Hankel operators that commute with second-order differential operators

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Abstract

Suppose that Γ is a continuous and self-adjoint Hankel operator on $L^2(0,\infty)$ with kernel $\phi(x+y)$ and that $Lf = -\frac{d}{dx}(a(x)\frac{df}{dx}) + b(x)f(x)$ with $a(0) = 0$. If a and b are both quadratic, hyperbolic or trigonometric functions, and ϕ satisfies a suitable form of Gauss's hypergeometric differential equation, or the confluent hypergeometric equation, then $\Gamma L = L\Gamma$. The paper catalogues the commuting pairs Γ and L, including important cases in random matrix theory. There are also results proving rapid decay of the singular numbers of Hankel integral operators with kernels that are analytic and of exponential decay in the right half-plane.

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1. Introduction

For $\phi \in L^2(0,\infty)$, we recall that the Hankel operator Γ_ϕ with kernel $\phi(x+y)$ is the integral operator

$$\Gamma_\phi f(s) = \int_0^\infty \phi(s+t)f(t)\,dt \tag{1.1}$$

from a subspace of $L^2(0,\infty)$ into $L^2(0,\infty)$.

Megretskiĭ, Peller and Treil [6] determined the possible spectral multiplicity function that a continuous and self-adjoint Hankel operator can have; however, their spectral theory does not yield much information about the eigenvectors. Tracy and Widom observed that some self-adjoint and compact Hankel operators commute with self-adjoint second-order differential operators L that have purely discrete spectrum, and hence Γ_ϕ and L have a common orthonormal basis of eigenfunctions; see [9, 10]. One can then use the WKB approximation to describe the asymptotic eigenvalue distribution of L, and the asymptotic behaviour of the eigenfunctions.

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In this paper we start with L and seek choices of ϕ, thus reversing the order of steps in [9, 10]. Suppose that

$$Lf = -\frac{d}{dx}\Big(a(x)\frac{df}{dx}\Big) + b(x)f(x), \tag{1.2}$$

where a and b are three times differentiable functions such that $a(0) = 0$. Suppose that $a''' = Aa'$ and $b''' = Bb'$, for some constants A and B. In section 2 we derive an explicit differential equation

$$\phi''(u) + \alpha(u)\phi'(u) - \beta(u)\phi(u) = 0 \tag{1.3}$$

which ensures that $\Gamma_\phi L = L\Gamma_\phi$.

The possibilities for a and b depend upon the sign of A, and in sections 3, 4 and 5 we consider the quadratic, hyperbolic and trigonometric cases in detail. In all cases, the differential equation reduces to a linear differential equation with rational functions as coefficients which has less than or equal to three singular points, and we determine the nature of the singularities. Thus we prove the following theorem.

Theorem 1.1. *The differential equation (1.3) may be transformed by change of variables to the hypergeometric equation*

$$x(1-x)\frac{d^2\phi}{dx^2} + \{\lambda - (\mu+\nu+1)x\}\frac{d\phi}{dx} - \mu\nu\phi = 0 \tag{1.4}$$

or confluent hypergeometric equation.

This result covers cases relating to standard models in physics. In [11] Tracy and Widom considered the integral operators that have kernels of the form

$$W(x,y) = \frac{f(x)g(y) - f(y)g(x)}{x-y} \tag{1.5}$$

where f and g satisfy

$$m(x)\frac{d}{dx}\begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} p(x) & q(x) \\ -r(x) & -p(x) \end{bmatrix}\begin{bmatrix} f(x) \\ g(x) \end{bmatrix} \tag{1.6}$$

for some real polynomial functions $p(x), q(x), r(x)$ and $m(x)$. Of particular interest are the cases where $m(x) = 1, x$ or $1 - x^2$. The main application of Tracy–Widom operators W is in random matrix theory, where they describe the eigenvalue distributions of $n \times n$ random matrices from the generalized unitary ensemble as $n \to \infty$.

Tracy and Widom introduced an indirect process for computing the spectrum of W, and applied it effectively to the Airy and Bessel kernels which are fundamentally important cases in random matrix theory; see [9, 10]. Specifically, the edge distribution is given by $\det(I - WP_{(x,\infty)})$, where $P_{(x,\infty)}$ is the orthogonal projection $L^2(0,\infty) \to L^2(x,\infty)$ for $x > 0$. Their first step was to introduce a self-adjoint Hankel operator Γ such that $\Gamma^2 = W$, then to use the spectral theory of Hankel operators to deduce information about the spectrum of W.

The identity $W = \Gamma^2$ is equivalent to the factorization of kernels

$$\frac{f(x)g(y) - f(y)g(x)}{x - y} = \int_0^\infty \phi(x+t)\phi(y+t)\,dt. \tag{1.7}$$

Now Γ_ϕ is Hilbert–Schmidt if and only if $\int_0^\infty t|\phi(t)|^2 dt < \infty$. If Γ_ϕ is Hilbert–Schmidt, then Γ_ϕ^2 is of trace class, and $\det(I - \Gamma_\phi^2)$ is defined.

In [2, 3] we considered several differential equations of the form (1.5) and resolved whether factorization of W takes place, giving explicit formulae for ϕ; several important examples involve the confluent hypergeometric equation, which may be reduced by change of variable to Whittaker's equation. For a more general statement about reducing systems, and how changes of variable can affect factorization, see [3, section 3].

By carrying out the reduction in Theorem 1.1 explicitly, we recover the Tracy–Widom kernels as special cases of a more general theory, and also obtain new examples of commuting Γ_ϕ and L; there remain a few residual cases that we have not been able to solve explicitly in terms of standard functions.

In section 6 we present results concerning the rate of decay of the singular numbers of Γ under hypotheses that are well suited to applications in random matrix theory.

We follow standard notation as used in [4], and let $L_n^{(\alpha)}$ be the Laguerre polynomial of degree n and order α, so

$$L_n^{(\alpha)}(s) = \frac{s^{-\alpha}}{n!} e^s \frac{d^n}{ds^n}\big(e^{-s} s^{n+\alpha}\big) \qquad (s > 0). \tag{1.8}$$

2. The main result

Theorem 2.1. *Let Γ be the Hankel operator that has kernel $\phi(x+y)$, let L be the differential operator*

$$Lf(x) = -\frac{d}{dx}\Big(a(x)\frac{df}{dx}\Big) + b(x)f(x), \tag{2.1}$$

where $a'''(x) = Aa'(x)$ and $a(0) = 0$.

