Dispersion in space-time transformation optics

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***Abstract*—The use of spacetime cloaking to hide events is an intriguing trick, but the unavoidable presence of dispersion limits the performance of any implementation, and needs to be accounted for. We show how the dispersion changes under transformation.**

I. Introduction

The electromagnetic implementation of cloaking, the hiding of *objects* from sight by diverting and reassembling illuminating electromagnetic fields has now been with us ten years [1]. The notion of hiding *events* is now five years old [2]. Both schemes as initially introduced, however, neglected one crucial component that, ironically, made them possible in the first place.

In order to achieve the graduated and controllable modulation of material properties that are a necessary part of any transformation device, we need to understand the underlying behaviour which generates them. From a fundamental (microscopic) perspective, all non-vacuum material properties are dynamic in nature, resulting from the reaction of atom, molecules, or more complex structures (metamaterials) to the impinging electromagnetic field, and thus changing how that field propagates. It is then an effective - and most likely homogenized [3], [4] – version of this dynamic process which we can often simplify into macroscopic permittivity and permeability functions, or perhaps even just a refractive index. The sole remaining symptom of the original dynamics is then the frequency dependence of these constitutive quantities.

In an ordinary spatial cloak, the intrinsically dynamic nature of material responses might not be too much of a problem - we can specify an operating frequency and bandwidth, and hope that our expertise at metamaterial construction allows us to achieve the necessary material properties [5]–[7].

In an event cloak [2], or any other spacetime transformation device [8], [9], the time dependence of the material response is more problematic: the spacetime transformation not only affects the required material parameters, but also the underlying dynamics of the material response. So either we will need to adjust our material design to compensate for that extra complication, or engineer that extra complication so as to match our design specification. In practise this will probably reduce to an additional trade-off of the sort we already make when attempting to build an ordinary spatial-only transformation device – what degree of approximation can we tolerate when attempting to match our desired performance range? Indeed, in the experimental spacetime cloaking realization of Fridman et al in 2012 [10] they used dispersion to engineer an effective controllable speed profile to achieve their aim – but in doing so ignored how a 'perfect' event cloak would have to adjust the dispersion mechanisms.

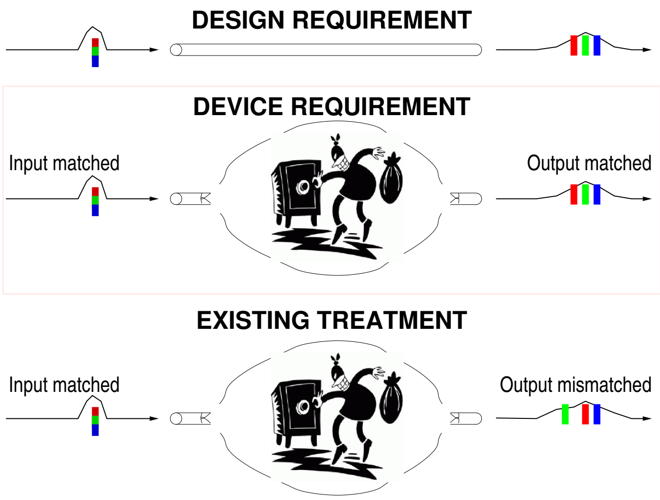

Fig. 1. Whilst the visible behaviour of the device should only be to (top) alter any incident illumination by the expected dispersion properties, the actual transformation device (middle) also must hide a chosen event from any observer. Existing treatments, which ignore the effects of spacetime transformations on the dispersion properties, will not perfectly match the design requirement – even if all the beam steering and scattering suppression is still implemented correctly.

II. Scheme

In this work we envisage a simple event cloak scheme, but in contrast to the original formulation based on a homogeneous, isotropic, and dispersion-free background, we want our *device* to hide an event inside a dispersive medium, albeit still an homogeneous and isotropic one. Further, we are going to *design* the device so that the dispersive medium appears to be acting like the standard simple Lorentz model.

