

ON A MOLLIFIER OF THE PERTURBED RIEMANN ZETA-FUNCTION

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ABSTRACT. The mollification $\zeta(s) + \zeta'(s)$ put forward by Feng is computed by analytic methods coming from the techniques of the ratios conjectures of L -functions. The current situation regarding the percentage of non-trivial zeros of the Riemann zeta-function on the critical line is then clarified.

1. INTRODUCTION

1.1. Statement of the results. The Riemann zeta-function $\zeta(s)$ is defined by the Dirichlet series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s = \sigma + it$, $\sigma > 1$ and $t \in \mathbb{R}$. The functional equation of $\zeta(s)$ is given by

$$\zeta(s) = \xi(1-s),$$

where

$$\xi(s) = H(s)\zeta(s) \quad \text{and} \quad H(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

This allows us to perform a meromorphic continuation to the whole complex plane except at $s = 1$ where $\zeta(s)$ has a simple pole with residue equal to 1. The connection with number theory comes from the Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

for $\operatorname{Re}(s) > 1$, and where the product is taken over all the primes p . It is well-known from Riemann and from von Mangoldt that the non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ are located inside the critical strip $0 < \beta < 1$. Moreover, if $N(T)$ denotes the number of such zeros up to height $0 \leq \gamma < T$ then

$$N(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - 1 \right) + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

where

$$S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right) \ll \log T,$$

as $T \rightarrow \infty$, see e.g. [13, 16] for properties of $\zeta(s)$. To state the results, we let $N_0(T)$ denote the number of non-trivial zeros up to height $T > 0$ such that $\beta = 1/2$. Similarly, let $N_0^*(T)$ denote the number such zeros which are also simple. We then define

$$\kappa = \liminf_{T \rightarrow \infty} \frac{N_0(T)}{N(T)} \quad \text{and} \quad \kappa^* = \liminf_{T \rightarrow \infty} \frac{N_0^*(T)}{N(T)}.$$

The history behind the value of κ can be found in [3, 8, 14]. The main breakthroughs were as follows. In 1942, Selberg [15] established that $0 < \kappa \leq 1$. Levinson later showed in 1974 that $\kappa \geq .3474$. This was improved by Conrey to $\kappa \geq .4088$ in 1989 and later refined by Bui, Conrey and Young [3] to $\kappa \geq .4105$, and shortly afterward by Feng [8] to $\kappa \geq .4127$. It should be noted that both results are improvements of $\kappa \geq .4088$ and are independent of each other.

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The second author, Roy and Zaharescu [14] as well as Bui [2] brought up a point regarding the strength of Feng's result. In [14], it was explained that $\kappa \geq .4107$, unconditionally, using Feng's mollifier. However, the computation of the mixed terms of the mollifiers of Conrey and of Feng was not carried through explicitly.

It should also be remarked that Bui [2] suggests that the bound obtained in this paper can be attained using the twisted second moment of the Riemann zeta-function due to Balasubramanien *et. al.* [1] and that he also suggests an alternative argument that could lead to the bound $\kappa > .41098$.

In this paper, we close this gap and we explain Feng's brilliant choice in the context of the powerful technology developed in [3, 17]. These ideas come from the ratios conjectures of L -functions due to Conrey, Farmer and Zirnbauer [6] as well as to Conrey and Snaith [7]. It should be noted that Feng's methodology to obtain the main terms of his theorem consisted on an ingenious combination of elementary methods, namely induction and Mertens' formula, applied to Conrey's result [5]. On the other hand, this choice of methods blurred a bit the length the mollifier was allowed to take. Other than choosing the same mollifier, our computations do not overlap and the methods are quite different.

Lastly, the closing of this gap will clarify the situation of the percentage of non-trivial zeros on the critical line when one attaches Feng's second-piece mollifier to Conrey's.

1.2. Choice of mollifiers. Let $Q(x)$ be a real polynomial satisfying $Q(0) = 1$, $Q(x) + Q(1-x) = \text{constant}$, and define

$$(1.1) \quad V(s) = Q\left(-\frac{1}{L} \frac{d}{ds}\right) \zeta(s),$$

where for large T ,

$$L = \log T.$$

If $\psi(s)$ is a mollifier, then it is well-known from the work of Levinson [12] and of Conrey [5] that Littlewood's lemma [16, §9.9] followed by the arithmetic and geometric mean inequalities yields

$$(1.2) \quad \kappa \geq 1 - \frac{1}{R} \log \left(\frac{1}{T} \int_1^T |V\psi(\sigma_0 + it)|^2 dt \right) + o(1),$$

where $\sigma_0 = 1/2 - R/L$, and R is a bounded positive real number to be chosen later. Following Feng [8], we will choose a mollifier of the form

$$\psi(s) = \psi_1(s) + \psi_2(s),$$

where ψ_1 is the mollifier considered by Conrey. Let $P_1(x) = \sum_j a_j x^j$ be a certain polynomial satisfying $P_1(0) = 0$, $P_1(1) = 1$, and let $y_1 = T^{\theta_1}$ where $0 < \theta_1 < 4/7$. We adopt the notation

$$P_1[n] = P_1\left(\frac{\log(y_1/n)}{\log y_1}\right)$$

for $1 \leq n \leq y_1$. By convention, we set $P_1[x] = 0$ for $x \geq y_1$. Then $\psi_1(s)$ is given by

$$(1.3) \quad \psi_1(s) = \sum_{h \leq y_1} \frac{\mu(h) h^{\sigma_0 - 1/2}}{h^s} P_1[h],$$

where $\mu(n)$ is the Möbius function. For the second mollifier, we take

$$(1.4) \quad \psi_2(s) = \sum_{k \leq y_2} \frac{\mu(k) k^{\sigma_0 - 1/2}}{k^s} \sum_{\ell=2}^K \sum_{p_1 \cdots p_\ell | k} \frac{\log p_1 \cdots \log p_\ell}{\log^\ell y_2} P_\ell[k].$$

Here $K \geq 2$ is an integer of our choice and p_1, \dots, p_ℓ are distinct primes. Also we need $P_\ell(0) = 0$ for $\ell = 2, \dots, K$. In this case $y_2 = T^{\theta_2}$ where $0 < \theta_2 < 1/2$.

Remark 1.1. It will become clear in the calculation of the crossterm integral between ψ_1 and ψ_2 that one needs $\theta_1 + \theta_2 < 1 - \varepsilon$. Therefore, if θ_1 increases, then θ_2 decreases unless some difficult work is done to push θ_2 back to its original (or higher) value. See the comments between Theorem 1.1 and Theorem 1.2 for more details.

The reason behind this choice is that Feng wishes to mollify not only $\zeta(s)$ but also $\frac{\zeta'(s)}{\log T}$, which is the second term coming from (1.1). This is accomplished by looking at

$$(1.5) \quad \frac{1}{\zeta(s) + \frac{\zeta'(s)}{\log T}} = \frac{1}{\zeta(s)} - \frac{1}{\log T} \frac{\zeta'}{\zeta^2}(s) + \frac{1}{\log^2 T} \frac{(\zeta')^2}{\zeta^3}(s) - \frac{1}{\log^3 T} \frac{(\zeta')^3}{\zeta^4}(s) + \dots$$

When k is a square-free positive integer, then one has

$$(\mu * \Lambda^{*\ell})(k) = (-1)^\ell \mu(k) \sum_{p_1 \cdots p_\ell | k} \log p_1 \cdots \log p_\ell,$$

where $f * g$ denotes the Dirichlet convolution of arithmetic functions f and g . Here $\Lambda^{*\ell}$ stands for convolving the von Mangoldt function $\Lambda(n)$ with itself exactly ℓ times. If k contains a square divisor, then, as remarked by Feng [8], the coefficients a_j resulting from (1.5) contribute a lower order to the mean value integrals I_{11} , I_{12} and I_{22} related to κ in (1.2) (see below for exact definitions of these I -integrals).

1.3. Numerical evaluations. We will prove the following.

Theorem 1.1. *We obtain with $\theta_1 = \theta_2 = 1/2 - \varepsilon$*

$$\kappa \geq .369927 \quad \text{and} \quad \kappa^* \geq .359991,$$

unconditionally.

Using the work of Iwaniec and Deshouillers [10, 11], Conrey [5] was able to push the size of the mollifier ψ_1 to $\theta_1 < 4/7 - \varepsilon$. In the light of Lemma 2.1 and (3.9) below, we require $\theta_1 + \theta_2 < 1$ in our argumentation. The points brought up in [2] and [14] show that some difficult work is needed if one takes $\theta_1 + \theta_2 > 1$. Theorem 1.1 utilizes $\theta_1, \theta_2 < 1/2 - \varepsilon$. However, if we take $\theta_1 < 4/7 - \varepsilon$ and $\theta_2 < 3/7 - \varepsilon$, then we get

Theorem 1.2. *We obtain with $\theta_1 < 4/7 - \varepsilon$ and $\theta_2 < 3/7 - \varepsilon$*

$$\kappa \geq .410725 \quad \text{and} \quad \kappa^* \geq .403211,$$

unconditionally.

It should therefore be stressed that Theorem 1.1 is an improvement of the last theorem to ever use $\theta_1 = 1/2 - \varepsilon$, namely the first corollary of [4], where it was shown that $\kappa \geq .3658$.

The method sketched in [3, 14] to deal with multiple piece mollifiers carries through and our main result is as follows.

Theorem 1.3. *Suppose that $\theta_1 + \theta_2 = 1 - \varepsilon$ with $\theta_1 < 4/7$ and $\theta_2 < 1/2$ and $\varepsilon > 0$ small. Then*

$$\frac{1}{T} \int_1^T |V\psi(\sigma_0 + it)|^2 dt = c(P_1, P_\ell, Q, R, \theta_1, \theta_2) + o(1),$$

where $c(P_1, P_\ell, Q, R, \theta_1, \theta_2) = c_{11} + 2c_{12} + c_{22}$ and the c_{ij} are given by (1.6), (1.7) and (1.8).

We use `Mathematica` to numerically evaluate $c(P_1, P_\ell, Q, R, 1/2, 1/2)$ with the following choices of parameters. With $K = 3$, $R = 1.3$,

$$\begin{aligned} Q(x) &= .481936 + .632349(1 - 2x) - .144698(1 - 2x)^3 + .0304136(1 - 2x)^5, \\ P_1(x) &= x + .225339x(1 - x) - 1.01137x(1 - x)^2 + .174004x(1 - x)^3 - .100235x(1 - x)^4, \\ P_2(x) &= 1.05138x + .284201x^2, \\ P_3(x) &= .222032x - .13254x^2, \end{aligned}$$

we have $\kappa \geq .369927$. To get $\kappa^* \geq .359991$, we take $K = 3$, $R = 1.2$,

$$\begin{aligned} Q(x) &= .476202 + .523798(1 - 2x), \\ P_1(x) &= x + .0531913x(1 - x) - .594999x(1 - x)^2 - .00107597x(1 - x)^3 - .0761954x(1 - x)^4, \\ P_2(x) &= .896567x - .0297464x^2, \\ P_3(x) &= .0699271x - .108964x^2. \end{aligned}$$

We also use `Mathematica` to numerically evaluate $c(P_1, P_\ell, Q, R, 4/7, 3/7)$ with the following choices of parameters. With $K = 3$, $R = 1.295$,

$$\begin{aligned} Q(x) &= .492203 + .621972(1 - 2x) - .148163(1 - 2x)^3 + .033988(1 - 2x)^5, \\ P_1(x) &= x + .229117x(1 - x) - 2.932318x(1 - x)^2 + 4.856163x(1 - x)^3 - 2.390999x(1 - x)^4 \\ P_2(x) &= -.072644x + 1.559440x^2 \\ P_3(x) &= .701568x - .554403x^2 \end{aligned}$$

we have $\kappa \geq .410725$. To get $\kappa^* \geq .403211$, we take $K = 3$, $R = 1.109$,

$$\begin{aligned} Q(x) &= .485034 + .514966(1 - 2x), \\ P_1(x) &= x + .0486916x(1 - x) - 2.02526x(1 - x)^2 + 3.43611x(1 - x)^3 - 1.62355x(1 - x)^4, \\ P_2(x) &= -.034431x + 1.09223x^2, \\ P_3(x) &= .479296x - 0.385868x^2. \end{aligned}$$

An interesting question to ask is: what would have happened if Feng had published his mollifier before Conrey's increment of θ_1 from $1/2$ to $4/7$. Since this has not been remarked before in the literature, we take the chance to answer it. If ψ_1 and ψ_2 are kept at $1/2 - \varepsilon$, then Feng's piece adds an additional 0.4127% to Conrey's 36.58% as shown in the table below.

θ_1	θ_2	%
1/2	1/2	36.58% + 0.4127%
4/7	3/7	40.88% + 0.1925%

TABLE 1. % according to sizes of θ

Since ψ_2 is the perturbation of ψ_1 , it behooves us to take θ_1 as large as possible ($4/7$) at the cost of sacrificing θ_2 to $3/7$ which only adds 0.1925%.

1.4. The smoothing argument. The idea of smoothing the mean value integrals was introduced in [3, 17] and it helps substantially in our calculations. Let $w(t)$ be a smooth function satisfying the following properties:

- (a) $0 \leq w(t) \leq 1$ for all $t \in \mathbb{R}$,
- (b) w has compact support in $[T/4, 2T]$,
- (c) $w^{(j)}(t) \ll_j \Delta^{-j}$, for each $j = 0, 1, 2, \dots$ and where $\Delta = T/L$.

This allows us to re-write Theorem 1.3 as follows.

Theorem 1.4. *Suppose that $\theta_1 = 1/2 - \varepsilon$ and $\theta_2 = 1/2 - \varepsilon$ for $\varepsilon > 0$ small. For any w satisfying conditions (a), (b) and (c) and $\sigma_0 = 1/2 - R/L$,*

$$\int_{-\infty}^{\infty} w(t)|V\psi(\sigma_0 + it)|^2 dt = c(P_1, P_\ell, Q, R, \theta_1, \theta_2)\widehat{w}(0) + O(T/L),$$

uniformly for $R \ll 1$, where $c(P_1, P_\ell, Q, R, \theta_1, \theta_2) = c_{11} + 2c_{12} + c_{22}$ and the c_{ij} are given by (1.6), (1.7) and (1.8).

How to deal with a two-piece mollifier was explained in [3, 8]. In [14] a 4-piece mollifier was studied. The idea is to open the square in the integrand to get

$$\begin{aligned} \int |V\psi|^2 &= \int |V\psi_1|^2 + \int |V|^2\psi_1\overline{\psi_2} + \int |V|^2\overline{\psi_1}\psi_2 + \int |V\psi_2|^2 \\ &= I_{11} + I_{12} + \overline{I_{12}} + I_{22}. \end{aligned}$$

We will compute these integrals in the next sections. The integral I_{12} is asymptotically real, thus I_{21} follows from I_{12} , i.e. $I_{12} \sim \overline{I_{21}}$.

1.5. The main terms. The main terms coming from integrals I_{11} , I_{12} and I_{22} are now stated as theorems.

Theorem 1.5 (Conrey). *Suppose $\theta_1 < 4/7$. Then*

$$\int_{-\infty}^{\infty} w(t)|V\psi_1(\sigma_0 + it)|^2 dt \sim c_{11}(P_1, Q, R, \theta_1)\widehat{w}(0) + O(T/L)$$

uniformly for $R \ll 1$, where

$$(1.6) \quad c_{11}(P_1, Q, R, \theta_1) = 1 + \frac{1}{\theta_1} \int_0^1 \int_0^1 e^{2Rv} \left(\frac{d}{dx} e^{R\theta x} P_1(x+u) Q(v+\theta x) \Big|_{x=0} \right)^2 dudv.$$

Let $(\ell)_k = \ell(\ell-1)\cdots(\ell-k+1)$ denote the Pochhammer symbol.