(i) Then there exists a real function α such that $\alpha(x+y) = (a'(x)-a'(y))/(a(x)-a(y))$.

(ii) Suppose further that $\beta(x+y) = (b(x)-b(y))/(a(x)-a(y))$ for some real function β and that

$$\phi''(u) + \alpha(u)\phi'(u) - \beta(u)\phi(u) = 0. \tag{2.2}$$

Then the operators Γ *and* L *commute on* $C_c^\infty(0,\infty)$.

Proof. Let $T_t : L^2(0,\infty) \to L^2(0,\infty)$ be the translation operator $T_t f(x) = f(x+t)$ and $T_t^\dagger$ the adjoint for $t > 0$. Then by the fundamental property of Hankel operators, we have $T_t\Gamma = \Gamma T_t^\dagger$ and hence $\frac{\partial}{\partial x}\Gamma = -\Gamma\frac{\partial}{\partial x}$ on $C_c^\infty(0,\infty)$. To exploit this, we introduce the expression

$$\Phi(x,y) = (a(y)-a(x))\phi''(x+y) + (a'(y)-a'(x))\phi'(x+y) - (b(y)-b(x))\phi(x+y); \quad (2.3)$$

then by successive integrations by parts, we obtain

$$(L\Gamma - \Gamma L)f(x) = \Big[\phi(x+y)a(y)f'(y) - \phi'(x+y)a(y)f(y)\Big]_0^\infty + \int_0^\infty \Phi(x,y)f(y)\,dy. \quad (2.4)$$

The term in square brackets vanishes since $a(0) = 0$; so the operators Γ and L commute if and only if $\Phi = 0$. The main idea is to reduce the condition $\Phi = 0$ to the differential equation

$$\big(a(x)-a(y)\big)\big(\phi''(x+y) + \alpha(x+y)\phi'(x+y) - \beta(x+y)\phi(x+y)\big) = 0, \quad (2.5)$$

involving functions α and β of only one variable $u = x+y$. The following lemma guarantees the existence of such α and β under suitable hypotheses, and hence gives the proof of Theorem 2.1.

Lemma 2.2. *Suppose that* a *and* b *are three times continuously differentiable and non constant real functions.*
(i) There exists a differentiable function α *such that*

$$\alpha(x+y) = \frac{a'(x)-a'(y)}{a(x)-a(y)} \quad (2.6)$$

if and only if $a'''(x) = Aa'(x)$ *for some constant* A.
(ii) If $a'''(x) = Aa'(x)$, *then* $a(x)-a(y) = f(x+y)g(x-y)$ *for some differentiable functions* f *and* g.
(iii) Suppose that, for some differentiable functions g *and* h,

$$\frac{b(x)-b(y)}{g(x-y)} = h(x+y). \quad (2.7)$$

Then there exists a constant B *such that* $b'''(x) = Bb'(x)$.

Proof. (i) ($\Rightarrow$) We have

$$0 =\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\Big)\Big(\frac{a'(x)-a'(y)}{a(x)-a(y)}\Big) \tag{2.8}$$

$$= \frac{(a''(x)a(x)-a'(x)^2)-(a''(y)a(y)-a'(y)^2)+(a(x)a''(y)-a''(x)a(y))}{(a(x)-a(y))^2}, \tag{2.9}$$

so we have

$$\frac{\partial^2}{\partial x\partial y}\Big(a(x)a''(y)-a''(x)a(y)\Big)=0, \tag{2.10}$$

which soon reduces to $a'''(x) = Aa'(x)$ for some constant A.

(i) ($\Leftarrow$) There are three main families of solutions of the differential equation $a'''(x) = Aa'(x)$, depending upon the sign of A:

(Q) quadratic case when $A = 0$ and $a(x) = q_2x^2 + q_1x$, with q_2 and q_1 not both zero, then

$$\alpha(u) = \frac{2q_2}{q_2u+q_1}; \tag{2.11}$$

(H) hyperbolic case where $A > 0$ and $a(x) = h_1 \cosh tx + h_2 \sinh tx + h_3$, with $t^2 = A$, $h_3 = -h_1$ and h_1 and h_2 not both zero, then $a(0) = 0$ and

$$\alpha(u) = \frac{th_1\cosh tu/2 + th_2 \sinh tu/2}{h_1 \sinh tu/2 + h_2 \cosh tu/2}; \tag{2.12}$$

(C) circular case when $A < 0$ and $a(x) = c_1 \cos tx + c_2 \sin tx + c_3$, where $t^2 = -A$, $c_3 = -c_1$, and c_1 and c_2 not both zero; then $a(0) = 0$ and

$$\alpha(u) = \frac{-tc_1\cos tu/2 - tc_2 \sin tu/2}{-c_1 \sin tu/2 + c_2 \cos tu/2}. \tag{2.13}$$

(ii) One considers cases (Q),(H) and (C), and applies the addition rule of trigonometry or hyperbolic trigonometry to factorize $a(x) - a(y)$.

(iii) When such g and h exist, we have

$$\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\Big)\Big(\frac{b(x)-b(y)}{g(x-y)}\Big)=0, \tag{2.14}$$

which reduces to

$$\frac{b'(x)+b'(y)}{b(x)-b(y)}=\frac{2g'(x-y)}{g(x-y)}, \tag{2.15}$$

and arguing as in (i), we deduce that $b'''(x) = Bb'(x)$ for some constant B.

□

Remark. Without loss of generality we can assume that a and b are bounded above or below on $(0,\infty)$. If $a(x) \geq 0$ for all $x > 0$ and b is bounded below, then the quadratic form associated with L is bounded below on $C_c^\infty(0,\infty)$, and hence L admits of a self-adjoint extension by Friedrichs's theorem.

3. Quadratic cases

In this section, we determine all the quadratic and linear choices of a and b and the corresponding ϕ such that L and Γ_ϕ commute. We look for α and β as in Theorem 2.1.

