Discrete Capacity Choice Problems in Repeated and Scaled Investment

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Declaration

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CONTENTS

Abstract

Most previous studies on real options are confined to the realm of continuous modelling while more practical discrete models are restricted by the methodology of solutions. This thesis applies the discount factor methodology to solve two scaling capacity choice problems: one is the scaling capacity expansions without switching options raised from Dxit and Pindyck (1994) and the other one is the case adding switching opportunities raised from Pindyck (1988). We respectively establish two discrete models for these two problems and examine their convergence to the continuous case by narrowing the discrete intervals. We verify with analytical inference and numerical results that the discrete models are soundly consistent with the continuous models, supporting the validity of the discount factor methodology.

I. Introduction

According to the real option theory (McDonald and Siegel, 1986; Dixit and Pindyck, 1994), irreversible investment using the investment opportunity involves an opportunity cost, i.e. the chance to invest in the future. Therefore, there exists an option of future investment for managers and only if the project or extra capacity value is larger than the sum of both the installation cost and the opportunity cost. Therefore, the naive investment rule of NPV is not valid and tends to cause overinvestment. The extra firm value required over the direct costs could be called the option premium. Some papers (Brennan and Schwartz, 1985; Majd and Pindyck, 1987) suggest that the opportunity cost is larger than the direct cost of a project in many cases.

The one step investment could be extended to multi-step investment by allowing firms to hold and operate a large number of capacity units and to add new capacity units. In other words, the firm chooses an investment policy to alter its capacity levels and the output flow depends on the installed capital stock. The series of critical points of the state variable to install the additional capacities are called expansion thresholds. In such case the principle for one step investment is still applicable that the payoff brought by the new capacities should cover both the installation cost and the opportunity cost to invest at any time in the future. Therefore, when a manager make the investment decisions of setting expansion thresholds, he or she should count the sequential flexibility to add new capacities, namely the expansion options. Nevertheless, unlike the one-step investment, the follow-on expansion options would affect the timing of prior exercise and the optimal expansion timings for multi-step investment should be driven by maximisation of the joint value of both initial and subsequent options. Therefore, the investment policy for the multi-step investment is different from that for the one-step investment. In the previous literature (e.g. Pindyck, 1988; Dixit and Pindyck, 1994) the investment decisions for multi-step investment are called capacity choice problem. With sequential flexibility to expand capacities in the future, the capacity choice problem concerns the optimal capacity for certain price levels or the optimal price that an amount of capacity could sustain.

The models of analyzing incremental investment and capacity choice in the presence of irreversibility and uncertainty begin with Pindyck (1988). Some papers (Bertola, 1998; Dixit and Pindyck, 1994) also shed light on the capacity expansion problem. Nevertheless, all these studies, as far as I know, assume continuously incremental investment, which entails unrealistic assumptions and inevitable limitations. The continuous capacity expansion models assume that firms can adjust the capacity to fit the dynamic price whenever necessary by infinitesimal amounts. In the real world, however, the variations of prices or capacities are usually discrete rather than continuous. The reason why previous literature prefers the hypothetical continuous models to the more realistic discrete models might arise from the restriction of the methodology. The capacity choice problems are essentially barrier control problems (Dixit, 1991), which controls the point representing the state of the controlled system from crossing a certain barrier curve. For an upward barrier curve, an increase of the price must be followed by an increase of the capacity to hold the optimal equilibrium. Such optimal polices could be conveniently modeled and solved by utilizing traditional real option techniques including dynamic programming and contingent claim approach. However, the techniques for real option problems in discrete cases are much less developed and prevent researchers to stepping into modelling the discrete capacity expansions.

Nevertheless, Ekern, Shackleton, and Sødal (2014) develops a methodology named the discount factor approach, which might provide us a viable instrument to establish and solve discrete capacity expansion models. The concept of the discount factor methodology is first supposed by Baily (1995). Dixit, Pindyck and Sødal (1999) takes the perspective further and provides the optimal investment rule for one-step investment. In such case, only the payoff is obtained when the option is exercised. However, in incremental investment a new option might be created when an old option is exercised, making the situation more complicated. Ekern, Shackleton, and Sødal (2014) develops the discount factor methodology to adapt it to the multi-step investment problem, which is just an appropriate instrument for the discrete capacity choice problem.

However, the approach proposed by Ekern, Shackleton, and Sødal (2014) could only be directly applied to multi-step investment problems with finite horizons. In other words, the incremental investment system should be closed, or bounded. On the other hand, the traditional continuous models (e.g. Pindyck, 1988) could solve infinite capacity choice problems by constructing the marginal condition of profit optimization that the marginal revenue is equal to the marginal cost without requiring the boundedness of the expressions.

To solve an unbounded discrete capacity choice model with infinite number of steps, self-similarities are required. That means all the value elements for each expansion step would grow geometrically at the same ratio. Then, the equations of the value matching conditions at all expansion thresholds would be linearly dependent. Thus, if we could solve the optimal capacity for any arbitrarily selected investment threshold, the optimal capacities for all other expansion thresholds could also be solved. When the scaling ratio of the self-similarities becomes infinitesimal, the discrete model converges to a continuous case.

In this thesis we respectively establish two discrete scaling capacity choice models. One is an irreversible capacity expansion model without switching options. That means the established capacities would operate forever regardless of the economic environment. The other one adds completely reversible switching options to the established capacities, which hence have repeated flexibilities to be temporarily turned off or turn on according to the economic status quo. We assume constant returns to scale and perfect competition and therefore convex costs are required to bind the size of the firm as indicated by Pindyck (1993). In the discrete scaling systems, installation costs per unit capacity are also scaled up for different capacity levels and therefore satisfy the requirement of the convex costs.

Using continuous models, Dxit and Pindyck (1994) and Pindyck (1988) respectively solve two irreversible capacity choice problems with and without switching options, which provide us the benchmarks to establish our discrete models and examine the validity of the discount factor methodology. However, both papers assume constant installation cost per unit capacity. Pindyck (1988) also assumes constant returns to scale and perfect competitions but solve the problem of indefinite expansions by introducing scaling operating costs to play the role of convex costs. Dxit and Pindyck (1994) directly avoid such problem by assuming decreasing returns to scale and imperfect competition. In order to make the continuous models perfectly comparable with our discrete models, we have to adjust their assumptions and models, and recalculate the numerical results. Both adjusted models assume constant returns to scale and perfect competition while we assume scaling installation costs as in our discrete models to play the role of convex costs to bind the firm size. Thus, the adjusted continuous models have exact the same assumptions and structures as the discrete models except for the continuous and discrete assumptions and are the continuous parallels of the discrete models. If the discrete method of discount factor methodology is valid to solve the capacity choice problems, then the numerical results when the discrete models converge to the continuous cases should be the same as those solved from the continuous counterparts.

The convergences of the scaling discrete models to the continuous case are implemented by making the scaling ratio between sequential thresholds and capacities infinitesimal. Then the results of the adjusted continuous models are compared with the numerical results of the discrete models when converge to the continuous cases. The numerical results show that the investment rules solved from the continuous models and from the discrete models in continuous limits are exactly the same. Thus, the discount factor methodology is soundly consistent with the traditional continuous methods for real options, and fills the gap of the real option solutions in discrete cases.

The rest of this thesis is structured as follows. Section II is the literature review which provides a thread running through capacity choice problems and the discount factor methodology. Section III review and adjust the continuous models of Dxit and Pindyck (1994) and Pindyck (1988) and calculate the corresponding numerical results. The next two sections respectively establish the discrete capacity choice models without and with switching options, examine their convergence to the continuous cases, and solve and compare the numerical results with those of the continuous counterparts. The last section is the conclusion.

II. Literature Review

2.1 Capacity choice problems

The capacity choice problems, as we mentioned above, possess three primary features: incremental investment, irreversibility, and uncertainty. Many previous studies focus on some of these elements but few combine all of them.

As to the former feature, it should be differentiated from the sequential investment. The sequential investment implies that the completion of a project requires a sequence of steps and it is not until installing all steps that the output is produced. Roberts and Weitzman (1981), Baldwin (1982), Bar-Ilan and Strange (1992), etc. explore the models of sequential investment. On the other hand, incremental investment allows a firm to hold many projects, each of which could operate and produce output independently. New capacities could be added if the business condition gets better while the existing ones could continue to operate. Old capacities might be temporarily shut down or retired if the business condition gets worse. Therefore, the investment policy of the capacity choice problem is to choose and alter a firm's capacity according to a stochastic state variable which affects the productivity.

The latter two features of irreversibility and uncertainty are ignored by the standard neoclassical theory of investment. Jorgenson (1963) implies that the capacity decision under such case follows the simple markup pricing rule that the marginal revenue equals the marginal cost. This is because the investment problem of installing capacities would be equivalent to that of renting capacity for a period if the capacities could be costlessly and instantaneously adjusted. Arrow (1968) and Nickell (1974) studied irreversible investment with certain expectations about the exogenous variables. They show that in such framework the marginal revenue of capacity equals the cost of capacity whenever gross investment is positive. Additionally, they indicate that irreversibility would drive a wedge between the cost of capacity and the marginal revenue.

However, when irreversibility and uncertainty are mutually interacted, the situation becomes complicated and the optimal capacity budgeting is not so straightforward. The results of previous literature diverge and are controversial as to the effects of irreversibility and uncertainty, provided different assumptions and models. Particularly, the parameter values could exert significant effects. This thesis doesn't aim to focus on the comparison of different models and the relationship between investment, irreversibility, and uncertainty but on the application of the discrete methodology on the capacity choice problems, specifically those raised in Pindyck (1988) and Dxit and Pindyck (1994). However, a comprehensive review of these capacity choice models could facilitate our understanding regarding to the assumptions made in these two works and in this thesis. Thus, we summarize below the general model of capacity choice used by most literature and then classify and discuss different assumptions and settings.

Nearly all literature (e.g. Hartman 1972, Pindyck 1993,Bertola 1998) assumes a firm endowed with a Cobb-Douglas production function:

$$Q_t = m(K_t, L_t) = \left(A_t {K_t}^{\lambda} {L_t}^{1-\lambda}\right)^{\xi} \quad 0 < \alpha \leq 1, \xi > 0 \tag{2.1}$$

where Q_t denotes production quantity at time t, ξ and λ respectively denote the return to scale in production and the labor share, K_t is the input of capacity, L_t is the input of labor and could be rented at the instantaneous price w_t, and A_t is an index of technological progress.

The assumptions of previous literature, even though some don't manifest explicitly, are mainly reflected by the parameter values in the equations above. A strand of literature (e.g. Hartman, 1972; Abel, 1983; Pindyck, 1993; Abel and Eberly, 1999) assumes, implicitly or explicitly, constant returns to scale, which is given by $\xi = 1$. Some other literature (e.g. Caballero, 1991; Bertola, 1998), on the contrary, assumes varying returns to scale. As to the paralleled models for this thesis, Pindyck (1988) assumes constant returns to scale while Dxit and Pindyck (1994) assumes decreasing returns to scale.

The demand function encountered by the firm could be expressed as

$$\theta_t = J_t G(Q_t) \tag{2.2}$$

where θ_t is the product price at time t, J_t is the exogenous shift variable, and $G(Q_t)$ is the function of production. Nearly all literature (e.g. Caballero, 1991; Pindyck, 1993; Abel and Eberly, 1997; Bertola, 1998) on this topic assumes isoelastic demand curve. Therefore, equation (2.2) could be expressed as

$$\theta_t = J_t Q_t^{\mu-1}, 0 < \xi\mu \leq 1 \tag{2.3}$$

where $\frac{1}{1-\mu}$ is the elasticity of demand. Additionally, μ indexes the monopoly power. The more competitive is a market, the larger is the demand elasticity of a firm in this market. When $\mu = 1$, the demand is perfectly elastic and the market price is not affected by Q_t and is equivalent to the exogenous shock J_t. Thus, a competitive firm is only a price taker. The paralleled model in Pindyck (1988) assume perfect competition while the one in Dxit and Pindyck (1994) assume downward-sloping industry demand curve, namely $G'(Q_t) < 0$.

Under the conditions (2.1) and (2.2), the function of the maximized operating profit $\Pi(K_t, J_t)$ is given by

$$\Pi(K_t, J_t) = \max[\theta_t Q_t(K_t, L_t) - w_t L_t] \tag{2.4}$$

This multivariate optimization could be solved using first order conditions. Second order conditions should also hold to ensure decreasing marginal returns regarding to the inputs. However, the analytical expression of the optimal profit depends on our assumption of the demand function in equation (2.2). Bertola (1998) provides the optimal profit function under isoelastic demand curve shown in equation (2.3):

$$\Pi(K_t, J_t) = \frac{1}{1+\eta} K_t^{1+\eta} H_t \tag{2.5}$$

where

$$\eta = \frac{\xi\mu - 1}{1 - (1-\lambda)\xi\mu}, -1 < \eta \leq 0 \tag{2.6}$$

$$H_t = \frac{\lambda\xi\mu}{1-(1-\lambda)\xi\mu}\left[\left((1-\lambda)\xi\mu\right)^{\frac{(1-\lambda)\xi\mu}{1-(1-\lambda)\xi\mu}} - \left((1-\lambda)\xi\mu\right)^{\frac{1}{1-(1-\lambda)\xi\mu}}\right] (1+\eta) J_t^{\frac{1}{1-(1-\lambda)\xi\mu}} w_t^{\frac{-(1-\lambda)\xi\mu}{1-(1-\lambda)\xi\mu}} A_t^{\frac{\xi\mu}{1-(1-\lambda)\xi\mu}} \tag{2.7}$$

$\Pi(K_t, J_t)$ is strictly concave in K_t as long as $\xi\mu < 1$. However, if $\xi\mu = 1$,

$$\Pi(K_t, J_t) = K_t H_t \tag{2.8}$$

$$\theta_t = J_t Q_t^{\mu-1} = J_t \tag{2.9}$$

where

$$H_t = \left[(1-\lambda)^{\frac{1-\lambda}{\lambda}} - (1-\lambda)^{\frac{1}{\lambda}}\right] (A_t J_t)^{\frac{1}{\lambda}} w_t^{\frac{-(1-\lambda)}{\lambda}} \tag{2.10}$$

Therefore, if we assume constant returns to scale and perfect competition, namely $\mu = \xi = 1$, the profit $\Pi(K_t, T_t)$ is proportional to the capacity stock K_t. Pindyck (1988) also assume that one unit of capacity could produce one unit of output per time period. That means the production function is only determined by the input of capacity and $Q_t = K_t$. Thus, the labor share $\lambda = 1$ and $A_t = 1$. Therefore,

$$H_t = J_t \tag{2.11}$$

$$\Pi(K_t, J_t) = K_t J_t = K_t \theta_t \tag{2.12}$$

The profit function (2.12) is just the one used in Pindyck (1988) and therefore the general model of profit maximization for capacity choice shown above fits the specific model established in Pindyck (1988). To sum up, under the assumptions of constant returns to scale and perfect competition, the profit function equals to the product of capacity K_t and price θ_t.

Since the profit function is linear in capacity under the assumptions of constant returns to scale and perfect competitions, Pindyck (1993) indicates that convex costs of some kind are needed to bound the size of the firm, otherwise the firm would expand indefinitely if the present value of marginal profits from a unit of capacity exceeded the cost of the unit. Previous literature which assumes constant returns to scale and perfect competitions (e.g. Caballero, 1991; Abel and Eberly, 1997) usually solves this problem by introducing the adjustment costs, which are defined as the costs of changing capacities too rapidly (Dxit and Pindyck, 1994). In other words, the adjustment cost is the cost which depends on the rate of investment in a unit of time. However, since the adjustment costs are a function of only the rate of investment but not the existing capacity level, investment in a period is independent of investment or the capacity level in any other period. Then, investment in future only depends on the realization of demand that period and on the adjustment cost function while is independent of current investment. Correspondingly, the firm only need compare the marginal cost of investment with current and expected future marginal profits. Thus, the adjustment cost models ignore the effect of the irreversibility proposed in Pindyck (1988) and Bertola (1998), which implies that the current investment decisions are affected by the options in the future and uncertainty would decrease the current investment undertaken. Additionally, Pindyck (1993) claim that the adjustment cost is not the sole determinant of firm size in equilibrium. A pure adjustment-cost model with constant returns to scale is also inconsistent with competitive market equilibrium because many small firms will enter the industry with very small adjustment costs. In the limit, the industry would be composed of an infinite number of infinitesimally small firms with no adjustment costs. Bertola (1998) also indicates that models of investment based on convex adjustment costs have not been very successful empirically (Abel & Blanchard, 1986; Hall, 1987). The realism of smooth adjustment costs as the source of investment dynamics is doubtful. However, as far as we know, there are no previous literature construct the irreversible capacity choice models with constant returns to scale and perfect competition in the absence of adjustment costs. This is just the gap this thesis aims to fill.

2.2 Development of the methodology towards capacity choice problems

The traditional solutions to the optimal investment policies of capacity choice problems discussed above involve two boundary conditions: the value matching condition and the smooth pasting condition. The value matching condition means that the value functions of the option and of the net payoff should be continuous at the optimal transition point while the smooth pasting condition indicates that the first derivative of the value functions should be continuous at the optimal transition point. In other words, the value functions should converge tangentially at the transition point. Before solving the optimal thresholds, we should first obtain the explicit value functions of the investment options using dynamic programming or contingent claims analysis. Then, the optimal thresholds could be solved by substituting the explicit option expressions into the value matching condition and the smooth pasting condition. Therefore, one prerequisite for solving the optimal thresholds using traditional continuous method is the existence of explicit function of the option values. Previous literature using the continuous method (e.g. Pindyck, 1988) usually assumes the underlying state variable follows a geometric Brownian motion which ensures that the analytical solutions of the option values exist. However, if an alternative stochastic process is utilized, analytical solutions of the option value might not exist. In such case the continuous method is no longer available and thus encounters many restrictions. Additionally, the investment in the real world is unlikely to be continuous but discrete. Therefore, the discrete models might provide more realistic simulations of investment process in real life and more appropriate guide for investment decision making. Nevertheless, the discrete models are much less developed than the continuous model due to the restriction of the methodology.

The discount factor methodology is a viable approach to the discrete models. The concept of this methodology is first proposed by Baily in 1995. Assuming constant interest rates, Baily (1995) implies that the value of an option follows the trade-off between a larger versus later net benefit. If the stochastic process of the underlying asset has a positive drift which is smaller than the discount rate, waiting for option exercise would lead to a higher expected value but simultaneously lower discount factor. The higher the investment threshold is, the larger the investment value is. However, the payoff is also expected to be received at a more distant future and will be discounted more heavily. Therefore the optimal choice of the threshold is a trade-off. As a result, the optimization of the expected present option value requires an appropriate exercise time.

2.3 Discount factor methodology for one-step investment

Dixit, Pindyck and Sødal (1999) takes the perspective further and provides the optimal investment rule for one-step investment by developing an analogy with the trade-off in the pricing decision of a downward-sloping demand curve, namely the trade-off between a higher profit margin and a lower volume of sales. It suggests that the option value is like the profit margin while the discount factor is like the demand curve. With this analogy, Dixit, Pindyck and Sødal (1999) indicates that the one threshold optimal investment rule is similar to the markup pricing rule, i.e. the monopoly price rule that equals the marginal revenue with the marginal cost. The markup pricing rule is reviewed in **Appendix I**.

Now suppose the initial value of the state variable P is P_0 and consider an arbitrary threshold $\tilde{P}$ where $P_0 < \tilde{P}$. Thus the firm will wait until the first time T at which P has reached $\tilde{P}$ and then invest. Ekern, Shackleton, and Sødal (2014) indicates that valuation can occur using stochastic stopping time methods for time homogeneous problems. Therefore, though the time T taken for the state variable P to move from the current value P_0 to the threshold $\tilde{P}$ is a random variable, expectations can be formed with respect to T and its random continuous discount factor $e^{-\rho T}$ where ρ is the risk-adjusted discount rate and $P(T) = \tilde{P}$. Therefore, the expectation of the net present value of the investment is given by

$$E[e^{-\rho T}](\tilde{P} - X) \tag{2.13}$$

where X is the installation cost of the investment. Alternatively, we could also use the risk-free rate r instead of ρ and take the risk-neutral expectation, which is given by

$$E^{RN}[e^{-rT}](\tilde{P} - X) \tag{2.14}$$

Since the expected discount factor is independent of the stopping time T but only affected by the initial value P_0 and the threshold value $\tilde{P}$ for a given diffusion of P, we could denote the expected discount factor as:

$$D(P_0, \tilde{P}) = E^{RN}[e^{-rT} | P_T = \tilde{P}] \tag{2.15}$$

When the value of the state variable P is far away from the threshold in terms of P and time T, the discount factor should be smaller than one. However, the discount factor will approach unit value as P approaches the threshold in a stochastic way. When P reaches the threshold, the discount factor will become 1 and the present value of the payoff converges to the current value of the payoff. The optimal investment threshold, P^*, should be the value which could maximizes

$$D(P_0, \tilde{P})(\tilde{P} - X) \tag{2.16}$$

Similar to the markup pricing rule, we take the first-order condition for the optimal P^*.

$$D(P_0, P^*) + D_{\tilde{P}}(P_0, P^*)P^* - D_{\tilde{P}}(P_0, P^*)X = 0 \tag{2.17}$$

where $D_{\tilde{P}}(P_0, P^*)$ is the partial derivative of D with respect to the second argument $\tilde{P}$ and we are evaluating it at $\tilde{P} = P^*$. The condition (2.17) indicates that the optimal threshold is reached when the marginal discounted benefit from the investment, i.e. $D(P_0, P^*) + D_{\tilde{P}}(P_0, P^*)P^*$, equals the expected marginal discounted cost $D_{\tilde{P}}(P_0, P^*)X$. The equation (2.17) could be rewritten in the following form:

$$\frac{P^* - X}{P^*} = \left[-\frac{P^* D_{\tilde{P}}(P_0, P^*)}{D(P_0, P^*)}\right]^{-1} = 1/\epsilon_D \tag{2.18}$$

where ϵ_D is the elasticity of the discount factor $D(P_0, P^*)$ with respect to P^*, i.e., $\epsilon_D \equiv -P^* D_{\tilde{P}}(P_0, P^*)/D(P_0, P^*)$. $D_{\tilde{P}}(P_0, P^*) < 0$, $D(P_0, P^*) > 0$, and therefore $P^* > X$. With exogenous sunk cost X, equation (2.18) indicates that the trade-off that determines the optimal threshold is governed by the elasticity of the discount factor with respect to the threshold. A higher threshold P^* provides a higher margin $(P^* - X)$ of benefits over costs but a smaller discount factor $D(P_0, P^*)$ because the process is expected to take longer to reach the higher threshold. Comparing the investment problem to the markup pricing rule, the investment threshold is analogous to the inverse demand function while the discount factor is analogous to the quantity sold. The optimal threshold P^* is therefore reached when the marginal benefit $D(P_0, P^*)/D_{\tilde{P}}(P_0, P^*) + P^*$ equals the marginal cost X.

