesquilinear form on the vector space

$$V := \Big\{ f : \mathcal{U}^{\mathcal{E}} \to \mathbb{C} \,\Big|\, \operatorname{supp} f \subset\subset \mathcal{U}^{\mathcal{E}}, \sum\nolimits_{u \in \mathcal{U}^{\mathcal{E}}} f(u) = 0 \Big\}.$$

Quotienting out by the null space $\{f \in V : \langle f, f\rangle = 0\}$ and completing yields a Hilbert space $\mathsf{k}(\mathcal{E})$, called the *index space of* $\mathcal{E}$; its dimension, denoted $\operatorname{ind}\mathcal{E}$, is called the *index of* $\mathcal{E}$.

The index is an isomorphism invariant for Arveson systems: if $\mathcal{E}^1 \cong \mathcal{E}^2$ then $\mathsf{k}(\mathcal{E}^1) \cong \mathsf{k}(\mathcal{E}^2)$.

Example 1.8. Our notations for Fock Arveson systems are given in the appendix. The covariance function of the Fock Arveson system $\mathcal{F}^{\mathsf{k}}$ is given by

$$\gamma\big((e^{\lambda t}\varepsilon^c_t)_{t>0}, (e^{\mu t}\varepsilon^d_t)_{t>0}\big) = \overline{\lambda} + \mu + \langle c, d\rangle \qquad (c, d \in \mathsf{k},\ \lambda, \mu \in \mathbb{C}).$$

These Arveson systems are of type I and satisfy

$$\mathsf{k}(\mathcal{F}^{\mathsf{k}}) \cong \mathsf{k}.$$

Thus $\mathcal{F}^{\mathsf{k}_1} \cong \mathcal{F}^{\mathsf{k}_2}$ implies $\mathsf{k}_1 \cong \mathsf{k}_2$. Conversely, $\mathcal{E}^I \cong \mathcal{F}^{\mathsf{k}(\mathcal{E})}$ for any Arveson system $\mathcal{E}$.

The following notion plays a central role in this paper, from Section 3 onwards.

Definition 1.9. A *pointed Arveson system* is an ordered pair $(\mathcal{E}, u)$ consisting of a spatial Arveson system $\mathcal{E}$ and a fixed normalised unit u, which we refer to as the *reference unit.*

Remarks. Our terminology is a refinement of Liebscher's (in [15]); his is in conflict with the now-common use of the term spatial Arveson system.

There is an obvious notion of *isomorphism* for pointed Arveson systems.

By means of Fock Weyl operators (see the appendix), it is easily seen that, for a type I Arveson system $\mathcal{E}$, the family of pointed systems $\{(\mathcal{E}, u) : u \in \mathcal{U}^{\mathcal{E}}_1\}$ are all isomorphic. However, in view of a theorem of Tsirelson ([36]), this need not be true for type II Arveson systems.

2. Inclusion systems

In this section we introduce notations for inclusion systems and recall their basic theory. We also describe the Fock inclusion systems. Inclusion systems are defined like product systems except that their structure maps are only required to be isometric. They arise very often in quantum dynamics. For instance, the product system associated with a completely positive semigroup on the algebra of bounded operators on a separable Hilbert space is in fact the product system generated by an inclusion system derived from the semigroup ([9],[18],[17],[24],[8]). Our basic reference is [8], where inclusion systems were introduced. Shalit and Solel also studied them, in a more abstract setting, under the name *subproduct systems* ([24]).

Definition 2.1. An *inclusion system* E is a family of Hilbert spaces $(E_t)_{t>0}$ together with isometric *structure maps* $\beta^E_{s,t} : E_{s+t} \to E_s \otimes E_t$ $(s, t > 0)$ satisfying

$$(I^E_r \otimes \beta^E_{s,t})\, \beta^E_{r,s+t} = (\beta^E_{r,s} \otimes I^E_t)\, \beta^E_{r+s,t} \qquad (r, s, t > 0),$$

where $I^E_s := I_{E_s}$ $(s > 0)$.

Remark. Thus a product system is an inclusion system whose structure maps are unitary.

Definition 2.2. Let E be an inclusion system. If, for all $t > 0$, F_t is a closed subspace of E_t, and, for all $s, t > 0$, $\beta^E_{s,t}(F_{s+t}) \subset F_s \otimes F_t$, then the isometries $\beta^F_{s,t} : F_{s+t} \to F_s \otimes F_t$ $(s, t > 0)$ induced by compression render F an inclusion system. Such systems are called *inclusion subsystems of* E.

We now define the product system generated by an inclusion system. It is an inductive limit construction.

Notation. For $T > 0$, set

$$J_T := \bigcup_{n=1}^{\infty} J^{(n)}_T \quad \text{where} \quad J^{(n)}_T := \Big\{ \mathbf{t} \in (\mathbb{R}_{>0})^n : \sum t_i = T \Big\},$$

and for $S, T > 0$, $m, n \in \mathbb{N}$, $\mathbf{s} \in J^{(m)}_S$ and $\mathbf{t} \in J^{(n)}_T$, set

$$\mathbf{s} \smile \mathbf{t} := (s_1, \cdots, s_m, t_1, \cdots, t_n) \in J^{(m+n)}_{S+T}.$$

A partial order on J_T is defined as follows. For $\mathbf{r} \in J_T^{(m)}$ and $\mathbf{s} \in J_T$,

$$\mathbf{s} \geqslant \mathbf{r} \ \text{ if } \ \mathbf{s} = \mathbf{r}_1 \smile \cdots \smile \mathbf{r}_m \ \text{ where } \ \mathbf{r}_i \in J_{r_i} \text{ for } i = 1, \cdots, m.$$

Thus $(T) \leqslant \mathbf{t}$ for all $\mathbf{t} \in J_T$. The partially ordered set J_T is directed:

$$\forall_{\mathbf{r},\mathbf{s} \in J_T} \exists_{\mathbf{t} \in J_T} : \mathbf{t} \geqslant \mathbf{r} \text{ and } \mathbf{t} \geqslant \mathbf{s}.$$

Let E be an inclusion system and fix $T > 0$ for now. For $\mathbf{t} \in J_T^{(n)}$, set $E_{\mathbf{t}} := E_{t_1} \otimes \cdots \otimes E_{t_n}$, thus $E_{(T)} = E_T$. Define isometries $\big(\beta^E_{\mathbf{s},\mathbf{t}} : E_{\mathbf{s}} \to E_{\mathbf{t}}\big)_{\mathbf{s} \leqslant \mathbf{t} \text{ in } J_T}$ as follows: for $p \in \mathbb{N}$ and $\mathbf{r} \in J_R^{(p)}$ set

$$\beta^E_{R,\mathbf{r}} = \begin{cases} I^E_R & \text{if } \mathbf{r} = (R) \\ \big(I^E_{r_1} \otimes \cdots \otimes I^E_{r_{p-2}} \otimes \beta^E_{r_{p-1},r_p}\big) \cdots \big(I^E_{r_1} \otimes \beta^E_{r_2, r_3+\cdots+r_p}\big)\beta^E_{r_1, r_2+\cdots+r_p} & \text{otherwise} \end{cases},$$

and for $\mathbf{s} \leqslant \mathbf{t}$ with $\mathbf{s} \in J_T^{(m)}$ and $\mathbf{t} = \mathbf{s}_1 \smile \cdots \smile \mathbf{s}_m$,

$$\beta^E_{\mathbf{s},\mathbf{t}} := \beta^E_{s_1,\mathbf{s}_1} \otimes \cdots \otimes \beta^E_{s_m,\mathbf{s}_m}.$$

Thus $\beta^E_{\mathbf{s},\mathbf{s}} = I^E_{\mathbf{s}} := I_{E_{\mathbf{s}}}$.

For $T > 0$, $\big((E_{\mathbf{t}})_{\mathbf{t} \in J_T}, (\beta^E_{\mathbf{r},\mathbf{s}})_{\mathbf{r} \leqslant \mathbf{s} \in J_T}\big)$ forms an inductive system of Hilbert spaces:

$$\beta^E_{\mathbf{t},\mathbf{t}} = I^E_{\mathbf{t}} \quad (\mathbf{t} \in J_T) \ \text{ and } \ \beta^E_{\mathbf{s},\mathbf{t}} \beta^E_{\mathbf{r},\mathbf{s}} = \beta^E_{\mathbf{r},\mathbf{t}} \quad (\mathbf{r} \leqslant \mathbf{s} \leqslant \mathbf{t} \text{ in } J_T).$$

Let $\big(\mathcal{E}_T, (\imath^E_{\mathbf{t}} : E_{\mathbf{t}} \to \mathcal{E}_T)_{\mathbf{t} \in J_T}\big)$ denote its inductive limit. For ease of reference, we list its key properties next.

(i) *Minimality.* $\mathcal{E}_T$ is a Hilbert space satisfying $\mathcal{E}_T = \vee_{\mathbf{t} \in J_T} \operatorname{Ran} \imath^E_{\mathbf{t}}$.

(ii) *Isometry.* $\imath^E_{\mathbf{t}}$ is an isometry ($\mathbf{t} \in J_T$) and $\imath^E_{\mathbf{s}} \circ \beta^E_{\mathbf{r},\mathbf{s}} = \imath^E_{\mathbf{r}}$ ($\mathbf{r} \leqslant \mathbf{s}$ in J_T).

(iii) *Subnet property.* For any $K \subset J_T$ such that $\forall_{\mathbf{s} \in J_T} \exists_{\mathbf{t} \in K} : \mathbf{t} \geqslant \mathbf{s}$, the inductive limit of $\big((E_{\mathbf{t}})_{\mathbf{t} \in K}, (\beta^E_{\mathbf{r},\mathbf{s}})_{\mathbf{r} \leqslant \mathbf{s} \text{ in } K}\big)$ equals $\big(\mathcal{E}_T, (\imath^E_{\mathbf{t}} : E_{\mathbf{t}} \to \mathcal{E}_T)_{\mathbf{t} \in K}\big)$.

(iv) *Universal property.* For $K \subset J_T$ as in (iii), and any family of Hilbert space isometries $(\jmath_{\mathbf{t}} : E_{\mathbf{t}} \to \mathsf{H})_{\mathbf{t} \in K}$ satisfying $\jmath_{\mathbf{s}} \circ \beta^E_{\mathbf{r},\mathbf{s}} = \jmath_{\mathbf{r}}$ ($\mathbf{r} \leqslant \mathbf{s}$ in K), there is a unique isometry $\jmath : \mathcal{E}_T \to \mathsf{H}$ such that $\jmath_{\mathbf{t}} = \jmath \circ \imath^E_{\mathbf{t}}$ ($\mathbf{t} \in K$).

Now let $R, S > 0$ and set $J_R \smile J_S := \{\mathbf{r} \smile \mathbf{s} : \mathbf{r} \in J_R, \mathbf{s} \in J_S\}$. For $\mathbf{t} \in J_{R+S}$ there are $\mathbf{r} \in J_R$ and $\mathbf{s} \in J_S$ such that $\mathbf{r} \smile \mathbf{s} \geqslant \mathbf{t}$. Therefore, by the subnet property (iii), the inductive limit of $\big((E_{\mathbf{t}})_{\mathbf{t} \in J_R \smile J_S}, (\beta^E_{\mathbf{u},\mathbf{v}})_{\mathbf{u} \leqslant \mathbf{v} \text{ in } J_R \smile J_S}\big)$ equals $\big(\mathcal{E}_T, (\imath^E_{\mathbf{t}} : E_{\mathbf{t}} \to \mathcal{E}_T)_{\mathbf{t} \in J_R \smile J_S}\big)$ where $T = R + S$. For $\mathbf{r}, \mathbf{r}' \in J_R$ and $\mathbf{s}, \mathbf{s}' \in J_S$ such that $\mathbf{r} \smile \mathbf{s} \leqslant \mathbf{r}' \smile \mathbf{s}'$, necessarily $\mathbf{r} \leqslant \mathbf{r}'$ and $\mathbf{s} \leqslant \mathbf{s}'$ so

$$(\imath^E_{\mathbf{r}'} \otimes \imath^E_{\mathbf{s}'}) \circ \beta^E_{\mathbf{r} \smile \mathbf{s}, \mathbf{r}' \smile \mathbf{s}'} = (\imath^E_{\mathbf{r}'} \circ \beta^E_{\mathbf{r},\mathbf{r}'}) \otimes (\imath^E_{\mathbf{s}'} \circ \beta^E_{\mathbf{s},\mathbf{s}'}) = \imath^E_{\mathbf{r}} \otimes \imath^E_{\mathbf{s}}.$$

The family $\big(\widetilde{\imath}_{\mathbf{t}} : E_{\mathbf{t}} \to \mathcal{E}_R \otimes \mathcal{E}_S)_{\mathbf{t} \in J_R \smile J_S}\big)$, in which $\widetilde{\imath}_{\mathbf{r} \smile \mathbf{s}} := \imath^E_{\mathbf{r}} \otimes \imath^E_{\mathbf{s}}$, satisfies $\widetilde{\imath}_{\mathbf{t}'} \circ \beta^E_{\mathbf{t},\mathbf{t}'} = \widetilde{\imath}_{\mathbf{t}}$ for $\mathbf{t} \leqslant \mathbf{t}'$ in $J_R \smile J_S$. Therefore, by the universal property (iv), there is a unique isometry $B^{\mathcal{E}}_{R,S} : \mathcal{E}_{R+S} \to \mathcal{E}_R \otimes \mathcal{E}_S$ such that $\widetilde{\imath}_{\mathbf{t}} = B^{\mathcal{E}}_{R,S} \circ \imath^E_{\mathbf{t}}$ ($\mathbf{t} \in J_R \smile J_S$), equivalently $\imath^E_{\mathbf{r}} \otimes \imath^E_{\mathbf{s}} = B^{\mathcal{E}}_{R,S} \circ \imath^E_{\mathbf{r} \smile \mathbf{s}}$ ($\mathbf{r} \in J_R$, $\mathbf{s} \in J_S$). It follows from the minimality property (i) that $\operatorname{Ran} B^{\mathcal{E}}_{R,S} = \mathcal{E}_R \otimes \mathcal{E}_S$, so $B^{\mathcal{E}}_{R,S}$ is unitary. It is now easily verified that, for $R, S, T > 0$, $\mathbf{r} \in J_R$, $\mathbf{s} \in J_S$ and $\mathbf{t} \in J_T$,

$$(B^{\mathcal{E}}_{R,S} \otimes I^{\mathcal{E}}_T) B^{\mathcal{E}}_{R+S,T} \circ \imath^E_{\mathbf{r} \smile \mathbf{s} \smile \mathbf{t}} \ \text{ and } \ (I^{\mathcal{E}}_R \otimes B^{\mathcal{E}}_{S,T}) B^{\mathcal{E}}_{R,S+T} \circ \imath^E_{\mathbf{r} \smile \mathbf{s} \smile \mathbf{t}}$$

both equal $\imath^E_{\mathbf{r}} \otimes \imath^E_{\mathbf{s}} \otimes \imath^E_{\mathbf{t}}$. Since $\bigcup_{\mathbf{u} \in J_R \smile J_S \smile J_T} \operatorname{Ran} \imath^E_{\mathbf{u}}$ is total in $\mathcal{E}_{R+S+T}$, it follows that $(B^{\mathcal{E}}_{R,S} \otimes I^{\mathcal{E}}_T) B^{\mathcal{E}}_{R+S,T} = (I^{\mathcal{E}}_R \otimes B^{\mathcal{E}}_{S,T}) B^{\mathcal{E}}_{R,S+T}$ ($R, S, T > 0$). In the above notations, we have established the following theorem.

Theorem 2.3 ([8], Theorem 5)**.** *The family $(\mathcal{E}_T)_{T>0}$ defined above forms a product system with respect to the structure maps $(B^{\mathcal{E}}_{S,T})_{S,T>0}$.*

As mentioned above, this is called the *product system generated by E*.

Theorem 2.4. *Let $\mathcal{E}$ be a product system and let F be an inclusion subsystem. Then the product system generated by F may be viewed as a product subsystem of $\mathcal{E}$.*

Proof. Let $\mathcal{F}$ be the product system generated by F. We need to obtain an isometric morphism of product systems $\jmath : \mathcal{F} \to \mathcal{E}$.

Let $T > 0$. Consider the isometries $(B^{\mathcal{E}}_{T,\mathbf{t}})^*|_{F_{\mathbf{t}}} : F_{\mathbf{t}} \to \mathcal{E}_T$ ($\mathbf{t} \in J_T$). For $\mathbf{r} \leqslant \mathbf{s}$ in J_T,

$$(B^{\mathcal{E}}_{T,\mathbf{s}})^*|_{F_{\mathbf{s}}} \circ \beta^F_{\mathbf{r},\mathbf{s}} = (B^{\mathcal{E}}_{T,\mathbf{s}})^* B^{\mathcal{E}}_{\mathbf{r},\mathbf{s}}|_{F_{\mathbf{r}}} = (B^{\mathcal{E}}_{T,\mathbf{r}})^*|_{F_{\mathbf{r}}}$$

Therefore, by the universal property (iv), there is a unique isometry $\jmath_T : \mathcal{F}_T \to \mathcal{E}_T$, such that $\jmath_T \circ \imath^F_{\mathbf{t}} = (B^{\mathcal{E}}_{T,\mathbf{t}})^*|_{F_{\mathbf{t}}}$ for the canonical maps $\imath^F_{\mathbf{t}} : F_{\mathbf{t}} \to \mathcal{F}_T$ ($\mathbf{t} \in J_T$).