Lemma 3.1. *Suppose that $a(x) = q_2x^2 + q_1x + q_0$. Then β of Theorem 2.1 exists if and only if b is quadratic or linear; so that $b(x) = b_2x^2 + b_1x + b_0$, and then*

$$\beta(x) = \frac{b_2x + b_1}{q_2x + q_1}. \tag{3.1}$$

Proof. We need to find β and b such that

$$\frac{b(x) - b(y)}{x - y} = (q_2(x + y) + q_1)\beta(x + y), \tag{3.2}$$

so $b'''(x) = Bb'(x)$ by Lemma 2.2. Hence b must belong to one of the types (Q), (H) and (C), and evidently by (2.15) only a quadratic has the right form.

□

Proposition 3.2. *The differential equation (2.2) for ϕ is*

$$-\phi''(u) - \frac{2q_2}{q_2u + q_1}\phi'(u) + \frac{b_2u + b_1}{q_2u + q_1}\phi(u) = 0, \tag{3.3}$$

which may be transformed by change of variables to an elementary or confluent form of the hypergeometric differential equation.

Proof. This formula follows from Theorem 2.1 and Lemma 3.1. The differential equation (3.3) has less than or equal to two singular points, the only possibilities being $-q_1/q_2$ and infinity. Determining the nature of the singularities in all possible cases below, we find that cases Q(i) and Q(ii) are trivial; cases Q(iii) and Q(iv) are elementary; in cases Q(iv)-(viii), infinity is an irregular singular point; in cases Q(vi)-(viii) $-q_1/q_2$ is a regular singular point. We deduce that as a special case of [12, p. 352] and [4, p. 1084], the solution is either an elementary function or may be expressed in terms of confluent hypergeometric functions.

□

Examples 3.3. *(Quadratic cases).* We now carry out a systematic reduction of the (3.3), bearing in mind that we wish to have a continuous Hankel operator Γ_ϕ.

Q(i) If $q_1 = q_2 = 0$, then $a(x) = 0$ and $b(x)$ is constant by (2.3).

After translating the variable u, we assume henceforth without losing generality that exactly one of q_1 and q_2 is non-zero.

Q(ii) If $q_2 = b_2 = b_1 = 0$, then ϕ is linear, so does not give a continuous Hankel operator.

Q(iii) If $b_2 = b_1 = 0$, $q_2 > 0$ and $q_1 \geq 0$, then

$$\phi(u) = C_1 + \frac{C_2}{q_2 u + q_1}. \tag{3.4}$$

In particular, when $q_2 = 1$, $q_1 = 0$, $C_1 = 0$ and $C_2 = 1$, we thus obtain Carleman's operator Γ with kernel $1/(x + y)$. Carleman's operator is continuous on $L^2(0, \infty)$ and has spectrum $[0, \pi]$ with multiplicity two by a result due to Power [6]. Both Γ and the differential operator $L = -\frac{d}{dx}\left(x^2 \frac{d}{dx}\right)$ are multipliers for the Mellin transform, so this case of Theorem 2.1 was to be expected.

Q(iv) If $q_2 = b_2 = 0$ and $b_1, q_1 \neq 0$, then ϕ is hyperbolic or trigonometric according to the sign of b_1/q_1. In particular, $b_1 = q_1$ gives $\phi(u) = e^{-u}$, and Γ_ϕ is a rank-one continuous Hankel operator. We defer the case of $b_1/q_1 < 0$ until section 5.

Q(v) If $q_2 = 0$ and $b_2 \neq 0$, then we obtain the Airy equation. In particular, if $a(x) = x$ and $b(x) = x(x + s)$, then ϕ satisfies $\phi''(u) = (u + s)\phi(u)$, so the Airy function $\phi(u) = \mathrm{Ai}(u + s)$ gives a solution. Here the associated Hankel operator Γ_ϕ is Hilbert–Schmidt, and L has discrete spectrum as described by Titchmarsh's theory of oscillations [8]; Tracy and Widom discuss the asymptotics of the common sequence of eigenfunctions in various ranges [9]. The Hankel operator Γ_ϕ has square

$$\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}, \tag{3.5}$$

which is the famous Airy kernel as used in [9] to describe the soft edge of eigenvalues distributions from the Gaussian unitary ensemble.

When $q_1 = 1$ and $b_2 < 0$, we obtain $\phi''(u) + u\phi(u) = 0$ by translation and rescaling, and the general solution of this is

$$\phi(u) = C_1 u^{1/2} J_{1/3}\Big(\frac{2}{3}u^{3/2}\Big) + C_2 u^{1/2} J_{-1/3}\Big(\frac{2}{3}u^{3/2}\Big), \tag{3.6}$$

for typical C_1 and C_2 with asymptotics [5, p.133]

$$\phi(u) \asymp C u^{-1/4} \cos\big(\frac{2}{3}u^{3/2} + c\big) \qquad (u \to \infty).$$

Q(vi) If $b_2 = 0$ and $b_1, q_2 \neq 0$, then we obtain after translation and rescaling the equation

$$-\phi''(u) - \frac{2}{u}\phi'(u) \pm \frac{1}{4u}\phi(u) = 0 \tag{3.7}$$

with solutions including [5, p.250]

$$\phi_+(u) = u^{-1/2}K_1(\sqrt{u}), \qquad \phi_-(u) = u^{-1/2}J_1(\sqrt{u}). \tag{3.8}$$

We have by [5, p. 435]

$$\phi_+(u) \asymp \begin{cases} \frac{1}{u} & \text{as } u \to 0+ \\ \left(\frac{\pi}{2}\right)^{1/2} u^{-3/4} e^{-\sqrt{u}} & \text{as } u \to \infty, \end{cases} \tag{3.9}$$

so Γ_{ϕ_+} defines a continuous linear operator on $L^2(0,\infty)$ which is not Hilbert–Schmidt; but the compression of Γ_{ϕ_+} to $L^2(x,\infty)$ is Hilbert–Schmidt for $x > 0$. Further by [5],

$$\phi_-(u) \asymp \begin{cases} \frac{1}{2} & \text{as } u \to 0+ \\ \left(\frac{\pi}{2}\right)^{1/2} u^{-3/4} \cos(\sqrt{u} - 3\pi/4) & \text{as } u \to \infty, \end{cases} \tag{3.10}$$

so Γ_{ϕ_-} is not Hilbert–Schmidt. As in [3, 2.3], the associated integrable operator is

$$\frac{x\phi_-(x)(y\phi_-(y))' - (x\phi_-(x))' y\phi_-(y)}{x-y} = \frac{1}{4}\int_0^\infty \phi_-(x+t)\phi_-(y+t)\,dt.$$

