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Supplementary Material for "Convergence of Regression Adjusted Approximate Bayesian Computation"

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1. NOTATIONS AND SET-UP

First some limit notations and conventions are given. For two sets A and B, the sum of integrals $\int_A f(x)\,dx + \int_B f(x)\,dx$ is written as $(\int_A + \int_B) f(x)\,dx$. For a constant $d \times p$ matrix A, let the minimum and maximum eigenvalues of $A^T A$ be $\lambda_{\min}^2(A)$ and $\lambda_{\max}^2(A)$ where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are non-negative. Obviously, for any p-dimension vector x, $\lambda_{\min}(A)\|x\| \leq \|Ax\| \leq \lambda_{\max}(A)\|x\|$. For two matrices A and B, we say A is bounded by B or $A \leq B$ if $\lambda_{\max}(A) \leq \lambda_{\min}(B)$. For a set of matrices $\{A_i : i \in I\}$ for some index set I, we say it is bounded if $\lambda_{\max}(A_i)$ are uniformly bounded in i. Denote the identity matrix with dimension d by I_d. Notations from the main text will also be used.

The following basic asymptotic results (Serfling, 2009) will be used throughout.

LEMMA 6. *(i) For a series of random variables Z_n, if $Z_n \to Z$ in distribution as $n \to \infty$, $Z_n = O_p(1)$. (ii) (Continuous mapping) For a series of continuous function $g_n(x)$, if $g_n(x) = O(1)$ almost everywhere, then $g_n(Z_n) = O_p(1)$, and this also holds if $O(1)$ and $O_p(1)$ are replaced by $\Theta(1)$ and $\Theta_p(1)$.*

Some notations regarding the posterior distribution of approximate Bayesian computation are given. For $A \subset \mathbb{R}^p$ and a scalar function $h(\theta, s)$, let

$$\pi_A(h) = \int_A \int_{\mathbb{R}^d} h(\theta,s)\pi(\theta) f_n(s \mid \theta) K\{\varepsilon_n^{-1}(s - s_{\text{obs}})\}\varepsilon_n^{-d}\, ds d\theta,$$

and

$$\widetilde{\pi}_A(h) = \int_A \int_{\mathbb{R}^d} h(\theta,s)\pi_\delta(\theta) \widetilde{f}_n(s \mid \theta) K\{\varepsilon_n^{-1}(s - s_{\text{obs}})\}\varepsilon_n^{-d}\, ds d\theta.$$

Then $\Pi_\varepsilon(\theta \in A \mid s_{\text{obs}}) = \pi_A(1)/\pi_{\mathcal{P}}(1)$ and its normal counterpart $\widetilde{\Pi}_\varepsilon(\theta \in A \mid s_{\text{obs}}) = \widetilde{\pi}_A(1)/\widetilde{\pi}_{\mathcal{P}}(1)$.

The following results from Li & Fearnhead (2015) will be used throughout.

LEMMA 7. *Assume Conditions 1–4. Then as $n \to \infty$,*

(i) if Condition 5 also holds then, for any $\delta < \delta_0$, $\pi_{B_\delta^c}(1)$ and $\widetilde{\pi}_{B_\delta^c}(1)$ are $o_p(1)$, and $O_p(e^{-a_{n,\varepsilon}^{\alpha_\delta} c_\delta})$ for some positive constants c_δ and α_δ depending on δ;

(ii) $\pi_{B_\delta}(1) = \widetilde{\pi}_{B_\delta}(1)\{1 + O_p(\alpha_n^{-1})\}$ *and* $\sup_{A\subset B_\delta} |\pi_A(1) - \widetilde{\pi}_A(1)| / \widetilde{\pi}_{B_\delta}(1) = O_p(\alpha_n^{-1})$*;*

(iii) if $\varepsilon_n = o(a_n^{-1/2})$*,* $\widetilde{\pi}_{B_\delta}(1)$ *and* $\pi_{B_\delta}(1)$ *are* $\Theta_p(a_{n,\varepsilon}^{d-p})$*, and thus* $\widetilde{\pi}_{\mathcal{P}}(1)$ *and* $\pi_{\mathcal{P}}(1)$ *are* $\Theta_p(a_{n,\varepsilon}^{d-p})$*;*

(iv) if $\varepsilon_n = o(a_n^{-1/2})$ *and Condition 5 holds,* $\theta_\varepsilon = \widetilde{\theta}_\varepsilon + o_p(a_{n,\varepsilon}^{-1})$*. If* $\varepsilon_n = o(a_n^{-3/5})$*,* $\theta_\varepsilon = \widetilde{\theta}_\varepsilon + o_p(a_n^{-1})$*.*

Proof. (i) is from Li & Fearnhead (2015, Lemma 3) and a trivial modification of its proof when Condition 5 does no hold; (ii) is from Li & Fearnhead (2015, equation 13 of supplements); (iii) is from Li & Fearnhead (2015, Lemma 5 and equation 13 of supplements); and (iv) is from Li & Fearnhead (2015, Lemma 3 and Lemma 6). □

2. Proof for Results in Section 3·1

Proof of Lemma 1. For any fixed $v \in \mathbb{R}^d$, recall that $\widetilde{\Pi}(\theta \in A \mid s_{\text{obs}} + \varepsilon_n v)$ is the posterior distribution given $s_{\text{obs}} + \varepsilon_n v$ with prior $\pi_\delta(\theta)$ and the misspecified model $\widetilde{f}_n(\cdot \mid \theta)$. By Kleijn & van der Vaart (2012), if there exist Δ_{n,θ_0} and V_{θ_0} such that,

(KV1) for any compact set $K \subset t(B_\delta)$,

$$\sup_{t\in K}\left|\log\frac{\widetilde{f}_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0 + a_n^{-1}t)}{\widetilde{f}_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0)} - t^T V_{\theta_0}\Delta_{n,\theta_0} + \frac{1}{2}t^T V_{\theta_0} t\right| \to 0,$$

in probability as $n \to \infty$, and

(KV2) $E\{\widetilde{\Pi}(a_n\|\theta - \theta_0\| > M_n \mid s_{\text{obs}} + \varepsilon_n v)\} \to 0$ as $n \to \infty$ for any sequence of constants $M_n \to \infty$,

then

$$\sup_{A\in\mathscr{B}^p}\left|\widetilde{\Pi}\{a_n(\theta - \theta_0) \in A \mid s_{\text{obs}} + \varepsilon_n v\} - \int_A N(t; \Delta_{n,\theta_0}, V_{\theta_0}^{-1})\, dt\right| \to 0,$$

in probability as $n \to \infty$.

