Pure Higher-Order Effects in the Portfolio Choice Model

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Abstract

This paper examines the effects of higher-order risk attitudes and statistical moments on the optimal allocation of risky assets within the standard portfolio choice model. We derive the expressions for the optimal proportion of wealth invested in the risky asset to show they are functions of portfolio returns third- and fourth-order moments as well as on the investor's risk preferences of prudence and temperance. We illustrate the relative importance that the introduction of those higher-order effects have in the decision of expected utility maximizers using data for the US.

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1 Introduction

Following the theoretical contribution of Eeckhoudt and Schlesinger (2006) which set out lottery preference definitions, experimental studies were reported which examined the apportion of risks consistent with the higher-order risk preferences of prudence and temperance.[1] The reported experimental results revealed that a significant proportion of individuals make prudent and temperate choices consistent with standard expected utility theory (Ebert and Wiesen, 2014; Deck and Schlesinger, 2010).[2] Ebert and Wiesen (2014) provide an excellent review of the theoretical literature which examines the role of higher-order risk attitudes such as prudence and temperance on decision making in areas such as precautionary savings, monetary policy, insurance demand, and bidding in auctions. Despite these contributions, the solution of the classical portfolio choice model, i.e., the optimal proportion of wealth that an agent invests in the risky asset, has typically been obtained through a first-order Taylor approximation around a portfolio risk of zero (see Gollier, 2001). As a consequence the higher-order risk preferences play no role in portfolio choice that depends only on the mean and variance of returns and the investor's first- and second-order risk attitudes. To the best of our knowledge only the papers by Athayde and Flôres (2004) and Zakamouline and Koekebakker (2009) provide a closed-form solution up to the third-order moment for the portfolio choice model.[3] Otherwise models of optimal portfolio weights that incorporate higher-order effects have generally been obtained either

[1] The importance of the third derivative of utility u ($u''' > 0$) in determining demand for precautionary savings defines prudence according to Kimball (1990). Behavioral aspects of investors have been related to the fourth-order derivative of the utility function ($u^{iv} < 0$) through the concept of temperance introduced by Kimball (1992).

[2] Those experimental results are not surprising given that most commonly used expected utility theory functions imply prudent and temperate choices. These utility functions exhibit mixed risk aversion, i.e., nth-degree risk aversion for all orders (Ebert, 2013).

[3] This approach allows Zakamouline and Koekebakker to present a theoretically sound portfolio performance measure that takes into account the skewness of the distribution of returns. Their Adjusted for Skewness Sharpe Ratio has a direct relation to the level of expected utility provided by the asset. This is in contrast to many other *arbitrary* –not theoretically founded– reward-to-risk ratios such as performance measures based on Value-at-Risk. In Athayde and Flôres (2004) the Markowitz's efficient frontier is extended to a three-moments multidimensional portfolio choice framework.

as implicit solution (see Guidolin and Timmerman, 2008; Jondeau and Rockinger, 2006) or using numerical optimization (see Kim et al., 2014).

This article contributes to the literature by providing expressions for the optimal asset allocation in the classical portfolio problem that give an explicit role to the effects of higher-order investor's risk preferences of prudence and temperance as well as higher-order moments. We present an example employing US data to provide an intuition on the relative importance that the introduction of those higher-order effects could have to interpret investors' decisions.

The remainder of the paper is organized as follows. Section 2 presents the standard portfolio choice model and our derivation of the optimal portfolio allocation using higher-order Taylor approximations. Section 3 is an illustrative example of the model using actual data for the US. Section 4 summarizes the conclusions.

