THE GENERALISED NILRADICAL OF A LIE ALGEBRA

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Abstract

A solvable Lie algebra L has the property that its nilradical N contains its own centraliser. This is interesting because gives a representation of L as a subalgebra of the derivation algebra of its nilradical with kernel equal to the centre of N. Here we consider several possible generalisations of the nilradical for which this property holds in any Lie algebra. Our main result states that for every Lie algebra L, $L/Z(N)$, where $Z(N)$ is the centre of the nilradical of L, is isomorphic to a subalgebra of $\mathrm{Der}(N^*)$ where N^* is an ideal of L such that N^*/N is the socle of a semisimple Lie algebra.
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1 Introduction

Throughout, L will be a finite-dimensional Lie algebra, over a field F, with nilradical N and radical R. If L is solvable, then N has the property that $C_L(N) \subseteq N$. This property supplies a representation of L as a subalgebra of $\mathrm{Der}(N)$ with kernel $Z(N)$. The purpose of this paper is to seek a larger ideal for which this property holds in all Lie algebras. The corresponding problem has been considered for groups (see, for example, Aschbacher [1, Chapter 11]). In group theory, the quasi-nilpotent radical (also called by some the generalised Fitting subgroup), $F^*(G)$, of a group G is defined to be $F(G) + E(G)$, where $F(G)$ is the Fitting subgroup and $E(G)$ is the set of *components* of G: that is, the quasi-simple subnormal subgroups of the group. It is also equal to the socle of $C_G(F(G))F(G)/F(G)$. The generalised Fitting subgroup, $\tilde{F}(G)$, is defined to be the socle of $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G (see, for example, [7]). Here we consider various possible analogues for Lie algebras.

First we introduce some notation that will be used. The *centre* of L is $Z(L) = \{x \in L : [x, y] = 0 \text{ for all } y \in L\}$; if S is a subalgebra of L, the *centraliser* of S in L is $C_L(S) = \{x \in L : [x, S] = 0\}$; the *Frattini ideal*, $\phi(L)$, of L is the largest ideal contained in all of the maximal subalgebras of L; we say that L is ϕ-free if $\phi(L) = 0$; the *socle* of S, $\mathrm{Soc}\, S$, is the sum of all of the minimal ideals of S; and the *L-socle* of S, $\mathrm{Soc}_L\, S$, is the sum of all of the minimal ideals of L contained in S. The symbol '$\oplus$' will be used to denote an algebra direct sum, whereas '$\dot{+}$' will denote a direct sum of the vector space structure alone.

We call L *quasi-simple* if $L^2 = L$ and $L/Z(L)$ is simple. Of course, over a field of characteristic zero a quasi-simple Lie algebra is simple, but that is not the case over fields of prime characteristic. For example, A_n where $n \equiv -1 (mod p)$ is quasi-simple, but not simple. This suggests using the quasi-simple subideals of a Lie algebra L to define a corresponding $E(L)$. However, first note that quasi-simple subideals of L are ideals of L. This follows from the following easy lemma.

Lemma 1.1 *If I is a perfect subideal (that is, $I^2 = I$) of L then I is a characteristic ideal of L.*

Proof. If I is perfect then $I = I^n$ for all $n \in \mathbb{N}$. It follows that $[L, I] = [L, I^n] \subseteq L\,(\mathrm{ad}\ I)^n \subseteq I$ for some $n \in \mathbb{N}$, and hence that I is an ideal of L. But now, if $D \in \mathrm{Der}(L)$, then $D([x_1, x_2]) = [x_1, D(x_2)] + [D(x_1), x_2] \in I$ for all $x_1, x_2 \in I$. Hence $D(I) = D(I^2) \subseteq I$. $\square$

Combining this with the preceding remark we have the following.

Lemma 1.2 *Let L be a Lie algebra over a field of characteristic zero. Then I is a quasi-simple subideal of L if and only if it is a simple ideal of L.*

We say that an ideal A of L is *quasi-minimal* in L if $A/Z(A)$ is a minimal ideal of $L/Z(A)$ and $A^2 = A$. Clearly a quasi-simple ideal is quasi-minimal. Over a field of characteristic zero, an ideal A of L is quasi-minimal if and only if it is simple. So an alternative is to define $E(L)$ to consist of the quasi-minimal ideals of L. We investigate these two possibilities in sections 3 and 5.

In sections 4 and 6 our attention turns to two further candidates for a generalised nilradical: the L-socle of $(N + C_L(N))/N$ and the socle of $L/\phi(L)$. All of these possibilities turn out to be related, but not always equal.

2 Preliminary results

Let L be a Lie algebra over a field F and let U be a subalgebra of L. If F has characteristic $p > 0$ we call U *nilregular* if the nilradical of U, $N(U)$, has nilpotency class less than $p - 1$. If F has characteristic zero we regard every subalgebra of L as being nilregular. We say that U is *characteristic in L* if it is invariant under all derivations of L. Then we have the following result.

Theorem 2.1 *(i) If I is a nilregular ideal of L then $N(I) \subseteq N(L)$.*

(ii) If I is a nilregular subideal of L and every subideal of L containing I is nilregular, then $N(I) \subseteq N(L)$.

Proof.

(i) We have that $N(I)$ is characteristic in I. This is well-known in characteristic zero, and is given by [6, Corollary 1] in characteristic p. Hence it is a nilpotent ideal of L and the result follows.

(ii) Let $I = I_0 < I_1 < \ldots < I_n = L$ be a chain of subalgebras of L with I_j an ideal of I_{j+1} for $j = 0, \ldots, n-1$. Then $N(I) \subseteq N(I_1) \subseteq .... \subseteq N(I_n) = N(L)$, by (i).

□

Similarly, we will call the subalgebra U *solregular* if the underlying field F has characteristic zero, or if it has characteristic p and the (solvable) radical of U, $R(U)$, has derived length less than $log_2 p$. Then we have the following corresponding result.

Theorem 2.2 *(i) If I is a solregular ideal of L then $R(I) \subseteq R(L)$.*

(ii) If I is a solregular subideal of L and every subideal of L containing I is solregular, then $R(I) \subseteq R(L)$.

Proof. This is similar to the proof of Theorem 2.1, using [8, Theorem 2]. □

We also have the following result which we will improve upon below, but by using a deeper result than is required here.

Theorem 2.3 *Let L be a Lie algebra over a field F, and let I be a minimal non-abelian ideal of L. Then either*

(i) I is simple or

(ii) F has characteristic p, $N(I)$ has nilpotency class greater than or equal to $p-1$, and $R(I)$ has derived length greater than or equal to $log_2 p$.

Proof. Let I be a non-abelian minimal ideal of L and let J be a minimal ideal of I. Then $J^2 = J$ or $J^2 = 0$. The former implies that J is an ideal of L by Lemma 1.1, and hence that I is simple. So suppose that $J^2 = 0$. Then $N(I) \neq 0$ and $R(I) \neq 0$. But if I is nilregular we have that $N(I) \subseteq N(L) \cap I = 0$, since I is non-abelian, a contradiction. Similarly, if I is solvregular, then $R(I) \subseteq R(L) \cap I = 0$, a contradiction. The result follows. □

As a result of the above we will call the subalgebra U *regular* if it is either nilregular or solregular; otherwise we say that it is *irregular*. Then we have the following corollary.

Corollary 2.4 *Let L be a Lie algebra over a field F. Then every minimal ideal of L is abelian, simple or irregular.*

Block's Theorem on differentiably simple rings (see [3]) describes the irregular minimal ideals as follows.

Theorem 2.5 *Let L be a Lie algebra over a field of characteristic $p > 0$ and let I be an irregular minimal ideal of L. Then $I \cong S \otimes \mathcal{O}_n$, where S is simple and $\mathcal{O}_n$ is the truncated polynomial algebra in n indeterminates. Moreover, $N(I)$ has nilpotency class $p-1$ and $R(I)$ has derived length $\lceil log_2 p \rceil$.*

Proof. Every non-abelian minimal ideal I of L is $\mathrm{ad}|_I$ (L)-simple, so the first assertion follows from [3, Theorem 1]. Now $N(I) = R(I) \cong S \otimes \mathcal{O}_n^+$, where $\mathcal{O}_n^+$ is the augmentation ideal of $\mathcal{O}_n$. It is then straightforward to check that the final assertion holds. □

Note that if $\mathcal{N}$ and $\mathcal{S}$ are the classes of Lie algebras that are themselves nilregular and solregular respectively, then $\mathcal{N} \not\subseteq \mathcal{S}$ and $\mathcal{S} \not\subseteq \mathcal{N}$, as the following examples show.

