Universality of testing ghost-free gravity

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In this paper we show that there is a universal prediction for the Newtonian potential for an *infinite derivative, ghost-free, quadratic curvature* gravity. We show that in order to make such a theory *ghost-free* at a perturbative level, the Newtonian potential always falls-off as $1/r$ in the infrared limit, while at short distances the potential becomes non-singular. We provide examples which can potentially test the scale of gravitational non-locality up to 0.01 eV.

Einstein's theory of General Relativity (GR) has passed successfully through innumerable tests from small scales to large scales [1]. One of its predictions, of the existence of gravitational waves, has recently been confirmed by the advanced Laser Interferometer Gravitational-Wave Observatory (LIGO), which has observed a transient gravitational-wave (GW) signal and tested the reliability of GR [2]. In all these examples, in the infrared (IR), the theory matches the Newtonian fall of $1/r$ potential. In spite of these great successes, the theory of GR is incomplete in the ultraviolet (UV), the classical solutions of GR exhibit black hole and cosmological type singularities, and at a quantum level the theory is not UV finite. GR definitely requires modifications in the UV; the question is what kind of corrections in the UV one would expect, which would make the theory well behaved in the classical and in quantum sense, and possibly resolve the short distance singularities.

For a massless graviton, in 4 dimensions, all the interactions in the UV can in *principle* be captured by incorporating higher derivatives allowed by the *diffeomorphism-invariance*. For instance, it is well-known that higher derivatives can ameliorate the UV behaviour, i.e. *4th* derivative gravity is renormalisable, but at a cost of introducing a *ghost* term in the spin-2 component of a graviton propagator [3]. Indeed, the presence of ghosts can lead to destabilising the classical vacuum, therefore rendering the theory unpredictable at both classical and at a quantum level.

Recently, the issue of ghosts has been addressed in the context of *quadratic* gravity - in order to make the theory generally covariant and *ghost-free* at the perturbative level, one would require infinite derivatives [4, 5]. Indeed, these *infinite* derivatives would modify the graviton propagator. However, if we capture the roots of these infinite derivatives by an *entire function*, then there will be no new degrees of freedom propagating in spacetime other than the massless transverse and traceless graviton, since *entire functions* do not introduce any new poles.

It has been demonstrated that these infinite derivatives with a graviton propagator modified by an entire function can indeed soften the quantum UV behaviour [6–9]. Furthermore, such a prescription also removes the cosmological Big Bang singularity [5, 11], and blackhole type singularity in a static limit [4], and in the dynamical context [12], in a linearised limit. One intuitive way to understand this is due to the fact that infinite derivatives render the gravitational interactions non-local [6, 9]. This non-locality also introduces an inherent new scale in 4 dimensions, i.e. $M \leq M_p \sim 2.4 \times 10^{18}$ GeV. Furthermore, an intriguing connection can be established between the gravitational entropy [14], and the propagating degrees of freedom in the spacetime. The gravitational entropy for ghost-free, infinite gravity does not get a contribution from the UV, but only from the Einstein-Hilbert action [15], and follows strictly the area - law for entropy for a Schwarzschild's black hole.

The aim of this paper is two-fold: first we show that for a wide class of infinite derivative theories of gravity which are *ghost-free*, it is possible to recover *not only* the $1/r$ fall of the Newtonian potential in a static limit in the IR, but also to ameliorate the short distance behaviour in the UV limit. Second, we wish to put a bound on the scale of non-locality, i.e. M, from the current table-top experiments from deviation of Newtonian gravity.

Let us first start by discussing the properties of GR in 4 dimensions. The linearised GR can be quantised around the Minkowski background, which is described by 2 massless degrees of freedom. The transverse and traceless components of the graviton propagator in 4 dimensions can be recast in terms of the spin projector operators, which involves the tensor $\mathcal{P}^{(2)}$, and only one of the scalar components, i.e. $\mathcal{P}_s^{(0)}$ [16]:

$$\Pi(k^2) \sim \frac{1}{k^2}\left(\mathcal{P}^{(2)} - \frac{1}{2}\mathcal{P}_s^{(0)}\right) , \qquad (1)$$

where k^μ is the 4-momentum vector, where we have suppressed the spacetime indices.