The amount by which P^* is over X also presents the extra value over cost required by the firm manager to keep the flexibility, namely the value of the investment option. In other words, at the optimal transition point, the net payoff of the investment, namely $(P^* - X)$, should equal the abandoned value of timing flexibility, namely the option value $V(P^*)$. Additionally, the present value of the net payoff at the transition is just the current value of flexibility, namely the current option value. Thus, the current option value at P_0 is the net present value of the option at the threshold P^*.

$$V(P_0) = D(P_0, P^*)(P^* - X) = D(P_0, P^*)V(P^*) \tag{2.19}$$

Dixit, Pindyck and Sødal (1999) proves that the elasticity ϵ_D, and in turn the optimal threshold P^* according to equation (2.18), is independent of the starting value P_0. Therefore, before reaching P^* we could connect the initial option value at any dynamic point P with the final option value at the investment threshold P^* using equation (2.19). For a certain investment threshold P^*, thus, a dynamic option value $V(P)$ could be expressed as a dynamic fraction of the final option value $V(P^*)$ which assumes the role of a scaling constant while the discount factor $D(P, P^*)$ is the dynamic fraction depending on the value of P.

$$V(P) = D(P, P^*)V(P^*) \tag{2.20}$$

As demonstrated above, in order to find the optimal investment threshold, we should solve the discount factor $D(P_0, P^*)$ given the stochastic process of P. For some stochastic processes, the probability distribution of T can be evaluated and an expected discount factor can be found analytically. Numerical methods are also possible in other cases. Dixit, Pindyck, and Sødal (1999) derives the discount factor of call options for several common stochastic processes while Ekern, Shackleton, and Sødal (2014) further provides the discount factor for put options. We summarize the derivations and supplement details in **Appendix II**. Particularly, if the underlying state variable P follows a GBM, the discount factors for call options and put options are respectively given by

Discount (call), $P \leq P^*$: $$D_c(P, P^*) = \left(\frac{P}{P^*}\right)^c \tag{2.21}$$

Discount (put), $P \geq P^*$: $$D_p(P, P^*) = \left(\frac{P}{P^*}\right)^p \tag{2.22}$$

With the assumption of GBM, therefore, it is easy to prove that $D(P, P^*)$ is iso-elastic with respect to P and the constant elasticities for $D_c(P, P^*)$ and $D_p(P, P^*)$ are respectively c and p. Therefore, the percentage changes in $D(P, P^*)$ caused by a percentage change in P is independent of the scale P. Such convenient iso-elastic feature of GBM makes equation (2.18) quite easy to be solved:

$$\frac{P^* - X}{P^*} = \frac{1}{\epsilon_D} = \frac{1}{c}, \qquad P^* = \frac{c}{c-1}X \tag{2.23}$$

The result of the optimal one-step investment threshold is analogous to the price-cost markup rule with an iso-elastic demand curve and the ratio of the markup is determined by the elasticity of the discount factor. Ekern, Shackleton, and Sødal (2014) also summarizes the discount factors for other process and we review them in

Appendix III. Though the elasticities of these processes are not iso-elastic like BGM, the techniques developed above are robust for other processes.

Equation (2.18) reveals that if we could find the elasticity of the discount factor, we could solve the optimal investment threshold even though we don't know the analytical expression of the option. In other words, the requirement for the value function of the option is transformed to that of the discount factor. The advantage of this transformation is not obvious in the one-step investment since the analytical expression of the option usually could be easily solved. For multi-step investment systems nevertheless, the analytical solutions in closed form to the option functions might not be available and the numerical solutions might be complicated. Such unique advantage makes the discount factor methodology a promising approach to solve the multi-step investment problems.

However, the discount factor methodology for one-step investment problems could not be directly applied to the multi-step investment problems. As shown in equation (2.24), in a case of one-step investment only the payoff, namely the net present value of the perpetual revenue $P^* - X$, is obtained when the option is exercised.

$$V(P^*) = P^* - X \tag{2.24}$$

Thus, the first-order condition as shown in equation (2.17) is easy to be solved. However, in a multi-step investment, a new option might be created when an old option is exercised. In such case managers should maximize the joint value of both initial and subsequent options rather than the single option value as shown in equation (2.19). The mutual dependent option values make it not possible to solve a single threshold alone without considering other thresholds simultaneously. Therefore, the simple investment rule presented in equation (2.23) could not adapt to the multi-step investment problems. In order to apply the discount factor approach to the multi-step investment problems, Ekern, Shackleton, and Sødal (2014) develops the methodology which is presented in the next subsection.

2.4 Discount factor methodology for multi-step investment

Though we could not adapt the simple policies shown above to the multi-step investment problems, Ekern, Shackleton, and Sødal (2014) indicates that two features of the discount factors could help to construct viable models of the discount factor approach to solve the multi-step investment problems.

First, as shown above in equation (2.20), the discount factors could connect the option values at the time of their creation with the option values at the end of their life. In other words, the option values at different thresholds could be connected, contributing an extra condition to the traditional boundary conditions of the value matching and the smooth pasting.

Second, the discount factors could be used to transform the smooth pasting conditions to the dollar beta matching conditions which don't depend on the analytical functions of options to find solutions.

Before interpreting the dollar beta matching condition, we first review two concepts, i.e. delta and beta, which play significant roles in the solution system. For an option $V(P)$, its first derivative with respect to the underlying variable P is known as delta.

$$\Delta(P) = \frac{\partial V(P)}{\partial P} \tag{2.25}$$

The elasticity of an option is known as beta, which tracks the sensitivity of an option's percentage change $\partial V(P)/V(P)$ to the return of P.

$$\beta(P) = \frac{P}{V(P)}\frac{\partial V(P)}{\partial P} \tag{2.26}$$

The smooth pasting condition requires that the project values at two sides of the threshold should be connected smoothly. That means the equality of the first derivative of the option value with respect to the underlying asset, namely the equality of delta. However, if the explicit function of the option value is not available, the delta is also theoretically unknown. Under such case, the optimal thresholds could not be solved.

Shackleton and Sødal (2005) indicates that the smooth pasting is equivalent to the rate of return equalization, which could be equally represented by a dollar beta matching condition. Thus, the requirement for delta could be transformed to that for beta.

The dollar beta is the regular beta scaled by the dollar value $V(P)$ of the option itself. It measures the dollar impact of the beta, i.e. weighted by its value and is equivalent to the delta scaled by the underlying P. The left hand side of equation (2.27) is the expression for the dollar beta and the right hand side is the delta scaled by P where $\beta(P)$ is the beta of $V(P)$ with respect to P.

$$\beta(P)V(P) = \frac{P}{V(P)}\frac{\partial V(P)}{\partial P}V(P) = P\frac{\partial V(P)}{\partial P} \tag{2.27}$$

Therefore, the dollar beta equals the delta multiplies by the value of P, and all smooth pasting conditions could be transformed to the dollar beta conditions. In other words, the delta equation is equivalent to the dollar beta equation. For example, if we take derivative at both sides of the value matching condition in equation (2.24), we could obtain the smooth pasting condition:

$$\frac{\partial V(P^*)}{\partial P^*} = 1 \tag{2.28}$$

Then, the dollar beta matching condition could be obtained by multiplying both sides of equation (2.28) by P:

$$\beta(P^*)V(P^*) = \frac{\partial V(P^*)}{\partial P^*}P^* = 1.P^* \tag{2.29}$$

The one on the right hand side of the equation above emphasizes that the beta of the payoff equals 1. This is intuitive that the elasticity of the payoff with respect to itself, namely $\frac{P}{P}\frac{\partial P}{\partial P}$, obviously equals one. With more complicated option flexibilities, the dollar beta matching condition will become more complicated. However, the principle remains and is simple that the dollar beta should be continuous at the threshold. As a result, if we know the beta of an option, namely the elasticity of an option with respect to the underlying price, we could construct the smooth pasting condition even we don't know the option's value function. However, the beta of an option value is also not straightforward. As illustrated later, the beta of an option value could be solved using the discount function. Alternatively, it could be yielded empirically, which is more practical for the investment in the real world. We focus on the theoretical solutions from the discount function.

As shown in equation (2.20), a dynamic option value $V(P)$ could be expressed as a dynamic fraction of the final option value $V(P^*)$ which assumes the role of a scaling constant. Therefore differentiating the option $V(P)$ or discount function $D(P,P^*)$ is equivalent and gives the same beta function. Therefore, the dollar beta of $V(P)$ could be expressed in terms of $D(P,P^*)$ and the beta of the call options and of the put options are respectively given by

$$\beta_c(P) = \frac{P}{V(P)}\frac{\partial V(P)}{\partial P} = \frac{P}{D_c(P,P^*)}\frac{\partial D_c(P,P^*)}{\partial P} = c \tag{2.30}$$

$$\beta_p(P) = \frac{P}{V(P)}\frac{\partial V(P)}{\partial P} = \frac{P}{D_p(P,P^*)}\frac{\partial D_p(P,P^*)}{\partial P} = p \tag{2.31}$$

Therefore, under GBM the beta of the option values is also iso-elastic and equals to the beta of the discount functions. Thus, for example, the smooth pasting condition in equation (2.28) could be transformed to a dollar beta matching condition which can be expressed as:

$$cV(P^*) = P^* \tag{2.32}$$

As shown above, the techniques to derive the beta of discount factors are robust to a substitution of process from GBM though they might not be iso-elastic as GBM. Though the expressions of these discount functions are different, they have similar solutions c, p to a fundamental quadratic.

To sum up, the discount function representation mainly plays two significant roles in the solution to the sequential capacity choice. First, it contributes to the third set of conditions that links the option values at different thresholds. Then, it could be used to solve the beta of the option value which is used to construct the dollar beta matching condition, the substitution of the smooth pasting condition. To quote the statement of Ekern, Shackleton, and Sødal (2014), “discount functions fulfil two useful roles; first they capture the dynamics of values and their betas and second they fix the ratios and scales of options at the time of their creation compared to their use.” The value matching and dollar beta matching at the transitions, together with the discount function relating the option values between transitions, contribute to the complete system which could be used to solve the optimal investment thresholds.

However, it is worth noting that the discount factor methodology developed by Ekern, Shackleton, and Sødal (2014) does not solve the exercise thresholds as a function of the installation costs but assume given thresholds which can infer the implied costs. We then numerically iterate the values of thresholds to let the costs match the target values. In essence, this method reverses the input and output variables. To make the abstract concept concrete, we make an analogy with the inverse function. If we could not directly solve the optimal threshold P^* as a function of the cost X, we could iterate the inverse function $X(P)$ until the value of X matches the target values X^*. The price P^* that makes the equation hold is the threshold we searched for.

$$X(P^*) = X^* \tag{2.33}$$

III. Results of adjusted Dxit and Pindyck (1994) and Pindyck (1988)

As shown in the literature review above, the adjustment cost is not a plausible assumption for the irreversible capacity choice problems under constant return to scale and perfect competition. Then how could we solve the issue of indefinite expansions without introducing the adjustment cost? In fact we could adopt increasing installation costs per unit of capacity instead of convex adjustment cost to bind the indefinite expansions. That means the installation costs per unit of capacity increases as the existing capacity level becomes larger. This kind of diseconomies of scale might arise from limited land available for plants (higher costs to purchase extra unit of land), higher technology requirement as the capacity increases, or other reasons. Therefore, convex installation cost per unit of capacity is economically sensible. Marginal installation cost proportional to the installed capacity level is one of the simplest specifications of the increasing installation cost per unit of capacity and we adopt the assumption in this thesis.

Dxit and Pindyck (1994) and Pindyck (1988) respectively introduce continuous capacity choice models without switching options and with switching options. Therefore, they could be treated as the continuous counterparts of our discrete models. However, these two papers both assume constant installation cost per unit of capacity. Additionally, Dxit and Pindyck (1994) also assumes decreasing returns to scale and imperfect competition. With different assumptions, our discrete models are not perfectly paralleled with Dxit and Pindyck (1994) and Pindyck (1988) and hence could not converge to the continuous models as the lumpy steps of investment shrink. Therefore, the assumptions of Dxit and Pindyck (1994) and Pindyck (1988) should be conformed to those of the discrete models except for the continuous and discrete settings, including scaling marginal installation costs respect to the capacity level, constant return to scales, and perfect competition. Then the numerical results of the adjusted continuous models should be calculated to compare with the results of discrete models.

In the paragraphs below we respectively review and modify the continuous models of Dxit and Pindyck (1994) and Pindyck (1988) and solve the corresponding numerical results.

3.1 Dxit and Pindyck (1994)

Dxit and Pindyck (1994) assumes that the production function and the industry demand function are respectively in the following forms:

$$Q_t = g(K_t) \tag{3.1}$$

$$\theta_t = J_t G(Q_t) \tag{3.2}$$

where Q_t is the flow of output at time t, K_t is the amount of capacity, θ_t is the price, and J_t is the shift variable, which follows the risk-neutral geometric Brownian motion

$$dJ_t = (r-\delta)J_t dt + \sigma J_t dz \tag{3.3}$$

where r is the risk free rate, δ is the yield, and σ is the volatility.

Assuming no variable costs (labor input L_t in the general model above), the profit flow is given by

$$\Pi(K_t, J_t) = \theta_t g(K_t) = J_t G\big(g(K_t)\big) g(K_t) = J_t h(K_t) \tag{3.4}$$

Dxit and Pindyck (1994) also assumes that the installation cost for a unit of capacity is constant and denoted by k. Diminishing returns to scale means $g''(K_t) < 0$ and imperfect competition means $G'(Q_t) < 0$. With these assumptions, Dxit and Pindyck (1994) solves the optimal investment threshold $J_t(K_t)$ and the marginal effect of capacity expansion on the value of the firm $v_K(K_t, J_t)$. Correspondingly we could obtain the optimal polices of the perpetual revenue which is defined as $P_t = \frac{\theta_t}{\delta}$[1].

$$J_t(K_t) = \frac{\beta_p}{\beta_p - 1} \frac{\delta k}{h'(K_t)} \tag{3.5}$$

$$P_t(K_t) = \frac{\theta_t}{\delta} = \frac{J_t G\big(g(K_t)\big)\beta_p k}{\big(\beta_p - 1\big)h'(K_t)} \tag{3.6}$$

$$v_K(K_t, J_t) = y'(K_t) J_t(K_t)^{\beta_p} + \frac{J_t(K_t) h'(K_t)}{\delta} = k \tag{3.7}$$

$$y'(K_t) = -\left(\frac{\beta_p - 1}{k}\right)^{\beta_p - 1} \left(\frac{h'(K_t)}{\beta_p \delta}\right)^{\beta_p} \tag{3.8}$$

where β_p is the positive root of the fundamental quadratic (3.9). The positive root β_p and the negative root β_n of equation (3.9) are given below.

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + (r - \delta)\beta - r = 0 \tag{3.9}$$

$$\beta_p = -\frac{\left(r - \delta - \frac{\sigma^2}{2}\right)}{\sigma^2} + \frac{1}{\sigma^2}\left[\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2\right]^{\frac{1}{2}} > 1 \tag{3.10}$$

$$\beta_n = -\frac{\left(r - \delta - \frac{\sigma^2}{2}\right)}{\sigma^2} - \frac{1}{\sigma^2}\left[\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2\right]^{\frac{1}{2}} < 0 \tag{3.11}$$

It is worth noting that the expressions of β_p and β_n are exactly the same as c and p in equations A2.9 and A2.10 in **Appendix II**. Therefore, $\beta_p = c$ and $\beta_n = p$ given the same values of r, δ, and σ. Therefore, in the following paragraphs we use c and p to replace β_p and β_n.

$y'(K_t)J_t(K_t)^c$ is the option value abandoned for expanding a unit of capacity. Therefore, $v_K(K_t, T_t)$ is the marginal perpetual revenue of capacity expansion minus the value of the option to invest, which should equal the marginal installation cost k at

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1 $P_t = \int_0^\infty E^{RN}[\theta_t]\, e^{-rt} dt = \int_0^\infty \theta_t\, e^{-\delta t} dt = \left[-\frac{\theta_t}{\delta} e^{-\delta t}\right]_0^\infty = \frac{\theta_t}{\delta}$

the threshold. Since $g''(K_t) < 0$, $G'(Q_t) < 0$, and $h(K_t) = G\big(g(K_t)\big)g(K_t)$, we could obtain $h''(K_t) < 0$. Therefore, $J_t(K_t)$ in equation (3.5) is convex in K_t while $v_K(K_t, T_t)$ is concave in K_t.

If we change the assumptions of Dxit and Pindyck (1994) to perfect competition and constant returns to scale with one unit of capacity produce one unit of output, then $Q_t = g(K_t) = K_t$ and $G(Q_t) = G(K_t) = 1$. Therefore, $\theta_t = J_t G\big(g(K_t)\big) = J_t$ and $h(K_t) = g(K_t) = K_t$. Thus,

$$J_t(K_t) = \frac{c\delta k}{c-1} \tag{3.12}$$

$$P_t(K_t) = \frac{\theta_t}{\delta} = \frac{J_t(K_t)}{\delta} = \frac{ck}{c-1} \tag{3.13}$$

$$y'(K_t) = -\left(\frac{c-1}{k}\right)^{c-1}\left(\frac{1}{c\delta}\right)^{c}\left(\frac{c\delta k}{c-1}\right)^{c} = \frac{-k}{c-1} \tag{3.14}$$

$$y'(K_t)J_t(K_t)^c = \frac{-k}{c-1}\left(\frac{c\delta k}{c-1}\right)^{c} = \frac{-(c\delta)^c k^{c+1}}{(c-1)^{c+1}} \tag{3.15}$$

Equations (3.12), (3.13), and (3.15) imply that the investment threshold $P_t(K_t)$ and the expansion option $y'(K_t)J_t(K_t)^c$ become constant and independent of capacity K_t under the assumptions of constant returns to scale and perfect competition. In other words, the threshold which could trigger the installation of the first unit capacity would trigger all the following capacity expansions. This is economically reasonable because constant returns to scale and inelastic demand curve (perfect competition) imply that marginal revenue contribution of a capacity is constant. Since the costs of investment, namely the sum of the option value and the installation cost, are also constant, the threshold that justify one unit of capacity would justify all capacity expansions. This is consistent with our analysis above that convex costs are required to bound the size of the firm which has constant returns to scale and infinitely elastic demand curve. In order to restrict the indefinite expansion, we assume the installation cost per unit capcity k_t is proportional to the capcity level K_t. Therefore, k_t could be expressed as

$$k_t = kK_t \tag{3.16}$$

Thus, the expressions of $J_t(K_t)$, $P_t(K_t)$, and $y'(K_t)$ are changed to

$$J_t(K_t) = \frac{c\delta kK_t}{c-1} \tag{3.17}$$

$$P_t(K_t) = \frac{J_t(K_t)}{\delta} = \frac{ck_t}{c-1} = \frac{ckK_t}{c-1} \tag{3.18}$$

$$y'(K_t) = -\left(\frac{c-1}{kK_t}\right)^{c-1}\left(\frac{1}{c\delta}\right)^{c}\left(\frac{c\delta kK_t}{c-1}\right)^{c} = \frac{-kK_t}{c-1} \tag{3.19}$$

$$y'(K_t)J_t(K_t)^c = \frac{-kK_t}{c-1}\left(\frac{c\delta kK_t}{c-1}\right)^{c} = \frac{-(c\delta)^c (kK_t)^{c+1}}{(c-1)^{c+1}} \tag{3.20}$$

Therefore, the investment thresholds become proportional to the capacity levels K_t. Then, the cumulative capital for capacity K_t should be equal to the sum of the additional capacity at each expansion, which is given by

$$X_t = \int_0^{K_t} kK_t \, dK_t = \frac{1}{2} kK_t^2 \tag{3.21}$$

If we substitute K_t in equation (3.21) by P_t according to equation (3.18), we could obtain the analytical expression of X_t as a function of P_t.

$$X_t = \frac{1}{2} kK_t^2 = \frac{1}{2k} \left(\frac{c-1}{c}\right)^2 P_t^2 \tag{3.22}$$

Therefore, the cumulative capital X_t is a quadratic function of the perpetual revenue P_t. With equations (3.22), we could now obtain the optimal investment policies, namely the investment threshold to input a given level of capital or the optimal cumulative capital for a given value of the perpetual revenue. According to equation (3.10), it is easy to derive that the values of c monotonically increase in the values of δ and monotonically decrease in the values of r and σ.

We now undertake numerical experiments for the adjusted model. Since the values of c are monotonically affected by r, δ, and σ, we could examine the parameter effects of r, δ, and σ by looking into the effect of c. For sensitive analysis, we respectively assume two sets of parameter values of r, δ, and σ and calculate the corresponding values of c. The first set is $r = 0.05$, $\delta = 0.05$, and $\sigma = 0.1$ and therefore $c = 3.70$. The second set is $r = 0.15$, $\delta = 0.1$, and $\sigma = 0.2$ and therefore $c = 2.09$.

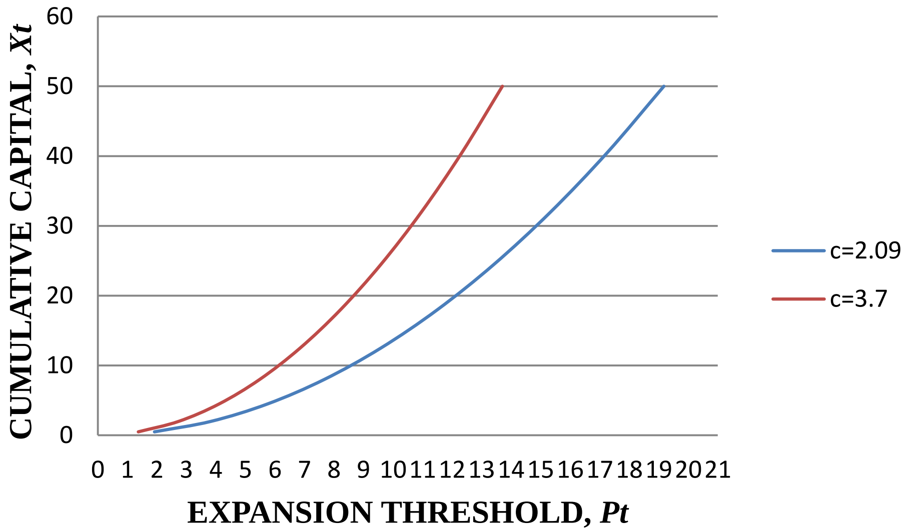
Table 3.1 reports the numerical results and we plot the cumulative capital X_t as a function of the perpetual revenue P_t in Figure 3.1 to make the expansion rule more intuitive.

Table 3.1 Optimal expansion rule for the adjusted model of Dxit and Pindyck (1994) ($k = 1$)

$c = 3.70$			$c = 2.09$		
P_t	K_t	X_t	P_t	K_t	X_t
1.37	1.00	0.50	1.92	1.00	0.50
2.74	2.00	2.00	3.83	2.00	2.00
4.11	3.00	4.50	5.75	3.00	4.50
5.48	4.00	8.00	7.67	4.00	8.00
6.85	5.00	12.50	9.59	5.00	12.50
8.22	6.00	18.00	11.50	6.00	18.00

This table reports the optimal expansion thresholds P_t against capacity levels K_t and the cumulative capitals X_t for the adjusted model of Dxit and Pindyck (1994). The results are solved using equations (3.18) and (3.22). Two sets of parameter values are assumed: $r = 0.05$, $\delta = 0.05$, and $\sigma = 0.1$ which lead to $c = 3.70$; $r = 0.15$, $\delta = 0.1$, and $\sigma = 0.2$ which lead to $c = 2.09$.

Figure 3.1 Expansion Thresholds and Cumulative Capitals



This figure plots the cumulative capital X_i against the expansion thresholds P_i respectively for $c = 3.7$ and $c = 2.09$. The results are solved using equations (3.15) and (3.17).

It could be observed from Figure 3.1 that the cumulative capital X_t convexly increases in the expansion threshold P_t, consistent with the quadratic equation (3.22). Additionally, higher value of c would always increase investment. This is economically sensible that the higher return r is required or the higher uncertainty σ is expected in future, the lower incentive of investment a firm has. On the contrary, if projects could generate higher yield δ, the firm is more willing to invest.