2 Higher-order risk preferences in the classical portfolio choice model

Consider an investor with a utility function u and initial wealth W that she can invest in risk-free and risky assets. Let r and $\widetilde{x}_0$ be the after-one-period sure and random net return of risk-free and risky assets, respectively. The problem of the agent is to choose the optimal amount of initial wealth invested in the risky asset, α, that maximizes her expected utility $V(\alpha)$

$$\underset{\{\alpha\}}{Max} V(\alpha) = Eu(\omega_0 + \alpha\widetilde{x}), \tag{1}$$

where $\widetilde{x} = \widetilde{x}_0 - r$ is the excess return, $\omega_0 = W(1+r)$ and $\alpha\widetilde{x}$ are after-one-period sure and random wealth, respectively. To determine the solution of eq. (1) we assume that the portfolio risk, k, is small, and as k is endogenous, we define the excess return, as is standard, as $\widetilde{x} = k\mu + \widetilde{y}$, where $E\widetilde{y} = 0$, $\mu > 0$, and $E\widetilde{x}$ is the risk premium.[4]

In order to employ the relevant information contained in returns' moments and investor's risk preferences up to the fourth-order, we use a 3rd-order Taylor expansion of $\alpha^*(k)$ around

[4] k may be negative, i.e., the model allows a short-sale of the risky asset; see proposition 6 in Gollier (2001, p. 54).

$k = 0$, after some calculations we obtain the optimal portfolio weight as:[5,6]

$$\alpha^*_{(3)} \simeq \left(\frac{\left(E(\widetilde{x}-E\widetilde{x})^3\right)^2}{2(V\widetilde{x})^5} \frac{P(\omega_0)^2}{A(\omega_0)^3} + \left(\frac{4}{3(V\widetilde{x})^2} + \frac{E(\widetilde{x}-E\widetilde{x})^3}{6(V\widetilde{x})^3} \right) \frac{P(\omega_0)}{A(\omega_0)^2} - \frac{E(\widetilde{x}-E\widetilde{x})^4}{6(V\widetilde{x})^4} \frac{T(\omega_0)P(\omega_0)}{A(\omega_0)^3} - \frac{1}{A(\omega_0)(V\widetilde{x})^2} \right) (E\widetilde{x})^3$$

$$+ \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P(\omega_0)}{A(\omega_0)^2} (E\widetilde{x})^2 + \frac{1}{A(\omega_0)V\widetilde{x}} E\widetilde{x}$$

$$= Z(\cdot)(E\widetilde{x})^3 + \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P(\omega_0)}{A(\omega_0)^2} (E\widetilde{x})^2 + \frac{1}{A(\omega_0)V\widetilde{x}} E\widetilde{x}, \tag{2}$$

where $A(\omega_0) = -Eu''(\omega_0)/Eu'(\omega_0)$, is the Arrow-Pratt index of absolute risk aversion, $P(\omega_0) = -Eu'''(\omega_0)/Eu''(\omega_0)$ and $T(\omega_0) = -Eu^{iv}(\omega_0)/Eu'''(\omega_0)$ are the investor's degree of absolute prudence and temperance, respectively, $V\widetilde{x}$ denotes the variance of $\widetilde{x}$, and $E(\widetilde{x}-E\widetilde{x})^3$ and $E(\widetilde{x}-E\widetilde{x})^4$ are the third- and fourth-order central moments of $\widetilde{x}$, respectively. We note that $Z(\cdot)$ is a function of $\widetilde{x}$'s four first-order moments and investor's risk preferences up to temperance.[7]

By dividing equation (2) by sure wealth, ω_0, we obtain the 3rd-order Taylor approximated optimal share of the portfolio invested in the risky asset as

$$\alpha^*_{P,(3)} \simeq \left(\frac{\left(E(\widetilde{x}-E\widetilde{x})^3\right)^2}{2(V\widetilde{x})^5} \frac{P_R(\omega_0)^2}{R(\omega_0)^3} + \left(\frac{4}{3(V\widetilde{x})^2} + \frac{E(\widetilde{x}-E\widetilde{x})^3}{6(V\widetilde{x})^3} \right) \frac{P_R(\omega_0)}{R(\omega_0)^2} - \frac{E(\widetilde{x}-E\widetilde{x})^4}{6(V\widetilde{x})^4} \frac{T_R(\omega_0)P_R(\omega_0)}{R(\omega_0)^3} - \frac{1}{R(\omega_0)(V\widetilde{x})^2} \right) (E\widetilde{x})^3$$