A optical pulse which has travelled through an ordinary dispersive medium will typically emerge with some chirp, which results from different frequency components experiencing different phase velocities, as well as generating a group velocity for the pulse as a whole. This situation is depicted at the top of fig. 1, and is how our spacetime cloaking device is designed to appear to an observer. We however, want to hide an event inside a different 'device' medium, whose spatial and temporal properties not only hide our chosen event, but *also* mimic an ordinary dispersive medium of our choice, as seen at the middle of fig. 1. If we do not properly consider *all* aspects of how the medium might need to be adapted to the true spacetime nature of our cloaking transformation, we will only partially cloak the event, as shown at the bottom of fig. 1.

This means the primary focus here is not the usually considered electromagnetic half of the transformation device design, but instead that of the dynamic material response responding to, and acting on, the electromagnetic fields. To avoid complicated mathematical expressions, here we will present a simple scalar-based approached to the material transformation; a more precise version will be given in the conference presentation.

A convenient choice of design polarization response – i.e. what we want an observer to infer is present – is the usual Lorentz-like temporal differential equation:

$$\partial_{\bar{t}}^2 P + \gamma \partial_{\bar{t}} P + \omega_p^2 P = \kappa E. \tag{1}$$

Here γ is the loss coefficient, ω_p the resonant frequency of the oscillator, and κ the coupling constant between the driving E field and the polarization response P. This response model provides the dispersion that we want the observer to see (or infer), even though our device is actually doing something much more tricksy.

The necessary complicated device behaviour can be described (or calculated) by first defining the design transformation or 'morphism' φ [8]. This might e.g. be that of the standard curtain-map event cloak [2], and acts between a *design* manifold mapped with (spanned by) coordinates $\bar{t}, \bar{z}$ and the *device* manifold mapped with (spanned by) coordinates $\hat{t}, \hat{z}$ [8]. It is

$$\bar{t} = \hat{t}, \quad \bar{x} = \hat{x}, \quad \bar{y} = \hat{y}, \quad \bar{z} + \alpha(\bar{t}, \bar{z}) = \hat{z}. \tag{2}$$

The result of this specification means that the polarization equation transforms in a non-trivial way; and it is worth noting that existing spacetime transformation designs [2], [10]–[13] implicitly assume that those non-trivial complications are negligible for the device operation. However, here we explicitly calculate the corrections. Fortunately there are only two altered derivative properties after the transformation, which are

$$\frac{d}{d\bar{t}} = \frac{\partial \hat{t}}{\partial \bar{t}} \frac{\partial}{\partial \hat{t}} + \frac{\partial \hat{z}}{\partial \bar{t}} \frac{\partial}{\partial \hat{z}} = \frac{\partial}{\partial \hat{t}} + \dot{\alpha}(\hat{t}, \bar{z}) \frac{\partial}{\partial \hat{z}}, \tag{3}$$

where $\dot{\alpha} = \partial_{\bar{t}} \alpha$ and

$$\frac{d}{d\bar{z}} = \frac{\partial \hat{t}}{\partial \bar{z}} \frac{\partial}{\partial \hat{t}} + \frac{\partial \hat{z}}{\partial \bar{z}} \frac{\partial}{\partial \hat{z}} = 0 + \alpha`(\hat{t}, \bar{z}) \frac{\partial}{\partial \hat{z}}, \tag{4}$$

where $\alpha` = \partial_{\bar{z}} \alpha$.

Our simple design transformation means that we can ignore the second of these, since our design polarization model contains purely temporal derivatives. We also need $\alpha' = \partial_{\hat{z}} \alpha$ which can be calculated by inverting the transformation, e.g. from $\bar{z} = \hat{z} - \beta(\hat{t}, \hat{z})$.

