A SINGULAR, ADMISSIBLE EXTENSION WHICH SPLITS ALGEBRAICALLY, BUT NOT STRONGLY, OF THE ALGEBRA OF BOUNDED OPERATORS ON A BANACH SPACE

TOMASZ KANIA, NIELS JAKOB LAUSTSEN, AND RICHARD SKILLICORN

In memoriam: Charles J. Read (1958–2015)

ABSTRACT. Let E be the Banach space constructed by Read (*J. London Math. Soc.* 1989) such that the Banach algebra $\mathscr{B}(E)$ of bounded operators on E admits a discontinuous derivation. We show that $\mathscr{B}(E)$ has a singular, admissible extension which splits algebraically, but does not split strongly. This answers a natural question going back to the work of Bade, Dales, and Lykova (*Mem. Amer. Math. Soc.* 1999), and complements recent results of Laustsen and Skillicorn (*C. R. Math. Acad. Sci. Paris* 2016).

1. Introduction and the main result

An *extension* of a Banach algebra $\mathscr{B}$ is a short-exact sequence of the form

$$\{0\} \longrightarrow \ker\varphi \longrightarrow \mathscr{A} \xrightarrow{\varphi} \mathscr{B} \longrightarrow \{0\}, \tag{1.1}$$

where $\mathscr{A}$ is a Banach algebra and $\varphi\colon \mathscr{A} \to \mathscr{B}$ is a continuous, surjective algebra homomorphism. The extension is *singular* if $\ker\varphi$ has the trivial product, that is, $ab = 0$ for each $a, b \in \ker\varphi$. We say that the extension *splits algebraically* (respectively, *splits strongly, is admissible*) if φ has a right inverse which is an algebra homomorphism (respectively, is a continuous algebra homomorphism, is bounded and linear). Every extension which splits strongly is evidently admissible and splits algebraically, and the admissibility of (1.1) is equivalent to $\ker\varphi$ being complemented in $\mathscr{A}$ as a Banach space.

Motivated by Bade, Dales, and Lykova's comprehensive study [1] of extensions of Banach algebras, the second- and third-named authors [6] investigated the interrelationship among the above properties for extensions of Banach algebras of the form $\mathscr{B}(E)$, that is, all bounded operators acting on a Banach space E, showing that there exist Banach spaces E_1 and E_2 such that:

(i) $\mathscr{B}(E_1)$ has a singular extension which splits algebraically, but the extension is not admissible, and so it does not split strongly;

2010 *Mathematics Subject Classification.* Primary 46H40, 46M18, 47L10; Secondary 16S70, 46B45, 46H10.

Key words and phrases. Bounded, linear operator; Banach space; Banach algebra; short-exact sequence; algebraic splitting; strong splitting; singular extension; admissible extension; pullback; separating space.

Kania acknowledges with thanks funding from the European Research Council/ERC Grant Agreement no. 291497.

(ii) $\mathscr{B}(E_2)$ has a singular extension which is admissible, but it does not split algebraically.

These results naturally raise the following question:

Does there exist a Banach space E such that $\mathscr{B}(E)$ has an admissible extension which splits algebraically, but not strongly?

The purpose of this note is to answer this question positively.

Theorem 1.1. *There exist a Banach space E and a continuous, surjective algebra homomorphism φ from a unital Banach algebra $\mathscr{A}$ onto $\mathscr{B}(E)$ such that the extension*

$$\{0\} \longrightarrow \ker\varphi \longrightarrow \mathscr{A} \xrightarrow{\ \varphi\ } \mathscr{B}(E) \longrightarrow \{0\} \tag{1.2}$$

is singular and admissible and splits algebraically, but it does not split strongly.

In fact, the Banach spaces E_1 and E_2 used to establish statements (i) and (ii) above are equal (but the extensions are obviously distinct), both being the Banach space E_{R} constructed by Read [10] with the property that there is a discontinuous derivation from $\mathscr{B}(E_{\mathrm{R}})$ into a one-dimensional $\mathscr{B}(E_{\mathrm{R}})$-bimodule. This Banach space will also play the role of E in Theorem 1.1.

It is no coincidence that the Banach algebra $\mathscr{B}(E)$ associated with the Banach space E in Theorem 1.1 admits a discontinuous derivation. Indeed, a Banach algebra $\mathscr{B}$ which has an extension that splits algebraically, but not strongly, obviously admits a discontinuous algebra homomorphism. This property is generally weaker than admitting a discontinuous derivation. However, if in addition we require that the extension be singular and admissible, then $\mathscr{B}$ must admit a discontinuous derivation.