$$+ \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P_R(\omega_0)}{R(\omega_0)^2} (E\widetilde{x})^2 + \frac{1}{R(\omega_0)V\widetilde{x}} E\widetilde{x}$$

$$= Z_R(\cdot)(E\widetilde{x})^3 + \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P_R(\omega_0)}{R(\omega_0)^2} (E\widetilde{x})^2 + \frac{1}{R(\omega_0)V\widetilde{x}} E\widetilde{x}, \tag{3}$$

where $Z_R(\cdot)$ depends on distributional moments of $\widetilde{x}$ up to the fourth order, and the relative measures of risk aversion, $R(\omega_0) = \omega_0 A(\omega_0)$, prudence, $P_R(\omega_0) = \omega_0 P(\omega_0)$, and temperance, $T_R(\omega_0) = \omega_0 T(\omega_0)$.

Expression (3) provides a direct and explicit relationship between the optimal portfolio

[5] Note that the optimal investment in the risky asset, $\alpha^*(k)$, depends on k, so if $E\widetilde{x} = 0$, i.e., $k = 0$, it is optimal to invest 0 units of wealth in the risky asset; $\alpha^*(0) = 0$. $\alpha^*(k)$ is obtained assuming $k > 0$.

[6] As is usual we assume that the moments of $\widetilde{y}$ are constant, i.e., $E\widetilde{y}^n u^{(n)}(\omega_0) = E\widetilde{y}^n Eu^{(n)}(\omega_0)\ \forall n$.

[7] To save space, the derivation of expression (2) is provided in the Appendix.

decision and both the distributional moments and risk attitudes up to fourth order.[8] In the context of applied research, (3) also provides an interpretation of the coefficients in a regression of the risky asset share on the moments of the return of the risky asset (see, e.g., Mitton and Vorkink, 2007).

In Table 1 we present the expressions of the marginal effect of the nth moment of $\widetilde{x}$ on $\alpha^*_{P,(\cdot)}$. This illustrates that the optimal portfolio choice depends on the whole risk profile of the investor as well as on the four first moments of the return distribution. The marginal effect of even the first two moments of the distribution on $\alpha^*_{P,(2)}$ and $\alpha^*_{P,(3)}$ depends on the level of both higher-order moments and risk attitudes.

Our formulae (3) nests previous results in the literature that have employed lower-order Taylor approximations to obtain the optimal solution of the portfolio choice model.[9] For example, the textbook first-order Taylor expansion of $\alpha^*(k)$ around $k = 0$, gives the optimal proportion invested in risky assets as a function of only mean, variance, and relative risk aversion:

$$\alpha^*_{P,(1)} \simeq \frac{E\widetilde{x}}{V\widetilde{x}} \frac{1}{R(\omega_0)}. \tag{4}$$

The expressions of the marginal effect of $\widetilde{x}$'s nth moment on $\alpha^*_{P,(1)}$ in Table 1 illustrate that omission of higher-order risk attitudes, such as prudence and temperance, may mislead the calculation of the demand for risky assets. For instance, the marginal effect of an increase in the variance of the risky asset on the investment allocated to that asset, $\frac{\partial \alpha^*_{P,(i)}}{\partial V\widetilde{x}}$, would be overestimated when the asset return exhibits negative skewness and $\alpha^*_{P,(1)}$ is employed instead of $\alpha^*_{P,(2)}$ or $\alpha^*_{P,(3)}$.

[8]Conditions exist for the class of HARA utility functions to ensure that higher-order Taylor expansions better approximate the exact expected utility. These conditions apply on the variable transformed to have a symmetric distribution; see Garlappi and Skoulakis (2011). This, in general, does not have to be the case for expected log utility, as shown in Hlawitschka (1994).