Now fix $S, T > 0$. In view of the identity

$$B^{\mathcal{E}}_{S+T,\mathbf{s}\smile\mathbf{t}} = (B^{\mathcal{E}}_{S,\mathbf{s}} \otimes B^{\mathcal{E}}_{T,\mathbf{t}}) B^{\mathcal{E}}_{S,T} \qquad (\mathbf{s} \in J_S, \mathbf{t} \in J_T),$$

which is not hard to verify,

$$B^{\mathcal{E}}_{S,T} \circ \jmath_{S+T} \circ \imath^F_{\mathbf{s}\smile\mathbf{t}} = B^{\mathcal{E}}_{S,T} \circ (B^{\mathcal{E}}_{S+T,\mathbf{s}\smile\mathbf{t}})^*|_{F_{\mathbf{s}\smile\mathbf{t}}} = (B^{\mathcal{E}}_{S,\mathbf{s}} \otimes B^{\mathcal{E}}_{T,\mathbf{t}})^*|_{F_{\mathbf{s}} \otimes F_{\mathbf{t}}}$$

and so, since

$$(\jmath_S \otimes \jmath_T) \circ B^{\mathcal{F}}_{S,T} \circ \imath^F_{\mathbf{s}\smile\mathbf{t}} = (\jmath_S \otimes \jmath_T) \circ (\imath^F_{\mathbf{s}} \otimes \imath^F_{\mathbf{t}}) = (B^{\mathcal{E}}_{S,\mathbf{s}})^*|_{F_{\mathbf{s}}} \otimes (B^{\mathcal{E}}_{T,\mathbf{t}})^*|_{F_{\mathbf{t}}},$$

the operators $B^{\mathcal{E}}_{S,T} \circ \jmath_{S+T}$ and $(\jmath_S \otimes \jmath_T) \circ B^{\mathcal{F}}_{S,T}$ agree on the set $\bigcup_{\mathbf{u} \in J_S \smile J_T} \operatorname{Ran} \imath^F_{\mathbf{u}}$ which is total in $\mathcal{F}_{S+T}$. It follows that the family of isometries $(\jmath_T : \mathcal{F}_T \to \mathcal{E}_T)_{T>0}$ forms a morphism of product systems, as required. □

Definition 2.5. Let E and F be inclusion systems. A *morphism* $E \to F$ is a family of bounded operators $A = (A_t : E_t \to F_t)_{t>0}$ satisfying the compatibility condition

$$A_{s+t} = (\beta^F_{s,t})^* (A_s \otimes A_t) \beta^F_{s,t} \qquad (s, t > 0), \tag{2.1}$$

and the quasicontractivity condition $e^{-kt}\|A_t\| \leqslant 1$ ($t > 0$) for some $k \in \mathbb{R}$. It is called a *strong morphism* if (2.1) is strengthened to $\beta^F_{s,t} A_{s+t} = (A_s \otimes A_t)\beta^F_{s,t}$ ($s, t > 0$).

A *unit* of E is a nonzero quasicontractive section u of E satisfying

$$u_{s+t} = (\beta^E_{s,t})^* u_s \otimes u_t \qquad (s, t > 0);$$

it is called a *strong unit* if this is strengthened to $\beta^E_{s,t} u_{s+t} = u_s \otimes u_t$ ($s, t > 0$).

Remark. A section x of an inclusion system E may be thought of as a family of bounded operators $X = (X_t := |x_t\rangle\langle 1| : \mathbb{C}_t \to E_t)_{t>0}$, where $(\mathbb{C}_t)_{t>0}$ is the one-dimensional inclusion system with $\mathbb{C}_t = \mathbb{C}$ ($t > 0$) and $\beta^{\mathbb{C}}_{s,t} : \lambda \mapsto \lambda \otimes 1 = \lambda$ ($s, t > 0$). Then x is a (strong) unit if and only if X is a (strong) morphism.

Theorem 2.6 ([8], Theorem 10)**.** *Let $\mathcal{E}$ be the product system generated by the inclusion system E. Then the family of canonical maps $\imath^E := (\imath^E_t : E_t \to \mathcal{E}_t)_{t>0}$ forms a strong isometric morphism of inclusion systems. Moreover $(\imath^E)^* := ((\imath^E_t)^* : \mathcal{E}_t \to E_t)_{t>0}$ restricts to a bijection from the set of units of $\mathcal{E}$ to the set of units of E, whose inverse is denoted by $u \mapsto \widehat{u}$.*

Remarks. (i) The quasicontractivity condition on units is crucial for the above result.

(ii) The unit $\widehat{u}$ of $\mathcal{E}$ is called the *lift* of the unit u of E; $u = (\imath^E)^*(\widehat{u})$.

(iii) For units u and v of E and $T > 0$,

$$\langle \widehat{u}_T, \widehat{v}_T \rangle = \lim_{\mathbf{t} \in J_T} \langle u_{\mathbf{t}}, v_{\mathbf{t}} \rangle$$

where, for $n \in \mathbb{N}$ and $\mathbf{t} \in J^{(n)}_T$, $u_{\mathbf{t}} := u_{t_1} \otimes \cdots \otimes u_{t_n}$. In particular, $\widehat{u}$ is normalised if u is.

(iv) Similarly (see [8], Theorem 11), every morphism of inclusion systems $A : E \to F$ lifts to a unique morphism $\widehat{A} : \mathcal{E} \to \mathcal{F}$ of the generated product systems. In terms of the corresponding canonical morphisms, $A_t = (\imath^F_t)^* \widehat{A}_t \imath^E_t$ ($t > 0$). The map $A \to \widehat{A}$ is a bijection between the corresponding spaces of morphisms which respects both isometry and unitarity.

We end this section with a key example.

Example 2.7. (Fock inclusion systems.) Let k be a separable Hilbert space. The Fock Arveson system over k, denoted $\mathcal{F}^{\mathsf{k}}$, is defined in the appendix, where the Guichardet picture of it is also described. In the notations used there, the *Fock inclusion system over* k, denoted F^{k}, is defined as follows:

$$F^{\mathsf{k}}_t = \widehat{\mathsf{K}_t} := \mathbb{C} \oplus \mathsf{K}_t$$
$$\subset \Gamma(\mathsf{K}_t) = \mathcal{F}^{\mathsf{k}}_t \qquad (t > 0)$$

and, in terms of the canonical identifications

$$\widehat{\mathsf{K}_s} \otimes \widehat{\mathsf{K}_t} = \mathbb{C} \oplus \mathsf{K}_s \oplus \mathsf{K}_t \oplus (\mathsf{K}_s \otimes \mathsf{K}_t), \quad \text{and}$$
$$\widehat{\mathsf{K}_r} \otimes \widehat{\mathsf{K}_s} \otimes \widehat{\mathsf{K}_t} = \mathbb{C} \oplus \mathsf{K}_r \oplus \mathsf{K}_s \oplus \mathsf{K}_t \oplus \mathsf{K}_{r,s,t} \quad \text{where}$$
$$\mathsf{K}_{r,s,t} := (\mathsf{K}_r \otimes \mathsf{K}_s \oplus \mathsf{K}_r \otimes \mathsf{K}_t \oplus \mathsf{K}_s \otimes \mathsf{K}_t) \oplus (\mathsf{K}_r \otimes \mathsf{K}_s \otimes \mathsf{K}_t) \qquad (r,s,t>0),$$

its structure maps are defined as follows: for $s,t>0$ and $(\lambda, g) \in F^{\mathsf{k}}_{s+t}$,

$$\beta^{F,\mathsf{k}}_{s,t}(\lambda, g) = \big(\lambda, g_{[0,s[}, (S^{\mathsf{k}}_s)^* g_{[s,s+t[}, 0\big) \in \mathbb{C} \oplus \mathsf{K}_s \oplus \mathsf{K}_t \oplus (\mathsf{K}_s \otimes \mathsf{K}_t)$$

or, in the notation $\Omega_t = (1,0) \in \widehat{\mathsf{K}_t}$,

$$\beta^{F,\mathsf{k}}_{s,t}(\lambda, g) = \lambda\, \Omega_s \otimes \Omega_t + \big(0, g_{[0,s[}\big) \otimes \Omega_t + \Omega_s \otimes \big(0, (S^{\mathsf{k}}_s)^* g_{[s,s+t[}\big) \in \widehat{\mathsf{K}_s} \otimes \widehat{\mathsf{K}_t}.$$

For $r,s,t>0$ and $(\lambda, g) \in F^{\mathsf{k}}_{r+s+t}$,

$$\big(\lambda, g_{[0,r[}, (S^{\mathsf{k}}_r)^* g_{[r,r+s[}, (S^{\mathsf{k}}_{r+s})^* g_{[r+s,r+s+t[}, 0\big) \in \mathbb{C} \oplus \mathsf{K}_r \oplus \mathsf{K}_s \oplus \mathsf{K}_t \oplus \mathsf{K}_{r,s,t},$$

is a common expression for

$$(\beta^{F,\mathsf{k}}_{r,s} \otimes I)\beta^{F,\mathsf{k}}_{r+s,t}(\lambda, g) \quad \text{and} \quad (I \otimes \beta^{F,\mathsf{k}}_{s,t})\beta^{F,\mathsf{k}}_{r,s+t}(\lambda, g).$$

In terms of the subspace inclusions $\jmath^{\mathsf{k}}_t : F^{\mathsf{k}}_t \to \mathcal{F}^{\mathsf{k}}_t$ $(t>0)$, the structure maps of the inclusion system F^{k} and Arveson system $\mathcal{F}^{\mathsf{k}}$ are related by

$$B^{\mathcal{F},\mathsf{k}}_{s,t} \circ \jmath^{\mathsf{k}}_{s+t} = (\jmath^{\mathsf{k}}_s \otimes \jmath^{\mathsf{k}}_t) \circ \beta^{F,\mathsf{k}}_{s,t} \qquad (s,t>0).$$

Thus F^{k} is an inclusion subsystem of $\mathcal{F}^{\mathsf{k}}$.

In Corollary 3.6 below, we verify that F^{k} generates the Fock Arveson system $\mathcal{F}^{\mathsf{k}}$.

Remark. The failure of the Fock inclusion system F^{k} to be a product system is clearly seen through the identity

$$\widehat{\mathsf{K}_s} \otimes \widehat{\mathsf{K}_t} \ominus \operatorname{Ran} \beta^{F,\mathsf{k}}_{s,t} = \mathsf{K}_s \otimes \mathsf{K}_t \qquad (s,t>0).$$

3. Addits of pointed Arveson systems

In this section we introduce the additive counterpart to the multiplicative notion of unit. This requires the fixing of a reference unit of the Arveson system and so is relevant to spatial Arveson systems. We show that the space of addits then has a natural Hilbert space structure with a one-dimensional subspace of 'trivial' addits. Elements of the orthogonal complement of this subspace are called roots and, when the reference unit is normalised, the subspace of roots is shown to be isomorphic to the index space of the Arveson system. This isomorphism is established by first revealing the root space of a Fock Arveson system with respect to the vacuum unit.

Definition 3.1. Let $(\mathcal{E}, u)$ be an Arveson system with unit. An *addit* of $(\mathcal{E}, u)$ is a measurable section a of $\mathcal{E}$ satisfying

$$a_{s+t} = a_s \cdot u_t + u_s \cdot a_t \qquad (s,t>0);$$

a *root* of $(\mathcal{E}, u)$ is an addit a satisfying

$$u_t \perp a_t \qquad (t>0).$$

Remarks. (i) The set of addits of $(\mathcal{E}, u)$ forms a subspace, denoted $A^{\mathcal{E}}_u$, of the space of measurable sections of $\mathcal{E}$, as does the set of roots, denoted $R^{\mathcal{E}}_u$.

(ii) *Normalisation.* Let $a \in A^{\mathcal{E}}_u$ and $\lambda \in \mathbb{C}$. Then

$$b := (e^{\lambda t} a_t)_{t>0} \in A^{\mathcal{E}}_v \quad \text{for the unit} \quad v := (e^{\lambda t} u_t)_{t>0}.$$

(iii) *Trivial addits.* For $\lambda \in \mathbb{C}$, $(\lambda t u_t)_{t>0} \in A^{\mathcal{E}}_u$. We refer to these as *trivial addits* of $(\mathcal{E}, u)$, and write $T^{\mathcal{E}}_u$ for the space of these. Note that

$$T^{\mathcal{E}}_u \cap R^{\mathcal{E}}_u = \{0\},$$

and, for $a, b \in T^{\mathcal{E}}_u$,

$$\langle a_t, b_t \rangle = t^2 \langle a_1, b_1 \rangle \|u_t\|^2 / \|u_1\|^2.$$

(iv) *Direct sum decomposition.* For $a \in A_u^{\mathcal{E}}$, define

$$a^{\mathrm{Triv}} := \Big(\frac{\langle u_t, a_t\rangle}{\|u_t\|^2}\, u_t\Big)_{t>0} \quad \text{and} \quad a^{\mathrm{Root}} := a - a^{\mathrm{Triv}}.$$

Claim. $a^{\mathrm{Triv}} \in T_u^{\mathcal{E}}$ and $a^{\mathrm{Root}} \in R_u^{\mathcal{E}}$, so $A_u^{\mathcal{E}} = T_u^{\mathcal{E}} \oplus R_u^{\mathcal{E}}$.

Since

$$\langle u_t, a_t^{\mathrm{Root}}\rangle = \langle u_t, a_t\rangle - \langle u_t, a_t\rangle = 0 \qquad (t > 0),$$

it remains to show that a^{Triv} is a trivial addit of $(\mathcal{E}, u)$. Since

$$\frac{\langle u_{s+t}, a_{s+t}\rangle}{\|u_{s+t}\|^2} = \frac{\langle u_s, a_s\rangle}{\|u_s\|^2} + \frac{\langle u_t, a_t\rangle}{\|u_t\|^2} \qquad (s, t > 0),$$

the measurable function $f_a : \mathbb{R}_{>0} \to \mathbb{C}$, $t \mapsto \langle u_t, a_t\rangle/\|u_t\|^2$ satisfies Cauchy's additive functional equation and so $f_a(t) = f_a(1)t$, in other words $a^{\mathrm{Triv}} = (\lambda t u_t)_{t>0}$ where $\lambda = \|u_1\|^{-2}\langle u_1, a_1\rangle$. In particular $a^{\mathrm{Triv}} \in T_u^{\mathcal{E}}$.

(v) Let $a, b \in R_u^{\mathcal{E}}$, and suppose that u is normalised. Then

$$\begin{aligned}\langle a_{s+t}, b_{s+t}\rangle &= \langle a_s, b_s\rangle\langle u_t, u_t\rangle + \langle u_s, u_s\rangle\langle a_t, b_t\rangle \\ &= \langle a_s, b_s\rangle + \langle a_t, b_t\rangle \qquad (s, t > 0).\end{aligned}$$

Therefore, appealing to measurability once more,

$$\langle a_t, b_t\rangle = t\langle a_1, b_1\rangle \qquad (t > 0).$$

(*Cf.* (iii)).

The above remarks indicate the usefulness of the notion of pointed Arveson system (Definition 1.9).

Notation. To a pointed Arveson system $(\mathcal{E}, u)$ we associate the family of bounded operators $(\theta_t^{\mathcal{E},u})_{t>0}$ defined by

$$\theta_t^{\mathcal{E},u} := tP_{\mathbb{C}u_1} + \sqrt{t}P_{\mathbb{C}u_1}^{\perp} \in B(\mathcal{E}_1) \qquad (t > 0).$$

Remarks. Let $a, b \in A_u^{\mathcal{E}}$ for a pointed Arveson system $(\mathcal{E}, u)$. For all $t > 0$,

$$\begin{aligned}&a_t^{\mathrm{Triv}} = ta_1^{\mathrm{Triv}} = t\langle u_1, a_1\rangle u_1 = tP_{\mathbb{C}u_1}a_1, \\ &\langle u_t, b_t^{\mathrm{Root}}\rangle = \langle u_t, b_t\rangle - \langle u_t, b_t^{\mathrm{Triv}}\rangle = 0, \text{ so } a_t^{\mathrm{Triv}} \perp b_t^{\mathrm{Root}}, \\ &\langle a_t, b_t\rangle = \langle a_t^{\mathrm{Triv}}, b_t^{\mathrm{Triv}}\rangle + \langle a_t^{\mathrm{Root}}, b_t^{\mathrm{Root}}\rangle \\ &\qquad = t^2\langle a_1^{\mathrm{Triv}}, b_1^{\mathrm{Triv}}\rangle + t\langle a_1^{\mathrm{Root}}, b_1^{\mathrm{Root}}\rangle = \langle \theta_t a_1, \theta_t b_1\rangle, \quad \text{where } \theta_t := \theta_t^{\mathcal{E},u}.\end{aligned}$$

Proposition 3.2. *Let $(\mathcal{E}, u)$ be a pointed Arveson system. Then the prescription*

$$\langle a, b\rangle := \langle a_1, b_1\rangle_{\mathcal{E}_1} \tag{3.1}$$

endows the vector space $A_u^{\mathcal{E}}$ with the structure of a Hilbert space for which the direct sum decomposition

$$A_u^{\mathcal{E}} = T_u^{\mathcal{E}} \oplus R_u^{\mathcal{E}}$$

is an orthogonal decomposition.

Proof. Set $\theta_t := \theta_t^{\mathcal{E},u}$ $(t > 0)$.