To see this, suppose contrapositively that every derivation from $\mathscr{B}$ into a Banach $\mathscr{B}$-bimodule is continuous. Then, by a theorem of Dales and Villena (see [4, Corollary 2.2], or [2, Corollary 2.7.7] for an exposition), every intertwining map (as defined on p. 4 below) from $\mathscr{B}$ into a Banach $\mathscr{B}$-bimodule is continuous, and therefore, using [1, Theorem 2.13] or [2, Theorem 2.8.16], we conclude that every singular, admissible extension of $\mathscr{B}$ which splits algebraically also splits strongly. This result implies in particular that every singular, admissible extension of the compact operators $\mathscr{K}(E_{\mathrm{R}})$ which splits algebraically also splits strongly, in contrast to Theorem 1.1, because Read showed that E_{R} has a Schauder basis [10, Lemma 5.3], and hence every derivation from $\mathscr{K}(E_{\mathrm{R}})$ into a $\mathscr{K}(E_{\mathrm{R}})$-bimodule is continuous (see [10, Theorem 5.2] or [2, Corollary 5.3.9]).

The usefulness of Read's Banach space E_{R} in the context of extensions originates from the following theorem, which strengthens and clarifies the main technical results underlying Read's construction of a discontinuous derivation from $\mathscr{B}(E_{\mathrm{R}})$ (see [10, p. 306 and Section 4]). In it, we denote by $\ell_2^\sim$ the unitization of the Hilbert space ℓ_2 endowed with the trivial product, so that $\ell_2^\sim = \ell_2 \oplus \mathbb{K}1$ as a vector space (where $\mathbb{K}$ denotes the scalar field, either $\mathbb{R}$ or $\mathbb{C}$, and 1 is the formal identity that we adjoin), and the product and norm on $\ell_2^\sim$ are given by

$$(\xi + s1)(\eta + t1) = s\eta + t\xi + st1 \quad \text{and} \quad \|\xi + s1\| = \|\xi\| + |s| \qquad (\xi, \eta \in \ell_2,\ s, t \in \mathbb{K}).$$

Theorem 1.2 ([7, Theorem 1.2]). *There exists a continuous, surjective algebra homomorphism $\psi\colon \mathscr{B}(E_{\mathrm{R}}) \to \ell_2^{\sim}$ with $\ker\psi = \mathscr{W}(E_{\mathrm{R}})$ (the ideal of weakly compact operators on E_{R}) such that the extension*

$$\{0\} \longrightarrow \mathscr{W}(E_{\mathrm{R}}) \longrightarrow \mathscr{B}(E_{\mathrm{R}}) \xrightarrow{\ \psi\ } \ell_2^{\sim} \longrightarrow \{0\} \tag{1.3}$$

splits strongly.

2. Proof of Theorem 1.1

Throughout, all Banach spaces and algebras are considered over the same scalar field $\mathbb{K}$, where either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 2.1. Let X and Y be Banach spaces, and let $S\colon X \to Y$ be a linear map. The *separating space* of S is given by

$$\mathfrak{S}(S) = \{y \in Y : \text{there is a null sequence } (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ such that } Sx_n \to y \text{ as } n \to \infty\}.$$

We shall require the following fundamental facts about the separating space; see, *e.g.*, [2, Propositions 5.1.2 and 5.2.2(ii)] or [11, Lemmas 1.2(i)–(ii) and 1.3(ii)]. The second clause is of course just a restatement of the Closed Graph Theorem. It explains why the separating space plays an important role in automatic continuity theory.

Proposition 2.2. *Let X and Y be Banach spaces, and let $S\colon X \to Y$ be a linear map. Then:*

(i) *$\mathfrak{S}(S)$ is a closed subspace of Y;*
(ii) *S is bounded if and only if $\mathfrak{S}(S) = \{0\}$.*
(iii) *Let Z be a Banach space, and suppose that $T\colon Y \to Z$ is bounded and linear. Then*

$$\mathfrak{S}(TS) = \overline{T[\mathfrak{S}(S)]}.$$

The pullback of Banach algebras. As in [6], the connection between the extension (1.3) of $\ell_2^{\sim}$ and the extension (1.2) of $\mathscr{B}(E_{\mathrm{R}})$ that we aim to establish goes via the pullback construction, which we shall now review. Let $\alpha\colon \mathscr{A} \to \mathscr{C}$ and $\beta\colon \mathscr{B} \to \mathscr{C}$ be continuous algebra homomorphisms between Banach algebras $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, and set