[9]In particular, the 2nd-order Taylor approximation for $\alpha^*(k)$ that would map the solution in Zakamouline and Koekebakker (2009) is as follows

$$\alpha^*_{(2)} \simeq \frac{E\widetilde{x}}{A(\omega_0)V\widetilde{x}} + \frac{E(\widetilde{x} - E\widetilde{x})^3}{2(V\widetilde{x})^3} \frac{P(\omega_0)}{A(\omega_0)^2} (E\widetilde{x})^2.$$

This solution only takes account of up to 3rd-order statistical moments and risk preferences.

Table 1

Pure effects of moments on optimal portfolio choice

	$\alpha^*_{P,(1)}$	$\alpha^*_{P,(2)}$	$\alpha^*_{P,(3)}$
$\frac{\partial \alpha^*_{P,(i)}}{\partial E\widetilde{x}}$	$\frac{1}{V\widetilde{x}R(\omega_0)}$	$\frac{1}{V\widetilde{x}R(\omega_0)} + \frac{E(\widetilde{x}-E\widetilde{x})^3}{(Var\widetilde{x})^3}\frac{P_R(\omega_0)}{R(\omega_0)^2}E\widetilde{x}$	$\frac{1}{V\widetilde{x}R(\omega_0)} + \left(\frac{E(\widetilde{x}-E\widetilde{x})^3}{(V\widetilde{x})^3}\frac{P_R(\omega_0)}{R(\omega_0)^2}\right)E\widetilde{x} + 3Z_R(\cdot)(E\widetilde{x})^2$
$\frac{\partial \alpha^*_{P,(i)}}{\partial V\widetilde{x}}$	$-\frac{E\widetilde{x}}{(V\widetilde{x})^2R(\omega_0)}$	$-\frac{E\widetilde{x}}{(V\widetilde{x})^2R(\omega_0)} - \frac{3E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^4}\frac{P_R(\omega_0)}{R(\omega_0)^2}(E\widetilde{x})^2$	$-\frac{E\widetilde{x}}{(V\widetilde{x})^2R(\omega_0)} - \frac{3E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^4}\frac{P_R(\omega_0)}{R(\omega_0)^2}(E\widetilde{x})^2 + \frac{\partial Z_R(\cdot)}{\partial V\widetilde{x}}(E\widetilde{x})^3$
$\frac{\partial \alpha^*_{P,(i)}}{\partial E(\widetilde{x}-E\widetilde{x})^3}$	0	$\frac{1}{2(V\widetilde{x})^3}\frac{P_R(\omega_0)}{R(\omega_0)^2}(E\widetilde{x})^2$	$\frac{(E\widetilde{x})^2}{2(V\widetilde{x})^3}\frac{P_R(\omega_0)}{R(\omega_0)^2} + \left(\frac{E(\widetilde{x}-E\widetilde{x})^3}{(V\widetilde{x})^5}\frac{P_R(\omega_0)^2}{R(\omega_0)^3} + \frac{1}{6(V\widetilde{x})^3}\frac{P_R(\omega_0)}{R(\omega_0)^2}\right)(E\widetilde{x})^3$
$\frac{\partial \alpha^*_{P,(i)}}{\partial E(\widetilde{x}-E\widetilde{x})^4}$	0	0	$-\frac{1}{6(V\widetilde{x})^4}\frac{T_R(\omega_0)P_R(\omega_0)}{R(\omega_0)^3}(E\widetilde{x})^3$
$\frac{\partial \alpha^*_{P,(i)}}{\partial E(\widetilde{x}-E\widetilde{x})^n}, \forall n > 4$	0	0	0

Notes: This table gathers the marginal effect of $\widetilde{x}$'s 1st-, 2nd- and higher-order moments on the optimal proportion of wealth invested in the risky asset, $\alpha^*_{P,(i)}$, approximated through ith-order (1st, 2nd and 3rd) Taylor expansions, for any utility function.

