ON SEMI-MODULAR SUBALGEBRAS OF LIE ALGEBRAS OVER FIELDS OF ARBITRARY CHARACTERISTIC

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Abstract

This paper is a further contribution to the extensive study by a number of authors of the subalgebra lattice of a Lie algebra. It is shown that, in certain circumstances, including for all solvable algebras, for all Lie algebras over algebraically closed fields of characteristic $p > 0$ that have absolute toral rank ≤ 1 or are restricted, and for all Lie algebras having the one-and-a-half generation property, the conditions of modularity and semi-modularity are equivalent, but that the same is not true for all Lie algebras over a perfect field of characteristic three. Semi-modular subalgebras of dimensions one and two are characterised over (perfect, in the case of two-dimensional subalgebras) fields of characteristic different from $2, 3$.

Keywords: Lie algebra; subalgebra lattice; modular; semi-modular, quasi-ideal.

AMS Subject Classification: 17B05, 17B50, 17B30, 17B20

1 Introduction

This paper is a further contribution to the extensive study by a number of authors of the subalgebra lattice of a Lie algebra, and is, in part, inspired by the papers of Varea ([15], [16]). A subalgebra U of a Lie algebra L is called

- *modular* in L if it is a modular element in the lattice of subalgebras of L; that is, if

  $$< U, B > \cap C = < B, U \cap C > \qquad \text{for all subalgebras } \; B \subseteq C,$$

  and

  $$< U, B > \cap C = < B \cap C, U > \qquad \text{for all subalgebras } \; U \subseteq C,$$

  (where, $< U, B >$ denotes the subalgebra of L generated by U and B);

- *upper modular* in L (um in L) if, whenever B is a subalgebra of L which covers $U \cap B$ (that is, such that $U \cap B$ is a maximal subalgebra of B), then $< U, B >$ covers U;
- *lower modular* in L (lm in L) if, whenever B is a subalgebra of L such that $< U, B >$ covers U, then B covers $U \cap B$;
- *semi-modular* in L (sm in L) if it is both u.m. and l.m. in L.

In this paper we extend the study of sm subalgebras started in [12]. In section two we give an example of a Lie algebra over a perfect field of characteristic three which has a sm subalgebra that is not modular. However, it is shown that for all solvable Lie algebras, and for all Lie algebras over an algebraically closed field of characteristic $p > 0$ that have absolute toral rank ≤ 1 or are restricted, the conditions of modularity, semi-modularity and being a quasi-ideal are equivalent. The latter extends results of Varea in [16] where the characteristic of the field is restricted to $p > 7$. It is then shown that for all Lie algebras having the one-and-a-half generation property the conditions of modularity and semi-modularity are equivalent.

In section three, sm subalgebras of dimension one are studied. These are characterised over fields of characteristic different from $2, 3$. This result generalises a result of Varea in [15] concerning modular atoms. In the fourth section we show that, over a perfect field of characteristic different from $2, 3$, the only Lie algebra containing a two-dimensional core-free sm subalgebra is $sl_2(F)$. It is also shown that, over certain fields, every sm subalgebra that is solvable, or that is split and contains the normaliser of each of its non-zero subalgebras, is modular.

Throughout, L will denote a finite-dimensional Lie algebra over a field F. There will be no assumptions on F other than those specified in individual results. The symbol '$\oplus$' will denote a vector space direct sum. If U is a subalgebra of L, the *core* of U, U_L, is the largest ideal of L contained in U; we say that U is *core-free* if $U_L = 0$. We denote by $R(L)$ the solvable radical of L, by $Z(L)$ the centre of L, and put $C_L(U) = \{x \in L : [x, U] = 0\}$.

2 General results

We shall need the following result from [12].

Lemma 2.1 *Let U be a proper sm subalgebra of a Lie algebra L over an arbitrary field F. Then U is maximal and modular in $< U, x >$ for all $x \in L \setminus U$.*

Proof: We have that U is maximal in $< U, x >$, by Lemma 1.4 of [12], and hence that U is modular in $< U, x >$, by Theorem 2.3 of [12]

In [12] it was shown that, over fields of characteristic zero, U is modular in L if and only if it is sm in L. This result does not extend to all fields of characteristic three, as we show next. Recall that a simple Lie algebra is *split* if it has a splitting Cartan subalgebra H; that is, if the characteristic roots of $\mathrm{ad}_L h$ are in F for every $h \in H$. Otherwise we say that it is *non-split*.