Theorem 1.6. *Suppose $\theta_1 + \theta_2 = 1 - \varepsilon$ where $\theta_1 < 4/7$ and $\theta_2 < 1/2$. Then*

$$\int_{-\infty}^{\infty} w(t)V\psi_1\overline{\psi_2}(\sigma_0 + it)dt \sim c_{12}(P_1, P_\ell, Q, R, \theta_1, \theta_2)\widehat{w}(0) + O(T/L)$$

uniformly for $R \ll 1$, where

$$(1.7) \quad \begin{aligned} c_{12}(P_1, P_\ell, Q, R, \theta_1, \theta_2) &= \sum_{\ell=2}^K \frac{(-1)^\ell}{(\ell-1)!} \int_0^1 (1-u)^{\ell-1} P_1(u) P_\ell(u) du \\ &\quad - \frac{\theta_1 - \theta_2}{\theta_1} \sum_{\ell=2}^K \frac{(-1)^\ell}{\ell!} \int_0^1 (1-u)^{\ell-1} P_1' \left(1 - (1-u)\frac{\theta_2}{\theta_1} \right) P_\ell(u) du \\ &\quad + \frac{1}{\theta_1} \sum_{\ell=2}^K \frac{(-1)^\ell}{\ell!} \frac{d^2}{dx dy} \left[e^{R(\theta_1 x + \theta_2 y)} \int_0^1 \int_0^1 e^{2Rv} (1-u)^\ell \right. \\ &\quad \left. \times P_1 \left(x + 1 - (1-u)\frac{\theta_2}{\theta_1} \right) P_\ell(y+u) Q(\theta_2 y + v) Q(\theta_1 x + v) dudv \Big|_{x=y=0} \right]. \end{aligned}$$

Theorem 1.7. *Suppose $\theta_2 < 1/2$. Then*

$$\int_{-\infty}^{\infty} w(t)|V\psi_2(\sigma_0 + it)|^2 dt \sim c_{22}(P_\ell, Q, R, \theta_2)\widehat{w}(0) + O(T/L)$$

uniformly for $R \ll 1$, where

$$\begin{aligned}
c_{22}(P_\ell, Q, R, \theta_2) &= \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{k=0}^{\min(\ell_1, \ell_2)} (-1)^{\ell_1+\ell_2-2k} \binom{\ell_1}{k} (\ell_2)_k \\
&\quad \times \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2-1)!} \int_0^1 (1-u)^{\ell_1+\ell_2-1} P_{\ell_1}(u) P_{\ell_2}(u) du \\
&\quad + \frac{1}{\theta_2} \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{k=0}^{\min(\ell_1, \ell_2)} (-1)^{\ell_1+\ell_2-2k} \binom{\ell_1}{k} (\ell_2)_k \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2)!} \frac{d^2}{dx dy} \left[e^{R\theta_2(x+y)} \right. \\
(1.8) \quad &\quad \times \left. \int_0^1 \int_0^1 e^{2Rv} (1-u)^{\ell_1+\ell_2} P_{\ell_1}(x+u) P_{\ell_2}(y+u) Q(v+\theta_2 x) Q(v+\theta_2 y) dudv \right]_{x=y=0}.
\end{aligned}$$

Remark 1.2. Note that in [8], c_{11} , c_{12} and c_{22} are all mixed into one single theorem and it is not immediately clear how to separate each individual c -term.

The smoothing argument is helpful because we can easily deduce Theorem 1.3 from Theorem 1.4 and so on. By having chosen $w(t)$ to satisfy conditions (a), (b) and (c) and in addition to being an upper bound for the characteristic function of the interval $[T/2, T]$, and with support $[T/2 - \Delta, T + \Delta]$, we get

$$\int_{T/2}^T |V\psi(\sigma_0 + it)|^2 dt \leq c(P_1, P_\ell, Q, R, \theta_1, \theta_2) \widehat{w}(0) + O(T/L).$$

Note that $\widehat{w}(0) = T/2 + O(T/L)$. We similarly get a lower bound. Summing over dyadic segments gives the full result.

1.6. The shift parameters α and β . Rather than working directly with $V(s)$, we shall instead consider the following three general shifted integrals

$$\begin{aligned}
I_{11}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \overline{\psi_1} \psi_1(\sigma_0 + it) dt, \\
I_{12}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \overline{\psi_1} \psi_2(\sigma_0 + it) dt, \\
I_{22}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \overline{\psi_2} \psi_2(\sigma_0 + it) dt.
\end{aligned}$$

The computation is now reduced to proving the following three lemmas.

Lemma 1.1. *We have*

$$I_{11} = c_{11}(\alpha, \beta) \widehat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, where

$$(1.9) \quad c_{11}(\alpha, \beta) = 1 + \frac{1}{\theta_1} \frac{d^2}{dx dy} \left[y_1^{-\beta x - \alpha y} \int_0^1 \int_0^1 T^{-v(\alpha+\beta)} P_1(x+u) P_1(y+u) dudv \right]_{x=y=0}.$$

Lemma 1.2. *We have*

$$I_{12} = c_{12}(\alpha, \beta) \widehat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, where

$$c_{12}(\alpha, \beta) = \sum_{\ell=2}^K \frac{(-1)^\ell}{(\ell-1)!} \int_0^1 (1-u)^{\ell-1} P_1(u) P_\ell(u) du$$

$$\begin{aligned}
& -\frac{\theta_1 - \theta_2}{\theta_1} \sum_{\ell=2}^K \frac{(-1)^\ell}{\ell!} \int_0^1 (1-u)^\ell P'_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(u) du \\
& + \frac{1}{\theta_1} \sum_{\ell=2}^K \frac{(-1)^\ell}{\ell!} \frac{d^2}{dx dy} \left[y_1^{-\beta x} y_2^{-\alpha y} \right. \\
(1.10) \quad & \left. \times \int_0^1 \int_0^1 T^{-v(\alpha+\beta)} (1-u)^\ell P_1 \left(x+1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(y+u) dudv \right]_{x=y=0}.
\end{aligned}$$

Lemma 1.3. *We have*

$$I_{22} = c_{22}(\alpha, \beta) \widehat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, where

$$\begin{aligned}
c_{22}(\alpha, \beta) &= \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{k=0}^{\min(\ell_1, \ell_2)} (-1)^{\ell_1+\ell_2-2k} \binom{\ell_1}{k} (\ell_2)_k \\
& \times \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2-1)!} \int_0^1 (1-u)^{\ell_1+\ell_2-1} P_{\ell_1}(u) P_{\ell_2}(u) du \\
& + \frac{1}{\theta_2} \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k (-1)^{\ell_1+\ell_2-2k} \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2)!} \\
(1.11) \quad & \times \frac{d^2}{dx dy} \left[y_2^{-\beta x - \alpha y} \int_0^1 \int_0^1 T^{-v(\alpha+\beta)} (1-u)^{\ell_1+\ell_2} P_{\ell_1}(x+u) P_{\ell_2}(y+u) dudv \right]_{x=y=0}.
\end{aligned}$$

To get Theorems 1.5, 1.6 and 1.7 we use the following technique. Let I_\star denote either of the integrals in questions, and note that

$$I_\star = Q \left(-\frac{1}{\log T} \frac{d}{d\alpha} \right) Q \left(-\frac{1}{\log T} \frac{d}{d\beta} \right) I_\star(\alpha, \beta) \Big|_{\alpha=\beta=-R/L}.$$

Since $I_\star(\alpha, \beta)$ and $c_\star(\alpha, \beta)$ are holomorphic with respect to α, β small, the derivatives appearing in the equation above can be obtained as integrals of radii $\asymp L^{-1}$ around the points $-R/L$, using Cauchy's integral formula. Since the error terms hold uniformly on these contours, the same error terms that hold for $I_\star(\alpha, \beta)$ also hold for I_\star . That the above differential operator on $c_\star(\alpha, \beta)$ does indeed give c_\star follows from

$$Q \left(\frac{-1}{\log T} \frac{d}{d\alpha} X^{-\alpha} \right) = Q \left(\frac{\log X}{\log T} \right) X^{-\alpha}.$$

Note that from the above equation we get

$$\begin{aligned}
Q \left(\frac{-1}{\log T} \frac{d}{d\alpha} \right) Q \left(\frac{-1}{\log T} \frac{d}{d\beta} \right) y_1^{-\beta x} y_2^{-\alpha y} T^{-v(\alpha+\beta)} &= Q \left(\frac{\log y_2^y T^v}{\log T} \right) Q \left(\frac{\log y_1^x T^v}{\log T} \right) y_1^{-\beta x} y_2^{-\alpha y} T^{-v(\alpha+\beta)} \\
&= Q(\theta_2 y + v) Q(\theta_1 x + v) y_1^{-\beta x} y_2^{-\alpha y} T^{-v(\alpha+\beta)},
\end{aligned}$$

as well as

$$\begin{aligned}
Q \left(\frac{-1}{\log T} \frac{d}{d\alpha} \right) Q \left(\frac{-1}{\log T} \frac{d}{d\beta} \right) y_2^{-\beta x - \alpha y} T^{-v(\alpha+\beta)} &= Q \left(\frac{\log y_2^y T^v}{\log T} \right) (y_2^y T^v)^{-\alpha} Q \left(\frac{\log y_2^x T^v}{\log T} \right) (y_2^x T^v)^{-\beta} \\
&= Q(\theta_2 y + v) Q(\theta_2 x + v) y_2^{-\beta x - \alpha y} T^{-v(\alpha+\beta)}.
\end{aligned}$$

Hence using the differential operators $Q((-1/\log T)d/d\alpha)$ and $Q((-1/\log T)d/d\beta)$ on the last line of $c_{12}(\alpha, \beta)$ we get

$$\frac{d^2}{dx dy} \left[y_1^{-\beta x} y_2^{-\alpha y} \int_0^1 \int_0^1 T^{-v(\alpha+\beta)} (1-u)^\ell P_1 \left(x+1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(y+u) Q(\theta_2 y+v) Q(\theta_1 x+v) dudv \Big|_{x=y=0} \right].$$

Theorem 1.6 then follows by setting $\alpha = \beta = -R/L$ and using $T^{z/L} = T^{z/\log T} = e^z$. Similarly, when we use the differential operators $Q((-1/\log T)d/d\alpha)$ and $Q((-1/\log T)d/d\beta)$ on the last line of $c_{22}(\alpha, \beta)$ it becomes

$$\frac{d^2}{dx dy} \left[e^{R\theta_2(x+y)} \int_0^1 \int_0^1 e^{2Rv} (1-u)^{\ell_1+\ell_2} P_{\ell_1}(x+u) P_{\ell_2}(y+u) Q(v+\theta_2 x) Q(v+\theta_2 y) dudv \Big|_{x=y=0} \right].$$

The same substitutions yield Theorem 1.7.

2. PRELIMINARY RESULTS

2.1. Results from complex analysis. The following results are needed throughout the paper.

Lemma 2.1. *Suppose that $w(t)$ satisfies conditions (a), (b) and (c) and that h, k are positive integers with $hk \leq T^{2\theta}$ with $\theta < 1/2$, and $\alpha, \beta \ll L^{-1}$. Moreover, set*

$$g_{\alpha,\beta}(s, t) = \pi^{-s} \frac{\Gamma(\frac{1}{2}(\frac{1}{2} + \alpha + s + it)) \Gamma(\frac{1}{2}(\frac{1}{2} + \beta + s - it))}{\Gamma(\frac{1}{2}(\frac{1}{2} + \alpha + it)) \Gamma(\frac{1}{2}(\frac{1}{2} + \beta - it))},$$

as well as

$$X_{\alpha,\beta,t} = \pi^{\alpha+\beta} \frac{\Gamma(\frac{1}{2}(\frac{1}{2} - \alpha - it)) \Gamma(\frac{1}{2}(\frac{1}{2} - \beta + it))}{\Gamma(\frac{1}{2}(\frac{1}{2} + \alpha + it)) \Gamma(\frac{1}{2}(\frac{1}{2} + \beta - it))}.$$

Then one has

$$\begin{aligned} \int_{-\infty}^{\infty} w(t) \left(\frac{h}{k} \right)^{-it} \zeta(\frac{1}{2} + \alpha + it) \zeta(\frac{1}{2} + \beta - it) dt &= \sum_{hm=kn} \frac{1}{m^{1/2+\alpha} n^{1/2+\beta}} \int_{-\infty}^{\infty} V_{\alpha,\beta}(mn, t) w(t) dt \\ &+ \sum_{hm=kn} \frac{1}{m^{1/2-\beta} n^{1/2-\alpha}} \int_{-\infty}^{\infty} V_{-\beta,-\alpha}(mn, t) X_{\alpha,\beta,t} w(t) dt + O_A(T^{-A}), \end{aligned}$$

where

$$V_{\alpha,\beta}(x, t) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} g_{\alpha,\beta}(s, t) x^{-s} ds, \quad G(s) = e^{s^2} p(s) \quad \text{and} \quad p(s) = \frac{(\alpha + \beta)^2 - (2s)^2}{(\alpha + \beta)^2}.$$

Proof. See Lemma 5 of [17]. The key point is that non-diagonal terms $hm \neq kn$ can safely be absorbed in the error terms. \square

Lemma 2.2. *Suppose $0 < \delta \asymp L^{-1}$, $\beta \ll L^{-1}$ and $\beta < \delta$. For some $\nu \asymp (\log \log y)^{-1}$ we have*

$$\begin{aligned} \Upsilon &:= \frac{1}{2\pi i} \int_{(\delta)} \frac{1}{\zeta(1+\beta+u)} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell-r} \left(\frac{y_\star}{n} \right)^u \frac{du}{u^{j+1}} \\ &= (-1)^{\ell-r} \frac{1}{2\pi i} \oint (\beta+u)^{1-\ell+r} \left(\frac{y_\star}{n} \right)^u \frac{du}{u^{j+1}} + O(L^{\ell-r-2+j}) + O\left(\left(\frac{y_\star}{n} \right)^{-\nu} L^\varepsilon \right), \end{aligned}$$

where $y_\star \geq n > 0$ and the contour is a circle of radius one enclosing the origin and $-\beta$.

Proof. This follows a similar procedure to Lemma 6.1 of [3] where the zero-free region of ζ is used. Let $Y = o(T)$ be a large parameter to be chosen later. By Cauchy's theorem, Υ is equal to the sum of residues at $u = 0$ and $u = -\beta$ plus integrals over the line segments $\gamma_1 = \{s = it : t \in \mathbb{R}, |t| \geq Y\}$, $\gamma_2 = \{s = \sigma \pm iY : -c/\log Y \leq \sigma \leq 0\}$, and $\gamma_3 = \{s = -c/\log Y + it : |t| \leq Y\}$, where c is some fixed positive constant such that $\zeta(1 + \beta + u)$ has no zeros in the region on the right-hand side of the contour determined by the γ_i 's. Another requirement on c is that the estimate (see [16,

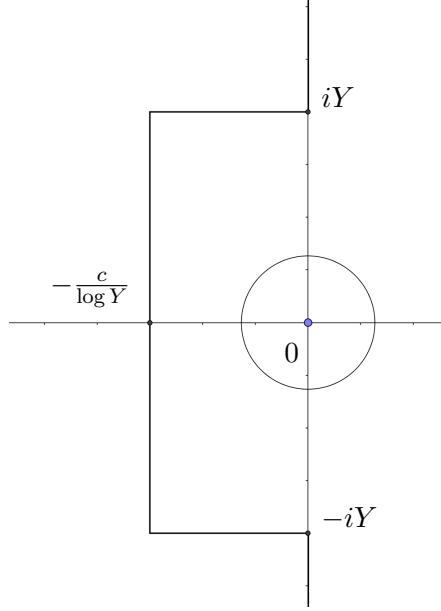


FIGURE 2.1. Curve γ in the proof of Lemma 2.2.