For (KV1), by the definition of $\widetilde{f}_n(s \mid \theta)$,

$$\log\frac{\widetilde{f}_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0 + a_n^{-1}t)}{\widetilde{f}_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0)} = \log\frac{N\{s_{\text{obs}} + \varepsilon_n v; s(\theta_0 + a_n^{-1}t), a_n^{-2}A(\theta_0 + a_n^{-1}t)\}}{N\{s_{\text{obs}} + \varepsilon_n v; s(\theta_0), a_n^{-2}A(\theta_0)\}}.$$

As $x^T Ax - y^T By = x^T(A - B)x + (x - y)^T B(x + y)$, for vectors x and y and matrices A and B, by applying a Taylor expansion on $s(\theta_0 + xt)$ and $A(\theta_0 + xt)$ around $x = 0$, the right hand side of above equation equals

$$\{Ds(\theta_0 + e_n^{(1)}t)t\}^T A(\theta_0)^{-1}\zeta_n(v,t) - \frac{a_n^{-1}}{2}\zeta_n(v,t)^T\left\{\sum_{i=1}^p D_{\theta_i}A^{-1}(\theta_0 + e_n^{(2)}t)t_i\right\}\zeta_n(v,t)$$
$$+ \frac{a_n^{-1}}{2}\left\{D\log\left|A(\theta_0 + e_n^{(3)}t)\right|\right\}^T t,$$

where $\zeta_n(v,t) = A(\theta_0)^{1/2}W_{\text{obs}} + a_n\varepsilon_n v - \frac{1}{2}Ds(\theta_0 + e_n^{(1)}t)t$ and for $j = 1, 2, 3$, $e_n^{(j)}$ is a function of t satisfying $|e_n^{(j)}| \le a_n^{-1}$ which is from the remainder of the Taylor expansions. Since

$Ds(\theta)$, $DA^{-1}(\theta)$ and $D\log|A(\theta)|$ are bounded in B_δ when δ is small enough,

$$\sup_{t\in K}\left|\log\frac{\widetilde{f}_n(s_{\text{obs}}+\varepsilon_n v\mid\theta_0+a_n^{-1}t)}{\widetilde{f}_n(s_{\text{obs}}+\varepsilon_n v\mid\theta_0)}-t^TI(\theta_0)\beta_0\{A(\theta_0)^{1/2}W_{\text{obs}}+c_\varepsilon v\}+\frac{1}{2}t^TI(\theta_0)t\right|\to 0,$$

in probability as $n\to\infty$, for any compact set K. Therefore (KV1) holds with $\Delta_{n,\theta_0}=\beta_0\{A(\theta_0)^{1/2}W_{\text{obs}}+c_\varepsilon v\}$ and $V_{\theta_0}=I(\theta_0)$.

For (KV2), let $r_n(s\mid\theta_0)=\alpha_n\{f_n(s\mid\theta_0)-\widetilde{f}_n(s\mid\theta_0)\}$. Since $r_n(s\mid\theta_0)$ is bounded by a function integrable in $\mathbb{R}^d$ by Condition 4,

$$E\{\widetilde{\Pi}(a_n\|\theta-\theta_0\|>M_n\mid s_{\text{obs}}+\varepsilon_n v)\}-\int_{\mathbb{R}^d}\widetilde{\Pi}(a_n\|\theta-\theta_0\|>M_n\mid s+\varepsilon_n v)\widetilde{f}_n(s\mid\theta_0)\,ds$$

$$\leq\alpha_n^{-1}\int_{\mathbb{R}^d}|r_n(s\mid\theta_0)|\,ds=o(1).$$

Then it is sufficient for the expectation under $\widetilde{f}_n(s\mid\theta_0)$ to be $o(1)$. For any constant $M>0$, with the transformation $\bar{v}=a_n\{s-s(\theta_0)\}$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\widetilde{\Pi}(a_n\|\theta-\theta_0\|>M_n\mid s+\varepsilon_n v)\widetilde{f}_n(s\mid\theta_0)\,ds\\ &\leq\int_{\|\bar{v}\|\leq M}\frac{\int_{\|t\|>M_n}\widetilde{\pi}(t,\bar{v}\mid v)\,dt}{\int_{t(B_\delta)}\widetilde{\pi}(t,\bar{v}\mid v)\,dt}N\{\bar{v};0,A(\theta_0)\}\,d\bar{v}+\int_{\|\bar{v}\|>M}N\{\bar{v};0,A(\theta_0)\}\,d\bar{v},\end{aligned}$$

where $\widetilde{\pi}(t,\bar{v}\mid v)=\pi_\delta(\theta_0+a_n^{-1}t)\widetilde{f}_n\{s(\theta_0)+a_n^{-1}\bar{v}+\varepsilon_n v\mid\theta_0+a_n^{-1}t\}$. For the first term in the above upper bound, it is bounded by a series which does not depend on M and is $o(1)$ as $M_n\to\infty$, as shown below. Obviously $\int_{t(B_\delta)}\widetilde{\pi}(t,\bar{v}\mid v)\,dt$ can be lower bounded for some constant $m_\delta>0$. Choose δ small enough such that $Ds(\theta)$ and $A(\theta)^{1/2}$ are bounded for $\theta\in B_\delta$. Let $\lambda_{\min}$ and $\lambda_{\max}$ be their common bounds. When $\|\bar{v}\|<M$ and M_n is large enough,

$$\{t:\|t\|>M_n\}\subset\left\{t:\frac{\sup_{\theta\in B_\delta}\|Ds(\theta)t\|}{2}\geq\|a_n\varepsilon_n v+\bar{v}\|\right\}.\tag{1}$$

Then since for any $\bar{v}$ satisfying $\|\bar{v}\|<M$, by a Taylor expansion,

$$\widetilde{f}_n\{s(\theta_0)+a_n^{-1}\bar{v}+\varepsilon_n v\mid\theta_0+a_n^{-1}t\}=a_n^dN\{Ds(\theta_0+e_n^{(1)}t)t;\bar{v}+a_n\varepsilon_n v,A(\theta_0+a_n^{-1}t)\},$$