Proposition 2.2 *Let L be a Lie algebra of dimension greater than three over an arbitrary field F, and suppose that every two linearly independent elements of L generate a three-dimensional non-split simple Lie algebra. Then there are maximal subalgebras M_1, M_2 of L such that $M_1 \cap M_2 = 0$.*

Proof: This is proved in Proposition 4 of [8].

Example

Let G be the algebra constructed by Gein in Example 2 of [7]. This is a seven-dimensional Lie algebra over a certain perfect field F of characteristic three. In G every linearly independent pair of elements generate a three-dimensional non-split simple Lie algebra. It follows from Proposition 2.2 above that there are two maximal subalgebras M, N in G such that $M \cap N = 0$. Choose any $0 \neq a \in M$. Then $< a, N > \cap M = M$, but $< N \cap M, a >= Fa$, so Fa is not a modular subalgebra of L. However, it is easy to see that all atoms of G are sm in G.

A subalgebra Q of L is called a *quasi-ideal* of L if $[Q, V] \subseteq Q + V$ for every subspace V of L. It is easy to see that quasi-ideals of L are always semi-modular subalgebras of L. When L is solvable the semi-modular subalgebras of L are precisely the quasi-ideals of L, as the next result, which is based on Theorem 1.1 of [15], shows.

Theorem 2.3 *Let L be a solvable Lie algebra over an arbitrary field F and let U be a proper subalgebra of L. Then the following are equivalent:*

(i) *U is modular in L;*

(ii) *U is sm in L; and*

(iii) *U is a quasi-ideal of L.*

Proof: (i) $\Rightarrow$ (ii) : This is straightforward.

(ii) $\Rightarrow$ (iii) : Let L be a solvable Lie algebra of smallest dimension containing a subalgebra U which is sm in L but is not a quasi-ideal of L. Then U is maximal and modular in L, by Lemma 2.1, and $U_L = 0$. Let A be a minimal ideal of L. Then $L = U + A$. Moreover, $U \cap A$ is an ideal of L, since A is abelian, whence $U \cap A = 0$ and $L = U \oplus A$. Now U is covered by $< U, A >$ so A covers $U \cap A = 0$. This yields that dim$A = 1$ and so U is a quasi-ideal of L, a contradiction.

(iii) $\Rightarrow$ (i) : This is straightforward.

Corollary 2.4 *Let L be a solvable Lie algebra over an arbitrary field F and let U be a core-free sm subalgebra of L. Then dim$(U) = 1$ and L is almost abelian.*

Proof: This follows from Theorem 2.3 and Theorem 3.6 of [1].

We now consider the case when L is not necessarily solvable. First we shall need the following result concerning $psl_3(F)$.

Proposition 2.5 *Let F be a field of characteristic 3 and let $L = psl_3(F)$. Then L has no maximal sm subalgebra.*

Proof: Let E_{ij} be the 3×3 matrix that has 1 in the (i, j)-position and 0 elsewhere, and denote by $\overline{E_{ij}}$ the canonical image of $E_{ij} \in sl_3(F)$ in $psl_3(F)$. Put $e_{-3} = \overline{E_{23}}$, $e_{-2} = \overline{E_{31}}$, $e_{-1} = \overline{E_{12}}$, $e_0 = \overline{E_{11}} - \overline{E_{22}}$, $e_1 = \overline{E_{21}}$, $e_2 = \overline{E_{13}}$, $e_3 = \overline{E_{32}}$. Then $e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3$ is a basis for $psl_3(F)$ with

$$[e_0, e_i] = e_i \text{ if } i > 0, \quad [e_0, e_i] = -e_i \text{ if } i < 0, \quad [e_{-i}, e_j] = \delta_{ij} e_0 \text{ if } i, j > 0 \quad \text{and}$$

$$[e_i, e_j] = e_{-k} \quad \text{for every cyclic permutation } (i, j, k) \text{ of } (1, 2, 3) \text{ or } (-3, -2, -1).$$

Put $B_{i,j} = Fe_0 + Fe_i + Fe_j$ for each non-zero i, j. If i, j are of opposite sign then $B_{i,j}$ is a subalgebra, every maximal subalgebra of which is two dimensional.

