Spectral Theory Using Linear Systems and Sampling from the Spectrum of Hill's Equation

Caroline Brett, MSci

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Abstract

This thesis, entitled *Spectral Theory Using Linear Systems and Sampling from the Spectrum of Hill's Equation* is submitted by Caroline Brett, Master of Science for the degree of Doctor of Philosophy, September 2015. It uses linear systems to solve various problems connected with Hill's equation, $-f'' + qf = \lambda f$ for $q \in C^2$, real-valued and π-periodic. Introducing a new operator, R_x constructed from a linear system, $(-A, B, C)$ allows us to solve Hill's equation and the inverse spectral problem. We use R_x to construct a function, $T(x, y)$ that satisfies a Gelfand–Levitan integral equation and then derive a PDE for $T(x, y)$. Solving this PDE recovers q. Extending Hill's work in [28], we show that there exist Hilbert–Schmidt operators, R_p and R_c analogous to R_x, such that the roots of their Carleman determinants are elements of the periodic spectrum of Hill's equation.

The latter half concerns sampling from entire functions in Paley–Wiener space. From the periodic spectrum of Hill's equation we derive a sampling sequence, $(t_n)_{n\in\mathbb{Z}}$. Whittaker, Kotel'nikov and Shannon give a sampling result for $(n)_{n\in\mathbb{Z}}$ where samples occur at a constant rate. Samples taken from the periodic spectrum do not occur at a constant rate, nevertheless we provide analogous results for this case. From $(t_n)_{n\in\mathbb{Z}}$ we also construct Riesz bases for $L^2[0, \pi]$ and $L^2[-\pi, \pi]$, the Fourier transform space of $PW(\pi)$. In $L^2[0, \pi]$ we construct the dual Riesz basis using linear systems. Furthermore, we show that the determinant of the Gram matrix associated with the Riesz basis is a Lipschitz continuous function of $(t_n)_{n\in\mathbb{Z}}$.

Finally, we look at an integral, I_a associated with Ramanujan and use it to create a basis for $PW\left(\frac{\pi}{2}\right)$. We conclude with an evaluation of various determinants associated with I_a.

Declaration

I, Caroline Brett, declare that the work submitted in this thesis is my own and has not been submitted for the award of a degree elsewhere. Some results contained within this thesis have previously been published in papers that form joint research with Professor Gordon Blower (Lancaster University) and Dr. Ian Doust (University of New South Wales). Much of the work in Section 4 of [6] has been carried out by myself. In particular, Theorem 5.2.0.17 of this thesis has already been published in [6] as Proposition 4.1(i). A further joint paper, [7] with Professor Gordon Blower and Dr. Ian Doust has been accepted for publication in Bulletin des Sciences Mathématiques. At the time of writing it can be found on Arxiv. The results in Section 5.5 of this thesis appear in [7] as Lemma 3.3 and Lemma 6.2. My contributions to [7] include Lemma 3.3 and the proof of Theorem 6.1(i),(iii). Also, aspects of the proof of Proposition 3.1 relating to Q are my own work.

Finally, I would like to acknowledge the EPSRC and extend my gratitude for their continued support and funding throughout this project.

Contents

Chapter 1

Introduction

One of the themes of this thesis is to calculate determinants. We therefore include a chapter detailing the necessary general theory to allow us to calculate the various determinants. Of particular interest will be the Carleman and Fredholm determinants defined for the Hilbert–Schmidt and trace class operators acting on some Hilbert space, H. It is with these operators that we pick up the story, stating various well-known results that will be required in subsequent proofs throughout the thesis. After providing some light background material on the Hilbert–Schmidt and trace class operators we then take a deeper look at their corresponding Carleman and Fredholm determinants. Carleman determinants are defined for $I+T$ where T is a Hilbert–Schmidt operator, while Fredholm determinants are defined for $I+S$ where S is a trace class operator. When giving results involving determinants in this thesis, we will often switch between the Carleman and Fredholm determinants. This is because, although Fredholm determinants appear to be a more natural choice, it is often easier to show that an operator is Hilbert–Schmidt than trace class, hence we use Carleman determinants when this is the case.

Much of the work we do will be carried out in Hilbert spaces. Of particular relevance are the Paley–Wiener spaces. Paley–Wiener spaces consist of band-limited functions and find uses in areas such as sampling. Naturally, our work on sampling in Chapter 5 will be conducted in Paley–Wiener spaces. In his papers McKean adopts the approach of constructing a space to meet his requirements. For example, McKean introduces special spaces of entire functions that depend upon the solutions of Hill's equation, and thus on the potential of Hill's equation. This method of constructing spaces is not ideal for problems such as the inverse spectral problem. We therefore choose to give results that work in standard spaces and the Paley–Wiener spaces provide a suitable setting for this.

We conclude our chapter on general theory with a discussion about linear systems, another theme of this thesis. One of our main ideas is to use linear systems to solve various problems via the introduction of a new operator, R_x. For example, we use linear systems to solve Gelfand–Levitan integral equations and differential equations such as Hill's equation. We also use linear systems on Carleman and Fredholm determinants to give conditions which, when satisfied, produce the periodic spectrum of Hill's equation. Generally, the linear systems used will be of the form $(-A, B, C)$ where $-A$ is the generator of a strongly continuous semigroup.

Given a differential equation, there are two problems that we can consider with respect to its spectrum. The spectral problem is to find the spectra of the differential equation given that the potential is known. The inverse spectral problem involves recovering a suitable potential from the spectral data. In this thesis we use Hill's equation as an example of a differential equation and consider aspects of the two spectral problems. Hill's equation takes the form

$$-f'' + qf = \lambda f$$

where q is real-valued, π-periodic and twice continuously differentiable. The function q is referred to as the potential and λ an eigenvalue. We take a particular interest in the periodic spectrum of Hill's equation, that is, the eigenvalues whose corresponding eigenfunctions are π or 2π-periodic. Indeed, in Section 4.5 we see that the roots of the Carleman determinants of the Hilbert–Schmidt operators, R_p and R_c are elements of the periodic spectrum of Hill's equation. In order to consider the inverse spectral problem we introduce a Gelfand–Levitan integral equation which is defined using a scattering function, ϕ. In this thesis we construct the scattering function from a known linear system, $(-A, B, C)$ and see that it arises as the inverse Laplace transform of the transfer function associated with $(-A, B, C)$. The construction of the scattering function from $(-A, B, C)$ allows us to solve problems such as the inverse spectral problem using linear systems.

As already mentioned, Hill's equation, a linear, second order differential equation will be central to this thesis. We seek to find the solutions of Hill's equation, the periodic spectrum and recover a potential of Hill's equation using linear systems. The use of linear systems in this context is a novel approach. Indeed we show that the linear system, $(-A, B, C)$ that was used to construct the function $T(x, y)$ where T satisfied the Gelfand–Levitan integral equation, can also be used to construct a solution to Hill's equation. Further, we seek to use linear systems to characterise the periodic spectrum of Hill's equation. The periodic spectrum consists of eigenvalues relating to periodic solutions of period π or 2π. Since the periodic spectrum is used to create a sampling sequence in Chapter 5, it will be necessary to have a way of deriving the periodic spectrum. This can be done using a modified system, $(-A, B, C, M)$. We show that the periodic spectrum of Hill's equation consists of the roots of various Carleman and Fredholm determinants of operators constructed from systems such as $(-A, B, C, M)$.

In order to recover the potential of Hill's equation we use a Gelfand–Levitan integral equation and we suppose that the scattering function, ϕ is even and twice continuously differentiable on the real line. If there exists a function $T(x, y)$ satisfying the Gelfand–Levitan integral equation then it can be shown that $T(x, y)$ satisfies a partial differential equation. Interestingly, the resulting partial differential equation is dependent upon the potential of Hill's equation. This provides a way in which we can solve the inverse spectral problem. For, if the scattering function is constructed from a linear system, $(-A, B, C)$ then that same linear system can be used to construct $T(x, y)$ providing T satisfies the Gelfand–Levitan integral equation. Since T satisfies the Gelfand–Levitan integral equation it therefore satisfies a partial differential equation dependent upon the potential of Hill's equation, hence the potential can be recovered using the linear system, $(-A, B, C)$. The new approach in this thesis to use $(-A, B, C)$ to construct an operator R_x of the form

$$R_x = \int_{-x}^{x} \left(e^{-zA} + e^{zA}\right) BC \left(e^{-zA} + e^{zA}\right) dz,$$

and use R_x to define the function $T(x, y)$.

We have touched upon Paley–Wiener spaces being the traditional setting for sampling theory. Also, we noted that the periodic spectrum of Hill's equation can be used to construct a sampling sequence. If λ_n is an element of the periodic spectrum of Hill's equation then it can be shown by Borg's estimates that λ_n is of order n^2 in size. By setting $t_n = \sqrt{\lambda_n}$ we create a sequence, $(t_n)_{n\in\mathbb{Z}}$ that we can compare with the sequence $(n)_{n\in\mathbb{Z}}$, showing that $(t_n)_{n\in\mathbb{Z}}$ is a sampling sequence. The importance of sampling sequences is that for a Hilbert space, H of entire functions and sampling sequence, $(t_n)_{n\in\mathbb{Z}}$, we can reconstruct a function, $f \in H$ from the values $(f(t_n))_{n\in\mathbb{Z}}$. This idea is expressed in a theorem by Whittaker, Kotel'nikov and Shannon. However, the Whittaker–Kotel'nikov–Shannon Sampling Theorem is valid for sampling sequences where the samples occur at a constant rate. Initially, the elements of the periodic spectrum of Hill's equation do not occur at regular intervals of some constant, set length, hence sampling from the periodic spectrum cannot occur at a constant rate. We therefore seek to find a result analogous to the Whittaker–Kotel'nikov–Shannon Sampling Theorem that holds for a sampling sequence derived from the periodic spectrum of Hill's equation and such that the rate of sampling is not constant.

The problem of reconstructing functions naturally leads us to consider bases. It is well known that the set $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ gives an orthonormal basis for $L^2[-\pi, \pi]$. Given $(t_n)_{n\in\mathbb{Z}}$ is a sampling sequence derived from the periodic spectrum and that $(t_n)_{n\in\mathbb{Z}}$ behaves like $(n)_{n\in\mathbb{Z}}$, it is natural to ask whether $\left\{e^{it_nx}\right\}_{n\in\mathbb{Z}}$ also forms an orthonormal basis for $L^2[-\pi, \pi]$. It turns out that this is not the case, however, we can show that $\left\{e^{it_nx}\right\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi, \pi]$. Furthermore, we also find that the space $L^2[0, \pi]$ has Riesz basis $\{\cos t_n x\}_{n\in\mathbb{N}}$ where, under certain circumstances, the dual Riesz basis can be constructed using the linear system, $(-A, B, C)$.

Furthering our discussion on sampling sequences, we also look for ways to calculate the Gram matrix of a sequence. Since the Gram matrix of an orthonormal sequence is equal to the identity matrix, this observation allows us to compare sequences with orthonormal sequences. We construct the Gram matrix of the Riesz basis $\left\{e^{it_nx}\right\}_{n\in\mathbb{Z}}$ and show that the resulting Gram matrix is a Lipschitz function of the sequence $(t_n)_{n\in\mathbb{Z}}$. This is another main result of this thesis. Furthermore, we show that the sampling sequence, $(t_n)_{n\in\mathbb{Z}}$ is associated with a Carleman determinant which depends in a Lipschitz continuous way on $(t_n)_{n\in\mathbb{Z}}$. This is a crucial technical point that Blower, Brett and Doust present in their paper [7].

The final aim of this thesis is to evaluate some determinants associated with an integral appearing in the work of Ramanujan. The integral, which we refer to as $I_a(t)$ takes the form

$$I_a(t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\, dx$$

and arises in sampling theory. The integral, I_a is a Paley–Wiener function and we show that $\{I_{2b}\}_{b\in\mathbb{N}}$ gives a basis for the even functions in the space $PW\left(\frac{\pi}{2}\right)$. Following this we analyse determinants with entries $I_a(t_j - k)$ for $j, k \in \mathbb{Z}$. Although we prove that the results hold for a general sequence, $(t_n)_{n\in\mathbb{Z}}$, they do in fact hold for the sampling sequence obtained from the periodic spectrum of Hill's equation. Of particular interest is the case in which $a \in \mathbb{N}$ and $t_j = j$. By Ramanujan's formula, $I_a(t_j - k)$ can be expressed in terms of Gamma functions. Furthermore, for $a \in \mathbb{N}$ and $t_j = j$, the Gamma functions become factorial expressions and so we can provide a formula for $I_a(t_j - k)$ involving factorials. The resulting matrix with entries $I_a(j-k)$ for $a \in \mathbb{N}$ has a Toeplitz shape. As far as we are aware, there has not been a previous attempt to construct such matrices and determinants.

Chapter 2

General Theory

This chapter is, as the title suggests, devoted to the general theory and background material that will be necessary to create the foundations of this thesis. We start with a gentle introduction to operators, focusing in particular on Hilbert–Schmidt and trace class operators. Dunford and Schwartz provide a detailed analysis of Hilbert–Schmidt operators in [15]. A gentler and somewhat basic approach is given by Young in [55]. Results that supplement the work of Young can be found in [49]. Readers wanting an alternative construction of the Hilbert–Schmidt and trace class operators should consult Nikolski [39] (Section 2.1, page 211).

This is followed by a brief introduction to some properties of entire integral functions as given by Titchmarsh in [54].

We then turn our attention to matrices and define the Vandermonde and Toeplitz matrices since these will be used in Chapter 6. Further, we define various determinants that will be used throughout the thesis. In particular, we define the Carleman determinant that is associated with the Hilbert–Schmidt operators and the Fredholm determinant that is associated with the trace class operators.

Next we show the reader how to construct the Paley–Wiener spaces. We recall the definition of the L^2 Fourier transform and use this to define the band-limited functions, which in turn make up the Paley–Wiener spaces. Most of our work on sampling in Chapter 5 will be conducted in Paley–Wiener spaces.

Following this we introduce two Hilbert–Schmidt operators U and U^* that are Fourier transforms defined on a restricted range of integration. We use these operators to construct another operator, S that has links with the Paley–Wiener spaces. Indeed we see that S is an integral operator whose kernel gives a reproducing kernel for the Paley–Wiener spaces.

The chapter concludes with an introduction to linear systems. Following the style of Chen in [8], we define a linear system, $(-A, B, C, D)$ and show the form of a solution of the differential equation associated with such a system. The solution requires a strongly continuous semigroup generated by $-A$. The final task being to give a specific example of a linear system.

2.1 Operators

Throughout this thesis we will work in a specific type of space known as a Hilbert space. We therefore begin by introducing Hilbert spaces and provide some basic definitions concerning operators that act on these spaces. For completeness we provide definitions that the reader should be familiar with such as linearity and boundedness, and we define what is meant by the adjoint of an operator. This short section culminates with a proposition that demonstrates some useful properties of the adjoint operation. These properties will be called upon in later proofs.

We first define an inner product space and use this to define a Hilbert space.

Definition 2.1.0.1 *Let V be a complex vector space and let the map, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be such that:*
(i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
(ii) $\langle cx, y \rangle = c\langle x, y \rangle$;
(iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;
(iv) $\langle x, x \rangle > 0$ *for* $x \neq 0$,
for all $x, y, z \in V$ and $c \in \mathbb{C}$. We say that $\langle \cdot, \cdot \rangle$ is an inner product *and the pair $(V, \langle \cdot, \cdot \rangle)$ is an* inner product space.

Definition 2.1.0.2 *Let H be a complex vector space with inner product, $\langle \cdot, \cdot \rangle$. If H is a complete metric space with respect to the metric induced by $\langle \cdot, \cdot \rangle$ then we say that $(H, \langle \cdot, \cdot \rangle)$ is a* Hilbert space.

Remark 2.1.0.3 *It should be noted that the reader is to assume that, unless otherwise stated, all Hilbert spaces, $(H, \langle \cdot, \cdot \rangle)$ are complex and separable.*

Next we turn our attention to operators. We briefly define linear and bounded operators before restricting our attention to integral operators.

Definition 2.1.0.4 *Let V and W be vector spaces over the same field, $\mathbb{F}$. If a map, $T : V \to W$ satisfies*

$$T(\lambda v_1 + \mu v_2) = \lambda T(v_1) + \mu T(v_2)$$

for all $v_1, v_2 \in V$ and $\lambda, \mu \in \mathbb{F}$, then we say that T is a linear *operator.*

Definition 2.1.0.5 *Let $T : V \to W$ be a linear operator with V, W normed spaces. If*

$$\|Tv\| \leq C \|v\|$$

for some $C \geq 0$ and for all $v \in V$, then T is a bounded *operator. We define the* norm *of the operator T, $\|T\|_{\text{op}}$ to be the smallest such C for which T is bounded.*

We now define a particular type of operator known as the integral operator.

Definition 2.1.0.6 *Let $a, b, c, d \in \mathbb{R}$. An* integral operator, *$T : L^2[a, b] \to L^2[c, d]$ is an operator satisfying,*

$$(Tf)(t) = \int_a^b k(t, x) f(x)\, dx \tag{2.1}$$

for $c \leq t \leq d$. We call $k : [c,d] \times [a,b] \to \mathbb{C}$ the kernel *and take $k(t,x)$ to be continuous for all t and x.*

Knowing that an operator is an integral operator can often simplify calculations. For example, in Section 2.1.1 if we have an integral operator then we can easily show that it is Hilbert–Schmidt by verifying that the kernel satisfies a given condition.

We conclude this section by defining the adjoint of an operator and stating some of its properties.

Definition 2.1.0.7 *Let H_1 and H_2 be complex Hilbert spaces and suppose that $T : H_1 \to H_2$ is a bounded linear operator. We define the* adjoint *of T to be the operator $T^* : H_2 \to H_1$ satisfying the equation*

$$\langle Tf, g\rangle_{H_2} = \langle f, T^* g\rangle_{H_1}. \tag{2.2}$$

If T is an integral operator with kernel $k(x,y)$ then the adjoint of $k(x,y)$ is $\overline{k(y,x)}$. An operator, T is said to be self-adjoint *if $T = T^*$ or equivalently, $k(x,y) = \overline{k(y,x)}$.*

Having an operator that is self-adjoint can simplify proofs greatly. In later sections we also see that it provides information about the eigenvalues of an operator. The following proposition provides a helpful list of properties that the adjoint satisfies. For a proof of the statement along with more detailed information regarding adjoint operators, we refer the reader to [49] (Section 56, page 262 and Theorem A, page 265).

Proposition 2.1.0.8 *Let T, T_1 and T_2 be operators with adjoint's T^*, T_1^* and T_2^* respectively. Then the following properties hold:*
(i) $(T_1 + T_2)^* = T_1^* + T_2^*$;
(ii) $(\lambda T)^* = \overline{\lambda} T^*$;
(iii) $(T_1 T_2)^* = T_2^* T_1^*$;
(iv) $T^{**} = T$;
(v) $\|T^*\| = \|T\|$.

2.1.1 Hilbert–Schmidt Operators

In this section we focus on a particular class of operators known as the Hilbert–Schmidt operators. We define what is meant by a Hilbert–Schmidt operator and state some of the basic properties. In Section 2.3 we will return to define the corresponding Carleman determinants. Knowing whether an operator is Hilbert–Schmidt will be crucial for our work on determinants in Section 4.5. There, we introduce for suitable systems of the form $(-A, B, C, M)$, operators R_p and R_c whose Carleman determinants have roots that are elements of the periodic spectrum of Hill's equation. This is one of the main new ideas of the thesis. More information regarding Hilbert–Schmidt operators and the details of the proofs that have been omitted here can be found in [15] (Section XI.6, page 1009).

First we introduce the Hilbert–Schmidt operators via the Hilbert–Schmidt norm.

Definition 2.1.1.1 *Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ a linear operator. Let $(e_n) \in H_1$ be an orthonormal basis then the* Hilbert–Schmidt norm, *$\|T\|_{HS}$ of the operator T is defined by*

$$\begin{aligned} \|T\|_{\mathrm{HS}}^2 &= \mathrm{tr}\,(T^*T) \\ &= \sum_n \|Te_n\|^2. \end{aligned}$$

Remark 2.1.1.2 *Note that the Hilbert–Schmidt norm is independent of the choice of basis. See [15] (Lemma 2, page 1010).*

Definition 2.1.1.3 *If $T : H_1 \to H_2$ is a bounded linear operator with finite Hilbert–Schmidt norm then we say that T is a* Hilbert–Schmidt *operator.*

Remark 2.1.1.4 *We observe that Definition 2.1.1.3 can be used to show that an operator is bounded. For if we can show that an operator is Hilbert–Schmidt then it follows from the definition that it must also be bounded.*

The following proposition shows that the set of Hilbert–Schmidt operators itself forms a Hilbert space. It also justifies the definition of Hilbert–Schmidt norm given in Definition 2.1.1.1.

Proposition 2.1.1.5 *Let* HS *be the set of Hilbert–Schmidt operators on a Hilbert space, $(H, \langle \cdot, \cdot \rangle)$ with orthonormal basis, (e_n). Then* HS *is itself a Hilbert space for the inner product given by*

$$\langle T, S \rangle_{\mathrm{HS}} = \sum_n \langle Te_n, Se_n \rangle.$$

Proof. By Definition 2.1.0.2, $(\mathrm{HS}, \langle \cdot, \cdot \rangle_{\mathrm{HS}})$ is a Hilbert space if it is a complete metric space. We use the definition of a metric space as provided by Simmons in [49] (Section 9, page 51) to verify that $(\mathrm{HS}, \langle \cdot, \cdot \rangle_{\mathrm{HS}})$ is indeed a metric space. Suppose that HS has metric d given by,

$$d(T, S) = \|T - S\|_{\mathrm{HS}}$$

where $T, S \in \mathrm{HS}$. First we check the positivity of d, this follows from the positivity of the norm, hence

$$\begin{aligned} d(T, S) &= \sqrt{\sum_n \|(T - S)e_n\|^2} \\ &\geq 0. \end{aligned}$$

Further, $d(T, S) = 0$ if and only if $\sum_n \|(T - S)e_n\|^2 = 0$. Clearly, $\sum_n \|(T - S)e_n\|^2 = 0$ if and only if $\|(T - S)e_n\| = 0$ for all n. It follows that $d(T, S) = 0$ if and only if $T - S = 0$ as required.

Next we check that d is symmetric. Now,

$$\begin{aligned} d(T, S) &= \sqrt{\sum_n \|(T - S)e_n\|^2} \\ &= \sqrt{\sum_n |-1|^2 \|(S - T)e_n\|^2} \\ &= d(S, T) \end{aligned}$$

proving the symmetry of d.

