Accepted Manuscript

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Please cite this article as: Zhou, S., Wan, F., Liu, W., Bretz, F., Computation of an exact confidence set for a maximum point of a univariate polynomial function in a given interval. *Statistics and Probability Letters* (2016), http://dx.doi.org/10.1016/j.spl.2016.10.032

Computation of an exact confidence set for a maximum point of a univariate polynomial function in a given interval

Sanyu Zhou[a], Fang Wan[b,*], Wei Liu[c], Frank Bretz[d]

[a]*School of Urban and Regional Science, Shanghai University of Finance and Economics, Shanghai 200433, China*
[b]*Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, UK*
[c]*S3RI and School of Mathematics, University of Southampton, Southampton SO17 1BJ, UK*
[d]*Novartis Pharma AG, Basel 4002, Switzerland*

Abstract

Construction of a confidence set for a maximum point of a function is an important statistical problem. Wan et al. (2015) provided an exact $1 - \alpha$ confidence set for a maximum point of a univariate polynomial function in a given interval. In this paper, we give an efficient computational method for computing the confidence set of Wan et al. (2015). We demonstrate with two examples that the new method is substantially more efficient than the proposals by Wan et al. (2015). Matlab programs have been written which make the implementation of the new method straightforward.

Keywords: Confidence set, Numerical quadrature, P-value, Statistical inference, Parametric regression, Semi-parametric regression

1. Introduction

Determination of the maximum point of a function is an important problem due to its wide applications; see e.g. Wan et al. (2015, 2016) and the references therein. Consider the function

$$l(x, \theta_0, \boldsymbol{\theta}) = \theta_0 + \theta_1 x + \cdots + \theta_p x^p = \theta_0 + f(x, \boldsymbol{\theta}), \qquad (1)$$

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*Corresponding author
Email address: `f.wan@lancaster.ac.uk` (Fang Wan)

where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)^T$ and $f(x, \boldsymbol{\theta}) = \theta_1 x + \cdots + \theta_p x^p$ contains all the information on the maximum points of $l(x, \theta_0, \boldsymbol{\theta})$. The interest is in the maximum points of $l(x, \theta_0, \boldsymbol{\theta})$ in a given interval of $x \in [a, b]$. It is clear that the maximum points of $l(x, \theta_0, \boldsymbol{\theta})$ do not depend on θ_0 and hence are the same as those of $f(x, \boldsymbol{\theta})$. If the value of $\boldsymbol{\theta}$ is known, then this is a simple calculus problem. The difficulty lies in that the value of $\boldsymbol{\theta}$ is unknown and only an estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ with a certain distribution, specified in (2) and (3) below, is available. Hence the maximum points of $l(x, \theta_0, \boldsymbol{\theta})$ can only be estimated based on $\hat{\boldsymbol{\theta}}$. We assume that the estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ has the normal distribution

$$\hat{\boldsymbol{\theta}} \sim N(\boldsymbol{\theta}, \sigma^2 \boldsymbol{\Sigma}), \tag{2}$$

where $\boldsymbol{\Sigma}$ is a known positive definite covariance matrix, and $\hat{\sigma}^2$ is an available estimator of the unknown error variance σ^2 with distribution $\hat{\sigma}^2 \sim \sigma^2 \chi_v^2/v$ independent of $\hat{\boldsymbol{\theta}}$, where v is the degrees of freedom (df) of the chi-square distribution. In the special case that σ^2 is a known constant, then $\hat{\sigma}^2 = \sigma^2$ for $v = \infty$ and we can assume without loss of generality that

$$\hat{\boldsymbol{\theta}} \sim N(\boldsymbol{\theta}, \boldsymbol{\Sigma}). \tag{3}$$