Let M be a maximal sm subalgebra of L. For each i, j of opposite sign, if $B_{i,j} \not\subseteq M$ then $M \cap B_{i,j}$ is two dimensional. Since M is at most five-dimensional, by considering the intersection with each of $B_{1,-1}, B_{2,-2}$ and $B_{3,-3}$ it is easy to see that $e_0 \in M$. But then, considering $B_{1,-1}$ again, we have either $e_1 \in M$ or $e_{-1} \in M$. Suppose the former holds. Taking the intersection of M with $B_{2,-3}$ shows that $e_{-3} \in M$; then with $B_{2,-1}$ gives $e_2 \in M$; next with $B_{3,-2}$ gives $e_{-2} \in M$; finally with $B_{3,-1}$ yields $e_3 \in M$.

But then $M = L$, a contradiction. A similar contradiction is easily obtained if we assume that $e_{-1} \in M$.

Let $(L_p, [p], \iota)$ be any finite-dimensional p-envelope of L. If S is a subalgebra of L we denote by S_p the restricted subalgebra of L_p generated by $\iota(S)$. Then the *(absolute) toral rank* of S in L, $TR(S, L)$, is defined by

$$TR(S, L) = \max\{\dim(T) : T \text{ is a torus of } (S_p + Z(L_p))/Z(L_p)\}.$$

This definition is independent of the p-envelope chosen (see [11]). We write $TR(L, L) = TR(L)$. Then, following the same line of proof, we have an extension of Lemma 2.1 of [16].

Lemma 2.6 *Let L be a Lie algebra over an algebraically closed field of characteristic $p > 0$ such that $TR(L) \leq 1$. Then the following are equivalent:*

(i) *U is modular in L;*

(ii) *U is sm in L; and*

(iii) *U is a quasi-ideal of L.*

Proof: We need only show that (ii) $\Rightarrow$ (iii). Let U be a sm subalgebra of L that is not a quasi-ideal of L. Then there is an $x \in L$ such that $< U, x > \neq U + Fx$. We have that U is maximal and modular in $< U, x >$, by Lemma 2.1, and $< U, x >$ is not solvable, by Theorem 2.3. Furthermore $TR(< U, x >) \leq TR(L) \leq 1$, by Proposition 2.2 of [11], and $< U, x >$ is not nilpotent so $TR(< U, x >) \neq 0$, by Theorem 4.1 of [11], which yields $TR(< U, x >) = 1$. We may therefore suppose that U is maximal and modular in L, of codimension greater than one in L, and that $TR(L) = 1$.

Put $L^\infty = \bigcap_{n \geq 1} L^n$. Suppose first that $R(L^\infty) \not\leq U$. Then $U \cap R(L^\infty)$ is maximal and modular in the solvable subalgebra $R(L^\infty)$, so $U \cap R(L^\infty)$ has codimension one in $R(L^\infty)$. Since U is maximal in L we have $L = U + R(L^\infty)$ and so $\dim(L/U) = 1$, which is a contradiction. This yields that $R(L^\infty) \leq U$. Moreover, $L^\infty \not\leq U$, since this would imply that U/L^∞ is maximal in the nilpotent algebra L/L^∞, giving $\dim(L/U) = 1$, a contradiction again. It follows that $(U \cap L^\infty)/R(L^\infty)$ is modular and maximal in $L^\infty/R(L^\infty)$. But now $L^\infty/R(L^\infty)$ is simple, by Theorem 2.3 of [17], and $1 = TR(L) \geq TR(L^\infty, L) \geq TR(L^\infty/R(L^\infty))$ by section 2 of [11], so $TR(L^\infty/R(L^\infty)) = 1$. This implies that

$$p \neq 2, \quad L^\infty/R(L^\infty) \in \{sl_2(F), W(1 : \underline{1}), H(2 : \underline{1})^{(1)}\} \text{ if } p > 3$$

$$\text{and} \quad L^\infty/R(L^\infty) \in \{sl_2(F), psl_3(F)\} \text{ if } p = 3,$$

by [9] and [10].