Finally we show that d satisfies the triangle inequality. Let $R \in \text{HS}$ then

$$\begin{aligned} d(T,S)^2 &= \|T-S\|_{\text{HS}}^2 \\ &= \|(T-R)+(R-S)\|_{\text{HS}}^2 \\ &= \langle (T-R)+(R-S),(T-R)+(R-S)\rangle_{\text{HS}}. \end{aligned}$$

Since inner products are distributive with respect to addition we have

$$\begin{aligned} d(T,S)^2 &= \langle T-R,T-R\rangle_{\text{HS}} + \langle T-R,R-S\rangle_{\text{HS}} + \langle R-S,T-R\rangle_{\text{HS}} + \langle R-S,R-S\rangle_{\text{HS}} \\ &= \|T-R\|_{\text{HS}}^2 + \langle T-R,R-S\rangle_{\text{HS}} + \langle R-S,T-R\rangle_{\text{HS}} + \|R-S\|_{\text{HS}}^2 . \end{aligned}$$

Now, by the Cauchy–Schwarz inequality we have

$$\begin{aligned} d(T,S)^2 &\leq \|T-R\|_{\text{HS}}^2 + \|T-R\|_{\text{HS}}\|R-S\|_{\text{HS}} + \|R-S\|_{\text{HS}}\|T-R\|_{\text{HS}} + \|R-S\|_{\text{HS}}^2 \\ &= \left(\|T-R\|_{\text{HS}} + \|R-S\|_{\text{HS}}\right)^2 \\ &= \left(d(T,R)+d(R,S)\right)^2. \end{aligned}$$

Upon taking the square root of both sides we see that d satisfies the triangle inequality. We have thus shown that $(\text{HS}, \langle\cdot,\cdot\rangle_{\text{HS}})$ is a metric space.

We finish the proof by showing that $(\text{HS}, \langle\cdot,\cdot\rangle_{\text{HS}})$ is complete. Let T be a Hilbert–Schmidt operator with matrix given by $A = [\langle Te_j, e_i\rangle]_{i,j}$ (see Definition 2.3.0.20). Then

$$\begin{aligned} \|T\|_{\text{HS}}^2 &= \text{tr}\,(A^*A) \\ &= \sum_{i,j} |\langle Te_j, e_i\rangle|^2, \end{aligned}$$

where the map $\|\cdot\|_{\text{HS}}^2 : \text{HS} \to \ell^2(\mathbb{N}\times\mathbb{N})$ is an isometry. The space $\ell^2(\mathbb{N}\times\mathbb{N})$ is complete by [55] (Theorem 3.2, page 21), therefore $(\text{HS}, \langle\cdot,\cdot\rangle_{\text{HS}})$ is also complete. ■

Definition 2.1.1.3 can sometimes be problematic to use in practice to show that a given operator is Hilbert–Schmidt. In the case of an integral operator there exists an easier method to determine whether an operator is Hilbert–Schmidt or not. The following result shows how this can be done. For a proof we refer the reader to [55] (Theorem 8.8, page 93).

Proposition 2.1.1.6 *Let $T : L^2[a,b] \to L^2[c,d]$ be an integral operator with kernel k as in (2.1). Then T is a Hilbert–Schmidt operator if and only if,*

$$\int_c^d \int_a^b |k(t,x)|^2 \, dx\, dt < \infty.$$

Staying with integral operators, the following proposition shows that the adjoint of a Hilbert–Schmidt operator is also a Hilbert–Schmidt operator. The proposition also gives the form that the adjoint takes.

Proposition 2.1.1.7 *Let $T : L^2[a,b] \to L^2[c,d]$ be a Hilbert–Schmidt operator defined by*

$$Tf(x) = \int_a^b k(x,y) f(y)\, dy.$$

Then the adjoint, $T^* : L^2[c,d] \to L^2[a,b]$ is given by

$$T^*g(y) = \int_c^d \overline{k(x,y)}\, g(x)\, dx.$$

Furthermore, T^* is also a Hilbert–Schmidt operator.

Proof. Let $T : L^2[a,b] \to L^2[c,d]$ be an integral Hilbert–Schmidt operator. Write $Tf(x) = \int_a^b k(x,y)f(y)\,dy$ then, by Definition 2.1.0.7 we have

$$\begin{aligned}\langle f, T^*g\rangle_{L^2[a,b]} &= \langle Tf, g\rangle_{L^2[c,d]} \\ &= \int_c^d Tf(x)\overline{g(x)}\,dx.\end{aligned}$$

Substituting in $Tf(x)$ and reversing the order of integration gives

$$\begin{aligned}\langle f, T^*g\rangle_{L^2[a,b]} &= \int_c^d \left(\int_a^b k(x,y)f(y)\,dy\right)\overline{g(x)}\,dx \\ &= \int_a^b \left(\int_c^d k(x,y)\overline{g(x)}\,dx\right) f(y)\,dy \\ &= \int_a^b \overline{\left(\int_c^d \overline{k(x,y)}g(x)\,dx\right)} f(y)\,dy.\end{aligned}$$

Hence,

$$T^*g(y) = \int_c^d \overline{k(x,y)}g(x)\,dx$$

as required.

We finish by showing that T^* is also a Hilbert–Schmidt operator. By Proposition 2.1.1.6, since T is a Hilbert–Schmidt operator, $\int_a^b \int_c^d |k(x,y)|^2\,dx\,dy < \infty$. Now, $\left|\overline{k(x,y)}\right|^2 = |k(x,y)|^2$ and so it follows that $\int_a^b \int_c^d \left|\overline{k(x,y)}\right|^2\,dx\,dy < \infty$. Thus T^* is also Hilbert–Schmidt. ■

It turns out that the above proposition holds for all Hilbert–Schmidt operators. That is to say that given any Hilbert–Schmidt operator the adjoint is also a Hilbert–Schmidt operator. This result is contained within the next proposition. It can be found, with proof in [15] (Lemma 2, page 1010).

Proposition 2.1.1.8 *Suppose that $T : H_1 \to H_2$ is a Hilbert–Schmidt operator and let $T^* : H_2 \to H_1$ denote the adjoint of T. Then $\|T\|_{\text{HS}} = \|T^*\|_{\text{HS}}$. Furthermore, T^* is a Hilbert–Schmidt operator.*

Proof. Note that by Proposition 2.1.0.8, $\|T\|_{\text{HS}} = \|T^*\|_{\text{HS}}$. Definition 2.1.1.3 then immediately tells us that T^* is a Hilbert–Schmidt operator. ■

We return to the case of a general Hilbert–Schmidt operator and provide some results that will be called upon in later sections. First we see that the product of a bounded operator and a Hilbert–Schmidt operator is a Hilbert–Schmidt operator.

Proposition 2.1.1.9 *Suppose that we have operators R and S such that R is bounded and S is Hilbert–Schmidt. Then RS and SR are Hilbert–Schmidt operators.*

Proof. We show that the operators RS and SR are Hilbert–Schmidt by calculating their Hilbert–Schmidt norms and showing that they are finite. First we calculate the norm of RS,

$$\|RS\|_{\text{HS}}^2 = \sum_n \|RSe_n\|^2.$$

Since R is bounded it follows that $\|RSe_n\| \leq \|R\|_{\text{op}} \|Se_n\|$, thus,

$$\begin{aligned} \|RS\|_{\text{HS}}^2 &\leq \|R\|_{\text{op}}^2 \sum_n \|Se_n\|^2 \\ &= \|R\|_{\text{op}}^2 \|S\|_{\text{HS}}^2 . \end{aligned}$$

Since S is Hilbert–Schmidt it follows that $\|RS\|_{\text{HS}}$ is finite and therefore RS is Hilbert–Schmidt.

Next we show that SR is Hilbert–Schmidt by showing that its adjoint, R^*S^* is Hilbert–Schmidt. That is, we show that

$$\|R^*S^*\|_{\text{HS}}^2 = \sum_n \|R^*S^*e_n\|^2$$

is finite. Now, by Proposition 2.1.0.8 we have $\|R^*\| = \|R\|$, therefore, given that R is bounded, it follows that R^* is also bounded. Thus, $\|R^*S^*e_n\| \leq \|R^*\|_{\text{op}} \|S^*e_n\|$. Furthermore, by Proposition 2.1.1.7, since S is Hilbert–Schmidt, its adjoint, S^* is also Hilbert–Schmidt. Hence,

$$\begin{aligned} \|R^*S^*\|_{\text{HS}}^2 &\leq \|R^*\|_{\text{op}}^2 \sum_n \|S^*e_n\|^2 \\ &= \|R^*\|_{\text{op}}^2 \|S^*\|_{\text{HS}}^2 \end{aligned}$$

from which it follows that $\|R^*S^*\|_{\text{HS}}$ is finite. This shows that the operator R^*S^* is Hilbert–Schmidt. Another application of Proposition 2.1.1.7 shows that the adjoint of R^*S^*, which is $(R^*S^*)^* = SR$, is also Hilbert–Schmidt. This completes the proof. ■

Finally, we conclude this section with a result about the eigenvalues of a Hilbert–Schmidt operator. The proof can be found in [15] (see proof of Theorem 25, page 1034).

Proposition 2.1.1.10 *Let T be a Hilbert–Schmidt operator with eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ listed according to multiplicity. Then*

$$\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty.$$

2.1.2 Trace Class Operators

As in the previous section we focus on a particular type of operator and provide some basic results that will be called upon in later sections. Here we introduce trace class operators which we define using the Hilbert–Schmidt operators. We return to trace class operators in Section 2.3 to define the corresponding Fredholm determinants.

Definition 2.1.2.1 *Let $T = RS$ where R and S are Hilbert–Schmidt operators. Then we say that T is a* trace class *operator.*

In the following proposition we note that the sum of two trace class operators is again trace class.

Proposition 2.1.2.2 *Let T and W be trace class operators. Then $T + W$ is also a trace class operator.*

Proof. Let T and W be trace class operators and suppose that $T = RS$ and $W = UV$ where R, S, U, V are Hilbert–Schmidt. Then

$$\begin{aligned} T + W &= RS + UV \\ &= \begin{bmatrix} R & U \end{bmatrix} \begin{bmatrix} S \\ V \end{bmatrix} \end{aligned}$$

where $\begin{bmatrix} R & U \end{bmatrix}$ and $\begin{bmatrix} S & V \end{bmatrix}^T$ are Hilbert–Schmidt since R, S, U and V are Hilbert–Schmidt. Therefore, $T + W$ is the product of two Hilbert–Schmidt operators and hence by Definition 2.1.2.1 is trace class. ■

We continue by defining the norm of a trace class operator.

Definition 2.1.2.3 *Let R and S be Hilbert–Schmidt operators so that $T = RS$ is a trace class operator. Then the* trace class norm, *$\|T\|_{\mathrm{TC}}$ is given by*

$$\|T\|_{\mathrm{TC}} = \inf \{\|R\|_{\mathrm{HS}} \|S\|_{\mathrm{HS}} : T = RS\}.$$

The following lemma gives a condition under which a matrix is trace class.

Lemma 2.1.2.4 *Let $[a_{ij}]_{i,j}$ be a complex matrix such that*

$$\sum_{i,j} |a_{ij}| < \infty.$$

Then $[a_{ij}]_{i,j}$ is trace class.

Proof. Let $[a_{ij}]_{i,j}$ be a complex matrix and let E_{ij} be the matrix with 1 in the $(i, j)^{\text{th}}$ position and zeros everywhere else. We can write any complex matrix as a sum of matrices of the form E_{ij}, where the coefficient of E_{ij} is given by the $(i, j)^{\text{th}}$ element of the original complex matrix. We therefore have

$$[a_{ij}]_{i,j} = \sum_{i,j} a_{ij} E_{ij}.$$

We want to calculate the trace class norm of $[a_{ij}]_{i,j}$ and show that it is finite. First note that since $E_{ij} = E_{ii} E_{ij}$ where E_{ii} and E_{ij} are Hilbert–Schmidt, it follows from Definition 2.1.2.3 that

$$\|E_{ij}\|_{\mathrm{TC}} \leq \|E_{ii}\|_{\mathrm{HS}} \|E_{ij}\|_{\mathrm{HS}}. \tag{2.3}$$

By Definition 2.1.1.1 we have

$$\|E_{ij}\|^2_{\mathrm{HS}} = \mathrm{tr}\left(E^*_{ij} E_{ij}\right),$$

and since E_{ij} is real it follows that

$$\begin{aligned} E^*_{ij} &= E^T_{ij} \\ &= E_{ji}. \end{aligned}$$

Thus for any $i, j \in \mathbb{Z}$ we have

$$\begin{aligned} \|E_{ij}\|^2_{\mathrm{HS}} &= \mathrm{tr}\,(E_{ji} E_{ij}) \\ &= \mathrm{tr}\,(E_{jj}) \\ &= 1. \end{aligned}$$

It now follows from (2.3) that $\|E_{ij}\|_{\mathrm{TC}} \leq 1$, hence,

$$\begin{aligned} \left\| \sum_{i,j} a_{ij} E_{ij} \right\|_{\mathrm{TC}} &\leq \sum_{i,j} |a_{ij}| \, \|E_{ij}\|_{\mathrm{TC}} \\ &\leq \sum_{i,j} |a_{ij}|. \end{aligned}$$

Since $\sum_{i,j} |a_{ij}| < \infty$ by hypothesis, we conclude that $[a_{ij}]_{i,j}$ is trace class. ■

As with Hilbert–Schmidt operators, it can be shown that the adjoint of a trace class operator is also trace class. We address this in the following proposition.

Proposition 2.1.2.5 *Suppose that T is a trace class operator then the adjoint, T^* is also trace class.*

Proof. Let T be a trace class operator then by Definition 2.1.2.1, $T = RS$ for some Hilbert–Schmidt operators, R and S. Now, using Proposition 2.1.0.8 we see that $T^* = S^* R^*$ where S^* and R^* are Hilbert–Schmidt by Proposition 2.1.1.8. Therefore, T^* is the product of two Hilbert–Schmidt operators and so T^* is trace class. ■

We finish this section by noting that every trace class operator is a Hilbert–Schmidt operator.

Proposition 2.1.2.6 *If T is a trace class operator then T is also a Hilbert–Schmidt operator.*

Proof. Suppose that T is a trace class operator. By Definition 2.1.2.1 we have $T = RS$ where R and S are Hilbert–Schmidt operators. It then follows from Definition 2.1.1.3 that R and S are both bounded. Therefore, by Proposition 2.1.1.9, T is also Hilbert–Schmidt. ■

2.2 Entire Functions

The purpose of this section is to show that any entire function of order $\frac{1}{2}$ has infinitely many zeros and to show that the function can be written as a convergent infinite product. This result will enable us to discover various properties of functions relating to Hill's equation in Chapter 4. Definitions and results contained within this section can be found in [54] (Chapter 8, page 246).

Definition 2.2.0.7 *An entire function, f is said to be of* order $p \geq 0$ *if, for all $\epsilon > 0$, there exists some constant C such that*

$$|f(z)| < Ce^{|z|^{p+\epsilon}}$$

for all $z \in \mathbb{C}$.

A function of finite order has associated with it a convergent infinite product known as the canonical product. Before defining the canonical product we give two preliminary definitions.

Definition 2.2.0.8 *Define the functions $E(\cdot, q)$ by*

$$\begin{aligned} E(x, 0) &= 1 - x; \\ E(x, q) &= (1 - x)e^{x + \frac{x^2}{2} + \cdots + \frac{x^q}{q}} \end{aligned}$$

for $q \in \mathbb{N}$. We call the functions $E(\cdot, q)$ the primary factors.

Definition 2.2.0.9 *Let f be an entire function of finite order with zeros $(z_n)_{n=1}^{\infty}$. The smallest non-negative integer, r for which*

$$\sum_{n=1}^{\infty}\left(\left|\frac{z}{z_n}\right|\right)^{r+1}$$

is convergent for all $z \in \mathbb{C}$ is known as the genus.

Remark 2.2.0.10 *Note that since $r \geq 0$, the sum in Definition 2.2.0.9 is concerned only with positive powers. Indeed we have $r + 1 \geq 1$.*

The following lemma can be found, with proof, in [54] (Section 8.22, page 249). It shows the relationship between the order of a function and its genus.

Lemma 2.2.0.11 *Let f be an entire function of order p and with zeros (z_n). Then the series*

$$\sum_{n}\frac{1}{|z_n|^a}$$

is convergent when $a > p$.

Combining Lemma 2.2.0.11 with Definition 2.2.0.9, we see that for a function with order p and genus r we must have $r + 1 > p$. We now continue to define the canonical product.

Definition 2.2.0.12 *Let f be an entire function of finite order with zeros $(z_n)_{n=1}^{\infty}$ and suppose that f has genus r. The* canonical product *is given by the convergent product*

$$\prod_{n=1}^{\infty} E\left[\frac{z}{z_n}, r\right].$$

The next proposition is known as *Hadamard's Factorisation Theorem*. It appears with proof in [54] (Section 8.24, page 250).

Proposition 2.2.0.13 *Let f be an entire function of order p such that $f(0) \neq 0$. Let f have zeros $(z_n)_{n=1}^{\infty}$ then*

$$f(z) = e^{P(z)} \prod_{n=1}^{\infty} E\left[\frac{z}{z_n}, r\right]$$

where $P(z)$ is a polynomial of degree m such that $m \leq p$.

We now come to the main result of this section. It states that for a function of order $\frac{1}{2}$, the function has infinitely many zeros and can be expressed as a convergent infinite product.

Proposition 2.2.0.14 *Let f be an entire function of order $\frac{1}{2}$. Then f has infinitely many zeros, $(z_n)_{n=1}^{\infty}$. Further, if $f(0) \neq 0$ then*

$$f(z) = C \prod_{n=1}^{\infty}\left[1 - \frac{z}{z_n}\right].$$

for some constant, C.

Proof. Suppose that f is an entire function of order $\frac{1}{2}$ and let f have zeros $(z_n)_{n=1}^{\infty}$. By Lemma 2.2.0.11, the series

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^a}$$

is convergent for $a > \frac{1}{2}$. Furthermore, by [54] (Section 8.26, page 252) the series is divergent for $a < \frac{1}{2}$. Since divergence occurs for $a < \frac{1}{2}$ we must have an infinite series. Thus f has infinitely many zeros.

Now suppose that $f(0) \neq 0$ and apply Hadamard's Factorisation Theorem 2.2.0.13. Since f has order $\frac{1}{2}$ it follows from Definition 2.2.0.9 and Lemma 2.2.0.11 that the genus, r is the smallest non-negative integer such that $r > -\frac{1}{2}$, hence $r = 0$. Given that f has zeros $(z_n)_{n=1}^{\infty}$, it now follows from Proposition 2.2.0.13 that

$$f(z) = e^{Q(z)} \prod_{n=1}^{\infty} E\left[\frac{z}{z_n}, 0\right]$$

where $Q(z)$ is a polynomial of degree less than $\frac{1}{2}$. Clearly we take $Q(z)$ to be a constant polynomial and so $e^{Q(z)} = C$ for some constant, C. Finally, by Definition 2.2.0.8 we see that $E\left[\frac{z}{z_n}, 0\right] = 1 - \frac{z}{z_n}$ and so

$$f(z) = C \prod_{n=1}^{\infty} \left[1 - \frac{z}{z_n}\right]$$

as required. ■

2.3 Matrices and Determinants

Here we introduce special types of matrices that will be used in later chapters. Specifically, we define the Vandermonde matrix and its determinant since this will be used explicitly in calculations throughout Chapter 6. We also describe a Toeplitz matrix so that the reader can become familiar with the shape. In Section 6.5 we show that matrices associated with Ramanujan's integral are Toeplitz. We also return to the subject of Hilbert–Schmidt and trace class operators, defining the Carleman and Fredholm determinants and their traces. Carleman determinants are defined for $I + T$ where T is a Hilbert–Schmidt operator, while Fredholm determinants are defined for $I + S$ where S is a trace class operator. The Carleman determinant is useful since it is easier to test whether an operator is Hilbert–Schmidt than to test if it is trace class. Carleman and Fredholm determinants will be used in Section 4.5 to give results regarding the periodic spectrum of Hill's equation. A detailed theory regarding the determinants and traces of Hilbert–Schmidt and trace class operators can be found in [51].

We begin by looking at two types of matrices; the Toeplitz matrices and the Vandermonde matrices.

Definition 2.3.0.15 *A matrix, $A = [a_{ij}]_{i,j=1}^{n}$ is said to be a* Toeplitz matrix *if it has the form*

$A = [a_{i-j}]_{i,j=1}^{n}$. *That is,*

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & & & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \\ a_2 & a_1 & a_0 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & a_{-2} \\ & & \ddots & \ddots & \ddots & a_{-1} \\ a_{n-1} & & & a_2 & a_1 & a_0 \end{bmatrix}.$$

The elements of a Toeplitz matrix satisfy the relation

$$a_{ij} = a_{i+1,j+1}.$$

Remark 2.3.0.16 *Note that the elements of a Toeplitz matrix are constant along the leading diagonals.*

Definition 2.3.0.17 *Let V be an $n \times n$ matrix of the form*

$$V = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{n-1} \end{bmatrix}$$

where $z_j \in \mathbb{C}$ for $j = 1, \ldots, n$. Then V is known as a Vandermonde matrix*. Furthermore, V has* Vandermonde determinant *given by*

$$\det V = \prod_{1 \le j < k \le n} (z_k - z_j).$$

The Toeplitz and Vandermonde matrices will be used in Chapter 6. Next we introduce another form of matrix known as Hill's type. As we shall see, a Hill's type matrix is related to a trace class operator.

Definition 2.3.0.18 *Let*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The matrix, $[a_{ij}]_{i,j}$ is said to be of Hill's type *if*

$$\sum_{i,j} |a_{ij} - \delta_{ij}| < \infty.$$

Remark 2.3.0.19 *Note that if A is of Hill's type then by Lemma 2.1.2.4, $A - I$ is trace class.*

Having defined some of the matrices that will appear in later chapters we now move on to define the determinants that will be used. We have previously described Hilbert–Schmidt and trace class operators and we now return to define their Carleman and Fredholm determinants respectively. First, for completeness, we define the matrix of an operator and the standard determinant.

Definition 2.3.0.20 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space with basis (e_i) and let $T: H \to H$ be an operator. For $Te_j \in H$ we may write*

$$Te_j = \sum_i \langle Te_j, e_i\rangle e_i. \tag{2.4}$$

We define the matrix of T with respect to the basis (e_i) *to be*

$$[\langle Te_j, e_i\rangle]_{i,j}\,,$$

where i denotes the row and j denotes the column. The coefficients of the basis elements in (2.4) form the j^{th} column of the matrix of T.

Definition 2.3.0.21 *Let S_n denote the symmetric group on $\{1,\ldots,n\}$ and let $A = [a_{ij}]_{i,j=1}^n$ be an $n \times n$ complex matrix. We denote by* $\det A$ *the standard* determinant *as given by the formula,*

$$\det A = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{\sigma(j),j}.$$

Remark 2.3.0.22 *If A has eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$ listed according to multiplicity then A has determinant*

$$\det A = \prod_{j=1}^{n} \lambda_j.$$

Having defined a basic determinant we now turn our attention to Fredholm determinants which arise in the presence of trace class operators. Note the similarity between the determinant given in Remark 2.3.0.22 and the Fredholm determinant as defined by Definition 2.3.0.24. It should be noted that the formula for a Fredholm determinant, which here we take as a definition, has been proved as a result in [51] (Theorem 3.7, page 35). In order for our Fredholm determinant to be well defined we shall first define the trace of a trace class matrix.

Definition 2.3.0.23 *Let T be a trace class operator with eigenvalues $\{\lambda_n\}$ counted with multiplicity. We define the* trace *of T to be the absolutely convergent series*

$$\operatorname{tr}(T) = \sum_n \lambda_n.$$

The formula for the trace of a trace class operator that appears in Definition 2.3.0.23 is a result credited to Lidskii. It can be found in [51] (equation (3.2), page 32 and Theorem 3.7, page 35). We also note that since $\operatorname{tr}(T)$ is a convergent sum, the product appearing in Definition 2.3.0.24 is also convergent, hence the Fredholm determinant is well defined.