Clearly (3.1) defines a nonnegative sesquilinear form on $A_u^{\mathcal{E}}$. Suppose that $a \in A_u^{\mathcal{E}}$ satisfies $\langle a, a\rangle = 0$. Then $a_1 = 0$ and so $\|a_t\| = \|\theta_t a_1\| = 0$ $(t > 0)$, so $a = 0$. Thus (3.1) defines an inner product on $A_u^{\mathcal{E}}$. Suppose next that $(a^{(n)})$ is a Cauchy sequence with respect to the induced metric on $A_u^{\mathcal{E}}$. Then, for all $t > 0$,

$$\|a_t^{(n)} - a_t^{(m)}\| = \|\theta_t a_1^{(n)} - \theta_t a_1^{(m)}\|_{\mathcal{E}_1} \leqslant \|\theta_t\|\|a_1^{(n)} - a_1^{(m)}\|_{\mathcal{E}_1} = \max\{t, \sqrt{t}\}\|a_1^{(n)} - a_1^{(m)}\|_{\mathcal{E}_1}$$

$(n, m \in \mathbb{N})$, so $(a_t^{(n)})$ is Cauchy, and thus convergent, in $\mathcal{E}_t$. Set $a_t := \lim_{n\to\infty} a_t^{(n)} \in \mathcal{E}_t$ $(t > 0)$. Then a is a measurable section of $\mathcal{E}$ satisfying

$$a_{s+t} = \lim_{n\to\infty} a_{s+t}^{(n)} = \lim_{n\to\infty} \big(a_s^{(n)}{\cdot}u_t + u_s{\cdot}a_t^{(n)}\big) = a_s{\cdot}u_t + u_s{\cdot}a_t \qquad (s, t > 0),$$

so $a \in A_u^{\mathcal{E}}$. Moreover

$$\|a^{(n)} - a\| = \|a_1^{(n)} - a_1\|_{\mathcal{E}_1} \to 0 \text{ as } n \to \infty.$$

Therefore $A^{\mathcal{E}}_u$ is complete and thus a Hilbert space with respect to the inner product (3.1).

It remains to show that $(T^{\mathcal{E}}_u)^\perp = R^{\mathcal{E}}_u$. It follows from remarks above that $R^{\mathcal{E}}_u \subset (T^{\mathcal{E}}_u)^\perp$; the reverse inclusion follow since

$$a \in (T^{\mathcal{E}}_u)^\perp \implies a_1^{\mathrm{Triv}} = \frac{\langle u_1, a_1\rangle}{\|u_1\|^2}\, u_1 = 0 \implies a^{\mathrm{Triv}} = 0 \implies a \in R^{\mathcal{E}}_u.$$

□

We next find the roots of the pointed Fock Arveson system $(\mathcal{F}^{\mathsf{k}}, \Omega^{\mathsf{k}})$, for a separable Hilbert space k, by working in the Guichardet picture (described in the appendix). Thus

$$\Omega^{\mathsf{k}}_t = \delta_\emptyset \in \mathcal{F}^{\mathsf{k}}_t \qquad (t > 0)$$

and, for $c \in \mathsf{k}$ we define the measurable section $\chi^c := (c_{[0,t[})_{t>0}$ of $\mathcal{F}^{\mathsf{k}}$, in which

$$c_{[0,t[}(\sigma) = \begin{cases} c & \text{if } \sigma \in \Gamma^{(1)}_{[0,t[} \\ 0 & \text{otherwise} \end{cases}.$$

Remark. Both Ω^{k} and each χ^c are actually sections of the Fock inclusion system F^{k}.

Proposition 3.3. *Let* k *be a separable Hilbert space. The prescription*

$$c \mapsto \chi^c \qquad (c \in \mathsf{k}) \tag{3.2}$$

defines an isometric isomorphism from k *to* $R^{\mathcal{F},\mathsf{k}}_\Omega$*, the space of roots of the pointed Arveson system* $(\mathcal{F}^{\mathsf{k}}, \Omega^{\mathsf{k}})$*.*

Proof. Abbreviate $(\mathcal{F}^{\mathsf{k}}, \Omega^{\mathsf{k}})$ to $(\mathcal{F}, \Omega)$, and $R^{\mathcal{F},\mathsf{k}}_\Omega$ to R_Ω, and let K_t be as in the appendix.

Claim 1. $\chi^c \in R_\Omega$ $(c \in \mathsf{k})$.

Fix $c \in \mathsf{k}$. Let $s, t > 0$, then for a.a. σ

$$\begin{aligned}
&(\chi^c_s \cdot \Omega_t + \Omega_s \cdot \chi^c_t)(\sigma) \\
&\quad = \chi^c_s\big(\sigma \cap [0,s[\big)\, \delta_\emptyset\big(\sigma \cap [s, s+t[\big) + \delta_\emptyset\big(\sigma \cap [0,s[\big)\, \chi^c_t\big((\sigma \cap [s,s+t[) - s\big) \\
&\quad = \begin{cases} c & \text{if } \sigma \in \Gamma^{(1)}, \text{ and either } \sigma \subset [0,s[\text{ or } \sigma \subset [s,s+t[\\ 0 & \text{otherwise} \end{cases} \\
&\quad = \begin{cases} c & \text{if } \sigma \in \Gamma^{(1)}_{s+t} \\ 0 & \text{otherwise} \end{cases} \\
&\quad = \chi^c_{s+t}(\sigma),
\end{aligned}$$

so $\chi^c_{s+t} = \chi^c_s \cdot \Omega_t + \Omega_s \cdot \chi^c_t$ $(s, t > 0)$. Since $\chi^c_t \perp \Omega_t$ $(t > 0)$, it follows that $\chi^c \in R_\Omega$.

Now let $a \in R_\Omega$ and set $c = V^* a_1 \in \mathsf{k}$ where V is the isometry $\mathsf{k} \to \mathsf{K}_1 \subset \mathcal{F}_1$, $c \mapsto \chi^c_1$.

Claim 2. $\operatorname{ess-supp} a_t \subset \Gamma^{(1)}_t$ $(t > 0)$.

Fix $t > 0$. For $q \in \mathbb{Q} \cap]0,t[$, $a_t = a_q \cdot \Omega_{t-q} + \Omega_q \cdot a_{t-q}$ so, for a.a. σ,

$$a_t(\sigma) = 1_{\Gamma_{[0,t[}}(\sigma)\big[a_q\big(\sigma \cap [0,q[\big)\, \delta_\emptyset\big((\sigma \cap [q,t[) - q\big) + \delta_\emptyset\big(\sigma \cap [0,q[\big)\, a_{t-q}\big((\sigma \cap [q,t[) - q\big)\big]$$

and therefore

$$a_t(\sigma) = 0 \text{ unless either } \sigma \subset [0,q[\text{ or } \sigma \subset [q,t[.$$

Thus, by the countability of $\mathbb{Q}$, there is a null set $\mathcal{N}$ of $\Gamma_{[0,t[}$ such that

$$\forall_{\sigma \in \Gamma_{[0,t[} \setminus \mathcal{N}}\, \forall_{q \in \mathbb{Q} \cap]0,t[} \colon\ a_t(\sigma) = 0 \text{ unless } \sigma \subset [0,q[\text{ or } \sigma \subset [q,t[.$$

For $\sigma = \{s_1 < \cdots < s_n\} \in \Gamma^{(\geqslant 2)}_{[0,t[} \setminus \mathcal{N}$, choosing $q \in \mathbb{Q}$ such that $s_1 < q < s_2$, we have $\sigma \not\subset [0,q[$ and $\sigma \not\subset [q,t[$ so $a_t(\sigma) = 0$. Thus $\operatorname{ess-supp} a_t \subset \Gamma^{(\leqslant 1)}_{[0,t[}$. Since a is a root of $(\mathcal{F}, \Omega)$, $0 = \langle \Omega_t, a_t\rangle = a_t(\emptyset)$, thus $\operatorname{ess-supp} a_t \subset \Gamma^{(1)}_{[0,t[}$.

Claim 3. $a = \chi^c$.

Fix $t > 0$. By the proven Claims 2 and 1, $a_t, \chi^c_t \in \mathsf{K}_t \subset \mathcal{F}_t$ and $a, \chi^c \in R_\Omega$. It follows that, for each $e \in \mathsf{k}$ and $s \in]0,t[$,

$$\begin{aligned}\langle a_t, e_{[0,s[}\rangle &= \langle a_s \cdot \Omega_{t-s} + \Omega_s \cdot a_{t-s}, \chi^e_s \cdot \Omega_{t-s}\rangle \\ &= \langle a_s, \chi^e_s\rangle \\ &= s\langle a_1, \chi^e_1\rangle = s\langle c, e\rangle = \langle c_{[0,t[}, e_{[0,s[}\rangle = \langle \chi^c_t, e_{[0,s[}\rangle.\end{aligned}$$

Therefore, since $a_t, \chi^c_t \in \mathsf{K}_t$ and the set $\{e_{[0,s[} : e \in \mathsf{k}, 0 < s < t\}$ is total in K_t, $a_t = \chi^c_t$. Thus $a = \chi^c$.

The prescription (3.2) therefore defines a bijection $\mathsf{k} \to R_\Omega$. The bijection is manifestly linear and, since $\|\chi^c\|_{R_\Omega} = \|\chi^c_1\| = \|c_{[0,1[}\| = \|c\|_{\mathsf{k}}$ $(c \in \mathsf{k})$, it is isometric too and thus an isometric isomorphism. □

Corollary 3.4. *Let* $(\mathcal{E}, u) = \big(\mathcal{F}^{\mathsf{k}_1} \otimes \mathcal{F}^{\mathsf{k}_2}, \Omega^{\mathsf{k}_1} \otimes \Omega^{\mathsf{k}_2}\big)$ *for separable Hilbert spaces* k_1 *and* k_2. *Then, in the above notation,*

$$R^{\mathcal{E}}_u = \big(R^{\mathcal{F},\mathsf{k}_1}_\Omega \otimes \Omega^{\mathsf{k}_2}\big) \oplus \big(\Omega^{\mathsf{k}_1} \otimes R^{\mathcal{F},\mathsf{k}_2}_\Omega\big).$$

Proof. Set $\mathsf{k} = \mathsf{k}_1 \oplus \mathsf{k}_2$.

Under the natural isomorphism of pointed Arveson systems $(\mathcal{F}^{\mathsf{k}}, \Omega^{\mathsf{k}}) \to (\mathcal{E}, u)$, the unit ε^e of $\mathcal{F}^{\mathsf{k}}$ maps to the unit $\varepsilon^{e_1} \otimes \varepsilon^{e_2}$ of $\mathcal{E}$, for $e = (e_1, e_2) \in \mathsf{k}$. Therefore, for $c = (c_1, c_2) \in \mathsf{k}$, the root χ^c of $(\mathcal{F}^{\mathsf{k}}, \Omega^{\mathsf{k}})$ maps to the root χ^{c_1,c_2} of $(\mathcal{E}, u)$ given by

$$\begin{aligned}\chi^{c_1,c_2}_t &= \lim_{\lambda\to 0} \lambda^{-1}\big(\varepsilon^{\lambda c_1}_t \otimes \varepsilon^{\lambda c_2}_t - \Omega^{\mathsf{k}_1} \otimes \Omega^{\mathsf{k}_2}\big) \\ &= \chi^{c_1}_t \otimes \Omega^{\mathsf{k}_2}_t + \Omega^{\mathsf{k}_1}_t \otimes \chi^{c_2}_t \qquad (t > 0).\end{aligned}$$

In view of the orthogonality relation

$$\chi^{c_1}_t \otimes \Omega^{\mathsf{k}_2}_t \perp \Omega^{\mathsf{k}_1}_t \otimes \chi^{c_2}_t \qquad (c_1 \in \mathsf{k}_1, c_2 \in \mathsf{k}_2, t > 0),$$

the result follows. □

Our goal now is to show that the addits of a pointed Arveson system generate the type I part of the Arveson system. We first show this for type I systems.

Lemma 3.5. *Let* k *be separable Hilbert space. Then the vacuum unit and its roots generate the Fock Arveson system* $\mathcal{F}^{\mathsf{k}}$.

Proof. Since the set of roots of $(\mathcal{F}^{\mathsf{k}}, \Omega^{\mathsf{k}})$ is $\{\chi^c : c \in \mathsf{k}\}$, and $\mathcal{F}^{\mathsf{k}}$ is generated by its units $\{(e^{\lambda t}\varepsilon^c_t)_{t>0} : c \in \mathsf{k}, \lambda \in \mathbb{C}\}$, it suffices to prove that

$$\big(\Omega^{\mathsf{k}}_{2^{-n}t} + \chi^c_{2^{-n}t}\big)^{\cdot\, 2^n} \to \varepsilon^c_t \;\text{ as }\; n \to \infty \qquad (c \in \mathsf{k}, t > 0).$$

Thus fix $c \in \mathsf{k}$ and $t > 0$, and set $x_n := \Omega^{\mathsf{k}}_{2^{-n}t} + \chi^c_{2^{-n}t}$ $(n \in \mathbb{N})$. Since $\|(x_n)^{\cdot\, 2^n}\|^2 = (1 + 2^{-n}t\|c\|^2)^{2^n} \leqslant e^{t\|c\|^2}$ $(n \in \mathbb{N})$, it suffices to prove that

$$\big\langle \varepsilon(g), (x_n)^{\cdot\, 2^n}\big\rangle \to \langle \varepsilon(g), \varepsilon^c_t\rangle \;\text{ as }\; n \to \infty$$

for all right continuous step functions $g \in \mathsf{K}_t$ whose discontinuities lie in the set $\{j2^{-N}t : j, N \in \mathbb{N}\}$. Thus fix such a step function $g = \sum_{i=1}^p d^{i-1}_{[s_{i-1},s_i[}$ in which $s_0 = 0$ and $s_p = t$. Then, for sufficiently large n,

$$s_i = 2^{-n}k_i(n)t \;\text{ for some }\; k_i(n) \in \mathbb{N} \qquad (i = 1, \cdots, p).$$

It therefore follows, by Euler's exponential formula, that

$$
\begin{aligned}
\langle \varepsilon(g), (x_n)^{\cdot 2^n} \rangle &= \prod_{i=1}^{p} \Big\langle \varepsilon(d^{i-1}_{[0,s_i - s_{i-1}[}), (x_n)^{\cdot (k_i^{(n)} - k_{i-1}^{(n)})} \Big\rangle \\
&= \prod_{i=1}^{p} \big(1 + 2^{-n} t \, \langle d^{i-1}, c \rangle\big)^{k_i^{(n)} - k_{i-1}^{(n)}} \\
&= \prod_{i=1}^{p} \Big(1 + \frac{s_i - s_{i-1}}{k_i(n) - k_{i-1}(n)} \langle d^{i-1}, c \rangle\Big)^{k_i^{(n)} - k_{i-1}^{(n)}} \\
&\to \prod_{i=1}^{p} e^{(s_i - s_{i-1}) \langle d^{i-1}, c \rangle} = \langle \varepsilon(g), \varepsilon_t^c \rangle \quad \text{as } n \to \infty,
\end{aligned}
$$

as required. □

Corollary 3.6. *Let k be a separable Hilbert space. Then the product system generated by the Fock inclusion system F^{k} is the Fock Arveson system $\mathcal{F}^{\mathsf{k}}$.*

Proof. In view of the remark which precedes Proposition 3.3, this follows from Lemma 3.5. □

Theorem 3.7. *Let $\mathcal{E}$ be a spatial Arveson system. Let $u \in \mathcal{U}_1^{\mathcal{E}}$ and let $\mathcal{F}$ be the product subsystem of $\mathcal{E}$ generated by u and all its roots. Then the following hold.*

(a) $\mathcal{F} = \mathcal{E}^I$.
(b) $R_u^{\mathcal{E}} = R_u^{\mathcal{E}^I}$.
(c) $\operatorname{ind} \mathcal{E} = \dim R_u^{\mathcal{E}}$.

Proof. Let h and k be the Hilbert spaces $R_u^{\mathcal{E}}$ and $\mathsf{k}(\mathcal{E})$ respectively, and let F be the inclusion subsystem of $\mathcal{E}$ generated by u and all of its roots. Thus $\dim \mathsf{k} = \operatorname{ind} \mathcal{E}$ and $\mathcal{F}$ is the product subsystem of $\mathcal{E}$ generated by F. Recall that, by Theorem 1.4, $\mathcal{F}$ is an Arveson subsystem of $\mathcal{E}$.

(a) We first show that $\mathcal{F}$ is a product subsystem of $\mathcal{E}^I$. By Proposition 3.3, the following prescription defines unitary operators

$$
A_t : F_t^{\mathsf{h}} \to F_t, \quad \lambda \Omega_t^{\mathsf{h}} + \chi_t^a \mapsto \lambda u_t + a_t \qquad (\lambda \in \mathbb{C}, a \in \mathsf{h} = R_u^{\mathcal{E}}, t > 0),
$$

and it is easily seen that $A = (A_t)_{t>0}$ is an isomorphism of inclusion systems. By Corollary 3.6, the product system generated by F^{h} is $\mathcal{F}^{\mathsf{h}}$. By Remark (iv) after Theorem 2.6, A lifts to an isomorphism of product systems $\widehat{A} : \mathcal{F}^{\mathsf{h}} \to \mathcal{F}$. Theorem 1.6, together with the remark following it, imply that $\widehat{A}$ is an isomorphism of Arveson systems, and so $\mathcal{F}$ is of type I. It follows that $\mathcal{F}$ is a product subsystem of $\mathcal{E}^I$.

We next show that $\mathcal{E}^I$ is a product subsystem of $\mathcal{F}$, equivalently that, for any normalised unit v of $\mathcal{E}$, $v_t \in \mathcal{F}_t$ $(t > 0)$. To this end, let $v \in \mathcal{U}_1^{\mathcal{E}}$ and fix an isomorphism of pointed Arveson systems $\psi : (\mathcal{F}^{\mathsf{k}}, \Omega^{\mathsf{k}}) \to (\mathcal{E}^I, u)$. Then $(e^{ixt} v_t)_{t>0} = \psi(\varpi^c)$ for some $c \in \mathsf{k}$ and $x \in \mathbb{R}$. Set $a := \psi(\chi^c) \in R_u^{\mathcal{E}^I}$. Since any root of $(\mathcal{E}^I, u)$ is a root of $(\mathcal{E}, u)$, $a_s \in F_s$ $(s > 0)$ and

$$
\psi_t(\varepsilon_t^c) = \lim_{n \to \infty} \psi_t\big(\Omega_{2^{-n}t}^{\mathsf{k}} + \chi_{2^{-n}t}^c\big)^{\cdot 2^n} = \lim_{n \to \infty} \big(u_{2^{-n}t} + a_{2^{-n}t}\big)^{\cdot 2^n} \in \mathcal{F}_t
$$

so $v_t = e^{-ixt} e^{-t\|c\|^2/2} \, \psi_t(\varepsilon_t^c) \in \mathcal{F}_t$ $(t > 0)$, as required.