Q(vii) If $b_2, q_2 \neq 0$, then we obtain after translation the equation

$$-\phi''(u) - \frac{2}{u}\phi'(u) + \Big(\frac{b_2}{q_2} + \frac{b_1}{q_2 u}\Big)\phi(u) = 0. \tag{3.11}$$

In particular, when $b_1 = 0$, we can rescale to the equation

$$-\phi''(u) - \frac{2}{u}\phi'(u) \pm \phi(u) = 0$$

with solutions

$$\phi_+(u) = \frac{e^{-u}}{u}, \qquad \phi_-(u) = \frac{\kappa_1 \cos u + \kappa_2 \sin u}{u}. \tag{3.12}$$

By Theorem 2.1 of [3], with $f(x) = e^{-x}$ and

$$g(x) = e^{-x}\int_0^\infty \frac{e^{-2t}\,dt}{x+t},$$

we have

$$\frac{f(x)g(y) - f(y)g(x)}{x-y} = \int_0^\infty \phi_+(x+t)\phi_+(y+t)\,dt. \tag{3.13}$$

This Γ_{ϕ_+} defines a continuous linear operator on $L^2(0,\infty)$, such that the compression to $L^2(x,\infty)$ is Hilbert–Schmidt for $x>0$.

Further, Γ_{ϕ_-} defines a continuous linear operator on $L^2(0,\infty)$ since the kernel is a sum of Schur multiples of the Carleman operator, namely

$$\phi_-(x+y)=\kappa_1\Bigl(\frac{\cos x\cos y}{x+y}-\frac{\sin x\sin y}{x+y}\Bigr)+\kappa_2\Bigl(\frac{\sin x\cos y}{x+y}+\frac{\cos x\sin y}{x+y}\Bigr). \tag{3.14}$$

With $f(x)=e^{ix}$ and

$$g(x)=e^{ix}\int_0^\infty \frac{e^{2it}dt}{x+t}, \tag{3.15}$$

we have by Theorem 2.1 of [3] the identity

$$\frac{f(x)g(y)-f(y)g(x)}{x-y}=\int_0^\infty \frac{e^{i(x+y+2t)}dt}{(x+t)(y+t)}. \tag{3.16}$$

Q(viii) Now we consider (3.11) when $q_2=b_2=1$, and $b_1=-2(n+1)$. Then $g(u)=u\phi(2^{-1}u)$ satisfies Laguerre's equation

$$g''(u)+\Bigl(\frac{-1}{4}+\frac{n+1}{u}\Bigr)g(u)=0, \tag{3.17}$$

so that, when $n+1$ is a positive integer, $g(u)=ue^{-u/2}L_n^{(1)}(u)$ and $\phi(u)=e^{-u}L_n^{(1)}(2u)$ gives a solution. See 2.2 of [3] for the corresponding Tracy–Widom operator.

This gives a complete catalogue of the possible quadratic cases.

4. Hyperbolic cases

In this section we consider the case in which a and b are hyperbolic functions, giving some L, and obtain Γ in terms of standard special functions such that L and Γ commute.

Without loss of generality we can choose $t=2$, since other cases occur by rescaling. The change of variables $x=e^{-2u}$ gives a unitary transformation $L^2((0,\infty);du)\to L^2((0,1);dx/x)$, and we modify the definition of the Hankel operator accordingly.

Definition *(Hankel operator).* For $\rho\in L^2((0,1);dx/x)$, the Hankel operator with kernel $\rho(xy)$ is

$$\Gamma_\rho h(x)=\int_0^1 \rho(xy)h(y)\frac{dy}{y}, \tag{4.1}$$

where h is in some subspace of $L^2((0,1);dy/y)$.

Lemma 4.1. *Suppose that a is hyperbolic, so $a(x)=h_1\cosh tx+h_2\sinh tx+h_3$ where h_1 and h_2 are not both zero. Then β of Theorem 2.1 exists if and only if $b(x)=h_4\cosh tx+h_5\sinh tx+h_6$ for some constants and then*

$$\beta(v)=\frac{h_4\sinh tv/2+h_5\cosh tv/2}{h_1\sinh tv/2+h_2\cosh tv/2}. \tag{4.2}$$

Proof. This follows from Lemma 2.2 since we need to find β and b such that

$$\frac{b(x)-b(y)}{\sinh t(x-y)/2} = 2\big(h_1 \sinh t(x+y)/2 + h_2 \cosh t(x+y)/2\big)\beta(x+y). \tag{4.3}$$

Hence $b'''(x) = Bb'(x)$, and of the types (Q), (C) and (H), only a hyperbolic b has the right form with $v = x+y$.

□

Proposition 4.2. *(i) With $t=2$ and $x = e^{-2u}$, the differential equation (2.2) for ϕ becomes in the new variable*

$$\frac{d^2\phi}{dx^2} + \frac{2(h_2-h_1)}{h_1+h_2+(h_2-h_1)x}\frac{d\phi}{dx} - \frac{h_4+h_5+(h_5-h_4)x}{4x^2(h_1+h_2+(h_2-h_1)x)}\phi = 0, \tag{4.4}$$

and the commuting differential operator L becomes

$$-\frac{d}{dx}\Big(2x\big\{(h_1-h_2)x^2 + 2h_3x + (h_1+h_2)\big\}\frac{df}{dx}\Big) + \Big(\frac{x}{2}(h_4-h_5) + h_6 + \frac{1}{2x}(h_4+h_5)\Big)f.$$

(ii) The equation (4.4) may be reduced by change of variables to the hypergeometric equation or the confluent hypergeometric equation.

Proof. (i) We have

$$\alpha = \frac{2h_1(1+x)+2h_2(1-x)}{h_1(1-x)+h_2(1+x)} \qquad (0<x<1), \tag{4.5}$$

and by Lemma 4.1 we have

$$\beta = \frac{h_4(1-x)+h_5(1+x)}{h_1(1-x)+h_2(1+x)} \qquad (0<x<1), \tag{4.6}$$

so (2.2) reduces as stated. One obtains the formula for L by changing variables.