$\widetilde{\pi}(t,\bar{v}\mid v)\leq cN(\lambda_{\max}^{-1}\lambda_{\min}\|t\|/2;0,1)$, where c is some positive constant, for t in the right hand side of (1). Then

$$\int_{\|\bar{v}\|\leq M}\frac{\int_{\|t\|>M_n}\widetilde{\pi}(t,\bar{v}\mid v)\,dt}{\int_{t(B_\delta)}\widetilde{\pi}(t,\bar{v}\mid v)\,dt}N\{\bar{v};0,A(\theta_0)\}\,d\bar{v}\leq m_\delta^{-1}c\int_{\|t\|>M_n}N(\lambda_{\max}^{-1}\lambda_{\min}\|t\|/2;0,1)\,dt,$$

the right hand side of which is $o(1)$ when $M_n\to\infty$. Meanwhile by letting $M\to\infty$, it can be seen that the expectation under $\widetilde{f}_n(s\mid\theta_0)$ is $o(1)$. Therefore (KV2) holds and the lemma holds.□

The following lemma is used for equations $\int_{\mathbb{R}^p}g_n(t,v)\,dt=|A(\theta_0)|^{-1/2}G_n(v)$ and $\int_{\mathbb{R}^p}g(t,v)\,dt=|A(\theta_0)|^{-1/2}G(v)$.

LEMMA 8. *For a rank-p $d\times p$ matrix A, a rank-d $d\times d$ matrix B and a d-dimension vector c,*

$$N(At;Bv+c,I_d)=N\left\{t;(A^TA)^{-1}A^T(c+Bv),(A^TA)^{-1}\right\}g(v;A,B,c),\tag{2}$$

where $P = A^T A$, and

$$g(v; A, B, c) = \frac{1}{(2\pi)^{(d-p)/2}} \exp\left\{-\frac{1}{2}(c+Bv)^T(I - A(A^TA)^{-1}A^T)(c+Bv)\right\}.$$

Proof. This can be verified easily by matrix algebra. □

The following lemma regarding the continuity of a certain form of integral will be helpful when applying the continuous mapping theorem.

LEMMA 9. *Let l_1, l_1', l_2, l_2' and l_3 be positive integers satisfying $l_1' \le l_1$ and $l_2' \le l_2$. Let A and B be $l_1 \times l_1'$ and $l_2 \times l_2'$ matrices, respectively, satisfying that A^TA and B^TB are positive definite. Let $g_1(\cdot)$, $g_2(\cdot)$ and $g_3(\cdot)$ be functions in $\mathbb{R}^{l_1}$, $\mathbb{R}^{l_2}$ and $\mathbb{R}^{l_3}$, respectively, that are integrable and continuous almost everywhere. Assume:*
(i) $g_j(\cdot)$ is bounded in $\mathbb{R}^{l_j}$ for $j = 1, 2$;
(ii) $g_j(w)$ depends on w only through $\|w\|$ and is a decreasing function of $\|w\|$, for $j = 1, 2$; and
(iii) there exists a non-negative integer l such that $\int_{\mathbb{R}^{l_3}} \prod_{k=1}^{l_1'+l_2'+l} w_{i_k} g_3(w)\, dw < \infty$ for any coordinates $(w_{i_1}, \ldots, w_{i_{l_1'+l_2'+l}})$ of w.
Then the function,

$$\int\int\int P_l(w_1, w_2, w_3) \left|g_1(Aw_1 + x_1w_2 + x_2w_3 + x_3) - g_1(Aw_1)\right| g_2(Bw_2 + x_4w_3 + x_5) g_3(w_3)\, dw_3 dw_2 dw_1,$$

where $x_1 \in \mathbb{R}^{l_1 \times l_2'}$, $x_2 \in \mathbb{R}^{l_1 \times l_3}$, $x_4 \in \mathbb{R}^{l_2 \times l_3}$, $x_3 \in \mathbb{R}^{l_1}$ and $x_5 \in \mathbb{R}^{l_2}$, is continuous almost everywhere.

Proof. Let m_A and m_B be the lower bound of A and B respectively. For any $(x_{01}, \ldots, x_{05}) \in \mathbb{R}^{l_1 \times l_2'} \times \mathbb{R}^{l_1 \times l_3} \times \mathbb{R}^{l_2 \times l_3} \times \mathbb{R}^{l_1} \times \mathbb{R}^{l_2}$ such that the integrand in the target integral is continuous, consider any sequence $(x_{n1}, \ldots, x_{n5})$ converging to $(x_{01}, \ldots, x_{05})$. It is sufficient to show the convergence of the target function at $(x_{n1}, \ldots, x_{n5})$. Let $V_A = \{w_1 : \|Aw_1\|/2 \ge \sup_{(x_{n1}, x_{n2}, x_{n3})} \|x_{n1}w_2 + x_{n2}w_3 + x_{n3}\|\}$, $V_B = \{w_2 : \|Bw_2\|/2 \ge \sup_{(x_{n4}, x_{n5})} \|x_{n4}w_3 + x_{n5}\|\}$, $U_A = \{w_1 : \|w_1\| \le 4m_A^{-1}(\|x_{01}w_2\| + \|x_{02}w_3\| + \|x_{03}\|)\}$ and $U_B = \{w_2 : \|w_2\| \le 4m_B^{-1}(\|x_{04}w_3\| + \|x_{05}\|)\}$. We have $V_A^c \subset U_A$ and $V_B^c \subset U_B$. Then according to the following upper bounds and condition (iii),

$$\begin{aligned}
&|g_1(Aw_1 + x_{n1}w_2 + x_{n2}w_3 + x_{n3}) - g_1(Aw_1)| \le g_1(Aw_1 + x_{n1}w_2 + x_{n2}w_3 + x_{n3}) + g_1(Aw_1),\\
&g_1(Aw_1 + x_{n1}w_2 + x_{n2}w_3 + x_{n3}) \le \bar{g}_1(m_A\|w_1\|/2)\mathbb{1}_{\{w_1 \in V_A\}} + \sup_{w \in \mathbb{R}^{l_1}} g_1(w)\mathbb{1}_{\{w_1 \in U_A\}},\\
&g_2(Bw_2 + x_4w_3 + x_5) \le \bar{g}_2(m_B\|w_2\|/2)\mathbb{1}_{\{w_2 \in V_B\}} + \sup_{w \in \mathbb{R}^{l_2}} g_2(w)\mathbb{1}_{\{w_2 \in U_B\}},
\end{aligned}$$

where $g_1(w) = \bar{g}_1(\|w\|)$ and $g_2(w) = \bar{g}_2(\|w\|)$, by applying the dominated convergence theorem, the target function at $(x_{n1}, \ldots, x_{n5})$ converges to its value at $(x_{01}, \ldots, x_{05})$. □