Therefore $\mathcal{F} = \mathcal{E}^I$, so (a) holds. (b) follows from (a).

(c) By (a) we have isomorphisms of Arveson systems

$$
\mathcal{F}^{\mathsf{h}} \cong \mathcal{F} = \mathcal{E}^I \cong \mathcal{F}^{\mathsf{k}}.
$$

This implies that $\mathsf{h} \cong \mathsf{k}$, and so $\operatorname{ind} \mathcal{E} = \dim \mathsf{k} = \dim \mathsf{h} = \dim R_u^{\mathcal{E}}$. □

4. Addits of pointed inclusion systems

In this section we extend notions of the previous section to inclusion systems, and show that, as with units, addits of an inclusion system lift to addits of the generated product system.

We call an ordered pair (E, u), consisting of an inclusion system E and a normalised unit u of E, a *pointed inclusion system.*

Definition 4.1. ([8]) Let (E,u) be a pointed inclusion system. An *addit of* (E,u) is a section a of E satisfying the additivity condition

$$a_{s+t} = (\beta^E_{s,t})^*(a_s \otimes u_t + u_s \otimes a_t) \qquad (s,t>0),$$

and the following boundedness condition: there is $k \in \mathbb{R}_+$ such that

$$\|a_t\|^2 \leqslant k(t+t^2) \qquad (t>0).$$

An addit a of (E,u) is a *root* if it satisfies

$$a_t \perp u_t \qquad (t>0).$$

We now establish the additive counterpart to (the second part of) Theorem 2.6, whose notations we continue to adopt.

Proposition 4.2. *Let (E,u) be a pointed inclusion system, and let $\mathcal{E}$ be the product system generated by E, and $\widehat{u}$ the lift of u. Then the following hold.*

(a) *The map $(\imath^E)^* : \big((\imath^E_t)^* : \mathcal{E}_t \to E_t\big)_{t>0}$ restricts to a bijection from the set of addits of $(\mathcal{E},\widehat{u})$ to the set of addits of (E,u), whose inverse is denoted by $a \mapsto \widehat{a}$.*

(b) *If a is a root of (E,u) then $\widehat{a}$ is a root of $(\mathcal{E},\widehat{u})$.*

Proof. Let us drop the superscripts on β^E, $B^{\mathcal{E}}$ and $\imath^E$.

(a) First let b be an addit of $(\mathcal{E},\widehat{u})$. Then $\imath^*(b)$ is an addit of (E,u) since

$$\begin{aligned}\beta^*_{s,t}\big(\imath^*_s b_s \otimes u_t + u_s \otimes \imath^*_t b_t\big) &= \big((\imath_s \otimes \imath_t)\circ \beta_{s,t}\big)^*\big(b_s \otimes \widehat{u}_t + \widehat{u}_s \otimes b_t\big)\\ &= \big(B_{s,t}\circ \imath_{s+t}\big)^*\big(b_s \otimes \widehat{u}_t + \widehat{u}_s \otimes b_t\big) = \imath^*_{s+t} b_{s+t} \qquad (s,t>0).\end{aligned}$$

Let α denote the resulting map from addits of $(\mathcal{E},\widehat{u})$ to addits of (E,u). Suppose that addits b^1 and b^2 of $(\mathcal{E},\widehat{u})$ satisfy $\alpha(b^1)=\alpha(b^2)$. Fix $T>0$. An induction on n confirms that, for any addit b of $(\mathcal{E},\widehat{u})$,

$$B_{T,\mathbf{t}}\, b_T = \sum_{j=1}^n \widehat{u}_{t_1} \otimes \cdots \otimes \widehat{u}_{t_{j-1}} \otimes b_{t_j} \otimes \widehat{u}_{t_{j+1}} \otimes \cdots \otimes \widehat{u}_{t_n}, \quad \text{and}$$

$$B_{T,\mathbf{t}} \circ \imath_{\mathbf{t}} = \imath_{t_1} \otimes \cdots \otimes \imath_{t_n} \qquad (n \in \mathbb{N}, \mathbf{t} \in J_T^{(n)}).$$

Therefore, for any addit b of $(\mathcal{E},\widehat{u})$,

$$\begin{aligned}\imath^*_{\mathbf{t}} b_T &= \big((\imath^*_{t_1} \otimes \cdots \otimes \imath^*_{t_n}) \circ B_{T,\mathbf{t}}\big) b_T\\ &= \sum_{j=1}^n u_{t_1} \otimes \cdots \otimes u_{t_{j-1}} \otimes \imath^*_{t_j} b_{t_j} \otimes u_{t_{j+1}} \otimes \cdots \otimes u_{t_n} \qquad (n \in \mathbb{N}, \mathbf{t} \in J_T^{(n)}).\end{aligned}$$

Now the RHS is the same for $b=b^1$ and $b=b^2$, therefore $\imath^*_{\mathbf{t}} b^1_T = \imath^*_{\mathbf{t}} b^2_T$ ($\mathbf{t} \in J_T$). Since the net $\big(\imath_{\mathbf{t}}\imath^*_{\mathbf{t}}\big)_{\mathbf{t}\in J_T}$ converges strongly to $I^{\mathcal{E}}_T$, it follows that $b^1_T = b^2_T$. Unfixing T we conclude that $b^1=b^2$, and so α is injective.

Since the trivial addits of $(\mathcal{E},\widehat{u})$ are clearly mapped by α onto the trivial addits of (E,u), in order to establish the surjectivity of α it suffices to fix a root a of (E,u) and find a root, $\widehat{a}$ say, of $(\mathcal{E},\widehat{u})$ such that $\imath^*(\widehat{a}) = a$. Accordingly, let a be a root of (E,u), with boundedness constant k and fix $T>0$.

Claim 1. Setting $a_{\mathbf{t}} := \sum_{j=1}^n u_{t_1} \otimes \cdots \otimes u_{t_{j-1}} \otimes a_{t_j} \otimes u_{t_{j+1}} \otimes \cdots \otimes u_{t_n}$ ($n \in \mathbb{N}, \mathbf{t} \in J_T^{(n)}$), the net $\big(\imath_{\mathbf{t}} a_{\mathbf{t}}\big)_{\mathbf{t}\in J_T}$ converges.

First note that the net is bounded since $a_t \perp u_t$ ($t>0$), so

$$\|\imath_{\mathbf{t}} a_{\mathbf{t}}\|^2 = \|a_{\mathbf{t}}\|^2 = \sum_{j=1}^n \|a_{t_j}\|^2 \leqslant k \sum_{j=1}^n (t_j + t_j^2) \leqslant k(T+T^2) \qquad (n \in \mathbb{N}, \mathbf{t} \in J_T^{(n)}).$$

Next note the identity

$$\imath^*_{\mathbf{s}} \imath_{\mathbf{t}} a_{\mathbf{t}} = \beta^*_{\mathbf{s},\mathbf{t}} a_{\mathbf{t}} = a_{\mathbf{s}} \qquad (\mathbf{s} \leqslant \mathbf{t} \text{ in } J_T).$$

Fix $x \in \mathcal{E}_T$ and $\varepsilon > 0$. Choose $\mathbf{r} \in J_T$ such that $\|x - \imath_{\mathbf{r}}\imath^*_{\mathbf{r}} x\| < \varepsilon$. Then, for $\mathbf{t} \geqslant \mathbf{r}$,

$$|\langle \imath_{\mathbf{t}} a_{\mathbf{t}} - \imath_{\mathbf{r}} a_{\mathbf{r}}, x\rangle|^2 = |\langle \imath_{\mathbf{t}} a_{\mathbf{t}}, (I - \imath_{\mathbf{r}}\imath^*_{\mathbf{r}})x\rangle|^2 \leqslant k(T+T^2)\varepsilon^2.$$

It follows that $\big(\imath_{\mathbf{t}} a_{\mathbf{t}}\big)_{\mathbf{t}\in J_T}$ is weakly Cauchy. Set $\widehat{a}_T :=$ weak-$\lim_{\mathbf{t}\in J_T} \imath_{\mathbf{t}} a_{\mathbf{t}}$. Now

$$\imath_{\mathbf{s}}\imath_{\mathbf{s}}^*\imath_{\mathbf{t}} a_{\mathbf{t}} = \imath_{\mathbf{s}} a_{\mathbf{s}} \qquad (\mathbf{s} \leqslant \mathbf{t} \text{ in } J_T),$$

therefore $\imath_{\mathbf{s}}\imath_{\mathbf{s}}^*\widehat{a}_T = \imath_{\mathbf{s}} a_{\mathbf{s}}$ ($\mathbf{s} \in J_T$). It follows that $\imath_{\mathbf{s}} a_{\mathbf{s}} \to \widehat{a}_T$ (in norm), as claimed.

Claim 2. Setting $\widehat{a} := (\widehat{a}_T)_{T>0}$, $\widehat{a}$ is an addit of $(\mathcal{E}, \widehat{u})$ such that $\imath^*(\widehat{a}) = a$.

Let $S, T > 0$. Write $u_{\mathbf{t}}$ for $\sum_{j=1}^n u_{t_1} \otimes \cdots \otimes u_{t_n}$ ($n \in \mathbb{N}, \mathbf{t} \in J_T^{(n)}$). Then, for $\mathbf{s} \in J_S$ and $\mathbf{t} \in J_T$,

$$(\imath_{\mathbf{s}} \otimes \imath_{\mathbf{t}})(a_{\mathbf{s}} \otimes u_{\mathbf{t}} + u_{\mathbf{s}} \otimes a_{\mathbf{t}}) = (\imath_{\mathbf{s}} \otimes \imath_{\mathbf{t}}) a_{\mathbf{s}\smile\mathbf{t}} = B_{S,T}\, \imath_{\mathbf{s}\smile\mathbf{t}} a_{\mathbf{s}\smile\mathbf{t}}.$$

Taking limits and using the fact that the net $(\imath_{\mathbf{r}} u_{\mathbf{r}})_{\mathbf{r}\in J_R}$ converges to $\widehat{u}_R$ ($R > 0$), we see that

$$\widehat{a}_S \otimes \widehat{u}_T + \widehat{u}_S \otimes \widehat{a}_T = B_{S,T} \widehat{a}_{S+T}.$$

Thus $\widehat{a}$ is an addit of $\widehat{u}$. Now, since

$$\imath_T^* \imath_{\mathbf{t}} a_{\mathbf{t}} = \beta_{T,\mathbf{t}}^* a_{\mathbf{t}} = a_T \qquad (\mathbf{t} \in J_T),$$

it follows that $\imath_T^* \widehat{a}_T = a_T$, and Claim 2 is established.

Therefore α is also surjective and so (a) follows.

(b) Let a be a root of (E, u). Then

$$\langle \imath_{\mathbf{t}} a_{\mathbf{t}}, \imath_{\mathbf{t}} u_{\mathbf{t}} \rangle = \langle a_{\mathbf{t}}, u_{\mathbf{t}} \rangle = 0 \qquad (T > 0, \mathbf{t} \in J_T).$$

Taking limits we see that $\langle \widehat{a}_T, \widehat{u}_T \rangle = 0$ ($T > 0$), so $\widehat{a}$ is a root of $\widehat{u}$. The proof is now complete. □

5. Amalgamation

The amalgamation of Arveson systems, via a contractive morphism, was introduced in [8]. This generalised a construction of Skeide which corresponds to the case where the morphism is given by Dirac dyads from normalised units ([28]). A formula for its index, in terms of that of the constituent systems, was given in [19]. In this section we first show how the root space of an amalgamated product of pointed Arveson systems (defined to be that given by the corresponding morphism of Dirac dyads) may be expressed in terms of the root spaces of its constituent systems, when the morphism is partially isometric. The amalgamated product of pointed Arveson systems may be realised as a product subsystem of the tensor product Arveson system ([19], Theorem 2.7); we give an explicit formula for the subsystem which shows, in particular, that it is independent of the fixed normalised units and so depends only on the underlying Arveson systems. The latter fact may alternatively be proved using random sets ([15]), or directly ([5]); see also [6]. The section ends with a new formula for the space of roots of the tensor product of two pointed Arveson systems.

To begin we quote a basic result.

Theorem 5.1 ([8], Section 3; [19], Theorem 2.7)**.** *Let $C : \mathcal{E}^2 \to \mathcal{E}^1$ be a contractive morphism between Arveson systems. Then there is a triple $(\mathcal{E}, J^1, J^2)$, unique up to isomorphism, consisting of a product system $\mathcal{E}$ and isometric morphisms of product systems $J^i : \mathcal{E}^i \to \mathcal{E}$ $(i = 1, 2)$ such that*

(i) *$(J_t^1)^* J_t^2 = C_t$ $(t > 0)$, and*
(ii) *$\mathcal{E} = J^1(\mathcal{E}^1) \bigvee J^2(\mathcal{E}^2)$.*

Notation: $\mathcal{E}^1 \otimes_C \mathcal{E}^2$. *Terminology*: *the* amalgamated product of $\mathcal{E}^1$ and $\mathcal{E}^2$ via C.

Conversely, let $\mathcal{E}^1$ and $\mathcal{E}^2$ be product subsystems of an Arveson system $\mathcal{F}$. Then $\mathcal{E}^1 \bigvee \mathcal{E}^2 = \mathcal{E}^1 \otimes_C \mathcal{E}^2$ where $C = \big((J_t^1)^* J_t^2\big)_{t>0}$ for the inclusion morphisms $J^i : \mathcal{E}^i \to \mathcal{F}$ $(i = 1, 2)$.

Remarks. The construction of $\mathcal{E}^1 \otimes_C \mathcal{E}^2$ is via an inclusion system; in case $\mathcal{E}^1$ and $\mathcal{E}^2$ are product subsystems of an Arveson system $\mathcal{F}$, $\mathcal{E}^1 \otimes_C \mathcal{E}^2$ is the product subsystem of $\mathcal{F}$ generated by the inclusion subsystem $(\mathcal{E}_t^1 \vee \mathcal{E}_t^2)_{t>0}$.

When C takes the form $\big(|u_t^1\rangle\langle u_t^2|\big)_{t>0}$ for normalised units u^i of $\mathcal{E}^i$ $(i = 1, 2)$, the case treated in [28], $\mathcal{E}^1 \otimes_C \mathcal{E}^2$ is denoted $\mathcal{E}^1 \otimes_{u^1,u^2} \mathcal{E}^2$.

The following proposition is a straight-forward consequence of Theorem 5.1.

Proposition 5.2. *Let $(\mathcal{E}, u)$ and $(\mathcal{F}, v)$ be pointed Arveson systems. Then*

$$\mathcal{E} \otimes_{u,v} \mathcal{F} \cong (\mathcal{E} \otimes v) \bigvee (u \otimes \mathcal{F}).$$

Notation. For a pointed Arveson system $(\mathcal{E}, u)$, we set

$$\mathcal{R}_u^{\mathcal{E}} := \{a_1 : a \in R_u^{\mathcal{E}}\}.$$

Thus $\mathcal{R}_u^{\mathcal{E}}$ is a closed subspace of the Hilbert space $\mathcal{E}_1$; by definition, $R_u^{\mathcal{E}} \cong \mathcal{R}_u^{\mathcal{E}}$.

Theorem 5.3. *Let $\mathcal{E} = \mathcal{E}^1 \otimes_C \mathcal{E}^2$ for spatial Arveson systems $\mathcal{E}^1$ and $\mathcal{E}^2$ and a partially isometric morphism $C : \mathcal{E}^2 \to \mathcal{E}^1$, and let $u^2 \in \mathcal{U}_1^{\mathcal{E}^2}$. Suppose that $\mathcal{E}$ is an Arveson system and that u^2 lies in the initial space of C: $C_t^* C_t u_t^2 = u_t^2$ $(t > 0)$. Then $u^1 := Cu^2$ is a unit which is identified with u^2 in $\mathcal{E}$ and, denoting the common unit in $\mathcal{E}$ by u,*

$$\mathcal{R}_u^{\mathcal{E}} = \mathcal{R}_{u^1}^{\mathcal{E}^1} \oplus_{C_1} \mathcal{R}_{u^2}^{\mathcal{E}^2}.$$

Proof. It follows from Theorem 5.1 that we may identify $\mathcal{E}^1$ and $\mathcal{E}^2$ with subsystems of $\mathcal{E}$, and C with $((J_t^1)^* J_t^2)_{t>0}$ where J^1 and J^2 are the corresponding inclusion morphisms. By Proposition 2.10 of [19], the projections $P_{\mathcal{E}_t^1}$ and $P_{\mathcal{E}_t^2}$ commute, so $P_{\mathcal{E}_t^1 \cap \mathcal{E}_t^2} = P_{\mathcal{E}_t^1} P_{\mathcal{E}_t^2}$ $(t > 0)$. Thus $\mathcal{E}^1 \cap \mathcal{E}^2 := \big(\mathcal{E}_t^1 \cap \mathcal{E}_t^2\big)_{t>0}$ is a product subsystem of $\mathcal{E}$. Under this identification u^2 and u^1 are identified, and $\mathcal{R}_{u^1}^{\mathcal{E}^1} \oplus_{C_1} \mathcal{R}_{u^2}^{\mathcal{E}^2}$ coincides with $\mathcal{R}_{u^1}^{\mathcal{E}^1} \vee \mathcal{R}_{u^2}^{\mathcal{E}^2}$ in $\mathcal{R}_u^{\mathcal{E}}$. The theorem is therefore proved once it is shown that $\mathcal{R}_{u^1}^{\mathcal{E}^1} \vee \mathcal{R}_{u^2}^{\mathcal{E}^2} = \mathcal{R}_u^{\mathcal{E}}$.