The distributional assumption (2) follows naturally from the standard pth order univariate polynomial regression model: $y = \theta_0 + f(x, \boldsymbol{\theta}) + e$, where e is the usual random error with distribution $N(0, \sigma^2)$. Based on n observations (y_i, x_i), $i = 1, \cdots, n$, $\boldsymbol{\Sigma}$ results from deleting the first row and the first column of $(X^T X)^{-1}$, where X is the usual $n \times (p+1)$ design matrix. The distributional assumption (3) holds asymptotically for many parametric and semi-parametric models (cf. Wan et al., 2015). A maximum point of $f(x, \boldsymbol{\theta})$ in a covariate interval of $x \in [a, b]$ may represent the dose in the range [a,b] that maximizes the response, or the amount of fertilizer that produces the hightest yield, etc.

Let k be a global maximum point of $f(x, \boldsymbol{\theta})$ in $x \in [a, b]$. Wan et al. (2015), denoted as WLHB henceforth, provides an exact $1 - \alpha$ level confidence set for k by inverting a family of exact $1 - \alpha$ level acceptance sets for testing $H_0 : k = k_0$ for each $k_0 \in [a, b]$. This method computes one critical constant $c(k_0)$ for each

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$k_0 \in [a, b]$, using numerical quadrature and search for $p = 2$ and using statistical simulation for $p \geq 3$.

In this paper, we first show that the acceptance probability used in the computation of a critical point can be computed using $p - 1$ dimensional numerical quadrature. Hence, when $p = 4$ for example, the acceptance probability can be computed using 3-dimensional numerical quadrature, which is usually much faster than statistical simulation to achieve the same computational accuracy. As polynomial regression models of order higher than 4 are rarely used in applications (cf. Liu et al., 2014), the computation method developed in this paper for computing the exact confidence set in WLHB is of considerable practical relevance.

We further show that, for testing $H_0 : k = k_0$, we only need to compute the p-value of the test, which requires the computation of one acceptance probability only. This requires substantially less computation time than the computation of the critical constant $c(k_0)$, which involves repeated computation of the acceptance probability for several candidate values of $c(k_0)$, whose corresponding acceptance probability is equal to $1 - \alpha$.

The paper is organized as follows. Section 2 gives a brief review of the WLHB confidence set. Section 3 considers how the acceptance probability can be computed efficiently at least for $p \leq 4$ using numerical quadrature. Section 4 then shows how to determine whether a $k_0 \in [a, b]$ belongs to the confidence set by computing just one acceptance probability. Section 5 illustrates the new computational method with two examples to demonstrate the substantial saving of computation time and the improved accuracy over the computation methods given in WLHB. Finally, section 6 contains discussions and conclusions.

2. WLHB method

Assume that k is a maximum point of $f(x; \boldsymbol{\theta})$ in $x \in [a, b]$. Let $\mathbf{Y}$ be the random observation based on which the estimate $\hat{\boldsymbol{\theta}}$ is computed. Wan et al. (2015) uses the following $1 - \alpha$ level acceptance set for testing $H_0 : k = k_0$ for

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each $k_0 \in [a, b]$:

$$A(k_0) = \{\mathbf{Y} : f(k_0; \hat{\boldsymbol{\theta}}) - f(x; \hat{\boldsymbol{\theta}}) \geq -c(k_0)\hat{v}(k_0, x), \ \forall x \in [a, b] \backslash k_0\},$$

where

$$
\begin{aligned}
\hat{v}(k_0, x) &= (\hat{\sigma}/\sigma)\sqrt{var(f(k_0; \hat{\boldsymbol{\theta}}) - f(x; \hat{\boldsymbol{\theta}}))} \\
&= \hat{\sigma}|k_0 - x|\sqrt{g(k_0, x, p)^T \boldsymbol{\Sigma} g(k_0, x, p)}
\end{aligned}
$$

with

$$g(k_0, x, p) = \frac{(k_0 - x, k_0^2 - x^2, \cdots, k_0^p - x^p)^T}{k_0 - x},$$

and $c(k_0)$ is the critical value chosen so that the acceptance probability is equal to $1 - \alpha$, that is,