Theorem 3.11]) $1/\zeta(\sigma + it) \ll \log(2 + |t|)$ holds in this region and $\zeta'/\zeta(\sigma + it) \ll \log(4 + |t|)$ (see [13, Theorem 6.7]). Then, one has

$$\int_{\gamma_1} \ll \int_Y^\infty \frac{\log(t)^{1+\ell-r}}{t^{j+1}} dt \ll \frac{\log(Y)^{1+\ell-r}}{Y^j},$$

since $j \geq 3$. Moreover, since $n \leq y_\star$

$$\int_{\gamma_2} \ll \int_{-c/\log Y}^0 \log(Y)^{1+\ell-r} \left(\frac{y_\star}{n}\right)^x \frac{1}{Y^{j+1}} dx \ll \frac{\log(Y)^{\ell-r}}{Y^{j+1}},$$

and finally

$$\int_{\gamma_3} \ll \int_{-Y}^Y \log(4 + |t|)^{\ell-r+1} \frac{(y_\star/n)^{-c/\log Y}}{c^2/\log^2 Y + t^2} dt \ll \log(Y)^{\ell-r+j} (y_\star/n)^{-c/\log Y}.$$

Appropriately choosing $Y \asymp (\log y_\star)$ gives an error of size $O((\log \log y_\star)^{\ell-r+j}) = O(\log y_\star)$. The next step is to sum the residues. This sum can now be expressed as

$$\frac{1}{2\pi i} \oint \frac{1}{\zeta(1 + \beta + u)} \left(\frac{\zeta'}{\zeta}(1 + \beta + u) \right)^{\ell-r} \left(\frac{y_\star}{n} \right)^u \frac{du}{u^{j+1}},$$

where the contour is now a small circle Ω of radius $\asymp 1/L$ around the origin such that $-\beta \in \Omega$. Since the radius of the circle is tending to zero, we can use the Laurent expansions

$$\frac{1}{\zeta(s)} = s - 1 + O((s - 1)^2) \quad \text{and} \quad \frac{\zeta'}{\zeta}(s) = -\frac{1}{s - 1} + \gamma + O(|s - 1|),$$

to finally obtain

$$\begin{aligned} & \frac{1}{2\pi i} \oint \frac{1}{\zeta(1+\beta+u)} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell-r} \left(\frac{y_\star}{n} \right)^u \frac{du}{u^{j+1}} \\ &= \frac{1}{2\pi i} \oint (\beta+u+O(u^2)) \left(\frac{-1}{\beta+u} + O(1) \right)^{\ell-r} \left(\frac{y_\star}{n} \right)^u \frac{du}{u^{j+1}}. \end{aligned}$$

Using the binomial theorem and a direct estimate gives, we get that the above is equal to

$$(-1)^{\ell-r} \oint (\beta+u)^{1-\ell+r} \left(\frac{y_\star}{n} \right)^u \frac{du}{u^{j+1}} + O(L^{j+\ell-r-2}),$$

which is the desired main term of the lemma. \square

This integral can be computed exactly. To do this, note that for any integer $k \geq 1$, one has

$$q^u(\beta+u)^k = \frac{d^k}{dy^k} e^{\beta y} (e^y q)^u \Big|_{y=0}.$$

Hence, one arrives at and where we temporarily set $q = y_\star/n$

$$\begin{aligned} \Upsilon &= (-1)^{\ell-r} \frac{1}{2\pi i} \oint \frac{d^{1-\ell+r}}{dy^{1-\ell+r}} e^{\beta y} (e^y q)^u \Big|_{y=0} \frac{du}{u^{j+1}} = (-1)^{\ell-r} \frac{d^{1-\ell+r}}{dy^{1-\ell+r}} e^{\beta y} \frac{1}{2\pi i} \oint (e^y q)^u \Big|_{y=0} \frac{du}{u^{j+1}} \\ (2.1) \quad &= \frac{(-1)^{\ell-r}}{j!} \frac{d^{1-\ell+r}}{dy^{1-\ell+r}} e^{\beta y} \left(y + \log \frac{y_\star}{n} \right)^j \Big|_{y=0}, \end{aligned}$$

by Cauchy's integral theorem.

2.2. Combinatorial results. When computing the crossterm of ψ_1 and ψ_2 the following result will be needed. This generalizes [8, Lemma 8] which is the particular case $h_1 = h_2 = h$.

Lemma 2.3. *For h_1 and h_2 square-free, we have*

$$\begin{aligned} \mathcal{Q}(\ell_1, \ell_2) &:= \sum_{p_1 p_2 \cdots p_{\ell_1} | h_1} \log p_1 \log p_2 \cdots \log p_{\ell_1} \sum_{q_1 q_2 \cdots q_{\ell_2} | h_2} \log q_1 \log q_2 \cdots \log q_{\ell_2} \\ &= \sum_{k=0}^{\min\{\ell_1, \ell_2\}} k! \binom{\ell_1}{k} \binom{\ell_2}{k} \sum_{\substack{p_1 p_2 \cdots p_k q_1 q_2 \cdots q_{\ell_1-k} r_1 r_2 \cdots r_{\ell_2-k} | h_1 h_2 / \gcd(h_1, h_2) \\ p_1 p_2 \cdots p_k | \gcd(h_1, h_2) \\ q_1 \cdots q_{\ell_1-k} | h_1 \\ r_1 \cdots r_{\ell_2-k} | h_2}} \\ &\quad \times \left(\prod_{f=1}^k \log^2 p_f \right) \left(\prod_{f=1}^{\ell_1-k} \log q_f \right) \left(\prod_{f=1}^{\ell_2-k} \log r_f \right), \end{aligned}$$

Here the p 's, the q 's and the r 's are all distinct primes.

Proof. We may write

$$\begin{aligned} \mathcal{Q}(\ell_1, \ell_2) &= \sum_{\substack{p_1 p_2 \cdots p_{\ell_1} | h_1 \\ p_a \neq p_b}} \log p_1 \log p_2 \cdots \log p_{\ell_1} \sum_{\substack{q_1 q_2 \cdots q_{\ell_2} | h_2 \\ q_a \neq q_b}} \log q_1 \log q_2 \cdots \log q_{\ell_2} \\ &= \ell_1! \ell_2! \sum_{\substack{p_1 p_2 \cdots p_{\ell_1} | h_1 \\ p_1 < p_2 < \cdots < p_{\ell_1}}} \log p_1 \log p_2 \cdots \log p_{\ell_1} \sum_{\substack{q_1 q_2 \cdots q_{\ell_2} | h_2 \\ q_1 < q_2 < \cdots < q_{\ell_2}}} \log q_1 \log q_2 \cdots \log q_{\ell_2} \end{aligned}$$

$$\begin{aligned}
&= \ell_1! \ell_2! \sum_{k=0}^{\min\{\ell_1, \ell_2\}} \sum_{\substack{p_1 p_2 \cdots p_k q_1 q_2 \cdots q_{\ell_1-k} r_1 r_2 \cdots r_{\ell_2-k} | h_1 h_2 / \gcd(h_1, h_2) \\ p_1 p_2 \cdots p_k | \gcd(h_1, h_2), q_1 \cdots q_{\ell_1-k} | h_1, r_1 \cdots r_{\ell_2-k} | h_2 \\ p_1 < p_2 < \cdots < p_k, q_1 < q_2 < \cdots < q_{\ell_1-k}, r_1 < r_2 < \cdots < r_{\ell_2-k}}} \\
&\quad \times (\log^2 p_1 \cdots \log^2 p_k) (\log q_1 \cdots \log q_{\ell_1-k}) (\log r_1 \cdots \log r_{\ell_2-k}) \\
&= \sum_{k=0}^{\min\{\ell_1, \ell_2\}} \frac{\ell_1! \ell_2!}{k! (\ell_1 - k)! (\ell_2 - k)!} \sum_{\substack{p_1 p_2 \cdots p_k q_1 q_2 \cdots q_{\ell_1-k} r_1 r_2 \cdots r_{\ell_2-k} | h_1 h_2 / \gcd(h_1, h_2) \\ p_1 p_2 \cdots p_k | \gcd(h_1, h_2), q_1 \cdots q_{\ell_1-k} | h_1, r_1 \cdots r_{\ell_2-k} | h_2}} \\
&\quad \times (\log^2 p_1 \cdots \log^2 p_k) (\log q_1 \cdots \log q_{\ell_1-k}) (\log r_1 \cdots \log r_{\ell_2-k}).
\end{aligned}$$

Using the definition of the binomial coefficient completes the proof. \square

2.3. Generalized von Mangoldt functions and Euler-MacLaurin summations. Recall that for a positive integer k , the generalized von Mangoldt function $\Lambda_k(n)$ is defined [9] by the Dirichlet convolution

$$\Lambda_k(n) = (\mu * \log^k)(n),$$

so that $\Lambda_1(n) = \Lambda(n)$. The generating series is

$$\sum_{n=1}^{\infty} \frac{\Lambda_k(n)}{n^s} = (-1)^k \frac{\zeta^{(k)}}{\zeta}(s),$$

for $\operatorname{Re}(s) > 1$ and here $\zeta^{(k)}$ stands for the k -th derivative of ζ with respect to s . By looking at

$$\frac{d}{ds} \left(\frac{\zeta^{(k)}}{\zeta}(s) \right) = \frac{\zeta^{(k+1)}}{\zeta}(s) - \frac{\zeta'}{\zeta}(s) \frac{\zeta^{(k)}}{\zeta}(s)$$

for $\operatorname{Re}(s) > 1$, we see that

$$\Lambda_{k+1}(n) = \Lambda_k(n) \log(n) + (\Lambda * \Lambda_k)(n),$$

and in particular for $k = 1$

$$(2.2) \quad \Lambda_2(n) = \Lambda(n) \log(n) + (\Lambda * \Lambda)(n).$$

Lemma 2.4. *We have for smooth functions F and G in the interval $[0, 1]$, $3 \leq z \leq x$, and $|s| \leq (\log x)^{-1}$*

$$\begin{aligned}
\sum_{n \leq z} \frac{\Lambda(n) \log n}{n^{1+s}} F\left(\frac{\log(x/n)}{\log x}\right) H\left(\frac{\log(z/n)}{\log z}\right) &= \frac{\log^2 z}{z^s} \int_0^1 (1-u) F\left(1 - (1-u) \frac{\log z}{\log x}\right) H(u) z^{us} du \\
&\quad + O(\log z).
\end{aligned}$$

Proof. Start by setting

$$\Psi(z, x) := \sum_{n \leq z} \frac{\Lambda(n) \log n}{n^{1+s}} F\left(\frac{\log(x/n)}{\log x}\right) H\left(\frac{\log(z/n)}{\log z}\right) \quad \text{and} \quad \psi(x) := \sum_{n \leq x} \Lambda(n).$$

By applying the Abel summation formula, one gets

$$\begin{aligned}
\Psi(z, x) &= \psi(z) \frac{\log z}{z^{1+s}} F\left(\frac{\log(x/z)}{\log x}\right) H(0) - \int_1^z \psi(u) \frac{d}{du} \left(\frac{\log u}{u^{1+s}} F\left(\frac{\log(x/u)}{\log x}\right) H\left(\frac{\log(z/u)}{\log z}\right) \right) du \\
&= - \int_1^z \psi(u) \frac{d}{du} \left(\frac{\log u}{u^{1+s}} F\left(\frac{\log(x/u)}{\log x}\right) H\left(\frac{\log(z/u)}{\log z}\right) \right) du + O(\log z) \\
&= - \int_1^z \psi(u) \frac{1 - (1+s) \log u}{u^{2+s}} F\left(\frac{\log(x/u)}{\log x}\right) H\left(\frac{\log(z/u)}{\log z}\right) du
\end{aligned}$$

$$\begin{aligned}
& - \int_1^z \psi(u) \frac{\log u}{u^{1+s}} \left(\frac{d}{du} F \left(\frac{\log(x/u)}{\log x} \right) \right) H \left(\frac{\log(z/u)}{\log z} \right) du \\
& - \int_1^z \psi(u) \frac{\log u}{u^{1+s}} F \left(\frac{\log(x/u)}{\log x} \right) \left(\frac{d}{du} H \left(\frac{\log(z/u)}{\log z} \right) \right) du + O(\log z) \\
& = \frac{\log^2 z (1+s)}{z^s} \int_0^1 \psi(z^{1-b})(1-b) F \left(1 - (1-b) \frac{\log z}{\log x} \right) H(b) z^{bs+b-1} db \\
& + O \left(\log z \int_1^z \psi(u) \frac{1}{u^{2+|s|}} du \right) \\
& + \frac{1}{\log x} \int_1^z \psi(u) \frac{\log u}{u^{2+s}} F' \left(\frac{\log(x/u)}{\log x} \right) H \left(\frac{\log(z/u)}{\log z} \right) du \\
& + \frac{1}{\log z} \int_1^z \psi(u) \frac{\log u}{u^{2+s}} F \left(\frac{\log(x/u)}{\log x} \right) H' \left(\frac{\log(z/u)}{\log z} \right) du + O(\log z) \\
& = \frac{\log^2 z (1+s)}{z^s} \int_0^1 \psi(z^{1-b})(1-b) F \left(1 - (1-b) \frac{\log z}{\log x} \right) H(b) z^{bs+b-1} db + O(\log z) \\
& = \frac{\log^2 z}{z^s} \int_0^1 \psi(z^{1-b})(1-b) F \left(1 - (1-b) \frac{\log z}{\log x} \right) H(b) z^{bs+b-1} db \\
& + O \left(\log z \int_0^1 \psi(z^{1-b})(1-b) z^{bs+b-1} db \right) + O(\log z) \\
& = \frac{\log^2 z}{z^s} \int_0^1 (1-b) F \left(1 - (1-b) \frac{\log z}{\log x} \right) H(b) z^{bs} db + O \left(\log z \int_0^1 (1-b) z^{bs} db \right) + O(\log z) \\
& = \frac{\log^2 z}{z^s} \int_0^1 (1-b) F \left(1 - (1-b) \frac{\log z}{\log x} \right) H(b) z^{bs} db + O(\log z),
\end{aligned}$$

since $\psi(x) = x + O(x \exp(-c\sqrt{\log x}))$ for $c > 1$ by the prime number theorem with remainder, see e.g. [16]. \square

Lemma 2.5. *We have for smooth functions F and G in the interval $[0, 1]$, $3 \leq z \leq x$, and $|s| \leq (\log x)^{-1}$*

$$\begin{aligned}
& \sum_{n \leq z} \frac{(d_k * \Lambda^{*l})}{n^{1+s}} F \left(\frac{\log x/n}{\log x} \right) H \left(\frac{\log z/n}{\log z} \right) \\
& = \frac{(\log z)^{k+l}}{(k+l-1)! z^s} \int_0^1 (1-u)^{k+l-1} F \left(1 - (1-u) \frac{\log z}{\log x} \right) H(u) z^{us} du + O((\log 3z)^{k+l-1}),
\end{aligned}$$

where $d_k(n)$ denotes the number of ways an integer n can be written as a product of $k \geq 2$ fixed factors. Note that $d_1(n) = 1$ and $d_2(n) = d(n)$, the number of divisors of n .

Proof. This can be proved by using induction over ℓ and Euler-Maclaurin summation. One starts with $\ell = 0$ and then uses [3, Lemma 4.4]. The exact details can be found in [14, Lemma 3.6]. \square

Lemma 2.6. *We have for smooth functions F and G in the interval $[0, 1]$, $3 \leq z \leq x$, and $|s| \leq (\log x)^{-1}$*

$$\begin{aligned}
& \sum_{n \leq z} \frac{(1 * \Lambda^{*a} * \Lambda \log)(n)}{n^{1+s}} F \left(\frac{\log(x/n)}{\log x} \right) H \left(\frac{\log(z/n)}{\log z} \right) \\
& = \frac{\log^{3+a} z}{(a+2)! z^s} \int_0^1 (1-u)^{a+2} F \left(1 - (1-u) \frac{\log z}{\log x} \right) H(u) z^{us} du + O(\log^{a+2} z).
\end{aligned}$$

Proof. Same as in the beginning of the proof of Lemma 2.5 but instead we use Lemma 2.4. \square

Lemma 2.7. *We have for smooth functions F and G in the interval $[0, 1]$, $3 \leq z \leq x$, and $|s| \leq (\log x)^{-1}$*

$$\begin{aligned} & \sum_{n \leq z} \frac{(\mathbf{1} * \Lambda^{*a} * \Lambda_2^{*b})(n)}{n^{1+s}} F\left(\frac{\log(x/n)}{\log x}\right) H\left(\frac{\log(z/n)}{\log z}\right) \\ &= 2^b \frac{\log^{1+a+2b} z}{(a+2b)! z^s} \int_0^1 (1-u)^{a+2b} F\left(1 - (1-u) \frac{\log z}{\log x}\right) H(u) z^{us} du + O(\log^{a+2b} z). \end{aligned}$$

Proof. This follows by induction on b and by using Lemma 2.6 combined with (2.2). \square

3. EVALUATION OF THE SHIFTED MEAN VALUE INTEGRALS $I_*(\alpha, \beta)$

3.1. Proof of Lemma 1.1. Although this was already explained in [17], the mean value integral $I_{22}(\alpha, \beta)$ builds up from $I_{12}(\alpha, \beta)$ which in turn is a refinement of $I_{11}(\alpha, \beta)$. Therefore, careful analysis will repay itself by going over the main points of the evaluation of $I_{11}(\alpha, \beta)$ briefly. For our purposes, we shall illustrate this for $\theta_1 < 1/2$; however, in [5] it was shown that one could take $\theta_1 < 4/7$. We start by inserting the definition of the mollifier ψ_1 in I_{11} so that

$$\begin{aligned} I_{11}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \psi_1 \overline{\psi_1}(\sigma_0 + it) dt \\ &= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \\ &\quad \times \sum_{h \leq y_1} \frac{\mu(h) h^{-1/2}}{h^{it}} P_1\left(\frac{\log y_1/h}{\log y_1}\right) \sum_{k \leq y_1} \frac{\mu(k) k^{-1/2}}{k^{-it}} P_1\left(\frac{\log y_1/k}{\log y_1}\right) dt \\ &= \sum_{h \leq y_1} \sum_{k \leq y_1} \frac{\mu(h) \mu(k)}{(hk)^{1/2}} P_1\left(\frac{\log y_1/h}{\log y_1}\right) P_1\left(\frac{\log y_1/k}{\log y_1}\right) \\ &\quad \times \int_{-\infty}^{\infty} w(t) \left(\frac{h}{k}\right)^{-it} \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) dt. \end{aligned}$$