In fact, in Refs. [4, 18] it has been shown that around the Minkowski background, in 4 dimensions, the most general quadratic order gravitational action which can be made *ghost-free* can be written in terms of the Ricci-scalar, R, the symmetric traceless tensor, $S_{\mu\nu} = R_{\mu\nu} -$

$\frac{1}{4}Rg_{\mu\nu}$, analog with the Einstein tensor, $R_{\mu\nu}$, which is the Ricci tensor, and $C_{\mu\nu\alpha\beta}$ is the Weyl tensor. The S-tensor vanishes on maximally symmetric backgrounds (Minkowski or (anti)-de Sitter) [18] [1]:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + \frac{\lambda}{2}\Bigg(R\mathcal{F}_1(\Box)R + S_{\mu\nu}\mathcal{F}_2(\Box)S^{\mu\nu} + C_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)C^{\mu\nu\lambda\sigma}\Bigg)\right], \quad (2)$$

where M_P^2 is the Planck mass, and λ is a dimensional coupling accounting for the higher curvature modification, and the $\mathcal{F}_i$ are Taylor expandable (i.e. analytic) functions of the covariant d'Alembertian [4], i.e. $\mathcal{F}_i(\Box) = \sum_{n=0} c_{i_n}\Box^n/M^{2n}$, where M is the scale of non-locality.

The equations of motion of this action have been worked out in Ref. [10]. As we shall show now, this class of infinite derivative theory indeed provides a unique platform to study departure from GR in future *table-top* experiments [19].

Physical excitations of this action, Eq. (2), around Minkwoski background have been studied very well. This can be computed by the second variation of the action, using $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Hereafter, bars denote background Minkowski quantities. A quick computation can be made by employing the covariant mode decomposition of the metric [20]:

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \bar{\nabla}_\mu A_\nu + \bar{\nabla}_\nu A_\mu + (\bar{\nabla}_\mu\bar{\nabla}_\nu - \frac{1}{4}\bar{g}_{\mu\nu}\bar{\Box})B + \frac{1}{4}\bar{g}_{\mu\nu}h, \quad (3)$$

where $\tilde{h}_{\mu\nu}$ is the transverse and traceless spin-2 excitation, A_μ is a transverse vector field, and B, h are two scalar degrees of freedom which mix. Upon linearization around maximally symmetric backgrounds, the vector mode and the double derivative scalar mode vanish identically, and we end up only with $\tilde{h}_{\mu\nu}$ and $\phi = h - \Box B$ [18]. Performing necessary computations (which are indeed straightforward around Minkowski as all derivatives commute), one gets [18]:

$$\delta^2 S(\tilde{h}_{\nu\mu}) = \int dx^4\sqrt{-\bar{g}}\frac{1}{2}\tilde{h}_{\mu\nu}\bar{\Box}a(\bar{\Box})\tilde{h}^{\mu\nu}, \qquad a(\bar{\Box}) = 1 + \frac{\lambda}{M_P^2}\bar{\Box}\left(\mathcal{F}_2(\bar{\Box}) + 2\mathcal{F}_3\left(\bar{\Box}\right)\right)$$
$$\delta^2 S(\phi) = -\int dx^4\sqrt{-\bar{g}}\frac{1}{2}\phi\bar{\Box}c(\bar{\Box})\phi, \qquad c(\bar{\Box}) = 1 - \frac{\lambda}{M_P^2}\bar{\Box}\left(6\mathcal{F}_1(\bar{\Box}) + \frac{1}{2}\mathcal{F}_2\left(\bar{\Box}\right)\right) \quad (4)$$

for the tensor component (where the field was rescaled by $M_P/2$ to become canonically normalised), and the scalar component (where the field was rescaled by $M_P\sqrt{3/32}$ to be canonically normalised), respectively.