Proof of Lemma 2. The first part holds according to Lemma 5 of Li & Fearnhead (2015). For the second part, when $c_\varepsilon = \infty$, by the transformation $v' = v'(v, t)$,

$$\int_{\mathbb{R}^d}\int_{t(B_\delta)} P_l(v) g_n(t, v)\, dt dv = \int_{\mathbb{R}^d}\int_{t(B_\delta)} P_l\left\{Ds(\theta_0)t + \frac{1}{a_n\varepsilon_n}v' - \frac{1}{a_n\varepsilon_n}A(\theta_0)^{1/2}W_{\rm obs}\right\} g_n'(t, v')\, dt dv'.$$

By applying Lemma 9 and the continuous mapping theorem in Lemma 6 to the right hand side of the above when $c_\varepsilon = \infty$, and to $\int_{\mathbb{R}^d}\int_{t(B_\delta)} P_l(v) g_n(t,v)\,dtdv$ when $c_\varepsilon < \infty$, and using $\int_{\mathbb{R}^p} g(t,v)\,dt = |A(\theta_0)|^{-1/2}G(v)$, the lemma holds. □

Proof of Lemma 3. (a), (b) and the first part of (c) hold immediately by Lemma 7. The second part of (c) is stated in the proof of Theorem 1 of Li & Fearnhead (2015). □

LEMMA 10. *Assume conditions 1–5.*

(i) If $c_\varepsilon \in (0,\infty)$ then $\Pi_\varepsilon\{a_n(\theta-\theta_\varepsilon)\in A \mid s_{\rm obs}\}$ and $\widetilde{\Pi}_\varepsilon\{a_n(\theta-\widetilde{\theta}_\varepsilon)\in A \mid s_{\rm obs}\}$ have the same limit in distribution.

(ii) If $c_\varepsilon = 0$ or $c_\varepsilon = 0\infty$ then

$$\sup_{A\in\mathscr{B}^p}\left|\Pi_\varepsilon\{a_{n,\varepsilon}(\theta-\theta_\varepsilon)\in A \mid s_{\rm obs}\} - \widetilde{\Pi}_\varepsilon\{a_{n,\varepsilon}(\theta-\widetilde{\theta}_\varepsilon)\in A \mid s_{\rm obs}\}\right| = o_p(1).$$

(iii) If Condition 6 holds then

$$\sup_{A\in\mathscr{B}^p}\left|\Pi_\varepsilon\{a_n(\theta^*-\theta^*_\varepsilon)\in A \mid s_{\rm obs}\} - \widetilde{\Pi}_\varepsilon\{a_n(\theta^*-\widetilde{\theta}^*_\varepsilon)\in A \mid s_{\rm obs}\}\right| = o_p(1).$$

Proof. Let $\lambda_n = a_{n,\varepsilon}(\theta_\varepsilon - \widetilde{\theta}_\varepsilon)$, and by Lemma 3(c), $\lambda_n = o_p(1)$. When $c_\varepsilon \in (0,\infty)$, for any $A\in\mathscr{B}^p$, decompose $\Pi_\varepsilon\{a_n(\theta-\theta_\varepsilon)\in A \mid s_{\rm obs}\}$ into the following three terms,

$$\begin{aligned}&\left[\Pi_\varepsilon\{a_n(\theta-\theta_\varepsilon)\in A \mid s_{\rm obs}\} - \widetilde{\Pi}_\varepsilon\{a_n(\theta-\theta_\varepsilon)\in A \mid s_{\rm obs}\}\right]\\ &+\left[\widetilde{\Pi}_\varepsilon\{a_n(\theta-\widetilde{\theta}_\varepsilon)\in A+\lambda_n \mid s_{\rm obs}\} - \widetilde{\Pi}_\varepsilon\{a_n(\theta-\widetilde{\theta}_\varepsilon)\in A \mid s_{\rm obs}\}\right]\\ &+\widetilde{\Pi}_\varepsilon\{a_n(\theta-\widetilde{\theta}_\varepsilon)\in A \mid s_{\rm obs}\}.\end{aligned}$$

For (i) to hold, it is sufficient that the first two terms in the above are $o_p(1)$. The first term is $o_p(1)$ by Lemma 3. For the second term to be $o_p(1)$, given the leading term of $\widetilde{\Pi}_\varepsilon\{a_n(\theta-\widetilde{\theta}_\varepsilon)\in A \mid s_{\rm obs}\}$ stated in the proof of Proposition 1 in the main text, it is sufficient that

$$\sup_{v\in\mathbb{R}^d}\left|\left(\int_{A+\lambda_n} - \int_A\right) N\{t;\mu_n(v), I(\theta_0)^{-1}\}\,dt\right| = o_p(1).$$

This holds by noting that the left hand side of the above is bounded by $(\int_{A+\lambda_n} - \int_A)c\,dt$ for some constant c and this upper bound is $o_p(1)$ since $\lambda_n = o_p(1)$. Therefore (i) holds.

When $c_\varepsilon = 0$ or ∞, $\sup_{A\in\mathscr{B}^p}\left|\Pi_\varepsilon\{a_{n,\varepsilon}(\theta-\theta_\varepsilon)\in A \mid s_{\rm obs}\} - \widetilde{\Pi}_\varepsilon\{a_{n,\varepsilon}(\theta-\widetilde{\theta}_\varepsilon)\in A \mid s_{\rm obs}\}\right|$ is bounded by

$$\begin{aligned}&\sup_{A\in\mathscr{B}^p}\left|\Pi_\varepsilon\{a_{n,\varepsilon}(\theta-\theta_\varepsilon)\in A \mid s_{\rm obs}\} - \widetilde{\Pi}_\varepsilon\{a_{n,\varepsilon}(\theta-\theta_\varepsilon)\in A \mid s_{\rm obs}\}\right|\\ &+\sup_{A\in\mathscr{B}^p}\left|\widetilde{\Pi}_\varepsilon\{a_{n,\varepsilon}(\theta-\widetilde{\theta}_\varepsilon)\in A+\lambda_n \mid s_{\rm obs}\} - \int_{A+\lambda_n}\psi(t)\,dt\right|\\ &+\sup_{A\in\mathscr{B}^p}\left|\widetilde{\Pi}_\varepsilon\{a_{n,\varepsilon}(\theta-\widetilde{\theta}_\varepsilon)\in A \mid s_{\rm obs}\} - \int_A\psi(t)\,dt\right|\\ &+\sup_{A\in\mathscr{B}^p}\left|\int_{A+\lambda_n}\psi(t)\,dt - \int_A\psi(t)\,dt\right|. \end{aligned} \tag{3}$$