Example **2.1** *Let L be a filiform nilpotent Lie algebra of dimension n over a field F. Then L has nilpotency class $n-1$ and derived length 2. Thus, if F has characteristic $p > 3$, and $n \geq p$, then L has nilpotency class greater than or equal to $p-1$, and so is not nilregular. However, it is solregular, since $2 < log_2 p$.*

Example **2.2** *Let $L = Fe_1 + Fe_2$ with product $[e_1, e_2] = e_2$ and let F have characteristic 3. The $N(L) = Fe_2$ has nilpotency class $1 < p-1$ and so L is nilregular. But $R(L) = L$, so L has derived length $2 > log_2 p$ and is not solregular.*

For every Lie algebra $L^{(n)} \subseteq L^{2^n}$, so any nilregular nilpotent Lie algebra of nilpotency class 2^n is solregular, since $2^n < p-1 < p$ implies that $n < log_2 p$. However, it is not true generally that a nilpotent nilregular Lie algebra is solregular, as the following example shows.

Example **2.3** *Let L be the seven-dimensional Lie algebra over a field F of characteristic $p = 7$ with basis $e_1, \ldots, e_7$ and products $[e_2, e_1] = e_4$, $[e_3, e_1] = e_5$, $[e_3, e_2] = e_5$, $[e_4, e_3] = -e_6$, $[e_5, e_1] = e_7$, $[e_5, e_2] = 2e_6$, $[e_5, e_4] = e_7$, $[e_6, e_1] = e_7$ and $[e_6, e_2] = e_7$ (see [4, page 87]). Then L has nilpotency class $5 < p-1$ and so is nilregular, but its derived length is $3 > log_2 p$, so it is not solregular.*

We also have the following result.

Corollary 2.6 *If L is a Lie algebra and A is a regular ideal of L, then A is quasi-minimal in L if and only if it is a quasi-simple ideal of L.*

However, the above result is not true for all ideals, as the following example shows.

EXAMPLE **2.4** *Let $L = sl(2) \otimes \mathcal{O}_m + 1 \otimes \mathcal{D}$, where $\mathcal{O}_m$ is the truncated polynomial algebra in m indeterminates, $\mathcal{D}$ is a non-zero solvable subalgebra of Der($\mathcal{O}_m$), $\mathcal{O}_m$ has no $\mathcal{D}$-invariant ideals, and the ground field is algebraically closed of characteristic $p > 5$. Then L is semisimple and $A = sl(2) \otimes \mathcal{O}_m$ is the unique minimal ideal of L (see [13, Theorem 6.4]). Since $Z(A) = 0$, A is clearly quasi-minimal but not quasi-simple.*

If S is a subalgebra of L, we denote by $R_c(S)$ the (solvable) characteristic radical of S; that is, the sum of all of the solvable characteristic ideals of L. (see Seligman [9]).

Theorem 2.7 *Let L be a Lie algebra over any field F. Then $R_c(C_L(N)) = Z(N)$. Moreover, if $C_L(N)$ is regular, then $R_c(C_L(N)) = R(C_L(N))$.*

Proof. Let $Z = Z(N)$, $\overline{L} = L/Z$ and $H = R_c(C_L(N))$. Then H is a characteristic ideal of $C_L(N)$, and hence an ideal of L. Assume that $\overline{H} \neq 0$. Then there exists $k \geq 1$ such that $H^{(k+1)} \subseteq Z$ but $X = H^{(k)} \not\subseteq Z$. Then $X^2 \subseteq Z$ and $X^3 \subseteq [N, C_L(N)] = 0$, since $X \subseteq C_L(N)$. It follows that X is a nilpotent ideal of L, and hence that $X \subseteq N$. But $[X, N] = 0$, giving $X \subseteq Z$, a contradiction.

Now suppose that $C_L(N)$ is nilregular. Then, clearly, $R_c(C_L(N)) \subseteq R(C_L(N))$. Suppose that $R(C_L(N)) \neq Z$. Let A/Z be a minimal ideal of $C_L(N)/Z$ with $A \subseteq R(C_L(N))$. Then $A^3 = 0$ and so $A \subseteq N(C_L(N)) \subseteq N(L)$, by Theorem 2.1 (i). Hence $A = Z$, a contradiction.

Finally, suppose that $C_L(N)$ is solregular. Then $R(C_L(N)) = R(L) \cap C_L(N)$ is an ideal of L, and arguing as in the first paragraph of this proof shows that $R(C_L(N)) = Z(N)$. □

This has the following useful corollary.

Corollary 2.8 *Let L be a Lie algebra over a field F, let N be its nilradical and let $C = C_L(N)$ be regular. Then*

(i) if $\phi(C) \cap Z(N) = 0$, $C = Z(N) \dot{+} B$ where B is a semisimple subalgebra of L and B^2 is an ideal of L;

(ii) if $\phi(L) \cap Z(N) = 0$, $C = Z(N) \oplus B$ where B is a maximal semisimple ideal of L; and

(iii) if F has characteristic zero, then $C = Z(N) \oplus S$ where S is the maximal semisimple ideal of L.

Proof.

(i) Suppose that $\phi(C) \cap Z(N) = 0$. Then $C = Z(N) \dot{+} B$ for some subalgebra B of C, by [10, Lemma 7.2]. Moreover, $B \cong C/Z(N)$ is semisimple, by Theorem 2.7, and $B^2 = C^2$ is an ideal of L.

(ii) Suppose that $\phi(L) \cap Z(N) = 0$. The $L = Z(N) \dot{+} U$ for some subalgebra U of L, by [10, Lemma 7.2] again. It follows that $C = Z(N) \oplus B$ where $B = C \cap U$, which is an ideal of L, and B is semisimple. Moreover, if S is a semisimple ideal of L with $B \subseteq S$, then $[S, N] \subseteq S \cap N = 0$, so $S \subseteq C$. Hence $S = B$.

(iii) So suppose now that F has characteristic zero. Then $C = Z(N) \dot{+} B$ where B is a Levi factor of C. Also, $B = B^2 = C^2$ is an ideal of L, so $C = Z(N) \oplus B$. Moreover, if S is the maximal semisimple ideal of L, then $B \subseteq S$ and $[S, N] \subseteq S \cap N = 0$, so $S \subseteq C$. It follows that $S = B$.

□

Finally, the following straightforward results will prove useful.

Lemma 2.9 *Let K be an ideal of L with $K \subseteq C_L(N)$. Then $Z(K) = Z(N) \cap K$.*

Proof. Clearly $Z(K)$ is an abelian ideal of L, so $Z(K) \subseteq N$. Moreover, $[Z(K), N] \subseteq [K, N] = 0$, so $Z(K) \subseteq Z(N) \cap K$. Also $[Z(N) \cap K, K] \subseteq [N, K] = 0$, so $Z(N) \cap K \subseteq Z(K)$. □

Lemma 2.10 *Let L be any Lie algebra and suppose that A is an ideal of L with $A^2 = A$. Then $Z(A) \subseteq \phi(L)$. If A is a quasi-minimal ideal of L, then $Z(A) = A \cap \phi(L)$.*

Proof. Suppose that $Z(A) \not\subseteq \phi(L)$. Then there is a maximal subalgebra U of L such that $L = Z(A) + U$. Thus $A = Z(A) + U \cap A$ and $U \cap A$ is an ideal of L. It follows that $A = A^2 = (U \cap A)^2 \subseteq U \cap A \subseteq A$, whence $Z(A) \subseteq U$, a contradiction. Hence $Z(A) \subseteq \phi(L)$.

Suppose now that A is a quasi-minimal ideal of L. Then $Z(A) \subseteq A \cap \phi(L) \subseteq A$, so $A \cap \phi(L) = A$ or $Z(A)$. The former implies that $A \subseteq \phi(L)$, which is impossible since $\phi(L)$ is nilpotent. Hence $A \cap \phi(L) = Z(A)$. □

3 The quasi-minimal radical

Here we construct a radical by adjoining the quasi-minimal ideals of L to its nilradical N.