$$\mathscr{D} = \{(a, b) \in \mathscr{A} \oplus \mathscr{B} : \alpha(a) = \beta(b)\}, \tag{2.1}$$

where $\mathscr{A} \oplus \mathscr{B}$ denotes the Banach-algebra direct sum of $\mathscr{A}$ and $\mathscr{B}$. For later reference, we note that $\mathscr{D}$ is unital in the case where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, α, and β are unital. Let γ and δ be the restrictions to $\mathscr{D}$ of the coordinate projections:

$$\gamma\colon\ (a, b) \mapsto a, \quad \mathscr{D} \to \mathscr{A}, \qquad \text{and} \qquad \delta\colon\ (a, b) \mapsto b, \quad \mathscr{D} \to \mathscr{B}. \tag{2.2}$$

They are continuous algebra homomorphisms, and the triple $(\mathscr{D}, \gamma, \delta)$ has the universal property of pullbacks, as observed in [6, Section 2], where the following connection with extensions is also established.

Proposition 2.3 ([6, Proposition 2.2])**.** *Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be Banach algebras such that there are extensions*

$$\{0\} \longrightarrow \ker\alpha \longrightarrow \mathscr{A} \xrightarrow{\ \alpha\ } \mathscr{C} \longrightarrow \{0\} \tag{2.3}$$

and

$$\{0\} \longrightarrow \ker\beta \longrightarrow \mathscr{B} \xrightarrow{\ \beta\ } \mathscr{C} \longrightarrow \{0\}, \tag{2.4}$$

and define $\mathscr{D}$, γ, and δ by (2.1) *and* (2.2) *above. Then δ is surjective, and the following statements concerning the extension*

$$\{0\} \longrightarrow \ker\delta \longrightarrow \mathscr{D} \xrightarrow{\ \delta\ } \mathscr{B} \longrightarrow \{0\} \tag{2.5}$$

hold true:

(i) (2.5) *is singular if and only if* (2.3) *is singular.*
(ii) *Suppose that* (2.4) *splits strongly (respectively, splits algebraically, is admissible). Then* (2.5) *splits strongly (respectively, splits algebraically, is admissible) if and only if* (2.3) *splits strongly (respectively, splits algebraically, is admissible).*

Derivations and intertwining maps. Throughout this paragraph, $\mathscr{A}$ denotes a Banach algebra, and X is a Banach $\mathscr{A}$-bimodule. The *annihilator* of $\mathscr{A}$ in X is given by

$$\operatorname{ann}_{\mathscr{A}} X = \{x \in X : a \cdot x = 0 = x \cdot a \text{ for each } a \in \mathscr{A}\}.$$

Let $S\colon \mathscr{A} \to X$ be a linear map, and consider the bilinear map

$$\delta^1 S\colon\ (a,b) \mapsto a\cdot(Sb) - S(ab) + (Sa)\cdot b, \quad \mathscr{A}\times\mathscr{A} \to X. \tag{2.6}$$

We say that S is a *derivation* if $\delta^1 S = 0$, while S is an *intertwining map* if $\delta^1 S$ is continuous.

Using $\delta^1 S$, we can define an algebra product on the vector space $\mathscr{A}\oplus X$ as follows:

$$(a,x)(b,y) = (ab, a\cdot y + x\cdot b + (\delta^1 S)(a,b)) \qquad (a,b\in\mathscr{A},\ x,y\in X). \tag{2.7}$$

Suppose that S is an intertwining map. Then the above product is continuous with respect to the norm $\|(a,x)\| = \|a\| + \|x\|$ for $a \in \mathscr{A}$ and $x \in X$, and so $\mathscr{A}\oplus X$ is a Banach algebra with respect to an equivalent norm. The map

$$\pi_{\mathscr{A}}\colon (a,x) \mapsto a, \quad \mathscr{A}\oplus X \to \mathscr{A},$$

is clearly a continuous, surjective algebra homomorphism with kernel $\{0\}\oplus X$, which has the trivial product, so that we obtain a singular extension of $\mathscr{A}$:

$$\{0\} \longrightarrow \{0\}\oplus X \longrightarrow \mathscr{A}\oplus X \xrightarrow{\ \pi_{\mathscr{A}}\ } \mathscr{A} \longrightarrow \{0\}. \tag{2.8}$$

Moreover, this extension is admissible because $a \mapsto (a,0)$, $\mathscr{A} \to \mathscr{A}\oplus X$, is a bounded, linear right inverse of $\pi_{\mathscr{A}}$, and it splits algebraically because $a \mapsto (a,-Sa)$, $\mathscr{A} \to \mathscr{A}\oplus X$, is an algebra homomorphism which is also a right inverse of $\pi_{\mathscr{A}}$. A direct calculation shows that (2.8) splits strongly if and only if there is a derivation $D\colon \mathscr{A} \to X$ such that $S - D$ is continuous. This is a well-known generalization of results in pure algebra, going back to Johnson [5, Theorem 2.1 and Corollary 2.2]; see also [2, Theorem 2.8.12] and the text preceding it.