$$
\begin{aligned}
&\inf_{H_0} P\{\mathbf{Y} \in A(k_0)\} \\
&= \inf_{H_0} P\{\mathbf{Y} : f(k_0, \hat{\boldsymbol{\theta}}) - f(x_0, \hat{\boldsymbol{\theta}}) \geq -c(k_0)\hat{v}(k_0, x), \ \forall x \in [a, b] \backslash k_0\} \\
&= \inf_{H_0} P\{\mathbf{Y} : \sup_{x \in [a,b] \backslash k_0} -\frac{(k_0 - x)}{|k_0 - x|} \frac{g(k_0, x, p)^T(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})}{\hat{\sigma}\sqrt{g(k_0, x, p)^T \boldsymbol{\Sigma} g(k_0, x, p)}} \leq c(k_0)\} \quad = 1 - \alpha.
\end{aligned}
\tag{4}
$$

The $1 - \alpha$ level confidence set of Wan et al. (2015, 2016) is then given by

$$\mathbf{C}_E(\mathbf{Y}) = \{k \in [a, b] : \mathbf{Y} \in A(k)\}.$$

The key of the WLHB method for the construction of $\mathbf{C}_E(\mathbf{Y})$ is the computation of the critical constant $c(k_0)$ for each $k_0 \in [a, b]$. A simulation-based method to obtain $c(k_0)$ is given for a general $p \geq 2$, and for the special case $p = 2$, a numerical method is provided involving one-dimensional numerical quadrature.

3. Computation of the acceptance probability

Throughout this section, we assume σ is unknown. Let $\mathbf{P}$ denote the unique positive definite square-root matrix of $\boldsymbol{\Sigma}$ and so $\mathbf{P}^2 = \boldsymbol{\Sigma}$. Then we have $\mathbf{N} :=$

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$\mathbf{P}^{-1}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})/\sigma \sim N(\mathbf{0}, \mathbf{I}_p)$. Furthermore, since $\hat{\sigma}/\sigma$ and $\hat{\boldsymbol{\theta}}$ are independent random variables and $\hat{\sigma}/\sigma \sim \sqrt{\chi_\nu^2/\nu}$, we have that $\mathbf{T} = \mathbf{P}^{-1}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})/\hat{\sigma} = \mathbf{N}/(\hat{\sigma}/\sigma)$ is a standard p-dimensional t random vector with ν degrees of freedom, and the Euclidean norm of $\mathbf{T}$ has the distribution $||\mathbf{T}|| \sim \sqrt{pF_{p,\nu}}$, where $F_{p,\nu}$ denotes an F distributed random variable with p and ν degrees of freedom. Define the polar coordinates $(R_T, \psi_{T_1}, \cdots, \psi_{T_{p-1}})^T$ of the vector $\mathbf{T} = (T_0, T_1, \cdots, T_{p-1})^T$ by

$$
\begin{cases}
T_0 = R_T \cos \psi_{T_1} \\
T_1 = R_T \sin \psi_{T_1} \cos \psi_{T_2} \\
T_2 = R_T \sin \psi_{T_1} \sin \psi_{T_2} \cos \psi_{T_3} \\
\cdots \quad\quad \cdots \quad\quad \cdots \\
T_{p-2} = R_T \sin \psi_{T_1} \sin \psi_{T_2} \cdots \sin \psi_{T_{p-2}} \cos \psi_{T_{p-1}} \\
T_{p-1} = R_T \sin \psi_{T_1} \sin \psi_{T_2} \cdots \sin \psi_{T_{p-2}} \sin \psi_{T_{p-1}} \, ,
\end{cases}
$$

where

$$
\begin{cases}
0 \le \psi_{T_1} \le \pi \\
\cdots \quad \cdots \\
0 \le \psi_{T_{p-2}} \le \pi \\
0 \le \psi_{T_{p-1}} < 2\pi \\
R_T \ge 0 \, .
\end{cases}
$$