Let $a \in R_u^{\mathcal{E}}$ and set $c := \big(J_t^1 b_t^1 + J_t^2 b_t^2 - J_t b_t\big)_{t>0}$ where $b_t^1 = (J_t^1)^* a_t$, $b_t^2 = (J_t^2)^* a_t$ and $b_t = (J_t)^* a_t$ $(t > 0)$, and J denotes the inclusion morphism $\mathcal{E}^1 \cap \mathcal{E}^2 \to \mathcal{E}$. Thus

$$c_t = \big(P_{\mathcal{E}_t^1} + P_{\mathcal{E}_t^2} - P_{\mathcal{E}_t^1 \cap \mathcal{E}_t^2}\big) a_t = P_{\mathcal{E}_t^1 \vee \mathcal{E}_t^2}\, a_t \qquad (t > 0).$$

Claim. $c \in R_u^{\mathcal{E}}$. First note that

$$\begin{aligned} b_s^1 \otimes u_t^1 + u_s^1 \otimes b_t^1 &= (J_s^1 \otimes J_t^1)^* (a_s \otimes u_t + u_s \otimes a_t) \\ &= (J_s^1 \otimes J_t^1)^* B_{s,t}^{\mathcal{E}} a_{s+t} = B_{s,t}^{\mathcal{E}^1} (J_{s+t}^1)^* a_{s+t} = B_{s,t}^{\mathcal{E}^1} b_{s+t}^1 \qquad (s, t > 0) \end{aligned}$$

so $b^1 \in R_{u^1}^{\mathcal{E}^1}$. Similarly, $b^2 \in R_{u^2}^{\mathcal{E}^2}$ and $b \in R_u^{\mathcal{E}^1 \cap \mathcal{E}^2}$. Thus $J^1 b^1, J^2 b^2, Jb \in R_u^{\mathcal{E}}$, so $c \in R_u^{\mathcal{E}}$.

Now $E := \big(\mathcal{E}_t^1 \vee \mathcal{E}_t^2\big)_{t>0}$ is an inclusion subsystem which generates the Arveson system $\mathcal{E}$, and

$$0 = P_{\mathcal{E}_t^1 \vee \mathcal{E}_t^2}(a_t - c_t) = P_t^E (a_t - c_t) = \imath_t^E (\imath_t^E)^* (a_t - c_t) \qquad (t > 0),$$

so $(\imath^E)^*(a - c) = 0$. Since $(a - c) \in R_u^{\mathcal{E}}$, it follows from Proposition 4.2 that $a - c = 0$. Thus

$$a_1 = c_1 = J_1^1 b_1^1 + J_1^2 b_1^2 - J_1 b_1 \in \mathcal{R}_u^{\mathcal{E}^1} + \mathcal{R}_u^{\mathcal{E}^2} \subset \mathcal{R}_u^{\mathcal{E}^1} \vee \mathcal{R}_u^{\mathcal{E}^2}.$$

The result follows. □

Corollary 5.4. *Let $(\mathcal{E}^1, u^1)$ and $(\mathcal{E}^2, u^2)$ be pointed Arveson systems. Then, identifying $\mathcal{E} = \mathcal{E}^1 \otimes_{u^1, u^2} \mathcal{E}^2$ with $(\mathcal{E}^1 \otimes u^2) \bigvee (u^1 \otimes \mathcal{E}^2)$, and letting u denote u^2 identified with u^1,*

$$\mathcal{R}_u^{\mathcal{E}} = \mathcal{R}_{u^1}^{\mathcal{E}^1} \oplus \mathcal{R}_{u^2}^{\mathcal{E}^2}.$$

Proof. For $a^1 \in R_{u^1}^{\mathcal{E}^1}$ and $a^2 \in R_{u^2}^{\mathcal{E}^2}$,

$$\langle a_1^1, a_1^2 \rangle_{C_1} = \langle a_1^1, C_1 a_1^2 \rangle = \langle a_1^1, u_1^1 \rangle \, \langle u_1^2, a_1^2 \rangle = 0.$$

The result follows. □

Remark. Root spaces need not behave well under amalgamation over contractive morphisms that are not partially isometric.

Example 5.5. Fix $\lambda \neq 0$. Set $\mathcal{E} = \mathcal{E}^1 \otimes_C \mathcal{E}^2$ where $\mathcal{E}^1 = \mathcal{E}^2 = \mathcal{F}^{\mathsf{k}}$ for the trivial Hilbert space $\mathsf{k} = \{0\}$, and $C = \big(|u_t^1\rangle\langle u_t^2|\big)_{t>0}$ for the units $u^1 := \Omega^{\{0\}}$ and $u^2 := \big(e^{-t\lambda^2/2} \Omega_t^{\{0\}}\big)_{t>0}$. Theorem 2.7 of [19] implies that $\mathcal{E}$ is isomorphic to the product system generated by the normalised units $\Omega^{\mathbb{C}}$ and ϖ^{λ} of $\mathcal{F}^{\mathbb{C}}$, in other words $\mathcal{E}$ is isomorphic to the Fock Arveson system $\mathcal{F}^{\mathbb{C}}$ itself. Thus

$$\mathcal{R}_{u^1}^{\mathcal{E}^1} = \{0\} = \mathcal{R}_{u^2}^{\mathcal{E}^2}, \ \text{ but, for any unit } u \text{ of } \mathcal{E}, \ \mathcal{R}_u^{\mathcal{E}} \cong \mathbb{C}.$$

For an inclusion subsystem F of an Arveson system $\mathcal{E}$, consider the following family of orthogonal projections in $B(\mathcal{E}_1)$:

$$P^F_{r,t} := \begin{cases} P_{F_t} \otimes I^{\mathcal{E}}_{1-t} & \text{if } 0 = r < t < 1 \\ P_{F_1} & \text{if } 0 = r \text{ and } t = 1 \\ I^{\mathcal{E}}_r \otimes P_{F_{t-r}} \otimes I^{\mathcal{E}}_{1-t} & \text{if } 0 < r < t < 1 \\ I^{\mathcal{E}}_r \otimes P_{F_{1-r}} & \text{if } 0 < r < t = 1. \end{cases}, \tag{5.1}$$

It follows from Proposition 3.18 of [14] that, for a product subsystem $\mathcal{F}$ of $\mathcal{E}$,

$$P^{\mathcal{F}}_{s,t} \to I^{\mathcal{E}}_1 \text{ as } (t-s) \to 0.$$

Theorem 5.6. *Let $\mathcal{E}$ and $\mathcal{F}$ be spatial Arveson systems. Then, for any normalised units u and v of $\mathcal{E}$ and $\mathcal{F}$ respectively,*

$$\mathcal{E} \otimes_{u,v} \mathcal{F} \cong (\mathcal{E} \otimes \mathcal{F}^I) \bigvee (\mathcal{E}^I \otimes \mathcal{F}).$$

Proof. Let $u \in \mathcal{U}^{\mathcal{E}}_1$ and $v \in \mathcal{U}^{\mathcal{F}}_1$. Set $\mathcal{G} := (\mathcal{E} \otimes v) \bigvee (u \otimes \mathcal{F})$ and, for $n \in \mathbb{N}$ and $i \in \{1, \cdots, n\}$, set $P^n_i := P^{\mathbb{C}u}_{s,t}$ where $\mathbb{C}u$ denotes the product subsystem of $\mathcal{E}$ generated by u, and $(s,t) = ((i-1)/n, i/n)$. By Proposition 5.2, $\mathcal{G} \cong \mathcal{E} \otimes_{u,v} \mathcal{F}$, and so, by symmetry, it suffices to show that $\mathcal{E} \otimes \mathcal{F}^I$ is a product subsystem of $\mathcal{G}$. By Theorem 3.7 it suffices to show that $z \otimes a_t \in \mathcal{G}_t$ for $t > 0$, $z \in \mathcal{E}_t$ and $a \in R^{\mathcal{F}}_v$. The argument we give, for the case $t = 1$ easily adjusts to deal with general $t > 0$. Thus let $z \in \mathcal{E}_1$ and $a \in R^{\mathcal{F}}_v$ with $\|a\| = 1$.

Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $\|z - P^{\mathbb{C}u}_{s,t} z\| \leqslant \varepsilon$ for $(t-s) \leqslant 1/n$ and take the root decomposition

$$a_1 = \sum_{i=1}^n x^i \quad \text{where} \quad x^i = (v_{1/n})^{\cdot (i-1)} \cdot a_{1/n} \cdot (v_{1/n})^{\cdot (n-i)} \qquad (i = 1, \cdots, n).$$

Thus $\|x^i\| = \|a_{1/n}\| = 1/\sqrt{n}$ for each i and, since $x^i \perp x^j$ for $i \neq j$,

$$\left\| z \otimes a_1 - \sum\nolimits_{i=1}^n P^n_i z \otimes x^i \right\|^2 = \left\| \sum\nolimits_{i=1}^n (z - P^n_i z) \otimes x^i \right\|^2 = \frac{1}{n} \sum_{i=1}^n \|z - P^n_i z\|^2 \leqslant \varepsilon^2.$$

We must therefore show that $P^n_i z \otimes x^i \in \mathcal{G}_1$ $(n \in \mathbb{N}, i = 1, \cdots, n)$. Accordingly, fix $n \in \mathbb{N}$ and $i \in \{1, \cdots, n\}$. Note that

$$P^n_i z \in \overline{\text{Lin}}\{c^1 \cdot \cdots \cdot c^{i-1} \cdot u_{1/n} \cdot c^{i+1} \cdot \cdots \cdot c^n : c^1, \cdots, c^n \in \mathcal{E}_{1/n}\}$$

and, for $c^1, \cdots, c^n \in \mathcal{E}_{1/n}$,

$$\left(c^1 \cdot \cdots \cdot c^{i-1} \cdot u_{1/n} \cdot c^{i+1} \cdot \cdots \cdot c^n\right) \otimes x^i =$$
$$(c^1 \otimes v_{1/n}) \cdot \cdots \cdot (c^{i-1} \otimes v_{1/n}) \cdot (u_{1/n} \otimes a_{1/n}) \cdot (c^{i+1} \otimes v_{1/n}) \cdot \cdots \cdot (c^n \otimes v_{1/n}),$$

whilst

$$c^j \otimes v_{1/n} \in \mathcal{E}_{1/n} \otimes v_{1/n} \subset \mathcal{G}_{1/n} \ (j \neq i) \quad \text{and} \quad u_{1/n} \otimes a_{1/n} \in u_{1/n} \otimes \mathcal{F}_{1/n} \subset \mathcal{G}_{1/n}.$$

It follows that $P^n_i z \otimes x^i \in (\mathcal{G}_{1/n})^{\cdot n} \subset \mathcal{G}_1$, as required. □

Remark. This result reaffirms justification for referring to the above (spatial) Arveson system as the *spatial product* of the spatial Arveson systems $\mathcal{E}$ and $\mathcal{F}$.

Corollary 5.7. *Let $\mathcal{E}$ and $\mathcal{F}$ be spatial Arveson systems. Then, for normalised units u and v of $\mathcal{E}$ and $\mathcal{F}$ respectively,*

$$(\mathcal{E}^I \otimes v) \bigvee (u \otimes \mathcal{F}^I) = \mathcal{E}^I \otimes \mathcal{F}^I = (\mathcal{E} \otimes \mathcal{F})^I.$$

Proof. The first identity follows from Proposition 5.2 and Theorem 5.6. The second is well-known; it is a consequence of the following identity (see [3], Corollary 3.7.3):

$$\mathcal{U}^{\mathcal{E} \otimes \mathcal{F}} = \{u \otimes v : u \in \mathcal{U}^{\mathcal{E}}, v \in \mathcal{U}^{\mathcal{F}}\}. \tag{5.2}$$

□

Our next result is the counterpart for roots of the identity (5.2) for units. It generalises Corollary 3.4.

Theorem 5.8. *Let $(\mathcal{E},u)$ and $(\mathcal{F},v)$ be pointed Arveson systems. Then*

$$R^{\mathcal{E}\otimes\mathcal{F}}_{u\otimes v} = (R^{\mathcal{E}}_u \otimes v) \oplus (u \otimes R^{\mathcal{F}}_v).$$

Proof. First note that, by Theorem 3.7 and the identity $(\mathcal{E}\otimes\mathcal{F})^I = \mathcal{E}^I \otimes \mathcal{F}^I$, we may suppose without loss that $\mathcal{E}$ and $\mathcal{F}$ are type I Arveson systems. Writing $(\mathcal{E}^1,u^1)$ and $(\mathcal{E}^2,u^2)$ for $(\mathcal{E},u)$ and $(\mathcal{F},v)$ respectively, and setting $\mathsf{k}^i := \mathsf{k}(\mathcal{E}^i)$ $(i=1,2)$, there are isomorphisms of pointed Arveson systems $\phi^i : (\mathcal{F}^{\mathsf{k}_i},\Omega^{\mathsf{k}_i}) \to (\mathcal{E}^i,u^i)$ $(i=1,2)$. Since the isomorphism $\phi^1\otimes\phi^2 : \big(\mathcal{F}^{\mathsf{k}_1}\otimes\mathcal{F}^{\mathsf{k}_2},\Omega^{\mathsf{k}_1}\otimes\Omega^{\mathsf{k}_2}\big) \to \big(\mathcal{E}^1\otimes\mathcal{E}^2, u^1\otimes u^2\big)$ restricts to a bijection of roots, and maps

$$R^{\mathcal{F},\mathsf{k}_1}_{\Omega}\otimes\Omega^{\mathsf{k}_2} \text{ to } R^{\mathcal{E}^1}_{u^1}\otimes u^2 \quad\text{and}\quad \Omega^{\mathsf{k}_1}\otimes R^{\mathcal{F},\mathsf{k}_2}_{\Omega} \text{ to } u^1\otimes R^{\mathcal{E}^1}_{u^2},$$

the result follows from Corollary 3.4. □

6. Cluster construction

In the first half of this section we develop a *cluster construction* for product subsystems of an Arveson system, and show how the construction leads to a new description of the type I part of a spatial Arveson system. In the second half we relate our construction to the Cantor–Bendixson derivative which sends a closed subset of the unit interval to its 'cluster', namely the collection of its accumulation points, via the connection to random sets elaborated in [14].

Notation. For an inclusion subsystem F of an Arveson system $\mathcal{E}$, and $t>0$, set

$$F^{\ominus\perp}_t := \mathcal{E}_t \ominus F^{\ominus}_t \quad\text{where}\quad F^{\ominus}_t := \bigvee_{0<r<t} (\mathcal{E}_r\ominus F_r)\otimes(\mathcal{E}_{t-r}\ominus F_{t-r}).$$

Proposition 6.1. *Let F be an inclusion subsystem of an Arveson system $\mathcal{E}$. Then $F^{\ominus\perp} := \big(F^{\ominus\perp}_t\big)_{t>0}$ is an inclusion subsystem of $\mathcal{E}$ containing F.*

The proof of this proposition is no easier than that of its generalisation, Proposition 6.9, which is given there (and does not depend on any of the intervening theory).

Definition 6.2. Let $\mathcal{F}$ be a product subsystem of an Arveson system $\mathcal{E}$. The *cluster of $\mathcal{F}$ in $\mathcal{E}$* is the product system generated by the inclusion system $\mathcal{F}^{\ominus\perp}$. We denote it $\mathcal{F}^{\breve{}}$.

Lemma 6.3. *Let $\mathcal{F}$ be a product subsystem of an Arveson system $\mathcal{E}$, and let $s,t>0$. Then the following hold.*

(a) $\mathcal{F}^{\ominus\perp}_s\otimes\mathcal{F}_t \subset \mathcal{F}^{\ominus\perp}_{s+t}$ *and* $\mathcal{F}_s\otimes\mathcal{F}^{\ominus\perp}_t \subset \mathcal{F}^{\ominus\perp}_{s+t}$.

(b) $(\mathcal{F}^{\ominus\perp}_s\ominus\mathcal{F}_s)\otimes\mathcal{F}_t \subset \mathcal{F}^{\ominus\perp}_{s+t}\ominus\mathcal{F}_{s+t}$ *and* $\mathcal{F}_s\otimes(\mathcal{F}^{\ominus\perp}_t\ominus\mathcal{F}_t) \subset \mathcal{F}^{\ominus\perp}_{s+t}\ominus\mathcal{F}_{s+t}$.

Proof. Let $s,t>0$. (a) Let $r>0$ satisfy $0<r<s+t$.