As we have already indicated, our strategy is to construct an extension with suitably chosen properties of the unitization of ℓ_2 endowed with the trivial product, and then apply Proposition 2.3 together with Theorem 1.2 to obtain the desired extension of $\mathscr{B}(E_{\mathrm{R}})$. Adjoining an identity to an extension is straightforward, in the sense that it does not alter any of the properties that we are interested in. Consequently, we shall henceforth focus our attention on the case where the Banach algebra $\mathscr{A}$ has the trivial product. It turns out that we may also simplify matters by supposing that the bimodule X has the trivial right action; that is, $ab = 0$ and $x \cdot a = 0$ for each $a, b \in \mathscr{A}$ and $x \in X$. In this case, derivations and intertwining maps have nice characterizations in terms of the annihilator and the separating space.

Lemma 2.4. *Let $\mathscr{A}$ be a Banach algebra with the trivial product, let X be a Banach $\mathscr{A}$-bimodule with the trivial right action, and let $S\colon \mathscr{A} \to X$ be a linear map. Then:*
(i) *S is a derivation if and only if $S[\mathscr{A}] \subseteq \operatorname{ann}_{\mathscr{A}} X$;*
(ii) *S is an intertwining map if and only if $\mathfrak{S}(S) \subseteq \operatorname{ann}_{\mathscr{A}} X$.*

Proof. We begin by noting that, by the hypotheses, (2.6) reduces to

$$(\delta^1 S)(a, b) = a \cdot (Sb) \qquad (a, b \in \mathscr{A}), \tag{2.9}$$

from which (i) follows immediately.

(ii). Suppose that S is an intertwining map, and let $y \in \mathfrak{S}(S)$ and $a \in \mathscr{A}$ be given. Since $y \cdot a = 0$ by hypothesis, it remains to show that $a \cdot y = 0$. Take a null sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that $Sx_n \to y$ as $n \to \infty$. On the one hand, the continuity of the left action of $\mathscr{A}$ on X implies that $a \cdot Sx_n \to a \cdot y$ as $n \to \infty$, while on the other the continuity of the map $\delta^1 S$ given by (2.9) shows that $a \cdot Sx_n = (\delta^1 S)(a, x_n) \to (\delta^1 S)(a, 0) = 0$ as $n \to \infty$, so that $a \cdot y = 0$.

Conversely, suppose that $\mathfrak{S}(S) \subseteq \operatorname{ann}_{\mathscr{A}} X$. A bilinear map is jointly continuous if (and only if) it is separately continuous, and (2.9) shows that $\delta^1 S$ is continuous in the first variable by the continuity of the module map. Now fix $a \in \mathscr{A}$, and consider the left action $L_a\colon x \mapsto a \cdot x$, $X \to X$, which is bounded and linear. The assumption means that $L_a[\mathfrak{S}(S)] = \{0\}$, so the composite map $L_a S$ is bounded by Proposition 2.2(ii)–(iii). Hence $\delta^1 S$ is continuous in the second variable because $(\delta^1 S)(a, b) = (L_a S)b$ for each $b \in \mathscr{A}$. □

For a Banach space Y, we denote by $\ell_\infty(\mathbb{N}, Y)$ the Banach space of all bounded sequences in Y.

Lemma 2.5. *Let $\mathscr{A}$ be an infinite-dimensional Banach algebra with the trivial product, and let Y be a closed subspace of a Banach space X. Endow X with the trivial right action of $\mathscr{A}$, and suppose that there exists a bounded, linear injection from the quotient space X/Y into $\ell_\infty(\mathbb{N}, Y)$. Then there exists a Banach left module action of $\mathscr{A}$ on X with $\operatorname{ann}_{\mathscr{A}} X = Y$. The converse is true in the case where $\mathscr{A}$ is separable.*