Now $H(2:\underline{1})^{(1)}$ has no modular and maximal subalgebras, by Corollary 3.5 of [15]; likewise $psl_3(F)$ by Proposition 2.5. It follows that $L^\infty/R(L^\infty)$ is isomorphic to $W(1:\underline{1})$, which has just one proper modular subalgebra and this has codimension one, by Proposition 2.3 of [15], or to $sl_2(F)$ in which the proper modular subalgebras clearly have codimension one. Hence $\dim(L^\infty/(U \cap L^\infty) = 1$. Since $L = U + L^\infty$ we conclude that $\dim(L/U) = \dim(L^\infty/(U \cap L^\infty) = 1$. This contradiction gives the claimed result.

We then have the following extension of Theorem 2.2 of [16]. The proof is virtually as given in [16], but as the restriction to characteristic > 7 has been removed the details need to be checked carefully. The proof is therefore included for the convenience of the reader.

Theorem 2.7 *Let L be a restricted Lie algebra over an algebraically closed field F of characteristic $p > 0$, and let U be a proper subalgebra of L. Then the following are equivalent:*

(i) *U is modular in L;*

(ii) *U is sm in L; and*

(iii) *U is a quasi-ideal of L.*

Proof: As before it suffices show that (ii) $\Rightarrow$ (iii). Let U be a sm subalgebra of L that is not a quasi-ideal of L. Then there is an $x \in L$ such that $< U, x > \neq U + Fx$. First note that $< U, x >$ is a restricted subalgebra of L. For, suppose not and pick $z \in < U, x >_p$ such that $z \notin < U, x >$. Since $< U, x >$ is an ideal of $< U, x >_p$ we have that $[z, U] \leq < U, x > \cap < U, z >$. But U is maximal in $< U, z >$, by Lemma 2.1, and so $< U, x > \cap < U, z > = U$, giving $[z, U] \leq U$. But U is self-idealizing, by Lemma 1.5 of [12], so $z \in U$. This contradiction proves the claim. So we may as well assume that $L = < U, x >$. Moreover, U is restricted since it is self-idealizing, whence $(U_L)_p \leq U$. As $(U_L)_p$ is an ideal of L we have that $U_L = (U_L)_p$. It follows that L/U_L is also restricted. We may therefore assume that U is a core-free modular and maximal subalgebra of L of codimension greater than one in L.

Now L is spanned by the centralizers of tori of maximal dimension, by Corollary 3.11 of [17], so there is such a torus T with $C_L(T) \not\leq U$. Let $L = C_L(T) \oplus \sum L_\alpha(T)$ be the decomposition of L into eigenspaces with respect to T. We have that $C_L(T)$ is a Cartan subalgebra of L, by Theorem 2.14 of [17]. It follows from the nilpotency of $C_L(T)$ and the modularity of U that $U \cap C_L(T)$ has codimension one in $C_L(T)$.

Now let $L^{(\alpha)} = \sum_{i \in P} L_{i\alpha}(T)$, where P is the prime field of F, be the 1-section of L corresponding to a non-zero root α. From the modularity of U we see that $U \cap L^{(\alpha)}$ is a modular and maximal subalgebra of $L^{(\alpha)}$. Since U is core-free and self-idealizing, $Z(L) = 0$. But then $TR(T, L) = TR(L)$, since T is a maximal torus, whence $TR(L^{(\alpha)}) \leq 1$, by Theorem 2.6 of [11]. It follows from Lemma 2.6 that $M \cap L^{(\alpha)}$ is a quasi-ideal of $L^{(\alpha)}$. As $U \cap L^{(\alpha)}$ is maximal in $L^{(\alpha)}$, we have that $\dim(L^{(\alpha)}/(U \cap L^{(\alpha)})) \leq 1$ and $L^{(\alpha)} = U \cap L^{(\alpha)} + C_L(T)$. This yields that $L = U + C_L(T)$ and hence that $\dim(L/U) = \dim(C_L(T)/(U \cap C_L(T))) = 1$, a contradiction. The result follows.