The naively transformed polarization equation is therefore given by a substitution of that device derivative combination from eqn. (3) that matches the design space $\partial_{\bar{t}}$. It is

$$\kappa E = \left[\partial_{\hat{t}} + \dot{\alpha} \partial_{\hat{z}}\right]^2 P + \gamma \left[\partial_{\hat{t}} + \dot{\alpha} \partial_{\hat{z}}\right] P + \omega_p^2 \tag{5}$$

$$\kappa E = \partial_{\hat{t}}^2 P + \left[2\dot{\alpha} \partial_{\hat{z}} + \gamma\right] \partial_{\hat{t}} P + \left[\dot{\dot{\alpha}} \partial_{\hat{z}} + \dot{\alpha}^2 \partial_{\hat{z}}^2 + \gamma \dot{\alpha} \partial_{\hat{z}} + \omega_p^2\right] P. \tag{6}$$

From this we can see the primary message resulting from a more systematic approach to spacetime transformation designs on dispersive media – our previously strictly spatially-local and temporal-only response models start to depend on spatial gradients. Notably, the dependence on spatial gradients can be related to notions of spatial dispersion [14].

In hindsight, such complications should not be surprising, since any spacetime transformation invariably mixes up our notions of space and time (coordinates). One feature of the above expression in eqn. (6) is the presence in the α-dependent terms of derivatives of both device and design coordinates. This unsatisfactory situation emphasizes another of the limitations of the simple approach given here, and a more rigorous and elegant method will be demonstrated in the conference presentation.

III. Discussion

We can see from the above that it is no simple matter to produce an exact dispersive spacetime transformation device, even in the simple scalar polarization model considered here. It is worth noting that a vectorial polarization model, as would be required in a true electromagnetic situation, would inherit more correction terms still. Notwithstanding those details, which we intend to cover in detail in the conference presentation, we can still investigate the limits that just the corrections listed here place on an transformation device we might make.

Notably, if we want to ignore the alterations to the dispersion that are required by the transformation, as calculated above, we need

1) that $2\dot{\alpha} \partial_{\hat{z}} P \ll \gamma P$; and if P varies on scales of wavelength λ, then $2\dot{\alpha} \ll \gamma \lambda$.
   I.e. the oscillator losses γ must dominate any contributions to the effective loss from the transformation gradient.
2) that $\dot{\dot{\alpha}} \partial_{\hat{z}} P + \dot{\alpha} \dot{\alpha}' \partial_{\hat{z}} P + \dot{\alpha}^2 \partial_{\hat{z}}^2 P + \gamma \dot{\alpha} \partial_{\hat{z}} P \ll \omega_p^2 P$;
   and if P varies on scales of wavelength λ, then
   $\dot{\dot{\alpha}} \lambda + \dot{\alpha} \dot{\alpha}' \lambda + \dot{\alpha}^2 + \gamma \dot{\alpha} \lambda \ll \omega_p^2$
   i.e. all frequency modulations (Doppler-like shifts) from the transformation must be small compared to the natural resonance frequency.

Both these constraints require the transformation to be smooth, to avoid singular contributions at transition regions, and ideally to have minimal first and second derivatives. However, typical schemes often neglect these smoothness requirements at the interface between the background (exterior) region and the transformed region of space (spacetime).

Indeed, although the local effect of non-smoothness might be large – a mathematical delta function – it is only the integrated contribution that counts.

IV. Conclusion

We have shown how a comprehensive description of a spacetime transformation device needs to not only modify the background material properties in space and in time, but must also allow for the alteration of the dynamic response of the medium itself. Notably, we showed in section III that such effects can only be neglected if constraints on the smoothness of the transformation are met. Further consequences for spacetime transformation devices, a more general method, and the linkage to electromagnetic field propagation, will all be discussed in the conference presentation.