Proof. Let $T\colon X/Y \to \ell_\infty(\mathbb{N}, Y)$ be a bounded, linear injection, and set

$$L_n = \pi_n T Q\colon\; X \to Y \qquad (n \in \mathbb{N}),$$

where $Q\colon X \to X/Y$ is the quotient map and $\pi_n\colon \ell_\infty(\mathbb{N}, Y) \to Y$ is the n^{th} coordinate projection. Since $\mathscr{A}$ is infinite-dimensional, we may choose a countable biorthogonal system $(a_n, f_n)_{n\in\mathbb{N}}$ in $\mathscr{A} \times \mathscr{A}^*$; that is, $a_n \in \mathscr{A}$ and f_n is a bounded, linear functional on $\mathscr{A}$ such that $\langle a_m, f_n\rangle = \delta_{m,n}$ (the Kronecker delta) for each $m, n \in \mathbb{N}$. Then the series

$$\sum_{n=1}^{\infty} \frac{\langle a, f_n\rangle}{2^n\|f_n\|(\|L_n\| + 1)} L_n x$$

converges absolutely in Y for each $a \in \mathscr{A}$ and $x \in X$, and its sum, which we shall denote by $a \cdot x$, has norm at most $\|a\|\,\|x\|$. It is easy to see that $(a, x) \mapsto a \cdot x$, $\mathscr{A} \times X \to Y$, is bilinear and that $a \cdot y = 0$ for each $y \in Y$, so that $a \cdot (b \cdot x) = 0 = (ab) \cdot x$ for each $a, b \in \mathscr{A}$ and $x \in X$. Hence we have defined a Banach left $\mathscr{A}$-module action on X such that $Y \subseteq \operatorname{ann}_{\mathscr{A}} X$. To prove the reverse inclusion, suppose that $x \in \operatorname{ann}_{\mathscr{A}} X$. Then we have

$$0 = a_m \cdot x = \frac{1}{2^m\|f_m\|(\|L_m\| + 1)} L_m x = \frac{1}{2^m\|f_m\|(\|L_m\| + 1)} \pi_m T Q x \qquad (m \in \mathbb{N}),$$

so that $TQx = 0$, and therefore $x \in \ker Q = Y$ because T is injective.

To prove the converse statement in the case where $\mathscr{A}$ is separable, suppose that X has a Banach left $\mathscr{A}$-module action with $\operatorname{ann}_{\mathscr{A}} X = Y$, and choose a sequence $(a_n)_{n\in\mathbb{N}}$ of unit vectors in $\mathscr{A}$ with $\overline{\operatorname{span}}\,\{a_n : n \in \mathbb{N}\} = \mathscr{A}$. For each $a \in \mathscr{A}$ and $x \in X$, we have $a \cdot x \in \operatorname{ann}_{\mathscr{A}} X = Y$ because $\mathscr{A}$ has the trivial product, so

$$T\colon x \mapsto (a_n \cdot x)_{n\in\mathbb{N}}, \quad X \to \ell_\infty(\mathbb{N}, Y),$$

defines a bounded, linear map with

$$\ker T = \{x \in X : a_n \cdot x = 0 \text{ for each } n \in \mathbb{N}\} = \operatorname{ann}_{\mathscr{A}} X = Y$$

because $\overline{\operatorname{span}}\,\{a_n : n \in \mathbb{N}\} = \mathscr{A}$. Hence the conclusion follows from the Fundamental Isomorphism Theorem. □

The following corollary summarizes what we have found so far, and what remains to be done to achieve our goal.

Corollary 2.6. *Let $\mathscr{A}$ be an infinite-dimensional Banach algebra with the trivial product, let Y be a closed subspace of a Banach space X such that there exists a bounded, linear injection from the quotient space X/Y into $\ell_\infty(\mathbb{N}, Y)$, and suppose that there exists a linear map $S\colon \mathscr{A} \to X$ with $\mathfrak{S}(S) \subseteq Y$. Then $\mathscr{A}$ has a singular, admissible extension which splits algebraically. Further, this extension splits strongly if and only if*

$$(S + T)[\mathscr{A}] \subseteq Y \tag{2.10}$$

for some bounded, linear map $T\colon \mathscr{A} \to X$.

Proof. By Lemma 2.5, we can endow X with a Banach $\mathscr{A}$-bimodule structure such that the right action is trivial and $\operatorname{ann}_{\mathscr{A}} X = Y$, and Lemma 2.4(ii) then implies that S is an intertwining map. Hence $\mathscr{A} \oplus X$ is a Banach algebra with respect to the product (2.7), and (2.8) is a singular, admissible extension of $\mathscr{A}$ which splits algebraically. As we stated, this extension splits strongly if and only if $S - D$ is continuous for some derivation $D\colon \mathscr{A} \to X$. By Lemma 2.4(i), the latter condition is equivalent to (2.10) (take $T = D - S$). □

We now come to our main technical lemma, which will ensure that we can apply Corollary 2.6 whenever the Banach algebra $\mathscr{A}$ satisfies the additional hypothesis that every bounded, linear map from $\mathscr{A}$ into ℓ_1 is compact. The statement of this lemma involves the following notation and terminology.