We shall say that the Lie algebra L has the *one-and-a-half generation property* if, given any $0 \neq x \in L$, there is an element $y \in L$ such that $< x, y >= L$. Then we have the following result.

Theorem 2.8 *Let L be a Lie algebra, over any field F, which has the one-and-a-half generation property. Then every sm subalgebra of L is a modular maximal subalgebra of L.*

Proof: Let U be a sm subalgebra of L and let $0 \neq u \in U$. Then there is an element $x \in L$ such that $L =< u, x >=< U, x >$. It follows from Lemma 2.1 that U is modular in L.

Corollary 2.9 *Let L be a Lie algebra over an infinite field F of characteristic different from $2, 3$ which is a form of a classical simple Lie algebra. Then every sm subalgebra of L is a modular maximal subalgebra of L.*

Proof: Under the given hypotheses L has the one-and-a-half generation property, by Theorem 2.2.3 and section 1.2.2 of [3], or by [5].

We also have the following analogue of a result of Varea from [15].

Corollary 2.10 *Let F be an infinite perfect field of characteristic $p > 2$, and assume that $p^n \neq 3$. Then the subalgebra $W(1 : \mathbf{n})_{\mathbf{0}}$ is the unique sm subalgebra of $W(1 : \mathbf{n})$.*

Proof: Let $L = W(1 : \mathbf{n})$ and let Ω be the algebraic closure of F. Then $L \otimes_F \Omega$ is simple and has the one-and-a-half generation property, by Theorem

4.4.8 of [3]. It follows that L has the one-and-a-half generation property (see section 1.2.2 of [3]). Let U be a sm subalgebra of L. Then U is modular and maximal in L by Theorem 2.8. Suppose that $U \neq L_0$. Then $L = U + L_0$ and $U \cap L_0$ is maximal in L_0. But L_0 is supersolvable (see Lemma 2.1 of [13] for instance) so $\dim(L_0/(L_0 \cap U)) = 1$. It follows that $\dim(L/U) = \dim(L_0/(L_0 \cap U)) = 1$, whence $U = L_0$, which is a contradiction.

3 Semi-modular atoms

We say that L is *almost abelian* if $L = L^2 \oplus Fx$ with $\operatorname{ad} x$ acting as the identity map on the abelian ideal L^2. A *μ-algebra* is a non-solvable Lie algebra in which every proper subalgebra is one dimensional. A subalgebra U of a Lie algebra L is a *strong ideal* (respectively, *strong quasi-ideal*) of L if every one-dimensional subalgebra of U is an ideal (respectively, quasi-ideal) of L; it is *modular** in L if it satisfies a dualised version of the modularity conditions, namely

$$< U, B > \cap C = < B, U \cap C > \qquad \text{for all subalgebras } \; B \subseteq C,$$

and

$$< U \cap B, C > = < B, C > \cap U \qquad \text{for all subalgebras } \; C \subseteq U.$$

Example

Let K be the three-dimensional Lie algebra with basis a, b, c and multiplication $[a, b] = c$, $[b, c] = b$, $[a, c] = a$ over a field of characteristic two. Then K has a unique one-dimensional quasi-ideal, namely Fc. Thus for each $0 \neq u \in Fc$ and $k \in K \setminus Fc$ we have that $< u, k >$ is two dimensional. However K is not almost abelian. In fact K is simple, Fc is core-free and is the Frattini subalgebra of K, and so any two linearly independent elements not in Fc generate K.