3.2 Pindyck (1988)

We now review and modify the continuous model of Pindyck (1988) and then calculate the corresponding numerical results. Dxit and Pindyck (1994) assumes that the firm only has the choice to expand its capacity but no choice to shut down or turn off, namely complete irreversibility. On the contrary, Pindyck (1988) consider a situation where firms not only hold infinite expansion options to add capacities in the future but also have the switching on/off options regarding to the existing capacities. That means firms could choose to temporarily turn off the existing capacities if the dynamic state variable, usually the price or the perpetual revenue, falls below certain switching thresholds. The switched off capacities could be switched on again if the state variable rebounds to certain switching on thresholds. The expansion thresholds are completely irreversible while the switching thresholds are completely reversible.

Pindyck (1988) also assumes that the shift variable of shock J_t follows a geometric Brownian motion in a form of equation (3.3). The installation cost per unit of capacity is also constant and is denoted as k. However, Pindyck (1988) assumes constant returns to scale and perfect competition. Therefore, price θ_t would be equal to shock J_t as discussed for equation (2.3) and also follows the same stochastic process as the shift variable of shock J_t. Additionally, Pindyck (1988) assumes that the operation of capacities would incur costs in a form of $(2\gamma + c_2)K + c_1$ where γ, c_1, and c_2 are all constant parameters. With these assumptions, Pindyck (1988) construct the marginal switching conditions and marginal expansion conditions. All denotations used in the last section for adjusted model of Dxit and Pindyck (1994) keep the same in this section while new introduced variables and parameters would be explained.

According to Pindyck (1988), the marginal switching conditions are given by

$$b_1{\theta_t}^c = b_2{\theta_t}^p + \frac{\theta_t}{\delta} - C_t \tag{3.23}$$

where

$$P_t = \frac{\theta_t}{\delta} \tag{3.24}$$

$$C_t = \frac{(2\tau + c_2)K_t + c_1}{r} \tag{3.25}$$

$$b_1 = \frac{r - p(r-\delta)}{r\delta(c-p)} \times [(2\tau + c_2)K + c_1]^{1-c} > 0 \tag{3.26}$$

$$b_2 = \frac{r - p(r-\delta)}{r\delta(c-p)} \times [(2\tau + c_2)K + c_1]^{1-p} > 0 \tag{3.27}$$

The switching thresholds of price are given by

$$\theta_t = (2\tau + c_2)K_t + c_1 \tag{3.28}$$

Therefore, the switching thresholds of perpetual revenue are given by

$$P_t^s = \frac{\theta_t}{\delta} = \frac{(2\tau + c_2)K_t + c_1}{\delta} \tag{3.29}$$

In equation (3.23), $b_1{\theta_t}^c$ presents the call option to switch on when the capacity stays idle while $b_2{\theta_t}^p$ presents the put option to switch off when the capacity is operating. P_t is the perpetual revenue while C_t is the perpetual operating cost[2]. Thus, $P_t - C_t$ is the perpetual net revenue produced by the capacity assuming it operates permanently. Therefore, when the capacity is at the transition of switching off, the right hand side of equation (3.23) is the option value and the expected net perpetual revenue abandoned while the left hand side is the value of the switching on call option obtained. Correspondingly, when the capacity is at the transition of switching on, the right hand side of equation (3.23) represents the perpetual net revenue and the switching off put option recovered while the left hand side is the switching on call option value given up for exercising the turn on decision. Equation (3.28) is just the markup pricing rule that equals the marginal revenue θ_t with the marginal operating cost $(2\tau + c_2)K_t + c_1$ for completely reversible switching decisions. Pindyck (1988) further provide the marginal expansion conditions.

$$a{\theta_t}^c = b_2{\theta_t}^p + P_t - C_t - k \tag{3.30}$$

where

$$a = \frac{pb_2}{c}{\theta_t}^{(p-c)} + \frac{1}{\delta c}{\theta_t}^{(1-c)} \tag{3.31}$$

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2 $C_t = \int_0^\infty E^{RN}[(2\tau + c_2)K_t + c_1]\, e^{-rt}dt = \left[-\frac{(2\tau+c_2)K_t+c_1}{r}e^{-rt}\right]_0^\infty = \frac{(2\tau+c_2)K_t+c_1}{r}$

$a\theta_t{}^c$ is the expansion option given up at the expansion thresholds while $b_2\theta_t{}^p$ and $P_t - C_t - k$ are respectively the switching off option obtained brought by the new added capacity and the expected perpetual revenue P_t by the new additional capacity net of the operating cost C_t and the installation cost k.

If we expend equation (3.30), we could obtain the function that could solve the expansion thresholds for given capacity levels.

$$\frac{b_2(c-p)}{c}(\theta_t)^p + \frac{(c-1)}{\delta c}\theta_t - \frac{(2\gamma + c_2)K_t + c_1}{r} - k = 0 \tag{3.32}$$

However, as discussed above, we should also modify the model of Pindyck (1988) and assume scaling installation costs per unit of capacity instead of constant ones if we would like the continuous model is paralleled with the discrete model constructed in this thesis. That means the installation cost per unit capcity k_t is proportional to the capcity level K_t and equal to kK_t. Additionally, we assume for the discrete model in section V that the perpetual operating cost is proportional to the capacity level. Therefore, we assume $c_1 = 0$ to keep the assumption the same for the discrete model and the continuous model. Thus, the operating cost per unit capacity is equal to $(2\tau + c_2)K_t$. With the new assumptions, we could obtain the new function of the optimal investment rule.

$$\frac{b_2(c-p)}{c}(\theta_t)^p + \frac{(c-1)}{\delta c}\theta_t - \frac{(2\tau + c_2)K_t}{r} - kK_t = 0 \tag{3.33}$$

Then, the cumulative capital for capacity K_t should be equal to the sum of the additional capacity at each expansion, which is given by

$$X_t = \int_0^{K_t} kK_t \, dK_t = \frac{1}{2}kK_t{}^2 \tag{3.34}$$

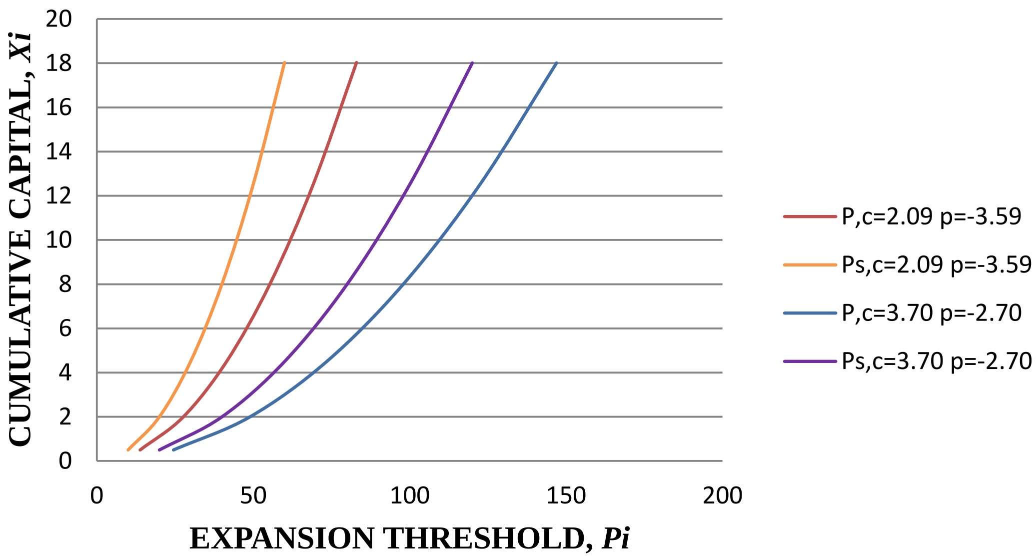
We now could calculate the new numerical results which are presented in Table 3.2 and Figure 3.2. The same parameters as in the adjusted model of Dxit and Pindyck (1994) are assumed the same values as those in the last section while the new parameters τ and c_2 are respectively assumed to equal 0.5 and 0.

Table 3.2 Optimal expansion rule for the adjusted model of Pindyck (1988)
($\tau = 0.5$, $c_2 = 0$, $k = 1$)

$c = 3.70, p = -2.70$				$c = 2.09, p = -3.59$			
P_t	P_t^s	K_t	X_t	P_t	P_t^s	K_t	X_t
24.49	20	1.0	0.5	13.84	10	1.0	0.5
33.35	40	2.0	2.0	15.50	20	2.0	2.0
73.47	60	3.0	4.5	23.25	30	3.0	4.5
97.96	80	4.0	8.0	31.00	40	4.0	8.0
122.45	100	5.0	12.5	38.76	50	5.0	12.5
146.94	120	6.0	18.0	46.50	60	6.0	18.0

This table reports the optimal expansion thresholds P_t, the switching thresholds P_t^s, the capacity levels K_t, and the cumulative capitals X_t for the adjusted model of Pindyck (1988). The results are solved using equations (3.29), (3.33), and (3.34).

Figure 3.1 Expansion Thresholds and Cumulative Capitals



This figure plots the cumulative capital X_i against the expansion thresholds P_i respectively for $c = 3.7$ and $c = 2.09$. The results are solved using equations (3.15) and (3.17).

It could be observed from Table 3.1 and Figure 3.1 that the cumulative capitals X_t convexly increase in the switching thresholds P_t^s, consistent with the linear function (3.29) and the quadratic function (3.34). Additionally, the cumulative capitals X_t also convexly increase in the expansion thresholds P_t. However, unlike the results in the last section, higher value of c would lead to higher expansion thresholds and correspondingly decrease investment. As mentioned above, c is monotonically decreasing in r. Higher value of r not only means higher required return for the investment, but also means lower perpetual operating cost according to equation (3.25), which would incent more investment. Therefore, the positive effect of higher r on the perpetual operating costs dominates and lead to opposite effect the value of c exerts on the expansion thresholds.

IV. Discrete Capacity Choice Problem without Switching Options

4.1 Model

In this section we establish the discrete counterpart of Dxit and Pindyck (1994), which provides the optimal capacity choice policies without switching options in continuous case. As we mentioned above, the discrete model is mainly based on the discount factor methodology of Ekern, Shackleton, and Sødal (2014). Besides the three sets of conditions used to solve the optimization, we also employ the matrix solution in Ekern, Shackleton, and Sødal (2014). Briefly, this method collects similar items into vectors and use matrix and vectors to represent the conditions. The matrix representation could facilitate the solution in large systems with many capacity choices. We illustrate the details in the following parts of this section.

We first let P be the perpetual revenue per unit output and follows a geometric Brownian motion under a risk-neutral measure:

$$dP = (r - \delta)Pdt + \sigma Pdz \tag{4.1}$$

As mentioned above, the perpetual revenue, namely the expected present value of all revenues in the future, equals the price θ divided by the yield δ. Therefore, the price θ follows the same diffusion as the perpetual revenue P.

$$\frac{d\theta}{\theta} = \frac{dP}{P} = (r - \delta)dt + \sigma dz \tag{4.2}$$

We assume that the installations of capacities are discrete and instantaneous. That means the output by the new capacity unit would be immediately produced once the expansion threshold is reached and the new capacity is established. The installations of capacity units are sequential and each capacity unit is identical.

In this discrete model of sequential capacity choice, we use the subscripts to indicate the ordinal number of the expansions. Therefore, we denote the ith expansion threshold by P_i; the cumulative capacity after the ith expansion by K_i; the cumulative installation cost, namely the cumulative capital, for K_i units of capacity by X_i; the installation cost for the ith expansion by $X_i - X_{i-1}$; the dynamic option values to expand capacities in the future with current capacity levels K_{i-1} and K_i by $V_{i-1}(P)$ and $V_i(P)$. The same as the assumption in the adjusted model of Dxit and Pindyck (1994), we assume constant returns to scale and perfect competition. We further assume constant returns to capacity which means a unit of capacity produces a unit of output per unit time. Therefore, K_i could also denote the output produced by K_i units of capacity. It is important to tell the difference between the amount of capacity K_i and the cumulative capital X_i. $X_i = \sum(K_i - K_{i-1})k_i$ where k_i is the installation cost per unit of capacity for the ith expansion. Therefore, the added capital at the ith expansion should be the difference between X_i and X_{i-1}, following the equation of $X_i - X_{i-1} = (K_i - K_{i-1})k_i$.

In the last section we show that the investment threshold P_i is proportional to the capacity level K_i in the adjusted model of Dxit and Pindyck (1994) as shown by equation (3.18). In order to keep consistent with the adjusted model and demonstrate the convergence of the discrete capacity choice model to the continuous counterpart, we assume that both the expansion thresholds and the capacity amounts increase at the same ratio of α, i.e.

$$P_{i+1} = (1 + \alpha)P_i \quad \text{and} \quad K_{i+1} = (1 + \alpha)K_i \tag{4.3}$$

With the assumption of equations (4.3), the capacity would immediately jump from K_{i-1} to $K_i = (1 + \alpha)K_{i-1}$ when P hit the threshold P_i. Thus, the perpetual revenue jump from $P_i K_{i-1}$ to $P_i K_i$. At this transition, installation cost $X_i - X_{i-1}$ is incurred for the capacity expansion. Besides, the option values the firm holds also change at the transition. When the ith capacity is installed, the expansion option of the ith capacity is exercised and abandoned. Therefore, the expansion option values shift from $V_{i-1}(P_i)$ with capacity K_{i-1} to $V_i(P_i)$ with capacity K_i. Having figured out all the value changes at the transition of capacity expansion, the value matching conditions at the thresholds could be written in the following way:

$$V_{i-1}(P_i) + P_i K_{i-1} = V_i(P_i) + P_i K_i - (X_i - X_{i-1}) \tag{4.5}$$

The terms at the left side of equation (4.5) are the values held before the expansion of the ith capacity while the terms at the right side are the values obtained after the expansion. Since we assume that the capacity expansion could be extended infinitely, then there should be infinite number of value matching conditions. We collect similar items into vectors and then use matrix and vector equations to represent their linkages.

$$\overset{\underline{W_\infty}}{\begin{bmatrix} \vdots \\ V_{i-1}(P_i) \\ V_i(P_{i+1}) \\ \vdots \end{bmatrix}} + \overset{\underline{Z_\infty}}{\begin{bmatrix} \vdots \\ P_i K_{i-1} \\ P_{i+1} K_i \\ \vdots \end{bmatrix}} = \overset{\underline{U_\infty}}{\begin{bmatrix} \vdots \\ V_i(P_i) \\ V_{i+1}(P_{i+1}) \\ \vdots \end{bmatrix}} + \overset{\underline{Y_\infty}}{\begin{bmatrix} \vdots \\ P_i K_i \\ P_{i+1} K_{i+1} \\ \vdots \end{bmatrix}} - \overset{\underline{X_\infty}}{\begin{bmatrix} \vdots \\ X_i - X_{i-1} \\ X_{i+1} - X_i \\ \vdots \end{bmatrix}} \tag{4.6}$$

The first and second columns of the left hand side respectively contain the expansion option values and perpetual payoffs before the ith capacity is installed. We denote them by $\underline{W_\infty}$ and $\underline{Z_\infty}$. The first and second columns of the right hand side respectively contain the expansion option values and the perpetual payoffs after the ith capacity is installed. We denote them by $\underline{U_\infty}$ and $\underline{Y_\infty}$. The third column of the right hand side contains the installation costs for each capacity expansion. It is denoted by $\underline{X_\infty}$. When a new capacity unit is installed, the option at the end of its life is abandoned but a new option is obtained. Therefore, $\underline{W_\infty}$ contains the option values at the end of their life while $\underline{U_\infty}$ contains the option values at the beginning of their life.

As discussed above, we also need to construct the discount functions to connect the expansion options at different thresholds. Since the expansion options are call options, we denote the discount function connecting the option values at P_i and P_{i+1} as $D_c(P_i, P_{i+1})$. Therefore,

$$V_i(P_i) = D_c(P_i, P_{i+1}) V_i(P_{i+1}) \tag{4.7}$$

Because we assume that P follows a geometric Brownian motion, the expression of $D_c(P_i, P_{i+1})$ is in the form of equation (2.21).

$$D_c(P_i, P_{i+1}) = \left(\frac{P_i}{P_{i+1}}\right)^c = \left(\frac{1}{1+\alpha}\right)^c \tag{4.8}$$

Therefore, the following relationship holds for the expansion option at any capacity levels.

$$V_i(P_i) = \left(\frac{1}{1+\alpha}\right)^c V_i(P_{i+1}) \tag{4.9}$$

We collect the discount factors in the matrix $\underline{D_\infty}$ and thus the matrix of the discount functions is given by

$$\overset{\underline{U_\infty}}{\begin{bmatrix} \vdots \\ \vdots \\ V_i(P_i) \\ V_{i+1}(P_{i+1}) \\ \vdots \\ \vdots \end{bmatrix}} = \overset{\underline{D_\infty}}{\begin{bmatrix} \vdots & \cdots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & D_c(P_i, P_{i+1}) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & D_c(P_{i+1}, P_{i+2}) & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \vdots & \vdots & \cdots & \vdots \end{bmatrix}} \overset{\underline{W_\infty}}{\begin{bmatrix} \vdots \\ \vdots \\ V_i(P_{i+1}) \\ V_{i+1}(P_{i+2}) \\ \vdots \\ \vdots \end{bmatrix}} \tag{4.10}$$

The matrix indicates that the system is unbounded at two sides without both the beginning and the ending.

In the last section we illustrate that the delta smooth pasting condition is equivalent to the dollar beta matching condition. Additionally, the beta for call options and put options under GBM are in the form of equations (A2.9) and (A2.10) in **Appendix II** and respectively denoted by c and p. We also demonstrate above that the beta of the payoffs equal to one. Therefore, both the perpetual payoffs before and after installing the ith capacity, namely $\underline{Z_\infty}$ and $\underline{Y_\infty}$, have unit beta. Thus, the dollar beta matching conditions according to the discount factor methodology discussed above should be

$$\begin{array}{ccccc} \underline{\beta_W W_\infty} & = & \underline{\beta_U U_\infty} & + & \left(\underline{\beta_Y Y_\infty} - \underline{\beta_Z Z_\infty}\right) \\ \begin{bmatrix} \vdots \\ cV_{i-1}(P_i) \\ cV_i(P_{i+1}) \\ \vdots \end{bmatrix} & = & \begin{bmatrix} \vdots \\ cV_i(P_i) \\ cV_{i+1}(P_{i+1}) \\ \vdots \end{bmatrix} & + & \begin{bmatrix} \vdots \\ P_i K_i - P_i K_{i-1} \\ P_{i+1}K_{i+1} - P_{i+1}K_i \\ \vdots \end{bmatrix} \end{array} \tag{4.11}$$

where $\underline{\beta_{W_\infty}}$, $\underline{\beta_{U_\infty}}$, $\underline{\beta_{Y_\infty}}$, and $\underline{\beta_{Z_\infty}}$ are respectively the vectors of beta for $\underline{W_\infty}$, $\underline{U_\infty}$, $\underline{Y_\infty}$, and $\underline{Z_\infty}$. So far we have established all three set of conditions required by the discount factor methodology to solve the capacity choice problem. However, the solution of the matrix system implicitly requires boundedness. This is because the existence of unique solution set requires the number of conditions equal to the number of unknown variables. However, all discount functions connect the expansion option values at the beginning of their life $V_i(P_i)$ with the option values at the end of their life $V_i(P_{i+1})$, which are determined in the conditions at the next thresholds. Therefore, we will always have one more unknown option value than the number of conditions included in the solution system. If we would like to solve the unknown option value, we should further include the conditions for the next threshold, introducing a new unknown option value which depends on the conditions at one more step. This will lead to an infinite loop that one more option value is always left unsolved.

For a closed matrix system with finite steps, such problem does not exist. There are two possible situations after the final capacity expansion is exercised. It could be either no any further expansion and exit opportunities or left with exit opportunities to a previous capacity state. The former case suggests that no new option value is created at the final expansion threshold and the latter case creates a new put option which is connected with a previous known option by a discount function. In other words, in the first case no new option is created and in the second case new discount factor relationship is created, both of which could equate the number of unknown option values and the number of discount functions. Therefore, the circulatory links between all expansion options ensure their values given certain thresholds values and all conditions could be determined simultaneously.

Then, how could we solve the discrete capacity choice problem in an infinite situation? As we shown below, such problem could be solved by self-similarity, which means all vectors in equation (4.6) are geometric sequences with the same scaling ratio.

We assume above that both the expansion thresholds and the optimal capacities have a geometric growth with the same growth rate α. Therefore, P_i and K_i would have a linear relationship. It is easy to derive that the payoff which equals the product of P_i and K_i would grow at a ratio of $(1+\alpha)^2$, namely $P_i K_i = (1+\alpha)^2 P_{i-1} K_{i-1}$. Similarly, the terms in vector $\underline{Z_\infty}$ also follow the equation $P_{i+1} K_i = (1+\alpha)^2 P_i K_{i-1}$.

We further assume that the option values would grow at the ratio of $(1+\alpha)^2$. Therefore,

$$V_i(P_{i+1}) = (1+\alpha)^2 V_{i-1}(P_i), \qquad V_{i+1}(P_{i+1}) = (1+\alpha)^2 V_i(P_i) \tag{4.12}$$

Since $P_iK_i - P_iK_{i-1}$, $V_i(P_i)$, and $V_{i-1}(P_i)$ are all scaled up at the ratio of $(1+\alpha)^2$, $X_i - X_{i-1}$ should also be scaled up at the same ratio according to equations (4.6). Thus, all terms in the value matching conditions are scaled up at the same ratio of $(1+\alpha)^2$, leading to a self-similarity system.

According to the discount factor relationship in equation (4.9) and the scaling relationship in equation (4.12), the following equation holds:

$$V_i(P_i) = (1+\alpha)^{-c} V_i(P_{i+1}) = (1+\alpha)^{2-c} V_{i-1}(P_i) \tag{4.13}$$

Combining equations (4.12) and (4.13), we could link $V_{i+1}(P_{i+1})$ with $V_{i-1}(P_i)$.

$$V_{i+1}(P_{i+1}) = (1+\alpha)^{-c} V_{i+1}(P_{i+2}) = (1+\alpha)^{4-c} V_{i-1}(P_i) \tag{4.14}$$

Therefore, the value of $V_{i+1}(P_{i+1})$, which should have been determined by the value of $V_{i+1}(P_{i+2})$ provided no scaling relationships, now could be connected with the value of $V_{i-1}(P_i)$, creating a new discount function. Thus, the value of $V_{i+1}(P_{i+1})$ could be determined by the boundary conditions of the ith and $i+1$th expansions and no longer need the boundary conditions of the $i+2$th expansion, which would create another unknown variable $V_{i+2}(P_{i+2})$. Hence, under the scaling assumption the number of unknown variables is equivalent to the number of boundary conditions, ensuring the existence of a unique solution.

Discount function (4.14) only connects the option values at two sequential expansion thresholds P_i and P_{i+1}. However, it could be extended to any size with arbitrary n sequential expansion thresholds. Due to the scaling relationship in equation (4.12), we could easily derive that $V_{i+n}(P_{i+n}) = (1+\alpha)^{2n} V_i(P_i)$. Then, equation (4.14) implies that

$$V_{i+n}(P_{i+n}) = (1+\alpha)^{2n} V_i(P_i) = (1+\alpha)^{2(n+1)-c} V_{i-1}(P_i) \tag{4.15}$$

Therefore, intercepting any size of the self-similarity system, equation (4.15) provides us the extra discount condition required for the solution to equal the number of unknown variables with the number of boundary conditions.

