Wagering on more than one Outcome in an Event in Cumulative Prospect Theory and Rank Dependent Utility [1]

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**Introduction**

In this letter we illustrate a seemingly previously not recognised implication of Cumulative Prospect Theory of Tversky and Kahneman (1992) (CPT) and Rank Dependent Utility of Quiggin (1982), (RDU) for optimal wagering. We demonstrate that the representative individual in both models can typically increase expected utility (called expected value in CPT) by wagering on more than one outcome in an event.[2] In general this betting strategy will not imply a constant positive expected return regardless of which of the competitors in the event wins - which is called a Dutch betting strategy. I will define betting on more than one horse in a race as a hedge wager. The hedge wager can take a variety of forms. It could constitute, for example, wagering on two or more horses in the same race to win. Alternatively it could constitute a win and lay bet. A lay wager introduced by Betfair allows one to bet that an outcome will not occur. So a hedge

[1] I am grateful for the helpful comments of a referee.

[2] Internet posts suggest that wagering on more than one outcome is popular with punters.

See e.g. https://betting.betfair.com/education/-generic-betting-principles/reverse-dutching-1-130111.html wager could constitute, for example, betting an outsider to win and laying the favourite to lose in a horse race.

Our analysis reveals that hedge wagering should be popular with the representative agent in CPT and RDU. For instance when playing European roulette we can demonstrate that the representative agent in CPT or RDU will always engage in hedge wagering. We can also demonstrate that hedge wagering in markets exhibiting a favorite–longshot bias can readily occur in both CPT and RDU. These implications contrast with models of wagering based solely on expected utility with a convex segment of the utility function since these risk-seeking individuals will not engage in hedge wagering. (See e.g. Golec and Tamarkin, (1998), Markowitz (1952) and Sauer (1998)).

The rest of the letter is structured as follows. In section one we demonstrate that the representative individual in CPT will engage in hedge wagering when playing European Roulette. We also explain why the representative CPT individual is very unlikely to engage in hedge wagering involving win and lay bets. In section two we demonstrate that the representative individual in RDU is likely to engage in hedge wagering including win and lay bets. The last section is a brief conclusion.

**Section 1 Cumulative Prospect Theory and Hedge Wagering**

In Cumulative Prospect Theory of Tversky and Kahneman (1992) there are three key assumptions. First, it is assumed that from a reference point the value function over gains is solely risk-averse and the value function over losses solely risk-seeking. Second, the representative agent's subjective probabilities of an outcome are assumed to differ from the objective probabilities via an inverted s-shaped probability weighting functions. The representative individual is therefore assumed to over-weight smaller probabilities and under-weight larger probabilities. Third, individuals are assumed to exhibit loss-aversion which implies that losses reduce utility, named value in CPT, by more than gains, of the same amount.

To illustrate that a hedge wager can exhibit higher expected value than a single wager in CPT we employ for simplicity and expositional purposes the parametric model of Tversky and Kahneman (1992).

For a single wager expected value, EV, is given by

$$EV = w^{+}(p)s^{\alpha}a^{\alpha} - w^{-}(1-p)\lambda s^{\beta} \qquad (1)$$

Where s is stake size, a is odds, p is the win probability and $\alpha$, $\beta$ and $\lambda$ are constants.

The probability weighting functions are given by[3]

$$w^{+}(p) = \frac{p^{\delta}}{[p^{\delta} + (1-p)^{\delta}]^{\frac{1}{\delta}}}, \; w^{-}(1-p) = \frac{(1-p)^{\rho}}{[p^{\rho} + (1-p)^{\rho}]^{\frac{1}{\rho}}}$$

The parameter values suggested by Tversky and Kahneman (1992) are given by

$\alpha = 0.88$, $\beta = 0.89$, $\delta = 0.61$, $\rho = 0.69$ and $\lambda = 2.25$.