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References

[1] J. B. Pendry, D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," *Science*, vol. 312, no. 5781, pp. 1780–1782, 2006. [Online]. Available: http://dx.doi.org/10.1126/science.1125907

[2] M. W. McCall, A. Favaro, P. Kinsler, and A. Boardman, "A spacetime cloak, or a history editor," *J. Opt.*, vol. 13, no. 2, p. 024003, 2011. [Online]. Available: http://dx.doi.org/10.1088/2040-8978/13/2/024003

[3] Special issue: Homogenization of metamaterials, *Photon. Nanostruct. Fundam. Appl.*, vol. 11, no. 4, pp. 362–477, November 2013. [Online]. Available: http://dx.doi.org/10.1016/j.photonics.2013.11.005

[4] D. R. Smith, D. C. Vier, T. Koschny, and C. M. Soukoulis, "Electromagnetic parameter retrieval from inhomogeneous metamaterials," *Phys. Rev. E*, vol. 71, p. 036617, 2005. [Online]. Available: http://link.aps.org/abstract/PRE/v71/e036617

[5] G. W. Milton, "Realizability of metamaterials with prescribed electric permittivity and magnetic permeability tensors," *New J. Phys.*, vol. 12, p. 033035, March 2010. [Online]. Available: http://dx.doi.org/10.1088/1367-2630/12/3/033035

[6] A. D. Boardman, V. V. Grimalsky, Y. S. Kivshar, S. V. Koshevaya, M. Lapine, N. M. Litchinitser, V. N. Malnev, M. Noginov, Y. G. Rapoport, and V. M. Shalaev, "Active and tunable metamaterials," *Laser Phot. Rev.*, vol. 5, pp. 287–307, March 2011. [Online]. Available: http://dx.doi.org/10.1002/lpor.201000012

[7] C. R. Simovski, P. A. Belov, A. V. Atrashchenko, and Y. S. Kivshar, "Wire metamaterials: Physics and applications," *Adv. Mat.*, vol. 24, no. 31, pp. 4229–4248, August 2012. [Online]. Available: http://dx.doi.org/10.1002/adma.201200931

[8] P. Kinsler and M. W. McCall, "The futures of transformations and metamaterials," *Photon. Nanostruct. Fundam. Appl.*, vol. 15, pp. 10–23, June 2015. [Online]. Available: http://dx.doi.org/10.1016/j.photonics.2015.04.005

[9] ——, "Cloaks, editors, and bubbles: applications of spacetime transformation theory," *Ann. Phys. (Berlin)*, vol. 526, pp. 51–62, January 2014. [Online]. Available: http://dx.doi.org/10.1002/andp.201300164

[10] M. Fridman, A. Farsi, Y. Okawachi, and A. L. Gaeta, "Demonstration of temporal cloaking," *Nature*, vol. 481, no. 7379, pp. 62–65, 2012. [Online]. Available: http://dx.doi.org/10.1038/nature10695

[11] J. M. Lukens, D. E. Leaird, and A. M. Weiner, "A temporal cloak at telecommunication data rate," *Nature*, vol. 498, no. 7453, pp. 205–208, 2013. [Online]. Available: http://www.nature.com/nature/journal/vaop/ncurrent/full/nature12224.html

[12] J. M. Lukens, A. J. Metcalf, D. E. Leaird, and A. M. Weiner, "Temporal cloaking for data suppression and retrieval," *Optica*, vol. 1, no. 6, pp. 372–375, 2014. [Online]. Available: http://dx.doi.org/10.1364/OPTICA.1.000372

[13] M. A. Lerma, "A mirror based event cloaking device," January 2012, originally released Jan 26, 2012 at http://www.math.northwestern.edu/~mlerma/papers/event_cloaking.pdf. [Online]. Available: http://arxiv.org/abs/1308.2606/

[14] J. Gratus and M. McCormack, "Spatially dispersive inhomogeneous electromagnetic media with periodic structure," *J. Opt.*, vol. 17, no. 2, p. 025105, January 2015. [Online]. Available: http://dx.doi.org/10.1088/2040-8978/17/2/025105