In this self-similar system, the number of the expansion steps should be infinite in both directions of $P \to 0$ and $P \to \infty$. Because P_i and K_i follow geometric sequences, both the threshold P_i and the capacity K_i could not become zero but infinitely approach zero. Nevertheless, in order to solve the infinite system, we have to assume the base points of the threshold and the capacity from where the investment begins. This is because the thresholds and capacities at any steps are evolved from the last step by scaling up. Therefore, we have to know at least one state of the threshold and capacity. Then all other thresholds P_i and capacities K_i before and after that point could be calculated by the scaling relationships. Given the values of all P_i and K_i, all terms in vector $\underline{Y_\infty}$ and $\underline{Z_\infty}$ could be solved.

We therefore assume the base points of P_i and K_i which are respectively denoted by P_0 and K_0. Then, all subsequent threshold P_i and the corresponding level of capacity

K_i could be solved according to the scaling relationships $P_i = (1+\alpha)^i P_0$ and $K_i = (1+\alpha)^i K_0$.

Since the scaling relationships exist between any two levels of expansion thresholds, the requirement for P_0 and K_0 to ensure unique existence of solution could be changed to that for the threshold and capacity P_i and K_i at any state. In other words, if we could observe any part of a self-similarly infinite system, the information for all the rest of the system could be inferred. Therefore, it does not matter how many thresholds are observed and used to solve the system. For simplicity and a clear illustration, we take an example of solving the system with two thresholds and equation (4.6) gives the value matching conditions of the two thresholds subsystem.

$$\begin{matrix} \underline{W} & = & \underline{U} & + & (\underline{Y}-\underline{Z}) & - & \underline{X} \\ \begin{bmatrix} V_{i-1}(P_i) \\ V_i(P_{i+1}) \end{bmatrix} & = & \begin{bmatrix} V_i(P_i) \\ V_{i+1}(P_{i+1}) \end{bmatrix} & + & \begin{bmatrix} \alpha(1+\alpha)^{i-1}P_iK_0 \\ \alpha(1+\alpha)^{i}P_{i+1}K_0 \end{bmatrix} & - & \begin{bmatrix} X_i - X_{i-1} \\ X_{i+1} - X_i \end{bmatrix} \end{matrix} \tag{4.16}$$

The two thresholds subsystem begins with $i-1$ units of capacity and ends with $i+1$ units of capacity. We then construct the discount functions. Equation (4.9) and equation (4.14) respectively provide the first and the second discount function in the subsystem. Therefore, the discount functions are given by

$$\begin{matrix} \underline{U} & = & \underline{D} & \underline{W} \\ \begin{bmatrix} V_i(P_i) \\ V_{i+1}(P_{i+1}) \end{bmatrix} & = & \begin{bmatrix} 0 & (1+\alpha)^{-c} \\ (1+\alpha)^{4-c} & 0 \end{bmatrix} & \begin{bmatrix} V_{i-1}(P_i) \\ V_i(P_{i+1}) \end{bmatrix} \end{matrix} \tag{4.17}$$

Based on the discount functions (4.13), we could obtain the restriction on c. Since a capacity expansion requires the abandonment of the flexibility to install the capacity in the future, the option value $V_i(P_i)$ should always be smaller than the option value $V_{i-1}(P_i)$. Therefore, $V_i(P_i) < V_{i-1}(P_i)$ always holds. Thus, $(1+\alpha)^{2-c} < 1$ and $c > 2$ is required. According to the function of c in equation (3.10),

$$c = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} > 2$$

Simplifying the inequation above, we could obtain

$$\delta - (r-\delta) = 2\delta - r > \sigma^2 \tag{4.18}$$

Inequation (4.18) implies that both the percentage drift $r-\delta$ and the percentage volatility σ of the underlying stochastic process should be small to make $c > 2$ hold.

With the value matching conditions in equation set (4.16), we could easily obtain the dollar beta matching conditions, as mentioned above, by differentiating the items line by line in P and then multiplying by P.

$$\begin{matrix} \underline{\beta_W W} & = & \underline{\beta_U U} & + & (\underline{\beta_Y Y} - \underline{\beta_Z Z}) \\ \begin{bmatrix} cV_{i-1}(P_i) \\ cV_i(P_{i+1}) \end{bmatrix} & = & \begin{bmatrix} cV_i(P_i) \\ cV_{i+1}(P_{i+1}) \end{bmatrix} & + & \begin{bmatrix} \alpha(1+\alpha)^{i-1}P_iK_0 \\ \alpha(1+\alpha)^{i}P_{i+1}K_0 \end{bmatrix} \end{matrix} \tag{4.19}$$

The solutions of three unknown vectors $\underline{W}$, $\underline{U}$, and $\underline{X}$ are given by the following matrix equations.

$$\underline{W} = \left(\underline{\beta_W} - \underline{\beta_U D}\right)^{-1}\left(\underline{\beta_Y Y} - \underline{\beta_Z Z}\right) \quad (4.20)$$

$$\underline{U} = \underline{DW} = \underline{D}\left(\underline{\beta_W} - \underline{\beta_U D}\right)^{-1}\left(\underline{\beta_Y Y} - \underline{\beta_Z Z}\right) \quad (4.21)$$

$$\underline{X} = \underline{U} + \left(\underline{Y} - \underline{Z}\right) - \underline{W} = \left(\underline{D} - \underline{I}\right)\left(\underline{\beta_W} - \underline{\beta_U D}\right)^{-1}\left(\underline{\beta_Y Y} - \underline{\beta_Z Z}\right) + \left(\underline{Y} - \underline{Z}\right) \quad (4.22)$$

Therefore, all expansion option values and installation costs could be expressed in terms of i, c, α, P_i and K_i. However, it is worth noting that in our discrete model, the additions to capacity are discrete while P follows a continuous stochastic process. Therefore, with existing i units of capacity the additions of capacity would not occur until P reaches the next threshold P_{i+1}. Since we assume the installations of the additional capacities are instantaneous, the dynamic perpetual revenue P could be regarded as constant when the capacity expansion happens. Therefore, the variation of P and K would not happen simultaneously. Thus, the growth of the payoff in a very short time interval could be expressed as:

$$\Delta(PK) = (P_i K_i)_t - (P_i K_i)_{t-\Delta t} = \Delta P K_i + \Delta K P_i \quad (4.23)$$

where either ΔP or ΔK should be zero when the other varies. Therefore, the relationship between P and K would be in a form of staircase. In a continuous model, however, P and K could change continuously and simultaneously. Therefore, the relationship between P and K should be a smooth curve. In the next subsection we will examine the situation where the discrete model converges to the continuous case.

4.2 The Convergence of the Discrete Model to the Continuous Case

In the discrete model of capacity choice shown above, we illustrate that the self-similar infinite system could be solved given values of i, c, α, K_0 and P_0. K_0 and P_0 describe the state at the initial state, and i, c and α describe the rule of scaling up and of discounting. Then what is the investment rule when the additions to capacity are continuous instead of discrete? In the continuous case α will become infinitesimal. Therefore, α is expected to disappear and only c, P_i, and K_i are left in the investment rule of continuous model, which is also the results from previous papers (e.g. Pindyck, 1988; Dixit and Pindyck, 1994) on continuous investment. We now derive the analytical solutions of $\underline{W}$, $\underline{U}$ and $\underline{X}$ when α becomes infinitesimal and P and K approximately vary continuously. As mentioned above, the following discount factor relationship holds between the expansion option values at two sequential thresholds:

$$V_i(P_{i+1}) = (1+\alpha)^c V_i(P_i) \quad (4.24)$$

To substitute the discount factor equation back into the beta matching condition, we could solve out the analytical expression of the option value $V_i(P_i)$:

$$V_{i-1}(P_i) = \frac{\alpha(1+\alpha)^{-1} P_i K_i}{c(1-(1+\alpha)^{2-c})} \quad (4.25)$$

$$V_i(P_i) = (1+\alpha)^{-c} V_i(P_{i+1}) = (1+\alpha)^{2-c} V_{i-1}(P_i) = \frac{\alpha(1+\alpha)^{1-c} P_i K_i}{c(1-(1+\alpha)^{2-c})} \quad (4.26)$$

As α becomes infinitesimal,

$$\begin{aligned}\lim_{\alpha\to 0} V_{i-1}(P_i) &= \lim_{\alpha\to 0}\frac{\alpha(1+\alpha)^{-1}P_iK_i}{c[1-(1+\alpha)^{2-c}]}\\ &= \lim_{\alpha\to 0}\frac{\alpha(1+\alpha)^{c-3}P_iK_i}{c\left[(c-2)\alpha+\frac{(c-2)(c-3)}{2}\alpha^2+\cdots\right]}\end{aligned} \tag{4.27}$$

According to l'Hôpital's rule, the equation above equals to

$$\lim_{\alpha\to 0} V_{i-1}(P_i) = \lim_{\alpha\to 0}\frac{P_iK_i}{c(c-2)} \tag{4.28}$$

Correspondingly,

$$\lim_{\alpha\to 0} V_i(P_i) = \lim_{\alpha\to 0}\frac{\alpha(1+\alpha)^{1-c}P_iK_i}{c(1-(1+\alpha)^{2-c})} = \lim_{\alpha\to 0}\frac{P_iK_i}{c(c-2)} = \lim_{\alpha\to 0} V_{i-1}(P_i) \tag{4.29}$$

This expression confirms the requirement of $c > 2$ to ensure the option values are positive. In this limit the option values will coalesce and the payoffs and costs will become infinitesimally small, but an uncountable number arise. That is to say that the rate of cost and benefit accrual per unit K_i remains bounded.

$$\begin{aligned}k_i = \lim_{\alpha\to 0}\frac{\Delta X_i}{\Delta K_i} = \lim_{\alpha\to 0}\frac{X_i - X_{i-1}}{K_i - K_{i-1}} &= \lim_{\alpha\to 0}\frac{V_i(P_i) - V_{i-1}(P_i) + \alpha(1+\alpha)^{-1}P_iK_i}{K_i - \frac{1}{1+\alpha}K_i}\\ &= \lim_{\alpha\to 0}\frac{[1-(1+\alpha)^{c-2}]V_i(P_i) + \alpha(1+\alpha)^{-1}P_iK_i}{\frac{\alpha}{1+\alpha}K_i}\\ &= \frac{(c-1)P_i}{c}\end{aligned} \tag{4.30}$$

This is just the installation cost per unit of capacity k_i for the ith expansion in the adjusted model of Dxit and Pindyck (1994) according to equation (3.18). Therefore, the investment rule converges when the discrete model converges to its continuous counterpart. In the next subsection we implement several numerical experiments to examine the validity of our inferences.

Given the analytical inference above, we implement the numerical experiments in the next subsection to examine their validity.

4.3 Numerical Experiments and Results

In this section we implement numerical experiments based on discount factor methodology to examine whether the discrete model are consistent with the continuous counterpart where the scaling ratio α approaches to zero.

In the last subsection we indicate that a pair of base points of threshold and capacity should be provided to solve all other investment rules and they are respectively denoted by P_0 and K_0. If these two models could perfectly converge, the expansion threshold P_i and the capacity level K_i should also follow equation (3.18) when α approaches zero. Therefore, we assume $P_0 = \frac{ckK_0}{c-1}$. Our purpose here is not to fix the value assignments of P_0 and K_0 but to demonstrate the consistence of the discrete model to the continuous model and the validity of the discrete model. In the real world a firm facing a non-optimal status of P_0 and K_0 usually could not continuously adjust the capital stock to the optimal point solved from a continuous model. In such case, we could utilize the discrete model to simulate the lumpy investment condition and correspondingly establish the investment policy under the discrete reality.

Since we only intercept a part of the infinite self-similarity system starting from P_0 and K_0, the installation costs for each capacity expansion $X_i - X_{i-1}$ calculated in the matrix solution also start from $X_1 - X_0$. Therefore, the matrix solution could only provide us the capital accumulated from X_0 to X_i which is equal to $X_i - X_0$, namely

$$\sum[(X_i - X_{i-1}) + (X_{i-1} - X_{i-2}) + \cdots + (X_1 - X_0)] = X_i - X_0 \tag{4.31}$$

In order to solve the total cumulative capital X_i, we have to solve X_0 first. Thanks to the infinite scaling system in which $X_i - X_{i-1}$ is scaled up at the ratio of $(1+\alpha)^2$, X_0 should be the sum of an infinite geometric progression with the common ratio $\frac{1}{(1+\alpha)^2}$ and the first term $\frac{1}{(1+\alpha)^2}(X_1 - X_0)$. Therefore,

$$\begin{aligned} X_0 &= \sum_{n=\infty}(X_1 - X_0)\left[\frac{1}{(1+\alpha)^2} + \frac{1}{(1+\alpha)^4} + \frac{1}{(1+\alpha)^6} + \cdots + \frac{1}{(1+\alpha)^{2n}}\right] \\ &= \frac{1}{\alpha(\alpha+2)}(X_1 - X_0) \end{aligned} \tag{4.32}$$

According to equation (4.30),

$$(X_1 - X_0) = k_1(K_1 - K_0) = \frac{(c-1)P_0}{c}(K_1 - K_0) = kK_0(K_1 - K_0) \tag{4.33}$$

Thus, the expression of X_0 is given by

$$X_0 = \frac{1}{\alpha(\alpha+2)}kK_0(K_1 - K_0) = \frac{1}{\alpha+2}kK_0{}^2 \tag{4.34}$$

When the discrete model converges to the continuous case with $\alpha \to 0$,

$$\lim_{\alpha\to 0} X_0 = \frac{1}{2}kK_0{}^2 \tag{4.35}$$

This is just the expression of the cumulative capital in equation (3.21) for the adjusted model of Dxit and Pindyck (1994), indicating the consistence of X_0 given the same P_0 and K_0.

As mentioned in section 3.1, the values of r, δ, and σ are monotonically reflected by the value of c. Therefore, we could examine the parameter effects of r, δ, and σ by examining the effect of c. Then, the gradual process of the discrete model converging to the continuous model could be reflected by changing the value of α.

Therefore, in the numerical experiments we undertake the sensitive analysis by assuming different values for c and α to examine their effects on the investment rules. In order to compare with the adjusted model of Dxit and Pindyck (1994), we assume $c = 3.70$ with $r = 0.05$, $\delta = 0.05$, $\sigma = 0.1$ and $c = 2.09$ with $r = 0.15$, $\delta = 0.1$, $\sigma = 0.2$. The value of the scaling ratio α is assumed to be 0.1, 0.01, and 0.0001, gradually approaching the continuous case.

Table 4.1 and Table 4.2 report values of P_i, K_i, X_i, $V_{i-1}(P_i)$, $V_{i-1}(P_i)$, $\Delta X_i/\Delta X_{i-1}$, and $\Delta X_i/\Delta K_i$ where $\Delta X_i/\Delta X_{i-1} = (X_i - X_{i-1})/(X_{i-1} - X_{i-2})$ is the growth rate of the installation costs for each expansion and $\Delta X_i/\Delta K_i = (X_i - X_{i-1})/(K_i - K_{i-1})$ is the installation cost per unit of capacity for the ith expansion.

Table 4.1 and Table 4.2 imply that $\Delta X_i/\Delta X_{i-1}$ keeps constant across different capacity levels and is equal to $(1+\alpha)^2$. This is consistent with the scaling structure that all terms in the value matching conditions are scaled up at the ratio of $(1+\alpha)^2$.

Then, we examine the effect of c. As indicated above, $c > 2$ is required to keep the inequality between $V_{i-1}(P_i)$ and $V_i(P_i)$. When inputting a value of c which is smaller than 2, the expansion option values all become negative, which is not possible in our capacity choice problem. The results in Table (4.1) and (4.2) indicate that larger values of c would lead to lower expansion thresholds for given levels of capital. Additionally, the expansion option values also become smaller as c increases. This is consistent with the inference above for the continuous model that higher required return r or higher uncertainty σ would undermine the investment incentives while higher yield δ would accelerate the investment process, providing c monotonically increases in δ while monotonically decreases in r and σ. Additionally, higher values of c will reduce the threshold for the installation of the first capacity because of $P_0 = \frac{ckK_0}{c-1}$, the thresholds for the following capacities would be correspondingly lower.

As to the effect of α on the investment rules, it could be observed that the expansion thresholds P_i for a given level of K_i and X_i approach to the results of the adjusted model of Dxit and Pindyck (1994) as α become smaller. When $\alpha = 0.0001$, the numerical result are nearly the same as that of the continuous counterpart in Table 3.1. Additionally, as α approaches zero the values of $\Delta X_i/\Delta K_i$ converge to kK_i, which is the expression for the installation cost per unit of capacity assumed in the adjusted model of Dxit and Pindyck (1994). The results are also consistent with equation (3.18) that $\frac{\Delta X_i}{\Delta K_i} = kK_i = \frac{(c-1)}{c} P_i$ as α approaches zero. Therefore, our discrete model without switching options solved by the discount factor methodology is perfectly consistent with the continuous model solved by dynamic programming or continent claims approach, supporting the validity of the discount factor methodology.

The convergence of the discrete model to the continuous model is intuitively illustrated in Figure 4.1 and 4.2 which plot the cumulative capital X_t against the expansion thresholds P_t in the adjusted continuous model of Dxit and Pindyck (1994), and the cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^s in the discrete model for $\alpha = 0.1$, $\alpha = 0.01$, and $\alpha = 0.0001$. Both figures show that the discrete model converges to the continuous model when α decreases. In fact, when $\alpha = 0.01$ and $\alpha = 0.0001$ we could hardly tell the difference between the continuous model and the discrete model by naked eyes.

Figure 4.1 and Figure 4.2 also show that the thresholds P_i for given levels of cumulative capital X_i are always larger when α is smaller. Correspondingly, the optimal value of X_i given P_i is smaller as α approaches zero. The results are economically sensible that under discrete assumption a firm could not adjust the capital stock to response the rising of P_i as precisely as it is able to change capital continuously and infinitesimally. The limitation on the minimum capital amount that could be installed in one expansion makes a firm overinvest under the discrete assumption compared with the optimal investment policies under the continuous assumption.

In order to scrutinize the convergence of the discrete model to the continuous case, we plot both the smooth curve that only connects the points at thresholds and the real route of the capital increases following the price increases in Figure 4.3, 4.4, and 4.5. Since the capacity increases only occur at discrete expansion thresholds while the price continuously vary along time, the real route of the capital increases are in a form of discrete staircase. Such staircase process well illustrate that our previous statement that the perpetual revenue P and the amount of capacity K could not change simultaneously. We put the curves for $\alpha = 0.01$ and $\alpha = 0.1$ in one figure for comparison. The smooth curve and discrete staircase for $\alpha = 0.01$ contain 100 thresholds ($i = 100$) and are presented by the red and blue lines while the smooth curve and discrete staircase for $\alpha = 0.1$ contain 10 thresholds ($i = 10$) and are presented by the green and grape lines.

Figure 4.3 shows that the discrete staircase becomes smaller and more closed to the smooth curve as α becomes smaller. Therefore, the cumulative capital after expanding 10 times at ratio of $\alpha = 0.1$ is near to the cumulative capital after expanding 100 times at ratio of $\alpha = 0.01$. Figure 4.4 plots the smooth curve and discrete staircase for $\alpha = 0.0001$ with 10000 thresholds. However, we could hardly distinguish by naked eyes between the smooth curve shown by the red line and the discrete staircase shown by the blue line, implying that the discrete staircase become quite closed to the smooth curve. Figure 4.5 “zooms in” the discrete staircase and focuses on 20 thresholds which only span from 1.37 to 1.3731, but we still could not differentiate the blue line from the red line. Therefore, it is expected that when α becomes infinitesimal, the intervals between sequential thresholds P_i and P_{i+1} and between sequential cumulative capitals X_i and X_{i+1} will also become infinitesimal. Thus, the expansion thresholds and corresponding optimal capitals will become continuous, and the staircase will approach without limit to the smooth curve, which meanwhile coincides with the adjusted continuous model of Dxit and Pindyck (1994) as shown in Figure 4.1 and 4.2.

Table 4.1 The Convergence of the Discrete Model to the Continuous Case
($r = 0.05$, $\delta = 0.05$, $\sigma = 0.1$, $c = 3.70$, $P_0 = 1.37$, $K_0 = 1$)

α	i	P_i	K_i	X_i	$\Delta X_i/\Delta X_{i-1}$	$\Delta X_i/\Delta K_i$	$V_{i-1}(P_i)$	$V_i(P_i)$
0.1	0	1.37	1.00	0.48	-	-	0.2724	0.2316
	8	2.94	2.00	2.36	1.2100	2.14	1.0343	0.8796
	12	4.30	3.00	5.12	1.2100	3.14	2.2171	1.8855
	15	5.72	4.00	9.10	1.2100	4.18	3.9278	3.3402
	17	6.93	5.00	13.34	1.2100	5.05	5.7506	4.8904
	19	8.38	6.00	19.55	1.2100	6.12	8.4195	7.1601
0.01	0	1.37	1.00	0.50	-	-	0.2230	0.2193
	70	2.75	2.00	2.02	1.0201	2.01	0.8804	0.8657
	111	4.14	3.00	4.57	1.0201	3.02	1.9909	1.9575
	140	5.52	4.00	8.15	1.0201	4.03	3.5455	3.4861
	162	6.87	5.00	12.62	1.0201	5.01	5.4932	5.4010
	180	8.22	6.00	18.06	1.0201	6.00	7.8594	7.7276
0.0001	0	1.37	1.00	0.50	-	-	0.2179	0.2178
	6932	2.74	2.00	2.00	1.0002	2.00	0.8715	0.8714
	10987	4.11	3.00	4.50	1.0002	3.00	1.9610	1.9606
	13864	5.48	4.00	8.00	1.0002	4.00	3.4862	3.4856
	16096	6.85	5.00	12.50	1.0002	5.00	5.4477	5.4468
	17919	8.22	6.00	18.00	1.0002	6.00	7.8442	7.8429

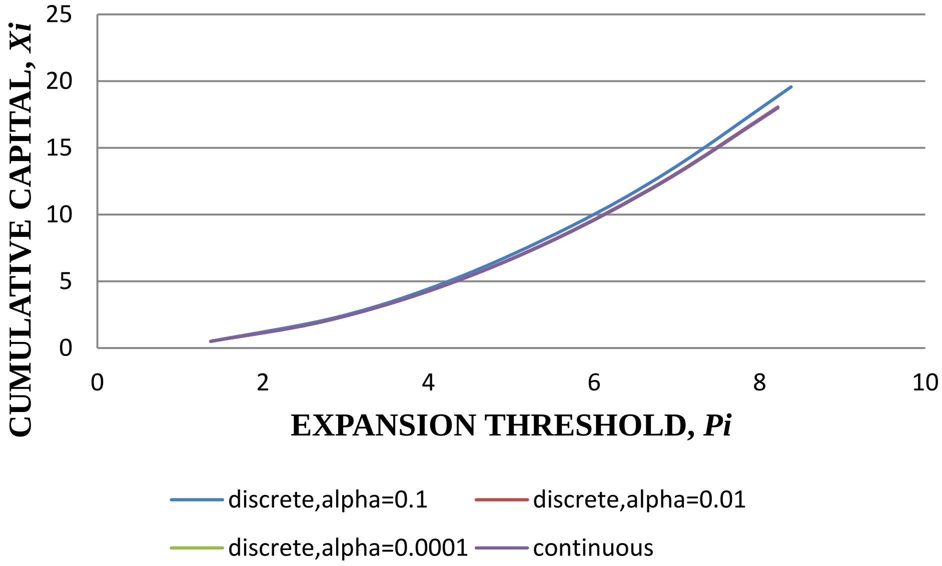
This table reports the effects of the discrete assumption and parameter values on the optimal investment rules. The data include the values of expansion threshold P_i, the amount of capacity K_i, the cumulative capital X_i, the growth rate of the installation costs for each expansion $\Delta X_i/\Delta X_{i-1}$, the option value at the end of its life $V_{i-1}(P_i)$, and the option value at the beginning of its life $V_i(P_i)$. The effects of r, δ, and σ are reflected in the value of c and the intervals of the discrete steps are reflected in the value of α. The values of r, δ, σ, and c are the same as those assumed in the continuous model in section 3.1 for comparison. The results are solved by discount factor methodology shown in equation (4.20), (4.21), and (4.22).