To illustrate that hedge wagering can readily occur we construct a hedge wager that exhibits only a single positive payoff and a loss payoff. This structure implies that the gain from wagering on the second outcome, if it wins, just compensates for the loss of the stake on the first outcome giving an overall break-even wager. We also assume that the positive payoff in the hedge wager has the same probability of winning as the optimal

[3] The results are robust to use of other weighting functions such as that of Prelec (1998) which have qualitatively similar properties.

single wager and that the total stake size is the same in the hedge and single wager and equal at one.[4][5]

The expected value, EV, of the hedge wager is therefore given by

$$EV = w^{+}(p)o^{\alpha} - w^{-}(1-p-q)\lambda \qquad (2)$$

Where o is the winning odds of the hedge wager and p and q are the probabilities of the gain and break-even payoff in the hedge wager. The weighting function over losses, $w^{-}(1-p-q)$, is defined in the same manner as the weighting functions for the single wager in (1) above.

For a hedge wager to occur the expected value of the hedge wager (2) has to exceed that of the single wager (1). From (2) and (1) this implies that

$$1-\frac{w^{-}(1-p-q)}{w^{-}(1-p)} > 1-\frac{o^{\alpha}}{a^{\alpha}} \qquad (3)$$

[4]Power value functions exhibit the property that individuals become infinitely gain seeking as symmetric gains and losses tend to zero. (See e.g. Kobberling and Wakker (2005)). This has the counterfactual implication that the individual will optimally wager on any outcome regardless of how negative the subjective expected return and under-weighted the higher probabilities. Consequently stake size has to be arbitrarily determined.

[5] In general the optimal stake sizes are different in the single and hedge wager when alternative specifications of the value functions such as exponential are assumed. This enhances the expected value of the hedge wager. Examples are available on request

The condition (3) is informative. The ratio of the subjective probabilities of losing of the hedge and single wager must be smaller than the ratio of the gains from the hedge and single wager. Consequently the more risk –averse the value function over gains,[6] given by a smaller value of $\alpha$, the more likely hedge wagering is to occur for any given degree of probability distortion.

We next illustrate how hedge wagers can increase expected value when wagering on European roulette[7]. With a stake size of one the single wager that maximizes expected value in (1) is to bet on a single number, say number 20 at odds of 35/1. This single wager has a positive expected value of 0.161.

Consider the simple hedge wager constituting a stake of 0.5 on number 20 with odds of 35/1 and a stake of 0.5 on all even numbers with odds of 1/1. Consequently the individual wins 18, so o=18, if number 20 comes up with p=1/37, and breaks even if any even number apart from 20 comes up with q= 17/37. The expected value of this hedge wager is 0.173 and 7.46% higher than the expected value of single wager.

This example demonstrates that the representative individual in CPT of Tversky and Kahneman will always engage in hedge wagering when playing European roulette.

[6] This is also the case with other value functions such as exponential.

[7] In European Roulette a wheel is spun which has 37 numbers 1-36 and a zero. Punters can wager on one or combinations of the numbers 1-36 when a ball dropped on to the spinning numbers comes up when the wheel is stationary. The odds paid depend on the numbers covered. For example one number has odds of 35/1 whilst all even numbers, 1-18, has odds of even, 1/1. All bets are lost if zero comes up. The house has an expected return to a unit stake of 1/37. All single or combination wagers have therefore a negative expected return of -1/37 to a unit stake.

Moreover the individual would never have a single wager on the second outcome since it has negative expected value[8].

We can also demonstrate that hedge wagers will also often dominate single wagers in betting markets that exhibit the favorite-long-shot bias such as horse racing. For reasons of space results are available on request.

Since low probability outcomes are assumed over weighted and higher probabilities under weighted a hedge wager constituting a win stake on a horse with longer odds and a lay stake on a favorite appears feasible in CPT. However computation reveals that for all reasonable parameter values of the probability weighting functions this is not the case. The expected value of the win and lay bet is smaller than a single win bet even when the objective expected return to a single lay bet is assumed as small as –0.01 which is much higher than the actual figure observed in the betting exchanges of around –0.06.

We next demonstrate that the representative individual in RDU may engage in win and lay bets.