With similar arguments as before, the first three terms are $o_p(1)$. For the fourth term, by transforming t to $t+\lambda_n$, it is upper bounded by $\int_{\mathbb{R}^p}|\psi(t-\lambda_n)-\psi(t)|\,dt$ which is $o_p(1)$ by the continuous mapping theorem. Therefore (ii) holds.

For (iii), the left hand side of the equation has the decomposed upper bound similar to (3), with θ, θ_ε, $\widetilde{\theta}_\varepsilon$ and $\psi(t)$ replaced by θ^*, θ_ε^*, $\widetilde{\theta}_\varepsilon^*$ and $N\{t; 0, I(\theta_0)^{-1}\}$. Then by Lemma 5, using the leading term of $\Pi_\varepsilon\{a_n(\theta^* - \theta_\varepsilon^*) \in A \mid s_{\text{obs}}\}$ stated in the proof of Theorem 1, and similar arguments to those used for the fourth term of (3), it can be seen that this upper bound is $o_p(1)$. Therefore (iii) holds. □

3. Proof for Results in Section 3·2

To prove Lemmas 4 and 5, some notation regarding the regression adjusted approximate Bayesian computation posterior, similar to those defined previously, are needed. Consider transformations $t = t(\theta)$ and $v = v(s)$. For $A \subset \mathbb{R}^p$ and the scalar function $h(t, v)$ in $\mathbb{R}^p \times \mathbb{R}^d$, let $\widetilde{\pi}_{A,tv}(h) = \int_{t(A)} \int_{\mathbb{R}^d} h(t, v) \widetilde{\pi}_{\varepsilon,tv}(t, v)\, dv dt$.

Proof of Lemma 4. Since $\beta_\varepsilon = \text{cov}_\varepsilon(\theta, s)\text{var}_\varepsilon(s)^{-1}$, to evaluate the covariance matrices, we need to evaluate $\pi_{\mathbb{R}^p}\{(\theta - \theta_0)^{k_1}(s - s_{\text{obs}})^{k_2}\}/\pi_{\mathbb{R}^p}(1)$ for $(k_1, k_2) = (0, 0), (1, 0), (1, 1), (0, 1)$ and $(0, 2)$.

First of all, we show that $\pi_{B_\delta^c}\{(\theta - \theta_0)^{k_1}(s - s_{\text{obs}})^{k_2}\}$ is ignorable for any $\delta < \delta_0$ by showing that it is $O_p(e^{-a_{n,\varepsilon}^{\alpha_\delta} c_\delta})$ for some positive constants c_δ and α_δ. By dividing $\mathbb{R}^d$ into $\{v : \|\varepsilon_n v\| \leq \delta'/3\}$ and its complement,

$$\sup_{\theta \in B_\delta^c} \int_{\mathbb{R}^d} (s - s_{\text{obs}})^{k_2} f_n(s \mid \theta) K\left(\frac{s - s_{\text{obs}}}{\varepsilon_n}\right) \varepsilon_n^{-d}\, ds$$
$$\leq \sup_{\theta \in B_\delta^c} \left\{ \sup_{\|s - s_{\text{obs}}\| \leq \delta'/3} f_n(s \mid \theta) \int_{\mathbb{R}^d} (s - s_{\text{obs}})^{k_2} K\left(\frac{s - s_{\text{obs}}}{\varepsilon_n}\right) \varepsilon_n^{-d}\, ds \right\}$$
$$+ \overline{K}\{\lambda_{\min}(\Lambda)\varepsilon_n^{-1}\delta'/3\}\varepsilon_n^{-d} \int_{\mathbb{R}^d} (s - s_{\text{obs}})^{k_2} f_n(s \mid \theta)\, ds. \quad (4)$$

By Condition 2(ii), Condition 6 and following the arguments in the proof of Lemma 3 of Li & Fearnhead (2015), the right hand side of (4) is $O_p(e^{-a_{n,\varepsilon}^{\alpha_\delta} c_\delta})$, which is sufficient for $\pi_{B_\delta^c}\{(\theta - \theta_0)^{k_1}(s - s_{\text{obs}})^{k_2}\}$ to be $O_p(e^{-a_{n,\varepsilon}^{\alpha_\delta} c_\delta})$.

For the integration over B_δ, by Lemma 7 (ii),

$$\frac{\pi_{B_\delta}\{(\theta - \theta_0)^{k_1}(s - s_{\text{obs}})^{k_2}\}}{\pi_{B_\delta}(1)} = a_{n,\varepsilon}^{-k_1}\varepsilon_n^{k_2} \left\{ \frac{\widetilde{\pi}_{B_\delta,tv}(t^{k_1}v^{k_2})}{\widetilde{\pi}_{B_\delta,tv}(1)} + \right.$$
$$\left. \alpha_n^{-1} \frac{\int_{t(B_\delta)} \int t^{k_1}v^{k_2}\pi(\theta_0 + a_{n,\varepsilon}^{-1}t) r_n(s_{\text{obs}} + \varepsilon_n v \mid \theta_0 + a_{n,\varepsilon}^{-1}t) K(v)\, dv dt}{\widetilde{\pi}_{B_\delta,tv}(1)} \right\} \{1 + O_p(\alpha_n^{-1})\}$$

where $r_n(s \mid \theta)$ is the scaled remainder $\alpha_n\{f_n(s \mid \theta) - \widetilde{f}_n(s \mid \theta)\}$. In the above, the second term in the first brackets is $O_p(\alpha_n^{-1})$ by the proof of Lemma 6 of Li & Fearnhead (2015). Then