(ii) In 4.3 below, we consider the nature of the singular points of the differential equation. Effectively there are four constants in (4.4), namely $h_1 \pm h_2$ and $h_4 \pm h_5$; by taking $h_3 = -h_1$, we ensure that $a(0) = 0$. One can easily verify that the effect of the change of variable $x = 1/y$ is to preserve the shape of the formula (4.4) and to interchange the constants $h_1 + h_2 \leftrightarrow h_2 - h_1$ and $h_4 + h_5 \leftrightarrow h_5 - h_4$.

In cases H(i), H(iii) and H(vii) below, there are two regular singular points, so the solution is elementary; in cases H(ii) and H(iv), the singular points are zero and infinity, and one of them is irregular, so the equation is confluent hypergeometric type; whereas in the remaining cases H(v) and H(vi), the three singular points are all regular, so the equation is of hypergeometric type.

$\square$

Examples 4.3 *(Hyperbolic cases).*

H(i) If $h_1 = h_2$ and $h_4 = h_5$, then the differential equation (4.4) reduces to

$$\phi''(u) - \frac{h_4}{4h_1u^2}\phi(u) = 0, \tag{4.7}$$

which has solutions $\phi(u) = u^p$ where $4p(p-1) = h_4/h_1$. When $p > 0$, the corresponding Hankel operator is Hilbert–Schmidt.

H(ii) If $h_1 = h_2$ and $h_4 \neq h_5$, we obtain after rescaling

$$\phi''(u) + \frac{1-\nu^2}{4u^2}\phi(u) = \pm\frac{1}{u}\phi(u), \tag{4.8}$$

with $\nu > 0$ where the solutions are

$$\phi_+(u) = \sqrt{u}K_\nu(2\sqrt{u}), \qquad \phi_-(u) = \sqrt{u}J_\nu(2\sqrt{u}). \tag{4.9}$$

Indeed, ϕ_+ emerges for the choice $h_1 = h_2 = -1$, $h_3 = 1$, $h_4 = 5 - \nu^2$ and $h_5 = -3 - \nu^2$; whereas ϕ_- emerges for $h_1 = h_2 = -1$, $h_3 = 1$, $h_4 = -3 - \nu^2$ and $h_5 = 5 - \nu^2$. We have

$$\phi_+(u) \asymp \begin{cases} 2^{-1}\Gamma(\nu)u^{(1-\nu)/2} & \text{as } u \to 0+ \\ 2^{-1}\pi^{1/2}u^{1/4}e^{-2\sqrt{u}} & \text{as } u \to \infty, \end{cases} \tag{4.10}$$

so Γ_{ϕ_+} defines a Hilbert–Schmidt operator on $L^2((0,1); dx/x)$ when $\nu < 1$.

Further, by [5, p.436] we have

$$\phi_-(u) \asymp \begin{cases} \Gamma(\nu+1)^{-1}u^{(1+\nu)/2} & \text{as } u \to 0+ \\ \pi^{-1/2}u^{1/4}\cos\left(2\sqrt{u} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) & \text{as } u \to \infty; \end{cases} \tag{4.11}$$

so Γ_{ϕ_-} defines a Hilbert–Schmidt operator on $L^2((0,1); dx/x)$. The associated Tracy–Widom operator on $L^2((0,1); dy/y)$ has kernel

$$\frac{\sqrt{x}J_\nu'(2\sqrt{x})J_\nu(2\sqrt{y}) - J_\nu(2\sqrt{x})\sqrt{y}J_\nu'(2\sqrt{y})}{x-y} = \int_0^1 J_\nu(2\sqrt{tx})J_\nu(2\sqrt{ty})dt, \tag{4.12}$$

and the commuting differential operator is

$$L_\nu f(x) = 4\frac{d}{dx}\Big(x(1-x)\frac{df}{dx}\Big) + \Big(-4x + \frac{1-\nu^2}{x} - 2\mu\Big)f(x) \tag{4.13}$$

with boundary conditions $f(0) = f(1) = 0$. In random matrix theory, this is associated with hard edges, such as occur with the Jacobi ensemble.

H(iii) If $h_1 = -h_2$ and $h_5 = -h_4$, then we obtain

$$\phi''(u) + \frac{2}{u}\phi'(u) - \frac{h_5}{4h_2u^2}\phi(u) = 0, \tag{4.14}$$

so we have solutions $\phi(u) = u^p$ where $4p^2 + 4p - h_5/h_2 = 0$.

H(iv) If $h_1 = -h_2$ and $h_4 \neq -h_5$, then we change variables to $\phi(x) = \psi(y)$ where $y = 1/x$ and obtain

$$\frac{d^2\psi}{dy^2} + \Big(\frac{h_4 + h_5}{8h_1y} + \frac{h_5 - h_4}{h_1y^2}\Big)\psi(y) = 0. \tag{4.15}$$

As in case H(ii), we thus we obtain solutions

$$\phi_+(x) = \frac{1}{\sqrt{x}}K_\nu\Big(\frac{2}{\sqrt{x}}\Big), \quad \phi_-(x) = \frac{1}{\sqrt{x}}J_\nu\Big(\frac{2}{\sqrt{x}}\Big).$$

By (4.10), ϕ_+ gives a Hilbert–Schmidt Hankel operator on $L^2((0,1); dx/x)$; whereas, by (4.11), ϕ_- gives a Hankel operator which is not Hilbert–Schmidt.

H(v) When $h_1 \neq 0$, $h_2 = 0$ and $h_4 = -h_5 \neq 0$, the equation (4.4) reduces to hypergeometric equation. In particular, if $h_1 = 1/4$, $h_2 = 0$, $h_4 = -h_5$, and μ and ν satisfy $\mu + \nu = 1$ and $\mu\nu = -2h_4$, then we have (1.4) with $\lambda = 0$ and nonzero parameters μ and ν; however, the usual series for the hypergeometric function $F(\mu, \nu, \lambda, x)$ is then undefined. By an identity from [4, p. 1073], the function

$$\phi(x) = xF(\mu + 1, \nu + 1, 2, x), \tag{4.16}$$

gives a power series solution which is analytic for $|x| < 1$, with $\phi(0) = 0$, and diverges everywhere on the circle $\{z : |z| = 1\}$ by [4, p. 1066].