Table 4.2 The Convergence of the Discrete Model to the Continuous Case
($r = 0.15$, $\delta = 0.1$, $\sigma = 0.2$, $c = 2.09$, $P_0 = 1.92$, $K_0 = 1$)

α	i	P_i	K_i	X_i	$\Delta X_i/\Delta X_{i-1}$	$\Delta X_i/\Delta K_i$	$V_{i-1}(P_i)$	$V_i(P_i)$
0.1	0	1.92	1.00	0.48	-	-	11.82	11.71
	8	4.11	2.00	2.36	1.2100	2.14	44.87	44.49
	12	6.02	3.00	5.12	1.2100	3.14	96.18	95.36
	15	8.01	4.00	9.10	1.2100	4.18	170.39	168.93
	17	9.69	5.00	13.34	1.2100	5.05	249.47	247.34
	19	11.73	6.00	19.55	1.2100	6.12	365.24	362.12
0.01	0	1.92	1.00	0.50	-	-	10.35	10.34
	70	3.85	2.00	2.02	1.0201	2.01	40.87	40.83
	111	5.79	3.00	4.57	1.0201	3.02	92.41	92.33
	140	7.72	4.00	8.15	1.0201	4.03	164.57	164.42
	162	9.61	5.00	12.62	1.0201	5.01	254.97	254.74
	180	11.50	6.00	18.06	1.0201	6.00	364.81	364.48
0.0001	0	1.92	1.00	0.50	-	-	10.20	10.20
	6932	3.83	2.00	2.00	1.0002	2.00	40.77	40.77
	10987	5.75	3.00	4.50	1.0002	3.00	91.74	91.74
	13864	7.67	4.00	8.00	1.0002	4.00	163.10	163.10
	16096	9.59	5.00	12.50	1.0002	5.00	254.87	254.87
	17919	11.50	6.00	18.00	1.0002	6.00	366.99	366.99

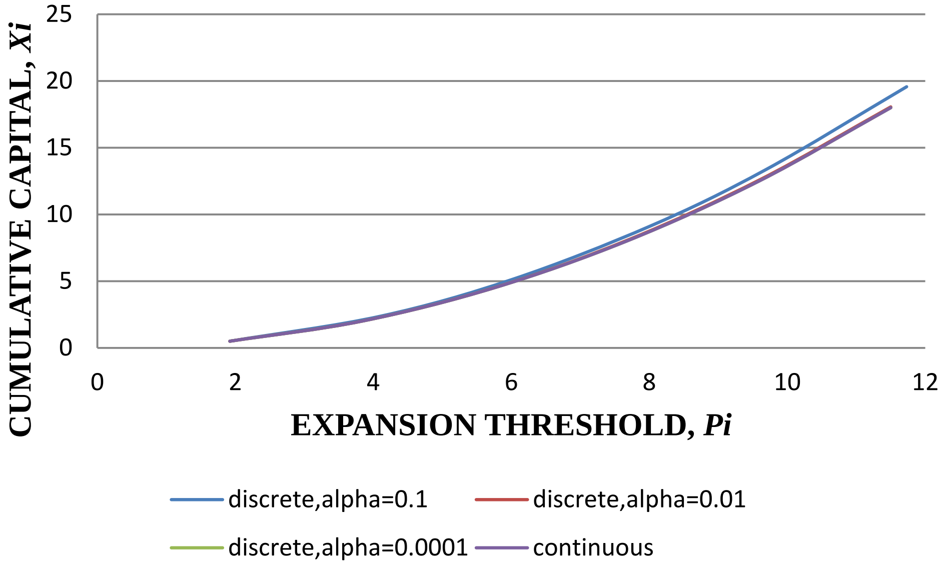
This table reports the effects of the discrete assumption and parameter values on the optimal investment rules. The data include the values of expansion threshold P_i, the amount of capacity K_i, the cumulative capital X_i, the growth rate of the installation costs for each expansion $\Delta X_i/\Delta X_{i-1}$, the option value at the end of its life $V_{i-1}(P_i)$, and the option value at the beginning of its life $V_i(P_i)$. The effects of r, δ, and σ are reflected in the value of c and the intervals of the discrete steps are reflected in the value of α. The values of r, δ, σ, and c are the same as those assumed in the continuous model in section 3.1 for comparison. The results are solved by discount factor methodology shown in equation (4.20), (4.21), and (4.22).

Figure 4.1 Expansion Thresholds and Cumulative Capitals
$(r = 0.05, \delta = 0.05, \sigma = 0.1, c = 3.70, P_0 = 1.37, \ K_0 = 1)$



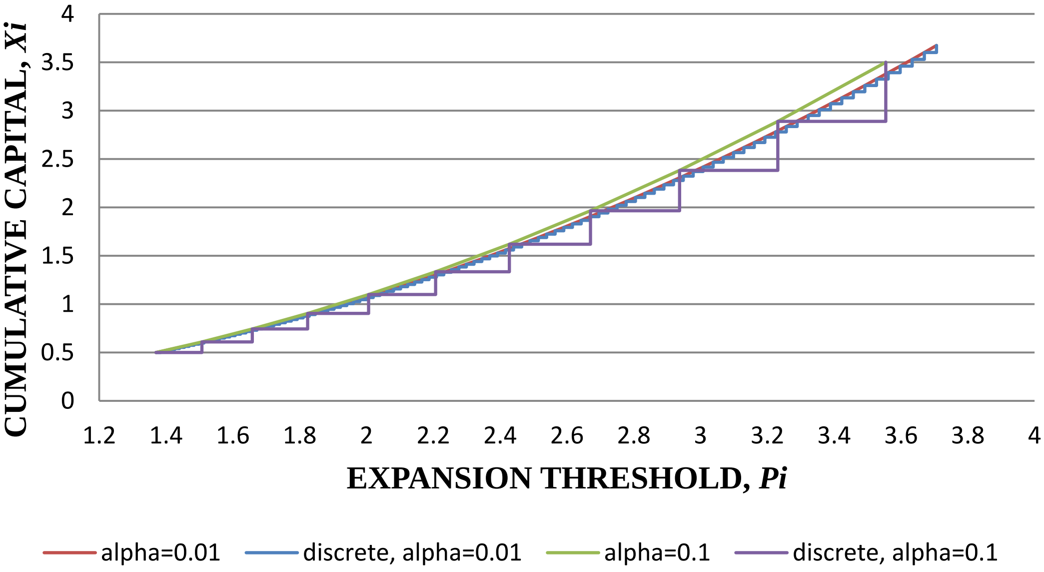
This figure respectively plots the cumulative capital X_t against the expansion thresholds P_t in the adjusted model of Dxit and Pindyck (1994), and the cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^S in the discrete model without switching options for $\alpha = 0.1$, $\alpha = 0.01$, and $\alpha = 0.0001$. The data for the continuous model is from Table 3.1 which is solved by equation (3.22) and the data for the discrete model is from Table 4.1 which is solved by discount factor methodology shown in equation (4.22).

Figure 4.2 Expansion Thresholds and Cumulative Capitals
$(r = 0.15, \delta = 0.1, \sigma = 0.2, c = 2.09, P_0 = 1.92, \ K_0 = 1)$



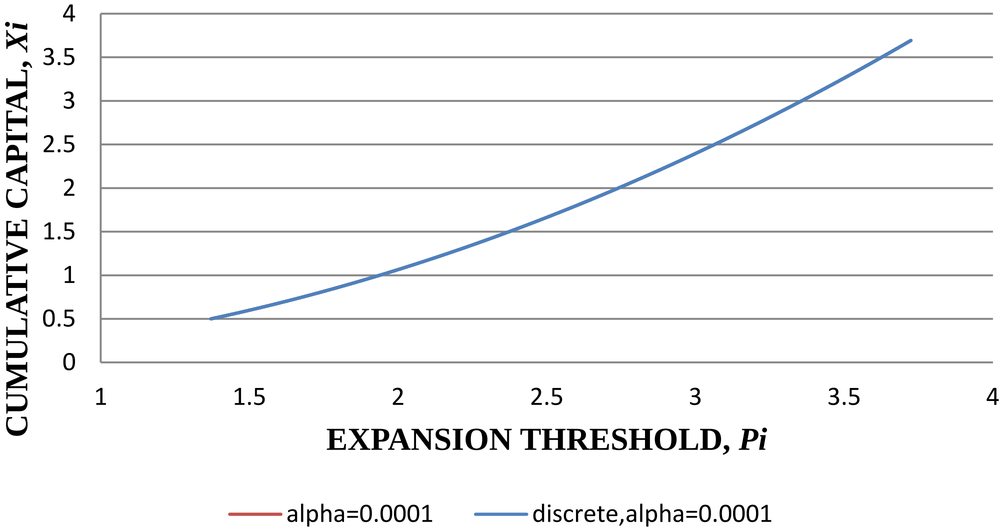
This figure respectively plots the cumulative capital X_t against the expansion thresholds P_t in the adjusted model of Dxit and Pindyck (1994), and the cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^S in the discrete model without switching options for $\alpha = 0.1$, $\alpha = 0.01$, and $\alpha = 0.0001$. The data for the continuous model is from Table 3.1 which is solved by equation (3.22) the data for the discrete model is from Table 4.1 which is solved by discount factor methodology shown in equation (4.22).

Figure 4.3 Expansion Thresholds and Cumulative Capitals
($r = 0.05$, $\delta = 0.05$, $\sigma = 0.1$, $c = 3.70$, $P_0 = 1.37$, $K_0 = 1$)



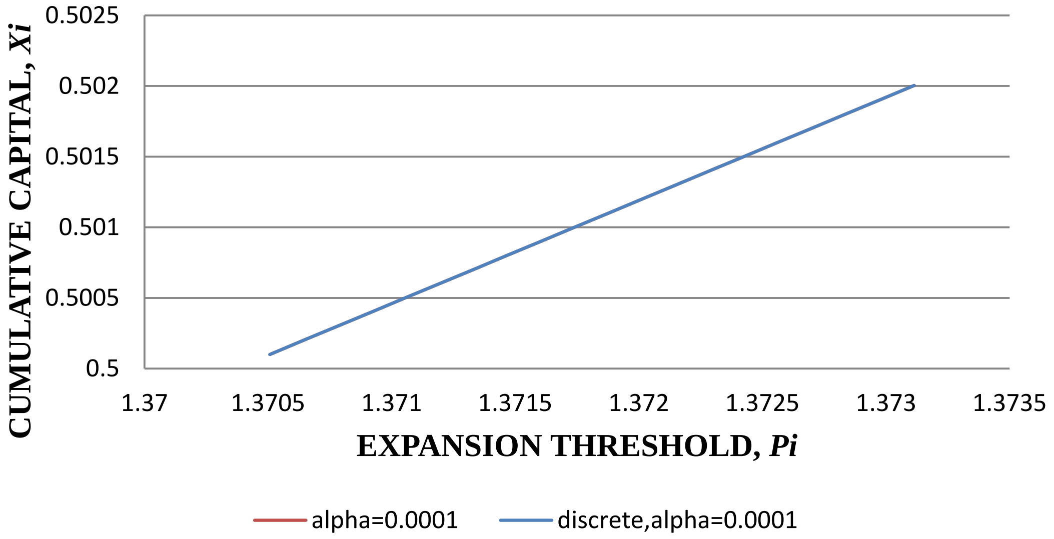
This figure plots the smooth curves and the discrete staircases of cumulative capital X_i against the expansion thresholds P_i for a discrete model with $\alpha = 0.01$ and $\alpha = 0.1$. The smooth curve and discrete staircase for $\alpha = 0.01$ contain 100 thresholds and are presented by the red and blue lines while the smooth curve and discrete staircase for $\alpha = 0.1$ contain 10 thresholds and are presented by the green and grape lines. The results are solved by discount factor methodology shown in equations (4.20), (4.21), and (4.22).

Figure 4.4 Expansion Thresholds and Cumulative Capitals
($r = 0.05$, $\delta = 0.05$, $\sigma = 0.1$, $c = 3.70$, $P_0 = 1.37$, $K_0 = 1$)



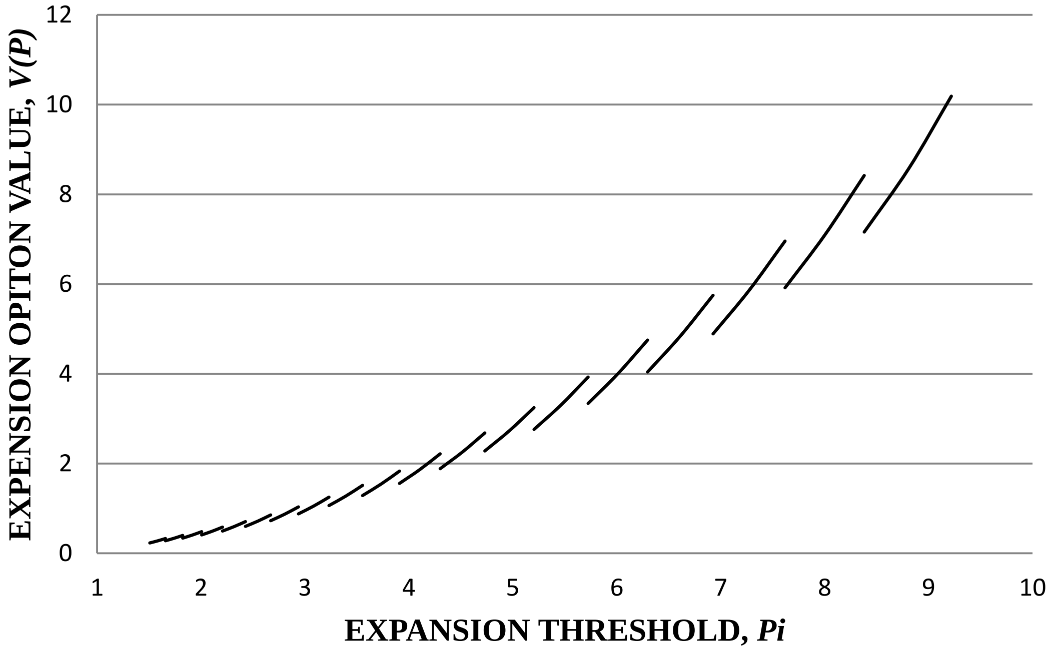
This figure plots the smooth curves and the discrete staircases of cumulative capital X_i against the expansion thresholds P_i for a discrete model with $\alpha = 0.0001$. The red line presents the smooth curves of X_i against P_i; the blue line presents the discrete staircases of X_i against P_i. The results are solved by discount factor methodology shown in equations (4.20), (4.21), and (4.22).

Figure 4.5 Expansion Thresholds and Cumulative Capitals
($r = 0.05$, $\delta = 0.05$, $\sigma = 0.1$, $c = 3.70$, $P_0 = 1.37$, $K_0 = 1$)



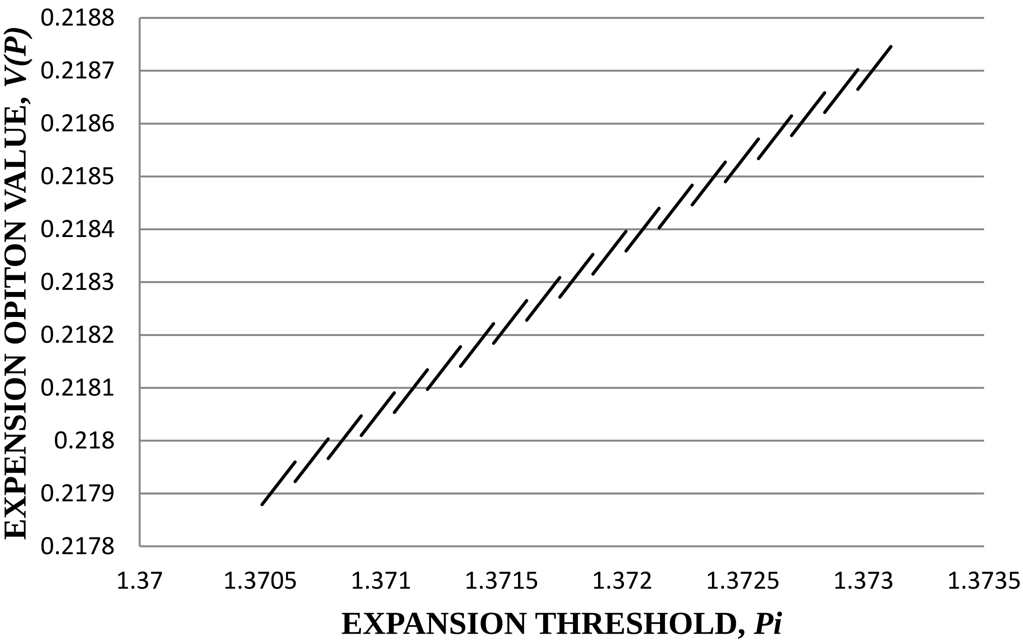
This figure plots the smooth curves and the discrete staircases of cumulative capital X_i against the expansion thresholds P_i for a discrete model with $\alpha = 0.0001$. The red line presents the smooth curves of X_i against P_i; the blue line presents the discrete staircases of X_i against P_i. The results are solved by discount factor methodology shown in equations (4.20), (4.21), and (4.22).

Figure 4.6 Expansion Thresholds and Expansion Options
($r = 0.05$, $\delta = 0.05$, $\sigma = 0.1$, $c = 3.70$, $P_0 = 1.37$, $K_0 = 1$, $\alpha = 0.1$)



This figure plots the values of the expansion options $V_{i-1}(P_i)$ and $V_i(P_i)$ against the expansion thresholds P_i. Each line segment presents the value of an expansion option from the beginning of its life to the end of its life. The results are solved by discount factor methodology shown in equations (4.20), (4.21), and (4.22).

Figure 4.7 Expansion Thresholds and Expansion Options
($r = 0.05$, $\delta = 0.05$, $\sigma = 0.1$, $c = 3.70$, $P_0 = 1.37$, $K_0 = 1$, $\alpha = 0.0001$)



This figure plots the values of the expansion options $V_{i-1}(P_i)$ and $V_i(P_i)$ against the expansion thresholds P_i. Each line segment presents the value of an expansion option from the beginning of its life to the end of its life. The results are solved by discount factor methodology shown in equations (4.20), (4.21), and (4.22).

We next examine the effect of the convergence to the continuous case on the values of expansion options. Table 4.1 and 4.2 show that the option value before expansion $V_{i-1}(P_i)$ is larger than the option value after expansion $V_i(P_i)$, verifying that each expansion would abandon the value of waiting to invest the additional capacity in the future. However, both $V_{i-1}(P_i)$ and $V_i(P_i)$ increase as P_i increases, implying the effect of P_i dominates the effect of K_i in the value of options. Table 4.1 and 4.2 also reveal that the values of $V_i(P_i)$ converge to the values of $V_{i-1}(P_i)$ as α decreases, consistent with our analytical inference in equation (4.29) that $V_i(P_i) = V_{i-1}(P_i)$ when α become infinitesimal.

As we discussed in section II, the option value at any dynamic point P before exercising at the threshold could be expressed as a fraction of the final option value at the threshold while the discount factor is the dynamic fraction depending on the value of P. Therefore, the option value at any dynamic point P before exercising under GBM could be expressed as:

$$V_i(P) = D_c(P, P_{i+1})V_i(P_{i+1}) = \left(\frac{P}{P_{i+1}}\right)^c V_i(P_{i+1}) \qquad (4.36)$$

Therefore, we could solve the expansion option value $V_i(P)$ at any point of P by discounting the option value at the end of its life $V_i(P_{i+1})$.

Figure 3.6 and Figure 3.7 respectively plot the values of the expansion options against the thresholds under $\alpha = 0.1$ and $\alpha = 0.0001$. In Figure 3.6 and Figure 3.7, we could observe many line segments, each of which presents the value of an expansion option from the beginning of its life to the end of its life. In the discussion above we regard the expansion options at different capacity levels as different options.

When an expansion threshold is reached, an additional capacity is installed and the existing expansion option is abandoned as it reaches the end of its life. Meanwhile another expansion option is created at the beginning of its life. When the next threshold is reached, the previously created option reaches the end of its life and is replaced by another option, etc.

Since the expansion of an additional capacity is always accompanied by the abandonment of some option values to invest in the future, $V_{i-1}(P_i)$ should always be larger than $V_i(P_i)$. This is consistent with Figure 3.6 and Figure 3.7 that the option values gap down at the expansion thresholds. Nevertheless, when α become smaller, the expected life of each option becomes shorter and the value gap between the options for two sequential capacity levels also becomes smaller, comparing Figure 3.6 and Figure 3.7. As α finally approaches zero, each line segment in figures is expected to become a dot and be continuously connected. This is consistent with our analysis above in equation (4.29) that $\lim_{\alpha\to 0} V_i(P_i) = \lim_{\alpha\to 0} V_{i-1}(P_i)$ when the discrete model converges to the continuous case.

To sum up, all the numerical results above support the consistency between the discrete model without switching options solved by the discount factor methodology and the adjusted continuous model of Dxit and Pindyck (1994). Thus, for continuous irreversible capacity choice problems, we could establish the corresponding discrete capacity choice model and solve it by the discount factor methodology, which provide us a new and deeper angle to look into the capacity choice problem. In the next section we would further examine the discrete model with switching options which is more complicated and its convergence to the continuous case.

V. Discrete Capacity Choice Problem with Switching Options

5.1 Model

In the last section we construct a scaling discrete model of capacity choice without switching options using the discount factor methodology and it could perfectly converge to the continuous model modified from Dxit and Pindyck (1994) in section 4.1. Pindyck (1988) consider a situation where firms not only hold infinite expansion options to add capacities in the future but also have the switching on/off options regarding to the existing capacities. That means firms could choose to temporarily turn off the existing capacities if the dynamic state variable, usually the price or the perpetual revenue, falls below certain switching thresholds. The switched off capacities could be switched on again if the state variable rebounds to certain switching on thresholds. The expansion thresholds are completely irreversible while the switching thresholds are completely reversible.
In this section we further establish a discrete capacity choice model with switching options to examine whether the discount factor methodology is consistent with the continuous solution in this situation.

Solutions for switching decisions

The paralleled continuous model in section 3.2 assumes a common switching threshold, which means the switching on and switching off thresholds are the same for a given capacity level. This is solvable in the continuous model since the specific expressions of the switching on/off options are solved first to establish the common switching boundary condition. However, since the discount factor methodology does require the expressions of the switching options, we need to respectively construct the boundary conditions for the switching on decision and the switching off decision, so that we have adequate conditions to solve the option expressions and the switching thresholds simultaneously. Therefore, in the discrete model we assume separate switching on and off thresholds and construct a two thresholds system for the switching decisions. The common switching thresholds are then simulated by arbitrarily making the switching on and switching off threshold converges.

We respectively denote the switching on/off thresholds for the ith additional capacity by P_{i+} and P_{i-}. When a unit of capacity is installed, a switching off option is simultaneously created. When the switching off threshold is reached, the switching off option is abandoned while a switching on option is obtained. If the switching on threshold then is reached, the switching on option is exercised and the idle capacity would be activated again. Meanwhile the switching off option is obtained again.