$$\frac{\pi_{B_\delta}\{(\theta - \theta_0)^{k_1}(s - s_{\text{obs}})^{k_2}\}}{\pi_{B_\delta}(1)} = a_{n,\varepsilon}^{-k_1}\varepsilon_n^{k_2} \left\{ \frac{\widetilde{\pi}_{B_\delta,tv}(t^{k_1}v^{k_2})}{\widetilde{\pi}_{B_\delta,tv}(1)} + O_p(\alpha_n^{-1}) \right\},$$

and the moments $\widetilde{\pi}_{B_\delta,tv}(t^{k_1}v^{k_2})/\widetilde{\pi}_{B_\delta,tv}(1)$ need to be evaluated. Theorem 1 of Li & Fearnhead (2015) gives the value of $\widetilde{\pi}_{B_\delta,tv}(t)/\widetilde{\pi}_{B_\delta,tv}(1)$, and this is obtained by substituting the leading term of $\widetilde{\pi}_{\varepsilon,tv}(t, v)$, that is $\pi(\theta_0)g_n(t, v)$ as stated in Lemma 2, into the integrands. The other moments can be evaluated similarly, and give

$$\frac{\widetilde{\pi}_{B_\delta,tv}(t^{k_1}v^{k_2})}{\widetilde{\pi}_{B_\delta,tv}(1)} = \begin{cases} b_n^{-1}\beta_0\{A(\theta_0)^{1/2}W_{\text{obs}} + a_n\varepsilon_n E_{G_n}(v)\}, & (k_1,k_2)=(1,0), \\ b_n^{-1}\beta_0\{A(\theta_0)^{1/2}W_{\text{obs}}E_{G_n}[v] + a_n\varepsilon_n E_{G_n}(vv^T)\}, & (k_1,k_2)=(1,1), \\ E_{G_n}(v), & (k_1,k_2)=(0,1), \\ E_{G_n}(vv^T), & (k_1,k_2)=(0,2), \end{cases}$$
$$+ O_p(a_{n,\varepsilon}^{-1}) + O_p(a_n^2\varepsilon_n^4), \qquad (5)$$

where $b_n = 1$ when $c_\varepsilon < \infty$, and $a_n\varepsilon_n$ when $c_\varepsilon = \infty$. By Lemma 2, $E_{G_n}(vv^T) = \Theta_p(1)$. Since $\alpha_n^{-1} = o(a_n^{-2/5})$, $\text{cov}_\varepsilon(\theta,s) = \varepsilon_n^2\beta_0\text{var}_{G_n}(v) + o_p(a_n^{-2/5}\varepsilon_n^2)$ and $\text{var}_\varepsilon(s) = \varepsilon_n^2\text{var}_{G_n}(v)\{1 + o_p(a_n^{-2/5})\}$. Thus

$$\beta_\varepsilon = \beta_0 + o_p(a_n^{-2/5}), \qquad (6)$$

and the lemma holds. □

For $A \subset \mathbb{R}^p$ and $B \subset \mathbb{R}^d$, let $\pi(A,B) = \int_A\int_B \pi(\theta)f_n(s \mid \theta)K\{\varepsilon_n^{-1}(s - s_{\text{obs}})\}\varepsilon_n^{-d}\,dsd\theta$ and $\widetilde{\pi}(A,B) = \int_A\int_B \pi(\theta)\widetilde{f}_n(s \mid \theta)K\{\varepsilon_n^{-1}(s - s_{\text{obs}})\}\varepsilon_n^{-d}\,dsd\theta$. Denote the marginal mean values of s for $\pi_\varepsilon(\theta, s \mid s_{\text{obs}})$ and $\widetilde{\pi}_\varepsilon(\theta, s \mid s_{\text{obs}})$ by s_ε and $\widetilde{s}_\varepsilon$ respectively.

Proof of Lemma 5. For (a), write $\Pi_\varepsilon(\theta^* \in B_\delta^c \mid s_{\text{obs}})$ as $\pi[\mathbb{R}^p, \{s : \theta^*(\theta,s) \in B_\delta^c\}]/\pi(\mathbb{R}^p,\mathbb{R}^d)$. By Lemma 7, $\pi(\mathbb{R}^p,\mathbb{R}^d) = \pi_{\mathcal{P}}(1) = \Theta_p(a_{n,\varepsilon}^{d-p})$. By the triangle inequality,

$$\pi[\mathbb{R}^p, \{s : \theta^*(\theta,s) \in B_\delta^c\}] \le \pi(B_{\delta/2}^c, \mathbb{R}^d) + \pi[B_{\delta/2}, \{s : \|\beta_\varepsilon(s - s_{\text{obs}})\| \ge \delta/2\}], \qquad (7)$$

and it is sufficient that the right hand side of the above inequality is $o_p(1)$. Since its first term is $\pi_{B_{\delta/2}^c}(1)$, by Lemma 7 the first term is $o_p(1)$.

When $\varepsilon_n = \Omega(a_n^{-7/5})$ or $\Theta(a_n^{-7/5})$, by (6), $\beta_\varepsilon - \beta_0 = o_p(1)$ and so β_ε is bounded in probability. For any constant $\beta_{\sup} > 0$ and $\beta \in \mathbb{R}^{p\times d}$ satisfying $\beta \le \beta_{\sup}$,

$$\pi[B_{\delta/2}, \{s : \|\beta(s - s_{\text{obs}})\| \ge \delta/2\}] \le K\left(\varepsilon^{-1}\frac{\delta}{2\beta_{\sup}}\right)\varepsilon_n^{-d},$$

and by Condition 2(iv), the second term in (7) is $o_p(1)$.