The full graviton propagator can then be written using a similar method to [16], barring the suppressed indices [2] [10, 17, 18]:

$$\Pi(k^2) = \frac{\mathcal{P}^{(2)}}{k^2 a(-k^2)} + \frac{\mathcal{P}^{(0)}}{k^2(a(-k^2) - 3c(-k^2))}, \quad (5)$$

where $\mathcal{P}^{(2),\,(0)}$ are the spin projection operators [16]. Note that the graviton propagator has two unknown functions $a(k^2)$ and $c(k^2)$, where all the information about the infinite derivatives is hiding.

In order to reduce the graviton propagator to that of GR one may demand that $a(\bar{\Box}) = c(\bar{\Box})$. In the IR limit then both $a(k^2 \to 0) = 1$, $c(k^2 \to 0) = 1$, such that Eq. (5) reduces to Eq. (1). In this limit the theory would match exactly GR's predictions in the IR, but would lead to modification in the UV. The entire modification can be summarised by one unknown function $a(\bar{\Box})$, which constrains the functions such that [10]:

$$12\mathcal{F}_1(\bar{\Box}) + 6\mathcal{F}_2(\bar{\Box}) + 4\mathcal{F}_3(\bar{\Box}) = 0.$$

Furthermore, both a and c must not contain any poles, i.e. no zeroes at all. This way the propagator, Eq. (5), will not contain any extra degrees of freedom propagating in the space-time other than the massless graviton with 2 helicity states. One possible choice is to assume $a(\bar{\Box})$ is an *entire function*. An entire function makes sure that in spite of infinite derivatives, there exist *no ghosts* at the perturbative level for a quadratic curvature gravity Eq. (2). One such example will be [4, 5, 7]

$$a(\bar{\Box}) = c(\bar{\Box}) = e^{-\bar{\Box}/M^2}. \quad (6)$$

This choice guarantees that in the UV the theory is softened, as for $k \to \infty$, $a(-k^2) = c(-k^2) = e^{k^2/M^2}$ suppresses the propagator in the UV, i.e. $\Pi(k^2) \to 0$ in Eq. (5), while $k \to 0$ yields the pure 4D GR propagator.

Our aim in this paper will be to generalise this to any entire function $\tau(-k^2)$, such that in the momentum space

[1] The original action was written in terms of $R_{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}$ in Ref. [4]. However there is no loss of generality in expressing the action as Eq. (2), see Ref. [18].

[2] In Refs. [16], the authors imposed 6 projection operators to decompose spin 2 and spin 0 component of the propagator, here we have employed a slightly different technique to decompose the 10 metric degrees of freedom.

we have:

$$a(-k^2) = c(-k^2) = e^{-\tau(-k^2/M^2)}\,. \tag{7}$$

The computation of the Newtonian potential, i.e. $\Phi(r)$, for the simplest choice, when $\tau(-k^2/M^2) = -k^2/M^2$ as in Eq. (6), was done already in Ref. [5], and the result is

$$\Phi(r) \sim -\frac{\mu}{M_P^2 r}\sqrt{\frac{\pi}{2}}\mathrm{erf}(Mr/2), \tag{8}$$

where μ is the mass of a δ-source. This potential is finite near $r \approx 0$ and decays as $1/r$ at distances above the non-locality scale, i.e. $r \gg M^{-1}$. The tests of $1/r$ fall of Newtonian gravity has been tested in the laboratory up to 5.6×10^{-5}m [21], which implies that the scale of non-locality should be bigger than $M > 0.01$ eV. Indeed, we know very little about the gravitational interaction above this limit! The cornerstone of this computation is the *sine Fourier* transform

$$f(r) = \int_{-\infty}^{+\infty} \frac{dp}{k} e^{\tau(-k^2)} \sin(kr), \tag{9}$$

where $\Phi(r) \sim -\mu f(r)/M_P^2 r$. When we consider the simplest choice, $\tau = -k^2/M^2$, the function $f(r)$ indeed gives an erf-function.