According to Lemma 2.1, we write $I_{11}(\alpha, \beta) = I'_{11}(\alpha, \beta) + I''_{11}(\alpha, \beta)$, where I'_{11} is given by

$$\begin{aligned} (3.1) \quad I'_{11}(\alpha, \beta) &= \sum_{h \leq y_1} \sum_{k \leq y_1} \frac{\mu(h) \mu(k)}{(hk)^{1/2}} P_1\left(\frac{\log y_1/h}{\log y_1}\right) P_1\left(\frac{\log y_1/k}{\log y_1}\right) \\ &\quad \times \sum_{hm=kn} \frac{1}{m^{1/2+\alpha} n^{1/2+\beta}} \int_{-\infty}^{\infty} V_{\alpha, \beta}(mn, t) w(t) dt. \end{aligned}$$

Notice that $I''_{11}(\alpha, \beta)$ is obtained by replacing α with $-\beta$, β with $-\alpha$ and multiplying inside the integrand by $X_{\alpha, \beta, t} = T^{-\alpha, \beta}(1 + O(L^{-1}))$. In other words,

$$I_{11}(\alpha, \beta) = I'_{11}(\alpha, \beta) + T^{-\alpha-\beta} I'_{11}(-\beta, -\alpha) + O(T/L).$$

Let us then look at I'_{11} more closely. Using the Mellin representations

$$P_1[h] = \sum_i \frac{a_i i!}{\log^i y_1} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_1}{h}\right)^s \frac{ds}{s^{i+1}} \quad \text{and} \quad P_1[k] = \sum_j \frac{a_j j!}{\log^j y_1} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_1}{k}\right)^u \frac{du}{u^{j+1}},$$

we then get

$$I'_{11}(\alpha, \beta) = \int_{-\infty}^{\infty} w(t) \sum_{i, j} \frac{a_i i! a_j j!}{\log^{i+j} y_1} \left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)} y_1^{s+u} g_{\alpha, \beta}(z, t) \frac{G(z)}{z}$$

$$\times \sum_{hm=kn} \frac{\mu(h)\mu(k)}{h^{1/2+s}k^{1/2+u}m^{1/2+\alpha+z}n^{1/2+\beta+z}} dz \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt.$$

We now evaluate the arithmetical sum $S = \sum_{hm=kn}$ in the integrand. This is done p -adically as follows. We denote by $\nu_p(n)$ the number of times the prime number p appears in n , and without risk of confusion we write $n' = \nu_p(n)$. This means that

$$\begin{aligned} S &= \sum_{hm=kn} \frac{\mu(h)\mu(k)}{h^{1/2+s}k^{1/2+u}m^{1/2+\alpha+z}n^{1/2+\beta+z}} \\ &= \prod_p \sum_{h'+n'=m'+k'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s}(p^{k'})^{1/2+u}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}} \\ &= \prod_p \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right) \\ &= \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha,\beta}(s,u,z), \end{aligned}$$

where the arithmetical factor $A_{\alpha,\beta}(s,u,z)$ is given by an absolutely convergent Euler product in some product of half-planes containing the origin. It will be important to remark that when $\alpha = \beta = 0$ and $s = u = z$ we have

$$(3.2) \quad A_{0,0}(z,z,z) = \sum_{hm=kn} \frac{\mu(h)\mu(k)}{h^{1/2+z}k^{1/2+z}m^{1/2+z}n^{1/2+z}} = \sum_{hm=kn} \frac{\mu(h)\mu(k)}{(hkmn)^{1/2+z}} = 1,$$

for all z , by the Möbius inversion formula. Inserting this into I'_{11} we get

$$\begin{aligned} I'_{11}(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{a_i i! a_j j!}{\log^{i+j} y_1} \left(\frac{1}{2\pi i} \right)^3 \int_{(1)} \int_{(1)} \int_{(1)} y_1^{s+u} g_{\alpha,\beta}(z,t) \frac{G(z)}{z} \\ &\quad \times \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha,\beta}(s,u,z) dz \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt. \end{aligned}$$

Now we deform the path of integration to $\operatorname{Re}(z) = -\delta + \varepsilon$ where $\delta > 0$ small and $\operatorname{Re}(s) = \operatorname{Re}(u) = \delta$. By doing this, we pick up a simple pole coming from $1/z$ at $z = 0$ only, since $G(z)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2z)$. The new path of integration with respect to z contributes an error of the size

$$\sum_{n \leq y_1} \frac{1}{n} \left(1 + \log \frac{y_1}{n} \right)^{-2} \ll 1 \ll L^{i+j-2}.$$

Thus, we end up with

$$\begin{aligned} I_{11}(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{a_i i! a_j j!}{\log^{i+j} y_1} \left(\frac{1}{2\pi i} \right)^2 \int_{(\delta)} \int_{(\delta)} \operatorname{res}_{z=0} y_1^{s+u} g_{\alpha,\beta}(z,t) \frac{G(z)}{z} \\ &\quad \times \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha,\beta}(s,u,z) \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt + O(L^{i+j-2}) \\ &= \widehat{w}(0) \zeta(1+\alpha+\beta) \sum_{i,j} \frac{a_i i! a_j j!}{\log^{i+j} y_1} J_{11} + O(L^{i+j-2}), \end{aligned}$$

where

$$J_{11} = \left(\frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} y_1^{s+u} \frac{\zeta(1+s+u)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)} A_{\alpha,\beta}(s,u,0) \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}}.$$

Using the Dirichlet series representation for $\zeta(1+s+u)$, we can separate the complex variables s and u . The next step is to use the Laurent expansion

$$\begin{aligned} \frac{A_{\alpha,\beta}(s,u,0)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)} &= (\alpha+s)(\beta+u)A_{0,0}(0,0,0) + O(L^{-3}) \\ &= (\alpha+s)(\beta+u) + O(L^{-3}) \end{aligned}$$

since $A_{0,0}(z,z,z) = 1$ for all z , in particular for $z = 0$. By the use of Lemma 2.2, we can deform the line integrals into contour integrals around circles of radius 1 around the origin. Thus,

$$J_{11} = \sum_{n \leq y_1} \left(\frac{1}{2\pi i} \right)^2 \oint \left(\frac{y_1}{n} \right)^s \frac{(s+\alpha)ds}{s^{i+1}} \oint \left(\frac{y_1}{n} \right)^s \frac{(u+\beta)du}{u^{j+1}} + O(L^{i+j-2}).$$

These integrals can be computed by the use of (2.1), so that

$$J_{11} = \frac{1}{i!j!} \frac{d^2}{dx dy} e^{\alpha x + \beta y} \sum_{n \leq y_1} \frac{1}{n} \left(x + \log \frac{y_1}{n} \right)^i \left(y + \log \frac{y_1}{n} \right)^j \Big|_{x=y=0} + O(L^{i+j-2}).$$

Let us note that

$$\frac{d}{dx} e^{\alpha x} \frac{1}{n} \left(x + \log \frac{y_1}{n} \right)^i \Big|_{x=0} = \frac{\log^i y_1}{\log y_1} \frac{d}{dx} y_1^{\alpha x} \left(x + \frac{\log(y_1/n)}{\log y_1} \right)^i \Big|_{x=0}.$$

Now sum over i to get (the two $i!$ cancel out)

$$P_1[n] = \sum_i a_i \left(x + \frac{\log(y_1/n)}{\log y_1} \right)^i$$

and similarly over j , so that

$$\begin{aligned} I'_{11}(\alpha, \beta) &= \widehat{w}(0) \zeta(1+\alpha+\beta) \sum_{i,j} \frac{a_i a_j}{\log^2 y_1} \\ &\quad \times \frac{d^2}{dx dy} \left[y_1^{\alpha x + \beta y} \sum_{n \leq y_1} \frac{1}{n} \left(x + \frac{\log(y_1/n)}{\log y_1} \right)^i \left(y + \frac{\log(y_1/n)}{\log y_1} \right)^j \Big|_{x=y=0} \right] + O(T/L) \\ &= \frac{\widehat{w}(0)}{(\alpha+\beta)\log^2 y_1} \frac{d^2}{dx dy} \left[y_1^{\alpha x + \beta y} \right. \\ &\quad \times \left. \sum_{n \leq y_1} \frac{1}{n} P \left(x + \frac{\log(y_1/n)}{\log y_1} \right) P \left(y + \frac{\log(y_1/n)}{\log y_1} \right) \Big|_{x=y=0} \right] + O(T/L) \\ &= \frac{\widehat{w}(0)}{(\alpha+\beta)\log^2 y_1} \frac{d^2}{dx dy} \left[y_1^{\alpha x + \beta y} \right. \\ &\quad \times \left. \int_1^{y_1} r^{-1} P \left(x + \frac{\log(y_1/r)}{\log y_1} \right) P \left(y + \frac{\log(y_1/r)}{\log y_1} \right) dr \Big|_{x=y=0} \right] + O(T/L) \\ &= \frac{\widehat{w}(0)}{(\alpha+\beta)\log y_1} \frac{d^2}{dx dy} \left[y_1^{\alpha x + \beta y} \int_0^1 P(x+u)P(y+u)du \Big|_{x=y=0} \right] + O(T/L). \end{aligned}$$

In the second equality we made use of $\zeta(1+\alpha+\beta) = 1/(\alpha+\beta) + O(1)$, in the third equality we used the Euler-MacLaurin formula, and in the fourth equality we employed the change of variables $r = M^{1-u}$. By adding and subtracting the same quantity we find that

$$(3.3) \quad I_{11}(\alpha, \beta) = [I'_{11}(\alpha, \beta) + I'_{11}(-\beta, -\alpha)] + I'_{11}(-\beta, -\alpha)(T^{-\alpha-\beta} - 1) + O(T/L).$$

For the term in square brackets we have

$$\begin{aligned} c'_{11}(\alpha, \beta) + c'_{11}(-\beta, -\alpha) &= \frac{1}{(\alpha + \beta) \log y_1} \int_0^1 (P'(u) + \alpha P(u) \log y_1)(P'(u) + \beta P(u) \log y_1) du \\ &\quad - \frac{1}{(\alpha + \beta) \log y_1} \int_0^1 (P'(u) - \beta P(u) \log y_1)(P'(u) - \alpha P(u) \log y_1) du \\ &= \int_0^1 2P'(u)P(u) du = 1. \end{aligned}$$

For the other term in (3.3) we have

$$\begin{aligned} c'_{11}(-\beta, -\alpha)(T^{-\alpha-\beta} - 1) &= \frac{T^{-\alpha-\beta} - 1}{(-\beta - \alpha) \log y_1} \frac{d^2}{dx dy} y_1^{-\beta x - \alpha y} \int_0^1 P(x+u)P(y+u) du \Big|_{x=y=0} \\ &= \frac{1 - T^{-\alpha-\beta}}{(\alpha + \beta) \log y_1} \frac{d^2}{dx dy} y_1^{-\beta x - \alpha y} \int_0^1 P(x+u)P(y+u) du \Big|_{x=y=0} \\ &= \frac{1}{\theta_1} \frac{d^2}{dx dy} y_1^{-\beta x - \alpha y} \int_0^1 \int_0^1 T^{-v(\alpha+\beta)} P(x+u)P(y+u) dudv \Big|_{x=y=0}, \end{aligned}$$

by the use of

$$(3.4) \quad \frac{1 - T^{-\alpha-\beta}}{(\alpha + \beta) \log y_1} = \frac{1}{\theta_1} \int_0^1 T^{-v(\alpha+\beta)} dv.$$

The additional restriction that $|\alpha + \beta| \gg L^{-1}$ is dealt with the holomorphy of $I(\alpha, \beta)$ and $c(\alpha, \beta)$ with $\alpha, \beta \ll L^{-1}$ which implies that the error term is also holomorphic in this region. The maximum modulus principle extends the error term to this enlarged domain. This proves Lemma 1.1.

3.2. Proof of Lemma 1.2. This is the term involving Conrey's and Feng's mollifiers. To compute this term, let us follow the same strategy as in $I_{11}(\alpha, \beta)$. We first insert the definitions of ψ_1 and ψ_2 into the mean value integral I_{12} so that

$$\begin{aligned} I_{12}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \overline{\psi_1} \psi_2(\sigma_0 + it) dt \\ &= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \\ &\quad \times \sum_{h \leq y_1} \frac{\mu(h)}{h^{1/2-it}} P_1[h] \sum_{k \leq y_2} \frac{\mu(k)}{k^{1/2+it}} \sum_{\ell=2}^K \sum_{p_1 \cdots p_\ell | k} \frac{\log p_1 \cdots \log p_\ell}{\log^\ell y_2} P_\ell[k] dt \\ &= \sum_{\ell=2}^K \sum_{h, k} \frac{\mu(h) \mu(k)}{(hk)^{1/2}} P_1[h] \sum_{p_1 \cdots p_\ell | k} \frac{\log p_1 \cdots \log p_\ell}{\log^\ell y_2} P_\ell[k] \\ &\quad \times \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \left(\frac{k}{h}\right)^{-it} dt. \end{aligned}$$

As for $I_{11}(\alpha, \beta)$, we use at this point Lemma 2.1 to write $I_{12}(\alpha, \beta) = I'_{12}(\alpha, \beta) + I''_{12}(\alpha, \beta) + E(\alpha, \beta)$, where $I'_{12}(\alpha, \beta)$ and $I''_{12}(\alpha, \beta)$ correspond to the two sums of Lemma 2.1 and $E(\alpha, \beta)$ is the error term. Specifically, one has

$$I'_{12}(\alpha, \beta) = \sum_{\ell=2}^K \sum_{h, k} \frac{\mu(h) \mu(k)}{(hk)^{1/2}} P_1[h] \sum_{p_1 \cdots p_\ell | k} \frac{\log p_1 \cdots \log p_\ell}{\log^\ell y_2} P_\ell[k]$$

$$(3.5) \quad \times \sum_{hm=kn} \frac{1}{m^{1/2+\alpha} n^{1/2+\beta}} \int_{-\infty}^{\infty} V_{\alpha,\beta}(mn, t) w(t) dt,$$

and for reasons of symmetry, $I''_{12}(\alpha, \beta)$ can be obtained from $I'_{12}(\alpha, \beta)$ by switching α and $-\beta$ and multiplying by

$$\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} = T^{-\alpha-\beta} + O(L^{-1}),$$

for $t \asymp T$. We thus see that it is enough to compute $I'_{12}(\alpha, \beta)$. The error term is given

$$\begin{aligned} E(\alpha, \beta) &\ll_{A, \theta_1, \theta_2} T^{-A} \sum_{\ell=2}^K \sum_{h, k} \frac{\mu(h)\mu(k)}{(hk)^{1/2}} P_1[h] \sum_{p_1 \cdots p_\ell | k} \frac{\log p_1 \cdots \log p_\ell}{\log^\ell y_2} P_\ell[k] \\ &\ll T^{-A} \sum_{\ell=2}^K \sum_{h \leq y_1} \sum_{k \leq y_2} \frac{1}{(hk)^{1/2}} \sum_{p_1 \cdots p_\ell | k} 1 \ll T^{-A} \sum_{\ell=2}^K \sum_{h \leq y_1} \sum_{k \leq y_2} \frac{1}{(hk)^{1/2}} (d(k))^\ell \\ &\ll T^{-A} \sum_{h \leq y_1} \frac{1}{h^{1/2-\varepsilon}} \sum_{k \leq y_2} \frac{1}{k^{1/2-\varepsilon}} \ll T^{-A} y_1^{1/2-\varepsilon} y_2^{1/2-\varepsilon} \\ &= T^{-A} T^{\theta_1(1/2-\varepsilon)\theta_2(1/2-\varepsilon)} = T^{-A+(\theta_1+\theta_2)/2-\varepsilon} \end{aligned}$$

for any $A > 2$. We remark that the above computation works for $\theta_1 + \theta_2$ arbitrarily large but the error term T^{-A} coming from Lemma 2.1 is only valid for $\theta_1 + \theta_2 < 1$. The next step is to use the Mellin integral representations of the polynomials P_1

$$P_1[h] = \sum_i \frac{a_i}{\log^i y_1} (\log(y_1/h))^i = \sum_i \frac{a_i i!}{\log^i y_1} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_1}{h}\right)^s \frac{ds}{s^{i+1}},$$

and P_ℓ

$$P_\ell[k] = \sum_j \frac{b_{\ell,j}}{\log^j y_2} (\log(y_2/k))^j = \sum_j \frac{b_{\ell,j} j!}{\log^j y_2} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_2}{k}\right)^u \frac{du}{u^{j+1}},$$

and the definition of $V_{\alpha,\beta}$ in Lemma 2.1 to write

$$\begin{aligned} I'_{12}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \sum_{\ell=2}^L \sum_{i,j} \frac{a_i b_{\ell,j} i! j!}{\log^i y_1 \log^{j+\ell} y_2} \\ &\quad \times \sum_{km=hn} \frac{\mu(h)\mu(k)}{(hk)^{1/2} m^{1/2+\alpha} n^{1/2+\beta}} \sum_{p_1 \cdots p_\ell | k} \log p_1 \cdots \log p_\ell \\ &\quad \times \left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)} \left(\frac{y_1}{h}\right)^s \left(\frac{y_2}{k}\right)^u \frac{g_{\alpha,\beta}(z, t) G(z)}{(mn)^z} \frac{dz}{z} \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt. \end{aligned}$$