The Jacobian of the transformation is

$$
|J| = R_T^{p-1} sin^{p-2}\psi_{T_1} sin^{p-3}\psi_{T_2} \cdots sin\psi_{T_{p-2}}.
$$

It is well known (cf. Liu et al., 2014, 2012) that $||\mathbf{T}|| = R_T$ and $(\psi_{T_1}, \cdots, \psi_{T_{p-1}})^T$ are independent, and the joint density function of $(\psi_{T_1}, \ldots, \psi_{T_{p-1}})^T$ is

$$
h(\psi_{T_1}, \cdots, \psi_{T_{p-1}}) = \frac{1}{2}\pi^{-p/2}\Gamma\left(p/2\right) sin^{p-2}\psi_{T_1} sin^{p-3}\psi_{T_2} \cdots sin\psi_{T_{p-2}},
$$

where $\Gamma(\cdot)$ denotes the gamma function.

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Now the acceptance probability in (4) can be expressed as

$$\inf_{H_0} P\{\mathbf{Y} : \sup_{x \in [a,b] \setminus k_0} -\frac{(k_0 - x)}{|k_0 - x|} \frac{[\mathbf{P}g(k_0, x, p)]^T \mathbf{P}^{-1}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})/\hat{\sigma}}{\sqrt{[\mathbf{P}g(k_0, x, p)]^T [\mathbf{P}g(k_0, x, p)]}} \leq c(k_0)\}$$

$$= P\{\mathbf{Y} : \sup_{x \in [a,b] \setminus k_0} -\frac{(k_0 - x)}{|k_0 - x|} \frac{[\mathbf{P}g(k_0, x, p)]^T \mathbf{T}}{\|\mathbf{P}g(k_0, x, p)\|} \leq c(k_0)\}$$

$$= P\{-\inf_{x \in [a,k_0)} \frac{[\mathbf{P}g(k_0, x, p)]^T \mathbf{T}}{\|\mathbf{P}g(k_0, x, p)\|} \leq c(k_0) \text{ and } \sup_{x \in (k_0,b]} \frac{[\mathbf{P}g(k_0, x, p)]^T \mathbf{T}}{\|\mathbf{P}g(k_0, x, p)\|} \leq c(k_0)\}$$

$$= P\{\|\mathbf{T}\| S_{k_0} \leq c(k_0)\}$$

$$= P\{\|\mathbf{T}\| \leq c(k_0)/S_{k_0}\}, \tag{5}$$

where $\mathbf{T} = \mathbf{P}^{-1}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})/\hat{\sigma}$ and $S_{k_0} = max(Q_{k_0}, R_{k_0})$ with

$$Q_{k_0} = -\inf_{x \in [a,k_0)} \frac{[\mathbf{P}g(k_0, x, p)]^T \boldsymbol{u}}{\|\mathbf{P}g(k_0, x, p)\|}, \ R_{k_0} = \sup_{x \in (k_0,b]} \frac{[\mathbf{P}g(k_0, x, p)]^T \boldsymbol{u}}{\|\mathbf{P}g(k_0, x, p)\|}$$

and $\boldsymbol{u} = \mathbf{T}/\|\mathbf{T}\|$. Note that Q_{k_0} and R_{k_0} depend on $\mathbf{T}$ only through $(\psi_{T_1}, \cdots, \psi_{T_{p-1}})^T$. Since $\|\mathbf{T}\|$ and $(\psi_{T_1}, \cdots, \psi_{T_{p-1}})^T$ are independent and $\|\mathbf{T}\| \sim \sqrt{pF_{p,\nu}}$, the expression in (5) can be written as

$$\int_{\psi_{T_1}=0}^{\pi} \cdots \int_{\psi_{T_{p-2}}=0}^{\pi} \int_{\psi_{T_{p-1}}=0}^{2\pi} h(\psi_{T_1}, \cdots, \psi_{T_{p-2}}, \psi_{T_{p-1}})$$
$$\times F_{p,\nu}(c_{k_0}^2/(pS_{k_0}^2(\psi_{T_1}, \cdots, \psi_{T_{p-2}}, \psi_{T_{p-1}})))d\psi_{T_1} \cdots d\psi_{T_{p-2}} d\psi_{T_{p-1}}, \tag{6}$$

85 where $F_{p,\nu}(\cdot)$ denotes the cumulative distribution function of an F random variable.