Definition 2.3.0.24 *Let T be a trace class operator on Hilbert space, H. Suppose that T has eigenvalues $\{\lambda_n\}$ counted with multiplicity. We define the* Fredholm determinant *of T to be*

$$\det(I+T) = \prod_n [1+\lambda_n].$$

Our next task will be to define the Carleman determinant for Hilbert–Schmidt operators. We first take some time to define the trace of T^2 where T is a Hilbert–Schmidt operator before addressing the issue of Carleman determinants. Note that as well as defining the trace, Definition 2.3.0.25 shows that the eigenvalues of a Hilbert–Schmidt operator are square summable. Again, the formula provided in Definition 2.3.0.25 can be found in [51] (equation (3.3), page 32).

Definition 2.3.0.25 *Let T be a Hilbert–Schmidt operator with eigenvalues $\{\lambda_n\}$ counted with multiplicity. Then the trace of T^2 is given by the absolutely convergent sum*

$$\operatorname{tr}\left(T^2\right) = \sum_n \lambda_n^2.$$

Remark 2.3.0.26 *Note that the formula for* $\operatorname{tr}\left(T^2\right)$ *follows directly from Definition 2.3.0.23, for if T is Hilbert–Schmidt then T^2 is trace class by Definition 2.1.2.1.*

We now state the definition of a Carleman determinant. First note that it follows from [15] (Theorem 26, page 1036) that the product given in Definition 2.3.0.27 is convergent, hence well defined.

Definition 2.3.0.27 *Let T be a Hilbert–Schmidt operator on a Hilbert space, H and suppose that T has eigenvalues $\{\lambda_n\}$ counted with multiplicity. We define the* Carleman determinant *to be*

$$\det_2(I+T) = \prod_n [1+\lambda_n] e^{-\lambda_n}.$$

The following proposition shows the relationship between the Carleman and Fredholm determinants, it will be used in Section 4.5 to switch between the two. That is to say, if we find a condition for one determinant then, assuming we have a trace class operator, the same condition holds for the other determinant.

Proposition 2.3.0.28 *Let T be a trace class operator then the following relation holds,*

$$\det_2(I+T) = \det(I+T)e^{-\operatorname{tr} T}.$$

Proof. We first note that since T is a trace class operator, $\det(I+T)$ is defined. Also, by Proposition 2.1.2.6, T is Hilbert–Schmidt and therefore $\det_2(I+T)$ is defined. By Definition 2.3.0.27 we have the convergent product

$$\det_2(I+T) = \prod_n [1+\lambda_n] e^{-\lambda_n}.$$

Rearranging the terms so that we collect all of the exponential terms together, we thus obtain

$$\det_2(I+T) = \left(\prod_n [1+\lambda_n]\right) e^{-\sum_n \lambda_n}.$$

It now follows from Definition 2.3.0.23 and Definition 2.3.0.24 that

$$\det_2(I+T) = \det(I+T)e^{-\operatorname{tr} T}$$

as required. ■

The following result is a consequence of Proposition 2.3.0.28. Corollary 2.3.0.29 will be used in Section 4.5 to show that the Carleman and Fredholm determinants can be used interchangeably, in the sense that the determinants have the same zeros.

Corollary 2.3.0.29 *Let T be a trace class operator. Then*

$$\det_2(I+T) = 0 \iff \det(I+T) = 0.$$

Proof. Let T be a trace class operator then by Proposition 2.1.2.6, T is also Hilbert–Schmidt. Therefore, $\det_2(I+T)$ is defined. Now suppose that $\det_2(I+T)=0$. By Proposition 2.3.0.28, this happens if and only if $\det(I+T)e^{-\operatorname{tr}T}=0$. Since the exponential term is never zero, we conclude that $\det_2(I+T)=0$ if and only if $\det(I+T)=0$. ∎

Next we state and prove *Sylvester's Determinant Theorem.* The result is commonly used with determinants of the form $\det(I+AB)$, that is, it is defined for Fredholm determinants with A,B Hilbert–Schmidt. Here we present the usual case and also show that the result holds for Carleman determinants.

Proposition 2.3.0.30 *Suppose that A and B are Hilbert–Schmidt operators then*

$$\det(I+AB)=\det(I+BA).$$

Further,

$$\det_2(I+AB)=\det_2(I+BA).$$

Proof. We approximate the Hilbert–Schmidt operators by a sequence of finite rank operators. Let A_n be a finite rank operator corresponding to a matrix of size $j_n\times k_n$. Similarly, let B_n be a finite rank operator corresponding to a matrix of size $k_n\times j_n$. Let A,B be Hilbert–Schmidt operators and suppose that $A_n\to A$ and $B_n\to B$ as $n\to\infty$. Then

$$\begin{aligned}\begin{bmatrix} I_{j_n}+A_nB_n & -A_n \\ 0 & I_{k_n}\end{bmatrix}\begin{bmatrix} I_{j_n} & 0 \\ B_n & I_{k_n}\end{bmatrix} &= \begin{bmatrix} I_{j_n} & -A_n \\ B_n & I_{k_n}\end{bmatrix} \\ &= \begin{bmatrix} I_{j_n} & 0 \\ B_n & I_{k_n}\end{bmatrix}\begin{bmatrix} I_{j_n} & -A_n \\ 0 & I_{k_n}+B_nA_n\end{bmatrix}. \end{aligned}\tag{2.5}$$

Taking the determinant of both sides of (2.5) then gives

$$\det\begin{bmatrix} I_{j_n}+A_nB_n & -A_n \\ 0 & I_{k_n}\end{bmatrix}\det\begin{bmatrix} I_{j_n} & 0 \\ B_n & I_{k_n}\end{bmatrix}=\det\begin{bmatrix} I_{j_n} & 0 \\ B_n & I_{k_n}\end{bmatrix}\det\begin{bmatrix} I_{j_n} & -A_n \\ 0 & I_{k_n}+B_nA_n\end{bmatrix}.\tag{2.6}$$

Notice that each matrix in (2.6) is triangular and so it follows that

$$\det(I_{j_n}+A_nB_n)\det(I_{k_n})\det(I_{j_n})\det(I_{k_n})=\det(I_{j_n})\det(I_{k_n})\det(I_{j_n})\det(I_{k_n}+B_nA_n).$$

Hence,

$$\det(I_{j_n}+A_nB_n)=\det(I_{k_n}+B_nA_n).\tag{2.7}$$

Taking the limits of both sides of (2.7) as $n\to\infty$ now gives

$$\det(I+AB)=\det(I+BA)$$

completing the first part of the proof.

Now consider the Carleman determinant. By Proposition 2.3.0.28 we have

$$\det_2(I+AB)=\det(I+AB)e^{-\operatorname{tr}AB}.$$

Now, since A, B are Hilbert–Schmidt, it follows from [15] (Lemma 14(b), page 1098) that $\operatorname{tr} AB = \operatorname{tr} BA$. Using this together with the first part of the proposition we see that

$$\begin{aligned} \det_2(I + AB) &= \det(I + BA)e^{-\operatorname{tr} BA} \\ &= \det_2(I + BA) \end{aligned}$$

as required. ■

Finally we present *Andréief's Identity.* This identity, appearing in Lemma 2.3.0.31 will be used in the evaluation of Ramanujan's integral in Chapter 6. It also makes an appearance in Section 5.4 to help with the evaluation of determinants of Gram matrices. For a proof of Andréief's Identity see [3] (Lemma 2.2.2(i), page 51).

Lemma 2.3.0.31 *Let B be a bounded interval. For $j \in \{1, \dots, n\}$, let f_j and g_j be continuous complex functions defined on B. Then,*

$$\det \left[\int_B f_j(x) g_k(x)\, dx \right]_{j,k=1}^n = \frac{1}{n!} \int \cdots \int_{B^n} \det [f_j(x_k)]_{j,k=1}^n \det [g_l(x_k)]_{l,k=1}^n \, dx_1 \dots dx_n.$$

2.4 The Construction of the Paley–Wiener Spaces

This section is devoted to the construction of the Paley–Wiener spaces. Paley–Wiener spaces consist of functions whose Fourier transforms have compact support. If the support of the Fourier transform is a fixed interval then the corresponding Paley–Wiener space consists of the band-limited functions. We therefore construct the Paley–Wiener spaces by first defining Fourier transforms and band-limited functions. Once we have defined a Paley–Wiener space we then introduce the Paley–Wiener Theorem which states that any function, $f \in L^2(\mathbb{R})$ that is entire and of exponential type belongs to a Paley–Wiener space.

2.4.1 Fourier Transforms

We introduce the L^2 Fourier transform which is a type of integral operator. It can be shown that the L^2 Fourier transform can be derived from the L^1 Fourier transform by taking the limit in L^2. Both the L^1 and L^2 Fourier transforms are in fact equivalent. Details of how to construct the L^2 Fourier transform from the L^1 Fourier transform can be found in [46] (Chapter 9, page 178).

Definition 2.4.1.1 *For $f \in L^2(\mathbb{R})$, we define the* Fourier transform *of f to be,*

$$\hat{f}(t) = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(x) e^{-itx}\, dx$$

where the limit exists in the L^2 sense.

Fourier transforms are invertible and we define the inverse Fourier transform as follows.

Definition 2.4.1.2 *Given $f \in L^2(\mathbb{R})$ we define the* inverse Fourier transform *to be,*

$$\check{f}(x) = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(t) e^{itx}\, dt$$

where again the limit exists in the L^2 sense.

We complete this short section by giving two results that will be used in later proofs. First an intuitive result that shows that the inverse Fourier transform is indeed the inverse of the Fourier transform. The result, often referred to as the L^2 *Inversion Theorem*, allows us to write a function, f as the inverse Fourier transform of its Fourier transform. See [46] (Theorem 9.13 (d), page 186) for details.

Proposition 2.4.1.3 *Let $f \in L^2(\mathbb{R})$ and suppose that f has Fourier transform $\hat{f}$. Then*

$$f(x) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(t)e^{itx}\,dt$$

where the limit exists in the L^2 sense.

Finally we give another result from [46] (Theorem 9.13 (b), page 186) that states that if $f \in L^2$ then f has Fourier transform also in L^2 and such that their norms are equal. The latter part of this result, that the norms of f and its Fourier transform are equal, is known as *Plancherel's formula*. We use Plancherel's formula to prove the converse of the Paley–Wiener Theorem in Section 2.4.3.

Proposition 2.4.1.4 *Suppose that $f \in L^2(\mathbb{R})$ and let $\hat{f}$ be the Fourier transform of f as defined in Definition 2.4.1.1. Then $\hat{f} \in L^2(\mathbb{R})$ and*

$$\|f\|_{L^2(\mathbb{R})} = \left\|\hat{f}\right\|_{L^2(\mathbb{R})}.$$

2.4.2 Band-limited Functions

The set of band-limited functions are defined by properties of their Fourier transforms. Specifically a band-limited function is one in which the density of its Fourier transform lies in a given interval. Further information on band-limited functions can be found in [18] (Section 2.9, page 121).

Definition 2.4.2.1 *Suppose that $f \in L^2(\mathbb{R})$ and let $\hat{f}$ denote its Fourier transform. We say that f is a* band-limited *function if for some fixed b,*

$$\hat{f}(t) = 0 \quad \textit{for} \quad t \in \mathbb{R}\backslash[-b, b].$$

It is easily seen from Definition 2.4.2.1 that a function is band-limited if its Fourier transform is supported on the interval $[-b, b]$. Note that by Proposition 2.4.1.4, since $f \in L^2(\mathbb{R})$ we also have $\hat{f} \in L^2(\mathbb{R})$, hence f is band-limited if $\hat{f} \in L^2[-b, b]$ for some fixed b.

The following proposition gives an alternative formulation for a band-limited function. It makes clear the idea that band-limited functions are the functions of Paley–Wiener spaces as will become apparent in Section 2.4.3. The proof is simple and relies on the inverse Fourier transform.

Proposition 2.4.2.2 *Let $f \in L^2(\mathbb{R})$. Then f is a band-limited function if and only if*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} g(t)e^{ixt}\,dt \tag{2.8}$$

for some $g \in L^2[-b, b]$ where $b > 0$.

Proof. Suppose that $f \in L^2(\mathbb{R})$ is band-limited so that $\hat{f}$ has support on $[-b, b]$ for some fixed b. By Proposition 2.4.1.3 we have

$$f(x) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(t)e^{ixt}\, dt.$$

Choose $R > b$ then, since $\hat{f}(t) = 0$ for $t \notin [-b, b]$,

$$\begin{aligned} \int_{-R}^{R} \hat{f}(t)e^{ixt}\, dt &= \int_{-R}^{-b} \hat{f}(t)e^{ixt}\, dt + \int_{-b}^{b} \hat{f}(t)e^{ixt}\, dt + \int_{b}^{R} \hat{f}(t)e^{ixt}\, dt \\ &= \int_{-b}^{b} \hat{f}(t)e^{ixt}\, dt. \end{aligned}$$

Thus $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} \hat{f}(t)e^{ixt}\, dt$.

Now suppose that $f \in L^2(\mathbb{R})$ takes the form given by (2.8) for some $g \in L^2[-b, b]$. We extend the function g to the whole real line by setting $g(t) = 0$ for $t \in \mathbb{R}\backslash[-b, b]$. By the uniqueness of Fourier transforms, $g(t) = \hat{f}(t)$, thus f is band-limited. ■

In practical applications, when determining whether a function is band-limited or not, it is often useful to see if it can be written in the form of (2.8).

2.4.3 Paley–Wiener Spaces and the Paley–Wiener Theorem

Having laid the foundations we can now define a Paley–Wiener space. Sampling theory is traditionally carried out on Paley–Wiener space. McKean and Trubowitz have used special function spaces to do sampling related to Hill's equation, so in this thesis we want to work in a more standard context. We focus on defining a Paley–Wiener space using the concepts of Sections 2.4.1 and 2.4.2 and also find a reproducing kernel for a Paley–Wiener space over an interval $[-b, b]$. We will return to the concept of reproducing kernels in Section 5.4 where we create a sequence of reproducing kernels based on sampling points and then construct their Gram matrix. Continuing with the current section, we introduce the Paley–Wiener Theorem which gives a set of conditions which, once satisfied, will ensure that a given function lies in a specified Paley–Wiener space. Should the reader wish to find more information regarding Paley–Wiener spaces and the properties of Paley–Wiener functions, they should consult [41] (Section 7.1, page 204).

Definition 2.4.3.1 *Let $C \subseteq \mathbb{R}$ be a compact set. The* Paley–Wiener space*, $PW(C)$ is defined to be*

$$PW(C) = \left\{ f \in L^2(\mathbb{R}) : f(t) = \frac{1}{\sqrt{2\pi}} \int_C \hat{f}(x)e^{itx}\, dx, \forall t \in \mathbb{R}, \hat{f} \in L^2(C) \right\}.$$

Remark 2.4.3.2 *If $b > 0$ is real then we use the notation $PW(b)$ to denote $PW[-b, b]$.*

The space $PW(C)$ is therefore the space of functions whose Fourier transforms are supported on C, that is, for $f \in PW(C)$, $\hat{f}(x) = 0$ for $x \notin C$. Paley–Wiener spaces thus consist of band-limited functions.

In the following definition we define a reproducing kernel. A reproducing kernel for a Hilbert space, H is a function which, when taken as an inner product with any $f \in H$ will evaluate the function f at a desired point. More detailed information regarding reproducing kernels and examples of reproducing kernels for various spaces can be found in [41] (Section 5.1, page 144).

Definition 2.4.3.3 *Let S be a non-empty set. Further, let H be a Hilbert space such that for all $f \in H$, $f : S \to \mathbb{C}$. A function, $k_s \in H$ is a* reproducing kernel *for H, if for all $f \in H$*

$$f(s) = \langle f, k_s \rangle_H.$$

An example of a reproducing kernel can be found in the following proposition.

Proposition 2.4.3.4 *Let $t \in \mathbb{R}$. The function*

$$k_t(x) = \frac{\sin b(t-x)}{\pi(t-x)}$$

is a reproducing kernel for $PW(b)$.

Proof. Let $f \in PW(b)$ then by Definition 2.4.3.1 we have

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} \hat{f}(x) e^{itx}\, dx \tag{2.9}$$

for all $t \in \mathbb{R}$. Also, by Definition 2.4.1.1, the Fourier transform of f is given by

$$\hat{f}(x) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(y) e^{-ixy}\, dy. \tag{2.10}$$

Substituting equation (2.10) into (2.9) and reversing the order of integration we obtain

$$\begin{aligned} f(t) &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-b}^{b} \int_{-R}^{R} f(y) e^{i(t-y)x}\, dy\, dx \\ &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} f(y) \int_{-b}^{b} e^{i(t-y)x}\, dx\, dy. \end{aligned}$$

Note that

$$\begin{aligned} \int_{-b}^{b} e^{i(t-y)x}\, dx &= \left[\frac{e^{i(t-y)x}}{i(t-y)} \right]_{x=-b}^{b} \\ &= \frac{e^{i(t-y)b} - e^{-i(t-y)b}}{i(t-y)} \\ &= 2\frac{\sin b(t-y)}{t-y}. \end{aligned}$$

Thus,

$$\begin{aligned} f(t) &= \lim_{R\to\infty} \int_{-R}^{R} f(y) \frac{\sin b(t-y)}{\pi(t-y)}\, dy \\ &= \langle f, k_t \rangle_{L^2(\mathbb{R})} \end{aligned}$$

where $k_t(x) = \frac{\sin b(t-x)}{\pi(t-x)}$. Hence k_t is indeed a reproducing kernel for $PW(b)$. ■

We end this section by looking at the Paley–Wiener Theorem. The theorem, which we label here as a proposition, provides a way in which we can identify functions that belong to Paley–Wiener spaces. It states that if a function is entire and of exponential type then it must be band-limited, hence belongs to a Paley–Wiener space. The compact set over which the Paley–Wiener space is defined is dependent upon the bound in the definition of exponential type.

Definition 2.4.3.5 *An entire function, f is said to be of* exponential type M *if there exist constants C, M such that,*

$$|f(z)| \leq Ce^{M|z|}$$

for all $z \in \mathbb{C}$.

The following proposition is known as the *Paley–Wiener Theorem*. It can be found with proof in [46] (Theorem 19.3, page 375) and [41] (Theorem 7.1.3, page 205).

Proposition 2.4.3.6 *Suppose that f is an entire function of exponential type M. Further, suppose that $f|_{\mathbb{R}} \in L^2(\mathbb{R})$ where $f|_{\mathbb{R}}$ denotes the restriction of f to the real axis. Then there exists a function $g \in L^2(-M, M)$ such that*

$$f(z) = \int_{-M}^{M} g(t)e^{itz}\, dt \tag{2.11}$$

for all $z \in \mathbb{C}$.

As previously stated, the Paley–Wiener Theorem is typically used to characterise band-limited functions. It shows us that for an entire function, f of exponential type M, we can find a function g such that g is the Fourier transform of f. Notice the similarity between (2.11) and Proposition 2.4.1.3. Moreover, $g \in L^2(-M, M)$ so by Proposition 2.4.2.2, f is a band-limited function. Alternatively, one can think of the Paley–Wiener Theorem as showing us that for a given entire function f, of exponential type M, f is, up to a constant, the Fourier transform of the function $\mathbb{I}_{[-M,M]}g$. This is easily seen as follows,

$$\begin{aligned} f(z) &= \int_{-M}^{M} g(t)e^{itz}\, dt \\ &= \lim_{R\to\infty} \int_{-R}^{R} \mathbb{I}_{[-M,M]}(t)g(t)e^{itz}\, dt \\ &= \lim_{R\to\infty} \int_{-R}^{R} \left[\mathbb{I}_{[-M,M]}(-t)g(-t)\right] e^{-itz}\, dt. \end{aligned} \tag{2.12}$$

Comparison of (2.12) with Definition 2.4.1.1 now shows that $f(z)$ is the Fourier transform of

$$\sqrt{2\pi}\mathbb{I}_{[-M,M]}(-t)g(-t).$$

It should be noted that the converse of the Paley–Wiener Theorem is also true. It tells us that a band-limited function is square integrable, entire and of exponential type. Alternatively, it states that a function, $f \in PW(M)$ is entire and of exponential type M. We state the converse of the Paley–Wiener theorem with proof in the following result.

Proposition 2.4.3.7 *Suppose that a function f has the form,*

$$f(z) = \int_{-M}^{M} g(t)e^{itz}\, dt \tag{2.13}$$

for some $g \in L^2(-M, M)$. Then f is entire, of exponential type M and $f|_{\mathbb{R}} \in L^2(\mathbb{R})$ where $f|_{\mathbb{R}}$ denotes the restriction of f to the real axis.

Proof. Let f be defined as in equation (2.13). We first show that f is entire. Clearly f is differentiable at all points $z \in \mathbb{C}$ with derivative

$$f'(z) = i \int_{-M}^{M} t g(t) e^{itz}\, dt.$$

It now follows that f is entire since e^{itz} is entire.

To see that $f|_{\mathbb{R}} \in L^2(\mathbb{R})$ it suffices to show that $f|_{\mathbb{R}}$ has finite L^2 norm over the real line. Let x be real and write f as follows,

$$f(-x) = \lim_{R \to \infty} \int_{-R}^{R} \mathbb{I}_{(-M,M)}(t) g(t) e^{-itx}\, dt.$$

Clearly, $f(-x)$ is the Fourier transform of $\sqrt{2\pi}\mathbb{I}_{(-M,M)}(t)g(t)$, hence by Plancherel's Formula, 2.4.1.4 we have,

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R})} &= \left\|\mathbb{I}_{(-M,M)}(t)g(t)\right\|^2_{L^2(\mathbb{R})} \\ &= \int_{-\infty}^{\infty} \left|\mathbb{I}_{(-M,M)}(t)g(t)\right|^2\, dt \\ &= \int_{-M}^{M} |g(t)|^2\, dt. \end{aligned}$$

As $g \in L^2(-M,M)$ it follows that

$$\int_{-M}^{M} |g(t)|^2\, dt < \infty,$$

hence $\|f\|_{L^2(\mathbb{R})}$ is finite.

To prove that f is of exponential type, first set $z = u + iv$ for $u, v \in \mathbb{R}$ then

$$\begin{aligned} |f(z)| &= |f(u+iv)| \\ &= \left| \int_{-M}^{M} g(t) e^{itu} e^{-tv}\, dt \right| \\ &\leq \int_{-M}^{M} |g(t)| \left|e^{-tv}\right|\, dt \end{aligned} \tag{2.14}$$

We now split into 3 cases. Case (i) in which $v > 0$; case (ii) in which $v < 0$; and case (iii) where $v = 0$.

Case (i) ($v > 0$): For $-M \leq t \leq M$ we have $-M \leq -t \leq M$ and so

$$e^{-Mv} \leq e^{-tv} \leq e^{Mv}.$$

It now follows from (2.14) that

$$|f(z)| \leq \int_{-M}^{M} e^{Mv} |g(t)|\, dt.$$

Using the Cauchy–Schwarz inequality and evaluating the first integral we see that

$$\begin{aligned} |f(z)| &\leq \left(\int_{-M}^{M} e^{2Mv}\, dt \right)^{\frac{1}{2}} \left(\int_{-M}^{M} |g(t)|^2\, dt \right)^{\frac{1}{2}} \\ &= \sqrt{2M} e^{Mv} \left(\int_{-M}^{M} |g(t)|^2\, dt \right)^{\frac{1}{2}} \\ &\leq Ce^{M|z|} \end{aligned}$$

where $C = \left(2M\int_{-M}^{M} |g(t)|^2 \, dt\right)^{\frac{1}{2}}$.