Lemma 3.1 *Quasi-minimal ideals of L are characteristic in L.*

Proof. This follows from Lemma 1.1. □

Lemma 3.2 *Let $A/Z(A)$ be a minimal ideal of $L/Z(A)$. Then $A = A^2 + Z(A)$ and A^2 is quasi-minimal in L.*

Proof. Let $P = A^2$ and $\overline{L} = L/Z(A)$. Then $\overline{P}$ is an ideal of $\overline{L}$ and $\overline{A}$ is minimal, so $\overline{P} = \overline{0}$ or $\overline{A}$. The former implies that A is abelian, a contradiction. Hence $\overline{P} = \overline{A}$, so $A = P + Z(A) = A^2 + Z(A)$. Also, $P = A^2 = P^2$ and $[Z(P), A] = [Z(P), P] + [Z(P), Z(A)] = 0$, so $P \cap Z(A) = Z(P)$. Thus $P/Z(P) = P/P \cap Z(A) \cong P + Z(A)/Z(A) = A/Z(A)$ is a minimal ideal of $L/Z(P)$. □

Proposition 3.3 *Let A be quasi-minimal in L and B be an ideal of L. Then either $A \subseteq B$ or $A \subseteq C_L(B)$.*

Proof. Clearly $A \cap B + Z(A)/Z(A)$ is an ideal of $L/Z(A)$ contained in $A/Z(A)$, so $A \cap B + Z(A) = A$ or $A \cap B + Z(A) = Z(A)$. The former implies that $A = A^2 \subseteq A \cap B \subseteq A$, whence $A = A \cap B$ and $A \subseteq B$. The latter yields that $A \cap B \subseteq Z(A)$, giving $[A, B] = [A^2, B] \subseteq [A, [A, B]] \subseteq [A, A \cap B] \subseteq [A, Z(A)] = 0$ and so $A \subseteq C_L(B)$. □

The *quasi-minimal components* of L are its quasi-minimal ideals. Write MComp(L) for the set of quasi-minimal components of L, and let $E^{\dagger}(L)$ be the subalgebra generated by them. Then $E^{\dagger}(L)$ is a characteristic ideal of L, by Lemma 1.1.

Corollary 3.4 *$E^{\dagger}(L) \subseteq C_L(R)$.*

Proof. Let $A \in \text{MComp}(L)$ and put $B = R$ in Proposition 3.3. Then either $A \subseteq R$ or $A \subseteq C_L(R)$. But the former is impossible, since $A^2 = A$, whence $A \subseteq C_L(R)$. □

Corollary 3.5 *Distinct quasi-minimal components of L commute, so*

$$E^{\dagger}(L) = \sum_{P \in MComp(L)} P,$$

where $[P, Q] = 0$ and $P \cap Q \subseteq Z(R)$ for all $P, Q \in MComp(L)$.

Proof. This first assertion follows directly from Proposition 3.3. But then $P \cap Q \subseteq Z(P) \cap Z(Q) \subseteq N$ and $[P, R] = [Q, R] = 0$, using Corollary 3.4. Hence $P \cap Q \subseteq Z(R)$. □

Lemma 3.6 *If B is an ideal of L, then $MComp(B) \subseteq MComp(L) \cap B$. Moreover, if B is regular, then this is an equality.*

Proof. Let A be a quasi-minimal ideal of B. Then A is a quasi-minimal ideal of L, by Lemma 3.1. Thus $\text{MComp}(B) \subseteq \text{MComp}(L) \cap B$.

Now suppose that B is regular, and let $A \in \text{MComp}(L) \cap B$, so A is a quasi-minimal ideal of L and $A \subseteq B \cap C_L(N)$, by Corollary 3.4. Let $C/Z(A)$ be a minimal ideal of $B/Z(A)$ with $C \subseteq A$. Then $C^2 \subseteq Z(A)$ or $C^2 + Z(A) = C$. The former implies that $C^3 = 0$, and hence that C is a nilpotent ideal of B. If B is nilregular, it follows from Theorem 2.1 that $C \subseteq N$, whence $[C, A] = 0$ and $C \subseteq Z(A)$, a contradiction. Similarly, if B is solregular, then $C \subseteq R(B) \subseteq R(L)$, by Theorem 2.2. But then $[C, A] = 0$, by Corollary 3.4, since $A \in E^\dagger(L)$, leading to the same contradiction. Hence $C^2 + Z(A) = C$. But now

$$[L, C] = [L, C^2 + Z(A)] \subseteq [[L, C], C] + Z(A) \subseteq [B, C] + Z(A) \subseteq C,$$

so C is an ideal of L. But $A/Z(A)$ is a minimal ideal of $L/Z(A)$, so $C = Z(A)$ or $C = A$. It follows that $A/Z(A)$ is a minimal ideal of $B/Z(A)$ and $A^2 = A$. Thus $A \in \text{MComp}(B)$. □

Example **3.1** *Note that if B is not regular then the inclusion in Lemma 3.6 can be strict. For, let L be as in Example 2.4. Then $\mathcal{O}_m$ has a unique maximal ideal $\mathcal{O}_m^+$ and $A^+ = sl(2) \otimes \mathcal{O}_m^+$ is the unique maximal ideal of A (and is nilpotent). Hence $MComp(A) \subseteq A^+ \neq A$, whereas $MComp(L) = A$.*

Proposition 3.7 *Let L be a Lie algebra in which $C_L(N)$ is regular. Put $Z = Z(N)$, $\overline{L} = L/Z$, $\overline{S} = Soc(\overline{C_L(N)})$. Then $E^\dagger(L) = S^2$ and $S = E^\dagger(L) + Z$.*

Proof. Let $H = C_L(N)$. Then $R(\overline{H}) = 0$, by Theorem 2.7. Hence each minimal ideal of $\overline{H}$ is quasi-minimal in $\overline{H}$, and so is a quasi-minimal component of $\overline{H}$. Thus $\overline{S} \subseteq E^\dagger(\overline{H})$. Let $\overline{K} \in \text{MComp}(\overline{H}) \subseteq \text{MComp}(\overline{L})$, by Lemma 3.6. Hence $K/Z(K)$ is a quasi-minimal ideal of $L/Z(K)$, by Lemma 2.9. Then $K = K^2 + Z$ with K^2 quasi-minimal in L, since $Z(K) = Z$ by Lemma 3.2. Hence $K^2 \in \text{MComp}(L)$, so $S \subseteq E^\dagger(L) + Z$.

Let $P \in \text{MComp}(L)$. Then $P \subseteq H$ since $E^\dagger(L) \subseteq H$, by Corollary 3.4. Hence $P \in \text{MComp}(L) \cap H = \text{MComp}(H)$, by Lemma 3.6. Hence $\overline{P}$ is a minimal ideal of $\overline{H}$, so $P \subseteq S$. Thus $S = E^\dagger(L) + Z$ and $E^\dagger(L) = S^2$. □

We define the *quasi-minimal radical* of L to be $N^\dagger(L) = N + E^\dagger(L)$. From now on we will denote $N^\dagger(L)$ simply by $N^\dagger$. Then this has the property we are seeking.

Theorem 3.8 *If L is a Lie algebra, over any field F, with nilradical N, then $C_L(N^\dagger) = Z(N)$. In particular, $C_L(N^\dagger) \subseteq N^\dagger$.*

Proof. Let $C = C_L(N^\dagger)$. Then $Z(N) \subseteq C$, by Corollary 3.4. Suppose that $Z(N) \neq C$ and let $A/Z(N)$ be a minimal ideal of $L/Z(N)$ with $A \subseteq C$. Then $[A, Z(N)] \subseteq [C, N^\dagger] = 0$, so $Z(N) \subseteq Z(A)$. Thus $A = Z(A)$ or $Z(A) = Z(N)$. The former implies that $A \subseteq N$. But $[A, N] \subseteq [C, N^\dagger] = 0$, so $A \subseteq Z(N)$, a contradiction. The latter implies that $A^2 \subseteq E^\dagger \subseteq N^\dagger$, by Lemma 3.2. Hence $A^3 \subseteq [C, N^\dagger] = 0$, so $A \subseteq N$, which leads to the same contradiction as before. The result follows. □

Proposition 3.9 *Let L be a Lie algebra in which $N^\dagger$ is regular. Then $N^\dagger(N^\dagger) = N^\dagger$.*

Proof. Clearly $N^\dagger(N^\dagger) \subseteq N^\dagger$. But $E^\dagger(L) \subseteq E^\dagger(N^\dagger)$, by putting $B = N^\dagger$ in Lemma 3.6, and, clearly, $N \subseteq N(N^\dagger)$, giving the reverse inclusion. □

Example **3.2** *Again, Proposition 3.9 does not hold if $N^\dagger$ is not regular. For, let L be as in Example 2.4. Then $N^\dagger = A$, but $N^\dagger(N^\dagger) = A^+$.*

Next we investigate the behaviour of $N^\dagger$ with respect to factor algebras, direct sums and ideals.