If $r<s$, then

$$\begin{aligned}\mathcal{F}^{\perp}_r\otimes\mathcal{F}^{\perp}_{s+t-r} &= \mathcal{F}^{\perp}_r\otimes(\mathcal{F}_{s-r}\otimes\mathcal{F}_t)^{\perp}\\ &= \mathcal{F}^{\perp}_r\otimes\big(\mathcal{F}^{\perp}_{s-r}\otimes\mathcal{F}_t \oplus \mathcal{E}_{s-r}\otimes\mathcal{F}^{\perp}_t\big) \subset \mathcal{F}^{\ominus}_s\otimes\mathcal{F}_t \oplus \mathcal{E}_s\otimes\mathcal{F}^{\perp}_t.\end{aligned}$$

If $r=s$, then

$$\mathcal{F}^{\perp}_r\otimes\mathcal{F}^{\perp}_{s+t-r} = \mathcal{F}^{\perp}_s\otimes\mathcal{F}^{\perp}_t \subset \mathcal{E}_s\otimes\mathcal{F}^{\perp}_t.$$

If $r>s$, then

$$\mathcal{F}^{\perp}_r\otimes\mathcal{F}^{\perp}_{s+t-r} \subset \mathcal{E}_s\otimes\mathcal{E}_{r-s}\otimes\mathcal{F}^{\perp}_{s+t-r} \subset \mathcal{E}_s\otimes(\mathcal{F}_{r-s}\otimes\mathcal{F}_{s+t-r})^{\perp} = \mathcal{E}_s\otimes\mathcal{F}^{\perp}_t.$$

Therefore

$$\mathcal{F}^{\ominus}_{s+t} \subset \mathcal{F}^{\ominus}_s\otimes\mathcal{F}_t\oplus\mathcal{E}_s\otimes\mathcal{F}^{\perp}_t = (\mathcal{F}^{\ominus\perp}_s\otimes\mathcal{F}_t)^{\perp}.$$

The first inclusion follows. The second now follows by symmetry.

(b) Since $\mathcal{F}$ is a product subsystem of $\mathcal{E}$, the first inclusion in (b) follows from the first inclusion in (a):

$$(\mathcal{F}^{\ominus\perp}_s\ominus\mathcal{F}_s)\otimes\mathcal{F}_t = \mathcal{F}^{\ominus\perp}_s\otimes\mathcal{F}_t \ominus \mathcal{F}_s\otimes\mathcal{F}_t \subset \mathcal{F}^{\ominus\perp}_{s+t}\ominus\mathcal{F}_{s+t}.$$

The second inclusion in (b) follows similarly. □

Corollary 6.4. *Let $(\mathcal{E},u)$ be a pointed Arveson system, and set $\mathcal{F}=\mathbb{C}u$. Then, for $s,t>0$,*

$$\mathcal{F}^{\ominus\perp}_s\otimes u_t \subset \mathcal{F}^{\ominus\perp}_{s+t} \quad\text{and}\quad (\mathcal{F}^{\ominus\perp}_s\ominus\mathcal{F}_s)\otimes u_t \subset \mathcal{F}^{\ominus\perp}_{s+t}\ominus\mathcal{F}_{s+t};$$

$$u_s\otimes\mathcal{F}^{\ominus\perp}_t \subset \mathcal{F}^{\ominus\perp}_{s+t} \quad\text{and}\quad u_s\otimes(\mathcal{F}^{\ominus\perp}_t\ominus\mathcal{F}_t) \subset \mathcal{F}^{\ominus\perp}_{s+t}\ominus\mathcal{F}_{s+t}.$$

Notation. For a pointed Arveson system $(\mathcal{E}, u)$, set

$$X_t^{\mathcal{E},u} := (\mathbb{C}u_t)^{\ominus\perp} \ominus \mathbb{C}u_t \qquad (t > 0),$$

and define isometries

$$\jmath_{s,t}^{\mathcal{E},u} : X_s^{\mathcal{E},u} \to X_t^{\mathcal{E},u}, \quad x \mapsto x\cdot u_{t-s} \qquad (0 < s < t).$$

Then $\big((X_t^{\mathcal{E},u})_{t>0}, (\jmath_{r,s}^{\mathcal{E},u})_{0<r<s}\big)$ is easily seen to form an inductive system of Hilbert spaces. Let $\big(X^{\mathcal{E},u}, (\jmath_t^{\mathcal{E},u} : X_t^{\mathcal{E},u} \to X^{\mathcal{E},u})_{t>0}\big)$ denote its inductive limit, and write $x\cdot u_\infty$ for $\jmath_t^{\mathcal{E},u}(x)$ $(t > 0$, $x \in X_t^{\mathcal{E},u})$. Thus

$$(x\cdot u_r)\cdot u_\infty = x\cdot u_\infty \in X^{\mathcal{E},u} \qquad (r, t > 0, x \in X_t^{\mathcal{E},u}).$$

Finally, define isometries $(S_t^{\mathcal{E},u})_{t>0}$ on $X^{\mathcal{E},u}$ by the requirement

$$S_t^{\mathcal{E},u}(z\cdot u_\infty) = u_t\cdot z\cdot u_\infty \qquad \big(z \in \bigcup_{s>0} X_s^{\mathcal{E},u}\big),$$

and set $S_0^{\mathcal{E},u} = I_{X^{\mathcal{E},u}}$.

As usual, when it is expeditious to do so we identify $x\cdot y$ and $x \otimes y = B_{s,t}^{\mathcal{E}}(x\cdot y)$, for $x \in \mathcal{E}_s$, $y \in \mathcal{E}_t$ and $s, t > 0$.

Lemma 6.5. *Let $(\mathcal{E}, u)$ be a pointed Arveson system. Then*

$$X_{s+t}^{\mathcal{E},u} \otimes u_\infty = X_s^{\mathcal{E},u} \otimes u_\infty + S_s^{\mathcal{E},u}(X_t^{\mathcal{E},u} \otimes u_\infty), \text{ and}$$
$$X^{\mathcal{E},u} = X_s^{\mathcal{E},u} \otimes u_\infty + S_s^{\mathcal{E},u} X^{\mathcal{E},u}, \qquad (s, t > 0).$$

Proof. We drop the superscripts. Let $s, t > 0$ and set $\mathcal{F} = \mathbb{C}u$. Then, by Proposition 6.1,

$$X_{s+t} = \mathcal{F}_{s+t}^{\ominus\perp} \ominus \mathbb{C}u_{s+t} \subset (\mathcal{F}_s^{\ominus\perp} \otimes \mathcal{F}_t^{\ominus\perp}) \ominus \mathbb{C}(u_s \otimes u_t) = X_s \otimes u_t \,\oplus\, u_s \otimes X_t \,\oplus\, X_s \otimes X_t,$$

but

$$X_s \otimes X_t \subset \{u_s\}^\perp \otimes \{u_t\}^\perp \subset \mathcal{F}_{s+t}^{\ominus} \subset X_{s+t}^\perp,$$

so $X_{s+t} \subset X_s \otimes u_t \oplus u_s \otimes X_t$. The reverse inclusion also holds since

$$\begin{aligned} X_s \otimes u_t \,\oplus\, u_s \otimes X_t &= (\mathcal{F}_s^{\ominus\perp} \ominus \mathbb{C}u_s) \otimes u_t \,\oplus\, u_s \otimes (\mathcal{F}_t^{\ominus\perp} \ominus \mathbb{C}u_t) \\ &= (\mathcal{F}_s^{\ominus\perp} \otimes u_t \,\oplus\, u_s \otimes \mathcal{F}_t^{\ominus\perp}) \ominus \mathbb{C}u_{s+t} \subset \mathcal{F}_{s+t}^{\ominus\perp} \ominus \mathbb{C}u_{s+t} = X_{s+t}. \end{aligned}$$

The first identity follows. The second follows from the first. □

Lemma 6.6. *Let $(\mathcal{E}, u)$ be a pointed Arveson system. Then $S^{\mathcal{E},u} := (S_t^{\mathcal{E},u})_{t\geqslant 0}$ is a strongly continuous one-parameter semigroup of isometries. Moreover it is purely isometric.*

Proof. Clearly $S^{\mathcal{E},u}$ is a one-parameter semigroup of isometries. Let $x \in X_p^{\mathcal{E},u}$ and $y \in X_q^{\mathcal{E},u}$ where $p, q > 0$. Fix $T > 0$ such that $T > \max\{p, q + 1\}$. Then, for $0 \leqslant t \leqslant 1$,

$$\langle x \otimes u_\infty, u_t \otimes y \otimes u_\infty\rangle = \langle x \otimes u_{T-p}, u_t \otimes y \otimes u_{T-q-t}\rangle = \langle x \otimes u_{T-p}, U_t^{\mathcal{E},T}(y \otimes u_{T-q})\rangle$$

where $U^{\mathcal{E},T} = (U_t^{\mathcal{E},T})_{t\in\mathbb{R}}$ is the unitary flip group on $\mathcal{E}_T$. Weak continuity of the semigroup $S^{\mathcal{E},u}$ therefore follows from the strong continuity of $U^{\mathcal{E},T}$. Since weak continuity implies strong continuity for one-parameter semigroups on Banach spaces, the first part follows.

For the last part, let $s, t > 0$. Then

$$u_t \otimes z \otimes u_\infty \perp x \otimes u_s \otimes u_\infty = x \otimes u_\infty \qquad (z \in X_s^{\mathcal{E},u}, x \in X_t^{\mathcal{E},u}).$$

It follows that $\operatorname{Ran} S_t^{\mathcal{E},u} \perp \operatorname{Ran} \jmath_t^{\mathcal{E},u}$ $(t > 0)$, so

$$\bigcap_{t>0} \operatorname{Ran} S_t^{\mathcal{E},u} \subset \bigcap_{t>0} (\operatorname{Ran} \jmath_t^{\mathcal{E},u})^\perp = \Big(\bigcup_{t>0} \operatorname{Ran} \jmath_t^{\mathcal{E},u}\Big)^\perp = \{0\},$$

and therefore $S^{\mathcal{E},u}$ is purely isometric. □

By Cooper's Theorem ([10]; see Theorem 9.3, Chapter III of [30]), it follows from Lemma 6.6 that, for any pointed Arveson system $(\mathcal{E}, u)$ there is a Hilbert space $\mathsf{k}(\mathcal{E}, u)$ and unitary operator $V^{\mathcal{E},u} : X^{\mathcal{E},u} \to \mathsf{K}^{\mathcal{E},u} := L^2(\mathbb{R}_+; \mathsf{k}(\mathcal{E}, u))$ such that $V^{\mathcal{E},u} S_t^{\mathcal{E},u} = S_t^{\mathsf{k}(\mathcal{E},u)} V^{\mathcal{E},u}$ $(t \geqslant 0)$. Moreover $\mathsf{k}(\mathcal{E}, u)$ is separable since $X^{\mathcal{E},u}$ is.

Recall our notation $\mathsf{K}_t^{\mathcal{E},u} := \{ g \in \mathsf{K}^{\mathcal{E},u} : \text{ess-supp}\, g \subset [0, t] \}$ $(t > 0)$.

Lemma 6.7. *Let $(\mathcal{E}, u)$ be a pointed Arveson system. Set $F^{\mathcal{E},u} = (\mathcal{F}_t^{\ominus\perp})_{t>0}$ where $\mathcal{F} = \mathbb{C}u$. For $t > 0$, define the operator*

$$\phi_t^{\mathcal{E},u} : F_t^{\mathcal{E},u} = \mathcal{F}_t \oplus X_t^{\mathcal{E},u} \to F_t^{\mathsf{k}(\mathcal{E},u)} = \mathbb{C} \oplus \mathsf{K}_t^{\mathcal{E},u}, \quad \lambda u_t + x \mapsto (\lambda, J_t^* V^{\mathcal{E},u} x \cdot u_\infty)$$

where J_t denotes the inclusion map $\mathsf{K}_t^{\mathcal{E},u} \to \mathsf{K}^{\mathcal{E},u}$. Then $\phi^{\mathcal{E},u} = (\phi_t^{\mathcal{E},u})_{t>0}$ is an isomorphism of inclusion systems.

Proof. Drop the superscripts from $F_r^{\mathcal{E},u}$, $\mathsf{K}_r^{\mathcal{E},u}$, $X_r^{\mathcal{E},u}$, $\phi_r^{\mathcal{E},u}$, $\jmath_r^{\mathcal{E},u}$, $S_r^{\mathcal{E},u}$ $(r > 0)$ and $V^{\mathcal{E},u}$, and abbreviate $\mathsf{k}(\mathcal{E}, u)$ to k.

Each operator ϕ_t is easily seen to be unitary. Fix $s, t > 0$. Then

$$\begin{aligned}\big(\beta_{s,t}^{F,\mathsf{k}} \circ \phi_{s+t}\big)(u_{s+t}) = \beta_{s,t}^{F,\mathsf{k}}(1, 0) = (1, 0) \otimes (1, 0) \\ = \phi_s(u_s) \otimes \phi_t(u_t) = (\phi_s \otimes \phi_t)(\beta_{s,t}^F u_{s+t}).\end{aligned}$$

Also, if $z = x_s \cdot u_t + u_s \cdot x_t = \jmath_{s+t}^*\big(\jmath_s(x_s) + S_s \jmath_t(x_t)\big)$ where $x_s \in X_s$ and $x_t \in X_t$, then

$$\begin{aligned}\big(\beta_{s,t}^{F,\mathsf{k}} \circ \phi_{s+t}\big)(z) &= \beta_{s,t}^{F,\mathsf{k}}\big(0, J_{s+t}^* V(x_s \cdot u_\infty + S_s(x_t \cdot u_\infty))\big) \\ &= \beta_{s,t}^{F,\mathsf{k}}\big(0, J_{s+t}^*(V(x_s \cdot u_\infty) + S_s^{\mathsf{k}} V(x_t \cdot u_\infty))\big) \\ &= (0, J_s^* V(x_s \cdot u_\infty)) \otimes (1, 0) + (1, 0) \otimes (0, J_t^* V(x_t \cdot u_\infty)) \\ &= \phi_s(x_s) \otimes \phi_t(u_t) + \phi_s(u_s) \otimes \phi_t(x_t) = (\phi_s \otimes \phi_t)(\beta_{s,t}^F z).\end{aligned}$$

Since $F_{s+t} = \mathbb{C}u_{s+t} \oplus \jmath_{s+t}^*\big(\jmath_s(X_s) + S_s \jmath_t(X_t)\big)$, it follows that $\beta_{s,t}^{F,\mathsf{k}} \circ \phi_{s+t} = (\phi_s \otimes \phi_t) \circ \beta_{s,t}^F$. Therefore ϕ is an isomorphism of inclusion systems. □

Theorem 6.8. *Let $\mathcal{E}$ be a spatial Arveson system. Then, for any normalised unit u of $\mathcal{E}$,*

$$(\mathbb{C}u)\breve{} = \mathcal{E}^I.$$

Proof. Let $u \in \mathcal{U}_1^{\mathcal{E}}$ and set $\mathcal{F} = \mathbb{C}u$.

The isomorphism of inclusion systems $\phi^{\mathcal{E},u}$, defined in Lemma 6.7, lifts to an isomorphism of product systems $\psi : \breve{\mathcal{F}} \to \mathcal{F}^{\mathsf{k}(\mathcal{E},u)}$. Theorems 1.6 and 1.4 imply that ψ is an isomorphism of Arveson systems. Thus $\breve{\mathcal{F}}$ is of type I, and so is contained in $\mathcal{E}^I$.

Now let $a \in R_u^{\mathcal{E}}$ and $t > 0$. Then

$$a_t = a_r \otimes u_{t-r} + u_r \otimes a_{t-r} \in \mathcal{F}_r^\perp \otimes \mathcal{F}_{t-r} \,\oplus\, \mathcal{F}_r \otimes \mathcal{F}_{t-r}^\perp \subset \big(\mathcal{F}_r^\perp \otimes \mathcal{F}_{t-r}^\perp\big)^\perp \qquad (0 < r < t),$$

so $a_t \in \mathcal{F}_t^{\ominus\perp}$. By Theorem 3.7, the product subsystem of $\mathcal{E}$ generated by u and all of its roots is $\mathcal{E}^I$, therefore $\breve{\mathcal{F}}$ contains $\mathcal{E}^I$. The result follows. □

Before turning to its connection with the Cantor–Bendixson derivative applied to random closed sets (in the closed unit interval), we briefly mention a natural generalisation of our cluster construction. For an ordered pair of inclusion subsystems $F = (F^1, F^2)$ of an Arveson system $\mathcal{E}$, and $t > 0$, set

$$F_t^{\ominus\perp} := \mathcal{E}_t \ominus F_t^{\ominus} \quad \text{where} \quad F_t^{\ominus} := \vee_{0<r<t} (\mathcal{E}_r \ominus F_r^1) \otimes (\mathcal{E}_{t-r} \ominus F_{t-r}^2).$$

this extends the earlier construction (for a single inclusion subsystem F of $\mathcal{E}$) as follows:

$$(F, F)_t^{\ominus\perp} = F_t^{\ominus\perp} \qquad (t > 0).$$

Proposition 6.9. *Let $F = (F^1, F^2)$ be an ordered pair of inclusion subsystems of an Arveson system $\mathcal{E}$. Then $F^{\ominus\perp} := \big(F_t^{\ominus\perp}\big)_{t>0}$ is an inclusion subsystem of $\mathcal{E}$ containing F^1 and F^2.*

Proof. Let $s,t>0$. For $0<r<t$,

$$(F_r^1)^\perp \otimes (F_{t-r}^2)^\perp \subset (F_r^1)^\perp \otimes \mathcal{E}_{t-r} \subset (F_r^1 \otimes F_{t-r}^1)^\perp \subset (F_t^1)^\perp,$$

so $F_t^\ominus \subset (F_t^1)^\perp$, thus $F_t^1 \subset F_t^{\ominus\perp}$; also

$$\mathcal{E}_s \otimes (F_r^1)^\perp \subset (F_s^1 \otimes F_r^1)^\perp \subset (F_{s+r}^1)^\perp,$$

so

$$\mathcal{E}_s \otimes (F_r^1)^\perp \otimes (F_{t-r}^2)^\perp \subset (F_{s+r}^1)^\perp \otimes (F_{t-r}^2)^\perp \subset F_{s+t}^\ominus,$$

thus $\mathcal{E}_s \otimes F_t^\ominus \subset F_{s+t}^\ominus$.