We now have to compute the arithmetical sum $\sum_{km=hn}$. Further details on this procedure can be found in [14]. Let us define

$$S_\ell = S_{\ell, \alpha, \beta}(s, u, z) = \sum_{km=hn} \frac{\mu(h)\mu(k)}{(hk)^{1/2} m^{1/2+\alpha+z} n^{1/2+\beta+z}} \sum_{p_1 \cdots p_\ell | k} \log p_1 \cdots \log p_\ell.$$

We start by inverting the order of the sum so that

$$S_\ell = (-1)^\ell \sum_{\substack{p_i \neq p_j \\ i < j}} \log p_1 \cdots \log p_\ell \sum_{\substack{hn=p_1 \cdots p_\ell \tilde{k} m \\ (p_1 \cdots p_\ell, \tilde{k})=1}} \frac{\mu(h)\mu(\tilde{k})}{h^{1/2+s} \tilde{k}^{1/2+u} m^{1/2+\alpha+z} n^{1/2+\beta+z}} \frac{1}{(p_1 \cdots p_\ell)^{1/2+u}}$$

$$(3.6) \quad = (-1)^\ell \sum_{\substack{p_i \neq p_j \\ i < j}} \frac{\log p_1 \cdots \log p_\ell}{(p_1 \cdots p_\ell)^{1/2+u}} \tilde{S}_{\ell, \alpha, \beta}(s, u, z),$$

where $k = \tilde{k} p_1 \cdots p_\ell$ and where we define the inner sum to be

$$\tilde{S}_\ell = \tilde{S}_{\ell, \alpha, \beta}(s, u, z) = \sum_{\substack{h, \tilde{k}, m, n \\ hn = p_1 \cdots p_\ell \tilde{k} m \\ (p_1 \cdots p_\ell, \tilde{k}) = 1}} \frac{\mu(h)\mu(\tilde{k})}{h^{1/2+s} \tilde{k}^{1/2+u} m^{1/2+\alpha+z} n^{1/2+\beta+z}}.$$

Recall that $\nu_p(n) = n'$ denotes the number of times the prime number p appears in n . We can write the above as

$$(3.7) \quad \begin{aligned} \tilde{S}_\ell &= \prod_{p \in \{p_1, \dots, p_\ell\}} \sum_{h' + n' = m' + 1} \frac{\mu(p^{h'})}{(p^{h'})^{1/2+s} (p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\ &\quad \times \prod_{p \notin \{p_1, \dots, p_\ell\}} \sum_{h' + n' = k' + m'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s} (p^{k'})^{1/2+u} (p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\ &= \frac{\Pi_1(\alpha, \beta, s, u, z)}{\Pi_2(\alpha, \beta, s, u, z)} \Pi_3(\alpha, \beta, s, u, z), \end{aligned}$$

where we define

$$\begin{aligned} \Pi_1(\alpha, \beta, s, u, z) &= \prod_p \sum_{h' + n' = k' + m'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s} (p^{k'})^{1/2+u} (p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\ &= \prod_p \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right), \end{aligned}$$

as well as

$$\begin{aligned} \Pi_2(\alpha, \beta, s, u, z) &= \prod_{p \in \{p_1, \dots, p_\ell\}} \sum_{h' + n' = k' + m'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s} (p^{k'})^{1/2+u} (p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\ &= \prod_{p \in \{p_1, \dots, p_\ell\}} \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right), \end{aligned}$$

and finally

$$\begin{aligned} \Pi_3(\alpha, \beta, s, u, z) &= \prod_{p \in \{p_1, \dots, p_\ell\}} \sum_{h' + n' = m' + 1} \frac{\mu(p^{h'})}{(p^{h'})^{1/2+s} (p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\ &= \prod_{p \in \{p_1, \dots, p_\ell\}} \left(\frac{1}{p^{1/2+\beta+z}} - \frac{1}{p^{1/2+s}} + O(p^{-2+\varepsilon}) \right). \end{aligned}$$

Hence we arrive at the following expression for \tilde{S}_ℓ

$$\begin{aligned} \tilde{S}_\ell &= \prod_p \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right) \\ &= \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)} A_{\alpha, \beta}(s, u, z), \end{aligned}$$

where the arithmetical factor $A_{\alpha,\beta}(s, u, z)$ is given by an absolutely convergent Euler product in some product of half-planes containing the origin. Therefore, when we go back to the expression for S_ℓ in (3.6), we obtain the following

$$(3.8) \quad S_\ell = \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)} A_{\alpha,\beta}(s, u, z) (-1)^\ell \sum_{\substack{p_i \neq p_j \\ i < j}} \log p_1 \cdots \log p_\ell \\ \times \prod_{p \in \{p_1, \dots, p_\ell\}} \frac{E(p) + O(p^{-2+\varepsilon})}{1 - \frac{1}{p^{1+s+\alpha+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} - E(p) + O(p^{-2+\varepsilon})},$$

where

$$E(p) = \frac{1}{p^{1/2+u}} \left(-\frac{1}{p^{1/2+s}} + \frac{1}{p^{1/2+\beta+z}} \right) = -\frac{1}{p^{1+s+u}} + \frac{1}{p^{1+\beta+u+z}}.$$

At this stage, we compare (3.8) in its exact form (that is, with big- O terms replaced by their exact expressions) against (3.6) and (3.7) in its exact form, and we use the fact that for $\alpha = \beta = 0$ and $s = u = z$, the ratio of zeta functions

$$\frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)}$$

reduces to 1. In other words, reverting the p -adic analysis in

$$\frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)} A_{\alpha,\beta}(s, u, z) \\ = \prod_p \sum_{h'+n'=k'+m'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s}(p^{k'})^{1/2+u}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}},$$

we find that

$$\frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)} A_{\alpha,\beta}(s, u, z) = \sum_{hn=km} \frac{\mu(h)\mu(k)}{h^{1/2+s}k^{1/2+u}m^{1/2+\alpha+z}n^{1/2+\beta+z}}.$$

Following (3.2), we know that

$$A_{0,0}(z, z, z) = \sum_{km=hn} \frac{\mu(h)\mu(k)}{(hkmn)^{1/2+z}},$$

and thus, we find that

$$A_{0,0}(z, z, z) = 1$$

for all z . Let us denote the last part of (3.8) by H_ℓ , specifically

$$H_\ell = (-1)^\ell \sum_{\substack{p_i \neq p_j \\ i < j}} \prod_{p \in \{p_1, \dots, p_\ell\}} (E(p) + O(p^{-2+\varepsilon})) \log p \left(1 + E(p) + \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right) \\ = (-1)^\ell \sum_{\substack{p_i \neq p_j \\ i < j}} \prod_{p \in \{p_1, \dots, p_\ell\}} \left(E(p) \log p + O\left(\frac{\log p}{p^{2-\varepsilon}}\right) \right).$$

We now employ the principle of inclusion-exclusion to write

$$H_\ell = (-1)^\ell \left(\sum_p E(p) \log p + O\left(\frac{\log p}{p^{2-\varepsilon}}\right) \right)^\ell + \sum_p B(p),$$

where

$$B(p) \ll_{\alpha,\beta,s,u,z,\varepsilon} \frac{1}{p^{2-\varepsilon}}.$$

To end the computation, we must identify the logarithms of the prime numbers with the signature of the von Mangoldt function $\Lambda(n)$ and hence match the resulting expressions to logarithmic derivatives of the Riemann zeta-function by the use of

$$\frac{\zeta'}{\zeta}(s) = - \sum_{n=1}^{\infty} \Lambda(n)n^{-s} = - \sum_p \frac{\log p}{p^s} \left(1 - \frac{1}{p^s}\right)^{-1} = - \sum_p \frac{\log p}{p^s} + O\left(\frac{\log p}{p^{2s}}\right),$$

for $\operatorname{Re}(s) > 1$. With this in mind, H_ℓ becomes

$$\begin{aligned} H_\ell &= (-1)^\ell \left(\frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\beta+u+z) + O(1) \right)^\ell + D(\alpha, \beta, s, u, z) \\ &= (-U)^\ell + \sum_{m=0}^{\ell-1} U^m B_m(\alpha, \beta, s, u, z) + D(\alpha, \beta, s, u, z), \end{aligned}$$

where $D(\alpha, \beta, s, u, z)$ are terms of smaller order and where

$$U = -\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\beta+u+z).$$

We also have that

$$B_m(\alpha, \beta, s, u, z) \ll_{\alpha, \beta, s, u, z} \sum_p \frac{\log p}{p^{2-\varepsilon}}.$$

All of these terms are analytic in a larger region of the complex plane, thus we are only interested in the term U^ℓ . Consequently, the end result of this is that

$$\begin{aligned} I'_{12}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \sum_{\ell=2}^K \sum_{i,j} \frac{a_i b_{\ell,j} i! j!}{\log^i y_1 \log^{j+\ell} y_2} \left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)} \\ &\quad \times \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)} A_{\alpha,\beta}(s, u, z) \left(\frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\beta+u+z) \right)^\ell \\ &\quad \times (-1)^\ell y_1^s y_2^u \frac{G(z)}{z} g_{\alpha,\beta}(z, t) \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt. \end{aligned}$$

The next step is to deform the path of integration to $\operatorname{Re}(z) = -\delta + \varepsilon$ where $\delta > 0$ is small, fixed and $\delta < \varepsilon$ as well as $\operatorname{Re}(s) = \operatorname{Re}(u) = \delta$. By doing this, we pick up the contribution of the residue of the simple pole of $1/z$ at $z = 0$ only, since, as before in the $I_{11}(\alpha, \beta)$ case, $G(z)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2z)$. The new path of integration with respect to z contributes

$$(3.9) \quad \ll T^{1+\varepsilon} \left(\frac{y_1 y_2}{T} \right)^\delta \ll T^{1-\varepsilon}.$$

by keeping $\theta_1 + \theta_2 = 1 - \varepsilon$ (since $y_1 = T^{\theta_1}$ and $y_2 = T^{\theta_2}$). We now write

$$I'_{12}(\alpha, \beta) = I'_{120}(\alpha, \beta) + O(T^{-1+\varepsilon}),$$

where $I'_{120}(\alpha, \beta)$ corresponds to the residue at $z = 0$. Then

$$\begin{aligned} I'_{120}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \sum_{\ell=2}^K \sum_{i,j} \frac{a_i b_{\ell,j} i! j!}{\log^i y_1 \log^{j+\ell} y_2} \\ &\quad \times \left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)} \int_{(\delta)} \operatorname{res}_{z=0} \frac{G(z)}{z} g_{\alpha,\beta}(z, t) y_1^s y_2^u \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha,\beta}(s, u, z) \\ &\quad \times (-1)^\ell \left(\frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\beta+u+z) \right)^\ell \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt \end{aligned}$$

$$(3.10) \quad = \widehat{w}(0)\zeta(1+\alpha+\beta) \sum_{\ell=2}^K (-1)^\ell \sum_{i,j} \frac{a_i b_{\ell,j} i! j!}{\log^i y_1 \log^{j+\ell} y_2} J_{12},$$

where

$$J_{12}(\alpha, \beta) = \left(\frac{1}{2\pi i} \right)^2 \int_{(\delta)} \int_{(\delta)} \frac{\zeta(1+s+u) A_{\alpha, \beta}(s, u, 0)}{\zeta(1+u+\beta) \zeta(1+s+\alpha)} \left(\frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\beta+u) \right)^\ell y_1^s y_2^u \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}}.$$

Let us now use the binomial theorem to write

$$\begin{aligned} J_{12}(\alpha, \beta) &= \left(\frac{1}{2\pi i} \right)^2 \int_{(\delta)} \int_{(\delta)} \frac{\zeta(1+s+u) A_{\alpha, \beta}(s, u, 0)}{\zeta(1+\beta+u) \zeta(1+\alpha+s)} \\ &\quad \times \sum_{r=0}^{\ell} \binom{\ell}{r} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell-r} \left(-\frac{\zeta'}{\zeta}(1+s+u) \right)^r y_1^s y_2^u \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} \\ &= \left(\frac{1}{2\pi i} \right)^2 \int_{(\delta)} \int_{(\delta)} \frac{A_{\alpha, \beta}(s, u, 0)}{\zeta(1+\beta+u) \zeta(1+\alpha+s)} \sum_{r=0}^{\ell} \binom{\ell}{r} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell-r} \\ &\quad \times \sum_{n=1}^{\infty} \frac{(\mathbf{1} * \Lambda^{*r})(n)}{n^{1+s+u}} y_1^s y_2^u \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} \\ &= \sum_{n \leq \min(y_1, y_2)} \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(\mathbf{1} * \Lambda^{*r})(n)}{n} \left(\frac{1}{2\pi i} \right)^2 \\ &\quad \times \int_{(\delta)} \int_{(\delta)} \frac{A_{\alpha, \beta}(s, u, 0)}{\zeta(1+\beta+u) \zeta(1+\alpha+s)} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell-r} \left(\frac{y_1}{n} \right)^s \left(\frac{y_2}{n} \right)^u \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}}, \end{aligned}$$

where we have used the Dirichlet convolution of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

for $\text{Re}(s) > 1$. Here $\mathbf{1}(n) = 1$ for all n denotes the identity function. Next, we take $\delta \asymp L^{-1}$ and bound the integral trivially to get $J_{12} \ll L^{i+j-1}$. This means that we can use a Taylor series so that $A_{\alpha, \beta}(s, u, 0) = A_{0,0}(0, 0, 0) + O(|s| + |u|)$ to write $J_{12}(\alpha, \beta) = J'_{12}(\alpha, \beta) + O(L^{i+j-2})$, say. We recall that we have shown earlier that $A_{0,0}(z, z, z) = 1$ for all z , in particular $A_{0,0}(0, 0, 0) = 1$. This implies that the complex variables s and u are now separated as

$$J'_{12}(\alpha, \beta) = \sum_{n \leq \min(y_1, y_2)} \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(\mathbf{1} * \Lambda^{*r})(n)}{n} L_{12,1} L_{12,2},$$

where

$$L_{12,1} = \frac{1}{2\pi i} \int_{(\delta)} \frac{1}{\zeta(1+\alpha+s)} \left(\frac{y_1}{n} \right)^s \frac{ds}{s^{i+1}},$$

and

$$(3.11) \quad L_{12,2} = \frac{1}{2\pi i} \int_{(\delta)} \frac{1}{\zeta(1+\beta+u)} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell-r} \left(\frac{y_2}{n} \right)^u \frac{du}{u^{j+1}}.$$

The first of these two integrals was dealt with in the $I_{11}(\alpha, \beta)$ case and its main term is

$$L_{12,1} = \frac{1}{2\pi i} \oint (\alpha+s) \left(\frac{y_1}{n} \right)^s \frac{ds}{s^{i+1}} = \frac{1}{i!} \frac{d}{dx} e^{\alpha x} \left(x + \log \frac{y_1}{n} \right)^i \Big|_{x=0}.$$

For the second integral we will need the following Lemma 2.2 and equation (2.1). Hence, one gets

$$L_{12,2} = \frac{(-1)^{\ell-r}}{2\pi i} \oint (\beta + u)^{1-\ell+r} \left(\frac{y_2}{n}\right)^u \frac{du}{u^{j+1}} = \frac{(-1)^{\ell-r}}{j!} \frac{d^{1-\ell+r}}{dy^{1-\ell+r}} e^{\beta y} \left(y + \log \frac{y_2}{n}\right)^j \Big|_{y=0}.$$

This means that when we insert these results into J'_{12} we obtain

$$\begin{aligned} J'_{12}(\alpha, \beta) &= \frac{1}{i!} \frac{1}{j!} \sum_{n \leq \min(y_1, y_2)} \sum_{r=0}^{\ell} (-1)^{\ell-r} \binom{\ell}{r} \frac{(\mathbf{1} * \Lambda^{*r})(n)}{n} \\ &\quad \times \frac{d}{dx} \frac{d^{1-\ell+r}}{dy^{1-\ell+r}} e^{\alpha x + \beta y} \left(x + \log \frac{y_1}{n}\right)^i \Big|_{x=0} \left(y + \log \frac{y_2}{n}\right)^j \Big|_{y=0} + O(L^{i+j-2}). \end{aligned}$$