For a given $(\psi_{T_1}, \cdots, \psi_{T_{p-1}})$, the method given in WLHB can be used to accurately and quickly compute Q_{k_0}, R_{k_0} and so S_{k_0}. The function $F_{p,v}(\cdot)$ can be computed efficiently by using the Matlab built-in function $fcdf$. Hence the
90 integral in expression (6) can be computed quickly at least for small values of p. For example, expression (6) involves one dimensional integration for $p = 2$ and three dimensional integration for $p = 4$.

WLHB provided an alternative expression for the acceptance probability when $p = 2$, which also involves one-dimensional integration. As the derivation
95 of that expression uses the geometry of the acceptance region, the method is difficult to be generalized. On the other hand, the expression (6) is derived

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using only algebra. From the expression (6), one can compute the acceptance probability for each given value of $c(k_0)$. A numerical searching method, such as the bisection method, can then be used to find the critical constant $c(k_0)$ so

100 that the acceptance probability is equal to $1 - \alpha$. This is used in Wan et al. (2015) for $p = 2$ only to find $c(k_0)$, which is then used to determine whether each k_0 belongs to the confidence set $C_E(Y)$. Note, however, that in the process of finding $c(k_0)$, the acceptance probability in (6) needs to be computed several times for different candidate values of $c(k_0)$.

105 In the next section, it is shown that the acceptance probability in (6) needs to be computed only once for one particular value of $c(k_0)$ in order to determine whether k_0 belongs to the confidence set $C_E(Y)$.

4. P-value Method

From the definition of the acceptance set $A(k_0)$ and the confidence set $\mathbf{C}_E(\mathbf{Y})$ in Section 2, a given point $k_0 \in [a, b]$ belongs to $\mathbf{C}_E(\mathbf{Y})$ if and only if

$$P\{ \sup_{x \in [a,b] \setminus k_0} -\frac{(k_0 - x)}{|k_0 - x|} \frac{[\mathbf{P}g(k_0, x, p)]^T \mathbf{T}}{\sqrt{g(k_0, x, p)^T \mathbf{\Sigma} g(k_0, x, p)}} \leq c^* \} \leq 1 - \alpha, \qquad (7)$$

where

$$c^* = \sup_{x \in [a,b] \setminus k_0} -\frac{(k_0 - x)}{|k_0 - x|} \frac{g(k_0, x, p)^T \hat{\boldsymbol{\theta}}/\hat{\sigma}}{\sqrt{g(k_0, x, p)^T \mathbf{\Sigma} g(k_0, x, p)}}$$

with $\hat{\boldsymbol{\theta}}$ and $\hat{\sigma}$ being the estimates of $\boldsymbol{\theta}$ and σ based on the observed data. For

110 the given $\hat{\boldsymbol{\theta}}$ and $\hat{\sigma}$, c^* can be computed accurately and quickly by using the method given in WLHB. Hence we need to compute the acceptance probability in (7) only once and compare it with $1 - \alpha$ in order to determine whether k_0 is in $\mathbf{C}_E(\mathbf{Y})$. This is much faster than the computation of $c(k_0)$, which requires the computation of several acceptance probabilities.

115 In essence, this method uses the p-value of the test based on the acceptance set $A(k_0)$ for testing $H_0 : k = k_0$ to determine whether H_0 is accepted or rejected. We therefore call this the p-value method.