By symmetry, $F_t^2 \subset F_t^{\ominus\perp}$ and $F_s^\ominus \otimes \mathcal{E}_t \subset F_{s+t}^\ominus$. Therfore

$$F_{s+t}^{\ominus\perp} \subset (\mathcal{E}_s \otimes F_t^{\ominus\perp}) \cap (F_s^{\ominus\perp} \otimes \mathcal{E}_t) = F_s^{\ominus\perp} \otimes F_t^{\ominus\perp}.$$

It follows that $F^{\ominus\perp}$ is an inclusion system containing F^1 and F^2. □

This completes the treatment of our cluster construction for product subsystems. In order to relate it to random closed sets we summarise the basic relevant properties of hyperspaces next. Thus let X be a topological space. The Vietoris topology on $K(X)$, the collection of compact subsets of X, has $\{H_U : U \text{ open in } X\} \cup \{M_F : F \text{ closed in } X\}$ as sub-base ([12]); the *hit sets* and *miss sets* of $K(X)$ being defined as follows:

$$H_A := \big\{Z \in K(X) : Z \cap A \neq \emptyset\big\} \text{ and } M_A := \big\{Z \in K(X) : Z \cap A = \emptyset\big\} \qquad (A \subset X).$$

Note that, for $A, B \subset X$ and $\mathcal{A} \subset \mathcal{P}(X)$, the following hold: $\{Z \in K(X) : Z \subset A\} = M_{A^\complement}$,

$$M_A = (H_A)^\complement,\ H_{\bigcup \mathcal{A}} = \bigcup\nolimits_{A\in\mathcal{A}} H_A,\ H_\emptyset = \emptyset \text{ and } \{\emptyset\} = M_X, \text{ so}$$

$$A \subset B \implies H_A \subset H_B,\ M_{\bigcup \mathcal{A}} = \bigcap\nolimits_{A\in\mathcal{A}} M_A,\ M_\emptyset = K(X) \text{ and } \{\emptyset\} = (H_X)^\complement.$$

Thus $\emptyset$ is an isolated point of $K(X)$, and a nonempty basic open set of $K(X)$ takes the form $B = M_F \cap H_{U_1} \cap \cdots \cap H_{U_n}$ for some set F closed in X, $n \in \mathbb{N}$ and sets $U_1, \cdots, U_n$ open in X such that $F^\complement \cap U_i \neq \emptyset$ for $i = 1, \cdots, n$. Note also that, for a sequence (F_n) of closed sets of X,

$$F_n \downarrow F \implies M_F = \bigcup_{n=1}^\infty M_{F_n}. \tag{6.1}$$

For any dense subset D of X, $K_{00}(X) \cap \mathcal{P}(D)$ is dense in $K(X)$, where $K_{00}(X)$ denotes the collection of subsets of X having finite cardinality. If X has compatible metric d (with diameter at most one) then the induced Hausdorff metric d_{H} on $K(X)$ (for which $d_{\mathrm{H}}(Z,\emptyset) = 1 = d_{\mathrm{H}}(\emptyset, Z)$ for all $Z \in K(X) \setminus \{\emptyset\}$) is compatible with the Vietoris topology, and is complete if d is. If $\varepsilon > 0$ and $F \subset\subset X$ is an ε-net ([31], Definition 7.2.8) with respect to a compatible metric d for X with diameter at most one, then $\mathcal{P}(F)$ is an ε-net for d_{H}, so $K(X)$ is totally bounded with respect to d_{H} if X is totally bounded with respect to d. It follows from these basic facts that $K_{00}(X)$ is dense in $K(X)$, and $K(X)$ is separable, metrisable, completely metrisable, Polish, or compact metrisable, if X has that property. When X is compact Hausdorff (so that $K(X)$ equals the collection of closed subsets of X), the Vietoris topology coincides with another well-known hyperspace topology, namely the Fell topology.

For a subset A of X we denote by A' its derived set, consisting of its points of accumulation, $\{x \in X : x \in \overline{A \setminus \{x\}}\}$. Note that (1) $A' \subset \overline{A}$, (2) A' is closed if A is, (3) if X is a T_1-space then $A' = \overline{A}\,'$, and so A' is closed. Note the further elementary properties (assuming, for (5), that X is T_1): for $A \subset B \subset X$, $C \subset X$, U open in X and $K \in K(X)$,

$$(4)\ (A \cap C)' \subset B' \cap C',\ (5)\ A' \cap U \neq \emptyset \implies \#(A \cap U) = \infty;\ (6)\ K' = \emptyset \iff \#K < \infty.$$

Thus, for X Hausdorff, the prescription $Z \mapsto Z'$ defines a map $\Delta_X : K(X) \to K(X)$, the *Cantor–Bendixson derivative* (whose study, as an operator, was initiated by Kuratowski; see [13]).

We now turn to the connection with random closed sets. Set $\mathcal{C} := K(\mathbb{I})$, $\mathcal{C}_{00} := K_{00}(\mathbb{I})$ and $\Delta := \Delta_{\mathbb{I}}$, where $\mathbb{I}$ denotes the unit interval $[0,1]$ with its standard topology. Thus $\mathcal{C}$ is compact and metrised by the Hausdorff metric of the standard metric of $\mathbb{I}$, in particular it is second countable, with countable dense subset $\mathcal{C}_{00} \cap \mathcal{P}(\mathbb{I} \cap \mathbb{Q})$, and $\Delta^{-1}(\{\emptyset\}) = \mathcal{C}_{00} \subsetneq \mathcal{C}$. By a *random closed subset of* $\mathbb{I}$ is meant simply a $\mathcal{C}$-valued random variable, in other words a measurable map from Ω to $\mathcal{C}$, for a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

Lemma 6.10. *Let $F, U \subset \mathbb{I}$, with F closed, U open and $F \supset U$. Then, the following hold.*

(a) $\Delta^{-1}(M_F) \subset \{Z \in \mathcal{C} : \#(Z \cap F) < \infty\} \subset \{Z \in \mathcal{C} : \#(Z \cap U) < \infty\} \subset \Delta^{-1}(M_U)$.

(b) *Let ∂F denote the topological boundary of F. Then*

$$\begin{aligned}\Delta^{-1}(M_F) \cup H_{\partial F} &= \{Z \in \mathcal{C} : \#(Z \cap F) < \infty\} \cup H_{\partial F} \\ &= \{Z \in \mathcal{C} : \#(Z \cap \operatorname{Int} F) < \infty\} \cup H_{\partial F} = \Delta^{-1}(M_{\operatorname{Int} F}) \cup H_{\partial F}.\end{aligned}$$

Proof. (a) follows from (4), (6) and (5) above.

(b) For $Z \in \mathcal{C}$, (a) implies that the middle sets are sandwiched by the outer sets. Let $Z \in \mathcal{C} \setminus H_{\partial F} = M_{\partial F}$. Then $Z \cap \partial F = \emptyset$, so $Z \cap F = Z \cap \operatorname{Int} F$ and so $Z' \cap F = Z' \cap \operatorname{Int} F$. Thus $Z \in \Delta^{-1}(M_F)$ if and only if $Z \in \Delta^{-1}(M_{\operatorname{Int} F})$. Therefore the outer sets coincide, as required. □

The contents of the following proposition are known; we include their short proofs since they are instructive and do not seem to be readily available.

Proposition 6.11. *The following hold.*

(a) $\operatorname{Borel}(\mathcal{C}) = \sigma\{M_J : J$ *is a closed subinterval of* $\mathbb{I}\}$.

(b) Δ *is Borel measurable.*

Proof. (a) Denote the RHS σ-algebra by Σ. Let U be open in $\mathbb{I}$ and let F be closed in $\mathbb{I}$. Then $U = \bigcup J_n$ for a sequence (J_n) of closed subintervals of $\mathbb{I}$, and $F = \bigcap F_n$ for a sequence (F_n) of closed sets of $\mathbb{I}$ such that $F_n \downarrow F$ and, for each $n \in \mathbb{N}$, $F_n = \bigcup_{i=1}^{k(n)} J_n^i$ for some closed subintervals $J_n^1, \cdots, J_n^{k(n)}$ of $\mathbb{I}$. Therefore, using (6.1),

$$H_U = \bigcup_{n=1}^{\infty} H_{J_n} = \bigcup_{n=1}^{\infty} \left(M_{J_n}\right)^{\complement} \in \Sigma \quad \text{and} \quad M_F = \bigcup_{n=1}^{\infty} M_{F_n} = \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{k(n)} M_{J_n^i} \in \Sigma.$$

Since $\mathcal{C}$ is second countable it follows from Lindelöf's Theorem that every open set of $\mathcal{C}$ is a countable union of basic open sets, so $\Sigma \supset \operatorname{Borel}(\mathcal{C})$. The reverse inclusion is clear.

(b) Let J be a closed subinterval of $\mathbb{I}$, say $[a, b]$. For U open in $\mathbb{I}$ and $p \in \mathbb{N}$, since $\mathbb{I}$ is Hausdorff, the set $\{Z \in \mathcal{C} : \#(Z \cap U) \geqslant p\}$ equals the open set

$$\bigcup \{H_{V_1} \cap \cdots \cap H_{V_p} : V_1, \cdots, V_p \text{ disjoint open subsets of } U\}.$$

It follows from (6.1) that $M_J = \bigcup_{n=1}^{\infty} M_{U_n} = \bigcup_{n=1}^{\infty} M_{\overline{U_n}}$, where $U_n :=]a - \frac{1}{n}, b + \frac{1}{n}[\cap \mathbb{I}$ $(n \in \mathbb{N})$. Now, by part (a) of Lemma 6.10,

$$\Delta^{-1}\big(M_{\overline{U_n}}\big) \subset \{Z \in \mathcal{C} : \#(Z \cap \overline{U_n}) < \infty\} \subset \{Z \in \mathcal{C} : \#(Z \cap U_n) < \infty\} \subset \Delta^{-1}\big(M_{U_n}\big)$$

for each $n \in \mathbb{N}$. It follows that

$$\Delta^{-1}(M_J) = \bigcup_{n \in \mathbb{N}} \{Z \in \mathcal{C} : \#(Z \cap U_n) < \infty\} = \bigcup_{n, p \in \mathbb{N}} \{Z \in \mathcal{C} : \#(Z \cap U_n) \geqslant p\}^{\complement} \in \operatorname{Borel}(\mathcal{C}).$$

The Borel measurability of Δ therefore follows from (a). □

Remark. Δ is not continuous, since $\{\emptyset\}$ is closed in $\mathcal{C}$ but $\Delta^{-1}(\{\emptyset\})$ is not closed because it equals $\mathcal{C}_{00}$ which is a dense proper subset of $\mathcal{C}$.

For the convenience of the reader we quote the key propositions upon which our next result depends. recall that in Liebscher's approach the parameter set of an Arveson system $\mathcal{E}$ is extended to $\mathbb{R}_+$, with $\mathcal{E}_0 := \mathbb{C}$.

Theorem 6.12 ([14], Theorem 3.16, Proposition 3.18, Corollary 3.21)**.** *Let $\mathcal{E}$ be an Arveson system, let $P = (P_{r,t})_{0 \leqslant r < t \leqslant 1}$ be a family of nonzero orthogonal projections in the von Neumann algebra $B(\mathcal{E}_1)$ satisfying the evolution and bi-adaptedness conditions*

$$P_{r,s} P_{s,t} = P_{r,t} \quad \text{and} \quad P_{r,t} \in I_r^{\mathcal{E}} \otimes B(\mathcal{E}_{t-r}) \otimes I_{1-t}^{\mathcal{E}} \qquad (0 \leqslant r < s < t \leqslant 1), \tag{6.2}$$

and let ω and φ be faithful normal states on $B(\mathcal{E}_1)$. Then the following hold:

(a) *The map $(r, t) \mapsto P_{r,t}$ is strongly continuous, with $P_{r,t} \to I_1^{\mathcal{E}}$ as $(r, t) \to (s, s)$ for $0 < s < 1$.*

(b) *There is a unique Borel probability measure* $\mathbb{P}^P_\omega$ *on* $\mathcal{C}$ *satisfying*

$$\mathbb{P}^P_\omega\Big(\bigcap_{i=1}^N M_{[s_i,t_i]}\Big) = \omega\Big(\prod_{i=1}^N P_{s_i,t_i}\Big)$$

($N \in \mathbb{N}, 0 \leqslant s_i < t_i \leqslant 1$ *for* $i = 1, \cdots, N$).

(c) $\mathbb{P}^P_\omega(H_{\{a\}}) = 0$ $(a \in \mathbb{I})$.

(d) *The correspondence* $1_{M_{[s,t]}} \mapsto P_{s,t}$ $(0 \leqslant s < t \leqslant 1)$, *extends to an injective normal unital representation* $\pi^P : L^\infty(\mathbb{P}^P_\omega) \to B(\mathcal{E}_1)$. *Moreover,*

$$\operatorname{Ran} \pi^P = \{P_{s,t} : 0 \leqslant s < t \leqslant 1\}''.$$

(e) $\mathbb{P}^P_\varphi \sim \mathbb{P}^P_\omega$.

Remarks. (i) For a product subsystem $\mathcal{F}$ of $\mathcal{E}$, the family $P^{\mathcal{F}} = (P^{\mathcal{F}}_{r,t})_{0\leqslant r<t\leqslant 1}$, as defined in (5.1), satisfies (6.2).

(ii) By (e), the space $L^\infty(\mathbb{P}^P_\omega)$, and therefore also the representation π^P, is independent of the choice of faithful normal state ω on $B(\mathcal{E}_1)$.

(iii) For a faithful normal state ω on $B(\mathcal{E}_1)$, we write $\mathbb{P}^{\mathcal{F}}_\omega$ and $\pi^{\mathcal{F}}$ respectively for the Borel probability measure $\mathbb{P}^P_\omega$ and representation π^P, when $P = P^{\mathcal{F}}$. Let $\mathcal{M}^{\mathcal{F}}$ denote the probability measure equivalence type of $\mathbb{P}^{\mathcal{F}}_\omega$. By (e), $\mathcal{M}^{\mathcal{F}}$ is independent of the choice of faithful normal state ω.

We need the following extension of [14], Corollary 6.2.

Theorem 6.13. *Let* $\mathcal{F}$ *be a product subsystem of an Arveson system* $\mathcal{E}$. *Then*

$$\mathcal{M}^{\mathcal{F}} = \big\{\mathbb{P}^{\mathcal{F}}_\omega : \omega \text{ is a faithful normal state on } B(\mathcal{E}_1)\big\}.$$

Proof. The proof in [14], for the case where $\mathcal{F}$ is generated by a unit of $\mathcal{E}$, works equally well for an arbitrary product subsystem. □

We are now ready to give our generalisation of Proposition 3.33 of [14].

Theorem 6.14. *Let* $\mathcal{F}$ *be a product subsystem of an Arveson system* $\mathcal{E}$. *Then the following hold.*

(a) $\pi^{\mathcal{F}}\big(1_{\Delta^{-1}(M_{[s,t]})}\big) = P^{\mathcal{F}^\smile}_{s,t}$ $(0 \leqslant s < t \leqslant 1)$.

(b) $\mathbb{P}^{\mathcal{F}}_\omega \circ \Delta^{-1} = \mathbb{P}^{\mathcal{F}^\smile}_\omega$, *for any faithful normal state* ω *on* $B(\mathcal{E}_1)$.

(c) $\mathcal{M}^{\mathcal{F}} \circ \Delta^{-1} = \mathcal{M}^{\mathcal{F}^\smile}$.