In the discrete model without switching options above there is an implicit assumption that the operation of capacities incurs no cost. Thus, the expansions only involve the fixed installation costs. However, if we introduce the switching flexibilities, the operating costs could not be left out. As shown in section II, the completely reversible switching decisions are governed by the markup pricing rule that the marginal revenue equals the marginal cost. The marginal revenue in our discrete model is the revenue created by the additional capacity per period while the marginal cost is the operating cost of the additional capacity. When a unit of capacity produces a unit of output per period, it will incur a unit of operating cost. When a capacity is switched off and become idle, the operating cost will be cut since the unit is no longer utilized. However, the operating cost will be restored if it is switched on later. Following the assmptions of Pindyck (1988), the operating cost per unit capacity should be proportional to the installed capacity. Therefore, for a certain level of capacity, the operating cost per unit capacity should be constant no matter it is the saved operating cost when the capacity is switched off, the incurred operating cost when the capacity is switched on, or the operating cost taken when new capacity is installed. However, in order to establish the two-way switching on/off system, we have to first assume different operating costs for different status.

If we sum up the operating cost incurred by a unit of capacity across time assuming it operates permanently, we could obtain its perpetual operating cost C and it equals the operating cost divided by the risk-free rate r shown in footnote 2. We then denote the perpetual operating cost per unit capacity saved when the ith additional capacity is switched off by C_{i-} and the perpetual operating cost per unit capacity incurred when the ith additional capacity is switched on by C_{i+}. The perpetual operating cost per unit capacity incurred after the ith capacity first installed is denoted by C_i. When the switching on/off thresholds converge, our discrete model simulates the assumption of common switching threshold in Pindyck (1988) and become its discrete counterpart. Such convergence could be approximated by making $C_{i+} = C_{i-} + \varepsilon$ with ε given a value closed to zero such as 0.0001.

The total perpetual operating costs for the ith additional capacity should be the product of the perpetual operating cost and the amount of the ith capacity expansion. Thus, the total perpetual operating costs with respect to C_{i-}, C_{i+}, and C_i are respectively $C_{i+}(K_i - K_{i-1})$, $C_{i-}(K_i - K_{i-1})$, and $C_i(K_i - K_{i-1})$.

We then respectively denote the switching on options and switching off options for the ith additional capacity by $I_i(P)$ and $F_i(P)$. Since we assume separate switching on/off thresholds P_{i+} and P_{i-}, there should be two sets of boundary conditions at switching transitions for any unit of capacity, with one for the switching on threshold and one for the switching off threshold. Therefore, for the ith added capacity, the value matching conditions are given by

$$\begin{array}{ccccccc} \underline{N_i} & = & \underline{M_i} & + & \underline{T_i} & - & \underline{C_i} \\ \begin{bmatrix} I_i(P_{i+}) \\ F_i(P_{i-}) \end{bmatrix} & = & \begin{bmatrix} F_i(P_{i+}) \\ I_i(P_{i-}) \end{bmatrix} & + & \begin{bmatrix} P_{i+}(K_i - K_{i-1}) \\ -P_{i-}(K_i - K_{i-1}) \end{bmatrix} & - & \begin{bmatrix} C_{i+}(K_i - K_{i-1}) \\ -C_{i-}(K_i - K_{i-1}) \end{bmatrix} \end{array} \tag{5.1}$$

where $\underline{N_i}$ and $\underline{M_i}$ respectively represent the vectors of switching options at the end of their life and at the beginning of their life. $\underline{T_i}$ and $\underline{C_i}$ respectively denote the vectors of payoffs and perpetual operating costs for the ith added capacity. Using the discount factor methodology, we now could easily obtain the dollar beta matching conditions by take derivative at both sides of the equations and then simultaneously multiply by P. Therefore,

$$\begin{array}{ccccc} \underline{\beta_N N_i} & = & \underline{\beta_M M_i} & + & \left(\underline{\beta_T T_i} - \underline{\beta_C C_i}\right) \\ \begin{bmatrix} cI_i(P_{i+}) \\ pF_i(P_{i-}) \end{bmatrix} & = & \begin{bmatrix} pF_i(P_{i+}) \\ cI_i(P_{i-}) \end{bmatrix} & + & \begin{bmatrix} P_{i+}(K_i - K_{i-1}) \\ -P_{i-}(K_i - K_{i-1}) \end{bmatrix} \end{array} \tag{5.2}$$

where $\underline{\beta_N}$, $\underline{\beta_M}$, $\underline{\beta_T}$, and $\underline{\beta_C}$ respectively denote the beta matrix for $\underline{N_i}$, $\underline{M_i}$, $\underline{T_i}$, and $\underline{C_i}$. The discount function connects the values of switching options at the end of their life with those at the beginning of their life. For a put option to switch off, it is created when the capacity unit is switched on while for a call option to switch on, it is created when the capacity unit is switched off. Thus, the discount functions for switching options for the ith capacity are given by

$$\begin{array}{ccccc} \underline{M_i} & = & \underline{DS_i} & & \underline{N_i} \\ \begin{bmatrix} F_i(P_{i+}) \\ I_i(P_{i-}) \end{bmatrix} & = & \begin{bmatrix} 0 & D_p(P_{i+}, P_{i-}) \\ D_c(P_{i-}, P_{i+}) & 0 \end{bmatrix} & & \begin{bmatrix} I_i(P_{i+}) \\ F_i(P_{i-}) \end{bmatrix} \end{array} \tag{5.3}$$

where $\underline{DS_i}$ denotes the matrix of the discount functions connecting the switching on/off options for the ith capacity. $D_p(P_{i+}, P_{i-})$ is the discount function for put options in the form of equation (2.22).

With the three sets of conditions in equation (5.1), (5.2), and (5.3), we now could solve the perpetual operating costs $\underline{C_i}$ and the values of switching options $\underline{N_i}$ and $\underline{M_i}$ given switching thresholds. Therefore, the solutions of $\underline{C_i}$, $\underline{N_i}$, and $\underline{M_i}$ are given by

$$\underline{N_i} = \left(\underline{\beta_N} - \underline{\beta_M DS_i}\right)^{-1} \underline{\beta_T T_i} \tag{5.4}$$

$$\underline{M_i} = \underline{DS_i N_i} = \underline{DS_i} \left(\underline{\beta_N} - \underline{\beta_M DS_i}\right)^{-1} \underline{\beta_T T_i} \tag{5.5}$$

$$\underline{C_i} = \left(\underline{DS_i} - \underline{I}\right) \left(\underline{\beta_N} - \underline{\beta_M DS_i}\right)^{-1} \underline{\beta_T T_i} + \underline{T_i} \tag{5.6}$$

where I is an two by two identity matrix. The existence of a discount mapping between $\underline{N_i}$ and $\underline{M_i}$ ensures uniqueness of the solution. Therefore, the set of perpetual operating costs are one-to-one correspond to the set of switching thresholds and we could obtain a certain set of switching threshold values if given a set of perpetual operating costs. This could be achieved by presuming initial values $\underline{T_i}$ and then numerically iterating the values of $\underline{T_i}$ until $\underline{C_i}$ match with the target values. Then, the values of $\underline{T_i}$ that map $\underline{C_i}$ to specific values are the switching thresholds we search for.

Solution for expansion decisions

Though the switching thresholds and the expansion thresholds could be solved independently, the switching options would affect the expansion decisions since the capacity expansions would create switching off options. The capacity expansions now would also bring in operating costs for the new added capacities. Therefore, the value matching conditions for the discrete model with switching flexibilities are given by:

$$\begin{matrix} \underline{W} & = & \underline{U} & + & \underline{F} & + & (\underline{Y}-\underline{Z}) & - & \underline{C} & - & \underline{X} \\ \begin{bmatrix} V_{i-1}(P_i) \\ V_i(P_{i+1}) \end{bmatrix} & = & \begin{bmatrix} V_i(P_i) \\ V_{i+1}(P_{i+1}) \end{bmatrix} & + & \begin{bmatrix} F_i(P_i) \\ F_{i+1}(P_{i+1}) \end{bmatrix} & + & \begin{bmatrix} P_i(K_i-K_{i-1}) \\ P_{i+1}(K_{i+1}-K_i) \end{bmatrix} & - & \begin{bmatrix} C_i(K_i-K_{i-1}) \\ C_{i+1}(K_{i+1}-K_i) \end{bmatrix} & - & \begin{bmatrix} X_i-X_{i-1} \\ X_{i+1}-X_i \end{bmatrix} \end{matrix} \quad (5.7)$$

where $\underline{F}$ represents the vector collecting the switching off put options for the ith added capacity at expansion thresholds P_i; $\underline{C}$ denotes the perpetual operating costs for the added capacities. The denotations and definitions of other vectors keep the same as those in the discrete model without switching options in the last section. It is worth noting that $F_i(P_{i+})$ in $\underline{M_i}$, $F_i(P_{i-})$ in $\underline{N_i}$, and $F_i(P_i)$ in $\underline{F}$ are all defined as the switching off put option for the ith capacity but respectively present the values at P_{i+}, P_{i-}, and P_i. Thus, the values of $F_i(P_i)$ in $\underline{F}$ could be solved by discounting the values of $F_i(P_{i+})$ in $\underline{M_i}$ or $F_i(P_{i-})$ in $\underline{N_i}$ according the discount function shown in equation (2.8). Therefore, though the switching thresholds P_{i+} and P_{i-} don't not appear in equation (5.7) directly, they would affect the conditions via $\underline{F}$.

Pindyck (1988) assumes that the operating cost rC_i per unit capacity is proportional to the capacity level K_i. Such assumption is economically sensible that the production might follow the diseconomies of scale. In order to keep consistent with the continuous counterpart, rC_i is assumed to be proportional to the capacity level K_i. This assumption also matches our discrete scaling system. Since K_i grows at the rate of $(1+\alpha)$, the perpetual operating cost C_i should also grow at the rate of $(1+\alpha)$, i.e. $C_{i+1}=(1+\alpha)C_i$. Thus, $C_{i+1}(K_{i+1}-K_i)=(1+\alpha)^2C_i(K_i-K_{i-1})$ and grows at the same scaling ratio as other terms in the value matching conditions. If we denote the proportion ratio between rC_i and K_i by γ, then

$$rC_i=\gamma K_i \quad (5.8)$$

$$C_i(K_i-K_{i-1})=\frac{\gamma}{r}K_i(K_i-K_{i-1})=\alpha(1+\alpha)^{-1}\frac{\gamma}{r}{K_i}^2 \quad (5.9)$$

where r is the risk free rate. Similar to the model without switching options in the last section, the installation cost could be expressed as the product of the average installation cost per unit capacity and the capacity amount, which is given by

$$X_i-X_{i-1}=(K_i-K_{i-1})k_i \quad (5.10)$$

where k_i is the average installation cost per unit capacity for the ith expansion from K_{i-1} to K_i. If k_i is constant against capacity K_i, namely $k_i=k_{i-1}\ \forall\ i\geq 1$, X_i-X_{i-1}

should be proportional to $K_i - K_{i-1}$ and also scaled up at the ratio of $1 + \alpha$. This contradicts the requirement for scaling system that $X_{i+1} - X_i = (1+\alpha)^2(X_i - X_{i-1})$. Therefore, k_i should also be scaled up at the ratio of $1 + \alpha$ to ensure $(K_i - K_{i-1})k_i$ are scaled up at the ratio of $(1+\alpha)^2$.

Now four vectors of unknown variables are left to be solved, which are $\underline{W}$, $\underline{U}$, $\underline{X}$, and $\underline{F}$. Compared with the value matching conditions in the discrete model without switching options in the last sections, we still have to solve one more unknown vector, namely $\underline{F}$, before we use the discount factor methodology with three sets of conditions to calculate the rest three unknown vectors including $\underline{W}$, $\underline{U}$, and $\underline{X}$.

As shown above, the put option values at the expansion thresholds, namely $\underline{F}$, could be solved by discounting the switching off put option values at the switching thresholds, namely $\underline{N_i}$ and $\underline{M_i}$, which have been solved in equations (5.4) and (5.5). To solve $F_i(P_i)$, we could discount from the switching options at either the switching off thresholds or the switching on thresholds since the discount relationship exists for both cases.

Discount functions with the switching on thresholds

$$F_i(P_i) = D_p(P_i, P_{i+})F_i(P_{i+}) \tag{5.11}$$

Discount functions with the switching off thresholds

$$F_i(P_i) = D_p(P_i, P_{i-})F_i(P_{i-}) \tag{5.12}$$

According to equations (5.4), (5.5), and (5.6), we could obtain the switching thresholds P_{i+} and P_{i-} by numerical iteration given the target values of the perpetual operating cost at the switching thresholds C_{i-} and C_{i+}. Correspondingly, we could solve the values of $F_i(P_i)$ from the discount function (5.11) or (5.12). Then, how could we determine the target values for C_{i-} and C_{i+}?

We prove in **Appendix IV** that when the switching on/off thresholds converge, namely $\lim_{\varepsilon\to 0} P_{i-} + \varepsilon = P_{i+}$, the perpetual operating cost at the switching on/off thresholds would also converge, namely $\lim_{\varepsilon\to 0} C_{i-} + \varepsilon = C_{i+}$. Additionally, in such case the switching thresholds follow the markup pricing rule that equals the marginal revenue with the marginal operating cost. If we denote the common switching threshold and the perpetual operating cost after convergence by P_i^s and C_i^s, then

$$C_i^s = \frac{\delta}{r} P_i^s = \frac{(1-c)(1-p)}{cp} P_i^s \tag{5.13}$$

where r and δ are respectively the risk free rate and the yield in equation (4.1). This is consistent with the continuous counterpart of Pindyck (1988) which assumes markup pricing rule for the switching decisions. We assume above that the operating cost only depends on the capacity level following Pindyck (1988), thus we should have $C_i^s = C_i$ when the switching thresholds converge. Since we assume $C_i = \frac{\gamma}{r} K_i$ as the function of the perpetual operating cost at expansion thresholds, we could obtain the target values of C_{i-} and C_{i+} when the switching thresholds converge.

$$\lim_{\varepsilon\to 0} C_{i-} + \varepsilon = C_{i+} = C_i^s = C_i = \frac{\gamma}{r} K_i \tag{5.14}$$

Therefore, the optimal switching thresholds when $\lim_{\varepsilon\to 0} P_{i-} + \varepsilon = P_{i+}$ are the values that hold equation (5.6) given equation (5.14). Since the switching decisions follow the markup pricing rule of equation (5.13), the common switching thresholds P_i^s should be proportional to the capacity level K_i. Additionally, since P_i and K_i increase at the same ratio of $(1+\alpha)$, P_i^s should also be proportional to P_i in such case. If we denote the proportional coefficient between P_i^s and P_i by φ, then

$$\frac{(1-c)(1-p)}{cp} P_i^s = \frac{(1-c)(1-p)}{cp} \varphi P_i = C_i^s = C_i = \frac{\gamma K_i}{r} \tag{5.15}$$

Therefore,

$$\varphi = \frac{cp\gamma K_i}{(1-c)(1-p)rP_i} = \frac{cp\gamma K_0}{(1-c)(1-p)rP_0} \tag{5.16}$$

$$P_i^s = \varphi P_i = \frac{cp\gamma K_0 P_i}{(1-c)(1-p)rP_0} = \frac{cp\gamma K_0 P_i}{(1-c)(1-p)rP_0} \tag{5.17}$$

Thus, φ is constant and determined by exogenous parameters c, p, γ, P_0 and K_0. In other words, the scaling ratio between the switching threshold and the expansion threshold is not arbitrarily selected but determined by the initial states, i.e. P_0 and K_0, beta of option values, i.e. c and p, and the relative size of perpetual operating cost to capacity, i.e. γ. Therefore, the switching policies are not exogenously determined but are connected with the expansion policies. Meanwhile, the expansion polices are affected by the switching opportunities since the capacity expansions are accompanied with the creations of the switching off put options for the added capacities.

It is obvious that the switching threshold should be smaller than the expansion threshold for a certain capacity level, implying $0 < \varphi < 1$. Therefore,

$$\varphi = \frac{cp\gamma K_0}{(1-c)(1-p)rP_0} = \frac{\gamma K_0}{\delta P_0} < 1 \tag{5.18}$$

$$C_0^s = \frac{\gamma K_0}{r} = \frac{\delta}{r} P_0^s < \frac{\delta}{r} P_0 \tag{5.19}$$

Therefore, the inequality $0 < \varphi < 1$ is consistent with our model assumption in equation (5.13). According to equations (5.15) and (5.17), the switching thresholds P_i^s are certain given either the capacity level K_i or the expansion thresholds P_i. Correspondingly, $\underline{F}$ could be exclusively solved from equation (5.11) or (5.12) for certain K_i or P_i. Hence, in equation (5.7) only $\underline{W}$, $\underline{U}$, and $\underline{X}$ are left unknown and unsolved, equivalent to the case without switching options in equation (4.16).

Following the discount factor methodology in the last section, we could then obtain the dollar beta matching conditions from equation (5.7).

$$\begin{array}{ccccccc} \underline{\beta_W W} & = & \underline{\beta_U U} & + & \underline{\beta_F F} & + & \left(\underline{\beta_Y Y} - \underline{\beta_Z Z}\right) \\ \begin{bmatrix} cV_{i-1}(P_i) \\ cV_i(P_{i+1}) \end{bmatrix} & = & \begin{bmatrix} cV_i(P_i) \\ cV_{i+1}(P_{i+1}) \end{bmatrix} & + & \begin{bmatrix} pF_i(P_i) \\ pF_{i+1}(P_{i+1}) \end{bmatrix} & + & \begin{bmatrix} \alpha(1+\alpha)^{-1} P_i K_i \\ \alpha(1+\alpha)^{-1} P_{i+1} K_{i+1} \end{bmatrix} \end{array} \tag{5.20}$$

Since both the discount relationships and the self-similarity hold for the new model with switching options, the discount function for the expansion options is the same as that of the discrete model without switching options.

$$\underset{\underline{U}}{\begin{bmatrix} V_i(P_i) \\ V_{i+1}(P_{i+1}) \end{bmatrix}} \overset{=}{=} \underset{\underline{D}}{\begin{bmatrix} 0 & (1+\alpha)^{-c} \\ (1+\alpha)^{4-c} & 0 \end{bmatrix}} \underset{\underline{W}}{\begin{bmatrix} V_{i-1}(P_i) \\ V_i(P_{i+1}) \end{bmatrix}} \tag{5.21}$$

With value matching conditions, dollar beta matching conditions, and discount functions, we now could solve the expansion installation costs $\underline{X}$ and the expansion option values $\underline{W}$ and $\underline{U}$. The solutions of $\underline{W}$, $\underline{U}$, and $\underline{X}$ are given by the following matrix equations.

$$\underline{W} = \left(\underline{\beta_W} - \underline{\beta_U D}\right)^{-1} \left[\underline{\beta_F F} + \left(\underline{\beta_Y Y} - \underline{\beta_Z Z}\right)\right] \tag{5.22}$$

$$\underline{U} = \underline{DW} = \underline{D}\left(\underline{\beta_W} - \underline{\beta_U D}\right)^{-1} \left[\underline{\beta_F F} + \left(\underline{\beta_Y Y} - \underline{\beta_Z Z}\right)\right] \tag{5.23}$$

$$\underline{X} = \underline{U} - \underline{W} + \underline{F} + \left(\underline{Y} - \underline{Z}\right) - \underline{C} \tag{5.24}$$

$$= \left(\underline{D} - \underline{I}\right)\left(\underline{\beta_W} - \underline{\beta_U D}\right)^{-1} \left[\underline{\beta_F F} + \left(\underline{\beta_Y Y} - \underline{\beta_Z Z}\right)\right] + \underline{F} + \left(\underline{Y} - \underline{Z}\right) - \underline{C}$$

Comparing the solutions above with those of the discrete model without switching options in equations (4.20), (4.21), and (4.22), we could find that they are similar but involve new factors of switching option values $\underline{F}$. Therefore the switching polices exert effect on the expansion policies, which should be deviated from those in the discrete model without switching options in the last section. The next subsection therefore examines the convergence of the discrete model to the continuous case when switching opportunities are added.

5.2 The Convergence of the discrete Model to the Continuous Case

In this part we examine the analytical expressions of the switching and expansion options when the discrete model converges to the continuous case with $\alpha \to 0$. Since $\underline{F}$ are determined by the switching options at the switching thresholds, we first look into the analytical expressions of $F_i(P_{i+})$ and $F_i(P_{i-})$ as $\alpha \to 0$. Since we assume common switching thresholds for our discrete model, we substitute P_i^s for P_{i+} and P_{i-} in equations (5.1), (5.2), and (5.3) to provide the analytical expressions by taking advantage of equation (5.15). Using Gaussian Elimination for equations (5.1), (5.2), and (5.15) we could derive that

$$F_i(P_i^s) = \left[\frac{c-1}{p-c}\varphi P_i - \frac{c\gamma}{r(p-c)}K_i\right]\alpha(1+\alpha)^{-1}K_i \tag{5.25}$$
$$= \frac{c\gamma}{r(c-p)(1-p)}\alpha(1+\alpha)^{-1}{K_i}^2$$

Then, the expression of $F_i(P_i)$ could be obtained from equation (5.11) or (5.12).

$$F_i(P_i) = \left(\frac{1}{\varphi}\right)^p \frac{c\gamma}{r(c-p)(1-p)} \alpha(1+\alpha)^{-1} {K_i}^2 \tag{5.26}$$

It is easy to prove that both $F_i(P_i^s)$ and $F_i(P_i)$ are scaled up at the ratio of $(1+\alpha)^2$, which is consistent with our setting of the scaling system. Additionally, equation (5.25) and (5.26) imply that both $F_i(P_i^s)$ and $F_i(P_i)$ converge to zero as $\alpha \to 0$. With the expressions of $F_i(P_i)$, we could then obtain the expressions of $V_i(P_i)$ and $V_{i-1}(P_i)$ by substituting equations (5.26) and (5.21) into equation (5.20).

$$V_i(P_i) = \left[p\left(\frac{1}{\varphi}\right)^p \frac{c\gamma K_i}{r(c-p)(1-p)} + P_i\right] \frac{\alpha(1+\alpha)^{-1} K_i}{c[(1+\alpha)^{c-2} - 1]} \tag{5.27}$$

$$V_{i-1}(P_i) = \left[p\left(\frac{1}{\varphi}\right)^p \frac{c\gamma K_i}{r(c-p)(1-p)} + P_i\right] \frac{\alpha(1+\alpha)^{c-3} K_i}{c[(1+\alpha)^{c-2} - 1]} \tag{5.28}$$

As α becomes infinitesimal,

$$\lim_{\alpha \to 0} V_i(P_i) = \lim_{\alpha \to 0} \left[p\left(\frac{1}{\varphi}\right)^p \frac{c\gamma K_i}{r(c-p)(1-p)} + P_i\right] \frac{\alpha(1+\alpha)^{-1} K_i}{c[(1+\alpha)^{c-2} - 1]} \tag{5.29}$$

According to l'Hôpital's rule, equation (5.27) is equal to

$$\lim_{\alpha \to 0} V_i(P_i) = \left[p\left(\frac{1}{\varphi}\right)^p \frac{c\gamma K_i}{r(c-p)(1-p)} + P_i\right] \frac{K_i}{c-2} \tag{5.30}$$

Similarly, the expression of $V_{i-1}(P_i)$ as α becomes infinitesimal could also be obtained using l'Hôpital's rule.

$$\lim_{\alpha \to 0} V_{i-1}(P_i) = \left[p\left(\frac{1}{\varphi}\right)^p \frac{c\gamma K_i}{r(c-p)(1-p)} + P_i\right] \frac{K_i}{c-2} \tag{5.31}$$

Therefore,

$$\lim_{\alpha \to 0} V_i(P_i) = \lim_{\alpha \to 0} V_{i-1}(P_i) \tag{5.32}$$

Equations (5.30) and (5.31) also imply that $c > 2$ should hold in the case with switching options to ensure the option values are positive.