Proposition 3.10 *Let L be a Lie algebra over any field, and let I be an ideal of L. Then*

$$\frac{N^\dagger(L) + I}{I} \subseteq N^\dagger\left(\frac{L}{I}\right).$$

Proof. Clearly $N(L) + I/I \subseteq N(L/I)$. Let A be a quasi-minimal ideal of L, so $A/Z(A)$ is a minimal ideal of $L/Z(A)$ and $A^2 = A$. Put $C = C_L(A + I/I)$. Then $Z(A) \subseteq C \cap A \subseteq A$, so $C \cap A = A$ or $C \cap A = Z(A)$. The former implies that $A = A^2 \subseteq I$, whence $A + I/I \subseteq N(L/I)$. If the latter holds, then $C = C \cap (A + I) = C \cap A + I = Z(A) + I$ and $A \cap I \subseteq A \cap C = Z(A)$, whence

$$\frac{A + I/I}{Z(A + I/I)} \cong \frac{A + I}{C} = \frac{A + I}{Z(A) + I} \cong \frac{A}{Z(A) + A \cap I} = \frac{A}{Z(A)}$$

and

$$\left(\frac{A + I}{I}\right)^2 = \frac{A + I}{I}.$$

Thus $A + I/I$ is a quasi-minimal ideal of L/I and

$$\frac{E^\dagger(L) + I}{I} \subseteq E^\dagger\left(\frac{L}{I}\right).$$

The result follows. □

The above inclusion can be strict, as we shall see later.

Proposition 3.11 *Let L be a Lie algebra over any field, and suppose that $L = I \oplus J$, where I, J are ideals of L. Then $N^\dagger(L) = N^\dagger(I) \oplus N^\dagger(J)$.*

Proof. It is easy to see that $N^\dagger(I) \oplus N^\dagger(J) \subseteq N^\dagger(L)$. Let π_I, π_J be the projection maps onto I, J respectively. Then $N(L) = \pi_I(N(L)) \oplus \pi_J(N(L))$. Clearly $\pi_I(N(L)) \subseteq N(I)$ and $\pi_J(N(L)) \subseteq N(J)$, so $N(L) \subseteq N(I) \oplus N(J)$.

Let A be a quasi-minimal ideal of L, so $A/Z(A)$ is a minimal ideal of L and $A^2 = A$. Then

$$A = A^2 \subseteq [A, I \oplus J] = [A, I] \oplus [A, J] \subseteq A,$$

so $A = [A, I] \oplus [A, J]$. Since $A = A^2 = [A, I]^2 + [A, J]^2$, we also have that $[A, I]^2 = [A, I]$ and $[A, J]^2 = [A, J]$. Now $[A, I] + Z(A) = Z(A)$ or A. The former implies that $[A, I] \subseteq Z(A)$, which gives that $[A, I] = [A, I]^2 = 0$. The latter yields that $A/Z(A) \cong [A, I]/Z(A) \cap [A, I]$. Now $Z(A) \cap [A, I] \subseteq Z([A, I])$, so $Z([A, I]) = [A, I]$ or $Z(A) \cap [A, I]$. The former gives $[A, I] = [A, I]^2 = 0$ again, whereas the latter yields that $[A, I]/Z[A, I]$ is quasi-minimal and $[A, I] \in E^\dagger(I)$.

Similarly $[A, J] = 0$ or else $[A, J] \in E^\dagger(J)$. It follows that $E^\dagger(L) \subseteq E^\dagger(I) \oplus E^\dagger(J)$, whence the result. □

Proposition 3.12 *Let L be a Lie algebra over any field, and let I be a nilregular ideal of L. Then $N^\dagger(I) \subseteq N^\dagger(L)$.*

Proof. Since I is nilregular, we have that $N(I) \subseteq N(L)$, by Theorem 2.1 (i). Also, $E^{\dagger}(I) \subseteq E^{\dagger}(L)$, by Lemma 3.6, whence the result. □

The following result describes the ideals of L contained in $E^{\dagger}$.

Proposition 3.13 *Let A be an ideal of L with $A \subseteq E^{\dagger}(L)$. Then $A = P_1 + \ldots + P_k + Z(A)$, where P_i is a quasi-minimal component of L for $1 \leq i \leq k$.*

Proof. Let $E^{\dagger}(L) = P_1 + \ldots + P_n$, where P_i is a quasi-minimal component of L for each $1 \leq i \leq n$. Then $P_i \subseteq A$ or $P_i \subseteq C_L(A)$ for each $i = 1, \ldots, n$, by Proposition 3.3. Let $P_i \subseteq A$ for $1 \leq i \leq k$ and $P_i \not\subseteq A$ for $k+1 \leq i \leq n$. Then $A \cap (P_{k+1} + \ldots + P_n) \subseteq Z(A)$, so $A = (P_1 + \ldots + P_k) + Z(A)$. □

Finally we give two further characterisations of $N^{\dagger}$, valid over any field. Recall that A/B is a chief factor of L if B is an ideal of L and A/B is a minimal ideal of L/B.

Theorem 3.14 *Let L be a Lie algebra, over any field F, with radical R. Then*

$$N^{\dagger} = \cap\{A + C_L(A/B) \mid A/B \text{ is a chief factor of } L\}.$$

Proof. Denote the given intersection by I, let A/B be a chief factor of L and let P be a quasi-minimal component of L. Then $P \subseteq A$ or $P \subseteq C_L(A)$, by Proposition 3.3. Hence $E^{\dagger} \subseteq I$. Moreover, $N \subseteq I$, by [2, Lemma 4.3], so $N^{\dagger} \subseteq I$.

If P is a quasi-minimal component of L then $P/Z(P)$ is a chief factor of L. Also, if $C = C_L(P/Z(P))$ we have $[C, P] = [C, P^2] \subseteq [[C, P], P] \subseteq [Z(P), P] = 0$, so $C = C_L(P)$ and $N \subseteq C$, by Corollary 3.4. Hence $I \subseteq P + C_L(P/Z(P)) = P + C_L(P)$. Now, if P, Q are quasi-minimal components of L, then

$$(P + C_L(P)) \cap (Q + C_L(Q)) = P + Q + C_L(P) \cap C_L(Q),$$

since $P \subseteq C_L(Q)$ and $Q \subseteq C_L(P)$. It follows that $I \subseteq N^{\dagger} + C_L(E^{\dagger})$ and $I = N^{\dagger} + I \cap C_L(E^{\dagger})$.

If

$$0 = N_0 \subset N_1 \subset \ldots \subset N_k = N$$

is part of a chief series for L then $I \subseteq \cap_{i=1}^{k} C_L(N_i/N_{i-1})$, so I acts nilpotently on N. Suppose that $N \subset I \cap C_L(E^{\dagger})$. Let A/N be a minimal ideal of L/N with $A \subseteq I \cap C_L(E^{\dagger})$. Then $A^2 \subseteq N$ or $A^2 + N = A$. The former implies that $A \subseteq N$, since A acts nilpotently on N, a contradiction. Hence $A = A^2 + N \subseteq A^r + N$ for all $r \geq 1$. But now

$$[A, N] \subseteq [A^r + N, N] \subseteq N(\text{ad}, A)^r + N^r,$$

so $[A, N] = 0$, whence $A \subseteq C_L(E^\dagger) \cap C_L(N) = C_L(N^\dagger) = Z(N)$, by Theorem 3.8, a contradiction again. Thus $I \cap C_L(E^\dagger) = N$ and $I = N^\dagger$. $\square$

We put

$$I_L(A/B) = \{x \in L \mid \text{ad}\,(x + B)|_{A/B} = \text{ad}\,(a + B)|_{A/B} \text{ for some } a \in A\}.$$

The map $\text{ad}\,(x+B)|_{A/B}$ is called the inner derivation *induced* by x on A/B. Then $I_L(A/B) = A + C_L(A/B)$, by [11, Lemma 1.4 (i)], so we have the following corollary.