**Section 2 Rank Dependent Utility and Hedge Wagering**

Rank dependent utility (RDU) of Quiggin (1982) has the same specification as standard expected utility theory, with an everywhere-concave utility function, except that the objective probability of an event is replaced by the subjective probability. As with CPT low probabilities are assumed over-weighted and larger probabilities are under-weighted.

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[8] Employing alternative value functions, such as the exponential, we can demonstrate that the expected value of the hedge wager and its total stake size can exceed those of the optimal single wager by large percentages. Results are available on request.

The same cumulative weighting of subjective probabilities for multiple outcomes is as subsequently assumed in CPT of Tversky and Kahneman (1992) except the worst outcome rather than the best outcome is weighted first. We can show that individuals with RDU preferences will always engage in hedge bets when playing European roulette and may also engage in hedge bets in wagering markets that exhibit a favourite long shot bias such as horse racing. However unlike in CPT an individual with RDU preferences may engage in hedge wagers of the win and lay type. To illustrate we assume logarithmic utility though the results are qualitatively similar for any concave utility function.

The expected utility, EU, of a win and lay wager with betting wealth of one is given by

$$\mathrm{EU} = x\ln(1-s-tb) + (y-x)\ln(1+t-s) + (1-y)\ln(1+sa+t) \qquad (4)$$

Where s and t are the win and lay stakes and a and b are the win and lay odds.

The probability weighting functions are given by

$$x = (4-4c)(q)^3 - (6-6c)(q)^2 + (3-2c)(q)$$
$$y = (4-4c)(1-p)^3 - (6-6c)(1-p)^2 + (3-2c)(1-p)$$

Where p is the win probability of a horse, say a long shot, q the win probability of the favorite and c is a constant<1.

The probability weighting functions, x and y, are those proposed by Rieger and Wang (2006). With c<1 they exhibit over weighting of small probabilities and under weighting of larger probabilities. However they do not exhibit probability distortion when the objective probability is 0.5. This was an assumption of Quiggin (1982). We observe in (4) that there are three payoffs in a win and lay wager. The worst payoff is if the favorite wins with a negative payoff of s+tb. This is the stake on the outsider, s, plus the stake on the favorite, t, times the odds against the favorite not winning. The intermediate payoff is neither the favorite nor the outsider winning, t-s, and the best payoff is the outsider winning, sa+t.

The optimal stakes derived from (4) are given by

$$s = \frac{-yb(1+a) + ab - 1 + x(1+b)}{ab - 1}, \; t = \frac{y(1+a) - xa(1+b)}{ab - 1} \tag{5}$$

Clearly if the stakes s and t in (5) are positive the expected utility of the hedge wager must be greater than the expected utility of either the optimal single win or single lay wager at the same respective odds and probabilities. As an example we suppose c=0.5, win odds a=25/1, objective win probability p= 0.03615, lay odds b=0.68 and objective win probability q =0.63095. The objective expected returns per unit stake to either the single win or lay wager are approximately –0.06, typical of the expected return per unit stake in Betfair after tax on winning bets of 0.05. The optimal hedge stakes are $s = 0.0305$ and $t = 0.0175$ with expected utility $EU = 0.01$. The optimal single lay wager has a stake $t = 0.0624$ with expected utility $EU = 0.0013$. The optimal single win wager has a stake $s = 0.0312$ with expected utility $EU = 0.0099$.

In this example the optimal hedge wager only increases expected utility by a small amount more than the optimal single win wager. However the example is interesting since it reveals that the optimal hedge wager can exhibit a lower total stake size than the optimal single lay wager size and with a considerable increase in expected utility over the single lay wager.

## Conclusion

Our analysis reveals that the representative agents in CPT and RDU may obtain higher expected value or utility from betting on more than one outcome in an event - which we called a hedge wager - than can be obtained from a single wager. In CPT the representative agent will seemingly only engage in hedge wagers which constitute solely win bets. This differs from the representative individual in RDU who will engage in all types of hedge wagers including win and lay bets on the betting exchanges. . Consequently RDU unlike CPT is consistent with all the patterns of betting that punters engage in.

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