Comparing equations (5.30) and (5.31) with equations (4.28) and (4.29), we could find that the expansion options with switching options have an extra negative value of $p\left(\frac{1}{\varphi}\right)^p \frac{c\gamma {K_i}^2}{r(c-p)(1-p)(c-2)}$ than the expansion options without switching options. Nevertheless, we could not jump to conclude that the expansion options become smaller after adding the switching options for a certain level of capacity K_i since the expansion option values are also affected by P_i. In our scaling system, the values of P_i and K_i depend on the values of P_0 and K_0, and with different assumptions of P_0 and K_0 we would have different P_i given a certain value of K_i.

Thus, the conclusion that switching options reduce the values of expansion options could be made as long as assuming the same values of P_0 and K_0 for both models with and without switching options. This is economically sensible that the opportunity costs abandoned for each expansion, namely the expansion options, would be lower since switching options introduce some reversibility to the irreversible expansion process.

We now calculate the installation cost per unit capacity when the discrete staircases become infinitesimal. According to equation (5.7), it could be expressed as

$$\begin{aligned} k_i &= \lim_{\alpha \to 0} \frac{\Delta X_i}{\Delta K_i} = \lim_{\alpha \to 0} \frac{X_i - X_{i-1}}{K_i - K_{i-1}} \\ &= \lim_{\alpha \to 0} \frac{V_i(P_i) - V_{i-1}(P_i) + F_i(P_i) + \alpha(1+\alpha)^{-1} P_i K_i - C_i(K_i - K_{i-1})}{\alpha(1+\alpha)^{-1} K_i} \\ &= \frac{c-1}{c} P_i + \left[\left(\frac{1}{\varphi} \right)^p \frac{1}{1-p} - 1 \right] \frac{\gamma K_i}{r} \end{aligned} \tag{5.33}$$

Comparing equation (5.33) with equation (4.30), we could find that the installation cost per unit capacity after adding switching options have an extra value of $\left[\left(\frac{1}{\varphi} \right)^p \frac{1}{1-p} - 1 \right] \frac{\gamma K_i}{r}$. Since $0 < \varphi < 1$ and $p < 0$, it is easy to prove that

$$0 < \left(\frac{1}{\varphi} \right)^p \frac{1}{1-p} < 1 \tag{5.34}$$

Since the value of k_i also depends on both P_i and K_i, the comparison between equation (5.33) and equation (4.30) should also be made with the prerequisite of the same P_0 and K_0. Therefore, with the same P_0 and K_0, the installation cost per unit capacity k_i would become smaller after introducing the switching options. Correspondingly, the installed capital stock for a certain capacity level should be smaller for the expansion model with switching options. This is against the economic intuition that the more reversibility brought by the switching options should encourage a firm to invest more capital which could be switched off if economic condition deteriorates.

However, the exercise of expanding the ith capacity with the switching options not only incurs the installation cost $k_i(K_i - K_{i-1})$ but also incurs the perpetual operating cost $C_i(K_i - K_{i-1})$. Therefore, the total cost for an expansion with the switching options should be $(k_i + C_i)(K_i - K_{i-1})$ and correspondingly the total cost per unit capacity is given by

$$\begin{aligned} k_i + C_i &= \frac{c-1}{c} P_i + \left[\left(\frac{1}{\varphi} \right)^p \frac{1}{1-p} - 1 \right] \frac{\gamma K_i}{r} + \frac{\gamma}{r} K_i \\ &= \frac{(c-1)P_i}{c} + \left(\frac{1}{\varphi} \right)^p \frac{1}{1-p} \frac{\gamma K_i}{r} \end{aligned} \tag{5.35}$$

Therefore, unlike the expansion without switching options in which all investment manifests as assets of installed capitals, the expansion with switching options takes some investment as expenses of operating costs. Nevertheless, the net effect of the switching options on the total cost per unit capacity, namely $\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p}\frac{\gamma K_i}{r}$, is positive. Thus, the switching opportunities could increase the incentive of a firm to install more capitals.

Similar to the last section, we should also calculate X_0, which is the cumulative capital for P_0 and K_0, as the base point of the discrete expansions. According to equation (5.33),

$$(X_1 - X_0) = k_1(K_1 - K_0) = \left\{\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r}\right\} K_1(K_1 - K_0) \tag{5.36}$$

Then, equation (4.32) shows that $X_0 = \frac{1}{\alpha(\alpha+2)}(X_1 - X_0)$. Therefore, the expression of X_0 is given by

$$\begin{aligned} X_0 &= \frac{1}{\alpha(\alpha+2)}\left\{\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r}\right\} K_1(K_1 - K_0) \\ &= \frac{(1+\alpha)}{(2+\alpha)}\left\{\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r}\right\} {K_0}^2 \end{aligned} \tag{5.37}$$

When the discrete model converges to the continuous case with $\alpha \to 0$,

$$\begin{aligned} \lim_{\alpha\to 0} X_0 &= \lim_{\alpha\to 0}\frac{1}{(1+\alpha)(\alpha+2)}\left\{\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r}\right\} {K_0}^2 \\ &= \frac{1}{2}\left\{\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r}\right\} {K_0}^2 \end{aligned} \tag{5.38}$$

Given the analytical inference above, we implement the numerical experiments in the next subsection to examine their validity.

5.3 Numerical Experiments and Results

In this numerical experiment, the results of the discrete model with switching option are compared with those of the adjusted continuous model of Pindyck (1988) in section 3.2. We summarize the numerical results in Table 5.1 and Table 5.2, which report the values of P_i, P_i^s, K_i, X_i, $V_{i-1}(P_i)$, $V_{i-1}(P_i)$, $F_i(P_i)$, $\Delta X_i/\Delta X_{i-1}$, and $\Delta X_i/\Delta K_i$ where $\Delta X_i/\Delta X_{i-1} = (X_i - X_{i-1})/(X_{i-1} - X_{i-2})$ is the growth rate of the installation costs for each expansion and $\Delta X_i/\Delta K_i = (X_i - X_{i-1})/(K_i - K_{i-1})$ is the installation cost per unit of capacity for the ith expansion.

To keep the discrete model consistent with the continuous counterpart, all parameter values are kept the same as those assumed in the continuous model in section 3.2. Therefore, we assume the same P_0 and K_0 as those assumed in section 3.2, namely $P_0 = 24.49$ and $K_0 = 1$ for $c = 3.70$, and $P_0 = 13.84$ and $K_0 = 1$ for $c = 2.09$. The initial capital X_0 is calculated according to equation (5.37).

We assume in section 3.2 that $C_t = \frac{(2\tau + c_2)K_t}{r}$, therefore $\gamma = (2\tau + c_2) = 1$ to ensure the proportional coefficients between the operating cost and the capacity level are the same for the discrete model and the continuous model. Table 5.1 and Table 5.2 investigate the convergence of the discrete model to the continuous case by gradually decreasing α. Table 5.1 assumes $c = 3.70$, $p = -2.70$, $\gamma = 1$, $P_0 = 24.49$, and $K_0 = 1$ while Table 5.2 assumes $c = 2.09$, $p = -3.59$, $\gamma = 1$, $P_0 = 13.84$, and $K_0 = 1$. If we substitute these two sets of parameter values into equation (5.37), we could obtain the values of X_0. Since we assume $K_0 = 1$ always hold, the effects of the parameters on X_0 are all reflected on the values of $\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r}$.

Parameter values in Table 5.1

$$\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r} = 1.00 \tag{5.39}$$

Parameter values in Table 5.2

$$\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r} = 1.00 \tag{5.40}$$

This is exactly equal to the value of scale factor k in the scaling installation cost per unit capacity kK_t assumed in the adjusted model of Pindyck (1988) in section 3.2. Therefore, the initial cumulative capital X_0 should be equal to

$$X_0 = \frac{(1+\alpha)}{(2+\alpha)} k{K_0}^2 \tag{5.41}$$

$$\lim_{\alpha \to 0} X_0 = \frac{1}{2} k{K_0}^2 \tag{5.42}$$

Therefore, in terms of the initial cumulative capital X_0 the discrete model with switching options could perfectly converge to its continuous counterpart. It is easy to prove that the inference in equation (5.38) and (5.42) could be generalized to any values of the cumulative capital X_i, thus

$$\lim_{\alpha \to 0} X_i = \frac{1}{2}\left\{\frac{(c-1)P_0}{cK_0} + \left[\left(\frac{1}{\varphi}\right)^p \frac{1}{1-p} - 1\right]\frac{\gamma}{r}\right\}{K_i}^2 = \frac{1}{2} k{K_i}^2 \tag{5.43}$$

Therefore, in the discrete models with and without switching options the cumulative capital X_i have the same quadratic function of the capacity amount K_i. Thus, the cumulative capital X_i would be the same in these two discrete models for given capacity level K_i. This is supported by the numerical results in Table 5.1 and Table 5.2 that values of X_i keep the same as those in Table 4.1 for the discrete model without switching options.

The results in Table (5.1) and (5.2) indicate that larger values of c would lead to higher expansion thresholds P_i for given levels of cumulative capital X_i. This is consistent with the implication of the adjusted model of Pindyck (1988) in section 3.2.

Additionally, the expansion option values also become smaller as c increases. This is consistent with the results for the discrete model without switching options, giving the same implications of r, δ, and σ as discussed in section 4.3.

When the value of α decreases and converges to zero, Table 5.1 and 5.2 shows that the values of P_i, P_i^s, and X_i also converge to the numerical results of the continuous model in Table 3.2. When $\alpha = 0.0001$, the values of P_i, P_i^s, and X_i are nearly the same as P_t, P_t^s, and X_t in the continuous model for a certain capacity level K_i or K_t.

Therefore, the investment rules solved from the discount factor methodology could soundly converge the investment rules solved from the continuous method. Additionally, the values of $\Delta X_i/\Delta X_{i-1}$ are equal to $(1+\alpha)^2$ and therefore the solution from the discount factor methodology could keep the scaling system. Then, the values of $\Delta X_i/\Delta K_i$ converges to kK_i as the value of α approaches to zero. Thus, the installation costs per unit capacity are also consistent between the discrete model and the continuous model.

Similar to the numerical results in section 4.3, the value of $V_{i-1}(P_i)$ is also larger than the value of $V_i(P_i)$ for the discrete model with switching options, consistent with the economic sense that the exercise of expansions would reduce the expansion option values. The values of the expansion options $V_{i-1}(P_i)$ and $V_i(P_i)$ also converge as $\alpha \to 0$, supporting the implication from equation (5.32).

In order to provide more intuitive illustration, we further plot the discrete model and the case when the discrete model converges to the continuous case in Figure 5.1, Figure 5.2, Figure 5.3, Figure 5.4, and Figure 5.

Figure 5.1 and Figure 5.2 plot the cumulative capital X_t against the expansion thresholds P_t and the switching thresholds P_t^s in the adjusted continuous model of Pindyck (1988), and the cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^s in the discrete model with switching options for $\alpha = 0.1$, $\alpha = 0.01$, and $\alpha = 0.0001$. Similar to Figure 4.1 and 4.2, the discrete model converges to the continuous model as α decreases. The lines presenting the discrete model with $\alpha = 0.01$ and $\alpha = 0.0001$ are nearly overlapped with the line presenting the continuous model.

Similar to Figure 4.3, 4.4, and 4.5, Figure 5.3, 5.4, and 5.5 show the staircases of the discrete expansions but also the staircases of the discrete switching on/off. The implications are also similar to the last section. The discrete staircase indicates that the perpetual revenue P could not move simultaneously with the amount of capacity K in either the expansion transitions or the switching transitions, at which P remain fixed while X shift to new values. The results in Figure 5.3, 5.4, and 5.5 are solved by discount factor methodology respectively in a 20, 180, and 18000 thresholds system but their absolute scales that the thresholds and capitals span are closed. As the value of α decreases, the expansion discrete staircases presented by the blue line and the switching discrete staircases presented by the grape and orange lines become smaller and respectively be more closed to the smooth curves of the red and green lines. When α infinitely approaches zero at the last, the staircases will disappear and the blue line and the grape line will be overlapped with the red line and the green line, which meanwhile converge to the continuous model as shown in Figure 5.1 and 5.2. It could be observed that the absolute differences between the expansion threshold and the switching threshold become larger as the perpetual revenue P become larger. This is due to the proportional relationships between the expansion thresholds and the switching thresholds, i.e. $P_i^s = \varphi P_i$.

To sum up, all numerical results above support our analytical implications for the convergence of the discrete capacity choice model to the continuous adjusted model of Pindyck (1988), supporting the validity of the discount factor methodology in solving discrete capacity choice problems.

Table 5.1 The Convergence of the Discrete Model to the Continuous Case
($r = 0.05$, $\delta = 0.05$, $\sigma = 0.1$, $c = 3.70$, $p = -2.70$, $\gamma = 1$, $P_0 = 24.49$, $K_0 = 1$)

α	i	P_i	P_i^s	K_i	X_i	$\Delta X_i/\Delta X_{i-1}$	$\Delta X_i/\Delta K_i$	$V_{i-1}(P_i)$	$V_i(P_i)$	$F_i(P_i)$
0.1	0	24.49	20.00	1.00	0.52	-	-	3.90	3.31	0.20
	8	52.50	42.88	2.00	2.40	1.2100	2.14	14.80	12.58	0.76
	12	76.86	62.78	3.00	5.16	1.2100	3.14	31.72	26.97	1.62
	15	102.30	83.56	4.00	9.14	1.2100	4.18	56.19	47.79	2.87
	17	123.78	101.10	5.00	13.38	1.2100	5.06	82.27	69.97	4.20
	19	149.78	122.33	6.00	19.59	1.2100	6.12	120.46	102.44	6.15
0.01	0	24.49	20.00	1.00	0.50	-	-	3.19	3.14	0.02
	70	49.15	40.14	2.00	2.02	1.0201	2.01	12.60	12.38	0.07
	111	73.90	60.36	3.00	4.57	1.0201	3.02	28.48	28.00	0.16
	140	98.62	80.55	4.00	8.15	1.0201	4.03	50.72	49.87	0.29
	162	122.76	100.26	5.00	12.63	1.0201	5.01	78.59	77.27	0.45
	180	146.84	119.93	6.00	18.07	1.0201	6.00	112.44	110.56	0.64
0.0001	0	24.49	20.00	1.00	0.50	-	-	3.12	3.12	0.0002
	6932	48.98	40.01	2.00	2.00	1.0002	2.00	12.47	12.47	0.0007
	10987	73.47	60.01	3.00	4.50	1.0002	3.00	28.05	28.05	0.0016
	13864	97.96	80.01	4.00	8.00	1.0002	4.00	49.88	49.87	0.0029
	16096	122.46	100.02	5.00	12.51	1.0002	5.00	77.94	77.92	0.0045
	17919	146.95	120.02	6.00	18.01	1.0002	6.00	112.22	112.20	0.0065

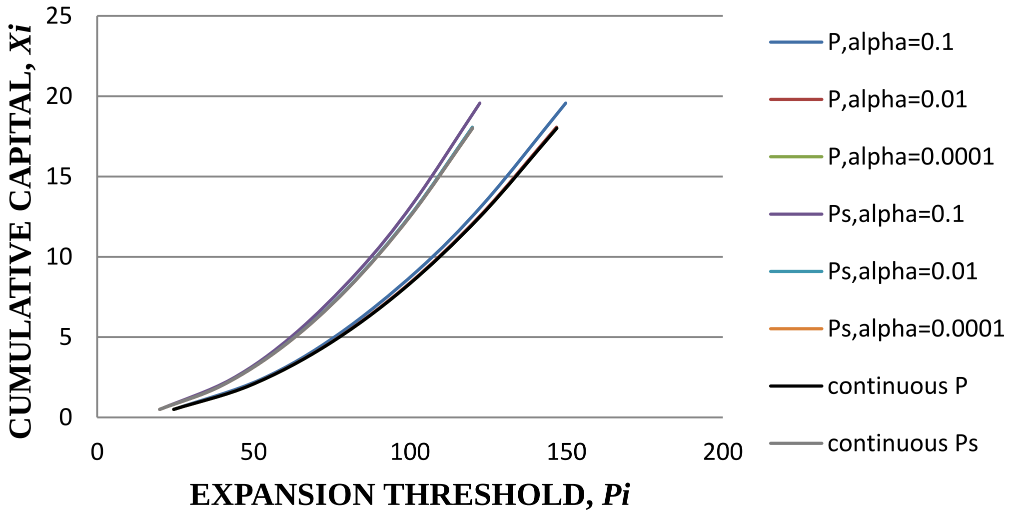
This table reports the effects of the discrete assumption and parameter values on the optimal investment rules of capacity choice problem with switching options. The data include the values of expansion threshold P_i, the switching thresholds P_i^s, the amount of capacity K_i, the cumulative capital X_i, the growth rate of the installation costs for each expansion $\Delta X_i/\Delta X_{i-1}$, the installation cost per unit capacity $\Delta X_i/\Delta K_i$, the option value at the end of its life $V_{i-1}(P_i)$, and the option value at the beginning of its life $V_i(P_i)$. The effects of r, δ, and σ are reflected in the value of c and the intervals of the discrete steps are reflected in the value of α. The values of r, δ, σ, c, p, γ, P_0, and K_0 are the same as those assumed in the continuous model in section 3.2 for comparison. The results are solved from discount factor methodology presented in equation (5.22), (5.23), and (5.24).

Table 5.2 The Convergence of the Discrete Model to the Continuous Case
($r = 0.15$, $\delta = 0.1$, $\sigma = 0.2$, $c = 2.09$, $p = -3.59$, $\gamma = 1$, $P_0 = 13.84$, $K_0 = 1$)

α	i	P_i	P_i^s	K_i	X_i	$\Delta X_i/\Delta X_{i-1}$	$\Delta X_i/\Delta K_i$	$V_{i-1}(P_i)$	$V_i(P_i)$	$F_i(P_i)$
0.1	0	13.84	10.00	1.00	0.52	-	-	81.57	80.87	0.02
	8	29.66	21.43	2.00	2.41	1.2100	2.15	309.75	307.11	0.07
	12	43.42	31.38	3.00	5.16	1.2100	3.14	663.98	658.31	0.15
	15	57.79	41.77	4.00	9.15	1.2100	4.18	1176.2	1166.2	0.26
	17	69.93	50.54	5.00	13.40	1.2100	5.06	1722.1	1707.4	0.39
	19	84.61	61.15	6.00	19.62	1.2100	6.12	2521.4	2499.9	0.57
0.01	0	13.84	10.00	1.00	0.50	-	-	71.46	71.40	0.0017
	70	27.76	20.07	2.01	2.02	1.0201	2.01	282.12	281.86	0.0066
	111	41.75	30.17	3.02	4.58	1.0201	3.02	637.94	637.37	0.0150
	140	55.71	40.27	4.03	8.16	1.0201	4.03	1136.1	1135.1	0.0267
	162	69.35	50.12	5.01	12.64	1.0201	5.02	1760.2	1758.6	0.0414
	180	82.95	59.95	6.00	18.09	1.0201	6.00	2518.4	2516.1	0.0593
0.0001	0	13.84	10.00	1.00	0.50	-	-	70.38	70.38	0.0000
	6932	27.67	20.00	2.00	2.00	1.0002	2.00	281.49	281.48	0.0001
	10987	41.51	30.00	3.00	4.51	1.0002	3.00	633.36	633.36	0.0001
	13864	55.34	40.00	4.00	8.01	1.0002	4.01	1125.9	1125.9	0.0003
	16096	69.18	50.00	5.00	12.52	1.0002	5.01	1759.5	1759.5	0.0004
	17919	83.01	60.00	6.00	18.03	1.0002	6.01	2533.5	2533.5	0.0006

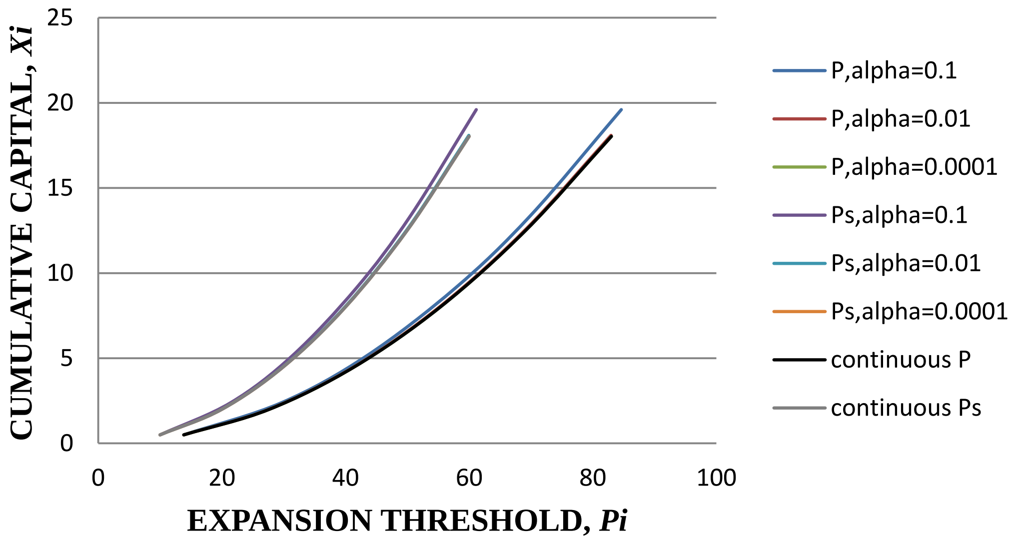
This table reports the effects of the discrete assumption and parameter values on the optimal investment rules of capacity choice problem with switching options. The data include the values of expansion threshold P_i, the switching thresholds P_i^s, the amount of capacity K_i, the cumulative capital X_i, the growth rate of the installation costs for each expansion $\Delta X_i/\Delta X_{i-1}$, the installation cost per unit capacity $\Delta X_i/\Delta K_i$, the option value at the end of its life $V_{i-1}(P_i)$, and the option value at the beginning of its life $V_i(P_i)$. The effects of r, δ, and σ are reflected in the value of c and the intervals of the discrete steps are reflected in the value of α. The values of r, δ, σ, c, p, γ, P_0, and K_0 are the same as those assumed in the continuous model in section 3.2 for comparison. The results are solved from discount factor methodology presented in equation (5.22), (5.23), and (5.24).