3 Illustrative example

In this section we illustrate our results with actual data on US household portfolio allocation for a range of values of risk preferences used in previous studies. Data from the Panel Study of Income Dynamics (PSID) from 1983 to 2003 that was employed by Brunnermeier and Nagel (2008) to compute the proportion of wealth that US households invest in risky assets suggests that around fifty six percent of the liquid wealth is invested in risky assets, $\alpha_P^* = 0.56$. To obtain the distributional moments of the excess return of risky assets we employ real returns of the S&P500 from Shiller's website and the real returns on the three month Treasury bill.[10]

In order to analyse the effect of higher-order moments on the optimal portfolio allocation we need to assume values for the coefficients of risk attitudes. Values of the relative risk aversion found in the literature that are consistent with rationality typically lie between one and ten, and sometimes go higher (see, e.g., Tsay and Wu, 2014; Cocco et al, 2005; and Mehra and Prescott, 1985). Studies that use aggregated household data to infer coefficients of relative prudence have found values between four and eleven (Eisenhauer and Ventura, 2003), while others have elicited direct measures through experimental methods and found values close to two (Noussair et al., 2013). In Table 2 we report the optimal proportion of wealth invested in the risky asset, $\alpha_{P,(i)}^*$, $i = 1, 2, 3$, for a range of values of risk attitudes covering most of the coefficients typically used in the literature. We confirm that $\alpha_{P,(\cdot)}^*$ declines as the levels of relative risk aversion, prudence and temperance increase. Our study indicates that higher-order moments exert an effect on the optimal portfolio allocation for a given value of $(R(\omega_0), P_R(\omega_0), T_R(\omega_0))$, and that this effect decreases with the levels of risk attitudes.

For instance, employing a parameterization of our model with levels of relative risk aversion, prudence and temperance, such as, $R(\omega_0) = 3$, $P_R(\omega_0) = 4$, and $T_R(\omega_0) = 5$, would yield optimal investments of 0.68 for $\alpha_{P,(1)}^*$, 0.65 for $\alpha_{P,(2)}^*$ and 0.63 for $\alpha_{P,(3)}^*$. These are proportions of investment in risky assets near the ones observed for the US PSID micro

[10]We assume that agents do not hold information beyond the time period covered in the PSID data and therefore we use the S&P500 and Treasury bill returns data from 1926 and up to 2003.

data in Brunnermeier and Nagel (2008).[11] These results show that higher-order risk attitudes and distributional moments play a significant, yet moderate, role on optimal portfolio choice.

Another set of values worth considering is the one where risk attitudes are consistent with log utility, i.e. $(R(\omega_0), P_R(\omega_0), T_R(\omega_0)) = (1, 2, 3)$. In this case, investors would allocate all their wealth in risky assets, $\alpha^*_{P,(i);i=1,2,3} \geq 1$. This is in line with the classic paper of Feldstein (1969) that analyzed the optimal allocation of wealth between a risk free and a risky asset. He demonstrated that the investor's decision to *plunge,* i.e., optimally allocating all wealth in the risky asset, could occur for reasonable values of the expected and variance of the portfolio return assuming log utility and a log-normal distribution of asset returns.[12]

Table 2

Optimal weights for different levels of risk attitudes

(R, P, T)	(1,2,3)	(3,4,5)	(5,6,7)	(7,8,9)	(9,10,11)	(11,12,13)
$\alpha^*_{P,(1)}$	2.0497	0.6832	0.4099	0.2928	0.2277	0.1863
$\alpha^*_{P,(2)}$	1.9133	0.6529	0.3936	0.2817	0.2193	0.1796
$\alpha^*_{P,(3)}$	1.5957	0.6325	0.3852	0.2764	0.2154	0.1765

Notes: This table displays the optimal proportion of wealth invested in the risky asset, $\alpha^*_{P,(i);i=1,2,3}$, for values of the relative risk aversion, prudence and temperance, $(R, P, T) \equiv (R(\omega_0), P_R(\omega_0), T_R(\omega_0))$, ranging from (1,2,3) to (11,12,13).