First, the *density character* of a Banach space X is the smallest cardinality of a dense subset of X.

Second, for a non-empty set Ξ, we denote by $\ell_1(\Xi)$ the Banach space of all absolutely summable, scalar-valued functions defined on Ξ. For $\xi \in \Xi$, we write e_ξ for the function which takes the value 1 at ξ and 0 elsewhere, and we write e'_ξ for the corresponding coordinate functional, so that e'_ξ is given by $\langle f, e'_\xi\rangle = f(\xi)$ for each $f \in \ell_1(\Xi)$. As usual, we write ℓ_1 for $\ell_1(\mathbb{N})$.

Lemma 2.7. *Let Z be an infinite-dimensional Banach space such that every bounded, linear map from Z into ℓ_1 is compact, let Ξ be a normalized Hamel basis for Z, and consider the linear map $S\colon Z \to \ell_1(\Xi)$ given by $S\xi = e_\xi$ for each $\xi \in \Xi$. Then*

$$(S + T)[Z] \not\subseteq \mathfrak{S}(S) \tag{2.11}$$

for each bounded, linear map $T\colon Z \to \ell_1(\Xi)$, and the density character of the quotient space $\ell_1(\Xi)/\mathfrak{S}(S)$ is no greater than the density character of Z.

Proof. No infinite-dimensional Banach space has a countable Hamel basis, so the set Ξ is necessarily uncountable. Assume towards a contradiction that $(S+T)[Z] \subseteq \mathfrak{S}(S)$ for some bounded, linear map $T\colon Z \to \ell_1(\Xi)$.

To verify that T is compact, consider a bounded sequence $(z_n)_{n\in\mathbb{N}}$ in Z. Then the set

$$\Gamma = \bigcup_{n\in\mathbb{N}} \{\xi \in \Xi : \langle Tz_n, e'_\xi\rangle \neq 0\}$$

is countable. Let $P\colon \ell_1(\Xi) \to \ell_1(\Gamma)$ be the canonical projection. By the hypothesis, PT is compact, so $(z_n)_{n\in\mathbb{N}}$ has a subsequence $(z_{n_j})_{j\in\mathbb{N}}$ such that $(PTz_{n_j})_{j\in\mathbb{N}}$ is convergent. Since

$$\|Tz_m - Tz_n\| = \|PTz_m - PTz_n\| \qquad (m, n \in \mathbb{N})$$

by the choice of Γ, we conclude that $(Tz_{n_j})_{j\in\mathbb{N}}$ is also convergent, and hence T is compact.

In particular, T has separable range and is strictly singular (see, *e.g.*, [9, Propositions 1.11.7 and 1.11.9]), so $T[Z] \subseteq \overline{\operatorname{span}}\,\{e_\xi : \xi \in \Xi_0\}$ for some countable subset Ξ_0 of Ξ, and the infinite-dimensional subspace $\operatorname{span}(\Xi \setminus \Xi_0)$ of Z contains a unit vector z such that $\|Tz\| \leqslant 1/2$. Take a finite subset Υ of $\Xi \setminus \Xi_0$ such that $z = \sum_{\xi\in\Upsilon} s_\xi \xi$ for some scalars s_ξ $(\xi \in \Upsilon)$. By the assumption, $(S + T)z \in \mathfrak{S}(S)$, so we can find a null sequence $(x_n)_{n\in\mathbb{N}}$ in Z such that $Sx_n \to (S + T)z$ as $n \to \infty$. Write x_n as $x_n = \sum_{\xi\in\Upsilon} s_{n,\xi}\xi + y_n$, where $s_{n,\xi} \in \mathbb{K}$ $(\xi \in \Upsilon)$ and $y_n \in \operatorname{span}(\Xi \setminus \Upsilon)$. For each $\xi \in \Upsilon$, we have

$$Sy_n \in \operatorname{span}\{e_\eta : \eta \in \Xi \setminus \Upsilon\} \subseteq \ker e'_\xi$$

by the definition of S, and therefore

$$s_{n,\xi} = \langle Sx_n, e'_\xi\rangle \to \langle (S+T)z, e'_\xi\rangle = \langle Sz, e'_\xi\rangle + \langle Tz, e'_\xi\rangle = s_\xi \quad \text{as} \quad n\to\infty$$