Corollary 3.15 *Let L be a Lie algebra over any field F. Then $N^\dagger$ is the set of all elements of L which induce an inner derivation on every chief factor of L.*

4 The generalised nilradical of L

We define the *generalised nilradical* of L, $N^*(L)$, by

$$\frac{N^*(L)}{N} = \text{Soc}_{L/N}\left(\frac{N + C_L(N)}{N}\right)$$

As usual we denote $N^*(L)$ simply by N^*. The following result shows that this is, in fact, the same as the quasi-nilpotent radical.

Theorem 4.1 *Let L be a Lie algebra with nilradical N over any field. Then $N^* = N^\dagger$.*

Proof. Put $C = C_L(N)$. Let $A/Z(A)$ be a minimal ideal of $L/Z(A)$ for which $A^2 = A$. Then $Z(A) \subseteq A \cap N$, so $A \cap N = A$ or $A \cap N = Z(A)$. the former implies that $A \subseteq N$, which is a contradiction, so the latter holds. It follows that $(A + N)/N \cong A/A \cap N = A/Z(A)$, so $(A + N)/N$ is a minimal ideal of L/N. Moreover, $[A, N] = [A^2, N] \subseteq [A, [A, N] \subseteq [A, Z(A)] = 0$, so $A \subseteq C$ and $(A + N)/N \subseteq N^*/N$. Hence $N^\dagger \subseteq N^*$.

Now let A/N be a minimal ideal of L/N with $A \subseteq N + C$. Then $A = N + A \cap C$. Now $Z(A \cap C) = Z(N)$, by Lemma 2.9, so $A/N \cong A \cap C/N \cap C = A \cap C/Z(N) = A \cap C/Z(A \cap C)$. It follows that $A \cap C/Z(A \cap C)$

is a minimal ideal of $L/Z(A \cap C)$. Thus $(A \cap C)^2$ is a quasi-minimal ideal of L, by Lemma 3.2. Moreover, $(A \cap C)^2 + Z(N) = Z(N)$ or $A \cap C$. The former implies that $(A \cap C)^2 \subseteq Z(N)$, which yields that $(A \cap C)^3 = 0$ and $A \cap C \subseteq N$, a contradiction. Hence $A \cap C = (A \cap C)^2 + Z(N) \subseteq N^\dagger$, and so $A \subseteq N^\dagger$. This shows that $N^* \subseteq N^\dagger$. □

This last result together with Theorem 3.8 gives the following.

Theorem 4.2 *Let L be a Lie algebra over any field F. Then $L/Z(N)$ is isomorphic to a subalgebra of $Der(N^*)$, and N^*/N is a direct sum of minimal ideals of L/N which are simple or irregular.*

Proof. The isomorphism results from the map $\theta : L \to \mathrm{Der}(N^*)$ given by $\theta(x) = \mathrm{ad}\, x\,|_{N^*}$. Let A/N be a minimal ideal of L/N with $A \subseteq A + C$. The $A = N + A \cap C$ and, as in the second paragraph of the proof of Theorem 4.1, $(A \cap C)^2$ is quasi-minimal in L, which implies that A/N cannot be abelian. It follows from Corollary 2.4 that A/N is simple or irregular. □

Proposition 4.3 *Let L be a Lie algebra with nilradical N over a field F, and suppose that $C_L(N)$ is nilregular in L. Then*

$$\frac{N^*}{N} = Soc\left(\frac{N + C_L(N)}{N}\right).$$

Proof. Put $C = C_L(N)$, $D = N + C$. Let A/N be a minimal ideal of D/N. Then $A^2 + N = N$ or A. The former implies that $A^2 \subseteq N$, whence $A^3 \subseteq [N, N + C] \subseteq N^2$, and an easy induction shows that $A^{n+1} \subseteq N^n = 0$ for some $n \in \mathbb{N}$. It follows that A is a nilpotent ideal of D, which is an ideal of L, and thus that $A \subseteq N(D) = N + N(C) \subseteq N$, by Theorem 2.1, a contradiction. Hence $A = A^2 + N$ and

$$[L, A] = [L, A^2 + N] \subseteq [[L, A], A] + [L, N] \subseteq [D, A] + N \subseteq A,$$

so A/N is a minimal ideal of L/N inside D/N.

Now suppose that B/N is a minimal ideal of L/N inside D/N, and let A/N be a minimal ideal of D/N inside B/N. Then, by the argument in the paragraph above, A/N is an ideal of L/N, and so $A = B$. The result follows. □

Proposition 4.4 *(i) If $C_L(N)$ is regular and $\phi(L) \cap Z(N) = 0$ then $N^*(L) = N(L) \oplus S$, where S is the socle of a maximal semisimple ideal of L.*

(ii) Over a field of characteristic zero, $N^*(L) = N(L) \oplus S = N(L) + C_L(N)$, where S is the biggest semisimple ideal of L.

Proof. This follows from Corollary 2.8. □

Proposition 4.5 *Let L be a Lie algebra over a field of characteristic zero and let $I \subseteq N^*(L)$ be an ideal of L. Then*

$$\frac{N^*(L)}{I} \subseteq N^*\left(\frac{L}{I}\right).$$

Proof. This is a special case of Proposition 3.10. □

As a result of Example 3.2 we define, for each non-negative integer n, N_n^*, inductively by

$$N_0^*(L) = L \text{ and } N_n^* = N^*(N_{n-1}^*(L)) \text{ for } n > 0.$$

Clearly the series

$$L = N_0^*(L) \supseteq N_1^*(L) \supseteq \dots$$

will terminate in an equality, so we put $N_\infty^*(L)$ equal to the minimal subalgebra in this series. It is easy to see that $N_\infty^*(N_\infty^*(L)) = N_\infty^*(L)$. Then we have

Proposition 4.6 *Let $n \in \mathbb{N} \cup \{0\}$, and let I, J be ideals of the Lie algebra L over the field F. Then*

(i) if $N_k^*(I)$ is a nilregular ideal of $N_k^*(L)$ then $N_{k+1}^*(I)$ is a characteristic ideal of $N_k^*(L)$ for $k \geq 0$;

(ii) if $I \subseteq N_n^*(L))$ is an ideal of L then $N_{n+1}^*(L)/I \subseteq N_{n+1}^*(L/I)$.

(iii) if $L = I \oplus J$, then $N_k^*(L) = N_k^*(I) \oplus N_k^*(J)$ for all $k \geq 0$.

Proof.

(i) This follows from Theorem 2.1 (i) and Lemma 3.1.

(ii) The case $n = 1$ is given by Proposition 4.5. So suppose that the case $n = k$ holds, where $k \geq 1$, and let $I \subseteq N_k^*(L)$. Then $I \subseteq N_{k-1}^*(L)$. Hence

$$\frac{N_{k+1}^*(L)}{I} = \frac{N^*(N_k^*(L))}{I} \subseteq N^*\left(\frac{N_k^*(L)}{I}\right) \subseteq N^*\left(N_k^*\left(\frac{L}{I}\right)\right) = N_{k+1}^*\left(\frac{L}{I}\right).$$

The result now follows by induction

(iii) This is a straightforward induction proof: the case $k = 1$ is given by Proposition 3.11

□

Corollary 4.7 *Let $n \in \mathbb{N}$, and let I, J be ideals of the Lie algebra L over the field F. Then*

(i) if $N^*_\infty(I)$ is nilregular, it is a characteristic ideal of $N^*_\infty(L)$;

(ii) if $I \subseteq N^*_\infty(L)$ is an ideal of L then $N^*_\infty(L)/I \subseteq N^*_\infty(L/I)$.

(iii) if $L = I \oplus J$, then $N^*_\infty(L) = N^*_\infty(I) \oplus N^*_\infty(J)$.

5 The quasi-nilpotent radical

Here we construct a radical by adjoining the quasi-simple ideals of L to the nilradical N. Since quasi-simple ideals are quasi-minimal they are characteristic in L.