By making the changes

$$x \rightarrow \frac{x}{\log y_1} \quad \text{and} \quad y \rightarrow \frac{y}{\log y_2},$$

we can write this in the more convenient form

$$\begin{aligned} J'_{12}(\alpha, \beta) &= \frac{\log^{i-1} y_1 \log^{j-1} y_2}{i! j!} \sum_{n \leq \min(y_1, y_2)} \sum_{r=0}^{\ell} (-1)^{\ell-r} \binom{\ell}{r} \frac{(\mathbf{1} * \Lambda^{*r})(n)}{n} \\ &\quad \times \frac{d}{dx} \frac{d^{1-\ell+r}}{dy^{1-\ell+r}} y_1^{\alpha x} y_2^{\beta y} \left(x + \frac{\log(y_1/n)}{\log y_1}\right)^i \Big|_{x=0} \left(y + \frac{\log(y_2/n)}{\log y_2}\right)^j \Big|_{y=0} + O(L^{i+j-2}). \end{aligned}$$

Telescoping back to (3.10) we obtain that

$$\begin{aligned} I'_{120}(\alpha, \beta) &= \frac{\widehat{w}(0)}{(\alpha + \beta) \log y_1 \log y_2} \frac{d^2}{dx dy} \left[y_1^{\alpha x} y_2^{\beta y} \right. \\ &\quad \times \sum_{\ell=2}^L (-1)^\ell \frac{1}{\log^\ell y_2} \sum_{r=0}^{\ell} (-1)^{\ell-r} \binom{\ell}{r} \frac{d^{r-\ell}}{dy^{r-\ell}} \\ &\quad \left. \times \sum_{n \leq \min(y_1, y_2)} \frac{(\mathbf{1} * \Lambda^{*r})(n)}{n} P_1 \left(x + \frac{\log(y_1/n)}{\log y_1}\right) P_\ell \left(y + \frac{\log(y_2/n)}{\log y_2}\right) \Big|_{x=y=0} \right] + O(T/L), \end{aligned}$$

where the sum over i has been identified to the polynomial P_1 , and the sum over j to the polynomials P_ℓ . We now perform the summation over n by using Lemma 2.5. To do so, we now set $y_1 \geq y_2$. The lemma yields

$$\begin{aligned} \sum_{n \leq y_2} \frac{(\mathbf{1} * \Lambda^{*r})(n)}{n^{1+s}} P_1 \left(x + \frac{\log(y_1/n)}{\log y_1}\right) P_\ell \left(y + \frac{\log(y_2/n)}{\log y_2}\right) \\ = \frac{\log^{r+1} y_2}{y_2^s} \int_0^1 (1-u)^r P_1 \left(x + 1 - (1-u) \frac{\theta_2}{\theta_1}\right) P_\ell(y+u) y_2^{us} du + O(\log(3y_2)^r). \end{aligned}$$

Therefore, the resulting expression for I'_{120} is

$$\begin{aligned} I'_{120}(\alpha, \beta) &= \frac{\widehat{w}(0)}{(\alpha + \beta) \log y_1 \log y_2} \frac{d^2}{dx dy} \left[\int_0^1 \right. \\ &\quad \times y_1^{\alpha x} y_2^{\beta y} \sum_{\ell=2}^L (-1)^\ell \frac{1}{\log^\ell y_2} \sum_{r=0}^{\ell} (-1)^{\ell-r} \binom{\ell}{r} \frac{d^{r-\ell}}{dy^{r-\ell}} \\ &\quad \left. \times \frac{\log^{r+1} y_2}{r!} (1-u)^r P_1 \left(x + 1 - (1-u) \frac{\theta_2}{\theta_1}\right) P_\ell(y+u) du \Big|_{x=y=0} \right] + O(T/L) \end{aligned}$$

Now we must go back to I_{12} . We recall that $I_{12}(\alpha, \beta)$ was formed by adding $I'_{12}(\alpha, \beta)$ and $I''_{12}(\alpha, \beta)$, where I''_{12} is formed by taking I'_{12} , switching α and $-\beta$, and then multiplying by $T^{-\alpha-\beta}$. Note that $r \leq \ell$ and thus only the case $r = \ell$ contributes to the main term. Therefore

$$I'_{12}(\alpha, \beta) = \frac{\widehat{w}(0)}{(\alpha + \beta) \log y_1} \sum_{\ell=2}^L (-1)^\ell \frac{1}{\ell!} \frac{d^2}{dx dy} \left[y_1^{\alpha x} y_2^{\beta y} \right. \\ \left. \times \int_0^1 (1-u)^\ell P_1 \left(x+1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(y+u) du \Big|_{x=y=0} \right] + O(T/L).$$

We now use

$$I_{12}(\alpha, \beta) = I'_{12}(\alpha, \beta) + T^{-\alpha-\beta} I'_{12}(-\beta, -\alpha) + O(T/L) \\ = (I'_{12}(\alpha, \beta) + I'_{12}(-\beta, -\alpha)) + (T^{-\alpha-\beta} - 1) I'_{12}(-\beta, -\alpha) + O(T/L).$$

We first take a look at the first term in the brackets

$$\frac{d^2}{dx dy} \left[\left(y_1^{\alpha x} y_2^{\beta y} - y_1^{-\beta x} y_2^{-\alpha y} \right) \int_0^1 (1-u)^\ell P_1 \left(x+1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(y+u) du \Big|_{x=y=0} \right] \\ = (\alpha + \beta) \log y_1 \int_0^1 (1-u)^\ell P_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P'_\ell(y) du \\ + (\alpha + \beta) \log y_2 \int_0^1 (1-u)^\ell P'_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(y) du \\ = (\alpha + \beta) \log y_1 \left(\int_0^1 (1-u)^\ell P_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P'_\ell(y) du + \int_0^1 (1-u)^\ell P'_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(y) du \right) \\ - (\alpha + \beta)(\theta_1 - \theta_2) \log T \int_0^1 (1-u)^\ell P'_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(y) du$$

Since we had that $P_1(0) = P_\ell(0) = 0$ it follows that

$$0 = (1-u)^\ell P_1(u) P_\ell(u) \Big|_{u=0}^1 = \int_0^1 \left((1-u)^\ell P_1(u) P_\ell(u) \right)' du.$$

We can therefore write

$$\ell \int_0^1 (1-u)^{\ell-1} P_1(u) P_\ell(u) du = \int_0^1 (1-u)^\ell P_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P'_\ell(u) du \\ + \int_0^1 (1-u)^\ell P'_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(u) du.$$

Combining these observations, we see that

$$I'_{12}(\alpha, \beta) + I'_{12}(-\beta, -\alpha) = \widehat{w}(0) \sum_{\ell=2}^L \frac{(-1)^\ell}{(\ell-1)!} \int_0^1 (1-u)^{\ell-1} P_1(u) P_\ell(u) du \\ - \widehat{w}(0) \frac{\theta_1 - \theta_2}{\theta_1} \sum_{\ell=2}^L \frac{(-1)^\ell}{\ell!} \int_0^1 (1-u)^\ell P'_1 \left(1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(u) du.$$

For the expression $(T^{-\alpha-\beta} - 1) I'_{12}(-\beta, -\alpha)$, we use (3.4) to get

$$(T^{-\alpha-\beta} - 1) I'_{12}(-\beta, -\alpha) \\ = \frac{\widehat{w}(0)}{\theta_1} \sum_{\ell=2}^L \frac{(-1)^\ell}{\ell!} \frac{d^2}{dx dy} \left[y_1^{-\beta x} y_2^{-\alpha y} \int_0^1 \int_0^1 T^{-v(\alpha+\beta)} (1-u)^\ell P_1 \left(x+1 - (1-u) \frac{\theta_2}{\theta_1} \right) P_\ell(y+u) dudv \Big|_{x=y=0} \right]$$

+ $O(T/L)$.

By using similar arguments for the holomorphy of the error terms as in the Section 3.1, we end the proof of Lemma 1.2.

3.3. Proof of Lemma 1.3. This is the hardest case. Once again, we insert the definitions of the Feng mollifiers ψ_2 in the mean value integral $I_{22}(\alpha, \beta)$ so that

$$\begin{aligned}
I_{22}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \psi_F \overline{\psi_F}(\sigma_0 + it) dt \\
&= \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \sum_{h_1 \leq y_2} \frac{\mu(h_1)}{h_1^{1/2+it}} \\
&\quad \times \sum_{\ell_1=2}^K \sum_{p_1 p_2 \cdots p_{\ell_1} | h_1} \frac{\log p_1 \log p_2 \cdots \log p_{\ell_1}}{\log^{\ell_1} y_2} P_{\ell_1}[h_1] \\
&\quad \times \sum_{h_2 \leq y_2} \frac{\mu(h_2)}{h_2^{1/2-it}} \sum_{\ell_2=2}^K \sum_{q_1 q_2 \cdots q_{\ell_2} | h_2} \frac{\log q_1 \log q_2 \cdots \log q_{\ell_2}}{\log^{\ell_2} y_2} P_{\ell_2}[h_2] dt \\
&= \sum_{h_1, h_2 \leq y_2} \frac{\mu(h_1) \mu(h_2)}{\sqrt{h_1 h_2}} \\
&\quad \times \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{p_1 p_2 \cdots p_{\ell_1} | h_1} \sum_{q_1 q_2 \cdots q_{\ell_2} | h_2} \frac{\log p_1 \log p_2 \cdots \log p_{\ell_1} \log q_1 \log q_2 \cdots \log q_{\ell_2}}{\log^{\ell_1 + \ell_2} y_2} \\
&\quad \times P_{\ell_1}[h_1] P_{\ell_2}[h_2] \int_{-\infty}^{\infty} w(t) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) \left(\frac{h_1}{h_2}\right)^{-it} dt.
\end{aligned}$$

We already explained in the computation of $I_{12}(\alpha, \beta)$ how to deal with this integral, namely write $I_{22}(\alpha, \beta) = I'_{22}(\alpha, \beta) + I''_{22}(\alpha, \beta)$, where $I''_{22}(\alpha, \beta)$ can be obtained from I'_{22} by switching α and $-\beta$ and multiplying by

$$\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} = T^{-\alpha-\beta} + O(L^{-1}).$$

We now use the Mellin integral representations of the polynomials

$$P_{\ell_1}[h_1] = \sum_i \frac{b_{i, \ell_1}}{\log^i y_2} (\log(y_2/h_1))^i = \sum_i \frac{b_{i, \ell_1} i!}{\log^i y_2} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_2}{h_1}\right)^u \frac{du}{u^{i+1}},$$

and

$$P_{\ell_2}[h_2] = \sum_j \frac{b_{j, \ell_2}}{\log^j y_2} (\log(y_2/h_2))^j = \sum_j \frac{b_{j, \ell_2} j!}{\log^j y_2} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_2}{h_2}\right)^s \frac{ds}{s^{j+1}}.$$

This leaves us with

$$\begin{aligned}
I'_{22}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{i, j} \frac{b_{i, \ell_1} i!}{\log^{i+j} y_2} \frac{b_{j, \ell_2} j!}{\log^{\ell_1 + \ell_2} y_2} \\
&\quad \times \left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)} y_2^{s+u} g_{\alpha, \beta}(z, t) \frac{G(z)}{z} \sum_{mh_1 = nh_2} \frac{\mu(h_1) \mu(h_2)}{h_1^{1/2+u} h_2^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}} \\
&\quad \times \sum_{p_1 p_2 \cdots p_{\ell_1} | h_1} \sum_{q_1 q_2 \cdots q_{\ell_2} | h_2} \log p_1 \log p_2 \cdots \log p_{\ell_1} \log q_1 \log q_2 \cdots \log q_{\ell_2} dz \frac{du}{u^{j+1}} \frac{ds}{u^{i+1}} dt.
\end{aligned}$$

We now have to compute the arithmetical sum $\sum_{mh_1=nh_2}$ with p -adic analysis. The first step is to consolidate the two sums over primes into a single sum. This is accomplished by the use of Lemma 2.3. Let us define

$$(3.12) \quad S_{\ell_1, \ell_2, k} = \sum_{mh_1=nh_2} \frac{\mu(h_1)\mu(h_2)}{h_1^{1/2+u} h_2^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}}$$

$$\times \sum_{\substack{p_1 \cdots p_k q_1 \cdots q_{\ell_1-k} r_1 \cdots r_{\ell_2-k} | h_1 h_2 / \gcd(h_1, h_2) \\ p_1 \cdots p_k | \gcd(h_1, h_2) \\ q_1 \cdots q_{\ell_1-k} | h_1 \\ r_1 \cdots r_{\ell_2-k} | h_2}} \log^2 p_1 \cdots \log^2 p_k \log q_1 \cdots \log q_{\ell_1-k} \cdots \log r_1 \cdots \log r_{\ell_2-k}.$$

The next step is to swap the order of the sums. For this, we making the changes

$$h_1 = \tilde{h}_1 p_1 \cdots p_k q_1 \cdots q_{\ell_1-k},$$

$$h_2 = \tilde{h}_2 p_1 \cdots p_k r_1 \cdots r_{\ell_2-k},$$

implying that

$$(\tilde{h}_1, p_1 \cdots p_k q_1 \cdots q_{\ell_1-k}) = 1,$$

$$(\tilde{h}_2, p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}) = 1,$$

$$(q_1 \cdots q_{\ell_1-k}, r_1 \cdots r_{\ell_2-k}) = 1,$$

so that

$$S_{\ell_1, \ell_2, k} = (-1)^{\ell_1 + \ell_2} \sum_{\substack{p_i \neq p_j \\ q_i \neq q_j \\ r_i \neq r_j \\ p_i \neq q_i \neq r_i}} \frac{\log^2 p_1 \cdots \log^2 p_k \log q_1 \cdots \log q_{\ell_1-k} \log r_1 \cdots \log r_{\ell_2-k}}{(p_1 \cdots p_k q_1 \cdots q_{\ell_1-k})^{1/2+u} (p_1 \cdots p_k r_1 \cdots r_{\ell_2-k})^{1/2+s}}$$

$$\times \sum_{\substack{m \tilde{h}_1 p_1 \cdots p_k q_1 \cdots q_{\ell_1-k} = n \tilde{h}_2 p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}, \\ (\tilde{h}_1, p_1 \cdots p_k q_1 \cdots q_{\ell_1-k}) = 1, \\ (\tilde{h}_2, p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}) = 1}} \frac{\mu(\tilde{h}_1)\mu(\tilde{h}_2)}{(\tilde{h}_1)^{1/2+u} (\tilde{h}_2)^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}}.$$

Here the p 's, the q 's and the r 's are all distinct primes. Let us define the inner sum to be $\tilde{S}_{\ell_1, \ell_2, k}$ and let us recall that $\nu_p(n) = n'$ is the number of times the prime p appears in n so that

$$\tilde{S}_{\ell_1, \ell_2, k} = \sum_{\substack{m \tilde{h}_1 p_1 \cdots p_k q_1 \cdots q_{\ell_1-k} = n \tilde{h}_2 p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}, \\ (\tilde{h}_1, p_1 \cdots p_k q_1 \cdots q_{\ell_1-k}) = 1 \\ (\tilde{h}_2, p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}) = 1}} \frac{\mu(\tilde{h}_1)\mu(\tilde{h}_2)}{(\tilde{h}_1)^{1/2+u} (\tilde{h}_2)^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}}$$

$$= \prod_{p \in \{p_1, \dots, p_k\}} \sum_{n'=m'} \frac{1}{(p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}}$$

$$\times \prod_{q \in \{q_1, \dots, q_{\ell_1-k}\}} \sum_{1+m' = n' + \tilde{h}_2} \frac{\mu(q^{\tilde{h}_2})}{(q^{\tilde{h}_2})^{1/2+s} (q^{m'})^{1/2+\alpha+z} (q^{n'})^{1/2+\beta+z}}$$

$$\times \prod_{r \in \{r_1, \dots, r_{\ell_2-k}\}} \sum_{\tilde{h}_1 + m' = n' + 1} \frac{\mu(r^{\tilde{h}_1})}{(r^{\tilde{h}_1})^{1/2+u} (r^{m'})^{1/2+\alpha+z} (r^{n'})^{1/2+\beta+z}}$$

$$\begin{aligned}
& \times \prod_{p \notin \{p_1, \dots, p_k\} \cup \{q_1, \dots, q_{\ell_1-k}\} \cup \{r_1, \dots, r_{\ell_2-k}\}} \\
& \times \sum_{\tilde{h}'_1+m'=n'+\tilde{h}'_2} \frac{\mu(p^{\tilde{h}'_1})\mu(p^{\tilde{h}'_2})}{(p^{\tilde{h}'_1})^{1/2+u} (p^{\tilde{h}'_2})^{1/2+s} (p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\
& = \frac{\Pi_1(\alpha, \beta, s, u, z) \Pi_2(\alpha, \beta, s, u, z) \Pi_3(\alpha, \beta, s, u, z) \Pi_4(\alpha, \beta, s, u, z)}{\Pi_5(\alpha, \beta, s, u, z)}.
\end{aligned}$$