Case (ii) ($v < 0$): For $-M \le t \le M$ and $v < 0$ we now have $Mv \le -tv \le -Mv$, hence

$$e^{Mv} \le e^{-tv} \le e^{-Mv}.$$

Therefore, by (2.14) we now have

$$|f(z)| \le \int_{-M}^{M} e^{-Mv} \, |g(t)| \, dt.$$

Again, applying the Cauchy–Schwarz inequality and evaluating the first integral gives,

$$\begin{aligned} |f(z)| &\le \left(\int_{-M}^{M} e^{-2Mv} \, dt\right)^{\frac{1}{2}} \left(\int_{-M}^{M} |g(t)|^2 \, dt\right)^{\frac{1}{2}} \\ &= \sqrt{2M} e^{-Mv} \left(\int_{-M}^{M} |g(t)|^2 \, dt\right)^{\frac{1}{2}} \\ &\le Ce^{M|z|} \end{aligned}$$

where $C = \left(2M\int_{-M}^{M} |g(t)|^2 \, dt\right)^{\frac{1}{2}}$.

Case (iii) ($v = 0$): Finally, let $v = 0$ then by (2.14),

$$|f(z)| \le \int_{-M}^{M} |g(t)| \, dt.$$

In keeping with the previous calculations we apply the Cauchy–Schwarz inequality, thus

$$\begin{aligned} |f(z)| &\le \left(\int_{-M}^{M} 1^2 \, dt\right)^{\frac{1}{2}} \left(\int_{-M}^{M} |g(t)|^2 \, dt\right)^{\frac{1}{2}} \\ &= \sqrt{2M} \left(\int_{-M}^{M} |g(t)|^2 \, dt\right)^{\frac{1}{2}}. \end{aligned}$$

Clearly, taking $C = \left(2M\int_{-M}^{M} |g(t)|^2 \, dt\right)^{\frac{1}{2}}$ we have $|f(z)| \le Ce^{M|z|}$. Therefore $|f(z)| \le Ce^{M|z|}$ for all z. ∎

2.5 The Operators U and U^*

In this section we introduce two operators, U and U^* that are related to the Fourier transform and the inverse Fourier transform respectively. We show that the functions Uf and U^*f are band-limited while the operators U and U^* are Hilbert–Schmidt. The operators will then be used to construct a further operator, S. We see that S is a self-adjoint trace class operator and an integral operator with kernel given by the reproducing kernel for $PW(b)$.

Definition 2.5.0.8 *Let $t \in \mathbb{R}$. Define the operator $U : L^2[-a,a] \to L^2[-b,b]$ to be such that*

$$Uf(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-itx} \, dx. \tag{2.15}$$

Likewise, define the operator $U^* : L^2[-b,b] \to L^2[-a,a]$ to be such that

$$U^*f(t) = \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} f(x) e^{itx} \, dx. \tag{2.16}$$

Remark 2.5.0.9 *Note that equations (2.15) and (2.16) define Uf and U^*f for t real. We can easily extend this definition to the complex plane by taking*

$$\begin{aligned} Uf(z) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-izx}\, dx; \\ U^*f(z) &= \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} f(x) e^{izx}\, dx \end{aligned}$$

for $z \in \mathbb{C}$.

We see that Uf is actually the Fourier transform of f defined on a restricted range of integration. Also, the operator U acts on the band-limited functions. It is easy to see that $f \in L^2[-a,a]$ is band-limited since $\hat{f} = Uf$ has support on $[-b,b]$. Similarly, we see that U^*f is the inverse Fourier transform of f on a restricted range of integration. Notice also that $U^*f = \check{f}$ takes the form of (2.8) and so $\check{f} \in L^2[-a,a]$ is band-limited. Finally, the notation U^* is appropriate since U^* is the adjoint of U as the following proposition shows.

Proposition 2.5.0.10 *The operator U as defined by equation (2.15) has adjoint U^* as defined by equation (2.16).*

Proof. We use Definition 2.1.0.7 to find the adjoint of the operator U. Thus,

$$\begin{aligned} \langle f, U^*g \rangle_{L^2[-a,a]} &= \langle Uf, g \rangle_{L^2[-b,b]} \\ &= \int_{-b}^{b} Uf(t) \overline{g(t)}\, dt. \end{aligned}$$

Using Definition 2.5.0.8 and then reversing the order of integration we see that

$$\begin{aligned} \langle f, U^*g \rangle_{L^2[-a,a]} &= \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} \left(\int_{-a}^{a} f(x) e^{-itx}\, dx \right) \overline{g(t)}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) \int_{-b}^{b} \overline{g(t)} e^{-itx}\, dt\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) \overline{\int_{-b}^{b} g(t) e^{itx}\, dt}\, dx, \end{aligned}$$

hence $U^*g(x) = \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} g(t) e^{itx}\, dt$ as required. ∎

If we extend the definition of Uf and U^*f to the complex plane then we can apply Proposition 2.4.3.7 to show that Uf and U^*f are entire and exponentially bounded. This is summarised in the following proposition.

Proposition 2.5.0.11 *Let Uf and U^*f as given in Definition 2.5.0.8 be defined over the complex plane as in Remark 2.5.0.9. Then the function Uf is entire, of exponential type a and satisfies*

$$\int_{-\infty}^{\infty} |Uf(t)|^2\, dt < \infty.$$

Similarly, the function U^*f is entire, of exponential type b and satisfies

$$\int_{-\infty}^{\infty} |U^*f(t)|^2\, dt < \infty.$$

Proof. First note that by Definition 2.5.0.8 and Remark 2.5.0.9, for $x \in \mathbb{R}$ and $z \in \mathbb{C}$ we have

$$Uf(-z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{izx}\, dx$$

for $f \in L^2[-a,a]$. Hence Uf satisfies the conditions of Proposition 2.4.3.7 and so we deduce that $Uf \in L^2(\mathbb{R})$ is entire and of exponential type a.

Similarly,

$$U^*f(z) = \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} f(x) e^{izx}\, dx$$

also satisfies the conditions of Proposition 2.4.3.7 and so $U^*f \in L^2(\mathbb{R})$ is entire and of exponential type b. ∎

Remark 2.5.0.12 *Proposition 2.5.0.11 shows that the functions Uf and U^*f are band-limited. Further, it shows that $Uf \in PW(a)$ and $U^*f \in PW(b)$.*

Next we turn our attention to proving some properties of U and U^*. An important feature of the operators U and U^* is that they are both Hilbert–Schmidt operators. We prove this in the following lemma.

Lemma 2.5.0.13 *The operators U and U^* as stated in Definition 2.5.0.8 are Hilbert–Schmidt for finite a and b.*

Proof. The operator U has kernel e^{-itx} for $x \in [-a,a]$ and $t \in [-b,b]$. Now,

$$\begin{aligned} \int_{-b}^{b} \int_{-a}^{a} \left| e^{-itx} \right|^2 dx\, dt &= \int_{-b}^{b} \int_{-a}^{a} 1\, dx\, dt \\ &= 4ab. \end{aligned}$$

Clearly, $\int_{-b}^{b} \int_{-a}^{a} \left| e^{-itx} \right|^2 dx\, dt$ is finite when a and b are both finite. Therefore, for $a, b < \infty$, it follows from Proposition 2.1.1.6 that U is a Hilbert–Schmidt operator.

Similarly U^* has kernel e^{itx} such that for a and b finite,

$$\begin{aligned} \int_{-a}^{a} \int_{-b}^{b} \left| e^{itx} \right|^2 dx\, dt &= 4ab \\ &< \infty. \end{aligned}$$

So for $a, b < \infty$, U^* is also a Hilbert–Schmidt operator by Proposition 2.1.1.6. ∎

We close this section by using U and U^* to define a new operator, S which is self-adjoint and trace class. Recall from Proposition 2.1.2.6 that any trace class operator is also a Hilbert–Schmidt operator and so S will also be Hilbert–Schmidt. Notice that in Definition 2.5.0.14, S is defined as an integral operator with kernel, $k(t,x) = \frac{\sin b(t-x)}{\pi(t-x)}$. Recalling Proposition 2.4.3.4, the reproducing kernel for $PW(b)$ is $k_t(x) = \frac{\sin b(t-x)}{\pi(t-x)}$. Therefore, S has kernel given by the reproducing kernel for the space $PW(b)$.

Definition 2.5.0.14 *Let $S : L^2[-a,a] \to L^2[-a,a]$ be the operator defined by*

$$Sf(t) = \frac{1}{\pi} \int_{-a}^{a} f(x) \frac{\sin b(t-x)}{t-x}\, dx.$$

Remark 2.5.0.15 *Given* $k_t(x) = \frac{\sin b(t-x)}{\pi(t-x)}$ *we know that* $k_t \in L^2(\mathbb{R})$ *since* k_t *is a reproducing kernel for* $PW(b)$*. Thus,*

$$\begin{aligned} Sf(t) &= \int_{-a}^{a} f(x)k_t(x)\,dx \\ &= \langle f, k_t\rangle_{L^2[-a,a]} \end{aligned}$$

and so Sf *is well defined.*

In the following proposition we see that $S = U^*U$ where U and U^* are given by Definition 2.5.0.8. This fact justifies the domain and codomain of S being as stated in Definition 2.5.0.14, for $U^*U : L^2[-a,a] \to L^2[-b,b] \to L^2[-a,a]$.

Proposition 2.5.0.16 *Let* U *and* U^* *be the operators given in Definition 2.5.0.8 and let* S *be the operator defined in Definition 2.5.0.14. Then,*

$$S = U^*U.$$

Furthermore, S *is self-adjoint and trace class.*

Proof. Using Definition 2.5.0.8 we have

$$\begin{aligned} U^*Uf(t) &= \frac{1}{\sqrt{2\pi}}\int_{-b}^{b} Uf(y)e^{ity}\,dy \\ &= \frac{1}{2\pi}\int_{-b}^{b}\left(\int_{-a}^{a} f(x)e^{-iyx}\,dx\right)e^{ity}\,dy \\ &= \frac{1}{2\pi}\int_{-b}^{b}\int_{-a}^{a} f(x)e^{i(t-x)y}\,dx\,dy. \end{aligned}$$

Changing the order of integration gives

$$U^*Uf(t) = \frac{1}{2\pi}\int_{-a}^{a} f(x)\int_{-b}^{b} e^{i(t-x)y}\,dy\,dx. \tag{2.17}$$

Evaluating the inner integral in (2.17) we obtain,

$$\begin{aligned} \int_{-b}^{b} e^{i(t-x)y}\,dy &= \left[\frac{e^{i(t-x)y}}{i(t-x)}\right]_{y=-b}^{b} \\ &= \frac{e^{i(t-x)b} - e^{-i(t-x)b}}{i(t-x)} \\ &= 2\frac{\sin b(t-x)}{t-x}. \end{aligned}$$

Therefore,

$$\begin{aligned} U^*Uf(t) &= \frac{1}{\pi}\int_{-a}^{a} f(x)\frac{\sin b(t-x)}{t-x}\,dx \\ &= Sf(t) \end{aligned}$$

as required.

Next we prove that S is self-adjoint. Note that S is an integral operator with kernel $S(t,x) = \frac{\sin b(t-x)}{\pi(t-x)}$. We check that the condition on the kernel given in Definition 2.1.0.7 holds. First note that for x, t and b real, the sine function is real and so $S(t,x)$ is real. Furthermore, $S(t,x)$ is an even function, thus

$$\begin{aligned}\overline{S(x,t)} &= \overline{\frac{\sin b(x-t)}{\pi(x-t)}} \\ &= \frac{\sin b(x-t)}{\pi(x-t)} \\ &= \frac{\sin b(t-x)}{\pi(t-x)} \\ &= S(t,x).\end{aligned}$$

It follows from Definition 2.1.0.7 that S is self-adjoint.

Finally we show that S is a trace class operator. By Lemma 2.5.0.13, the operators U and U^* are Hilbert–Schmidt. By the first part of the result $S = U^*U$, so S is the product of two Hilbert–Schmidt operators. It follows from Definition 2.1.2.1 that S is trace class. ■

2.6 Linear Systems

This final section provides some background material relating to linear systems. Detailed constructions of linear systems can be found in [8] and [42]. Here we define a linear system, $(-A, B, C, D)$ by way of a generator, $-A$ relating to a semigroup and we then proceed to demonstrate the solution of such a linear system. Linear systems will appear in Chapter 3 where they are used to solve the Gelfand–Levitan integral equation through the construction of the operator R_x. Finally we give an example of a linear system that will be used to prove numerous results in Chapter 4.

We begin by constructing a strongly continuous semigroup. In order to create a strongly continuous semigroup we first need to define a generator which we do via a translation operator.

Definition 2.6.0.17 *Let E be an open or closed, finite or infinite interval of the real line. Let $f \in L^2(E)$ then for $t \in \mathbb{R}$ we define the* translation operator *by*

$$T_t f(x) = \begin{cases} f(x+t) & \text{for } x+t \in E, \\ 0 & \text{for } x+t \notin E. \end{cases}$$

Definition 2.6.0.18 *Suppose that $t \in \mathbb{R}$ and let T_t be the translation operator as defined in Definition 2.6.0.17. Let E be a finite or infinite interval on the real line. We introduce the* generator, *A defined by*

$$-Af = \lim_{t \to 0} \frac{T_t f - f}{t}, \tag{2.18}$$

where A has domain, $\mathcal{D}_E(A) = \{f : f, f' \in L^2(E),\ f \text{ absolutely continuous}\}$.

Remark 2.6.0.19 *For the definition of an absolutely continuous function see [54] (Section 11.7, page 364).*

In the above definition we introduced $\mathcal{D}_E(A)$ and so we take this opportunity to define a norm on $\mathcal{D}_E(A)$.

Definition 2.6.0.20 *Let $\mathcal{D}_E(A) = \{f : f, f' \in L^2(E),\ f$ absolutely continuous$\}$ where E is some interval of the real line. We define the norm on $\mathcal{D}_E(A)$ to be*

$$\|f\|^2_{\mathcal{D}_E(A)} = \|f\|^2_{L^2(E)} + \|f'\|^2_{L^2(E)}\,.$$

The terminology *generator* in Definition 2.6.0.18 may at first seem peculiar, however it is fully justified as the reader will understand upon reaching Proposition 2.6.0.24. We also note the similarity between (2.18) and the formal definition of a derivative. Indeed, the following proposition shows that the action of the generator is equivalent to the operation of differentiation.

Proposition 2.6.0.21 *Let A be a generator as in equation (2.18). Then $-Af = f'$ for $f \in \mathcal{D}_E(A)$.*

Proof. Let $f \in \mathcal{D}_E(A)$ then by (2.18) we have

$$-Af(x) = \lim_{t\to 0} \frac{T_t f(x) - f(x)}{t}.$$

The function f is defined on the interval E which may be some proper subset of the real line. We therefore extend the definition of f to the real line as follows,

$$f(x) = \begin{cases} f(x) & \text{for } x \in E, \\ 0 & \text{for } x \notin E. \end{cases}$$

Now suppose that $x + t \in E$ then by Definition 2.6.0.17 we have $T_t f(x) = f(x + t)$. Therefore

$$\begin{aligned} -Af(x) &= \lim_{t\to 0} \frac{f(x+t) - f(x)}{t} \\ &= f'(x) \end{aligned}$$

where the final line follows from the definition of differentiation. Now suppose that $x + t \notin E$ so that $T_t f(x) = 0$. Then

$$-Af(x) = \lim_{t\to 0} \frac{-f(x)}{t}.$$

By extension to the real line, $f(x + t) = 0$, and so

$$\begin{aligned} -Af(x) &= \lim_{t\to 0} \frac{f(x+t) - f(x)}{t} \\ &= f'(x) \end{aligned}$$

as required. In both cases, $-Af = f'$. ■

The following proposition provides an alternative notation for the translation map. When calculating various operators in Chapters 3 and 4 it will be more convenient to use the notation of Proposition 2.6.0.22 which we credit to Lagrange.

Proposition 2.6.0.22 *Let $T_t : \mathcal{D}_E(A) \to \mathcal{D}_E(A)$ be the translation operator defined by Definition 2.6.0.17. Then $T_t = e^{-tA}$ where A is the generator given by (2.18).*

Proof. Let $x + t \in E$ then, using Definition 2.6.0.17 and differentiating both sides of the translation map with respect to t gives

$$T'_t f(x) = f'(x + t).$$

From Proposition 2.6.0.21 we know that $-A$ is the differentiation operator, thus $-Af(x+t) = f'(x+t)$. Therefore

$$\begin{aligned} T_t' f(x) &= -Af(x+t) \\ &= -AT_t f(x) \end{aligned}$$

where the last line follows from Definition 2.6.0.17. Thus T_t satisfies the first order differential equation

$$T_t' + AT_t = 0. \tag{2.19}$$

Now suppose that $x + t \notin E$ then $T_t f(x) = 0$. Further, $T_t' f(x) = 0$ and so again T_t satisfies the first order differential equation, (2.19). It is clear that we can solve (2.19) by taking $T_t = e^{-tA}$, hence the result. ■

Next we define a strongly continuous semigroup and show that the set of operators $\{T_t\}_{t\geq 0}$ does indeed form one. Should the reader require further information regarding semigroups they should consult [12].

Definition 2.6.0.23 *A* strongly continuous semigroup *is a set* $\{T(t) : t \in \mathbb{R}^+\}$ *of bounded linear operators on a Hilbert space, H satisfying:*
(i) $T(0) = I$ *where I is the identity operator on H;*
(ii) $T(s+t) = T(s)T(t)$ *for all* $s, t \geq 0$*;*
(iii) For all $h \in H$, $\|T(t)h - h\| \to 0$ *as* $t \to 0^+$.

The following theorem shows that $\{T_t\}_{t\geq 0} = \{e^{-tA}\}_{t\geq 0}$ forms a strongly continuous semigroup. We also note that the use of the terminology *generator* in Definition 2.6.0.18 is now justified since A generates the semigroup $\{e^{-tA}\}_{t\geq 0}$.

Theorem 2.6.0.24 *Let A be the generator defined by (2.18) and let $t \geq 0$. For T_t operating on $L^2(\mathbb{R})$, T_t is bounded and satisfies $\|T_t\|_{\text{op}} \leq 1$. Furthermore, the set $\{T_t\}_{t\geq 0} = \{e^{-tA}\}_{t\geq 0}$ forms a strongly continuous semigroup of operators on $L^2(E)$, where E is some interval of the real line.*

Proof. Firstly we show that the T_t are bounded. Given T_t operates on $L^2(\mathbb{R})$ we have

$$\begin{aligned} \|T_t f\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |T_t f(x)|^2 \, dx \\ &= \int_{-\infty}^{\infty} |f(x+t)|^2 \, dx \\ &= \int_{-\infty}^{\infty} |f(x)|^2 \, dx \\ &= \|f\|^2_{L^2(\mathbb{R})} . \end{aligned}$$

This shows that T_t is bounded with $\|T_t\|_{\text{op}} \leq 1$.

Now suppose that E is some interval of the real line. We show that $\{T_t\}_{t\geq 0}$ forms a strongly continuous semigroup on $L^2(E)$ by checking that the conditions of Definition 2.6.0.23 are satisfied. We know that the $\{T_t\}$ are bounded so we start by showing that they are linear operators. Let $t \geq 0$ then as T_t denotes translation we can write

$$T_t[\lambda f + \mu g](x) = [\lambda f + \mu g](x+t)$$

for $x + t \in E$. Note that $L^2(E)$ is a vector space and so is distributive with respect to addition and associative with respect to multiplication. Since T_t operates on $L^2(E)$ we thus have

$$\begin{aligned} T_t[\lambda f + \mu g](x) &= (\lambda f)(x+t) + (\mu g)(x+t) \\ &= \lambda f(x+t) + \mu g(x+t). \end{aligned}$$

The elements in the final line above are translations and so it follows that,

$$T_t[\lambda f + \mu g](x) = \lambda T_t f(x) + \mu T_t g(x).$$

The case in which $x + t \notin E$ is trivial thus proving that T_t is linear.

We now check that conditions (i)-(iii) of Definition 2.6.0.23 are satisfied by T_t. Firstly, $T_0 = e^0 = I$ and so condition (i) is satisfied.

Next we note that for $s, t \geq 0$ and $x + s + t \in E$ we must have $x + t \in E$ and so

$$\begin{aligned} T_{s+t} f(x) &= f(x+s+t) \\ &= T_s f(x+t) \\ &= T_s T_t f(x). \end{aligned}$$

Now suppose that $x + s + t \notin E$ so that $T_{s+t} f(x) = 0$. We split into two cases: $x + r \in E$ and $x + r \notin E$ where $r = s, t$. Without loss of generality, suppose that $x + t \in E$, then $T_t f(x) = f(x+t)$. Thus,

$$\begin{aligned} T_{s+t} f(x) &= 0 \\ &= T_s f(x+t) \\ &= T_s T_t f(x). \end{aligned}$$

For the second case, again without loss of generality, suppose that $x + t \notin E$ then $T_t f(x) = 0$. Hence,

$$\begin{aligned} T_{s+t} f(x) &= 0 \\ &= T_s T_t f(x). \end{aligned}$$

This shows that condition (ii) is satisfied.