Figure 5.1 Expansion Thresholds and Cumulative Capitals
$(r = 0.05, \delta = 0.05, \sigma = 0.1, c = 3.70, p = -2.70, \gamma = 1, P_0 = 24.49,\ K_0 = 1)$



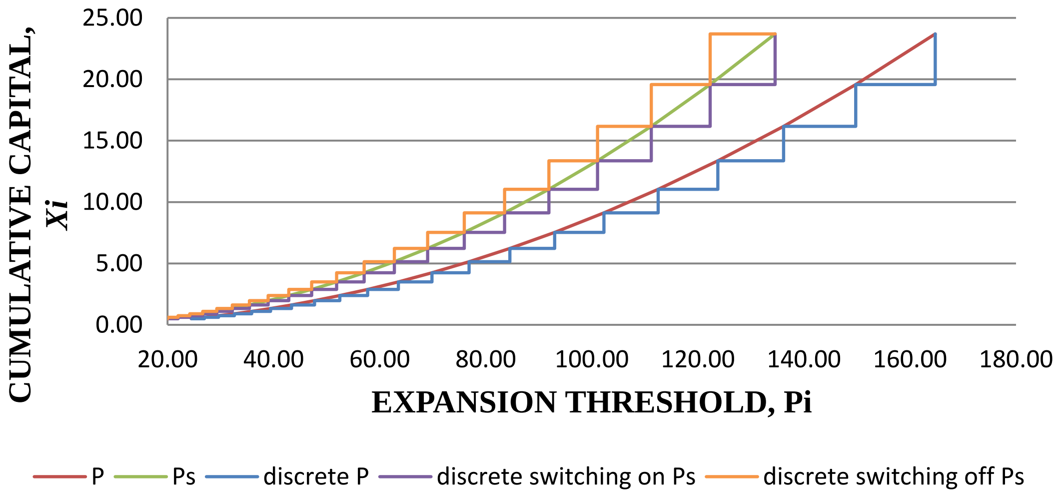
This figure respectively plots the cumulative capital X_t against the expansion thresholds P_t and the switching thresholds P_t^s in the adjusted continuous model of Pindyck (1988), and the cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^s in the discrete model with switching options for $\alpha = 0.1$, $\alpha = 0.01$, and $\alpha = 0.0001$. The data for the continuous model is from Table 3.1 which is solved by equation (3.32) and (3.33); the data for the discrete model is from Table 5.2 which is solved by discount factor methodology shown in equations (5.17) and (5.24).

Figure 5.2 Expansion Thresholds and Cumulative Capitals
$(r = 0.15, \delta = 0.1, \sigma = 0.2, c = 2.09, p = -3.59, \gamma = 1, P_0 = 13.84,\ K_0 = 1)$



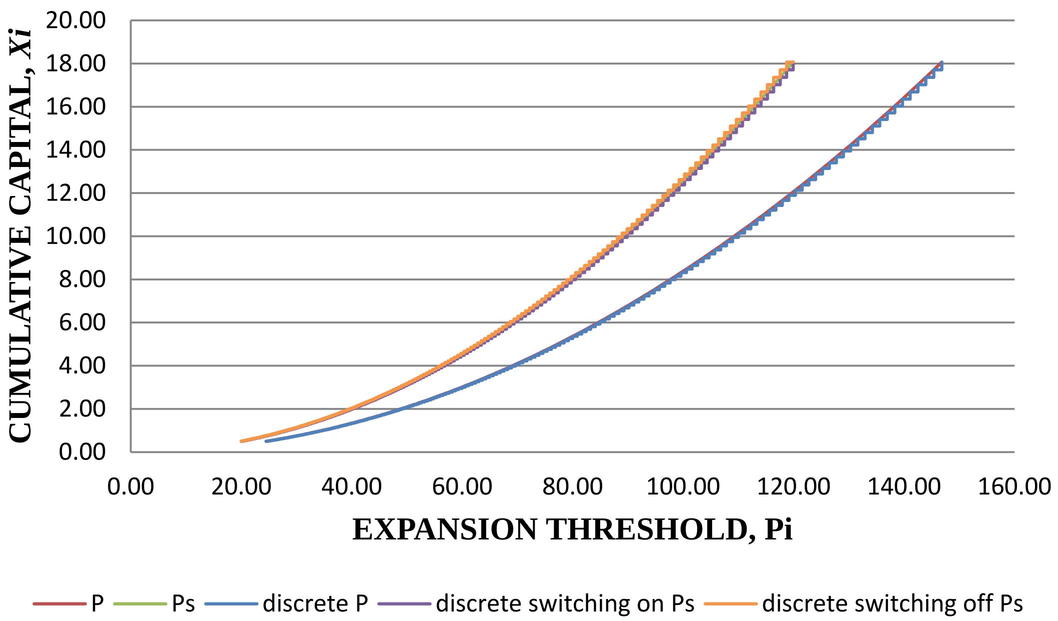
This figure respectively plots the cumulative capital X_t against the expansion thresholds P_t and the switching thresholds P_t^s in the adjusted continuous model of Pindyck (1988), and the cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^s in the discrete model with switching options for $\alpha = 0.1$, $\alpha = 0.01$, and $\alpha = 0.0001$. The data for the continuous model is from Table 3.1 which is solved by equation (3.32) and (3.33); the data for the discrete model is from Table 5.2 which is solved by discount factor methodology shown in equations (5.17) and (5.24).

Figure 5.3 Expansion and Switching Thresholds and Cumulative Capitals
($r = 0.05, \delta = 0.05, \sigma = 0.1, c = 3.70, p = -2.70, \gamma = 1, P_0 = 24.49, K_0 = 1$)



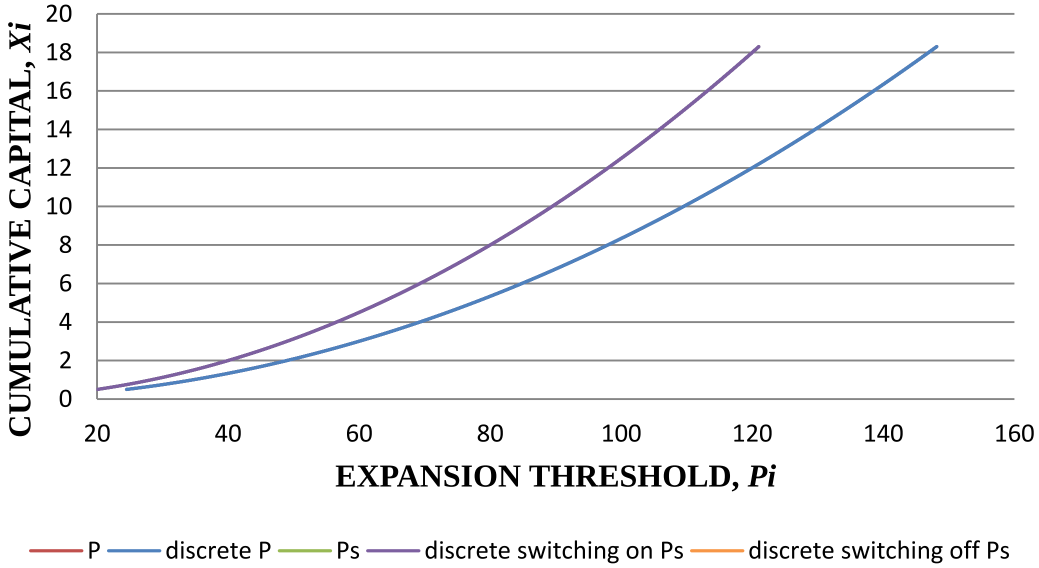
This figure plots the smooth curves and the discrete staircases of cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^s for a discrete model with $\alpha = 0.1$. The red line presents the smooth curves of X_i against P_i; the blue line presents the discrete staircases of X_i against P_i; the green line presents the smooth curves of X_i against P_i^s; the grape line presents the switching on staircases of X_i against P_i^s; The orange line presents the switching off staircases of X_i against P_i^s. The data are solved by discount factor methodology shown in equations (5.17) and (5.24).

Figure 5.4 Expansion and Switching Thresholds and Cumulative Capitals
($r = 0.05, \delta = 0.05, \sigma = 0.1, c = 3.70, p = -2.70, \gamma = 1, P_0 = 24.49, K_0 = 1$)



This figure plots the smooth curves and the discrete staircases of cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^s for a discrete model with $\alpha = 0.01$. The red line presents the smooth curves of X_i against P_i; the blue line presents the discrete staircases of X_i against P_i; the green line presents the smooth curves of X_i against P_i^s; the grape line presents the switching on staircases of X_i against P_i^s; The orange line presents the switching off staircases of X_i against P_i^s. The data are solved by discount factor methodology shown in equations (5.17) and (5.24).

Figure 5.5 Expansion and Switching Thresholds and Cumulative Capitals
$(r = 0.05, \delta = 0.05, \sigma = 0.1, c = 3.70, p = -2.70, \gamma = 1, P_0 = 24.49,\ K_0 = 1)$



This figure plots the smooth curves and the discrete staircases of cumulative capital X_i against the expansion thresholds P_i and the switching thresholds P_i^s for a discrete model with $\alpha = 0.0001$. The red line presents the smooth curves of X_i against P_i; the blue line presents the discrete staircases of X_i against P_i; the green line presents the smooth curves of X_i against P_i^s; the grape line presents the switching on staircases of X_i against P_i^s; The orange line presents the switching off staircases of X_i against P_i^s. The data are solved by discount factor methodology shown in equations (5.17) and (5.24).

VI. Conclusion

Most previous studies on real options are confined to the realm of continuous modelling. However, investment activities in real world are usually discrete, making it necessary and valuable to look into discrete models. Nevertheless, the techniques for real option problems in discrete cases are much less developed and prevent researchers from stepping into modelling the discrete capacity choice problems. The discount factor methodology provides us a valuable approach to solve a discrete model while self-similarity is required to solve for an unbounded system with infinite steps. Using the discount factor methodology and self-similarity, we design and solve two discrete capacity choice models respectively without and with switching options. Dxit and Pindyck (1994) and Pindyck (1988) also respectively propose two continuous models for the capacity choice problems without and with switching options. Modifying the assumptions of Dxit and Pindyck (1994) and Pindyck (1988), we adjust the models in these two works and establish two paralleled continuous models which have exact the same assumptions and structures as our discrete models except for the continuous and discrete settings. Our numerical results show that the discrete models could soundly converge to the paralleled continuous models when the discrete threshold intervals diminish, supporting the validity of the discount factor methodology.

The discount factor methodology in conjunction with the scaling method could be extended to more situations such as disinvestment. In such case, two-way discount factor methodology (developed by Ekern, Shackleton, and Sødal, 2014) is required to establish the discrete model. Additionally, the reversibility of the switching opportunities might also be restricted as proposed by Ekern (1993). The discount factor methodology has more potential in future studies on exploring the discrete capacity choice problems.

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Appendix

Appendix I

For a monopoly firm, it could reduce quantity to raise price and the profit has a form of

$$\pi = P(Q) \times Q - C(Q) \quad \text{(A1.1)}$$

where π is the economic profit, $P(Q)$ is the underlying price, Q is the quantity sold, and $C(Q)$ is the total cost of producing Q. To maximize the profit π, the first order condition with respect to Q should equal 0:

$$P'(Q) \times Q + P(Q) - C'(Q) = 0 \quad \text{(A1.2)}$$

where $P'(Q)$ is the marginal revenue and $C'(Q)$ is the marginal cost. Therefore,

$$P'(Q) \times Q + P(Q) = C'(Q) \quad \text{(A1.3)}$$

The right hand side of the above equation represents the marginal revenue and the left hand side represents the marginal cost. Transforming the equation, we yield:

$$\frac{P - C'(Q)}{P} = -\frac{P'(Q) \times Q}{P} = 1/\epsilon_P \quad \text{(A1.4)}$$

where $\epsilon_P = -P/(P'\,(Q) \times Q)$ is the price elasticity of demand.

Appendix II

Suppose that P follows a general Ito process of the form

$$dP = f(P)dt + g(P)dz \quad \text{(A2.1)}$$

We want $D(P, P^*) = E[e^{-rT}]$, where T is the hitting time to P^*. Over an interval dt, P will change by a small, random amount dP. Therefore (suppressing P^*),

$$D(P) = e^{-rt}E[D(P + dP)] \quad \text{(A2.2)}$$

Expanding $D(V + dV)$ using Ito's Lemma, noting that $e^{-rt} = 1 - rdt$ for small dt, and substituting (I) for dV yields the following differential equation for the discount factor:

$$\frac{1}{2}g^2(P)D_{PP} + f(P)D_P - \rho D \quad \text{(A2.3)}$$

For simplicity, suppose P follows a GBM under a risk-neutral measure:

$$dP = (r - \delta)Pdt + \sigma Pdz \quad \text{(A2.4)}$$

where r is the risk free rate and δ is the yield. Over an interval dt, P will change by a small, random amount dP. Therefore,

$$D(P,P^*) = e^{-rdt}E^{RN}[D(P+dP,P^*)] \quad \text{(A2.5)}$$

Expanding $D(P+dP,P^*)$ using Ito's Lemma, the following differential equation for the discount factor can be yielded:

$$\frac{1}{2}\sigma^2P^2\frac{\partial^2D(P,P^*)}{\partial P^2} + (r-\delta)P\frac{\partial D(P,P^*)}{\partial P} - rD(P,P^*) = 0 \quad \text{(A2.6)}$$

This is a homogeneous linear equation of second order, so its solution is a linear combination of any two linear independent solutions:

$$D(P,P^*) = A_1P^c + A_2P^p \quad \text{(A2.7)}$$

where A_1 and A_2 are constants to be determined. c and p are respectively the positive and the negative root of the following quadratic equation in β:

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0 \quad \text{(A2.8)}$$

Therefore,

$$c = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} > 1 \quad \text{(A2.9)}$$

$$p = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} < 0 \quad \text{(A2.10)}$$

Since we have three unknowns, including A_1, A_2, and P^*, we need three conditions to complete the solution.

For a call option as demonstrated in Dixit, Pindyck and Sødal (1999), the option should be almost worthless when P equals zero. This is derived from the stochastic process (A2.4) that P will stay at zero when P goes to zero. Economically the threshold P^* is too remote to reach in such case. Correspondingly, the discount factor should also be zero in such case. To ensure the condition holds, the coefficient of the negative power of P should be equal to zero, namely $A_2 = 0$. Additionally, the solution should satisfy the condition at the transition point P^* that the discount factor will become unit when P reaches the threshold, namely $D(P^*,P^*) = 1$. Therefore,

$$D(P^*,P^*) = A_1P^{*c} = 1, \qquad A_1 = \frac{1}{P^{*c}} \quad \text{(A2.11)}$$

Thus, the discount factor for a call option under a GBM is in a form of

$$D(P,P^*) = \left(\frac{P}{P^*}\right)^c \quad \text{(A2.12)}$$

For a put option, in contrast, the likelihood of abandonment or suspension should become extremely small as P goes to infinity. Therefore, the option value should approach zero as $P \to \infty$. Correspondingly, the discount factor should also be zero in such case. To ensure the condition holds, the coefficient A_1 corresponding to the positive root c should be zero. Additionally, the solution should satisfy the condition at the transition point P^* that the discount factor will become unit when P reaches the threshold, namely $D(P^*, P^*) = 1$. Therefore,

$$D(P^*, P^*) = A_2 P^{*p} = 1, \qquad A_2 = \frac{1}{P^{*p}} \tag{A2.13}$$

Thus, the discount factor for a put option under a GBM is in a form of

$$D(P, P^*) = \left(\frac{P}{P^*}\right)^p \tag{A2.14}$$

Appendix III

Processes	RN diffusion dP	Call/put discount functions	β quadratic for c, p
Geometric Brownian motion	$(r-\delta)Pdt + \sigma P dz$	$D_c(P,P^*) = \left(\frac{P}{P^*}\right)^c : P \le P^*$ $D_p(P,P^*) = \left(\frac{P}{P^*}\right)^p : P \ge P^*$	$\frac{\sigma^2}{2}\beta(\beta-1) + (r-\delta)\beta - r = 0$
Arithmetic Brownian notion	$\alpha dt + \sigma dz$	$D_c(P,P^*) = e^{c(P-P^*)}: P \le P^*$ $D_p(P,P^*) = e^{p(P-P^*)}: P \ge P^*$	$\frac{\sigma^2}{2}\beta^2 + \alpha\beta - r = 0$
Mean reverting	$\kappa(\bar{P} - P)dt + \sigma P dz$	$D_p(P,P^*) = \frac{\left(\frac{P}{P^*}\right)^p M\left(-p, 2+2p+\frac{2\kappa}{\sigma^2}, P\frac{2\kappa\bar{P}}{\sigma^2}\right)}{M\left(-p, 2+2p+\frac{2\kappa}{\sigma^2}, \bar{P}\frac{2\kappa\bar{P}}{\sigma^2}\right)}$	$\frac{\sigma^2}{2}\beta(\beta-1) + \kappa\beta - r = 0$

Discount factors for Geometric, Arithmetic and Mean Reverting processes. M is the confluent hypergeometric function, used for example in Sarkar and Zapatero (2003).

Appendix IV

Equation (4.9) provides the solution for the operating costs at the switching thresholds.

$$\underline{C_i} = \left(\underline{DS_i} - \underline{I}\right)\left(\underline{\beta_N} - \underline{\beta_M}\,\underline{DS_i}\right)^{-1} \underline{\beta_T}\,\underline{T_i} + \underline{T_i} \tag{A4.1}$$

where

$$\underline{C_i} = \begin{bmatrix} C_{i+}(K_i - K_{i-1}) \\ -C_{i-}(K_i - K_{i-1}) \end{bmatrix}, \qquad \underline{DS_i} = \begin{bmatrix} 0 & D(P_{i+}, P_{i-}) \\ D(P_{i-}, P_{i+}) & 0 \end{bmatrix}, \qquad \underline{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\beta_W} = \begin{bmatrix} c & 0 \\ 0 & p \end{bmatrix}, \qquad \underline{\beta_U} = \begin{bmatrix} p & 0 \\ 0 & c \end{bmatrix}, \qquad \underline{\beta_T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{T_i} = \begin{bmatrix} P_{i+}(K_i - K_{i-1}) \\ -P_{i-}(K_i - K_{i-1}) \end{bmatrix}, \qquad D(P_-, P_+) = \left(\frac{P_{i-}}{P_{i+}}\right)^c, \qquad D(P_+, P_-) = \left(\frac{P_{i+}}{P_{i-}}\right)^p$$

If we assume $P_{i+} = P_{i-} + \varepsilon$, then

$$D(P_{i-}, P_{i+}) = \left(\frac{P_{i-}}{P_{i-} + \varepsilon}\right)^c, \qquad D(P_{i+}, P_{i-}) = \left(\frac{P_{i-} + \varepsilon}{P_{i-}}\right)^p \tag{A4.2}$$

When $\varepsilon \to 0$, the two way switching thresholds would converge to one reversible switching threshold. Solving the matrix expression above, we could respectively obtain the expressions for C_+ and for C_-.

$$\begin{aligned} &\lim_{\varepsilon\to 0} C_{i+} \\ &= \lim_{\varepsilon\to 0}\left\{\frac{p(c-1) + c(1-p)P_{i-}{}^{c-p}(P_{i-}+\varepsilon)^{p-c} + (p-c)P_{i-}{}^{-p}(P_{i-}+\varepsilon)^p}{[1-(1-p)P_{i-}{}^{c-p}(P_{i-}+\varepsilon)^{p-c}]cp} P_{i-} \right. \\ &\left. + \left\{\frac{-p + cP_{i-}{}^{c-p}(P_{i-}+\varepsilon)^{p-c}}{[1-(1-p)P_{i-}{}^{c-p}(P_{i-}+\varepsilon)^{p-c}]cp} + 1\right\}\varepsilon\right\} \end{aligned} \tag{A4.3}$$

$$\begin{aligned} &\lim_{\varepsilon\to 0} C_{i-} \\ &= \lim_{\varepsilon\to 0}\left\{\frac{c(1-p) + p(c-1)P_{i-}{}^{c-p}(P_{i-}+\varepsilon)^{p-c} + (p-c)P_{i-}{}^{c}(P_{i-}+\varepsilon)^{-c}}{[1-(1-p)P_{i-}{}^{c-p}(P_{i-}+\varepsilon)^{p-c}]cp} P_{i-} \right. \\ &\left. + \frac{(p-c)P_{i-}{}^{c}(P_{i-}+\varepsilon)^{-c}}{[1-(1-p)P_{i-}{}^{c-p}(P_{i-}+\varepsilon)^{p-c}]cp}\varepsilon\right\} \end{aligned} \tag{A4.4}$$

The expressions above could be expanded while only the terms with the lowest power of ε matter as to the values of C_{i+} and C_{i-} when ε converges to zero. Therefore, the terms with higher power of ε could be omitted after the expansion. If the terms in numerator have higher power than the terms in denominator, the value of the expression should converge to zero. In the contrary, if the terms in numerator have lower power of ε than the terms in denominator, the value of the expression would become infinitely large. However, if the terms of numerator and denominator have the same power of ε, the value depends on the ratio of the terms' coefficients. Thus, the expressions of C_{i+} and C_{i-} could be simplified in the following way:

$$\begin{aligned} \lim_{\varepsilon\to 0} C_{i+} = \lim_{\varepsilon\to 0}&\left\{\left[\frac{-p(c-p)P_{i-}{}^{c-p-1}\varepsilon + (p-c)cP_{i-}{}^{c-p-1}\varepsilon}{cp(c-p)P_{i-}{}^{c-p-1}\varepsilon} + 1\right]P_{i-} \right. \\ &\left. + \frac{(c-p)P_{i-}{}^{c-p}\varepsilon}{cp(c-p)P_{i-}{}^{c-p-1}\varepsilon}\right\} \end{aligned} \tag{A4.5}$$

Therefore,

$$\begin{aligned} \lim_{\varepsilon\to 0} C_{i+} &= \left[\frac{-p(c-p) + (p-c)c}{cp(c-p)} + 1\right]P_{i-} + \frac{(c-p)}{cp(c-p)}P_{i-} \\ &= \lim_{\varepsilon\to 0}\left[\frac{(1-c)(1-p)}{cp}(P_{i+} - \varepsilon)\right] = \frac{(1-c)(1-p)}{cp}P_{i+} \end{aligned} \tag{A4.6}$$

Therefore,

$$\frac{C_{i+}}{P_{i+}} = \frac{(1-c)(1-p)}{cp} \tag{A4.7}$$

Since c and p are respectively two roots of the following quadratic equation,

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0 \tag{A4.8}$$

therefore,

$$c, p = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \pm \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{A4.9}$$

$$\frac{C_{i+}}{P_{i+}} = \frac{(1-c)(1-p)}{cp} = \frac{1-(c+p)+cp}{cp} = \frac{1-\left(1+2\frac{r-\delta}{\sigma^2}\right)+\frac{2r}{\sigma^2}}{\frac{2r}{\sigma^2}} = \frac{\delta}{r} \tag{A4.10}$$

Similarly, the expression of C_{i-} could be expanded and left the terms with lowest power of ε.

$$\begin{aligned} C_{i-} &= \lim_{\varepsilon\to 0}\left\{\left[-\frac{-p(p-c){P_{i-}}^{c-p-1}\varepsilon + (c-p)c{P_{i-}}^{c-p-1}\varepsilon}{cp(c-p){P_{i-}}^{c-p-1}\varepsilon} + 1\right]P_{i-} + \frac{1}{cp}P_{i-}\right\} \\ &= \left[-\frac{-p(p-c)+(c-p)c}{cp(c-p)} + 1\right]P_{i-} + \frac{1}{cp}P_{i-} \\ &= \frac{cp-c-p+1}{cp}P_{i-} \end{aligned} \tag{A4.11}$$

Therefore,

$$\frac{C_{i-}}{P_{i-}} = \frac{(1-c)(1-p)}{cp} = \frac{\delta}{r} \tag{A4.12}$$

Equations (A5.10) and (A5.12) are just the markup pricing rule that the revenue is equal to the operating cost at the switching thresholds according to footnotes 1 and 2. Since P is the dynamic perpetual revenue and C is the dynamic perpetual operating cost, the markup pricing rule could be expressed as

$$\delta P = rC \tag{A4.13}$$

Therefore, when P_{i+} and P_{i-} converge, C_{i+} and C_{i-} also converge and follows the markup pricing rule, keeping a constant scaling ratio with the switching threshold.