4 Conclusions

This paper implements higher-order Taylor expansions to derive an explicit relationship between the optimal portfolio choice of expected utility maximizers and high-order risk attitudes, such as prudence and temperance, as well as high-order statistical moments,

[11]Similar empirical studies to Brunnermeier and Nagel (2008) that use household panel micro data for Italy and the UK find evidence of a relative risk aversion coefficient of approximately 3 (see Chiappori and Paiella (2011) and Paya and Wang (2016)).

[12]This analysis was a counter example to the result of Tobin (1958) who demonstrated the sufficiency of risk aversion, under quadratic utility or two-parameter distributions, to ensure diversification. See Ñíguez et al. (2015) for a more recent comprehensive analysis of optimal portfolio choice under log utility and the role of higher-order moments.

such as skewness and kurtosis. Our results facilitate the analysis of marginal changes in higher-order distributional moments and preferences in the determination of portfolio choice decisions. We provide an intuition about the degree of relevance of those effects based on panel micro data for the US. Comparing these new solutions to the baseline mean-variance model we find important reductions in risky positions depending on the risk attitudes parameterization. Higher-order preferences and moments of the returns' distribution modify the portfolio choice even within this simple, yet relevant, theoretical framework.

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This study analyzed existing data on the US stock market and US Treasury bill rates. Further documentation about the data that were used in this paper is available from the Lancaster University data archive at https://dx.doi.org/10.17635/lancaster/researchdata/88

Appendix

Fully differentiating the first order condition (FOC) of the investor's maximization problem, equation (1), with respect to k, evaluating the resulting equation at $k = 0$, $V''(\alpha^*(k))|_{k=0}$, and clearing for $\alpha^{*\prime}(0)$ gives

$$\alpha^{*\prime}(0) = \frac{\mu}{E\widetilde{y}^2}\frac{1}{A(\omega_0)}. \tag{A.1}$$

In order to embody the risky asset return's third-order moment and the investor's third-order risk attitude of prudence in the standard portfolio model we now consider a 2nd-order

Taylor expansion for $\alpha^*(k)$ around $k=0$, i.e. $\alpha^*(k) \simeq \alpha^*(0) + k\alpha^{*\prime}(0) + \frac{1}{2}k^2\alpha^{*\prime\prime}(0)$; following the same procedure used to obtain expression (A.1), i.e., starting by fully differentiating $V''(\alpha^*(k))$ we obtain

$$\begin{aligned} V'''(\alpha^*(k)) &= 2\mu^2\alpha^*(k)Eu''(\widetilde{\omega}) + 4\mu\alpha^{*\prime}(k)E(k\mu+\widetilde{y})u''(\widetilde{\omega}) \\ &\quad +\mu^2\alpha^*(k)^2E(k\mu+\widetilde{y})u'''(\widetilde{\omega}) + 2\mu\alpha^*(k)\alpha^{*\prime}(k)E(k\mu+\widetilde{y})^2u'''(\widetilde{\omega}) \\ &\quad +\alpha^{*\prime}(k)^2E(k\mu+\widetilde{y})^3u'''(\widetilde{\omega}) + \alpha^{*\prime\prime}(k)E(k\mu+\widetilde{y})^2u''(\widetilde{\omega}) \\ &= 0. \end{aligned} \tag{A.2}$$

Evaluating equation (A.2) at $k=0$ yields

$$\alpha^{*\prime}(0)^2E\widetilde{y}^3u'''(\omega_0) + \alpha^{*\prime\prime}(0)E\widetilde{y}^2u''(\omega_0) = 0. \tag{A.3}$$