Finally we check condition (iii). For any $f \in L^2(E)$ we can approximate f by a simple function, f_s. Let $E = \bigcup_{j=1}^n E_j$ where the E_j are disjoint intervals, then we can approximate f by

$$f_s(x) = \sum_{j=1}^n c_j \mathbb{I}_{E_j}(x),$$

as in the construction of the Lebesgue integral of f. Therefore, given $\epsilon > 0$ we can choose f_s such that

$$\|f - f_s\|_{L^2(E)} < \epsilon.$$

Observe that we may use the triangle inequality to show that

$$\begin{aligned} \|T_t f - f\|_{L^2(E)} &= \|[T_t f - T_t f_s] + [T_t f_s - f_s] + [f_s - f]\|_{L^2(E)} \\ &\leq \|T_t(f - f_s)\|_{L^2(E)} + \|T_t f_s - f_s\|_{L^2(E)} + \|f - f_s\|_{L^2(E)} \\ &\leq 2\|f - f_s\|_{L^2(E)} + \|T_t f_s - f_s\|_{L^2(E)}. \end{aligned}$$

The last line follows from the fact that T_t is bounded and $\|T_t\|_{\text{op}} \leq 1$. Now consider the term $\|T_t f_s - f_s\|_{L^2(E)}$. Suppose that $x+t \in E$ for all $x \in E$, then

$$\begin{aligned}
\|T_t f_s - f_s\|_{L^2(E)}^2 &= \int_E |[T_t f_s - f_s](x)|^2 \, dx \\
&= \int_E |f_s(x+t) - f_s(x)|^2 \, dx \\
&= \int_E \left| \sum_{j=1}^n c_j \left[\mathbb{I}_{E_j}(x+t) - \mathbb{I}_{E_j}(x)\right] \right|^2 dx. \qquad (2.20)
\end{aligned}$$

Applying the Cauchy–Schwarz inequality to the sum in (2.20) gives

$$\left| \sum_{j=1}^n c_j \left[\mathbb{I}_{E_j}(x+t) - \mathbb{I}_{E_j}(x)\right] \right|^2 \leq \left(\sum_{j=1}^n |c_j|^2 \right) \left(\sum_{k=1}^n |\mathbb{I}_{E_k}(x+t) - \mathbb{I}_{E_k}(x)|^2 \right).$$

Therefore,

$$\begin{aligned}
\|T_t f_s - f_s\|_{L^2(E)}^2 &\leq \int_E \left(\sum_{j=1}^n |c_j|^2 \right) \left(\sum_{k=1}^n |\mathbb{I}_{E_k}(x+t) - \mathbb{I}_{E_k}(x)|^2 \right) dx \\
&= \left(\sum_{j=1}^n |c_j|^2 \right) \sum_{k=1}^n \int_E |\mathbb{I}_{E_k}(x+t) - \mathbb{I}_{E_k}(x)|^2 \, dx.
\end{aligned}$$

Suppose that $E_k = (a_k, b_k)$ then

$$\begin{aligned}
\int_E |\mathbb{I}_{E_k}(x+t) - \mathbb{I}_{E_k}(x)|^2 \, dx &= \int_E \left|\mathbb{I}_{(a_k - t, b_k - t)}(x) - \mathbb{I}_{(a_k, b_k)}(x)\right|^2 dx \\
&= \int_{a_k - t}^{a_k} |1|^2 \, dx + \int_{a_k}^{b_k - t} |1-1|^2 \, dx + \int_{b_k - t}^{b_k} |0-1|^2 \, dx \\
&= [x]_{a_k - t}^{a_k} + [x]_{b_k - t}^{b_k} \\
&= 2t.
\end{aligned}$$

Note that the same argument holds if E_k is a half-open or closed interval. It therefore follows that

$$\begin{aligned}
\|T_t f_s - f_s\|_{L^2(E)}^2 &\leq \left(\sum_{j=1}^n |c_j|^2 \right) \sum_{k=1}^n 2t \\
&= 2tn \sum_{j=1}^n |c_j|^2 \\
&\to 0
\end{aligned}$$

as $t \to 0^+$. Now suppose that some of the $x+t$ fall outside of the set E. Suppose also without loss of generality that $E = (a,b)$. In this case we may write

$$\begin{aligned}
\|T_t f_s - f_s\|_{L^2(E)}^2 &= \int_a^b |[T_t f_s - f_s](x)|^2 \, dx \\
&= \int_a^{b-t} |f_s(x+t) - f_s(x)|^2 \, dx + \int_{b-t}^b |f_s(x)|^2 \, dx.
\end{aligned}$$

The same argument that was used to show $\|T_t f_s - f_s\|_{L^2(E)}^2 \to 0$ as $t \to 0^+$ for all $x+t \in E$, can be used to show that

$$\int_a^{b-t} |f_s(x+t) - f_s(x)|^2 \, dx \to 0$$

as $t \to 0^+$. It is also clear that

$$\int_{b-t}^{b} |f_s(x)|^2 \, dx \to 0$$

as $t \to 0^+$. Hence, in all cases

$$\|T_t f - f\|_{L^2(E)} \to 0$$

as $t \to 0^+$. ■

Notice that when we defined the translation map, it was defined for all real t. However, we have only shown that $\{e^{-tA}\}$ is a semigroup for $t \geq 0$. We can in fact also find a semigroup for $t \leq 0$. Suppose that $t \leq 0$ then we can write $t = -|t|$ where $|t| \geq 0$. By Definition 2.6.0.17 we have

$$\begin{aligned} T_t &= f(x+t) \\ &= f(x-|t|) \end{aligned}$$

where $T_t = e^{|t|A}$. This prompts the following theorem.

Theorem 2.6.0.25 *Let A be the generator as in (2.18) and let $t \geq 0$. For T_{-t} operating on $L^2(\mathbb{R})$, T_{-t} is bounded and satisfies $\|T_{-t}\|_{\text{op}} \leq 1$. Furthermore, the set $\{T_{-t}\}_{t\geq 0} = \{e^{tA}\}_{t\geq 0}$ forms a strongly continuous semigroup of operators on $L^2(E)$, where E is some interval of the real line.*

Proof. The proof follows exactly the same method as that of Theorem 2.6.0.24. First we show that T_{-t} is bounded. Let T_{-t} operate on $L^2(\mathbb{R})$ then

$$\begin{aligned} \|T_{-t}f\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |T_{-t}f(x)|^2 \, dx \\ &= \int_{-\infty}^{\infty} |f(x-t)|^2 \, dx \\ &= \int_{-\infty}^{\infty} |f(x)|^2 \, dx \\ &= \|f\|^2_{L^2(\mathbb{R})} . \end{aligned}$$

This shows that T_{-t} is bounded with $\|T_{-t}\|_{\text{op}} \leq 1$.

To show that $\{T_{-t}\}_{t\geq 0}$ forms a strongly continuous semigroup on $L^2(E)$, we again check that the set $\{T_{-t}\}_{t\geq 0}$ satisfies Definition 2.6.0.23. We have already shown that T_{-t} is bounded so it remains to check linearity and that conditions (i)-(iii) hold. In order to ascertain linearity, we first note that $L^2(E)$ is a vector space and hence is distributive with respect to addition and associative with respect to multiplication. Thus for $x - t \in E$,

$$\begin{aligned} T_{-t}[\lambda f + \mu g](x) &= [\lambda f + \mu g](x-t) \\ &= (\lambda f)(x-t) + (\mu g)(x-t) \\ &= \lambda f(x-t) + \mu g(x-t). \end{aligned}$$

As T_{-t} represents translation it follows that

$$T_{-t}[\lambda f + \mu g](x) = \lambda T_{-t} f(x) + \mu T_{-t} g(x).$$

Again, the case in which $x - t \notin E$ is trivial. This proves that T_{-t} is linear.

Next we show that T_{-t} satisfies conditions (i)-(iii) as stated in Definition 2.6.0.23. Clearly we have $T_0 = e^0 = I$ and so condition (i) is satisfied.

Now let $s, t \geq 0$ and suppose that $x - s - t \in E$ we must have $x - t \in E$ and so

$$\begin{aligned} T_{-s-t}f(x) &= f(x-s-t) \\ &= T_{-s}f(x-t) \\ &= T_{-s}T_{-t}f(x). \end{aligned}$$

Now suppose that $x - s - t \notin E$ so that $T_{-s-t}f(x) = 0$ and split into two cases: $x - r \in E$ and $x - r \notin E$ where $r = s, t$. Without loss of generality, suppose that $x - t \in E$, then $T_{-t}f(x) = f(x-t)$. Thus,

$$\begin{aligned} T_{-s-t}f(x) &= 0 \\ &= T_{-s}f(x-t) \\ &= T_{-s}T_{-t}f(x). \end{aligned}$$

For the second case, again without loss of generality, suppose that $x - t \notin E$ then $T_{-t}f(x) = 0$. Hence,

$$\begin{aligned} T_{-s-t}f(x) &= 0 \\ &= T_{-s}T_{-t}f(x). \end{aligned}$$

This shows that condition (ii) is satisfied.

Lastly, for any $f \in L^2(E)$ we can approximate f by a simple function, f_s. Let $E = \bigcup_{j=1}^{n} E_j$ where the E_j are disjoint intervals. Then, given $\epsilon > 0$ we can choose

$$f_s(x) = \sum_{j=1}^{n} c_j \mathbb{I}_{E_j}(x)$$

such that

$$\|f - f_s\| < \epsilon.$$

Again, using the triangle inequality,

$$\begin{aligned} \|T_{-t}f - f\|_{L^2(E)} &= \|[T_{-t}f - S(t)f_s] + [T_{-t}f_s - f_s] + [f_s - f]\|_{L^2(E)} \\ &\leq \|T_{-t}(f - f_s)\|_{L^2(E)} + \|T_{-t}f_s - f_s\|_{L^2(E)} + \|f - f_s\|_{L^2(E)} \\ &\leq 2\|f - f_s\|_{L^2(E)} + \|T_{-t}f_s - f_s\|_{L^2(E)}. \end{aligned}$$

The last line being true since T_{-t} is bounded with $\|T_{-t}\|_{\text{op}} \leq 1$. Note that if $x - t \in E$ for all $x \in E$ we have

$$\begin{aligned} \|T_{-t}f_s - f_s\|_{L^2(E)}^2 &= \int_E |(T_{-t}f_s - f_s)(x)|^2 \, dx \\ &= \int_E |f_s(x-t) - f_s(x)|^2 \, dx \\ &= \int_E \left| \sum_{j=1}^{n} c_j \left[\mathbb{I}_{E_j}(x-t) - \mathbb{I}_{E_j}(x) \right] \right|^2 dx. \end{aligned}$$

By the Cauchy–Schwarz inequality

$$\left| \sum_{j=1}^{n} c_j \left[\mathbb{I}_{E_j}(x-t) - \mathbb{I}_{E_j}(x) \right] \right|^2 \le \left(\sum_{j=1}^{n} |c_j|^2 \right) \left(\sum_{k=1}^{n} |\mathbb{I}_{E_k}(x-t) - \mathbb{I}_{E_k}(x)|^2 \right),$$

therefore

$$\begin{aligned} \|T_{-t} f_s - f_s\|_{L^2(E)}^2 &\le \int_E \left(\sum_{j=1}^{n} |c_j|^2 \right) \left(\sum_{k=1}^{n} |\mathbb{I}_{E_k}(x-t) - \mathbb{I}_{E_k}(x)|^2 \right) dx \\ &= \left(\sum_{j=1}^{n} |c_j|^2 \right) \sum_{k=1}^{n} \int_E |\mathbb{I}_{E_k}(x-t) - \mathbb{I}_{E_k}(x)|^2. \end{aligned}$$

Now suppose, without loss of generality, that $E_k = (a_k, b_k)$. Then

$$\begin{aligned} \int_E |\mathbb{I}_{E_k}(x-t) - \mathbb{I}_{E_k}(x)|^2 \, dx &= \int_{a_k}^{a_k+t} |-1|^2 \, dx + \int_{a_k+t}^{b_k} |1-1|^2 \, dx + \int_{b_k}^{b_k+t} |1|^2 \, dx \\ &= 2t. \end{aligned}$$

It follows that

$$\begin{aligned} \|T_{-t} f_s - f_s\|_{L^2(E)}^2 &\le \left(\sum_{j=1}^{n} |c_j|^2 \right) \sum_{k=1}^{n} 2t \\ &= 2tn \sum_{j=1}^{n} |c_j|^2 \\ &\to 0 \end{aligned}$$

as $t \to 0^+$. Now suppose that some of the $x - t$ fall outside of the set E. Suppose also without loss of generality that $E = (a, b)$. In this case we may write

$$\begin{aligned} \|T_{-t} f_s - f_s\|_{L^2(E)}^2 &= \int_a^b |[T_{-t} f_s - f_s](x)|^2 \, dx \\ &= \int_a^{a+t} |f_s(x)|^2 \, dx + \int_{a+t}^{b} |f_s(x-t) - f_s(x)|^2 \, dx. \end{aligned}$$

Again, the same argument that was used to show $\|T_{-t} f_s - f_s\|_{L^2(E)}^2 \to 0$ as $t \to 0^+$ for all $x - t \in E$, can be used to show that

$$\int_{a+t}^{b} |f_s(x-t) - f_s(x)|^2 \, dx \to 0$$

as $t \to 0^+$. Also, we clearly have

$$\int_a^{a+t} |f_s(x)|^2 \, dx \to 0$$

as $t \to 0^+$. Hence in all cases,

$$\|T_{-t} f - f\|_{L^2(E)} \to 0$$

as $t \to 0^+$ and we have satisfied condition (iii). ■

Having constructed a strongly continuous semigroup we are now able to define a linear system. First we give a broad definition of a linear system and state a theorem that shows the solutions of such a system. We then close the section with an example of a linear system that will be used in Section 4.5.

Definition 2.6.0.26 *Let E be an open or closed interval of the real line. Let the state space be $L^2(E)$ and let the input and output spaces be $\mathbb{C}$. For time $t \in (0,\infty)$, let $U : (0,\infty) \to \mathbb{C}$ denote the input, $X : (0,\infty) \to L^2(E)$ the state and $Y : (0,\infty) \to \mathbb{C}$ the output. Also let U, X, Y be continuous. Take A to be the generator of the strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on $L^2(E)$, where $T_t = e^{-tA}$. Then define the following operators:*

$$\begin{aligned} &T_t : L^2(E) \to L^2(E), \\ &B : \mathbb{C} \to L^2(E), \\ &C : \mathcal{D}_E(A) \to \mathbb{C}, \\ &D : \mathbb{C} \to \mathbb{C} \end{aligned}$$

where B, C and D are bounded linear operators. Then $(-A, B, C, D)$ determines the linear, time-invariant system

$$\frac{d}{dt}X(t) = -AX(t) + BU(t) \tag{2.21}$$

$$Y(t) = CX(t) + DU(t). \tag{2.22}$$

Remark 2.6.0.27 *It will sometimes be necessary to restrict the codomain of the operator B. In this case we will take $B_r : \mathbb{C} \to \mathcal{D}_E(A)$ and speak of the linear system $(-A, B_r, C, D)$.*

The reader should think of the linear system defined by Definition 2.6.0.26 as proposing a problem. That is, given the equations (2.21) and (2.22), can we find a suitable system $(-A, B, C, D)$ such that (2.21) and (2.22) hold? In the case that $X : (0,\infty) \to \mathcal{D}_E(A)$, we are able to construct a solution to the linear system. The following theorem presents such a solution.

Theorem 2.6.0.28 *Suppose that $X : (0,\infty) \to \mathcal{D}_E(A)$ and $B_r : \mathbb{C} \to \mathcal{D}_E(A)$. Then the linear system as stated by Definition 2.6.0.26 and determined by $(-A, B_r, C, D)$ has solution*

$$X(t) = T_t X(0) + \int_0^t T_{t-s} B_r U(s)\, ds, \tag{2.23}$$

$$Y(t) = CT_t X(0) + \int_0^t CT_{t-s} B_r U(s)\, ds + DU(t). \tag{2.24}$$

Remark 2.6.0.29 *The integrals in (2.23) and (2.24) are defined as Lebesgue integrals. This follows from [32] (Theorem 3.7.4, page 80).*

Proof. First note that since $0 \leq s \leq t$ in equations (2.23) and (2.24), we have $t - s \geq 0$ so $T_{t-s} = e^{-(t-s)A}$ does indeed belong to the semigroup $\left\{e^{-tA}\right\}_{t\geq 0}$. Now let $X(t) = T_t X(0) + \int_0^t T_{t-s} B_r U(s)\, ds$ and note that since $T_t = e^{-tA}$,

$$\begin{aligned} \frac{d}{dt}T_t &= \frac{d}{dt}e^{-tA} \\ &= -Ae^{-tA} \\ &= -AT_t. \end{aligned}$$

A rigorous proof of this fact can be found in [14] (Lemma 7(b), page 619). It therefore follows that

$$\begin{aligned} \frac{d}{dt}X(t) &= -AT_t X(0) + \int_0^t -AT_{t-s} B_r U(s)\, ds + T_0 B_r U(t) \\ &= -AX(t) + T_0 B_r U(t). \end{aligned}$$

Since $\{T_t\}_{t\geq 0}$ forms a strongly continuous semigroup, $T_0 = I$ and so

$$\frac{d}{dt}X(t) = -AX(t) + B_rU(t).$$

Now let $Y(t) = CT_tX(0) + \int_0^t CT_{t-s}B_rU(s)\,ds + DU(t)$. By [32] (Theorem 3.7.12, page 83) we can write

$$\int_0^t CT_{t-s}B_rU(s)\,ds = C\int_0^t T_{t-s}B_rU(s)\,ds$$

and so it follows that $Y(t) = CX(t) + DU(t)$. This completes the proof. ■

We now proceed to create an example of a linear system. The following system will reappear in Section 4.5 to enable us to calculate various operators.

Example 2.6.0.30

Let $\mathbb{C}$ be the input and output space and let $L^2[a,b]$ be the state space. Take the domain of A to be $\mathcal{D}_{[a,b]}(A)$ where A is the generator of the semigroups $\{e^{-tA}\}_{t\geq 0}$ and $\{e^{tA}\}_{t\geq 0}$. Define the operators

$$\begin{aligned} &A : \mathcal{D}_{[a,b]}(A) \to L^2[a,b], \\ &B : \mathbb{C} \to L^2[a,b], \\ &C : \mathcal{D}_{[a,b]}(A) \to \mathbb{C} \end{aligned}$$

by

$$\begin{aligned} &A : f(x) \mapsto -f'(x), \\ &B : \beta \mapsto \beta\psi(x), \\ &C : f \mapsto f(0) \end{aligned}$$

where $\psi \in L^2[a,b]$ is absolutely continuous. Then $(-A, B, C)$ defines a linear system.

Remark 2.6.0.31 *It will sometimes be necessary to restrict the codomain of the operator B. In the case of Example 2.6 we take $B_r : \mathbb{C} \to \mathcal{D}_{[a,b]}(A)$ where B_r is given by $\beta \mapsto \beta\psi(x)$ for $\psi \in \mathcal{D}_{[a,b]}(A)$. We then refer to the linear system $(-A, B_r, C)$.*

By Definition 2.6.0.26, when defining a linear system $(-A, B, C)$, we require that B and C are bounded linear maps. The following lemma shows that the operators B and C as defined by Example 2.6 are bounded (we note that they are clearly linear).

Lemma 2.6.0.32 *Suppose that $a \leq 0 < b$ where a and b are finite. Let $(-A, B, C)$ be the linear system defined in Example 2.6. Then the operators B and C are bounded.*

Proof. We begin by showing that B is bounded. Note that

$$\begin{aligned} \|B\beta\|_{L^2[a,b]} &= \|\beta\psi\|_{L^2[a,b]} \\ &= |\beta|\,\|\psi\|_{L^2[a,b]}\,. \end{aligned}$$

Since $\psi \in L^2[a,b]$ it follows that $\|\psi\|_{L^2[a,b]}$ exists and is finite. Therefore, B is bounded with bound $\|\psi\|_{L^2[a,b]}$.

Next we show that C is bounded. This requires a little more work. Let $f \in \mathcal{D}_{[a,b]}(A)$ where $a \leq 0 < b$. By the Fundamental Theorem of Calculus we have

$$\int_0^t f'(x)\,dx = f(t) - f(0) \tag{2.25}$$

for $t \leq b$. Integrating both sides of (2.25) with respect to t over the interval $[a, b]$ gives

$$\begin{aligned} \int_a^b \int_0^t f'(x)\,dx\,dt &= \int_a^b f(t)\,dt - \int_a^b f(0)\,dt \\ &= \int_a^b f(t)\,dt - (b-a)f(0). \end{aligned} \tag{2.26}$$

Now, rearranging equation (2.26) we obtain the following formula,

$$f(0) = \frac{1}{b-a}\left[\int_a^b f(t)\,dt - \int_a^b \int_0^t f'(x)\,dx\,dt\right]. \tag{2.27}$$

We show that $Cf = f(0)$ is bounded by showing that $\int_a^b f(t)\,dt - \int_a^b \int_0^t f'(x)\,dx\,dt$ is bounded. For $f \in \mathcal{D}_{[a,b]}(A)$ the map $f \mapsto \int_a^b f(t)\,dt$ is bounded since, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \left|\int_a^b f(t)\,dt\right|^2 &\leq \left(\int_a^b |f(t)|^2\,dt\right)\left(\int_a^b 1\,dt\right) \\ &= (b-a)\,\|f\|_{L^2[a,b]}^2 \end{aligned}$$

which is finite. Also, the map $f \mapsto \int_0^t f'(x)\,dx$ is bounded for $t \in [a, b]$. Again, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \left|\int_0^t f'(x)\,dx\right|^2 &\leq \left(\int_0^t |f'(x)|^2\,dx\right)\left(\int_0^t 1\,dx\right) \\ &\leq \left(\int_a^b |f'(x)|^2\,dx\right)\left(\int_a^b 1\,dx\right) \\ &= (b-a)\,\|f'\|_{L^2[a,b]}^2\,. \end{aligned}$$

By the definition of $\mathcal{D}_{[a,b]}(A)$, $f' \in L^2[a, b]$ and so it follows that $\|f'\|_{L^2[a,b]}^2$ is finite. Hence $f \mapsto \int_0^t f'(x)\,dx$ is bounded. Further, the map $f \mapsto \int_a^b \int_0^t f'(x)\,dx\,dt$ is also bounded because, again by the Cauchy–Schwarz inequality

$$\begin{aligned} \left|\int_a^b \int_0^t f'(x)\,dx\,dt\right|^2 &\leq \left(\int_a^b \left|\int_0^t f'(x)\,dx\right|^2 dt\right)\left(\int_a^b 1\,dt\right) \\ &\leq (b-a)\int_a^b (b-a)\,\|f'\|_{L^2[a,b]}^2\,dt \\ &= (b-a)^3\,\|f'\|_{L^2[a,b]}^2 \\ &< \infty. \end{aligned}$$

Therefore, by (2.27) we have

$$\begin{aligned} \|Cf\|_{\mathbb{C}} &= \left|\frac{1}{b-a}\left[\int_a^b f(t)\,dt - \int_a^b \int_0^t f'(x)\,dx\,dt\right]\right| \\ &\leq \frac{1}{b-a}\left(\left|\int_a^b f(t)\,dt\right| + \left|\int_a^b \int_0^t f'(x)\,dx\,dt\right|\right) \\ &\leq \frac{1}{b-a}\left[(b-a)^{\frac{1}{2}}\,\|f\|_{L^2[a,b]} + (b-a)^{\frac{3}{2}}\,\|f'\|_{L^2[a,b]}\right] \\ &= \frac{1}{\sqrt{b-a}}\,\|f\|_{L^2[a,b]} + \sqrt{b-a}\,\|f'\|_{L^2[a,b]}\,. \end{aligned}$$

As $b-a>0$ it follows that

$$\begin{aligned}\|Cf\|_{\mathbb{C}} &\leq \left(\frac{1}{\sqrt{b-a}}+\sqrt{b-a}\right)\left(\|f\|_{L^2[a,b]}+\|f'\|_{L^2[a,b]}\right)\\ &\leq \frac{2(1+b-a)}{\sqrt{b-a}}\left(\|f\|^2_{L^2[a,b]}+\|f'\|^2_{L^2[a,b]}\right)^{\frac{1}{2}}\\ &= \frac{2(1+b-a)}{\sqrt{b-a}}\,\|f\|_{D_{[a,b]}(A)}\,,\end{aligned}$$

hence C is bounded. ■

Chapter 3

The Gelfand–Levitan Integral Equation

Hill's equation is the differential equation

$$-f'' + qf = \lambda f$$

where q is real, twice continuously differentiable and π-periodic. The spectral problem is to find the spectra of this differential equation given q. In this thesis we deal with the periodic spectrum and the Bloch spectrum, as discussed in Chapter 4. It is also interesting to find the possible sequences that can occur as the periodic spectrum of such a Hill's equation. This involves recovering a suitable q from the spectral data, and is known as the inverse spectral problem. An important tool in the inverse spectral problem is the Gelfand–Levitan integral equation.