We shall need a result from [4]. However, because of the above example, there is a (slight) error in three results in this paper. The error comes from an incorrect use of Theorem 3.6 of [1]. The three corrected results are as follows:

Lemma 3.1 *(Lemma 2.2 of [4]) If Q is a strong quasi-ideal of L, then Q is a strong ideal of L, or L is almost abelian, or F has characteristic two, $L = K$ and $Q = Fc$.*

Proof: Assume that Q is a strong quasi-ideal and that there exists $q \in Q$ such that Fq is not an ideal of L. Then Theorem 3.6 of [1] gives that L is almost abelian, or F has characteristic two, $L = K$ and $Q = Fc$. The result follows.

The proof of the following result is the same as the original.

Proposition 3.2 *(Proposition 2.3 of [4]) Let Q be a proper quasi-ideal of a Lie algebra L which is modular* in L. Then Q is a strong quasi-ideal and so is given by Lemma 3.1.*

Lemma 3.3 *(Lemma 4.1 of [4]) Let L be a Lie algebra over an arbitrary field F. Let U be a core-free subalgebra of L such that $< u, z >$ is either two dimensional or a μ-algebra for every $0 \neq u \in U$ and $z \in L \setminus U$. Then one of the following holds:*

(i) *L is almost abelian;*

(ii) *$< u, z >$ is a μ-algebra for every $0 \neq u \in U$; and $z \in L \setminus U$*

(iii) *F has characteristic two, $L = K$ and $Fu = Fc$.*

Proof: This is the same as the original proof except that the following should be inserted at the end of sentence six: "or char$F = 2$ and $L = K$".

Using the above we now have the following result.

Lemma 3.4 *Suppose that Fu is sm in L but not an ideal of L. Then either*

(i) *L is almost abelian; or*

(ii) *$< u, x >$ is a μ-algebra for every $x \in L \setminus Fu$.*

(iii) *F has characteristic two, $L = K$ and $Fu = Fc$*

Proof: Pick any $x \in L \setminus Fu$. Then Fu is maximal in $< u, x >$, by Lemma 2.1. Now let M be a maximal subalgebra of $< u, x >$. If $u \in M$ then $M = Fu$. So suppose that $u \notin M$. Then Fu is a maximal subalgebra of $< u, x > = < u, M >$, whence $Fu \cap M = 0$ is maximal in M, since Fu is lm. It follows that every maximal subalgebra of $< u, x >$ is one dimensional. The claimed result now follows from Lemma 3.3.

We shall need the following result concerning 'one-and-a-half generation' of rank one simple Lie algebras over infinite fields of characteristic $\neq 2, 3$.

Theorem 3.5 *Let L be a rank one simple Lie algebra over an infinite field F of characteristic $\neq 2, 3$ and let Fx be a Cartan subalgebra of L. Then there is an element $y \in L$ such that $< x, y >= L$.*

Proof: Since L is rank one simple it is central simple. Let Ω be the algebraic closure of F and put $L_\Omega = L \otimes_F \Omega$, and so on. Then L_Ω is simple and Ωx is a Cartan subalgebra of L_Ω. Let

$$L_\Omega = \Omega x \oplus \sum_{\alpha \in \Phi} (L_\Omega)_\alpha$$

be the decomposition of L_Ω into its root spaces relative to Ωx. Then, with the given restrictions on the characteristic of the field, every root space $(L_\Omega)_\alpha$ is one dimensional (see [2]).

Let M be a maximal subalgebra of L containing x. Then M_Ω is a subalgebra of L_Ω and $\Omega x \subseteq M_\Omega$. So, M_Ω decomposes into root spaces relative to Ωx,

$$M_\Omega = \Omega x \oplus \sum_{\alpha \in \Delta} (M_\Omega)_\alpha.$$

We have that $\Delta \subseteq \Phi$ and $(M_\Omega)_\alpha \subseteq (L_\Omega)_\alpha$ for all $\alpha \in \Delta$. As $(L_\Omega)_\alpha$ is one dimensional for every $\alpha \in \Phi$, we have $(M_\Omega)_\alpha = (L_\Omega)_\alpha$ for every $\alpha \in \Delta$. Hence there are only finitely many maximal subalgebras of L containing x: $M_1, \ldots, M_r$ say. Since F is infinite, $\cup_{i=1}^r M_i \neq L$, so there is an element $y \in L$ such that $y \notin M_i$ for all $1 \leq i \leq r$. But now $< x, y >= L$, as claimed.