Each product is evaluated to

$$\begin{aligned}
\Pi_1(\alpha, \beta, s, u, z) &= \prod_p \sum_{\tilde{h}'_1+m'=n'+\tilde{h}'_2} \frac{\mu(p^{\tilde{h}'_1})\mu(p^{\tilde{h}'_2})}{(p^{\tilde{h}'_1})^{1/2+u} (p^{\tilde{h}'_2})^{1/2+s} (p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\
&= \prod_p \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right),
\end{aligned}$$

then

$$\begin{aligned}
\Pi_2(\alpha, \beta, s, u, z) &= \prod_{p \in \{p_1, \dots, p_k\}} \sum_{n'=m'} \frac{1}{(p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\
(3.13) \quad &= \prod_{p \in \{p_1, \dots, p_k\}} \left(1 + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right),
\end{aligned}$$

followed by

$$\begin{aligned}
\Pi_3(\alpha, \beta, s, u, z) &= \prod_{q \in \{q_1, \dots, q_{\ell_1-k}\}} \sum_{m'+1=n'+\tilde{h}'_2} \frac{\mu(q^{\tilde{h}'_2})}{(q^{\tilde{h}'_2})^{1/2+s} (q^{m'})^{1/2+\alpha+z} (q^{n'})^{1/2+\beta+z}} \\
&= \prod_{q \in \{q_1, \dots, q_{\ell_1-k}\}} \left(-\frac{1}{q^{1/2+s}} + \frac{1}{q^{1/2+\beta+z}} + O(q^{-2+\varepsilon}) \right),
\end{aligned}$$

as well as

$$\begin{aligned}
\Pi_4(\alpha, \beta, s, u, z) &= \prod_{r \in \{r_1, \dots, r_{\ell_2-k}\}} \sum_{\tilde{h}'_1+m'=n'+1} \frac{\mu(r^{\tilde{h}'_1})}{(r^{\tilde{h}'_1})^{1/2+u} (r^{m'})^{1/2+\alpha+z} (r^{n'})^{1/2+\beta+z}} \\
&= \prod_{r \in \{r_1, \dots, r_{\ell_2-k}\}} \left(-\frac{1}{r^{1/2+u}} + \frac{1}{r^{1/2+\alpha+z}} + O(r^{-2+\varepsilon}) \right),
\end{aligned}$$

and finally

$$\begin{aligned}
\Pi_5(\alpha, \beta, s, u, z) &= \prod_{p \in \{p_1, \dots, p_k\} \cup \{q_1, \dots, q_{\ell_1-k}\} \cup \{r_1, \dots, r_{\ell_2-k}\}} \\
& \times \sum_{\tilde{h}'_1+m'=n'+\tilde{h}'_2} \frac{\mu(p^{\tilde{h}'_1})\mu(p^{\tilde{h}'_2})}{(p^{\tilde{h}'_1})^{1/2+u} (p^{\tilde{h}'_2})^{1/2+s} (p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}} \\
& = \prod_{p \in \{p_1, \dots, p_k\} \cup \{q_1, \dots, q_{\ell_1-k}\} \cup \{r_1, \dots, r_{\ell_2-k}\}} \\
& \times \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right).
\end{aligned}$$

This leaves us with

$$\begin{aligned}\tilde{S}_{\ell_1, \ell_2, k} &= \prod_p \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right) \\ &= \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha, \beta}(s, u, z),\end{aligned}$$

where A is an arithmetical factor that is given by an absolutely convergent Euler product in some product of half-planes containing the origin. From our previous analysis of the $I_{12}(\alpha, \beta)$ case, we know that $A_{0,0}(z, z, z) = 1$ for all values of z . Therefore we end up with

$$\begin{aligned}S_{\ell_1, \ell_2, k} &= \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha, \beta}(s, u, z) \\ &\quad \times (-1)^{\ell_1+\ell_2} \sum_{\substack{p_i \neq p_j \\ q_i \neq q_j \\ r_i \neq r_j \\ p_i \neq q_i \neq r_i}} \log^2 p_1 \cdots \log^2 p_k \log q_1 \cdots \log q_{\ell_1-k} \log r_1 \cdots \log r_{\ell_2-k} \\ &\quad \times \prod_{p \in \{p_1, \dots, p_k\}} \frac{E_1(p) + O(p^{-2+\varepsilon})}{1 + \frac{1}{p^{1+s+\alpha+z}} + \frac{1}{p^{1+u+\beta+z}} - \frac{1}{p^{1+\alpha+\beta+2z}} + E_1(p) + O(p^{-2+\varepsilon})} \\ &\quad \times \prod_{q \in \{q_1, \dots, q_{\ell_1-k}\}} \frac{E_2(q) + O(q^{-2+\varepsilon})}{1 + \frac{1}{q^{1+s+\alpha+z}} - \frac{1}{q^{1+\alpha+\beta+2z}} - E_2(q) + O(q^{-2+\varepsilon})} \\ &\quad \times \prod_{r \in \{r_1, \dots, r_{\ell_2-k}\}} \frac{E_3(r) + O(r^{-2+\varepsilon})}{1 + \frac{1}{r^{1+u+\beta+z}} - \frac{1}{r^{1+\alpha+\beta+2z}} - E_3(r) + O(r^{-2+\varepsilon})},\end{aligned}$$

where

$$E_1(p) = \frac{1}{p^{1+s+u}},$$

and

$$E_2(q) = \frac{1}{q^{1/2+u}} \left(\frac{1}{q^{1/2+s}} - \frac{1}{q^{1/2+\beta+z}} \right) = \frac{1}{q^{1+s+u}} - \frac{1}{q^{1+\beta+u+z}},$$

and finally

$$E_3(r) = \frac{1}{r^{1/2+s}} \left(\frac{1}{r^{1/2+u}} - \frac{1}{r^{1/2+\alpha+z}} \right) = \frac{1}{r^{1+s+u}} - \frac{1}{r^{1+\alpha+s+z}}.$$

We define $H_{\ell_1, \ell_2, k}$ to be the last part of $S_{\ell_1, \ell_2, k}$. This means that

$$\begin{aligned}H_{\ell_1, \ell_2, k} &= (-1)^{\ell_1+\ell_2} \sum_{\substack{p_i \neq p_j \\ q_i \neq q_j \\ r_i \neq r_j \\ p_i \neq q_i \neq r_i}} \prod_{p \in \{p_1, \dots, p_k\}} (E_1(p) + O(p^{-2+\varepsilon})) \log^2 p \\ &\quad \times \prod_{p \in \{p_1, \dots, p_k\}} \left(1 - E_1(p) - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon}) \right) \\ &\quad \times \prod_{q \in \{q_1, \dots, q_{\ell_1-k}\}} (E_2(q) + O(q^{-2+\varepsilon})) \log q \left(1 + E_2(q) - \frac{1}{q^{1+s+\alpha+z}} + \frac{1}{q^{1+\alpha+\beta+2z}} + O(q^{-2+\varepsilon}) \right) \\ &\quad \times \prod_{r \in \{r_1, \dots, r_{\ell_2-k}\}} (E_3(r) + O(r^{-2+\varepsilon})) \log r \left(1 + E_3(r) - \frac{1}{r^{1+u+\beta+z}} + \frac{1}{r^{1+\alpha+\beta+2z}} + O(r^{-2+\varepsilon}) \right)\end{aligned}$$

$$\begin{aligned}
&= (-1)^{\ell_1+\ell_2} \sum_{\substack{p_i \neq p_j \\ q_i \neq q_j \\ r_i \neq r_j \\ p_i \neq q_i \neq r_i}} \prod_{p \in \{p_1, \dots, p_k\}} \left(E_1(p) \log^2 p + O\left(\frac{\log^2 p}{p^{2-\varepsilon}}\right) \right) \\
&\quad \times \prod_{q \in \{q_1, \dots, q_{\ell_1-k}\}} \left(E_2(q) \log q + O\left(\frac{\log q}{q^{2-\varepsilon}}\right) \right) \prod_{r \in \{r_1, \dots, r_{\ell_2-k}\}} \left(E_3(r) \log r + O\left(\frac{\log r}{r^{2-\varepsilon}}\right) \right) \\
&\quad + O(f(p^{-2+\varepsilon}, q^{-2+\varepsilon}, r^{-2+\varepsilon})),
\end{aligned}$$

for some polynomial f . Applying the inclusion-exclusion principle we then have

$$\begin{aligned}
H_{\ell_1, \ell_2, k} &= (-1)^{\ell_1+\ell_2} \left(\sum_p E_1(p) \log^2 p + O\left(\frac{\log^2 p}{p^{2-\varepsilon}}\right) \right)^k \\
&\quad \times \left(\sum_q E_2(q) \log q + O\left(\frac{\log q}{q^{2-\varepsilon}}\right) \right)^{\ell_1-k} \left(\sum_r E_3(r) \log r + O\left(\frac{\log r}{r^{2-\varepsilon}}\right) \right)^{\ell_2-k} + \sum_{p, q, r} B(p, q, r),
\end{aligned}$$

where

$$B(p, q, r) \ll_{\alpha, \beta, s, u, z, \varepsilon} f\left(\frac{1}{p^{2-\varepsilon}}, \frac{1}{q^{2-\varepsilon}}, \frac{1}{r^{2-\varepsilon}}\right).$$

As in the previous crossterm, we now need to identify the logarithms of the primes with the signature of the von Mangoldt functions $\Lambda(n)$ and $\Lambda_2(n)$. With this in mind, we first write

$$\frac{\zeta'}{\zeta}(s) = - \sum_{n=1}^{\infty} \Lambda(n) n^{-s} = - \sum_p \frac{\log p}{p^s} \left(1 - \frac{1}{p^s}\right)^{-1} = - \sum_p \frac{\log p}{p^s} + O\left(\frac{\log p}{p^{2s}}\right),$$

and

$$\frac{\zeta''}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda_2(n) n^{-s} = \sum_p \frac{\log^2 p}{p^s} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_p \frac{\log^2 p}{p^s} + O\left(\frac{\log^2 p}{p^{2s}}\right),$$

for $\operatorname{Re}(s) > 1$. This means that

$$\begin{aligned}
H_{\ell_1, \ell_2, k} &= (-1)^{\ell_1+\ell_2} \left(\frac{\zeta''}{\zeta}(1+s+u)\right)^k \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\beta+u+z)\right)^{\ell_1-k} \\
&\quad \times \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\alpha+s+z)\right)^{\ell_2-k} + D(\alpha, \beta, s, u, z) \\
&= (-1)^{\ell_1+\ell_2} (-V_1)^k (-V_2)^{\ell_1-k} (-V_3)^{\ell_2-k} \\
&\quad + \sum_{l=0}^{k-1} V_1^l A_l(\alpha, \beta, s, u, z) \sum_{m=0}^{\ell_1-k-1} V_2^m B_m(\alpha, \beta, s, u, z) \sum_{n=0}^{\ell_2-k-1} V_3^n C_n(\alpha, \beta, s, u, z),
\end{aligned}$$

where $D(\alpha, \beta, s, u, z)$ are terms of smaller order and where

$$V_1 = -\frac{\zeta''}{\zeta}(1+s+u), \quad V_2 = \frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\beta+u+z), \quad V_3 = \frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\alpha+s+z).$$

Moreover, we also have that

$$A_l(\alpha, \beta, s, u, z) \ll_{\alpha, \beta, s, u, z, \varepsilon} \sum_p \frac{\log^2 p}{p^{2-\varepsilon}}, \quad B_m(\alpha, \beta, s, u, z), C_n(\alpha, \beta, s, u, z) \ll_{\alpha, \beta, s, u, z, \varepsilon} \sum_p \frac{\log p}{p^{2-\varepsilon}}.$$

All of these terms are analytic in a larger region of the complex plane, thus we are only interested in the term $(-V_1)^k(-V_2)^{\ell_1-k}(-V_3)^{\ell_2-k}$. Consequently, the end result of this computation is that

$$\begin{aligned} I'_{22}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{i,j} \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k \frac{b_{i, \ell_1} i!}{\log^{i+j} y_2} \frac{b_{j, \ell_2} j!}{\log^{\ell_1 + \ell_2} y_2} \\ &\quad \times \left(\frac{1}{2\pi i} \right)^3 \int_{(1)} \int_{(1)} \int_{(1)} y_2^{s+u} g_{\alpha, \beta}(z, t) \frac{G(z)}{z} \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha, \beta}(s, u, z) \\ &\quad \times (-1)^{\ell_1 + \ell_2} \left(\frac{\zeta''}{\zeta}(1+s+u) \right)^k \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\beta+u+z) \right)^{\ell_1 - k} \\ &\quad \times \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\alpha+s+z) \right)^{\ell_2 - k} dz \frac{du}{u^{i+1}} \frac{ds}{s^{j+1}} dt. \end{aligned}$$

As in the calculation of I'_{12} , we now take the s, u, z contours of integration to $\delta > 0$ small and fixed with $\delta < \varepsilon$, and then move z to $-\delta + \varepsilon$, crossing a simple pole at $z = 0$ only (since, yet again, $G(z)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2z)$). The new line of integration with respect to z contributes

$$\ll T^{1+\varepsilon} \left(\frac{y_2^2}{T} \right)^\delta \ll T^{1-\varepsilon},$$

since $\theta_2 = 1/2 - \varepsilon$. Write $I'_{22}(\alpha, \beta) = I'_{220}(\alpha, \beta) + O(T^{1-\varepsilon})$, where $I'_{220}(\alpha, \beta)$ corresponds to the residue at $z = 0$, i.e.

$$\begin{aligned} I'_{220}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t) \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{i,j} \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k \frac{b_{i, \ell_1} i!}{\log^{i+j} y_2} \frac{b_{j, \ell_2} j!}{\log^{\ell_1 + \ell_2} y_2} \\ &\quad \times \left(\frac{1}{2\pi i} \right)^2 \int_{(\delta)} \int_{(\delta)} \operatorname{res}_{z=0} \frac{G(z)}{z} g_{\alpha, \beta}(z, t) y_2^{s+u} \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha, \beta}(s, u, z) \\ &\quad \times (-1)^{\ell_1 + \ell_2} \left(\frac{\zeta''}{\zeta}(1+s+u) \right)^k \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\beta+u+z) \right)^{\ell_1 - k} \\ &\quad \times \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\alpha+s+z) \right)^{\ell_2 - k} \frac{du}{u^{i+1}} \frac{ds}{s^{j+1}} dt \\ &= \widehat{w}(0) \zeta(1+\alpha+\beta) \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{i,j} \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k (-1)^{\ell_1 + \ell_2} \frac{b_{i, \ell_1} i!}{\log^{i+j} y_2} \frac{b_{j, \ell_2} j!}{\log^{\ell_1 + \ell_2} y_2} J_{22}, \end{aligned}$$

where

$$\begin{aligned} J_{22} &= \left(\frac{1}{2\pi i} \right)^2 \int_{(\delta)} \int_{(\delta)} y_2^{s+u} \frac{\zeta(1+s+u)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)} A_{\alpha, \beta}(s, u, 0) \\ &\quad \times \left(\frac{\zeta''}{\zeta}(1+s+u) \right)^k \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell_1 - k} \\ &\quad \times \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\alpha+s) \right)^{\ell_2 - k} \frac{du}{u^{i+1}} \frac{ds}{s^{j+1}}. \end{aligned}$$