This chapter provides the necessary preliminary material for problems that we explore in Chapter 4. In Chapter 4 we shall consider the problems of finding solutions to Hill's equation and reconstructing a potential of Hill's equation given that a linear system, $(-A, B, C)$ is known. The key to both of these problems is to know the scattering function, ϕ which we introduce in Section 3.1. We see that the scattering function can be constructed from a known linear system, $(-A, B, C)$, indeed its Laplace transform is the transfer function of $(-A, B, C)$. Much of the work throughout this thesis is concerned with even functions and so we modify the scattering function so that it is even.

For the remainder of the chapter we follow the method of Blower in [2], however, we make substantial changes to cover the periodic context. Once we have a known scattering function, ϕ we then introduce a twice continuously differentiable function, $T(x, y)$ such that ϕ and T satisfy a type of Gelfand–Levitan integral equation. In Section 3.3 we then see how the linear system, $(-A, B, C)$ is used to construct $T(x, y)$, thus solving the Gelfand–Levitan integral equation through the use of linear systems. From the Gelfand–Levitan integral equation, a partial differential equation for $T(x, y)$ arises that produces a potential of Hill's equation, q from $\frac{d}{dx}T(x, x)$. Thus in this chapter we construct $T(x, y)$ from the linear system, $(-A, B, C)$ such that T satisfies the Gelfand–Levitan integral equation and hence a partial differential equation dependent on q. We then solve the partial differential equation to ultimately recover q. Indeed this is the subject of Section 4.6. Further to this, in Section 4.2 we will see that once $T(x, y)$ has been derived from $(-A, B, C)$, it can be used to construct a solution of Hill's equation. The novel idea of this chapter is an operator R_x which is used to produce T from the linear system, $(-A, B, C)$.

The following maps should make the route taken through this chapter clear,

$$(-A, B, C) \mapsto \phi \mapsto R_x \mapsto T \mapsto q$$

where $\phi \mapsto T$ via the Gelfand–Levitan integral equation and $T \mapsto q$ via a partial differential equation.

3.1 The Scattering Function

The purpose of this section is to introduce the scattering function which will be necessary for our work throughout the current chapter. We do not concern ourselves with the intricacies of the scattering function here for these can be found in [2]. We simply note that the purpose of the scattering function is to recover the potential, q of Hill's equation. If the reader keeps this in mind then he will see that the partial differential equation arising in Section 3.2 is linked to the potential of Hill's equation, q appearing in Section 4.1.

This section therefore demonstrates that a scattering function can be obtained from a linear system, $(-A, B_r, C)$. In fact we shall see that the transfer function of a linear system, $(-A, B_r, C)$ is the Laplace transform of a scattering function.

Definition 3.1.0.33 *The* Laplace transform*, $\mathcal{L}$ of a function, f is given by*

$$\mathcal{L}[f(t); s] = \int_0^\infty f(t)e^{-st}\, dt.$$

Remark 3.1.0.34 *The Laplace transform is linear.*

Theorem 3.1.0.35 *Let $(-A, B_r, C, D)$ be a linear system as in Definition 2.6.0.26 with solutions given by Theorem 2.6.0.28. Suppose that $X(0) = 0$ and define the function, ϕ to be $\phi(t) = Ce^{-tA}B_r$. Then*

$$Y(t) = \int_0^t \phi(t-s)U(s)\, ds + DU(t).$$

Further,

$$\int_0^\infty e^{-tA}e^{-\lambda t}\, dt$$

converges to $(A + \lambda I)^{-1}$ for $\mathrm{Re}\,(\lambda) > 0$. Hence, Y has Laplace transform

$$\mathcal{L}[Y(t); \lambda] = (\mathcal{L}[\phi(t); \lambda] + D)\, \mathcal{L}[U(t); \lambda]$$

where

$$\mathcal{L}[\phi(t); \lambda] = C(A + \lambda I)^{-1}B_r.$$

Proof. Recall that $T_{t-s} = e^{-(t-s)A}$ and suppose that $X(0) = 0$. By Theorem 2.6.0.28 we have

$$\begin{aligned} Y(t) &= CT_t X(0) + \int_0^t CT_{t-s} B_r U(s)\, ds + DU(t) \\ &= \int_0^t Ce^{-(t-s)A} B_r U(s)\, ds + DU(t). \end{aligned}$$

Now let $\phi(t) = Ce^{-tA}B_r$, then

$$Y(t) = \int_0^t \phi(t-s)U(s)\, ds + DU(t),$$

proving the first part of the result.

Next we calculate the Laplace transform of Y. Since the Laplace transform is linear it follows that Y has Laplace transform given by

$$\mathcal{L}[Y(t);\lambda] = \mathcal{L}\left[\int_0^t \phi(t-s)U(s)\, ds; \lambda\right] + D\mathcal{L}[U(t);\lambda].$$

Now, by Definition 3.1.0.33

$$\mathcal{L}\left[\int_0^t \phi(t-s)U(s)\, ds; \lambda\right] = \int_0^\infty \left[\int_0^t \phi(t-s)U(s)\, ds\right] e^{-\lambda t}\, dt. \tag{3.1}$$

Reversing the order of integration in (3.1) we see that

$$\begin{aligned} \mathcal{L}\left[\int_0^t \phi(t-s)U(s)\, ds; \lambda\right] &= \int_0^\infty U(s) \left[\int_s^\infty \phi(t-s)e^{-\lambda t}\, dt\right] ds \\ &= \int_0^\infty U(s)e^{-\lambda s} \left[\int_s^\infty \phi(t-s)e^{-\lambda(t-s)}\, dt\right] ds \\ &= \mathcal{L}[\phi(t);\lambda]\mathcal{L}[U(t);\lambda]. \end{aligned}$$

Therefore,

$$\mathcal{L}[Y(t);\lambda] = \mathcal{L}[\phi(t);\lambda]\mathcal{L}[U(t);\lambda] + D\mathcal{L}[U(t);\lambda]$$

as required.

Finally let $\mathrm{Re}\,(\lambda) > 0$ and note that

$$\left\| \int_0^\infty e^{-tA}e^{-\lambda t}\, dt \right\| \le \int_0^\infty \left\| e^{-tA}e^{-\lambda t} \right\| dt.$$

By Theorem 2.6.0.24, $\left\| e^{-tA} \right\| \le 1$ and so

$$\begin{aligned} \left\| e^{-tA}e^{-\lambda t} \right\| &\le \left\| e^{-\lambda t} \right\| \\ &= e^{-\mathrm{Re}(\lambda)t}. \end{aligned}$$

Thus

$$\left\| \int_0^\infty e^{-tA}e^{-\lambda t}\, dt \right\| \le \int_0^\infty e^{-\mathrm{Re}(\lambda)t}\, dt.$$

Given $\mathrm{Re}\,(\lambda) > 0$

$$\begin{aligned} \int_0^\infty e^{-\mathrm{Re}(\lambda)t}\, dt &= \lim_{R\to\infty} \left[\frac{e^{-\mathrm{Re}(\lambda)t}}{-\mathrm{Re}\,(\lambda)}\right]_{t=0}^R \\ &= \frac{1}{\mathrm{Re}\,(\lambda)}. \end{aligned}$$

Hence

$$\left\| \int_0^\infty e^{-tA} e^{-\lambda t} \, dt \right\| \leq \frac{1}{\mathrm{Re}\,(\lambda)},$$

and so $\int_0^\infty e^{-tA} e^{-\lambda t}\, dt$ is convergent for $\mathrm{Re}\,(\lambda) > 0$. Given $\phi(t) = Ce^{-tA}B_r$ we observe that for $\mathrm{Re}\,(\lambda) > 0$,

$$\begin{aligned}
\mathcal{L}[\phi(t); \lambda] &= \int_0^\infty Ce^{-tA} B_r e^{-\lambda t}\, dt \\
&= \int_0^\infty Ce^{-t(A+\lambda I)} B_r \, dt \\
&= \lim_{R\to\infty} \left[-Ce^{-t(A+\lambda I)} (A+\lambda I)^{-1} B_r \right]_{t=0}^{R} \\
&= C(A+\lambda I)^{-1} B_r.
\end{aligned}$$

It now follows from [32] (Theorem 3.7.12, page 83) that

$$\int_0^\infty e^{-tA} e^{-\lambda t}\, dt = (A+\lambda I)^{-1}.$$

■

Remark 3.1.0.36 *In Theorem 3.1.0.35, the operator $(A+\lambda I)^{-1}$ is called the* resolvent *and the function,*

$$C(A+\lambda I)^{-1} B_r + D$$

is known as the transfer function*.*

The theorem above shows that the Laplace transform of a function, ϕ is the transfer function of the linear system $(-A, B_r, C)$, hence ϕ can indeed be obtained from $(-A, B_r, C)$. The function ϕ is known as the scattering function, however, for the purposes of this thesis we require a modified version.

Definition 3.1.0.37 *Let $(-A, B_r, C)$ be a linear system as in Definition 2.6.0.26. Define the* scattering function *to be*

$$\phi(x) = C\left(e^{-xA} + e^{xA}\right) B_r. \tag{3.2}$$

Proposition 3.1.0.38 *Given a linear system, $(-A, B_r, C)$, let ϕ be the scattering function defined in Definition 3.1.0.37. Then ϕ is continuously differentiable and bounded. Also, ϕ is an even function which is periodic if e^{-xA} is periodic.*

Proof. Let $(-A, B_r, C)$ be a linear system as given by Definition 2.6.0.26 and suppose that $\phi(x) = C\left(e^{-xA} + e^{xA}\right) B_r$. By Definition 2.6.0.26, B_r and C are bounded. Also, by Theorems 2.6.0.24 and 2.6.0.25, e^{-tA} is bounded with $\left\|e^{-tA}\right\|_{\mathrm{op}} \leq 1$ for $t \in \mathbb{R}$. Therefore,

$$\begin{aligned}
\left\| C\left(e^{-xA} + e^{xA}\right) B_r \right\| &\leq \|C\|_{\mathrm{op}} \left(\left\|e^{-xA}\right\|_{\mathrm{op}} + \left\|e^{xA}\right\|_{\mathrm{op}} \right) \|B_r\|_{\mathrm{op}} \\
&\leq 2\|C\|_{\mathrm{op}} \|B_r\|_{\mathrm{op}}
\end{aligned}$$

showing that ϕ is bounded.

We now consider the derivative of ϕ. Thus,

$$\phi'(x) = C\left(-e^{-xA} + e^{xA}\right) AB_r$$

and the continuity of ϕ' now follows from the continuity of the exponential terms. Hence ϕ is continuously differentiable.

Next we note that ϕ is an even function since

$$\begin{aligned}\phi(-x) &= C\left(e^{xA} + e^{-xA}\right) B_r \\ &= \phi(x).\end{aligned}$$

Finally, suppose that e^{-xA} is periodic with period p, then $e^{-(x+p)A} = e^{-xA}$. So,

$$\begin{aligned}\phi(x+p) &= C\left(e^{-(x+p)A} + e^{(x+p)A}\right) B_r \\ &= C\left(e^{-xA} + e^{xA}\right) B_r \\ &= \phi(x).\end{aligned}$$

Thus for e^{-xA} periodic, ϕ is also periodic. ■

Proposition 3.1.0.39 *Let $(-A, B_r, C)$ be the linear system given in Example 2.6. Then the scattering function, ϕ satisfies*

$$\phi(x) = \psi(x) + \psi(-x)$$

where $\psi \in \mathcal{D}_{[a,b]}(A)$ is absolutely continuous.

Proof. Let t be the variable and let $\beta \in \mathbb{C}$. Using the linear system given in Example 2.6 we have,

$$\begin{aligned}\phi(x)\beta &= C\left(e^{-xA} + e^{xA}\right) B_r\beta \\ &= C\left(e^{-xA} + e^{xA}\right) \psi(t)\beta \\ &= C\left[\psi(t+x) + \psi(t-x)\right]\beta \\ &= \left[\psi(x) + \psi(-x)\right]\beta.\end{aligned}$$

Hence, $\phi(x) = \psi(x) + \psi(-x)$ as required. ■

3.2 The Gelfand–Levitan Integral Equation and Derived Partial Differential Equation

In this section we suppose that the scattering function, ϕ is known. Further, we suppose that ϕ is even and twice continuously differentiable on the real line. We also assume the existence of a twice continuously differentiable function, $T(x, y)$. The scattering function, ϕ and the function, T are then used to construct an equation known as a Gelfand–Levitan integral equation. From the Gelfand–Levitan integral equation we will derive a partial differential equation for T. It will be seen in Chapter 4 that the partial differential equation resulting from the Gelfand–Levitan integral equation can be used to reconstruct a potential of Hill's equation.

We begin by introducing the Gelfand–Levitan integral equation. Note that the form of the Gelfand–Levitan integral equation proposed here differs from the standard form as presented by Gelfand and Levitan in [24].

Definition 3.2.0.40 *Suppose that the scattering function, ϕ is known. We define the function $G(x, y)$ to be,*

$$G(x, y) = \phi(x + y) + \phi(x - y) + T(x, y) + \mu \int_{-x}^{x} T(x, z)[\phi(z + y) + \phi(z - y)]\, dz$$

for some unknown function, $T(x, y)$ satisfying $-x \leq y \leq x$ and $\mu \in \mathbb{C}$. We call the equation

$$G(x, y) = 0, \tag{3.3}$$

the Gelfand–Levitan integral equation.

In the Gelfand–Levitan integral equation the function, $T(x, y)$ is unknown. We can find $T(x, y)$ by constructing a partial differential equation which we then solve for T. The following theorem shows how we can derive a partial differential equation for T from the Gelfand–Levitan integral equation.

Remark 3.2.0.41 *We note that the notation, $\frac{\partial}{\partial z} T(x, \pm x)$ means differentiate with respect to the second variable.*

Theorem 3.2.0.42 *Suppose that ϕ is even and twice continuously differentiable on the real line. Suppose further that $T(x, y)$ is twice continuously differentiable with bounded first and second partial derivatives for $x \geq 0$ and $-x \leq y \leq x$. If T satisfies the Gelfand–Levitan integral equation, (3.3), then T also satisfies the partial differential equation*

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) T(x, y) = q(x) T(x, y), \tag{3.4}$$

where $q(x) = 2\mu \frac{d}{dx} [T(x, x) + T(x, -x)]$.

Proof. The proof is completed in the following stages:
(1) Calculate the second derivative of (3.3) with respect to x;
(2) Calculate the second derivative of (3.3) with respect to y;
(3) Subtract the second derivative with respect to y from the second derivative with respect to x;
(4) Multiply (3.3) by the function, $q(x)$;
(5) Equate the equations formed in stages (3) and (4) to obtain a partial differential equation.

In the first stage we calculate the second derivative of (3.3) with respect to x. Taking the first derivative with respect to x of (3.3) we obtain,

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} G(x, y) \\ &= \frac{\partial}{\partial x} \left\{ \phi(x + y) + \phi(x - y) + T(x, y) + \mu \int_{-x}^{x} T(x, z)[\phi(z + y) + \phi(z - y)]\, dz \right\} \\ &= \phi'(x + y) + \phi'(x - y) + \frac{\partial}{\partial x} T(x, y) + \mu \int_{-x}^{x} \frac{\partial}{\partial x} T(x, z)[\phi(z + y) + \phi(z - y)]\, dz \\ &\quad + \mu T(x, x)[\phi(x + y) + \phi(x - y)] + \mu T(x, -x)[\phi(-x + y) + \phi(-x - y)]. \end{aligned}$$

Calculating the second derivative of (3.3) with respect to x yields,

$$
\begin{aligned}
0 &= \frac{\partial^2}{\partial x^2}G(x,y) \\
&= \frac{\partial^2}{\partial x^2}\left\{\phi(x+y)+\phi(x-y)+T(x,y)+\mu\int_{-x}^{x}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz\right\} \\
&= \phi''(x+y)+\phi''(x-y)+\frac{\partial^2}{\partial x^2}T(x,y)+\mu\int_{-x}^{x}\frac{\partial^2}{\partial x^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&\quad +\mu\frac{\partial}{\partial x}T(x,x)[\phi(x+y)+\phi(x-y)]+\mu\frac{\partial}{\partial x}T(x,-x)[\phi(-x+y)+\phi(-x-y)] \\
&\quad +\mu\left[\frac{d}{dx}T(x,x)\right][\phi(x+y)+\phi(x-y)]+\mu T(x,x)[\phi'(x+y)+\phi'(x-y)] \\
&\quad +\mu\left[\frac{d}{dx}T(x,-x)\right][\phi(-x+y)+\phi(-x-y)]-\mu T(x,-x)[\phi'(-x+y)+\phi'(-x-y)].
\end{aligned}
$$

As ϕ is even we can simplify the above, thus

$$
\begin{aligned}
0 &= \frac{\partial^2}{\partial x^2}G(x,y) \\
&= \phi''(x+y)+\phi''(x-y)+\frac{\partial^2}{\partial x^2}T(x,y)+\mu\int_{-x}^{x}\frac{\partial^2}{\partial x^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&\quad +\mu\frac{\partial}{\partial x}T(x,x)[\phi(x+y)+\phi(x-y)]+\mu\frac{\partial}{\partial x}T(x,-x)[\phi(x-y)+\phi(x+y)] \\
&\quad +\mu\left[\frac{d}{dx}T(x,x)\right][\phi(x+y)+\phi(x-y)]+\mu T(x,x)[\phi'(x+y)+\phi'(x-y)] \\
&\quad +\mu\left[\frac{d}{dx}T(x,-x)\right][\phi(x-y)+\phi(x+y)]+\mu T(x,-x)[\phi'(x-y)+\phi'(x+y)] \\
&= \phi''(x+y)+\phi''(x-y)+\frac{\partial^2}{\partial x^2}T(x,y)+\mu\int_{-x}^{x}\frac{\partial^2}{\partial x^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&\quad +\mu\left(\frac{\partial}{\partial x}+\frac{d}{dx}\right)[T(x,x)+T(x,-x)]\,[\phi(x+y)+\phi(x-y)] \\
&\quad +\mu\,[T(x,x)+T(x,-x)]\,[\phi'(x+y)+\phi'(x-y)]. \qquad (3.5)
\end{aligned}
$$

Next, we calculate the first derivative of (3.3) with respect to y. This gives,

$$
\begin{aligned}
0 &= \frac{\partial}{\partial y}G(x,y) \\
&= \frac{\partial}{\partial y}\left\{\phi(x+y)+\phi(x-y)+T(x,y)+\mu\int_{-x}^{x}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz\right\} \\
&= \phi'(x+y)-\phi'(x-y)+\frac{\partial}{\partial y}T(x,y)+\mu\int_{-x}^{x}T(x,z)[\phi'(z+y)-\phi'(z-y)]\,dz.
\end{aligned}
$$

The second derivative of (3.3) with respect to y produces,

$$
\begin{aligned}
0 &= \frac{\partial^2}{\partial y^2}G(x,y) \\
&= \frac{\partial^2}{\partial y^2}\left\{\phi(x+y)+\phi(x-y)+T(x,y)+\mu\int_{-x}^{x}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz\right\} \\
&= \phi''(x+y)+\phi''(x-y)+\frac{\partial^2}{\partial y^2}T(x,y)+\mu\int_{-x}^{x}T(x,z)[\phi''(z+y)+\phi''(z-y)]\,dz.
\end{aligned}
$$

We want the above equation to resemble what we found for the second derivative with respect to x. Notice that in the equation for x we have $\int_{-x}^{x}\frac{\partial^2}{\partial x^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz$, whereas in the equation for y we have $\int_{-x}^{x}T(x,z)[\phi''(z+y)+\phi''(z-y)]\,dz$. We perform two integrations by parts on $\int_{-x}^{x} T(x,z)[\phi''(z+y)+\phi''(z-y)]\,dz$ so that the integrand contains a partial derivative of T. Thus,

$$\begin{aligned}
&\int_{-x}^{x} T(x,z)[\phi''(z+y)+\phi''(z-y)]\,dz \\
&\quad= [T(x,z)(\phi'(z+y)+\phi'(z-y))]_{z=-x}^{x} - \int_{-x}^{x} \frac{\partial}{\partial z}T(x,z)[\phi'(z+y)+\phi'(z-y)]\,dz \\
&\quad= [T(x,z)(\phi'(z+y)+\phi'(z-y))]_{z=-x}^{x} - \left[\frac{\partial}{\partial z}T(x,z)(\phi(z+y)+\phi(z-y))\right]_{z=-x}^{x} \\
&\qquad+ \int_{-x}^{x} \frac{\partial^2}{\partial z^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&\quad= T(x,x)[\phi'(x+y)+\phi'(x-y)] - T(x,-x)[\phi'(-x+y)+\phi'(-x-y)] \\
&\qquad- \frac{\partial}{\partial z}T(x,x)[\phi(x+y)+\phi(x-y)] + \frac{\partial}{\partial z}T(x,-x)[\phi(-x+y)+\phi(-x-y)] \\
&\qquad+ \int_{-x}^{x} \frac{\partial^2}{\partial z^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz.
\end{aligned}$$

Again, using the fact that ϕ is even, we simplify the above to obtain

$$\begin{aligned}
&\int_{-x}^{x} T(x,z)[\phi''(z+y)+\phi''(z-y)]\,dz \\
&= T(x,x)[\phi'(x+y)+\phi'(x-y)] + T(x,-x)[\phi'(x-y)+\phi'(x+y)] \\
&\quad- \frac{\partial}{\partial z}T(x,x)[\phi(x+y)+\phi(x-y)] + \frac{\partial}{\partial z}T(x,-x)[\phi(x-y)+\phi(x+y)] \\
&\quad+ \int_{-x}^{x} \frac{\partial^2}{\partial z^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&= [T(x,x)+T(x,-x)]\,[\phi'(x+y)+\phi'(x-y)] - \frac{\partial}{\partial z}\,[T(x,x)-T(x,-x)]\,[\phi(x+y)+\phi(x-y)] \\
&\quad+ \int_{-x}^{x} \frac{\partial^2}{\partial z^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz.
\end{aligned}$$

Therefore, the second derivative of (3.3) with respect to y is

$$\begin{aligned}
0 &= \frac{\partial^2}{\partial y^2}G(x,y) \\
&= \phi''(x+y)+\phi''(x-y) + \frac{\partial^2}{\partial y^2}T(x,y) + \mu\,[T(x,x)+T(x,-x)]\,[\phi'(x+y)+\phi'(x-y)] \\
&\quad -\mu\frac{\partial}{\partial z}\,[T(x,x)-T(x,-x)]\,[\phi(x+y)+\phi(x-y)] \\
&\quad +\mu\int_{-x}^{x} \frac{\partial^2}{\partial z^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz. \qquad (3.6)
\end{aligned}$$

Now we subtract equation (3.6) from equation (3.5). This gives,

$$\begin{aligned}
0 &= \left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)G(x,y) \\
&= \left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y) + \mu\int_{-x}^{x}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}\right)T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&\quad +\mu\left(\frac{\partial}{\partial x}+\frac{d}{dx}\right)[T(x,x)+T(x,-x)]\,[\phi(x+y)+\phi(x-y)] \\
&\quad +\mu\frac{\partial}{\partial z}\,[T(x,x)-T(x,-x)]\,[\phi(x+y)+\phi(x-y)] \\
&= \left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y) + \mu\int_{-x}^{x}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}\right)T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&\quad +2\mu\frac{d}{dx}\,[T(x,x)+T(x,-x)]\,[\phi(x+y)+\phi(x-y)]. \qquad (3.7)
\end{aligned}$$

In the above, the last line follows since

$$\frac{d}{dx}\left[T(x,x)+T(x,-x)\right]=\frac{\partial}{\partial x}\left[T(x,x)+T(x,-x)\right]+\frac{\partial}{\partial z}\left[T(x,x)-T(x,-x)\right].$$

The penultimate step in this process involves multiplying (3.3) by the function q, thus

$$0=q(x)[\phi(x+y)+\phi(x-y)]+q(x)T(x,y)+\mu\int_{-x}^{x}q(x)T(x,z)[\phi(z+y)+\phi(z-y)]\,dz. \quad (3.8)$$

Finally, we compare equations (3.7) and (3.8) and note that if we take

$$q(x)=2\mu\frac{d}{dx}\left[T(x,x)+T(x,-x)\right],$$

then the functions $\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y)$ and $q(x)T(x,y)$ satisfy the same integral equation. Therefore, by uniqueness the two functions are equal and so we obtain the partial differential equation,

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y)=q(x)T(x,y)$$

where $q(x)=2\mu\frac{d}{dx}\left[T(x,x)+T(x,-x)\right]$. ■

The following theorem shows the equivalence of the functions $G(x,y)$ and $T(x,y)$ in that they both satisfy the same partial differential equation. The method of the proof is similar to that of Theorem 3.2.0.42.