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5. Examples

In this section two data examples, one from WLHB and the other from Liu
120 et al. (2014), are used to demonstrate that the p-value method is substantially
more efficient than the WLHB method. All the computations were done on
an ordinary windows laptop (Windows edition: Windows Vista^{TM} Home Basic,
Processor: Intel(R) Core(TM)2 Duo CPU T7100 @ 1.80GHz, Installed memory:
2534 MB, System type: 32-bit Operating System).

125 WLHB showed that $c_0 = \sqrt{pF(1-\alpha; p, v)}$ is a conservative critical value
which can be used to quickly compute the conservative confidence set $\mathbf{C}_0(\mathbf{Y})$.
Moreover, the exact $1-\alpha$ level confidence set $\mathbf{C}_E(\mathbf{Y})$ can be computed efficiently
by checking only the points k_0 in $\mathbf{C}_0(\mathbf{Y})$ to see whether they belong to $\mathbf{C}_E(\mathbf{Y})$.
This is used in our programs for the computation of the two examples.

130 The first example involves a fourth order polynomial regression model of the
transformed perinatal mortality rate (PMR), $Y = log(-log(PMR))$, and the
birth weight (BW) (x); see WLHB (Example 1) for more details. Based on the
data, we have $\hat{\boldsymbol{\theta}} = (4.18, 1.80, 0.42, -0.04)^T$,

$$\boldsymbol{\Sigma} = \begin{pmatrix} 175.673 & -116.729 & 31.771 & -3.039 \\ * & 78.591 & -21.628 & 2.087 \\ * & * & 6.010 & -0.585 \\ * & * & * & 0.057 \end{pmatrix},$$

$\hat{\sigma} = 0.0567$ and $v = 30$. It is of interest to identify the BW level in the observed
135 range $[a, b] = [0.85, 4.25]$ that maximizes the response Y (i.e., minimizes the
PMR). We compare the methods from WLHB with the proposed method in
this paper by constructing a $1 - \alpha = 95\%$ level confidence set for this optimal
BW level. We replace the interval $[a, b]$ by the grid of points that are $d = 0.01$
distance apart, and check each k_0 in the grid to see whether it belongs to the
140 confidence set.

We first construct the conservative confidence set $\mathbf{C}_0(\mathbf{Y})$ by using $c_0 = \sqrt{4F(0.95; 4, 30)} = 3.280$, which is given by $\mathbf{C}_0(\mathbf{Y}) = [3.72, 4.25]$ and takes
0.4194 seconds to compute. Next, we check whether each grid point k_0 in $\mathbf{C}_0(\mathbf{Y})$

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belongs to the exact confidence set $\mathbf{C}_E(\mathbf{Y})$. The WLHB method computes each

145 $c(k_0)$ based on $n = 100,000$ simulations; this takes about 22.0123 seconds for each k_0, and the accuracy of the acceptance probability is about $1 - \alpha \pm 0.002$ (Wan et al., 2015, pp. 564); The exact confidence set is $\mathbf{C}_E(\mathbf{Y}) = [3.75, 4.21]$, taking 3015.6904 seconds all together. The new method computes one acceptance probability for each grid point k_0 in $\mathbf{C}_0(\mathbf{Y})$, using 3-dimensional numerical

150 quadrature. Setting the accuracy of numerical quadrature at $1 - \alpha \pm 0.001$, the computation of one acceptance probability takes about 0.8570 seconds. The exact confidence set is $C_E = [3.76, 4.20]$, taking 117.8277 seconds all together. It is clear from this example that the computation time of the new method is only about 3.91% of that used by the WLHB method, while the new method

155 achieves a better accuracy than the WLHB method.