Substituting $\alpha^{*\prime}(0)$ and equation (A.1) in equation (A.3) yields

$$\begin{aligned} \alpha^{*\prime\prime}(0) &= \frac{E\widetilde{y}^3}{E\widetilde{y}^2}\left(-\frac{Eu'''(\omega_0)}{Eu''(\omega_0)}\right)\frac{\mu^2}{A(\omega_0)^2(E\widetilde{y}^2)^2} \\ &= \frac{E\widetilde{y}^3}{E\widetilde{y}^2}\frac{P(\omega_0)}{A(\omega_0)^2}\frac{\mu^2}{(E\widetilde{y}^2)^2}. \end{aligned} \tag{A.4}$$

For convenience, we substitute $\widetilde{y} = \widetilde{x} - E\widetilde{x}$ in the expression above and re-write $\alpha^{*\prime\prime}(0)$ as,

$$\alpha^{*\prime\prime}(0) = \frac{E(\widetilde{x}-E\widetilde{x})^3}{E(\widetilde{x}-E\widetilde{x})^2}\frac{P(\omega_0)}{A(\omega_0)^2}\frac{\mu^2}{\left(E(\widetilde{x}-E\widetilde{x})^2\right)^2}. \tag{A.5}$$

Substituting $\alpha^{*\prime}(0)$, equation (A.1), and $\alpha^{*\prime\prime}(0)$ in the 2nd-order Taylor approximation for $\alpha^*(k)$, we obtain the expression for the optimal level of investment in the risky asset, $\alpha^*_{(2)}$, as

$$\alpha^*_{(2)} \simeq \frac{E\widetilde{x}}{A(\omega_0)V\widetilde{x}} + \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{arx})^3}\frac{P(\omega_0)}{A(\omega_0)^2}(E\widetilde{x})^2. \tag{A.6}$$

Dividing equation (A.6) by after-one-period sure wealth, ω_0, we obtain a 2nd-order Taylor approximated optimal share of the portfolio invested in the risky asset, $\alpha^*_{P,(2)}$, as

$$\alpha^*_{P,(2)} \simeq \frac{E\widetilde{x}}{R(\omega_0)V\widetilde{x}} + \frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3}\frac{P_R(\omega_0)}{R(\omega_0)^2}(E\widetilde{x})^2. \tag{A.7}$$

We now proceed to incorporate the risky asset return's fourth-order moment and the investor's fourth-order risk attitude of temperance in the portfolio choice model. Following the procedure described above; full differentiation of equation (A.2) evaluated at $k=0$ yields

$$\begin{aligned} V^{iv}(\alpha^*(k))\big|_{k=0} &= 6\mu^2\alpha^{*\prime}(0)Eu''(\omega_0)+\mu\alpha^{*\prime}(0)^2E\widetilde{y}^3Eu'''(\omega_0) \\ &\quad +8\mu\alpha^{*\prime}(0)^2E\widetilde{y}^2Eu'''(\omega_0)+3\alpha^{*\prime\prime}(0)\alpha^{*\prime}(0)E\widetilde{y}^3u'''(\omega_0) \\ &\quad +\alpha^{*\prime\prime\prime}(0)E\widetilde{y}^2u''(\omega_0)+\alpha^{*\prime}(0)^3E\widetilde{y}^4u^{iv}(\omega_0) \\ &= 0. \end{aligned} \tag{A.8}$$