Theorem 3.2.0.43 *Suppose that ϕ is even and twice continuously differentiable on the real line. Suppose further that T has bounded first and second partial derivatives for $x \geq 0$ and $-x \leq y \leq x$. If T satisfies the partial differential equation*

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y)=q(x)T(x,y) \quad (3.9)$$

where $q(x)=2\mu\frac{d}{dx}\left[T(x,x)+T(x,-x)\right]$, then G also satisfies (3.9).

Proof. The proof is completed in the following stages:
(1) Calculate the second derivative of $G(x,y)$ with respect to x;
(2) Calculate the second derivative of $G(x,y)$ with respect to y;
(3) Subtract the second derivative of $G(x,y)$ with respect to y from the second derivative of $G(x,y)$ with respect to x to produce the desired partial differential equation.
Since the proof follows the same method as detailed in Theorem 3.2.0.42, we omit some of the details in the following calculations.

In the first stage we calculate the second derivative of $G(x,y)$ with respect to x. First,

$$\begin{aligned}
&\frac{\partial}{\partial x}G(x,y)\\
&\quad=\frac{\partial}{\partial x}\left\{\phi(x+y)+\phi(x-y)+T(x,y)+\mu\int_{-x}^{x}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz\right\}\\
&\quad=\phi'(x+y)+\phi'(x-y)+\frac{\partial}{\partial x}T(x,y)+\mu\int_{-x}^{x}\frac{\partial}{\partial x}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz\\
&\qquad+\mu T(x,x)[\phi(x+y)+\phi(x-y)]+\mu T(x,-x)[\phi(-x+y)+\phi(-x-y)].
\end{aligned}$$

The second derivative of $G(x,y)$ with respect to x is therefore,

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}G(x,y) \\
&\quad = \phi''(x+y) + \phi''(x-y) + \frac{\partial^2}{\partial x^2}T(x,y) + \mu\int_{-x}^{x}\frac{\partial^2}{\partial x^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&\qquad + \mu\left(\frac{\partial}{\partial x}+\frac{d}{dx}\right)[T(x,x)+T(x,-x)]\,[\phi(x+y)+\phi(x-y)] \\
&\qquad + \mu\,[T(x,x)+T(x,-x)]\,[\phi'(x+y)+\phi'(x-y)].
\end{aligned} \tag{3.10}$$

Next, we calculate the second derivative of $G(x,y)$ with respect to y. First,

$$\begin{aligned}
&\frac{\partial}{\partial y}G(x,y) \\
&\quad = \phi'(x+y) - \phi'(x-y) + \frac{\partial}{\partial y}T(x,y) + \mu\int_{-x}^{x}T(x,z)[\phi'(z+y)-\phi'(z-y)]\,dz.
\end{aligned}$$

The second derivative of $G(x,y)$ with respect to y is therefore,

$$\begin{aligned}
&\frac{\partial^2}{\partial y^2}G(x,y) \\
&\quad = \phi''(x+y) + \phi''(x-y) + \frac{\partial^2}{\partial y^2}T(x,y) + \mu\int_{-x}^{x}T(x,z)[\phi''(z+y)+\phi''(z-y)]\,dz.
\end{aligned}$$

Performing two integrations by parts on the latter integral we thus obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial y^2}G(x,y) \\
&\quad = \phi''(x+y) + \phi''(x-y) + \frac{\partial^2}{\partial y^2}T(x,y) + \mu\,[T(x,x)+T(x,-x)]\,[\phi'(x+y)+\phi'(x-y)] \\
&\qquad - \mu\frac{\partial}{\partial z}\,[T(x,x)-T(x,-x)]\,[\phi(x+y)+\phi(x-y)] \\
&\quad + \mu\int_{-x}^{x}\frac{\partial^2}{\partial z^2}T(x,z)[\phi(z+y)+\phi(z-y)]\,dz.
\end{aligned} \tag{3.11}$$

Now we subtract equation (3.11) from equation (3.10). This gives,

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)G(x,y) \\
&\quad = \left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y) + \mu\int_{-x}^{x}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}\right)T(x,z)[\phi(z+y)+\phi(z-y)]\,dz \\
&\qquad + 2\mu\frac{d}{dx}\,[T(x,x)+T(x,-x)]\,[\phi(x+y)+\phi(x-y)].
\end{aligned}$$

Since T satisfies the partial differential equation as specified by (3.9), it follows that

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)G(x,y) \\
&\quad = q(x)T(x,y) + \mu\int_{-x}^{x}q(x)T(x,z)[\phi(z+y)+\phi(z-y)]\,dz + q(x)[\phi(x+y)+\phi(x-y)] \\
&\quad = q(x)G(x,y),
\end{aligned}$$

hence G also satisfies the partial differential equation, (3.9). ■

3.3 The Use of Linear Systems in Solving Gelfand–Levitan Integral Equations

This section shows how we can solve a Gelfand–Levitan integral equation using linear systems. The first step is to assume that a scattering function, ϕ is known and that it arises from a linear system, $(-A, B, C)$. We then introduce a new operator which we call R_x and use it to construct a solution, T to the Gelfand–Levitan integral equation (3.3). Since the operator, R_x is defined in terms of the linear system, $(-A, B, C)$, this in turn allows us to construct T from the same linear system. We conclude this section by producing a simplified partial differential equation for T. This simpler partial differential equation will reappear in Chapter 4 to allow us to calculate the potential, q of Hill's equation. We also note here that q can be expressed in terms of the logarithm of a determinant.

Definition 3.3.0.44 *Let $(-A, B, C)$ be a linear system as in Definition 2.6.0.26. Define the operator $R_x : \mathcal{D}_E(A) \to L^2(E)$ to be*

$$R_x = \int_{-x}^{x} \left(e^{-zA} + e^{zA}\right) BC \left(e^{-zA} + e^{zA}\right) dz. \tag{3.12}$$

Remark 3.3.0.45 *Note that we can also define R_x for the linear system $(-A, B_r, C)$. In this case $R_x : \mathcal{D}_E(A) \to \mathcal{D}_E(A)$.*

Proposition 3.3.0.46 *Let $R_x : \mathcal{D}_E(A) \to L^2(E)$ be as defined by definition 3.3.0.44. If the operators*

$$\begin{aligned} P &= \int_{-x}^{x} \left(e^{-zA} + e^{zA}\right) B\, dz, \\ Q &= C\left(e^{-zA} + e^{zA}\right) \end{aligned}$$

are Hilbert–Schmidt then R_x is trace class.

Proof. Clearly we have $R_x = PQ$. If P and Q are Hilbert–Schmidt then R_x is trace class by Definition 2.1.2.1. ■

Recall Remark 3.1.0.36, we noted that for a linear system, $(-A, B_r, C, D)$ the function $C(A+\lambda I)^{-1}B_r + D$ was a transfer function and $(A + \lambda I)^{-1}$ a resolvent. Now, if we are using a linear system $(-A, B_r, C)$ then the transfer function becomes $C(A+\lambda I)^{-1}B_r$. The reader should note the similarity between this notation and the form of the function $T(x, y)$ in the following theorem. In Definition 3.3.0.44 we have used the notation R_x to suggest that $(I + \mu R_x)^{-1}$ is a type of resolvent. We note however that $(I + \mu R_x)^{-1}$ is different from $(A + \lambda I)^{-1} = \int_0^\infty e^{-z(A+\lambda I)}\, dz$.

Theorem 3.3.0.47 *Given the linear system, $(-A, B_r, C)$, let R_x be the operator given by (3.12) and assume $[I + \mu R_x]^{-1}$ exists. Further, let*

$$T(x, y) = -C\left(e^{-xA} + e^{xA}\right)[I + \mu R_x]^{-1}\left(e^{-yA} + e^{yA}\right) B_r. \tag{3.13}$$

If ϕ is as given by Definition 3.1.0.37 then T satisfies the Gelfand–Levitan integral equation, (3.3).

Proof. Let T be given by

$$T(x,y) = -C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}\left(e^{-yA}+e^{yA}\right)B_r$$

where R_x is as given by (3.12). Also, let $\phi(x) = C\left(e^{-xA}+e^{xA}\right)B_r$ as in Definition 3.1.0.37. We check that the Gelfand–Levitan integral equation, (3.3) holds. First note that

$$\begin{aligned}\phi(x+y)+\phi(x-y) &= C\left(e^{-(x+y)A}+e^{(x+y)A}\right)B_r + C\left(e^{-(x-y)A}+e^{(x-y)A}\right)B_r \\ &= C\left(e^{-xA}e^{-yA}+e^{xA}e^{yA}\right)B_r + C\left(e^{-xA}e^{yA}+e^{xA}e^{-yA}\right)B_r \\ &= C\left(e^{-xA}+e^{xA}\right)\left(e^{-yA}+e^{yA}\right)B_r.\end{aligned}$$

A simple substitution then gives

$$\begin{aligned}&G(x,y)\\ &= C\left(e^{-xA}+e^{xA}\right)\left(e^{-yA}+e^{yA}\right)B_r - C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}\left(e^{-yA}+e^{yA}\right)B_r \\ &\quad -\mu\int_{-x}^{x} C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}\left(e^{-zA}+e^{zA}\right)B_rC\left(e^{-zA}+e^{zA}\right)\left(e^{-yA}+e^{yA}\right)B_r\,dz \\ &= C\left(e^{-xA}+e^{xA}\right)\Big\{I-[I+\mu R_x]^{-1} \\ &\qquad -\mu[I+\mu R_x]^{-1}\int_{-x}^{x}\left(e^{-zA}+e^{zA}\right)B_rC\left(e^{-zA}+e^{zA}\right)dz\Big\}\left(e^{-yA}+e^{yA}\right)B_r.\end{aligned}$$

For the linear system, $(-A,B_r,C)$ we have $R_x = \int_{-x}^{x}\left(e^{-zA}+e^{zA}\right)B_rC\left(e^{-zA}+e^{zA}\right)dz$ and so

$$G(x,y) = C\left(e^{-xA}+e^{xA}\right)\Big\{I-[I+\mu R_x]^{-1}-\mu[I+\mu R_x]^{-1}R_x\Big\}\left(e^{-yA}+e^{yA}\right)B_r.$$

Clearly,

$$\begin{aligned}I-[I+\mu R_x]^{-1}-\mu[I+\mu R_x]^{-1}R_x &= I-[I+\mu R_x]^{-1}[I+\mu R_x] \\ &= 0,\end{aligned}$$

hence $G(x,y)=0$ and so T satisfies the Gelfand–Levitan integral equation. ■

The following corollary is a consequence of Theorems 3.2.0.42 and 3.3.0.47. It shows that if T takes the form given in Theorem 3.3.0.47 then we can produce a simplified partial differential equation for which T is a solution. In Chapter 4 we will see how the partial differential equation stated in Corollary 3.3.0.48 can be used to reconstruct the potential of Hill's equation.

Corollary 3.3.0.48 *Given the linear system, $(-A,B_r,C)$, let ϕ be as stated in Definition 3.1.0.37 and let R_x be as in Definition 3.3.0.44. Let*

$$T(x,y) = -C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}\left(e^{-yA}+e^{yA}\right)B_r,$$

then $T(x,-y)=T(x,y)$. Furthermore, T satisfies the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y) = q(x)T(x,y),$$

where $q(x) = 4\mu\frac{d}{dx}T(x,x)$.

Proof. Let $T(x,y) = -C\left(e^{-xA}+e^{xA}\right)\left[I+\mu R_x\right]^{-1}\left(e^{-yA}+e^{yA}\right)B_r$ then clearly,

$$\begin{aligned} T(x,-y) &= -C\left(e^{-xA}+e^{xA}\right)\left[I+\mu R_x\right]^{-1}\left(e^{yA}+e^{-yA}\right)B_r \\ &= T(x,y). \end{aligned}$$

Now, by Theorem 3.3.0.47, T satisfies the Gelfand–Levitan integral equation, (3.3). Therefore, by Theorem 3.2.0.42, T satisfies the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y) = q(x)T(x,y)$$

where $q(x) = 2\mu\frac{d}{dx}\left[T(x,x)+T(x,-x)\right]$. However, we have just shown that $T(x,-y) = T(x,y)$, so in fact we have

$$q(x) = 4\mu\frac{d}{dx}T(x,x)$$

as required. ∎

The final task in this section is to provide an alternative formula for q in terms of the operator R_x. The formula will allow us to write the function q in terms of the logarithm of a determinant. First we state a lemma that will be called upon in the proof.

Lemma 3.3.0.49 *Let S be a trace class operator depending on x so that $x \mapsto S(x)$ is a continuously differentiable function. Let S have eigenvalues $\{\lambda_j(x)\}$ and assume that $\mu \neq -\frac{1}{\lambda_j}$. Then*

$$\frac{d}{dx}\log\det(I+\mu S) = \operatorname{tr}\left[(I+\mu S)^{-1}\mu\frac{d}{dx}S\right].$$

Proof. Suppose that S is a trace class operator with eigenvalues $\{\lambda_j\}$. By Definition 2.3.0.24, S has Fredholm determinant

$$\det(I+\mu S) = \prod_j (1+\mu\lambda_j). \tag{3.14}$$

Taking the logarithm of each side of (3.14) gives

$$\begin{aligned} \log\det(I+\mu S) &= \log\left[\prod_j (1+\mu\lambda_j)\right] \\ &= \sum_j \log(1+\mu\lambda_j). \end{aligned}$$

By Definition 2.3.0.23, the trace of a trace class operator is equal to the sum of its eigenvalues, thus

$$\operatorname{tr}\left[\log(I+\mu S)\right] = \sum_j \log(1+\mu\lambda_j),$$

hence

$$\log\det(I+\mu S) = \operatorname{tr}\left[\log(I+\mu S)\right].$$

We now note that, in the style of Napier, we can define the logarithm of an operator in terms of an integral. Observe that

$$\begin{aligned}
&\int_0^\infty \left\{(tI+I)^{-1} - (tI+I+\mu S)^{-1}\right\} dt \\
&= \lim_{R\to\infty} \left\{[\log(tI+I)]_0^R - [\log(tI+I+\mu S)]_0^R\right\} \\
&= \lim_{R\to\infty} \{\log(RI+I) - \log(I) - \log(RI+I+\mu S) + \log(I+\mu S)\} \\
&= \log(I+\mu S).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dx}\log\det(I+\mu S) &= \frac{d}{dx}\operatorname{tr}\left[\int_0^\infty \left\{(tI+I)^{-1} - (tI+I+\mu S)^{-1}\right\} dt\right] \\
&= \operatorname{tr}\left[\frac{d}{dx}\int_0^\infty \left\{(tI+I)^{-1} - (tI+I+\mu S)^{-1}\right\} dt\right]
\end{aligned}$$

where the last line follows since S is trace class. An application of [32] (Theorem 3.7.12, page 83) now gives

$$\begin{aligned}
\frac{d}{dx}\log\det(I+\mu S) &= \operatorname{tr}\left[\int_0^\infty \left\{\frac{d}{dx}\left[(tI+I)^{-1} - (tI+I+\mu S)^{-1}\right]\right\} dt\right] \\
&= \operatorname{tr}\left[-\int_0^\infty \left\{\frac{d}{dx}(tI+I+\mu S)^{-1}\right\} dt\right].
\end{aligned}$$

In order to differentiate $(tI+I+\mu S)^{-1}$, first note that

$$I = (tI+I+\mu S)^{-1}(tI+I+\mu S). \tag{3.15}$$

Now differentiate both sides of (3.15) with respect to x, thus obtaining

$$\begin{aligned}
0 &= \frac{d}{dx}\left[(tI+I+\mu S)^{-1}(tI+I+\mu S)\right] \\
&= \left[\frac{d}{dx}(tI+I+\mu S)^{-1}\right](tI+I+\mu S) + (tI+I+\mu S)^{-1}\frac{d}{dx}(tI+I+\mu S). \qquad (3.16)
\end{aligned}$$

Upon rearranging (3.16) and completing the differentiation we see that

$$\begin{aligned}
\left[\frac{d}{dx}(tI+I+\mu S)^{-1}\right] &= -(tI+I+\mu S)^{-1}\left[\frac{d}{dx}(tI+I+\mu S)\right](tI+I+\mu S)^{-1} \\
&= -(tI+I+\mu S)^{-1}\left[\mu\frac{dS}{dx}\right](tI+I+\mu S)^{-1}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{dx}\log\det(I+\mu S) &= \operatorname{tr}\left[\int_0^\infty \left\{(tI+I+\mu S)^{-1}\mu\frac{dS}{dx}(tI+I+\mu S)^{-1}\right\} dt\right] \\
&= \operatorname{tr}\left[\int_0^\infty (tI+I+\mu S)^{-2}\mu\frac{dS}{dx}\, dt\right]. \qquad (3.17)
\end{aligned}$$

Here, the last line follows from [15] (Lemma 14(b), page 1098). Evaluating the integral in (3.17) we see that

$$\begin{aligned}
\int_0^\infty (tI+I+\mu S)^{-2}\mu\frac{dS}{dx}\, dt &= \lim_{R\to\infty}\left\{\left[-(tI+I+\mu S)^{-1}\right]_0^R \mu\frac{dS}{dx}\right\} \\
&= (I+\mu S)^{-1}\mu\frac{dS}{dx}.
\end{aligned}$$

Hence,

$$\frac{d}{dx}\log\det(I+\mu S) = \text{ tr }\left[(I+\mu S)^{-1}\mu\frac{dS}{dx}\right]$$

as required. ■

The following theorem shows that if T satisfies the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)T(x,y) = q(x)T(x,y)$$

then q can be expressed as the logarithm of a determinant involving the operator R_x. In Chapter 4 we will see that this means we can calculate the potential of Hill's equation given that the linear system, $(-A,B,C)$ and the operator, R_x are known.

Theorem 3.3.0.50 *Given the linear system, $(-A,B_r,C)$, let*

$$\begin{aligned} P &= \int_{-x}^{x}\left(e^{-zA}+e^{zA}\right)B_r\,dz, \\ Q &= C\left(e^{-zA}+e^{zA}\right) \end{aligned}$$

be Hilbert–Schmidt operators. Suppose that T takes the form given by equation (3.13). Then T satisfies the partial differential equation (3.4) and

$$q(x) = -2\frac{d^2}{dx^2}\log\det\left(I+\mu R_x\right).$$

Proof. Since T satisfies Theorem 3.3.0.47, it follows from Corollary 3.3.0.48 that

$$q(x) = 4\mu\frac{d}{dx}T(x,x).$$

Suppose that $e^{-xA}+e^{xA}$ and $\left[I+\mu R_x\right]^{-1}$ are $n\times n$ matrices. Further, let C be a $1\times n$ row vector and let B_r be a $n\times 1$ column vector. Then $T(x,x)$ can be considered as a 1×1 matrix and so we may take the trace of it. Since the trace operation is invariant under cyclic permutations, it follows from (3.13) that

$$\begin{aligned} T(x,x) &= \text{tr}\left[-C\left(e^{-xA}+e^{xA}\right)\left[I+\mu R_x\right]^{-1}\left(e^{-xA}+e^{xA}\right)B_r\right] \\ &= -\text{tr}\left[\left(e^{-xA}+e^{xA}\right)\left[I+\mu R_x\right]^{-1}\left(e^{-xA}+e^{xA}\right)B_rC\right] \\ &= -\text{tr}\left[\left[I+\mu R_x\right]^{-1}\left(e^{-xA}+e^{xA}\right)B_rC\left(e^{-xA}+e^{xA}\right)\right]. \end{aligned}$$

Now note that

$$\frac{d}{dx}R_x = 2\left(e^{-xA}+e^{xA}\right)B_rC\left(e^{-xA}+e^{xA}\right),$$

therefore

$$T(x,x) = -\frac{1}{2}\text{tr}\left[\left[I+\mu R_x\right]^{-1}\frac{d}{dx}R_x\right].$$

By hypothesis, the operators P and Q are Hilbert–Schmidt, thus $R_x = PQ$ is trace class by Proposition 3.3.0.46. It now follows from Lemma 3.3.0.49 that

$$T(x,x) = -\frac{1}{2\mu}\frac{d}{dx}\log\det(I+\mu R_x),$$

and hence,

$$\begin{aligned} q(x) &= 4\mu \frac{d}{dx}\left[-\frac{1}{2\mu}\frac{d}{dx}\log\det(I+\mu R_x)\right] \\ &= -2\frac{d^2}{dx^2}\log\det\left(I+\mu R_x\right). \end{aligned}$$

■

Chapter 4

Hill's Equation

Hill's equation, a linear, second order differential equation with π-periodic potential will be central to this thesis in that it will provide the foundations for the remainder of our work. Specifically, Chapters 5 and 6 suppose the existence of a sampling sequence that is derived from the periodic spectrum of Hill's equation. The novel approach in this thesis is to use linear systems to deal with Hill's equation. We show that some of the classical techniques such as Hill's discriminant fit naturally into the theory of linear systems. For a concise source of background information relating to Hill's equation see Magnus and Winkler [35].

In this chapter we introduce Hill's equation with a particular focus on the solutions of Hill's equation and the periodic spectrum. In Section 4.2 we see that we can construct a solution to Hill's equation using the function $T(x, y)$ found in Chapter 3. As found in Chapter 3, the function $T(x, y)$ satisfies the partial differential equation $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) T(x, y) = q(x)T(x, y)$, where $q(x) = 4\mu \frac{d}{dx} T(x, x)$ is the potential of Hill's equation. We used the linear system, $(-A, B, C)$ to construct an operator R_x that allowed us to write $T(x, y)$ in terms of $(-A, B, C)$. In the current chapter we see that the same function, $T(x, y)$ can be used to create one of the fundamental solutions to Hill's equation. Therefore, we show that solutions of Hill's equation can be constructed via linear systems.

Given a modified version of Hill's equation, whose potential has period $\frac{\pi i}{a}$, we are also able to show that when $\lambda = 0$ is an eigenvalue, the corresponding eigenfunctions are the spheroidal wave functions. Furthermore, we show that the spheroidal wave functions are also eigenfunctions of the operator S given in Definition 2.5.0.14.