When $\varepsilon_n = o(a_n^{-7/5})$, β_ε is unbounded and the above argument does not apply. Let δ_1 be a constant less than δ_0 such that $\inf_{\theta\in B_{\delta_1/2}} \lambda_{\min}\{A(\theta)^{-1/2}\} \ge m$ and $\inf_{\theta\in B_{\delta_1/2}} \lambda_{\min}\{Ds(\theta)\} \ge m$ for some positive constant m. In this case, it is sufficient to consider $\delta < \delta_1$. By Condition 4,

$$r_n(s \mid \theta) \le a_n^d|A(\theta)|^{1/2}r_{\max}[a_nA(\theta)^{-1/2}\{s - s(\theta)\}].$$

Using the transformation $t = t(\theta)$ and $v = v(s)$, $f_n(s \mid \theta) = \widetilde{f}_n(s \mid \theta) + \alpha_n^{-1}r_n(s \mid \theta)$ and applying the Taylor expansion of $s(\theta_0 + xt)$ around $x = 0$,

$$\pi[B_{\delta/2}, \{s : \|\beta_\varepsilon(s - s_{\text{obs}})\| \ge \delta/2\}] \le$$
$$c\int_{t(B_{\delta/2})}\int_{\|\beta_\varepsilon\varepsilon_n v\|\ge\delta/2} N[A(\theta_0 + a_n^{-1}t)^{-1/2}\{Ds(\theta_0 + e_n^{(1)}t)t - A(\theta_0)^{1/2}W_{\text{obs}} - a_n\varepsilon_n v\}; 0, I_d]K(v)\,dvdt$$
$$+ c\int_{t(B_{\delta/2})}\int_{\|\beta_\varepsilon\varepsilon_n v\|\ge\delta/2} r_{\max}[A(\theta_0 + a_n^{-1}t)^{-1/2}\{Ds(\theta_0 + e_n^{(1)}t)t - A(\theta_0)^{1/2}W_{\text{obs}} - a_n\varepsilon_n v\}]K(v)\,dvdt,$$

for some positive constant c. To show that the right hand side of the above inequality is $o_p(1)$, consider a function $g_4(\cdot)$ in $\mathbb{R}^d$ satisfying that $g_4(v)$ can be written as $\overline{g}_4(\|v\|)$ and $\overline{g}_4(\cdot)$ is decreasing. Let $A_n(t) = A(\theta_0 + a_n^{-1}t)^{-1/2}$, $C_n(t) = Ds(\theta_0 + \xi_1)$ and $c = A(\theta_0)^{1/2}W_{\text{obs}}$. For each n divide $\mathbb{R}^p$ into $V_n = \{t : \|C_n(t)t\|/2 \geq \|c + a_n\varepsilon_n v\|\}$ and V_n^c. In V_n, $\|A_n(t)\{C_n(t)t - c - a_n\varepsilon_n v\}\| \geq m^2\|t\|/2$ and in V^c, $\|t\| \leq 2m^{-1}\|c + a_n\varepsilon_n v\|$. Then

$$\int_{t(B_{\delta/2})}\int_{\|\beta_\varepsilon\varepsilon_n v\|\geq\delta/2} g_4[A_n(t)\{C_n(t)t - c - a_n\varepsilon_n v\}]K(v)\,dvdt$$
$$\leq \int_{\|\beta_\varepsilon\varepsilon_n v\|\geq\delta/2}\left\{\int_{\mathbb{R}^p}\overline{g}_4(m^2\|t\|/2)\,dt + \sup_{v\in\mathbb{R}^p} g_4(v)\int_{V_n^c} 1\,dt\right\}K(v)\,dv,$$

where $\int_{V_n^c} 1\,dt$ is the volume of V_n^c in $\mathbb{R}^p$. Then since $\beta_\varepsilon\varepsilon_n = o_p(1)$, $a_n\varepsilon_n = o_p(1)$ and $\int_{V_n^c} 1\,dt$ is proportional to $\|c + a_n\varepsilon_n v\|^p$, the right hand side of the above inequality is $o_p(1)$. This implies $\pi(B_{\delta/2}, \{s : \|\beta_\varepsilon(s - s_{\text{obs}})\| \geq \delta/2\}) = o_p(1)$.

Therefore in both cases $\Pi_\varepsilon(\theta^* \in B_\delta^c \mid s_{\text{obs}}) = o_p(1)$. For $\widetilde{\Pi}_\varepsilon(\theta^* \in B_\delta^c \mid s_{\text{obs}})$, since the support of its prior is B_δ, there is no probability mass outside B_δ , i.e. $\widetilde{\Pi}_\varepsilon(\theta^* \in B_\delta^c \mid s_{\text{obs}}) = 0$. Therefore (a) holds.

For (b),

$$\sup_{A\in\mathscr{B}^p}\left|\Pi_\varepsilon(\theta^* \in A_\theta\cap B_\delta \mid s_{\text{obs}}) - \widetilde{\Pi}_\varepsilon(\theta^* \in A_\theta\cap B_\delta \mid s_{\text{obs}})\right|$$
$$=\frac{\sup_{A\in\mathscr{B}^p}|\pi(\mathbb{R}^p, \{s : \theta^*(\theta, s) \in A_\theta\cap B_\delta\}) - \widetilde{\pi}(\mathbb{R}^p, \{s : \theta^*(\theta, s) \in A_\theta\cap B_\delta\})|}{\widetilde{\pi}_{B_\delta}(1)} + o_p(1)$$
$$\leq\alpha_n^{-1}\frac{\int_{B_\delta}\int_{\mathbb{R}^d}\pi(\theta)|r_n(s\mid\theta)|K\{\varepsilon_n^{-1}(s - s_{\text{obs}})\}\varepsilon_n^{-d}\,dsd\theta}{\widetilde{\pi}_{B_\delta}(1)} + o_p(1).$$

Then by the proof of Lemma 6 of Li & Fearnhead (2015), (b) holds.

For (c), to begin with, $a_n(\theta_\varepsilon^* - \widetilde{\theta}_\varepsilon^*) = a_n(\theta_\varepsilon - \widetilde{\theta}_\varepsilon) - a_n\beta_\varepsilon(s_\varepsilon - \widetilde{s}_\varepsilon)$. By Lemma 7, $a_n(\theta_\varepsilon - \widetilde{\theta}_\varepsilon) = o_p(1)$. For $a_n\beta_\varepsilon(s_\varepsilon - \widetilde{s}_\varepsilon)$, similar to the arguments of the proof of Lemma 4,

$$s_\varepsilon - s_{\text{obs}} = \varepsilon_n\left\{\frac{\widetilde{\pi}_{B_{\delta,tv}}(v)}{\widetilde{\pi}_{B_{\delta,tv}}(1)} + O_p(\alpha_n^{-1})\right\}\{1 + O_p(\alpha_n^{-1})\},\quad \widetilde{s}_\varepsilon - s_{\text{obs}} = \varepsilon_n\frac{\widetilde{\pi}_{B_{\delta,tv}}(v)}{\widetilde{\pi}_{B_{\delta,tv}}(1)}\{1 + O_p(\alpha_n^{-1})\}.$$