H(vi) In the generic case $h_1 \neq \pm h_2$ and $h_4 \neq \pm h_5$, we introduce the regular singular point $\zeta = -(h_1 + h_2)/(h_2 - h_1)$ and parameters $\alpha_1, \alpha_2, \beta_1$ and β_2 by the simultaneous quadratic equations

$$\alpha_1\alpha_2 = -\frac{h_4 + h_5}{4(h_1 + h_2)}, \qquad \beta_1\beta_2 = -\frac{h_5 - h_4}{4(h_2 - h_1)},$$

$$\alpha_1 + \alpha_2 = 1, \qquad \beta_1 + \beta_2 = 1. \tag{4.17}$$

Then the differential equation (4.4) is given in Riemann's notation [12, 5] by

$$\phi(x) = P\left\{\begin{matrix} 0 & \infty & \zeta & \\ \alpha_1 & \beta_1 & -1; & x \\ \alpha_2 & \beta_2 & 0 & \end{matrix}\right\}, \tag{4.18}$$

and hence by [5, p 156], may be reduced to the hypergeometric equation (1.4).

H(vii) Finally, when $h_1 \neq h_2$ and $h_4 = h_5 = 0$, as in Q(iii) we have elementary solutions

$$\phi(x) = C_1 + \frac{C_2}{(h_2 - h_1)x + h_2 + h_1}. \tag{4.19}$$

5. Circular cases

In this section we consider the remaining case, namely when ϕ, a and b are circular functions. The results of this section are quite analogous to those of H(v) and H(vi); although they are somewhat contrived, since the notion of a Hankel integral operator over the circle is not in common use.

Suppose that a, b and ϕ have period $2\pi/t$, and that $a(0) = a(2\pi/t) = 0$. We extend $f \in C_c^\infty(0,\infty)$ by $f(x) = 0$ for $x < 0$, and introduce $F(y) = \sum_{k=-\infty}^{\infty} f(y + 2\pi k/t)$, which is $2\pi/t$ periodic. Evidently

$$\int_0^\infty \phi(x+y)f(y)dy = \int_0^{2\pi/t} \phi(x+y)F(y)dy \tag{5.1}$$

is also $2\pi/t$ periodic.

Definition *(Hankel operator).* The Hankel operator on $L^2((0,2\pi/t);dy)$ with kernel $\phi(x+y)$ is

$$\Gamma F(x) = \int_0^{2\pi/t} \phi(x+y)F(y)\,dy.$$

We consider Γ as an operator on $C^\infty(0,2\pi/t)$, and look for a second-order differential operator L as in (1.2) such that $L\Gamma = \Gamma L$.

Lemma 5.1. *Suppose that $a(x) = c_1 \cos tx + c_2 \sin tx + c_3$ is circular, where c_1 and c_2 are not both zero. Then β of Theorem 2.1 exists if and only if $b(x) = c_4 \cos tx + c_5 \sin tx + c_6$ for some constants and then*

$$\beta(u) = \frac{-c_4 \sin tu/2 + c_5 \cos tu/2}{-c_1 \sin tu/2 + c_2 \cos tu/2}. \tag{5.2}$$

Proof. This follows from Lemma 2.2 since we need to find β and b such that

$$\frac{b(x) - b(y)}{\sin t(x-y)/2} = 2\Big(-c_1 \sin t(x+y)/2 + c_2 \cos t(x+y)/2\Big)\beta(x+y). \tag{5.3}$$

Hence $b'''(x) = Bb'(x)$, and of the types (Q), (H) and (C), only a circular b has the right form.

□

Proposition 5.2. *(i) Let $\tau = \tan u$ and $t = 2$. Then differential equation (2.2) for ϕ becomes in the new variable*

$$\frac{d^2\phi}{d\tau^2} + \frac{2c_1}{c_1\tau - c_2}\frac{d\phi}{d\tau} - \frac{-c_4\tau + c_5}{-c_1\tau + c_2}\frac{\phi(\tau)}{(1+\tau^2)^2} = 0, \tag{5.4}$$

and the commuting differential operator transforms to

$$-(1+\tau^2)\frac{d}{d\tau}\Big(\{c_1(1-\tau^2) + 2c_2\tau + c_3(1+\tau^2)\}\frac{df}{d\tau}\Big) + \Big(c_4\frac{1-\tau^2}{1+\tau^2} + c_5\frac{2\tau}{1+\tau^2} + c_6\Big)f.$$

(ii) The equation (5.4) may be reduced by change of variables to the hypergeometric equation.

Proof. (i) This follows from Theorem 2.1 and Lemma 5.1 by calculation.

(ii) The differential equation has regular singular points: at c_2/c_1, when $c_1 \neq 0$; at ∞, when $c_1 = 0$; and at $\pm i$, when $c_4 \neq 0$ or $c_5 \neq 0$. When $c_4 = c_5 = 0$, the equation has elementary solutions, as in H(vii) and Q(iii). The effect of the change of variable $\tau = 1/s$ is to preserve the shape of (5.4) and to interchange the constants $c_1 \leftrightarrow c_2$ and $c_4 \leftrightarrow c_5$. By [5, p. 156], the differential equation may be reduced by change of variables to Gauss's hypergeometric equation.

□

Examples 5.3 *(Circular case).* We present solutions of the differential equations in the special case where the singular points are $0, \pm i$. Let $c_1 = -1$, $c_2 = 0$, $c_4 = 3$ and $c_5 = 0$; so that, the differential equation in the original variables is

$$\phi''(x) + 2\cot x\, \phi'(x) + 3\phi(x) = 0, \tag{5.5}$$

which has general solution

$$\phi(x) = c_7 \cos x - c_8(\operatorname{cosec} x - 2\sin x). \tag{5.6}$$

The Hankel operator on $L^2((0,\pi); dx)$ with kernel $\cos(x+y)$ has rank two and eigenfunctions $\cos x$ and $\sin x$. In the new variable, the solution is the algebraic function

$$\phi(\tau) = \frac{c_7\tau + c_8(\tau^2 - 1)}{\tau\sqrt{\tau^2+1}}.$$

Proof of Theorem 1.1. The Propositions 3.2, 4.2 and 5.2 cover the three cases that together give the proof of Theorem 1.1.