Lemma 5.1 *Let $L/Z(L)$ be simple. Then $L = L^2 + Z(L)$ and L^2 is quasi-simple.*

Proof. Let $P = L^2$ and $\overline{L} = L/Z(L)$. Then $\overline{P}$ is an ideal of $\overline{L}$ and $\overline{L}$ is simple, so $\overline{P} = 0$ or $\overline{L}$. The former implies that L is abelian, a contradiction. Hence $\overline{P} = \overline{L}$, and so $L = P + Z(L) = L^2 + Z(L)$. Also, $P = L^2 = P^2$ and $P/Z(P) = P/P \cap Z(L) \cong (P + Z(L))/Z(L) = L/Z(L)$ is simple. □

Lemma 5.2 *Let A be a quasi-simple ideal of L and B an ideal of L. Then either $A \subseteq B$ or $A \subseteq C_L(B)$.*

Proof. Since quasi-simple ideals are quasi-minimal the result follows from Proposition 3.3. □

The *quasi-simple components* of L are its quasi-simple ideals. We will write SComp(L) for the set of quasi-simple components of L, and put $\hat{E}(L) = <\text{SComp}(L)>$, the subalgebra generated by the quasi-simple components of L. Clearly SComp(L) $\subseteq$ MComp(L), $\hat{E}(L) \subseteq E^\dagger(L)$ and $\hat{E}(L)$ is characteristic in L.

Lemma 5.3 *If B is an ideal of L, then SComp(B) = SComp(L) $\cap B$.*

Proof. If A is a quasi-simple ideal of B, it is an ideal of L since it is characteristic in B, and so $\mathrm{SComp}(B) \subseteq \mathrm{SComp}(L) \cap B$. The reverse inclusion is clear. □

Proposition 5.4 *Let $P \in SComp(L)$ and let B be an ideal of L. Then $P \in SComp(B)$ or $[P, B] = 0$.*

Proof. Suppose that $[P, B] \neq 0$. We have that P is a quasi-simple ideal of L, so $P \subseteq B$, by Lemma 5.2. Hence $P \in \mathrm{SComp}(B)$, by Lemma 5.3. □

Corollary 5.5 *Distinct quasi-simple components of L commute, so*

$$\hat{E}(L) = \sum_{P \in SComp(L)} P,$$

where $[P, Q] = 0$ and $P \cap Q \subseteq Z(R)$ for all $P, Q \in SComp(L)$.

Proof. This follows easily as in Corollary 3.5. □

Theorem 5.6 *Suppose that L is a Lie algebra in which $E^{\dagger}(L)$ is regular, then $\hat{E}(L) = E^{\dagger}(L)$.*

Proof. Let P be a quasi-simple ideal of L. Then $N(P)$ and $R(P)$ are ideals of $E^{\dagger}(L)$, by Corollary 5.5. It follows that P is a regular ideal of L and the result follows from Corollary 2.6. □

Clearly, if L is as in Example 2.4 we have $\hat{E}(L) = 0 \neq A = E^{\dagger}(L)$, so Theorem 5.6 does not hold for all Lie algebras.

Corollary 5.7 *Let L be a Lie algebra in which $E^{\dagger}(L)$ and $C_L(N)$ are regular. Put $Z = Z(N)$, $\overline{L} = L/Z$, $\overline{S} = Soc(\overline{C_L(N)})$. Then $\hat{E}(L) = S^2$ and $S = \hat{E}(L) + Z$.*

Proof. This follows from Proposition 3.7 and Theorem 5.6. □

We define the *quasi-nilpotent radical* of L to be $\hat{N}(L) = N + \hat{E}(L)$. From now on we will denote $\hat{N}(L)$ simply by $\hat{N}$. The following is an immediate consequence of Theorems 3.8 and 5.6.

Corollary 5.8 *Suppose that L is a Lie algebra in which $N^{\dagger}(L)$ is regular. Then $C_L(\hat{N}) = Z(N)$. In particular $C_L(\hat{N}) \subseteq \hat{N}$.*

Once more, Example 2.4 shows that the above result does not hold without some restrictions. For, if L is as in that example, then $\hat{N}(L) = 0$ and $C_L(\hat{N}(L)) = L$.

Proposition 5.9 *Let L be a Lie algebra a field F, and let B be a nilregular ideal of L. Then $\hat{N}(B) \subseteq \hat{N}$.*

Proof. Under the given hypotheses $N(B)$ is a characteristic ideal of B (see [6]), so $N(B) \subseteq N$. Moreover, $\hat{E}(B) \subseteq \hat{E}(L)$ by Lemma 5.3. □

Proposition 5.10 *Let L be a Lie algebra over any field. Then $\hat{N}(\hat{N}) = \hat{N}$.*

Proof. Clearly $\hat{N}(\hat{N}) \subseteq \hat{N}$. But $\hat{E}(\hat{N}) = \hat{E}(L)$, by Lemma 5.3, and, clearly, $N \subseteq N(\hat{N})$, giving the reverse inclusion. □

Proposition 5.11 *Let L be a Lie algebra over any field, and let I be an ideal of L. Then*

$$\frac{\hat{N}(L) + I}{I} \subseteq \hat{N}\left(\frac{L}{I}\right).$$

Proof. This follows exactly as in Proposition 3.10. □

6 Another generalisation of the nilradical

We put $\tilde{N}(L)/\phi(L) = \mathrm{Soc}(L/\phi(L))$. We write $\tilde{N}(L)$ simply as $\tilde{N}$. Then we see that this radical also has our desired property.

Theorem 6.1 *Let L be a Lie algebra over any field, with nilradical N. Then $C_L(\tilde{N}) \subseteq Z(N) \subseteq \tilde{N}$.*

Proof. Put $C = C_L(\tilde{N})$. Suppose first that $\phi(L) = 0$. Then $L = N \dot{+} U$ where $N = \mathrm{Asoc} L$ and U is a subalgebra of L, by [10, Theorems 7.3 and 7.4]. Then $C = N \dot{+} C \cap U$ and $C \cap U$ is an ideal of L. Suppose that $C \cap U \neq 0$ and let A be a minimal ideal of L with $A \subseteq C \cap U$. Then $A \subseteq \tilde{N}$, so $A^2 \subseteq [\tilde{N}, C] = 0$. Hence $A \subseteq N \cap U = 0$, a contradiction. It follows that $C = N$.

If $\phi(L) \neq 0$ we have

$$\frac{C + \phi(L)}{\phi(L)} \subseteq C_{L/\phi(L)}\left(\frac{\tilde{N}}{\phi(L)}\right) \subseteq \frac{N}{\phi(L)}.$$

Hence $C \subseteq N$, which yields $C \subseteq Z(N)$. □

Theorem 6.2 *Let L be a ϕ-free Lie algebra over any field F and suppose that $\tilde{N}(L)$ is nilregular. Then $L/C_L(\tilde{N}(L))$ is isomorphic to a subalgebra of*

$$\mathcal{M}_r^- \oplus \left(\bigoplus_{i=1}^{s} Der(A_i) \right)$$

where $\mathcal{M}_r$ is the set of $r \times r$ matrices over F, r is the dimension of the nilradical, and $A_1, \ldots, A_s$ are the simple minimal ideals of L.

Proof. Since L is ϕ-free we have that $\tilde{N}(L) = N(L) \oplus (\oplus_{i=1}^{r} A_i)$ where $A_1, \ldots, A_r$ are the non-abelian minimal ideals of L. Also, each A_i is nilregular and hence simple, by Corollary 2.4. The map $\theta : L \to \mathrm{Der}\,(\tilde{N}(L))$ given by $\theta(x) = \mathrm{ad}\, x \mid_{\tilde{N}(L)}$ is a homomorphism with kernel $C_L(\tilde{N}(L))$. But $N(L)$ is characteristic, since it is nilregular, and the A_i's are characteristic, since they are perfect, so

$$\mathrm{Der}\,(\tilde{N}(L)) = \mathrm{Der}\,(N(L)) \oplus \left(\bigoplus_{i=1}^{s} \mathrm{Der}\,(A_i) \right),$$

whence the result. $\square$

Proposition 6.3 $N^* \subseteq \tilde{N}$.