Then $a_n\beta_\varepsilon(s_\varepsilon - \widetilde{s}_\varepsilon) = O_p(\alpha_n^{-1}a_n\varepsilon_n)$ which is $o_p(1)$ if $\varepsilon_n = o(a_n^{-3/5})$. Therefore the first part of (c) holds. Since $\widetilde{\theta}_\varepsilon^* = \widetilde{\theta}_\varepsilon - \beta_\varepsilon(\widetilde{s}_\varepsilon - s_{\text{obs}})$, by the expansion of $\widetilde{\theta}_\varepsilon$ in Lemma 3(c), the above expansion of $\widetilde{s}_\varepsilon - s_{\text{obs}}$ and (5), the second part of (c) holds. □

4. PROOF FOR RESULTS IN SECTION 3·3

Proof of Theorem 2. The integrand of $p_{\text{acc},q}$ is similar to that of $\pi_{\mathbb{R}^p}(1)$. The expansion of $\pi_{\mathbb{R}^p}(1)$ is given in Lemma 7(ii), and following the same reasoning, $p_{\text{acc},q}$ can be expanded as $\varepsilon_n^d\int_{B_\delta}\int_{\mathbb{R}^d} q_n(\theta)\widetilde{f}(s_{\text{obs}} + \varepsilon_n v \mid \theta)K(v)\,dvd\theta\{1 + o_p(1)\}$. With transformation $t = t(\theta)$, plugging the expression of $q_n(\theta)$ and $\widetilde{\pi}_{\varepsilon,tv}(t, v)$ gives that

$$p_{\text{acc},q} = (a_{n,\varepsilon}\varepsilon_n)^d\int_{t(B_\delta)}(r_{n,\varepsilon})^{-p}q(r_{n,\varepsilon}^{-1}t - c_\mu)\frac{\widetilde{\pi}_{\varepsilon,tv}(t, v)}{\pi_\delta(\theta_0 + a_{n,\varepsilon}^{-1}t)}\,dvdt\{1 + o_p(1)\},$$

where $r_{n,\varepsilon} = \sigma_n / a_{n,\varepsilon}^{-1}$ and $c_{\mu,n} = \sigma_n(\mu_n - \theta_0)$. By the assumption of μ_n, denote the limit of $c_{\mu,n}$ by c_μ. Then by Lemma 2, $p_{\mathrm{acc},q}$ can be expanded as

$$p_{\mathrm{acc},q} = (a_{n,\varepsilon}\varepsilon_n)^d \int_{t(B_\delta)\times\mathbb{R}^d} (r_{n,\varepsilon})^{-p} q(r_{n,\varepsilon}^{-1} t - c_{\mu,n}) g_n(t,v)\, dv dt \{1 + o_p(1)\}. \tag{8}$$

Denote the leading term of the above by $Q_{n,\varepsilon}$.

For (1), when $c_\varepsilon = 0$, since $\sup_{t\in\mathbb{R}^p} g_n(t,v) \le c_1 K(v)$ for some positive constant c_1, $Q_{n,\varepsilon}$ is upper bounded by $(a_n\varepsilon_n)^d c_1$ almost surely. Therefore $p_{\mathrm{acc},q} \to 0$ almost surely as $n \to \infty$. When $r_{n,\varepsilon} \to \infty$, since $q(\cdot)$ is bounded in $\mathbb{R}^p$ by some positive constant c_2, $Q_{n,\varepsilon}$ is upper bounded by $(r_{n,\varepsilon})^{-p} c_2 (a_{n,\varepsilon}\varepsilon_n)^d \int_{\mathbb{R}^p\times\mathbb{R}^d} g_n(t,v)\, dv dt$. Therefore $p_{\mathrm{acc},q} \to 0$ in probability as $n \to \infty$ since $\int_{\mathbb{R}^p\times\mathbb{R}^d} g_n(t,v)\, dv dt = \Theta_p(1)$ by Lemma 2.

For (2), let $\tilde{t}(\theta) = r_{n,\varepsilon}^{-1} t(\theta) - c_{\mu,n}$ and $\tilde{t}(A)$ be the set $\{\phi : \phi = \tilde{t}(\theta) \text{ for some } \theta \in A\}$. Since $\tilde{t} = \sigma_n^{-1}(\theta - \theta_0) - c_{\mu,n}$ and $\sigma_n^{-1} \to \infty$, $\tilde{t}(B_\delta)$ converges to $\mathbb{R}^p$ in probability as $n \to \infty$. With the transformation $\tilde{t} = \tilde{t}(\theta)$,

$$Q_{n,\varepsilon} = \begin{cases} (a_n\varepsilon_n)^d \int_{\tilde{t}(B_\delta)\times\mathbb{R}^d} q(\tilde{t}) g_n\{r_{n,\varepsilon}(\tilde{t} + c_{\mu,n}), v\}\, d\tilde{t} dv, & c_\varepsilon < \infty, \\ \int_{\tilde{t}(B_\delta)\times\mathbb{R}^d} q(\tilde{t}) g_n'\{r_{n,\varepsilon}(\tilde{t} + c_{\mu,n}), v'\}\, d\tilde{t} dv', & c_\varepsilon = \infty. \end{cases}$$

By Lemma 9 and the continuous mapping theorem,

$$Q_{n,\varepsilon} \to \begin{cases} c_\varepsilon^d \int_{\mathbb{R}^p\times\mathbb{R}^d} q(\tilde{t}) g\{r_1(\tilde{t} + c_\mu), v\}\, d\tilde{t} dv, & c_\varepsilon < \infty, \\ \int_{\mathbb{R}^p\times\mathbb{R}^d} q(\tilde{t}) g\{r_1(\tilde{t} + c_\mu), v\}\, d\tilde{t} dv, & c_\varepsilon = \infty, \end{cases}$$

in distribution as $n \to \infty$. Since the limits above are $\Theta_p(1)$, $p_{\mathrm{acc},q} = \Theta_p(1)$.

For (3), when $c_\varepsilon = \infty$ and $r_1 = 0$, in the above, the limit of $Q_{n,\varepsilon}$ in distribution is $\int_{\mathbb{R}^p\times\mathbb{R}^d} q(\tilde{t}) g(0,v)\, d\tilde{t} dv = 1$. Therefore $p_{\mathrm{acc},q}$ converges to 1 in probability as $n \to \infty$. □

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