We now set out to prove that the leading behaviour of the potential at small distances, r, away from the source is always given by: $\Phi \approx \Phi_0 + \mathcal{O}(r)$, where Φ_0 is constant irrespectively of the form of an *entire function* $\tau(k^2)$, as long as it does not introduce any extra pole other than the massless graviton.

Note that for an *entire function*, we can always treat $f(r)$ as a polynomial function. As a warm-up exercise we note that the sine Fourier transformation for $\tau = -k^{2n}/M^{2n}$, gives

$$f(r) = \frac{Mr}{n}\sum_{p=0}(-1)^p \frac{\Gamma(\frac{p}{n}+\frac{1}{2n})}{(2p+1)!}(Mr)^{2p}\,, \tag{10}$$

using the Gamma function $\Gamma(x) \equiv (x-1)!$. The above result is a generalisation of Ref. [13], where the authors have analysed special cases for $n = 1, 2, 4$. From Fig. [1] we see that the Newtonian potential never blows up at $r = 0$.

An important observation here is that by increasing the value of n yields larger modulation for large r, giving us a clear deviation from predictions of GR at larger distances, and providing us with a glimpse of testing the non-locality scale M. We can see that by having higher modes we now switch on a new mechanism that can be falsifiable in a near-future experiment.

In Ref. [21], the inverse square law was shown to hold (with 95% confidence) down to a length scale of 5.6×10^{-5} m, which means that we can now constrain M for each $\tau(-k^2)$. Imagining the departure from Newtonian gravity to be within 1%, using

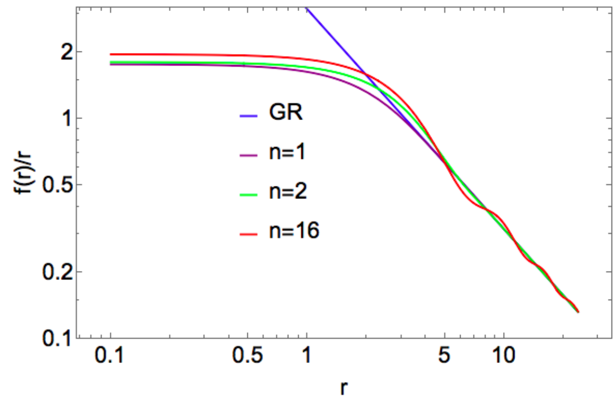

FIG. 1: We plot the Newtonian potential, $\Phi(r) \sim f(r)/r$ vs r for different n for Eq. (10), where $n = 1$ corresponds to the error function. For illustrative purpose we have taken $M = 1$ m^{-1}.

Eq. (10) we can set the scale of non-locality to be $M \sim 0.013,\ 0.025,\ 0.046,\ 0.70$ eV for $n = 1,\ 2,\ 4,\ 8$.

Now, let us illustrate the most general situation, when τ is not a monomial, we may represent it as

$$\tau(-k^2) = -\frac{k^2}{M^2} + \rho(k^2)\,. \tag{11}$$

If we expand $e^{\rho(k^2)} = \sum_m \rho_m k^{2m}/M^{2m}$ (clearly $\rho_0 = 1$), we yield the sine Fourier transformation of $e^{\tau}(k^2)$

$$f(r) = \sum_{m=0}^{\infty} \rho_m (-1)^m \frac{\partial}{\partial \alpha^m}\int \frac{dp}{p} e^{-\alpha\frac{p^2}{M^2}} \sin(pr), \tag{12}$$

which we can calculate either explicitly as

$$f(r) = \sum_{m,p=0}^{\infty} \rho_m (-1)^p \frac{\Gamma(m+p+\frac{1}{2})}{(2p+1)!}(Mr)^{2p+1}\,, \tag{13}$$

or using Hermitian polynomials $H_m(x)$ as

$$f(r) = \pi \mathrm{erf}\left(\frac{Mr}{2}\right) \\ -2\sqrt{\pi}e^{-\frac{M^2r^2}{4}}\sum_{m=1}^{\infty}\rho_m(-1)^m\frac{1}{4^m}H_{2m-1}\left(\frac{Mr}{2}\right)\,. \tag{14}$$