□

6. Singular numbers of Hankel integral operators

Let Γ_ϕ be the Hankel operator on $L^2(0,\infty)$ with kernel $\phi(x+y)$. In applications to random matrix theory, the ultimate aim of the analysis is to prove properties of $\det(I-\Gamma_\phi^2)$. If $W=\Gamma_\phi^2$, and W has eigenvalues $\lambda_1\geq\lambda_2\geq\dots$ listed according to multiplicity, then the singular numbers of Γ_ϕ are $s_j=\lambda_j^{1/2}$ for $(j=1,2,\dots)$. In this section we prove results which show that if ϕ is analytic on a suitable domain and satisfies various growth bounds, then Γ_ϕ is a trace-class operator and that its singular numbers are of rapid decay. An important feature of the results is that if $\phi(z)$ satisfies the hypotheses, then the translated functions $\phi_s(z)=\phi(s+z)$ also satisfy them for all $s>0$, up to modified constants. This reflects the fact that the compression of Γ_ϕ^2 to $L^2(s,\infty)$ is unitarily equivalent to $\Gamma_{\phi_s}^2$ on $L^2(0,\infty)$.

Definition *(Hankel matrix).* Suppose that $(\gamma_j)_{j=1}^\infty$ is a sequence in $\ell^2(\mathbf{Z}_+)$. Then the densely defined linear operator Γ with matrix $[\gamma_{j+k-1}]$ with respect to the standard orthonormal basis of $\ell^2(\mathbf{Z}_+)$ is a Hankel operator.

Theorem 6.1. *Let ϕ be an analytic function on the half plane $\{z=x+iy: x>-\delta\}$ for some $\delta>0$, and suppose that there exists $\varepsilon>0$ and $M_\varepsilon(\delta)>0$ such that*

$$|\phi(z)e^{\varepsilon z}|\leq M_\varepsilon(\delta)\qquad(\Re z>-\delta).\tag{6.1}$$

(i) Then Γ_ϕ has singular numbers s_j such that

$$j^p s_j\to 0\qquad(j\to\infty)\tag{6.2}$$

for all integers $p>0$.
(ii) For all integers $p>1$, there exists $C_p>0$ such that

$$\log\det(I+x\Gamma_\phi^2)\leq C_p x^{1/p}\qquad(x>0),\tag{6.3}$$

and the entire function $\det(I+z\Gamma_\phi^2)$ has order zero.

Proof. (i) The operator with kernel $\varepsilon^{-1}\phi(\varepsilon^{-1}(x+y))$ is unitarily equivalent to the operator with kernel $\phi(x+y)$, and $\varepsilon^{-1}\phi(z/\varepsilon)$ satisfies (6.1) with $\varepsilon=1$, $M_1(\varepsilon\delta)=\varepsilon^{-1}M_\varepsilon(\delta)$ on $\{z:\Re z>-\varepsilon\delta\}$. Hence we assume without loss of generality that ϕ has been rescaled so that it satisfies (6.1) with $\varepsilon=1$. Let $h_n(s)=e^{-s}L_n^{(0)}(2s)$, so that $(h_j)_{j=0}^\infty$ gives an orthonormal basis of $L^2(0,\infty)$, and is associated with a unitary equivalence between $L^2(0,\infty)$ and $\ell^2(\mathbf{Z}_+)$. Then the Hankel integral operator Γ_ϕ on $L^2(0,\infty)$ is unitarily equivalent to the Hankel matrix $\Gamma = [\gamma_{j+k-1}]_{j,k=1}^{\infty}$ on $\ell^2(\mathbf{Z}_+)$, where $\gamma_j = \int_0^\infty \phi(x)h_j(x)\,dx$ as in [6]. We shall show that the entries of Γ are of rapid decay as $j+k\to\infty$.

By integrating repeatedly by parts, we find that

$$\begin{aligned}\gamma_j &= \int_0^\infty \phi(x)h_j(x)\,dx\\ &= \frac{(-1)^j}{j!}\int_0^\infty \Big(\frac{d^j}{dx^j}\big(e^x\phi(x)\big)\Big)x^j e^{-2x}\,dx \end{aligned}\tag{6.4}$$

where by (6.1) $e^z\phi(z)$ is analytic and bounded inside the circle with centre x and radius $x+\delta$. Applying Cauchy's estimates, we obtain

$$x^j\Big|\frac{d^j}{dx^j}\big(e^x\phi(x)\big)\Big| \le j!M_1(\delta)\Big(\frac{x}{x+\delta}\Big)^j \tag{6.5}$$

and hence

$$|\gamma_j| \le M_1(\delta)\int_0^\infty \frac{x^j}{(x+\delta)^j}e^{-2x}\,dx, \tag{6.6}$$

and summing this estimate over j we obtain by the monotone convergence theorem

$$\begin{aligned}\sum_{j=1}^\infty j^{p+1}|\gamma_j| &\le M_1(\delta)\int_0^\infty \sum_{j=0}^\infty \frac{(j+p+1)!}{j!}\Big(\frac{x}{x+\delta}\Big)^j e^{-2x}dx\\ &\le \frac{M_1(\delta)}{\delta^{p+2}}\int_0^\infty (x+\delta)^{p+2}e^{-2x}dx,\end{aligned}\tag{6.7}$$

where the last step follows from the binomial theorem.

We approximate Γ by the Hankel matrix $\Gamma_N = [\gamma_{j+k-1}\mathbf{I}_{\{(j,k):j+k\le N\}}]$, which is zero outside the top left corner and has rank less than or equal to N. By considering approximation numbers [7], we find that

$$s_N \le \|\Gamma - \Gamma_N\|_{B(\ell^2)}, \tag{6.8}$$

and since the norm of a matrix is smaller than the absolute sum of its entries, we deduce that

$$\begin{aligned}s_N &\le \sum_{j=N+1}^\infty j|\gamma_j|\\ &\le N^{-p}\sum_{j=1}^\infty j^{p+1}|\gamma_j|\\ &\le \frac{M_1(\delta)}{2\delta^{p+2}N^p}\Big((2\delta)^{p+2}+\Gamma(p+3)\Big),\end{aligned}\tag{6.9}$$

where we have used (6.7) at the last step. We can repeat the argument with $p+1$ instead of p, and hence deduce that $j^p s_j \to 0$ as $j \to \infty$.