The second example involves a third order polynomial regression model for modelling the mean dose response; see Liu et al. (2014, Example 2) for more details. Based on the data, we have $\hat{\boldsymbol{\theta}} = (0.0953, -5.186 \times 10^{-4}, 7.5 \times 10^{-7})^T$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 8.6247 \times 10^{-5} & -6.3460 \times 10^{-7} & 1.0699 \times 10^{-9} \\ * & 5.0830 \times 10^{-9} & -8.8834 \times 10^{-12} \\ * & * & 1.5787 \times 10^{-14} \end{pmatrix}.$$

160 It is of interest to identify the dose level in the observed range $[a, b] = [0, 400]$ that maximizes the mean response. We use the methods of WLHB and this paper to construct a $1 - \alpha = 95\%$ level confidence set for this optimal dose level. We replace the interval $[a, b]$ by the grid of points that are $d = 1$ distance apart, and check each k_0 in the grid whether it belongs to the confidence set.

165 The conservative confidence set $\mathbf{C}_0(\mathbf{Y})$ with $c_0 = \sqrt{3F(0.95; 3, 38)} = 2.9249$ is $\mathbf{C}_0(\mathbf{Y}) = [109, 166]$, taking 0.2736 seconds. Next, we check whether each grid point k_0 in $\mathbf{C}_0(\mathbf{Y})$ belongs to the exact confidence set $\mathbf{C}_E(\mathbf{Y})$. The WLHB method computes each $c(k_0)$ based on $n = 100,000$ simulations; this takes about 54.6896 seconds for each k_0, and the accuracy of the acceptance probability is

170 about $1 - \alpha \pm 0.002$; The exact confidence set is $\mathbf{C}_E(\mathbf{Y}) = [111, 156]$, taking 3172

seconds all together. The new method computes one acceptance probability for each grid point k_0 in $\mathbf{C}_0(\mathbf{Y})$, using one 2-dimensional numerical quadrature. Setting the accuracy of numerical quadrature at $1 - \alpha \pm 0.001$, the computation of one acceptance probability takes about 0.0975 seconds. The exact confidence

175 set is $C_E = [112, 155]$, taking 7.48 seconds all together. It is clear from this example that the computation time of the new method is only about 0.23% of that used by the WLHB method, while the new method achieves a better accuracy than the WLHB method.

For quadratic polynomial regression, WLHB already used 1-dimensional

180 quadrature to compute the acceptance probability, employing the Matlab built-in function *fzero* to search for the critical constant $c(k_0)$ for each given k_0. As *fzero* combines efficiently the bisection, secant and inverse quadratic interpolation methods, it often takes no more than four computations of the acceptance probabilities to find the $c(k_0)$. Also, the computation of one acceptance proba-

185 bility, involving one 1-dimensional numerical quadrature, often takes a fraction of a second, e.g. 0.001 second, and the total computation time of $\mathbf{C}_E(\mathbf{Y})$ is just a few seconds, e.g. about 18 second for the example given in WLBH, when the computational error tolerance is set at 10^{-4}. Hence the computational saving of the new method of this paper over the WLBH method is not of practical

190 importance for quadratic polynomial regression.

From the examples above and several other data sets we have tried, the p-value method requires substantially less time to compute the confidence set $\mathbf{C}_E(\mathbf{Y})$ than the WLHB method for cubic and quartic polynomial regressions at least.

195 **6. Conclusions**

In this article, the new p-value method is given for computing the confidence set $\mathbf{C}_E(\mathbf{Y})$ of WLHB.

The p-value method hinges on the efficient computation of an acceptance probability, and this is accomplished by using numerical quadrature for at least

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200 $p \leq 4$, involving no more than a 3-dimensional integral, which is much faster than using statistical simulation to achieve a better accuracy. It is computationally efficient also because it needs to compute one acceptance probability only, while finding the critical constant $c(k_0)$ requires the computation of several acceptance probabilities, when judging whether k_0 belongs to $\mathbf{C}_E(\mathbf{Y})$.

205   As the polynomial regression of order higher than four is rarely used in applications, the new method is sufficient for most real problems. Matlab programs for implementing the new method for $p \leq 4$ have been written and are available from the authors, which make the new method easily applicable.

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