Note that Eq. (14) converges to a constant if ρ_m decreases at least as fast as $\frac{(-1)^m}{m!}$, i.e. $\rho = -k^2/M^2$.

In order to satisfy the low energy requirements of the underlying physics, we require that the function $e^{\tau(-k^2)}$ falls at least as fast as e^{-k^2/M^2}[4]. Any $e^{\tau(-k^2)}$ which does this will also fulfil the convergence condition for Eq. (14), meaning that any physically realistic $a(\Box)$ will give a Newtonian potential which returns to the GR $1/r$ potential in the IR limit.

Next, in order to graphically show the behaviour of Eq. (14), we take $\tau = -\frac{k^2}{M^2} - a_N\frac{k^{2N}}{M^{2N}}$ where the choice of a_N is motivated by the purpose of illustration of the oscillations that occur for $r \approx M^{-1}$. In this case,

$$\rho_m = \frac{(-a_N)^{m/N}}{(m/N)!} \text{ for } \frac{m}{N} \in \mathbb{N} \text{ and zero otherwise.} \quad (15)$$

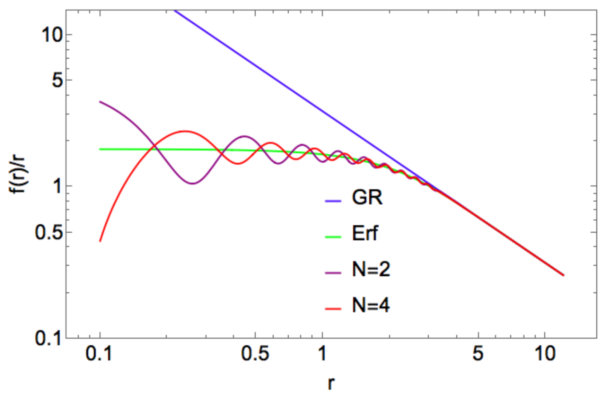

FIG. 2: We have plotted the Newtonian potential $\Phi(r) \sim f(r)/r$ vs r for Eq. (14) and (15), where we have chosen $a_2 = 4.65 \times 10^{-3}$ and $a_4 = 1.24 \times 10^{-7}$, and for illustrative purposes we have set $M = 1$ m^{-1}.

Let us conclude by pointing out that *infinite derivative, ghost-free* theories of gravity pose a real falsifiable feature compared to GR, which can be tested by measuring the Newtonian potential in near future experiments. We have shown that there exists a universal class of *entire* functions for which the theory is *ghost-free* as well as *singularity free* in the UV, while leaving some tantalisingly small effects in the IR. The current experimental limit puts the bound on non-locality to be around $M \sim 0.01$ eV. Indeed, it is intriguing to reiterate that we know very little about gravity and any modification from Newtonian potential can occur in the gulf of scales spanning some 30 orders of magnitude, but this window also provides an opportunity for testing gravity at short distances.

We would like to thank Tirthabir Biswas, Valeri Frolov, Saleh Qutub, Terry Tomboulis and Spyridon Talaganis for helpful discussions. AK is supported by the FCT Portugal fellowship SFRH/BPD/105212/2014 and in part by RFBR grant 14-01-00707. The work of A.M. is supported in part by the Lancaster-Manchester-Sheffield Consortium for Fundamental Physics under STFC grant ST/L000520/1.

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