by the continuity of e'_ξ and the fact that $Tz \in \overline{\operatorname{span}}\,\{e_\eta : \eta \in \Xi_0\} \subseteq \ker e'_\xi$. Using that the set Υ is finite, we obtain

$$y_n = x_n - \sum_{\xi\in\Upsilon} s_{n,\xi}\xi \to 0 - \sum_{\xi\in\Upsilon} s_\xi \xi = -z \quad \text{as} \quad n\to\infty \tag{2.12}$$

and

$$Sy_n = Sx_n - \sum_{\xi\in\Upsilon} s_{n,\xi}e_\xi \to (S+T)z - \sum_{\xi\in\Upsilon} s_\xi e_\xi = Tz \quad \text{as} \quad n\to\infty. \tag{2.13}$$

For each $n\in\mathbb{N}$, write y_n as $y_n = \sum_{\xi\in\Xi\setminus\Upsilon} t_{n,\xi}\xi$, where only finitely many of the scalars $t_{n,\xi}$ are non-zero. Then by the subadditivity of the norm on Z, we have

$$\|y_n\| \leqslant \sum_{\xi\in\Xi\setminus\Upsilon} |t_{n,\xi}| = \Bigl\| \sum_{\xi\in\Xi\setminus\Upsilon} t_{n,\xi}e_\xi \Bigr\| = \|Sy_n\| \to \|Tz\| \leqslant \frac{1}{2} \quad \text{as} \quad n\to\infty,$$

using (2.13). This, however, contradicts that $\|y_n\| \to \|-z\| = 1$ as $n\to\infty$ by (2.12), and consequently (2.11) follows.

To prove the final clause, let $Q\colon \ell_1(\Xi) \to \ell_1(\Xi)/\mathfrak{S}(S)$ be the quotient map. Proposition 2.2 implies that QS is bounded, so it will suffice to show that the range of QS is dense in $\ell_1(\Xi)/\mathfrak{S}(S)$. Let $\varepsilon > 0$. Each element $y \in \ell_1(\Xi)/\mathfrak{S}(S)$ has the form $y = Qf$ for some $f \in \ell_1(\Xi)$. Take a finite subset Υ of Ξ such that $\|f - \sum_{\xi\in\Upsilon} f(\xi)e_\xi\| \leqslant \varepsilon$, and set $z = \sum_{\xi\in\Upsilon} f(\xi)\xi \in Z$. Then we have

$$\|y - QSz\| \leqslant \|f - Sz\| = \Bigl\| f - \sum_{\xi\in\Upsilon} f(\xi)e_\xi \Bigr\| \leqslant \varepsilon,$$

from which the conclusion follows. □

Remark 2.8. Let Z be an infinite-dimensional Banach space which contains no subspace isomorphic to ℓ_1. Combining Rosenthal's ℓ_1-theorem with the Schur property of ℓ_1, we deduce that every bounded, linear map from Z into ℓ_1 is compact, and hence the hypothesis of Lemma 2.7 is satisfied.

We note in passing that the converse of this statement is not true. Indeed, the Banach space ℓ_∞ contains a subspace which is (isometrically) isomorphic to ℓ_1, and every bounded, linear map $T\colon \ell_\infty \to \ell_1$ is compact. To verify the latter fact, we observe that each such map T is weakly compact by a theorem of Pełczyński [8] because no subspace of its codomain is isomorphic to c_0. Using once more that ℓ_1 has the Schur property, we conclude that T is compact.

Corollary 2.9. *Let $\mathscr{A}$ be an infinite-dimensional, separable Banach algebra with the trivial product, and suppose that every bounded, linear map from $\mathscr{A}$ into ℓ_1 is compact. Then $\mathscr{A}$ has a singular, admissible extension which splits algebraically, but not strongly.*

Proof. Applying Lemma 2.7 with $Z = \mathscr{A}$ and taking $X = \ell_1(\Xi)$, where Ξ is a normalized Hamel basis for $\mathscr{A}$, we obtain a linear map $S\colon \mathscr{A} \to X$ such that the closed subspace $Y = \mathfrak{S}(S)$ of X satisfies $(S+T)[\mathscr{A}] \nsubseteq Y$ for each bounded, linear map $T\colon \mathscr{A} \to X$, and the quotient space X/Y is separable. Hence X/Y embeds into ℓ_∞, and thus into $\ell_\infty(\mathbb{N}, Y)$, and the conclusion follows from Corollary 2.6. □

Proof of Theorem 1.1. Our strategy is to apply Proposition 2.3 with $\mathscr{B} = \mathscr{B}(E_{\mathrm{R}})$, $\mathscr{C} = \ell_2^\sim$, and $\beta = \psi$, using the notation of Theorem 1.2. By (1.3), we have an extension of the form (2.4) which splits strongly.