If U is a subalgebra of L, then the *normaliser* of U in L is the set

$$N_L(U) = \{x \in L : [x, U] \subseteq U\}.$$

We can now give the following characterisation of one-dimensional semi-modular subalgebras of Lie algebras over fields of characteristic $\neq 2, 3$.

Theorem 3.6 *Let L be a Lie algebra over a field F, of characteristic $\neq 2, 3$ if F is infinite. Then Fu is sm in L if and only if one of the following holds:*

(i) *Fu is an ideal of L;*

(ii) *L is almost abelian and ad u acts as a non-zero scalar on L^2;*

(iii) *L is a μ-algebra.*

Proof: It is easy to check that (i), (ii), or (iii) hold then Fu is sm in L. So suppose that Fu is sm in L, but that (i), (ii) do not hold. First we claim that L is simple.

Suppose not, and let A be a minimal ideal of L. If $u \in A$, choose any $b \in L \setminus A$. Then $< u, b > \cap A$ is an ideal of $< u, b >$. Since $0 \neq u \in < u, b > \cap A$ and $b \notin A$, $< u, b >$ cannot be a μ-algebra. But then L is almost abelian, by Lemma 3.4, a contradiction. So $u \notin A$. By Lemma 3.3 of [12], $ua = \lambda a$ for all $a \in A$ and some $\lambda \in F$. But now $Fu + Fa$ is a two-dimensional subalgebra of $< u, a >$, a μ-algebra, which is impossible. Hence L is simple.

Now Fu is um in L and not an ideal of L, so $N_L(Fu) = Fu$, by Lemma 1.5 of [12]. Hence Fu is a Cartan subalgebra of L, and L is rank one simple. Now F cannot be finite, since there are no μ-algebras over finite fields, by Corollary 3.2 of [6]. Hence F is infinite. But then there is an element $y \in L$ such that $< u, y >= L$, by Theorem 3.5, and L is a μ-algebra. The result is established.

As a corollary to this we have a result of Varea, namely Corollary 2.3 of [14].

Corollary 3.7 *(Varea) Let L be a Lie algebra over a perfect field F, of characteristic $\neq 2, 3$ if F is infinite. If Fu is modular in L but not an ideal of L then L is either almost abelian or three-dimensional non-split simple.*

Proof: This follows from Theorem 3.6 and the fact that with the stated restrictions on F the only μ-algebras are three-dimensional non-split simple (Proposition 1 of [7]).

4 Semi-modular subalgebras of higher dimension

First we consider two-dimensional semi-modular subalgebras. We have the following analogue of Theorem 1.6 of [15].

Theorem 4.1 *Let L be a Lie algebra over a perfect field F of characteristic different from 2, 3, and let U be a two-dimensional core-free sm subalgebra of L. Then $L \cong sl_2(F)$.*

Proof: If U is modular then the result follows from Theorem 1.6 of [15], so we can assume that U is not a quasi-ideal of L. Thus, there is an element $x \in L$ such that $< U, x > \neq U + Fx$. Put $V = < U, x >$. Then $U_V = U$ implies that $< U, x >= U + Fx$, a contradiction; if $U_V = 0$ then $V \cong sl_2(F)$ by Lemma

2.1 and Theorem 1.6 of [15], and $< U, x >= U + Fx$, a contradiction. It follows that $\dim(U_V) = 1$. Put $U_V = Fu$. Now $\dim(U/U_V) = 1$ and V/U_V is three-dimensional non-split simple, by Theorem 3.6 and Proposition 1 of [7]. Thus $V = Fu \oplus S$, where S is three-dimensional non-split simple, by Lemma 1.4 of [15], and Fu, S are ideals of V.

Now we claim that $0 \neq Z(< U, y >) \subseteq U$ for every $y \in L \setminus U$. We have shown this above if $< U, y > \neq U + Fy$. So suppose that $< U, y >= U + Fy$. Then $< U, y >$ is three dimensional and not simple (since U is two dimensional and abelian), and so solvable. Then, by using Corollary 2.4, we have that U contains a one-dimensional ideal K of $U + Fy$ such that $(U + Fy)/K$ is two-dimensional non-abelian, and $K = Z(< U, y >)$.