(ii) We introduce the counting function $n(t) = \sharp\{j : ts_j^2 \geq 1\}$, and observe that by (6.9),

$$n(t) \leq c_p t^{1/p} \qquad (t > 0) \tag{6.10}$$

for some $c_p > 0$, since $s_j^2 \leq c_p j^{-p}$ for all j and $n(t) = 0$ for $0 < t < \|\Gamma_\phi\|^{-2}$. A standard summation formula then gives

$$\begin{aligned}\log\det(I + x\Gamma_\phi^2) &= \log\prod_{j=1}^\infty (1 + xs_j^2)\\ &= x\int_0^\infty \frac{n(t)dt}{t(x+t)}\\ &\leq c_p x^{1/p}\pi\mathrm{cosec}\,(\pi/p).\end{aligned} \tag{6.11}$$

This gives the asserted bound on the growth of $\log\det(I+x\Gamma_\phi^2)$, and shows that $\det(I+z\Gamma_\phi^2)$ has order zero.

□

The following applies to Q(vi), Q(vii) and Q(viii) of Examples 3.3.

Corollary 6.2. *Let ϕ be either:*
(i) $e^{-x}L_n^{(1)}(2x)$;
(ii) $(x+s)^{-1}e^{-(x+s)}$*; or*
(iii) $(x+s)^{-1/2}K_\nu(2\sqrt{x+s})$ *where* $s > 0$.
Then the eigenvalues of the Hankel operator Γ_ϕ on $L^2(0,\infty)$ are of rapid decay, so $j^p s_j \to 0$ as $j \to \infty$ for all integers p.

Proof. Theorem 6.1 applies directly to ϕ in cases (i) and (ii), and we can adapt the proof of Theorem 6.1 to deal with ϕ in case (iii). Let $z = re^{i\theta}$ and $0 < \delta < s$. One can show that there exists a constant C such that

$$|(z+s)^{1/2}K_\nu(2\sqrt{z+s})e^z| \leq C(s-\delta)^{-1/2}e^{r\cos\theta - (r\cos\theta)^{1/2}} \qquad (\Re z > -\delta), \tag{6.12}$$

and hence that

$$|\psi(z)e^z| \leq C(s-\delta)^{-1/2}e^{2x+\delta-(2x+\delta)^{1/2}} \tag{6.13}$$

when z lies on the circle with centre x and radius $x+\delta$. Now we can apply Cauchy's estimates as in (6.5), and then follow the proof of Theorem 6.1.

□

Let the Fourier transform of f be $\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx/\sqrt{2\pi}$.

Proposition 6.3. *Let ϕ be an analytic function on the strip $\{z = x + iy : |y| < \sigma\}$ and suppose that there exist K, $\delta > 0$ such that $|\phi(z)| \leq Ke^{-\delta|x|}$ for all real x and $|y| < \sigma$. Then there exist C_1 and $\kappa_2 > 0$ such that the singular numbers of the Hankel operator Γ_ϕ with kernel $\phi(x + y)$ all satisfy*

$$s_N(\Gamma_\phi) \leq C_1 e^{-\kappa_2 N^{1/3}}. \tag{6.14}$$

Proof. By shifting the line of integration in the Fourier integral, one can show that $\mathcal{F}\phi$ is analytic on the strip $\{\zeta = \xi + i\eta : |\eta| < \delta\}$. Further, there exist C_3 and $\varepsilon > 0$ such that $|\mathcal{F}\phi(\xi + i\eta)| \leq C_3 e^{-\varepsilon|\xi|}$ for all $\xi + i\eta$ on this strip.

The Fourier transform of h_n is easy to compute, and by Plancherel's theorem we have

$$\gamma_n = \int_0^\infty \phi(x) h_n(x)\, dx = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}\phi(\xi) \frac{(\xi - i)^n}{(\xi + i)^{n+1}} d\xi. \tag{6.15}$$

We shift the line of integration up to $\{\zeta = \xi + i\delta : -\infty < \xi < \infty\}$, and split the range of integration; the middle range of integration $[-\xi_0, \xi_0]$ is dominated by the rational factor, where we observe that

$$\Big| \frac{(\xi + i\delta - i)^n}{(\xi + i\delta + i)^{n+1}} \Big| \leq \Big(\frac{\xi_0^2 + (1-\delta)^2}{\xi_0^2 + (1+\delta)^2} \Big)^{n/2} \qquad (|\xi| \leq \xi_0); \tag{6.16}$$

meanwhile, the contributions from $|\xi| \geq \xi_0$ are bounded by

$$\Big| \int_{-\infty}^{-\xi_0} + \int_{\xi_0}^{\infty} \frac{(\xi + i\delta - i)^n}{(\xi + i\delta + i)^{n+1}} \mathcal{F}\phi(\xi + i\delta)\, d\xi \Big| \leq \int_{-\infty}^{-\xi_0} + \int_{\xi_0}^{\infty} |\mathcal{F}\phi(\xi + i\delta)| d\xi. \tag{6.17}$$

In particular, with $\xi_0 = n^{1/3}$, we obtain the bound

$$|\gamma_n| \leq 2n^{1/3} C_3 \Big(\frac{n^{2/3} + (1-\delta)^2}{n^{2/3} + (1+\delta)^2} \Big)^{n/2} + \frac{2C_3}{\varepsilon} e^{-\varepsilon n^{1/3}}. \tag{6.18}$$

Due to a similar approximation argument as (6.7), the singular numbers of Γ_ϕ consequently satisfy (6.14).

□

This result applies to an orthonormal sequence of functions which is used to analyse the Gaussian orthogonal ensemble as in [1].

Corollary 6.4. *Let ϕ_n be the n^{th} Hermite function. Then the singular numbers of the Hankel operator Γ_{ϕ_n} on $L^2(0,\infty)$ with kernel $\phi_n(x + y)$ are of rapid decay as in (6.14).*

Proof. The hypotheses of Proposition 6.3 hold for $\phi_n(z) = H_n(z)e^{-z^2/4}$, where H_n is the Hermite polynomial of degree n.

$\square$

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