Proof. Let $0 \leqslant s < t \leqslant 1$. First note that, by part (b) of Lemma 6.10 and part (c) of Theorem 6.12,

$$\pi^{\mathcal{F}}\big(1_{\Delta^{-1}(M_{[s,t]})}\big) = \pi^{\mathcal{F}}\big(1_{\{Z\in\mathcal{C}:\#(Z\cap[s,t])<\infty\}}\big). \tag{6.3}$$

For $Z \in \mathcal{C}$,

$$\#\big(Z \cap [s,t]\big) \geqslant 2 \iff \exists_{u\in]s,t[} : Z \in H_{[s,u[} \cap H_{[u,t]}.$$

and for $0 \leqslant a < b \leqslant 1$, $\pi^{\mathcal{F}}\big(1_{H_{[a,b[}}\big) = \pi^{\mathcal{F}}\big(1_{H_{[a,b]}}\big)$, and

$$\pi^{\mathcal{F}}\big(1_{H_{[a,b]}}\big) = I^{\mathcal{E}}_1 - P^{\mathcal{F}}_{a,b} = I^{\mathcal{E}}_a \otimes P_{\mathcal{F}^\perp_{b-a}} \otimes I^{\mathcal{E}}_{1-a},$$

so

$$\pi^{\mathcal{F}}\big(1_{H_{[s,u[}\cap H_{[u,t]}}\big) = I^{\mathcal{E}}_s \otimes P_{\mathcal{F}^\perp_{u-s}} \otimes P_{\mathcal{F}^\perp_{t-u}} \otimes I^{\mathcal{E}}_{1-t}, \qquad (s < u < t).$$

By the normality of $\pi^{\mathcal{F}}$, it follows that

$$\pi^{\mathcal{F}}\big(1_{\{Z\in\mathcal{C}:\#(Z\cap[s,t])\geqslant 2\}}\big) = \sup_{s<u<t} I^{\mathcal{E}}_s \otimes P_{\mathcal{F}^\perp_{u-s}\otimes\mathcal{F}^\perp_{t-u}} \otimes I^{\mathcal{E}}_{1-t} = I^{\mathcal{E}}_s \otimes P_V \otimes I^{\mathcal{E}}_{1-t}$$

where $V = \vee_{s<u<t}\big(\mathcal{F}^\perp_{u-s} \otimes \mathcal{F}^\perp_{t-u}\big) = F^{\ominus}_{t-s}$. By the evolution property,

$$P^\perp_V = \bigwedge_{s<u<t}\big(I^{\mathcal{E}}_1 - (I^{\mathcal{E}}_1 - P^{\mathcal{F}}_{s,u})(I^{\mathcal{E}}_1 - P^{\mathcal{F}}_{u,t})\big) = \bigwedge_{s<u<t}\big(P^{\mathcal{F}}_{s,u} + P^{\mathcal{F}}_{u,t} - P^{\mathcal{F}}_{s,t}\big).$$

It therefore follows that

$$\pi^{\mathcal{F}}\big(1_{\{Z\in\mathcal{C}:\#(Z\cap[s,t])\leqslant 1\}}\big) = I^{\mathcal{E}}_s \otimes P_{F^{\ominus\perp}_{t-s}} \otimes I^{\mathcal{E}}_{1-t} = P^{F^{\ominus\perp}}_{s,t}. \tag{6.4}$$

Now

$$\big\{Z \in \mathcal{C} : \#(Z \cap [s,t]) < \infty\big\} = \bigcup C_{\mathcal{P}} \tag{6.5}$$

where the union is over partitions $\mathcal{P} = \{s = s_0 < \cdots < s_N = t\}$ and $C_{\mathcal{P}} := \bigcap_{i=1}^N \big\{Z \in \mathcal{C} : \#(Z \cap [s_{i-1}, s_i]) \leqslant 1\big\}$. And so, applying (6.4) with $[s_{i-1}, s_i]$ in place of $[s,t]$,

$$\pi^{\mathcal{F}}(1_{C_{\mathcal{P}}}) = \prod_{i=1}^N P^{F^{\ominus\perp}}_{s_{i-1},s_i} \quad \text{for} \quad \mathcal{P} = \{s = s_0 < \cdots < s_N = t\}.$$

Therefore, by (6.5), the normality of $\pi^{\mathcal{F}}$, and the fact that the inclusion system $F^{\ominus\perp}$ generates the product system $\check{\mathcal{F}}$,

$$P^{\check{\mathcal{F}}}_{s,t} = \pi^{\mathcal{F}}\big(1_{\{Z\in\mathcal{C}:\#(Z\cap[s,t]<\infty\}}\big).$$

Combined with (6.3), this proves (a). Now (a) implies that

$$\omega\Big(\prod_{i=1}^N P^{\check{\mathcal{F}}}_{s_i,t_i}\Big) = (\omega\circ\pi^{\mathcal{F}})\Big(1_{\bigcap_{i=1}^N \Delta^{-1}(M_{[s_i,t_i]})}\Big) = \big(\mathbb{P}^{\mathcal{F}}_\omega \circ \Delta^{-1}\big)\Big(\bigcap_{i=1}^N M_{[s_i,t_i]}\Big),$$

for subintervals $[s_1,t_1],\cdots,[s_N,t_N]$ of $\mathbb{I}$, so (b) follows from part (b) of Theorem 6.12.

(c) In view of Theorem 6.13, this follows immediately from (b). □

Appendix. Fock Arveson systems and the Guichardet picture

The symmetric Fock space over a Hilbert space H is denoted $\Gamma(\mathsf{H})$. Its exponential vectors $\varepsilon(h) := \big((n!)^{-1/2}h^{\otimes n}\big)_{n\geqslant 0}$ $(h\in\mathsf{H})$ form a linearly independent and total set which witnesses the exponential property of symmetric Fock spaces, namely $\Gamma(\mathsf{H}_1\oplus\mathsf{H}_2) = \Gamma(\mathsf{H}_1)\otimes\Gamma(\mathsf{H}_2)$ via $\varepsilon(h_1,h_2)\mapsto\varepsilon(h_1)\otimes\varepsilon(h_2)$. For any contraction $C\in B(\mathsf{H})$, $\Gamma(C) := \bigoplus_{n\geqslant 0} C^{\otimes n}$ defines a contraction in $B(\Gamma(\mathsf{H}))$ satisfying $\Gamma(C)\varepsilon(h) = \varepsilon(Ch)$ $(h\in\mathsf{H})$; the map $C\to\Gamma(C)$ is a morphism of involutive semigroups with identity, in particular, $\Gamma(C)$ is isometric, respectively coisometric, if C is. For $h\in\mathsf{H}$, the *Fock Weyl operator* is the unitary operator $W(h)$ on $\Gamma(\mathsf{H})$ satisfying

$$W(h)\varpi(k) = e^{-i\,\mathrm{Im}\langle h,k\rangle}\varpi(h+k), \quad \text{where} \quad \varpi(k) := e^{-\|k\|^2/2}\varepsilon(k) \qquad (k\in\mathsf{H}).$$

Now let k be a separable Hilbert space. Set

$$\mathsf{K} := L^2(\mathbb{R}_+;\mathsf{k}) \;\text{ and }\; \mathsf{K}_t := \big\{g\in\mathsf{K} : \text{ess-supp}\, g\subset[0,t]\big\} \qquad (t>0),$$

and let $S^{\mathsf{k}} := (S^{\mathsf{k}}_t)_{t\geqslant 0}$ denote the one-parameter semigroup of unilateral shifts on K. The *Fock Arveson system over* k, denoted $\mathcal{F}^{\mathsf{k}}$, is defined by

$$\mathcal{F}^{\mathsf{k}}_t := \Gamma(L^2([0,t[;\mathsf{k}))\otimes\Omega^{\mathsf{k}}_{[t,\infty[} = \overline{\mathrm{Lin}}\{\varepsilon(g) : g\in\mathsf{K}_t\}, \qquad (t>0)$$

where $\Omega^{\mathsf{k}}_{[t,\infty[}$ denotes the vacuum vector $\varepsilon(0)$ in $\Gamma(L^2([t,\infty[;\mathsf{k}))$, with structure maps determined by the prescription

$$B^{\mathcal{F};\mathsf{k}}_{s,t} : \varepsilon(h)\mapsto\varepsilon(h_{[0,s[})\otimes\varepsilon((S^{\mathsf{k}}_s)^*h), \quad \text{for} \quad h\in\mathsf{K}_{s+t} \qquad (s,t>0).$$

It is an Arveson system consisting of an increasing family of subspaces of the Hilbert space $\mathcal{F}^{\mathsf{k}}_\infty = \Gamma(\mathsf{K})$. Its set of normalised units is given by $\{(e^{i\lambda t}\varpi^c_t)_{t>0} : c\in\mathsf{k}, \lambda\in\mathbb{R}\}$ where

$$\varpi^c := \big(e^{-\|c\|^2t/2}\varepsilon^c_t = \varpi(c_{[0,t[})\big)_{t>0} \;\text{ and }\; \varepsilon^c := \big(\varepsilon(c_{[0,t[})\big)_{t>0}.$$

The *vacuum unit* $\varpi^0 = \varepsilon^0$, of the Arveson system $\mathcal{F}^{\mathsf{k}}$, is denoted Ω^{k}.

In order to describe the Guichardet picture of the Fock Arveson system over k which (only here in this appendix) we denote by $\mathcal{G}^{\mathsf{k}}$, we need to introduce the symmetric measure space Γ_t over the Lebesgue space $[0,t[$, for $0<t\leqslant\infty$ ([11]). As a set,

$$\Gamma_t := \big\{\sigma\subset[0,t[: \#\sigma<\infty\big\}.$$

Thus, denoting $\{\sigma\subset[0,t[: \#\sigma = n\}$ by $\Gamma^{(n)}_t$, $\bigcup_{n\geqslant 0}\Gamma^{(n)}_t$ is a partition of Γ_t. Since, for each $n\in\mathbb{N}$, the map

$$\Delta^{(n)}_t := \big\{\mathbf{s}\in\mathbb{R}^n_+ : s_1<\cdots<s_n<t\big\}\to\Gamma^{(n)}_t, \quad \mathbf{s}\mapsto\{s_1,\cdots,s_n\}$$

is bijective, Lebesgue measure on $\Delta^{(n)}_t$ induces a measure on $\Gamma^{(n)}_t$ and thereby an isometric isomorphism $L^2(\Delta^{(n)}_t)\to L^2(\Gamma^{(n)}_t)$. Composing with the isometric isomorphism

$$L^2_{\mathrm{sym}}([0,t]^n;\mathsf{k}^{\otimes n})\to L^2(\Delta^{(n)}_t;\mathsf{k}^{\otimes n}), \quad F\mapsto\sqrt{n!}\,F|_{\Delta^{(n)}_t}$$

(and ampliating) gives an isometric isomorphism

$$L^2_{\mathrm{sym}}([0,t]^n;\mathsf{k}^{\otimes n})\to L^2(\Gamma^{(n)}_t;\mathsf{k}^{\otimes n}) \qquad (n\in\mathbb{N}).$$

By declaring that $\emptyset \in \Gamma_t^{(0)} \subset \Gamma_t$ is an atom of measure one, we have an isometric isomorphism

$$\mathcal{F}_t^{\mathsf{k}} \cong \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\text{sym}}([0,t]^n; \mathsf{k}^{\otimes n}) \cong \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2(\Gamma_t^{(n)}; \mathsf{k}^{\otimes n}) \cong \mathcal{G}_t^{\mathsf{k}},$$

where $\mathcal{G}_t^{\mathsf{k}} := \{G \in \mathcal{G}_\infty^{\mathsf{k}} : \text{ess-supp}\, G \subset \Gamma_t\}$ and, in terms of $\Phi(\mathsf{k})$, the full Fock space over k,

$$\mathcal{G}_\infty^{\mathsf{k}} := \{G \in L^2(\Gamma_\infty; \Phi(\mathsf{k})) : G(\sigma) \in \mathsf{k}^{\otimes \# \sigma} \text{ for a.a. } \sigma\}.$$

These isomorphisms are restrictions of a single isomorphism $\mathcal{F}_\infty^{\mathsf{k}} \to \mathcal{G}_\infty^{\mathsf{k}}$, under which $\varepsilon(g) \mapsto \pi_g$ where, for $g \in \mathsf{K}$,

$$\pi_g(\sigma) = \begin{cases} 1 & \text{if } \sigma = \emptyset, \\ g(s_1) \otimes \cdots \otimes g(s_n) & \text{if } \sigma = \{s_1 < \cdots < s_n\} \end{cases},$$

in particular, $\varepsilon(0) \mapsto \delta_\emptyset$. Moreover, for $G \in \mathcal{G}_\infty^{\mathsf{k}}$ and $t \geqslant 0$,

$$\big(\Gamma(S_t^{\mathsf{k}})G\big)(\sigma) = \begin{cases} G(\sigma - t) & \text{if } \sigma \subset [t, \infty[\\ 0 & \text{otherwise} \end{cases}.$$

The corresponding structure maps in the Guichardet picture are given by the prescription

$$B_{s,t}^{\mathcal{G},\mathsf{k}} H : (\alpha, \beta) \mapsto H\big(\alpha \cup (\beta + s)\big) 1_{\Gamma_{[0,s[} \times \Gamma_{[0,t[}}(\alpha, \beta) \qquad (H \in \mathcal{G}_{s+t}^{\mathsf{k}}).$$

For further details on Fock space and the Guichardet picture, see [16] and [20].

Acknowledgements

Valuable discussions with Volkmar Liebscher are gratefully acknowledged. This work was supported by the UK-India Education and Research Initiative(UKIERI), under the research collaboration grant *Quantum Probability, Noncommutative Geometry & Quantum Information.* The third-named author was also suported by DST-Inspire Fellowship IFA-13 MA 20.

References

1. W. Arveson, Continuous analogues of Fock space, *Mem. Amer. Math. Soc.* **80** (1989), no. 409.
2. ———, Continuous analogues of Fock space. IV. Essential states, *Acta Math.* **164** (1990), no. 3-4, 265–300.
3. ———, "Noncommutative dynamics and E-semigroups," *Springer Monographs in Mathematics*, Springer-Verlag, New York, 2003.
4. B.V.R. Bhat, Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^*-algebras, *J. Ramanujan Math. Soc.* **14** (1999), no. 2, 109–124.
5. B.V.R. Bhat, V. Liebscher, M. Mukherjee, and M. Skeide, The spatial product of Arveson systems is intrinsic, *J. Funct. Anal.* **260** (2011), no. 2, 566–573.
6. B.V.R. Bhat, V. Liebscher, and M. Skeide, A problem of powers and the product of spatial product systems, *in*, "Quantum Probability and Related Topics, QP–PQ **23** World Sci. Publ., Hackensack, NJ, 2008, pp. 93–106.
7. ———, Subsystems of Fock need not be Fock: spatial CP-semigroups, *Proc. Amer. Math. Soc.* **138** (2010), no. 7, 2443–2456.
8. B.V.R. Bhat and M. Mukherjee, Inclusion systems and amalgamated products of product systems, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13** (2010), no. 1, 1–26.
9. B.V.R. Bhat and M. Skeide, Tensor product systems of Hilbert modules and dilations of completely positive semigroups, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **3** (2000), no. 4, 519–575.
10. J.L.B. Cooper, One-parameter semigroups of isometric opertors in Hilbert space, *Ann. Math.* (2) **48** (1947), 827–842.
11. A. Guichardet, "Symmetric Hilbert spaces and related topics. Infinitely divisible positive definite functions. Continuous products and tensor products. Gaussian and Poissonian stochastic processes," *Lecture Notes in Mathematics* **261**, Springer, Berlin, 1972.
12. A.S. Kecheris, "Classical Descriptive Set Theory," Springer-Verlag, Berlin, 1995.
13. K. Kuratowski, Sur les décompositions semi-continus d'espaces métriques compacts, *Fund math.* **11** (1928), 169–183.
14. V. Liebscher, Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces, *Mem. Amer. Math. Soc.* **199** (2009), no. 930.
15. ———, The relation of spatial product and tensor product of arveson systems-the random set point of view, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **18** (2015), no. 4, 1550029, 22 pp.
16. J.M. Lindsay, Quantum stochastic analysis — an introduction, *in*, "Quantum Independent Increment Process, I: From Classical Probability to Quantum Stochastic Calculus" (Eds. U. Franz & M. Schürmann), *Lecture Notes in Mathematics* **1865**, Springer-Verlag, Heidelberg 2005, pp. 181–271.
17. D. Markiewicz, On the product system of a completely positive semigroup, *J. Funct. Anal.* **200** (2003), no. 1, 237–280.

18. P.S. Muhly and B. Solel, Quantum Markov processes (correspondences and dilations), *Internat. J. Math.* **13** (2002), no. 8, 863–906.
19. M. Mukherjee, Index computation for amalgamated products of product systems, *Banach J. Math. Anal.* **5** (2011), no. 1, 148–166.
20. K.R. Parthasarathy, "An Introduction to Quantum Stochastic Calculus," *Monographs in Mathematics* **85**, Birkhäuser, Basel 1992.
21. R.T. Powers, A nonspatial continuous semigroup of *-endomorphisms of $B(H)$, *Publ. Res. Inst. Math. Sci.* **23** (1987), no. 6, 1053–1069.
22. ———, New examples of continuous spatial semigroups of *-endomorphisms of $B(H)$, *Internat. J. Math.* **10** (1999), no. 2, 215–288.
23. ———, Addition of spatial E_0-semigroups, *in*, "Operator algebras, quantization and noncommutative geometry," *Contemporary Mathematics* **365** Amer. Math. Soc., Providence, RI, 2004, pp. 281–298.
24. O.M. Shalit and B. Solel, Subproduct systems, *Doc. Math.* **14** (2009), 801–868.
25. M. Skeide, Classification of e_0-semigroups by product systems, *arXiv*:0901.1798v3.
26. ———, Commutants of von Neumann modules, representations of $\mathcal{B}^a(E)$ and other topics related to product systems of Hilbert modules, *in*, "Advances in quantum dynamics," *Contemporary Mathematics* **335** Amer. Math. Soc., Providence, RI, 2003, pp. 253–262.
27. ———, A simple proof of the fundamental theorem about Arveson systems, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **9** (2006), no. 2, 305–314; Errata, **9** (2006), no. 3, 489.
28. ———, The index of (white) noises and their product systems, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **9** (2006), no. 4, 617–655.
29. ———, The Powers sum of spatial CPD-semigroups and CP-semigroups, "Noncommutative harmonic analysis with applications to probability II," *Banach Center Publications* textbf89 Polish Acad. Sci. Inst. Math., Warsaw, 2010, pp. 247–263.
30. B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, "Harmonic analysis of operators on Hilbert space," Second Edition, *Universitext*, Springer, New York, 2010.
31. W.A. Sutherland, "Introduction to Metric and Topological Spaces," *Oxford Science Publications*, Oxford University Press, 1975.
32. B. Tsirelson, Automorphisms of the type II_1 Arveson system of Warren's noise, *arXiv*:math/0612303v1.
33. ———, From random sets to continuous tensor products: answers to three questions of W. Arveson, *arXiv*:math/0001070v1.
34. ———, From slightly coloured noises to unitless product systems, *arXiv*:FA 0006165 vl.
35. ———, Non-isomorphic product systems, *in*, "Advances in quantum dynamics," *Contemporary Mathematics* **335** Amer. Math. Soc., Providence, RI, 2003, pp. 273–328.
36. ———, On automorphisms of type II Arveson systems (probabilistic approach), *New York J. Math.* **14** (2008), 539–576.

Stat-Math Unit, Indian Statistical Institute, R.V. College Post, Bangalore-560059, India
E-mail address: `bhat@isibang.ac.in`

Department of Mathematics & Statistics, Lancaster University, Lancaster LA1 4YF, UK
E-mail address: `j.m.lindsay@lancaster.ac.uk`

School of Mathematics, IISER Thiruvananthapuram, CET Campus, Kerala - 695016, India
E-mail address: `mithunmukh@iisertvm.ac.in`