Since U is maximal in $< U, x >$ we have $< U, x > \neq L$. Pick $y \in L \setminus < U, x >$. Then $0 \neq Z(< U, x + y >) \subseteq U$ by the above. Assume that $Z(< u, x >) \neq Z(< U, y >)$. Then $U = Z(< u, x >) \oplus Z(< U, y >)$. Let $0 \neq z \in Z(< U, x + y >)$ and write $z = z_1 + z_2$ where $z_1 \in Z(< U, x >)$, $z_2 \in Z(< U, y >)$. Then $0 = [z, (x + y)] = [z_2, x] + [z_1, y]$, so $[z_2, x] = -[z_1, y]$. Now, if $z_1 = 0$, then $[z_2, x] = 0$, whence $z_2 \in Z(< u, x >) \cap Z(< U, y >)$, a contradiction. Similarly, if $z_2 = 0$, then $[z_1, y] = 0$, whence $z_2 \in Z(< u, x >) \cap Z(< U, y >)$, a contradiction again. Hence $z_1, z_2 \neq 0$. Since $z_1, z_2 \in U$ we deduce that $[z_1, y] = -[z_2, x] \in < u, x > \cap < U, y >= U$. Thus $y \in N_L(U) = U$, a contradiction. It follows that $Z(< U, x >) = Z(< U, y >)$ for all $y \in L$, whence $[L, Z(< U, x >)] = 0$ and $Z(< U, x >)$ is an ideal of L, contradicting the fact that U is core-free.

Next we establish analogues of two results of Varea from [15].

Theorem 4.2 *Let L be a Lie algebra over an algebraically closed field F of characteristic $p > 5$. If U is a sm subalgebra of L such that U/U_L is solvable and $\dim(U/U_L) > 1$, then U is modular in L, and hence L/U_L is isomorphic to $sl_2(F)$ or to a Zassenhaus algebra.*

Proof: Let L be a Lie algebra of minimal dimension having a sm subalgebra U which is not modular in L, and such that U/U_L is solvable and $\dim(U/U_L) > 1$. Then $U_L = 0$ and U is solvable. Since U is not a quasi-ideal there is an element $x \in L \setminus U$ such that $S =< U, x > \neq U + Fx$. Let $K = U_S$. If $\dim(U/K) = 1$ then S/K is almost abelian, by Theorem 3.6, whence U is a quasi-ideal of S, a contradiction. It follows that $\dim(U/K) > 1$. If U/K is modular in S/K then $\dim(S/U) = 1$, by Theorem 2.4 of [15], a contradiction. The minimality of L then implies that $S = L$. This yields that U is modular in L, by Lemma 2.1. This contradiction establishes the result.

We say that the subalgebra U of L is *split* if $\mathrm{ad}_L x$ is split for all $x \in U$; that is, if $\mathrm{ad}_L x$ has a Jordan decomposition into semisimple and nilpotent parts for all $x \in U$.

Theorem 4.3 *Let L be a Lie algebra over a perfect field F of characteristic p different from 2. If U is a sm subalgebra of L which is split and which contains the normaliser of each of its non-zero subalgebras, then U is modular, and one of the following holds:*

(i) L is almost abelian and $\dim(U) = 1$;

(ii) $L \cong sl_2(F)$ and $\dim(U) = 2$;

(iii) L is a Zassenhaus algebra and U is its unique subalgebra of codimension one in L.

Proof: Let L be a Lie algebra of minimal dimension having a sm subalgebra U which is split and which contains the normaliser of each of its non-zero subalgebras, but which is not modular in L. Since U is not a quasi-ideal there is an element $x \in L \setminus U$ such that $S = < U, x > \neq U + Fx$. If $S \neq L$ then U is modular in S, by the minimality of L. It follows from Theorem 2.7 of [15] that U is a quasi-ideal of S, a contradiction. Hence $S = L$. Once again we see that U is modular in L, by Lemma 2.1. This contradiction establishes the result.

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