Proof. There is a subalgebra $U/\phi(L)$ of $L/\phi(L)$ such that $L/\phi(L) = N/\phi(L) \dot{+} U/\phi(L)$, by [10, Theorems 7.3 and 7.4]. Let A/N be a minimal ideal of L/N with $A \subseteq N + C_L(N)$. Then $A = N \dot{+} A \cap U$, so $[N, A] = [N, N + A \cap C] \subseteq \phi(L)$ and $A \cap U/\phi(L)$ is a minimal ideal of $L/\phi(L)$. Moreover, $N/\phi(L) \subseteq \mathrm{Soc}(L/\phi(L))$, by [10, Theorem 7.4]. Hence $A/N \subseteq \mathrm{Soc}(L/\phi(L))$, and so $N^* \subseteq \tilde{N}$. $\square$

In general we can have $N^* \subset \tilde{N}$ and $\tilde{N}(\tilde{N}) \subset \tilde{N}$, as we will show below. Recall that the category $\mathcal{O}$ is a mathematical object in the representation theory of semisimple Lie algebras. It is a category whose objects are certain representations of a semisimple Lie algebra and morphisms are homomorphisms of representations. The formal definition and its properties can be found in [5]. As in other artinian module categories, it follows from the existence of enough projectives that each $M \in \mathcal{O}$ has a *projective cover* $\pi : P \to M$. Here π is an epimorphism and is essential, meaning that no proper submodule of the projective module P is mapped onto M. Up to isomorphism the module P is the unique projective having this property (see [5, page 62]).

EXAMPLE **6.1** *So let S be a finite-dimensional simple Lie algebra over a field F of prime characteristic, let P be the projective cover for the trivial irreducible S-module and let R be the radical of P. Then R is a faithful irreducible S-module and P/R is the trivial irreducible S-module. Let $T = P \rtimes S$ be the semidirect sum of P and S. Then $T^2 = R \rtimes S$ is a primitive Lie algebra of type 1 and $\dim(T/T^2) = 1$, say $T = T^2 + Fx$. Put $L = T + Fy$ where $[x, y] = y$ and $[T^2, y] = 0$.*

Then $\phi(T) \subseteq T^2$, so $\phi(T)$ is an ideal of L and $\phi(T) \subseteq \phi(L)$, by [10, Lemma 4.1]. But $\phi(L) \subseteq T$ and, if M is a maximal subalgebra of T then $M + Fy$ is a maximal subalgebra of L, so $\phi(L) = \phi(T) = R$. Also $Soc(L/R) = (T^2 + Fy)/R$, so $\tilde{N}(L) = T^2 \oplus Fy$. However, $N(L) = R \oplus Fy$ and $C_L(N(L)) = N(L)$, so $N^*(L) = N(L) \neq \tilde{N}(L)$.

Moreover, $\phi(\tilde{N}(L)) = 0$, so $\tilde{N}(\tilde{N}(L)) = Soc(\tilde{N}(L)) = R \oplus Fy \neq \tilde{N}(L)$.

Notice that we also have $N^*(L)/\phi(L) = N(L)/R \cong Fy$, whereas $N^*(L/\phi(L)) = T^2 + Fy/R$. Hence the inclusions in Propositions 3.10, 4.5, 4.6 and Corollary 4.7 can be strict.

Note that a similar example can be constructed in characteristic p. Let L be a finite-dimensional restricted Lie algebra over a field F of prime characteristic, and let $u(L)$ denote the restricted universal enveloping algebra of L. Then every restricted L-module is a $u(L)$-module and vice versa, and so there is a bijection between the irreducible restricted L-modules and the irreducible $u(L)$-modules. In particular, as $u(L)$ is finite-dimensional, every irreducible restricted L-module is finite-dimensional. So, in the above example we could take S to be a restricted simple Lie algebra, as the projective cover of the trivial S-module again exists.

Proposition 6.4 *If I is an ideal of L then*

$$\frac{\tilde{N} + I}{I} \subseteq \tilde{N}\left(\frac{L}{I}\right).$$

Moreover, if $I \subseteq \phi(L)$, then $\tilde{N}(L)/I = \tilde{N}(L/I)$.

Proof. Let $A/\phi(L)$ be a minimal ideal of $L/\phi(L)$. Then

$$\frac{A + I/I}{\phi(L) + I/I} \cong \frac{A + I}{\phi(L) + I} \cong \frac{A}{A \cap (\phi(L) + I)}.$$

Now $\phi(L) \subseteq A \cap (\phi(L) + I)$, so $A \cap (\phi(L) + I) = A$ or $\phi(L)$. But $\phi(L) + I/I \subseteq \phi(L/I)$, so the former implies that $A + I/I = \phi(L/I)$ and $A + I/I \subseteq \tilde{N}(L/I)$.

If the latter holds then $A \cap I \subseteq \phi(L)$. But now, $\phi(L)/A \cap I = \phi(L/A \cap I)$, by [10, Proposition 4.3], so

$$\frac{A/A \cap I}{\phi(L/A \cap I)} = \frac{A/A \cap I}{\phi(L)/A \cap I} \cong \frac{A}{\phi(L)}.$$

It follows that $A/A \cap I \subseteq \tilde{N}(L/A \cap I)$, whence $A + I/I \subseteq \tilde{N}(L/I)$.

The second assertion follows from the definition of $\tilde{N}$ and the fact that $\phi(L/I) = \phi(L)/I$. □

Proposition 6.5 $\tilde{N}(L)/\phi(L) = N^*(L/\phi(L))$.

Proof. Suppose first that $\phi(L) = 0$. Then $\tilde{N}(L)$ is the socle of L. Now $N(L) = \text{Asoc}(L)$, by [10, Theorem 7.4]. Also, if A is a minimal ideal of L with $A \not\subseteq N(L) = N$, then $[A, N] \subseteq A \cap N = 0$, so $A \subseteq C_L(N)$. Hence $\tilde{N}(L) \subseteq N^*(L)$.

If $\phi(L) \neq 0$ the above shows that $\tilde{N}(L/\phi(L)) \subseteq N^*(L/\phi(L))$. The result now follows from Propositions 6.3 and 6.4. □

Proposition 6.6 *If* $L = I \oplus J$*, then* $\tilde{N}(L) = \tilde{N}(I) \oplus \tilde{N}(J)$.

Proof. We have that $N(L) = N(I) \oplus N(J)$ and $\phi(L) = \phi(I) \oplus \phi(J)$ by [10, Theorem 4.8]. Let $A/\phi(L)$ be a minimal ideal of $L/\phi(L)$ and suppose that $A \not\subseteq N(L)$. Then $A = A^2 + \phi(L)$. But $\phi(L) = \phi(I) \oplus \phi(J)$, by [10, Theorem 4.8], so

$$A = A^2 + \phi(I) + \phi(J) = [A, I] + \phi(I) + [A, J] + \phi(J).$$

Hence

$$\frac{A}{\phi(L)} \cong \frac{[A, I] + \phi(I)}{\phi(I)} \oplus \frac{[A, J] + \phi(J)}{\phi(J)}.$$

It is easy to see that the direct summands are minimal ideals of $I/\phi(I)$ and $J/\phi(J)$ respectively, so $\tilde{N}(L) \subseteq \tilde{N}(I) \oplus \tilde{N}(J)$. Also, if $A/\phi(I)$ is a minimal ideal of $I/\phi(I)$, then $A + \phi(J)/\phi(L)$ is a minimal ideal of $L/\phi(L)$, so $\tilde{N}(I) \subseteq \tilde{N}(L)$. Similarly $\tilde{N}(J)) \subseteq \tilde{N}(L)$, which gives the result. □

As a result of Example 6.1 we define, for each non-negative integer n, $\tilde{N}_n(L)$ inductively by

$$\tilde{N}_0(L) = L \text{ and } \tilde{N}_n(L) = \tilde{N}(\tilde{N}_{n-1}(L)) \text{ for } n > 0.$$

Clearly the series

$$L = \tilde{N}_0(L) \supseteq \tilde{N}_1(L) \supseteq \dots$$

will terminate in an equality, so we put $\tilde{N}_\infty(L)$ equal to the minimal subalgebra in this series. It is easy to see that $\tilde{N}_\infty(\tilde{N}_\infty(L)) = \tilde{N}_\infty(L)$.