From equation (A.8) we obtain

$$\begin{aligned} \alpha^{*\prime\prime\prime}(0) &= \left(\tfrac{8}{(E\widetilde{y}^2)^2}+\tfrac{E\widetilde{y}^3}{(E\widetilde{y}^2)^3}\right)\tfrac{P(\omega_0)}{A(\omega_0)^2}\mu^3+\tfrac{3\left(E\widetilde{y}^3\right)^2}{(E\widetilde{y}^2)^5}\tfrac{P(\omega_0)^2}{A(\omega_0)^3}\mu^3 \\ &\quad -\tfrac{E\widetilde{y}^4}{(E\widetilde{y}^2)^4}\tfrac{T(\omega_0)P(\omega_0)}{A(\omega_0)}\mu^3-\tfrac{6}{A(\omega_0)(E\widetilde{y}^2)^2}\mu^3. \end{aligned} \tag{A.9}$$

Substituting $\alpha^{*\prime}(0)$, equation (A.1), $\alpha^{*\prime\prime}(0)$, equation (A.6), and $\alpha^{*\prime\prime\prime}(0)$ in a 3rd-order Taylor expansion for $\alpha^*(k)$ around $k=0$ yields

$$\begin{aligned} \alpha^*_{(3)} &\simeq \left(\tfrac{\left(E(\widetilde{x}-E\widetilde{x})^3\right)^2}{2(V\widetilde{x})^5}\tfrac{P(\omega_0)^2}{A(\omega_0)^3}+\left(\tfrac{4}{3(V\widetilde{x})^2}+\tfrac{E(\widetilde{x}-E\widetilde{x})^3}{6(V\widetilde{x})^3}\right)\tfrac{P(\omega_0)}{A(\omega_0)^2}-\tfrac{E(\widetilde{x}-E\widetilde{x})^4}{6(V\widetilde{x})^4}\tfrac{T(\omega_0)P(\omega_0)}{A(\omega_0)}-\tfrac{1}{A(\omega_0)(V\widetilde{x})^2}\right)(E\widetilde{x})^3 \\ &\quad +\tfrac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3}\tfrac{P(\omega_0)}{A(\omega_0)^2}(E\widetilde{x})^2+\tfrac{1}{A(\omega_0)V\widetilde{x}}E\widetilde{x} \\ &= Z(\cdot)(E\widetilde{x})^3+\frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3}\frac{P_R(\omega_0)}{R(\omega_0)^2}(E\widetilde{x})^2+\frac{1}{R(\omega_0)V\widetilde{x}}E\widetilde{x}. \end{aligned} \tag{A.10}$$

Finally, we obtain the 3rd-order Taylor approximated optimal portfolio share invested in the risky asset, $\alpha^*_{P,(3)}\equiv\alpha^*_{(3)}/\omega_0$, as

$$\begin{aligned} \alpha^*_{P,(3)} &\simeq \left(\tfrac{\left(E(\widetilde{x}-E\widetilde{x})^3\right)^2}{2(V\widetilde{x})^5}\tfrac{P_R(\omega_0)^2}{R(\omega_0)^3}+\left(\tfrac{4}{3(V\widetilde{x})^2}+\tfrac{E(\widetilde{x}-E\widetilde{x})^3}{6(V\widetilde{x})^3}\right)\tfrac{P_R(\omega_0)}{R(\omega_0)^2}-\tfrac{E(\widetilde{x}-E\widetilde{x})^4}{6(V\widetilde{x})^4}\tfrac{T_R(\omega_0)P_R(\omega_0)}{R(\omega_0)^3}-\tfrac{1}{R(\omega_0)(V\widetilde{x})^2}\right)(E\widetilde{x})^3 \\ &\quad +\tfrac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3}\tfrac{P_R(\omega_0)}{R(\omega_0)^2}(E\widetilde{x})^2+\tfrac{1}{R(\omega_0)V\widetilde{x}}E\widetilde{x} \\ &= Z_R(\cdot)(E\widetilde{x})^3+\frac{E(\widetilde{x}-E\widetilde{x})^3}{2(V\widetilde{x})^3}\frac{P_R(\omega_0)}{R(\omega_0)^2}(E\widetilde{x})^2+\frac{1}{R(\omega_0)V\widetilde{x}}E\widetilde{x}. \end{aligned} \tag{A.11}$$

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