The next step is to employ the binomial theorem in the part of the integrand that involves ζ functions. Calling this part \mathcal{Z} , we then have

$$\mathcal{Z}(s, u) := \frac{\zeta(1+s+u)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)} \left(\frac{\zeta''}{\zeta}(1+s+u) \right)^k$$

$$\begin{aligned}
& \times \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell_1-k} \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\alpha+s) \right)^{\ell_2-k} \\
&= \frac{\zeta(1+s+u)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)} \left(\frac{\zeta''}{\zeta}(1+s+u) \right)^k \\
& \times \sum_{r_1=0}^{\ell_1-k} \binom{\ell_1-k}{r_1} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell_1-k-r_1} \left(-\frac{\zeta'}{\zeta}(1+s+u) \right)^{r_1} \\
& \times \sum_{r_2=0}^{\ell_2-k} \binom{\ell_2-k}{r_2} \left(\frac{\zeta'}{\zeta}(1+\alpha+s) \right)^{\ell_2-k-r_2} \left(-\frac{\zeta'}{\zeta}(1+s+u) \right)^{r_2} \\
&= \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} \binom{\ell_1-k}{r_1} \binom{\ell_2-k}{r_2} \sum_{n=1}^{\infty} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n^{1+s+u}} \\
& \times \frac{1}{\zeta(1+\beta+u)} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell_1-k-r_1} \frac{1}{\zeta(1+s+\alpha)} \left(\frac{\zeta'}{\zeta}(1+\alpha+s) \right)^{\ell_2-k-r_2},
\end{aligned}$$

where we have used the Dirichlet convolution of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \text{and} \quad \frac{\zeta''}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_2(n)}{n^s},$$

for $\text{Re}(s) > 1$. Now we take $\delta \asymp L^{-1}$ and bound the integral trivially to get $J_{22} \ll L^{i+j-1}$. This means that we can use a Taylor series expansion so that $A_{\alpha,\beta}(s, u, 0) = A_{0,0}(0, 0, 0) + O(|s| + |u|)$ to write $J_{22}(\alpha, \beta) = J'_{22}(\alpha, \beta) + O(L^{i+j-2})$, say. We recall that earlier we proved that $A_{0,0}(z, z, z) = 1$ for all z , and hence $A_{0,0}(0, 0, 0) = 1$. This has the effect of separating the complex variables s and u as follows

$$J'_{22} = \sum_{n \leq y_2} \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} \binom{\ell_1-k}{r_1} \binom{\ell_2-k}{r_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n} L_{22,1} L_{22,2},$$

where

$$(3.14) \quad L_{22,1} = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_2}{n} \right)^s \frac{1}{\zeta(1+s+\alpha)} \left(\frac{\zeta'}{\zeta}(1+\alpha+s) \right)^{\ell_2-k-r_2} \frac{ds}{s^{j+1}},$$

and

$$L_{22,2} = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_2}{n} \right)^u \frac{1}{\zeta(1+\beta+u)} \left(\frac{\zeta'}{\zeta}(1+\beta+u) \right)^{\ell_1-k-r_1} \frac{du}{u^{i+1}}.$$

These two integrals are identical, up to the symmetries in s/u , ℓ_1/ℓ_2 , α/β and r_1/r_2 and they were in fact treated in the $I_{12}(\alpha, \beta)$ case. The end results for the main terms are

$$L_{22,1} = \frac{(-1)^{\ell_2-k+r_2}}{2\pi i} \oint \left(\frac{y_2}{n} \right)^s (s+\alpha)^{1-\ell_2+k+r_2} \frac{ds}{s^{j+1}} = \frac{(-1)^{\ell_2-k+r_2}}{j!} \frac{d^{1-\ell_2+k+r_2}}{dx^{1-\ell_2+k+r_2}} e^{\alpha x} \left(x + \log \frac{y_2}{n} \right)^j \Big|_{x=0},$$

and

$$L_{22,2} = \frac{(-1)^{\ell_1-k+r_1}}{2\pi i} \oint \left(\frac{y_2}{n} \right)^u (u+\beta)^{1-\ell_1+k+r_1} \frac{du}{u^{i+1}} = \frac{(-1)^{\ell_1-k+r_1}}{i!} \frac{d^{1-\ell_1+k+r_1}}{dy^{1-\ell_1+k+r_1}} e^{\beta y} \left(y + \log \frac{y_2}{n} \right)^i \Big|_{y=0}.$$

Next, we insert these results into J_{22} and we end up with

$$J'_{22} = \frac{1}{i!} \frac{1}{j!} \sum_{n \leq y_2} \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} (-1)^{\ell_1+\ell_2-2k+r_1+r_2} \binom{\ell_1-k}{r_1} \binom{\ell_2-k}{r_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n}$$

$$\times \frac{d^{1-\ell_2+k+r_2}}{dx^{1-\ell_2+k+r_2}} \frac{d^{1-\ell_1+k+r_1}}{dy^{1-\ell_1+k+r_1}} e^{\alpha x + \beta y} \left(x + \log \frac{y_2}{n} \right)^j \Big|_{x=0} \left(y + \log \frac{y_2}{n} \right)^i \Big|_{y=0} + O(L^{i+j-2}).$$

To make matters easier, we again employ the change of variables

$$x \rightarrow \frac{x}{\log y_2} \quad \text{and} \quad y \rightarrow \frac{y}{\log y_2},$$

and this produces

$$\begin{aligned} J'_{22} &= \frac{\log^{i+j-2} y_2}{i!j!} \sum_{n \leq y_2} \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} (-1)^{\ell_1+\ell_2-2k+r_1+r_2} \binom{\ell_1-k}{r_1} \binom{\ell_2-k}{r_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n} \\ &\quad \times \frac{d^{1-\ell_2+k+r_2}}{dx^{1-\ell_2+k+r_2}} \frac{d^{1-\ell_1+k+r_1}}{dy^{1-\ell_1+k+r_1}} y_2^{\alpha x + \beta y} \left(x + \frac{\log(y_2/n)}{\log y_2} \right)^j \Big|_{x=0} \left(y + \frac{\log(y_2/n)}{\log y_2} \right)^i \Big|_{y=0} + O(L^{i+j-2}). \end{aligned}$$

We are now ready to insert this into I'_{220} so that

$$\begin{aligned} I'_{220}(\alpha, \beta) &= \frac{\widehat{w}(0)}{(\alpha + \beta) \log^2 y_2} \frac{d^2}{dx dy} \left[y_2^{\alpha x + \beta y} \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \frac{1}{\log^{\ell_1+\ell_2} y_2} \sum_{i,j} \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k a_{i, \ell_1} a_{j, \ell_2} \right. \\ &\quad \times \sum_{n \leq y_2} \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} (-1)^{r_1+r_2} \binom{\ell_1-k}{r_1} \binom{\ell_2-k}{r_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n} \\ &\quad \left. \times \frac{d^{k-\ell_2+r_2}}{dx^{k-\ell_2+r_2}} \frac{d^{k-\ell_1+r_1}}{dy^{k-\ell_1+r_1}} \left(x + \frac{\log(y_2/n)}{\log y_2} \right)^j \Big|_{x=0} \left(y + \frac{\log(y_2/n)}{\log y_2} \right)^i \Big|_{y=0} \right] + O(T/L), \end{aligned}$$

where we have used $\zeta(1 + \alpha + \beta) = 1/(\alpha + \beta) + O(1)$. We now sum over i and j , e.g.

$$P_{\ell_1} \left(x + \frac{\log(y_2/n)}{\log y_2} \right) = \sum_i b_{i, \ell_1} \left(x + \frac{\log(y_2/n)}{\log y_2} \right)^i,$$

thereby getting

$$\begin{aligned} I'_{220}(\alpha, \beta) &= \frac{\widehat{w}(0)}{(\alpha + \beta) \log^2 y_2} \frac{d^2}{dx dy} \left[y_2^{\alpha x + \beta y} \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \frac{1}{\log^{\ell_1+\ell_2} y_2} \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k \right. \\ &\quad \times \sum_{n \leq y_2} \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} (-1)^{r_1+r_2} \binom{\ell_1-k}{r_1} \binom{\ell_2-k}{r_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n} \\ &\quad \left. \times \frac{d^{k-\ell_2+r_2}}{dx^{k-\ell_2+r_2}} \frac{d^{k-\ell_1+r_1}}{dy^{k-\ell_1+r_1}} P_{\ell_1} \left(x + \frac{\log(y_2/n)}{\log y_2} \right) \Big|_{x=0} P_{\ell_2} \left(y + \frac{\log(y_2/n)}{\log y_2} \right) \Big|_{y=0} \right] + O(T/L). \end{aligned}$$

Lemma 2.7 gives us

$$\begin{aligned} &\sum_{n \leq y_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n} P_{\ell_1} \left(x + \frac{\log(y_2/n)}{\log y_2} \right) P_{\ell_2} \left(y + \frac{\log(y_2/n)}{\log y_2} \right) \\ &= \frac{2^{r_1+r_2} \log^{1+2k+r_1+r_2} y_2}{(1+r_1+r_2+2k)!} \int_0^1 (1-u)^{2k+r_1+r_2} P_{\ell_1}(x+u) P_{\ell_2}(y+u) du + O(\log^{2k+r_1+r_2} y_2), \end{aligned}$$

so that we are left with

$$I'_{220}(\alpha, \beta) = \frac{\widehat{w}(0)}{(\alpha + \beta) \log^2 y_2} \frac{d^2}{dx dy} \left[y_2^{\alpha x + \beta y} \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \frac{1}{\log^{\ell_1+\ell_2} y_2} \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k \right.$$

$$\begin{aligned} & \times \sum_{r_1=0}^{\ell_1} \sum_{r_2=0}^{\ell_2} (-1)^{r_1+r_2} \binom{\ell_1-k}{r_1} \binom{\ell_2-k}{r_2} \frac{d^{k-\ell_2+r_2}}{dx^{k-\ell_2+r_2}} \frac{d^{k-\ell_1+r_1}}{dy^{k-\ell_1+r_1}} \\ & \times \frac{2^{r_1+r_2} \log^{1+r_1+r_2+2k} y_2}{(1+r_1+r_2+2k)!} \int_0^1 (1-u)^{2k+r_1+r_2} P_{\ell_1}(x+u) P_{\ell_2}(y+u) du \Big|_{x=y=0} + O(T/L). \end{aligned}$$

Note that $r_1 \leq \ell_1 - k$ and $r_2 \leq \ell_2 - k$. Thus only the cases $r_1 = \ell_1 - k$ and $r_2 = \ell_2 - k$ contribute to the main term. We therefore have

$$\begin{aligned} I'_{220}(\alpha, \beta) &= \frac{\widehat{w}(0)}{(\alpha + \beta) \log y_2} \frac{d^2}{dx dy} \left[y_2^{\alpha x + \beta y} \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k (-1)^{\ell_1 + \ell_2 - 2k} \right. \\ & \times \left. \frac{2^{\ell_1 + \ell_2 - 2k}}{(\ell_1 + \ell_2)!} \int_0^1 (1-u)^{\ell_1 + \ell_2} P_{\ell_1}(x+u) P_{\ell_2}(y+u) du \Big|_{x=y=0} \right] + O(T/L). \end{aligned}$$

Recall that

$$I_{22}(\alpha, \beta) = I'_{22}(\alpha, \beta) + T^{-\alpha-\beta} I'_{22}(-\beta, -\alpha) + O(T/L),$$

and that

$$I'_{22}(\alpha, \beta) = I'_{220}(\alpha, \beta) + O(T^{1-\varepsilon}),$$

therefore

$$\begin{aligned} I_{22}(\alpha, \beta) &= I'_{220}(\alpha, \beta) + T^{-\alpha-\beta} I'_{220}(-\beta, -\alpha) + O(T/L) \\ &= (I'_{220}(\alpha, \beta) + I'_{220}(-\beta, -\alpha)) + (T^{-\alpha-\beta} - 1) I'_{220}(-\beta, -\alpha) + O(T/L). \end{aligned}$$

We first take a look at the first term in the brackets

$$\begin{aligned} & \frac{d^2}{dx dy} \left[(y_2^{\alpha x + \beta y} - y_2^{-\beta x - \alpha y}) \int_0^1 (1-u)^{\ell_1 + \ell_2} P_{\ell_1}(x+u) P_{\ell_2}(y+u) du \Big|_{x=y=0} \right] \\ &= (\alpha + \beta) \log y_2 \left(\int_0^1 (1-u)^{\ell_1 + \ell_2} P'_{\ell_1}(u) P_{\ell_2}(u) du + \int_0^1 (1-u)^{\ell_1 + \ell_2} P_{\ell_1}(u) P'_{\ell_2}(u) du \right). \end{aligned}$$

Since $P_{\ell_1}(0) = P_{\ell_2}(0) = 0$, we have also

$$0 = (1-u)^{\ell_1 + \ell_2} P_{\ell_1}(u) P_{\ell_2}(u) \Big|_{u=0}^1 = \int_0^1 \left((1-u)^{\ell_1 + \ell_2} P_{\ell_1}(u) P_{\ell_2}(u) \right)' du.$$

This implies

$$\begin{aligned} & (\ell_1 + \ell_2) \int_0^1 (1-u)^{\ell_1 + \ell_2 - 1} P_{\ell_1}(u) P_{\ell_2}(u) du \\ &= \int_0^1 (1-u)^{\ell_1 + \ell_2} P'_{\ell_1}(u) P_{\ell_2}(u) du + \int_0^1 (1-u)^{\ell_1 + \ell_2} P_{\ell_1}(u) P'_{\ell_2}(u) du. \end{aligned}$$

Combining these observations gives

$$\begin{aligned} I'_{220}(\alpha, \beta) + I'_{220}(-\beta, -\alpha) &= \widehat{w}(0) \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{k=0}^{\min(\ell_1, \ell_2)} (-1)^{\ell_1 + \ell_2 - 2k} \binom{\ell_1}{k} (\ell_2)_k \\ & \times \frac{2^{\ell_1 + \ell_2 - 2k}}{(\ell_1 + \ell_2 - 1)!} \int_0^1 (1-u)^{\ell_1 + \ell_2 - 1} P_{\ell_1}(u) P_{\ell_2}(u) du. \end{aligned}$$

For the expression $(T^{-\alpha-\beta} - 1) I'_{22}(-\beta, -\alpha)$, we again use (3.4) to get

$$\frac{\widehat{w}(0)}{\theta_2} \sum_{\ell_1=2}^K \sum_{\ell_2=2}^K \sum_{k=0}^{\min(\ell_1, \ell_2)} \binom{\ell_1}{k} (\ell_2)_k (-1)^{\ell_1 + \ell_2 - 2k} \frac{2^{\ell_1 + \ell_2 - 2k}}{(\ell_1 + \ell_2)!}$$

$$\times \frac{d^2}{dx dy} \left[y_2^{-\beta x - \alpha y} \int_0^1 \int_0^1 T^{-v(\alpha + \beta)} (1 - u)^{\ell_1 + \ell_2} P_{\ell_1}(x + u) P_{\ell_2}(y + u) dudv \Big|_{x=y=0} \right] + O(T/L).$$

By using similar arguments for the holomorphy of the error terms as in the Section 3.1, we end the proof of Lemma 1.3.

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REFERENCES

- [1] R. Balasubramanian, J. B. Conrey, and D. R. Heath-Brown *Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial*. J. reine angew. Math., (357):161–181, 1985.
- [2] H. M. Bui. *Critical zeros of the Riemann zeta-function*. <http://arxiv.org/abs/1410.2433>
- [3] H. M. Bui, B. Conrey, and M. P. Young. *More than 41% of the zeros of the zeta function are on the critical line*. Acta Arith., (150.1):35–64, 2011.
- [4] J. B. Conrey. *Zeros of derivatives of the Riemann’s ξ -function on the critical line*. J. Number Theory, (16):49–74, 1983.
- [5] J. B. Conrey. *More than two fifths of the zeros of the Riemann zeta function are on the critical line*. J. reine angew. Math., (399):1–26, 1989.
- [6] J. B. Conrey, D. W. Farmer and M. R. Zirnbauer. *Autocorrelation of ratios of L-functions*. Comm. Number Theory Phys., (2)94:593–636, 2008.
- [7] J. B. Conrey and N. C. Snaith. *Applications of the L-functions ratios conjectures*. Proc. London Math. Soc., (3)94:594–646, 2007.
- [8] S. Feng. *Zeros of the Riemann zeta function on the critical line*. J. Number Theory, (132):511–542, 2012.
- [9] A. Ivić. *On certain functions that generalize von Mangoldt’s function $\Lambda(n)$* . Mat. Vesnik (Belgrade), 12(27):361–366, 1975.
- [10] J. M. Deshouillers and H. Iwaniec. *Kloosterman sums and Fourier coefficients of cusp forms*. Invent. Math., (70):219–288, 1982.
- [11] J. M. Deshouillers and H. Iwaniec. *Power mean values of the Riemann zeta function II*. Acta Arith., (48):305–312, 1984.
- [12] N. Levinson. *More than One Third of Zeros of Riemann’s Zeta-Function are on $\sigma = \frac{1}{2}$* . Adv. Math., (13):383–436, 1974.
- [13] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory I: Classical Theory*. Cambridge Studies in Advanced Mathematics, 2007.
- [14] N. Robles, A. Roy, and A. Zaharescu. *Twisted second moments of the Riemann zeta-function and applications*. J. Math. Anal. Appl., (434):271–314, 2016.
- [15] A. Selberg. *On the zeros of Riemann’s zeta-function*. Skr. Norske Vid. Akad. Oslo I, (10):1–59, 1942.
- [16] E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function*. Oxford University Press, 2nd edition, 1986.
- [17] M. P. Young. *A short proof of Levinson’s theorem*. Arch. Math., (95):539–548, 2010.

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