Proposition 6.7 *Let $n \in \mathbb{N} \cup \{0\}$, and let I, J be ideals of the Lie algebra L over a field F.*

(i) *If $I \subseteq \phi(\tilde{N}_{n-1}(L))$ then $\tilde{N}_n(L/I) = \tilde{N}_n(L)/I$.*

(ii) *$N(\tilde{N}_n(L)) \subseteq N(\tilde{N}_{n+1}(L))$ for each $n \geq 0$.*

(iii) *If $\tilde{N}_\infty(L)$ is nilregular, then $\phi(\tilde{N}_{n+1}(L)) \subseteq \phi(\tilde{N}_n(L))$ for each $n \geq 0$.*

(iv) *If $\tilde{N}_\infty(L)$ is nilregular then $N(\tilde{N}_n(L)) = N(L)$ and $\tilde{N}_n(L)$ is an ideal of L for all $n \geq 0$.*

(v) *If $N^*(L)$ is nilregular then $N^*(L) \subseteq \tilde{N}_n(L)$ for each $n \geq 0$.*

(vi) *If $\tilde{N}_n(L)$ is nilregular and $\phi(\tilde{N}_n(L)) = 0$ then $\tilde{N}_{n+1}(L) = N^*(L)$.*

(vii) *If $N^*(L)$ is nilregular then $C_L(\tilde{N}_n(L)) = Z(N(L))$.*

(viii) *If F has characteristic zero, then $\tilde{N}_n(I) \subseteq \tilde{N}_n(L)$.*

(ix) *If F has characteristic zero, then $\tilde{N}_n(L) + I/I \subseteq \tilde{N}_n(L/I)$.*

(x) *If $L = I \oplus J$ then $\tilde{N}_n(L) = \tilde{N}_n(I) \oplus \tilde{N}_n(J)$.*

Proof.

(i) The case $n = 1$ is given by Proposition 6.4. A straightforward induction argument then yields the general case.

(ii) We have that $N(L) \subseteq \tilde{N}(L)$, by [10, Theorem 7.4], whence $N(L) \subseteq N(\tilde{N}(L))$. Thus $N(\tilde{N}(L)) \subseteq N(\tilde{N}_2(L))$, and a simple induction argument gives the general result.

(iii) Put $\tilde{N}_i = \tilde{N}_i(L)$. Then

$$\frac{\tilde{N}_{n+1}}{\phi(\tilde{N}_n)} = \bigoplus_{i=1}^{r} \frac{A_i}{\phi(\tilde{N}_n)},$$

where each direct summand is a minimal ideal of $\tilde{N}_n/\phi(\tilde{N}_n)$. Now

$$N\left(\frac{A_i}{\phi(\tilde{N}_n)}\right) \subseteq N\left(\frac{\tilde{N}_n}{\phi(\tilde{N}_n)}\right) = \frac{N(\tilde{N}_n)}{\phi(\tilde{N}_n)}$$

and $N(\tilde{N}_n) \subseteq N(\tilde{N}_\infty)$ by (ii), so the direct summands are nilregular, and hence are abelian or simple, by Corollary 2.4. It follows that they are ϕ-free, and thus, so is $\tilde{N}_{n+1}/\phi(\tilde{N}_n)$. The result follows.

(iv) Consider the first assertion: it clearly holds for $n = 0$. Suppose that $\tilde{N}_\infty(L)$ is nilregular and that the result holds for $k \leq n$ ($n \geq 0$). Then $\tilde{N}_k(L)$ is nilregular for all $k \geq 0$, by (ii). It follows from [8, Corollary 1] that $N(\tilde{N}_n(L))$ is a characteristic ideal of $\tilde{N}_n(L)$, and hence an ideal of $\tilde{N}_{n-1}(L)$. Thus $N(\tilde{N}_n(L)) = N(\tilde{N}_{n-1}(L))$, and so $N(\tilde{N}_n(L)) = N(L)$ by the inductive hypothesis, which proves the first assertion.

Put $\tilde{N}_n = \tilde{N}_n(L)$, $\phi_n = \phi(\tilde{N}_n)$ and let A/ϕ_n be a minimal ideal of $\tilde{N}_n/\phi_n$. If $A \not\subseteq N(\tilde{N}_n)$, then A/ϕ_n is a perfect subideal of L/ϕ_n and so an ideal of L/ϕ_n, by Lemma 1.1. The result follows.

(v) The case $n = 1$ is Proposition 6.3. So suppose that $N^*(L) \subseteq \tilde{N}_k(L)$ for some $k \geq 1$. Then

$$N^*(L) = N^*(N^*(L)) \subseteq N^*(\tilde{N}_k(L)) \subseteq \tilde{N}_{k+1}(L),$$

by Propositions 3.9, 3.12 and 6.3.

(vi) If $\phi(\tilde{N}_n(L)) = 0$ then

$$\tilde{N}_{n+1}(L) \subseteq N^*(\tilde{N}_n(L)) \subseteq N^*(L) \subseteq \tilde{N}_{n+1}(L),$$

since $\tilde{N}_n(L)$ is nilregular (and hence so is $N^*(L)$), by Propositions 6.5, 3.12 and (v) above.

(vii) Using (v) above we have that $C_L(\tilde{N}_n(L)) \subseteq C_L(N^*(L)) = Z(N)$, by Theorem 3.8.

(viii) We have $\phi(I) \subseteq \phi(L)$, by [10, Corollary 3.3], so $\tilde{N}(L/\phi(I)) = \tilde{N}(L)/\phi(I)$. Now

$$\tilde{N}(I)/\phi(I) = N^*(I/\phi(I) \subseteq N^*(L/\phi(I)) \subseteq \tilde{N}(L/\phi(I)) = \tilde{N}(L)/\phi(I),$$

by Propositions 6.5, 3.12 and 6.3. Hence $\tilde{N}(I) \subseteq \tilde{N}(L)$. Then a simple induction proof shows that $\tilde{N}_n(I) \subseteq \tilde{N}_n(L)$.

(ix) The case $n = 1$ is given by Proposition 6.4. Suppose it holds for some $k \geq 1$. Then

$$\frac{\tilde{N}_{k+1}(L) + I}{I} = \frac{\tilde{N}(\tilde{N}_k(L)) + I}{I} \subseteq \frac{\tilde{N}(\tilde{N}_k(L) + I) + I}{I}$$
$$\subseteq \tilde{N}\left(\frac{\tilde{N}_k(L) + I}{I}\right) \subseteq \tilde{N}\left(\tilde{N}_k\left(\frac{L}{I}\right)\right) = \tilde{N}_{k+1}\left(\frac{L}{I}\right),$$

by (viii) and Proposition 6.4.

(x) The case $n = 1$ is given by Proposition 6.6. A straightforward induction argument then gives the general result.

□

Corollary 6.8 *Let I, J be ideals of L.*

(i) If $I \subseteq \phi(\tilde{N}_\infty(L))$ then $\tilde{N}_\infty(L/I) = \tilde{N}_\infty(L)/I$.

(ii) If $\tilde{N}_\infty(L)$ is nilregular the $N(\tilde{N}_\infty(L)) = N(L)$ and $\tilde{N}_\infty(L)$ is an ideal of L.

(iii) If $N^*(L)$ is nilregular then $N^*(L) \subseteq \tilde{N}_\infty(L)$.

(iv) If $\tilde{N}_\infty(L)$ is nilregular and $\phi(\tilde{N}_\infty(L) = 0$ then $\tilde{N}_\infty(L) = N^*(L)$.

(v) If $N^*(L)$ is nilregular then $C_L(\tilde{N}_\infty(L)) = Z(N(L))$.

(vi) If F has characteristic zero, then $\tilde{N}_\infty(I) \subseteq \tilde{N}_\infty(L)$.

(vii) If F has characteristic zero, then $\tilde{N}_\infty(L) + I/I \subseteq \tilde{N}_\infty(L/I)$;

(viii) If $L = I \oplus J$ then $\tilde{N}_\infty(L) = \tilde{N}_\infty(I) \oplus \tilde{N}_\infty(J)$.

If S is a subalgebra of L the *core* of S, S_L, is the biggest ideal of L contained in S. The following is an analogue of a result for groups given by Vasil'ev et al. in [12].

Theorem 6.9 *Let L be a Lie algebra over any field. Then the core of the intersection of all maximal subalgebras such that $L = M + \tilde{N}(L)$ is equal to $\phi(L)$.*

Proof. Put P equal to the intersection of all maximal subalgebras such that $L = M + \tilde{N}$. Clearly $\tilde{N} \not\subseteq \phi(L)$ and $\phi(L) \subseteq P_L$. Factor out $\phi(L)$ and suppose that $P_L \neq 0$. Let A be a minimal ideal of L contained in P_L. Then $A \subseteq \tilde{N}(L)$.

Since $\phi(L) = 0$ there is a maximal subalgebra of L such that $A \not\subseteq M$. If $L = \tilde{N}(L) + M$ we have $A \subseteq P_L \subseteq M$, a contradiction. If not, then $A \subseteq \tilde{N}(L) \subseteq M$, a contradiction again. Hence $P_L = 0$.

It follows that $P_L \subseteq \phi(L)$, whence the result.

□

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