The Schur property of ℓ_1 implies that every bounded, linear map from ℓ_2 into ℓ_1 is compact, so by Corollary 2.9, we obtain a singular, admissible extension of ℓ_2

$$\{0\} \longrightarrow \ker\alpha_0 \longrightarrow \mathscr{E} \xrightarrow{\ \alpha_0\ } \ell_2 \longrightarrow \{0\} \tag{2.14}$$

which splits algebraically, but not strongly. Passing to the unitizations, and writing $\mathscr{A}$ for the unitization of the Banach algebra $\mathscr{E}$ (so that $\mathscr{A} = \mathscr{E} \oplus \mathbb{K}1$, with the product and norm defined in the usual way), we obtain an extension of the form (2.3) of $\mathscr{C} = \ell_2^\sim$, where

$$\alpha(a + s1) = \alpha_0(a) + s1 \qquad (a \in \mathscr{E},\ s \in \mathbb{K}),$$

and this extension clearly inherits the properties of (2.14), so that it is singular and admissible and splits algebraically, but it does not split strongly. Thus Proposition 2.3 produces an extension (2.5) of $\mathscr{B} = \mathscr{B}(E_{\mathrm{R}})$ which is singular and admissible and splits algebraically, but it does not split strongly. To complete the proof, we note that the algebra $\mathscr{D}$ in (2.5) is unital because the algebras $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ in (2.3) and (2.4) are unital, and hence so are the surjections α and β. □

Remark 2.10. The Banach space E_{DLW} constructed by Dales, Loy, and Willis [3] provides an interesting contrast to Read's space E_{R}, especially in relation to Theorem 1.1. Indeed, E_{DLW} shares with E_{R} the property that $\mathscr{B}(E_{\mathrm{DLW}})$ admits a discontinuous algebra homomorphism into a Banach algebra (under the assumption of the Continuum Hypothesis), but it differs in that every derivation from $\mathscr{B}(E_{\mathrm{DLW}})$ into a Banach $\mathscr{B}(E_{\mathrm{DLW}})$-bimodule is continuous. Hence every singular, admissible extension of $\mathscr{B}(E_{\mathrm{DLW}})$ which splits algebraically also splits strongly by a general result that was stated in the Introduction.

References

1. W. G. Bade, H. G. Dales, and Z. A. Lykova, Algebraic and strong splittings of extensions of Banach algebras, *Mem. Amer. Math. Soc.* **137** (1999).
2. H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monogr. Ser. 24, Clarendon Press, Oxford, 2000.
3. H. G. Dales, R. J. Loy, and G. A. Willis, Homomorphisms and derivations from $\mathscr{B}(E)$, *J. Funct. Anal.* **120** (1994), 201–219.
4. H. G. Dales and A. R. Villena, Continuity of derivations, intertwining maps, and cocycles from Banach algebras, *J. London Math. Soc.* **63** (2001), 215–225.
5. B. E. Johnson, The Wedderburn decomposition of Banach algebras with finite dimensional radical, *Amer. J. Math.* **90** (1968), 866–876.

6. N. J. Laustsen and R. Skillicorn, Splittings of extensions and homological bidimension of the algebra of bounded operators on a Banach space, *C. R. Math. Acad. Sci. Paris* **354** (2016), 459–463.
7. N. J. Laustsen and R. Skillicorn, Extensions and the weak Calkin algebra of Read's Banach space admitting discontinuous derivations, submitted; `arXiv:1602.08963`.
8. A. Pełczyński, Projections in certain Banach spaces, *Studia Math.* **19** (1960), 209–228.
9. A. Pietsch, *Operator ideals*, North Holland, Amsterdam–New York–Oxford, 1980.
10. C. J. Read, Discontinuous derivations on the algebra of bounded operators on a Banach space, *J. London Math. Soc.* **40** (1989), 305–326.
11. A. M. Sinclair, *Automatic continuity of linear operators*, London Math. Soc. Lecture Note Ser. 21, Cambridge University Press, Cambridge–New York–Melbourne, 1976.

Mathematics Institute, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, United Kingdom
E-mail address: `tomasz.marcin.kania@gmail.com`

Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster LA1 4YF, United Kingdom.
E-mail address: `n.laustsen@lancaster.ac.uk`

Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster LA1 4YF, United Kingdom.
E-mail address: `r.skillicorn@lancaster.ac.uk`