The remainder of this chapter focuses on the spectrum, in particular, the periodic spectrum and how we can characterise it through determinants. We define the periodic spectrum by way of the monodromy operator, M_π, showing that Hill's discriminant, Δ arises as the trace of M_π. Hence, we define the periodic spectrum by stating that an eigenvalue, λ of Hill's equation belongs to the periodic spectrum if and only if $\Delta^2(\lambda) - 4 = 0$. This condition on Hill's discriminant allows us to interpret Floquet's Theorem and shows that eigenvalues in the periodic spectrum correspond to solutions of Hill's equation that are periodic with period π or 2π. Using this information we follow the approach of Hill in [28] in search of a condition based upon determinants that will yield elements of the periodic spectrum. Roughly, we find that if an eigenvalue is a root of a particular Carleman or Fredholm determinant then it belongs to the periodic spectrum. Further, we extend Hill's method to include extra conditions based on the Carleman determinants of the operators, R_p and R_c that we express in terms of a linear system. That is to say, we use linear systems to create determinants whose roots belong to the periodic spectrum.

We conclude the chapter with a method for reconstructing a potential of Hill's equation given that the linear system, $(-A, B, C)$ is known. We use the results of Chapter 3 which show that given a known linear system and scattering function, there exists a function $T(x, y)$ satisfying the partial differential equation $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) T(x, y) = q(x)T(x, y)$ where $q(x) = 4\mu \frac{d}{dx} T(x, x)$ is the potential of Hill's equation. Our method is then to construct $T(x, y)$ from the linear system $(-A, B, C)$ and thus recover q.

4.1 Introduction to Hill's Equation

Here we define Hill's equation and its fundamental solutions. Also, we show that under certain conditions Hill's equation forms a Sturm–Liouville system. For more information on Sturm–Liouville systems see [55] (Chapter 9, page 105).

Definition 4.1.0.51 *Let q be a real-valued, π-periodic and twice continuously differentiable function. The linear second order differential equation*

$$-f''(x) + q(x)f(x) = \lambda f(x) \tag{4.1}$$

is known as Hill's equation. *We call q the* potential *and refer to λ as an* eigenvalue.

Remark 4.1.0.52 *We can also write Hill's equation in the homogeneous form*

$$f''(x) + [\lambda - q(x)]f(x) = 0. \tag{4.2}$$

As equation (4.1) is a second order differential equation it has two linearly independent solutions. We call these solutions the fundamental solutions. It is often helpful to write the fundamental solutions in matrix form. The following definition gives conditions under which a pair of solutions are fundamental and introduces their matrix notation.

Definition 4.1.0.53 *Let f_1 and f_2 be two continuously differentiable and linearly independent solutions of Hill's equation, (4.1) satisfying*

$$\begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix} (0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.3}$$

Then f_1 and f_2 are known as the first *and* second fundamental solutions *respectively. Furthermore, the matrix*

$$\begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix} (x)$$

is known as the fundamental solution matrix.

Remark 4.1.0.54 *The fundamental solution matrix is an entire function of λ. See [30].*

Hill's equation with some additional boundary conditions forms a Sturm–Liouville system. In the following definition we sate what is meant by a Sturm–Liouville system. Proposition 4.1.0.57 then imposes some boundary conditions on Hill's equation in order that it may form such a system.

Definition 4.1.0.55 *Let u, v, w be continuous real-valued functions on the finite interval $[a, b] \subset \mathbb{R}$, with $u, w > 0$. Suppose that u' exists and is continuous on $[a, b]$. Then the differential equation*

$$\frac{d}{dx}\left[u(x)\frac{d}{dx}f(x)\right] + [\lambda w(x) + v(x)]\, f(x) = 0$$

defined on $[a, b]$, together with one or both boundary conditions

$$\begin{aligned} 0 &= \alpha f(a) + \beta f'(a) \\ 0 &= \gamma f(b) + \delta f'(b), \end{aligned}$$

is known as a Sturm–Liouville system*. We exclude the trivial boundary conditions from this definition.*

Remark 4.1.0.56 *If the differential equation has both boundary conditions then the system is called a* regular *Sturm–Liouville system.*

Proposition 4.1.0.57 *Suppose that Hill's equation is defined on the interval, $[0, \pi]$. Then (4.2) together with the boundary conditions, $f'(0) = 0 = f'(\pi)$ forms a regular Sturm–Liouville system.*

Proof. Take Hill's equation to be in the form of (4.2) for $x \in [0, \infty]$. In the notation of Definition 4.1.0.55, take $u(x) = 1$, $w(x) = 1$ and $v(x) = -q(x)$. Now, u and w are obviously continuous, real-valued and positive. Further, u' exists and is continuous. Also, $v = -q$ is continuous and real-valued by Definition 4.1.0.51. Equation (4.2) is therefore in the form required by Definition 4.1.0.55. Hence, given the boundary conditions $f'(0) = 0 = f'(\pi)$, Hill's equation gives a Sturm–Liouville system. ■

Sturm–Liouville systems are interesting since it can be shown that their eigenvalues are real, see [55] (Theorem 9.8, page 114). Further, in the case of Hill's equation, the eigenfunctions form an orthogonal sequence. This latter statement is known as the *Sturm–Liouville Theorem* and it can be found with proof in [55] (Theorem 11.1, page 131). We return to the subject of eigenvalues in Section 4.4.

4.2 The Solutions of Hill's Equation

When considering differential equations, the most obvious question to ask is, can we find a solution? Further, can we find all solutions? This section is concerned with answering those questions. It is known that if we are able to find two linearly independent solutions to a second order, homogeneous differential equation, then the general solution (hence all solutions) will be a linear combination of the two linearly independent solutions. Therefore, if we can find two linearly independent solutions of Hill's equation, (4.2), i.e. if we find the fundamental solutions, then indeed we will have the general solution to Hill's equation. It turns out that finding two fundamental solutions is not such an easy task. We can however find a candidate for the first fundamental solution using linear systems. In Chapter 3 we saw that the linear system $(-A, B, C)$ can be used to construct a function $T(x, y)$ satisfying the partial differential equation $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) T(x, y) = q(x)T(x, y)$, where q is the potential of Hill's equation. In this section we see that the same function, $T(x, y)$ can be used to create a solution to Hill's equation.

First, we state a simple lemma that will be used in several proofs.

Lemma 4.2.0.58 *Suppose that $f(x)$ is a solution of Hill's equation, (4.1). Then $f(x + \pi)$ is also a solution.*

Proof. This is obvious since q is π-periodic. ■

The following theorem provides a solution to Hill's equation. Further, we propose that for a linear system, $(-A, B, C)$ such that $T(0, 0) = 0$ then the solution provided is in fact one of the fundamental solutions. The form of the solution given in Theorem 4.2.0.59 appears in [35] (page 47). A simplified version, appearing in Proposition 4.2.0.60 can also be found in [24] (equation (4), page 254).

Theorem 4.2.0.59 *Suppose that ϕ is even and twice continuously differentiable on the real line. Also suppose that $T(x, -y) = T(x, y)$ where $T(x, y)$ is twice continuously differentiable with bounded first and second partial derivatives for $x \geq 0$ and $-x \leq y \leq x$. Further, suppose that the Gelfand–Levitan integral equation, (3.3) holds with $\mu = 1$. Then*

$$f_1(x) = \cos x\sqrt{\lambda} + \int_{-x}^{x} T(x, y)e^{iy\sqrt{\lambda}}\, dy$$

is a solution of Hill's equation, (4.1). Moreover, if there exists a linear system such that $T(0, 0) = 0$ then f_1 is one of the fundamental solutions.

Proof. The proof simply involves verifying that f_1 satisfies the differential equation (4.2) and then checking that the conditions from (4.3) hold given $T(0, 0) = 0$. Let $f_1(x) = \cos x\sqrt{\lambda} + \int_{-x}^{x} T(x, y)e^{iy\sqrt{\lambda}}\, dy$. Differentiating f_1 with respect to x and noting that $T(x, -y) = T(x, y)$, we obtain

$$\begin{aligned} f_1'(x) &= -\sqrt{\lambda}\sin x\sqrt{\lambda} + \int_{-x}^{x} \frac{\partial}{\partial x} T(x, y)e^{iy\sqrt{\lambda}}\, dy + T(x, x)e^{ix\sqrt{\lambda}} + T(x, -x)e^{-ix\sqrt{\lambda}} \\ &= -\sqrt{\lambda}\sin x\sqrt{\lambda} + \int_{-x}^{x} \frac{\partial}{\partial x} T(x, y)e^{iy\sqrt{\lambda}}\, dy + T(x, x)\left(e^{ix\sqrt{\lambda}} + e^{-ix\sqrt{\lambda}}\right). \end{aligned}$$

We differentiate again to find the second derivative of f_1 with respect to x. Thus,

$$\begin{aligned} f_1''(x) &= -\lambda\cos x\sqrt{\lambda} + \int_{-x}^{x} \frac{\partial^2}{\partial x^2} T(x, y)e^{iy\sqrt{\lambda}}\, dy + \frac{\partial}{\partial x} T(x, x)e^{ix\sqrt{\lambda}} + \frac{\partial}{\partial x} T(x, -x)e^{-ix\sqrt{\lambda}} \\ &\quad + \left[\frac{d}{dx} T(x, x)\right]\left(e^{ix\sqrt{\lambda}} + e^{-ix\sqrt{\lambda}}\right) + i\sqrt{\lambda} T(x, x)\left(e^{ix\sqrt{\lambda}} - e^{-ix\sqrt{\lambda}}\right). \end{aligned}$$

Now note that T meets the conditions of Theorem 3.2.0.42, therefore T satisfies the partial differential equation, (3.4). Substituting this into the above and using the fact that $T(x, -y) = T(x, y)$, we obtain

$$\begin{aligned} f_1''(x) &= -\lambda\cos x\sqrt{\lambda} + \int_{-x}^{x} \left[\frac{\partial^2}{\partial y^2} + q(x)\right] T(x, y)e^{iy\sqrt{\lambda}}\, dy + \frac{\partial}{\partial x} T(x, x)\left(e^{ix\sqrt{\lambda}} + e^{-ix\sqrt{\lambda}}\right) \\ &\quad + \left[\frac{d}{dx} T(x, x)\right]\left(e^{ix\sqrt{\lambda}} + e^{-ix\sqrt{\lambda}}\right) + i\sqrt{\lambda} T(x, x)\left(e^{ix\sqrt{\lambda}} - e^{-ix\sqrt{\lambda}}\right). \end{aligned} \tag{4.4}$$

Next we perform integration by parts twice on $\int_{-x}^{x}\left[\frac{\partial^2}{\partial y^2}T(x,y)\right]e^{iy\sqrt{\lambda}}\,dy$, thus

$$\begin{aligned}
&\int_{-x}^{x}\left[\frac{\partial^2}{\partial y^2}T(x,y)\right]e^{iy\sqrt{\lambda}}\,dy\\
&=\left[e^{iy\sqrt{\lambda}}\frac{\partial}{\partial y}T(x,y)\right]_{y=-x}^{x}-i\sqrt{\lambda}\int_{-x}^{x}\left[\frac{\partial}{\partial y}T(x,y)\right]e^{iy\sqrt{\lambda}}\\
&=\left[e^{iy\sqrt{\lambda}}\frac{\partial}{\partial y}T(x,y)\right]_{y=-x}^{x}-i\sqrt{\lambda}\left[e^{iy\sqrt{\lambda}}T(x,y)\right]_{y=-x}^{x}-\lambda\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy\\
&=e^{ix\sqrt{\lambda}}\frac{\partial}{\partial y}T(x,x)-e^{-ix\sqrt{\lambda}}\frac{\partial}{\partial y}T(x,-x)-i\sqrt{\lambda}e^{ix\sqrt{\lambda}}T(x,x)+i\sqrt{\lambda}e^{-ix\sqrt{\lambda}}T(x,-x)\\
&\quad-\lambda\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy.
\end{aligned}$$

Again, using the fact that $T(x,-y)=T(x,y)$ we see that

$$\begin{aligned}
&\int_{-x}^{x}\left[\frac{\partial^2}{\partial y^2}T(x,y)\right]e^{iy\sqrt{\lambda}}\,dy\\
&=\frac{\partial}{\partial y}T(x,x)\left(e^{ix\sqrt{\lambda}}+e^{-ix\sqrt{\lambda}}\right)-i\sqrt{\lambda}T(x,x)\left(e^{ix\sqrt{\lambda}}-e^{-ix\sqrt{\lambda}}\right)-\lambda\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy.
\end{aligned}$$

Putting this into our equation for f_1'', (4.4) and noting that $\frac{d}{dx}T(x,x)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)T(x,x)$, we obtain

$$\begin{aligned}
f_1''(x) &= -\lambda\cos x\sqrt{\lambda}+\frac{\partial}{\partial y}T(x,x)\left(e^{ix\sqrt{\lambda}}+e^{-ix\sqrt{\lambda}}\right)-i\sqrt{\lambda}T(x,x)\left(e^{ix\sqrt{\lambda}}-e^{-ix\sqrt{\lambda}}\right)\\
&\quad-\lambda\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy+q(x)\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy+\frac{\partial}{\partial x}T(x,x)\left(e^{ix\sqrt{\lambda}}+e^{-ix\sqrt{\lambda}}\right)\\
&\quad+\left[\frac{d}{dx}T(x,x)\right]\left(e^{ix\sqrt{\lambda}}+e^{-ix\sqrt{\lambda}}\right)+i\sqrt{\lambda}T(x,x)\left(e^{ix\sqrt{\lambda}}-e^{-ix\sqrt{\lambda}}\right)\\
&= -\lambda\cos x\sqrt{\lambda}-\lambda\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy+q(x)\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy\\
&\quad+2\left[\frac{d}{dx}T(x,x)\right]\left(e^{ix\sqrt{\lambda}}+e^{-ix\sqrt{\lambda}}\right).
\end{aligned}$$

Now, as $\cos x=\frac{1}{2}\left(e^{ix}+e^{-ix}\right)$ it follows that

$$\begin{aligned}
&f_1''(x)\\
&=-\lambda\cos x\sqrt{\lambda}-\lambda\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy+q(x)\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy+4\left[\frac{d}{dx}T(x,x)\right]\cos x\sqrt{\lambda}.
\end{aligned}$$

Finally, by Corollary 3.3.0.48 we note that for $\mu=1$ we have $q(x)=4\frac{d}{dx}T(x,x)$, hence

$$\begin{aligned}
f_1''(x) &= -\lambda\cos x\sqrt{\lambda}-\lambda\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy+q(x)\int_{-x}^{x}T(x,y)e^{iy\sqrt{\lambda}}\,dy+q(x)\cos x\sqrt{\lambda}\\
&= \left[q(x)-\lambda\right]f_1(x).
\end{aligned}$$

Therefore, f_1 is a solution of Hill's equation.

Now suppose that there exists some linear system, $(-A,B,C)$ such that $T(0,0)=0$. If f_1 satisfies one of the conditions given by (4.3) then f_1 is one of the fundamental solutions. Clearly,

$$\begin{aligned}
f_1(0) &= \cos 0\\
&= 1.
\end{aligned}$$

Also, given $T(0,0) = 0$,

$$\begin{aligned} f_1'(0) &= -\sqrt{\lambda}\sin 0 + T(0,0)\left(e^0 + e^0\right) \\ &= 2T(0,0) \\ &= 0 \end{aligned}$$

and so f_1 is indeed one of the fundamental solutions. ■

We have therefore managed to show that f_1 as given by Theorem 4.2.0.59 is the first fundamental solution. As $T(x,-y) = T(x,y)$ we can use this fact to simplify the solution found in Theorem 4.2.0.59. This simplification is presented in the following proposition.

Proposition 4.2.0.60 *Suppose that ϕ is even and twice continuously differentiable on the real line. Also suppose that $T(x,-y) = T(x,y)$ where $T(x,y)$ is twice continuously differentiable with bounded first and second partial derivatives for $x \geq 0$ and $-x \leq y \leq x$. Further, suppose that the Gelfand–Levitan integral equation, (3.3) holds with $\mu = 1$. Let f_1 be the solution to Hill's equation found in Theorem 4.2.0.59, then*

$$f_1(x) = \cos x\sqrt{\lambda} + 2\int_0^x T(x,y)\cos y\sqrt{\lambda}\,dy.$$

Further, suppose that for a linear system, $(-A, B, C)$ we have

$$T(x,y) = -C\left(e^{-xA} + e^{xA}\right)[I + R_x]^{-1}\left(e^{-yA} + e^{yA}\right)B.$$

Then

$$f_1(x) = \cos x\sqrt{\lambda} - 2\int_0^x C\left(e^{-xA} + e^{xA}\right)[I + R_x]^{-1}\left(e^{-yA} + e^{yA}\right)B\cos y\sqrt{\lambda}\,dy.$$

Proof. By Theorem 4.2.0.59 we know that f_1 is a solution of Hill's equation where

$$f_1(x) = \cos x\sqrt{\lambda} + \int_{-x}^x T(x,y)e^{iy\sqrt{\lambda}}\,dy.$$

Given $e^{ix} = \cos x + i\sin x$ we may write

$$\begin{aligned} &f_1(x) \\ &= \cos x\sqrt{\lambda} + \int_{-x}^x T(x,y)\left[\cos y\sqrt{\lambda} + i\sin y\sqrt{\lambda}\right]\,dy \\ &= \cos x\sqrt{\lambda} + \int_{-x}^0 T(x,y)\left[\cos y\sqrt{\lambda} + i\sin y\sqrt{\lambda}\right]\,dy + \int_0^x T(x,y)\left[\cos y\sqrt{\lambda} + i\sin y\sqrt{\lambda}\right]\,dy \\ &= \cos x\sqrt{\lambda} + \int_0^x T(x,-y)\left[\cos y\sqrt{\lambda} - i\sin y\sqrt{\lambda}\right]\,dy + \int_0^x T(x,y)\left[\cos y\sqrt{\lambda} + i\sin y\sqrt{\lambda}\right]\,dy \\ &= \cos x\sqrt{\lambda} + \int_0^x [T(x,y) + T(x,-y)]\cos y\sqrt{\lambda}\,dy + i\int_0^x [T(x,y) - T(x,-y)]\sin y\sqrt{\lambda}\,dy. \end{aligned}$$

Since $T(x,-y) = T(x,y)$ it follows that

$$f_1(x) = \cos x\sqrt{\lambda} + 2\int_0^x T(x,y)\cos y\sqrt{\lambda}\,dy$$

as required.

Now let $T(x,y) = -C\left(e^{-xA} + e^{xA}\right)[I + R_x]^{-1}\left(e^{-yA} + e^{yA}\right)B$. Then clearly,

$$f_1(x) = \cos x\sqrt{\lambda} - 2\int_0^x C\left(e^{-xA} + e^{xA}\right)[I + R_x]^{-1}\left(e^{-yA} + e^{yA}\right)B\cos y\sqrt{\lambda}\,dy.$$

■

Having found the first fundamental solution to Hill's equation, we would like to find the second fundamental solution. Knowing both the first and second fundamental solutions would enable us to find the general solution. Unfortunately, finding the second fundamental solution, or indeed a second solution is not an easy task. The difficulty of this task in the case that the potential is a cosine function has already been proved by Ince. Let a, λ be complex constants. The second order differential equation,

$$\frac{d^2}{dz^2} f + [\lambda - 2a\cos 2z]\, f = 0$$

is known as *Mathieu's equation.* For $a \neq 0$, Ince proved that the general solution of Mathieu's equation is never periodic. This implies that if one periodic solution to Mathieu's equation exists then the second solution will not be periodic. More details of this fact can be found in [21] (page 119). Given that the potential for Hill's equation can be a great deal more complicated than a cosine function, the chances of finding a second solution to Hill's equation are slim.

4.3 Spheroidal Wave Functions Arising as Solutions of Hill's Equation

Spheroidal wave functions arise as solutions of the differential equation

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{d}{dz}f\right] + \left[\lambda - c^2z^2 - \frac{\mu^2}{1-z^2}\right] f = 0, \tag{4.5}$$

where $c \neq 0$. In particular, since (4.5) is a second order differential equation it will have two linearly independent solutions from which we can construct the general solution. Solutions of (4.5) are classified as either spheroidal wave functions of the *first kind* or spheroidal wave functions of the *second kind*. Spheroidal wave functions of the first kind are those that are finite at the points $z = \pm 1$. Spheroidal wave functions of the *second kind* have logarithmic singularities at $z = \pm 1$. Detailed information about the history, construction and applications of the spheroidal wave functions can be found in [22]. For more information regarding the difference between the functions see [22] (Section 2.3, page 12). In this section we are concerned with solving a particular case of Hill's equation, (4.1). We do this by taking a particular differential equation, of the form (4.5), whose solutions are known (they will be spheroidal wave functions) and transforming it into an equation that resembles Hill's equation. The resulting Hill's equation will, under certain conditions, have solutions that are spheroidal wave functions.

Definition 4.3.0.61 *Define the differential operator K_z to be*

$$K_z f = \frac{d}{dz}\left[\left(a^2-z^2\right)\frac{d}{dz}f\right] - b^2z^2 f.$$

We are going to consider the equation $-K_z f = \lambda f$. Comparing this with (4.5), the similarities are apparent for we have set $z = \frac{1}{a}w$, $\mu = 0$ and $c = ab$. It then follows that $-K_z f = \lambda f$ has eigenfunctions that are spheroidal wave functions. In the following theorem we transform $-K_z f = \lambda f$ into an equation of Hill's type. This then provides a way in which we can find solutions to certain types of Hill's equations.

Theorem 4.3.0.62 *Suppose that $-K_z f = \lambda f$ where f is a spheroidal wave function. Then f is also a solution of the equation,*

$$-\frac{d^2}{du^2}f + \left[a^4b^2\tanh^2 ua \operatorname{sech}^2 ua - \lambda a^2 \operatorname{sech}^2 ua\right] f = 0. \tag{4.6}$$

Remark 4.3.0.63 *Note that equation (4.6) has the general form of Hill's equation. The potential,*

$$q(u) = a^4b^2\tanh^2 ua \operatorname{sech}^2 ua - \lambda a^2 \operatorname{sech}^2 ua$$

has period $\frac{\pi i}{a}$ and the spheroidal wave functions are eigenfunctions associated with the eigenvalue zero.

Proof. Let f be a spheroidal wave function and suppose that f satisfies, $-K_z f = \lambda f$. Consider the change of variables given by $z = a\tanh ua$. Note that

$$\begin{aligned} \frac{dz}{du} &= a^2 \operatorname{sech}^2 ua \qquad (4.7) \\ &= a^2\left(1-\tanh^2 ua\right) \\ &= a^2 - z^2. \end{aligned}$$

Thus,

$$\begin{aligned} \left(a^2-z^2\right)\frac{d}{dz}f &= \frac{dz}{du}\frac{d}{dz}f \\ &= \frac{d}{du}f \end{aligned}$$

by the chain rule. Also by the chain rule we have

$$\frac{d}{dz}\frac{d}{du}f = \frac{du}{dz}\frac{d^2}{du^2}f.$$

The equation, $-K_z f = \lambda f$ therefore becomes

$$\begin{aligned} \lambda f &= -\frac{d}{dz}\left[\left(a^2-z^2\right)\frac{d}{dz}f\right] + b^2z^2 f \\ &= -\frac{d}{dz}\frac{d}{du}f + b^2z^2 f \\ &= -\frac{du}{dz}\frac{d^2}{du^2}f + b^2z^2 f \\ &= -\frac{\cosh^2 ua}{a^2}\frac{d^2}{du^2}f + a^2b^2\tanh^2 ua\, f, \qquad (4.8) \end{aligned}$$

where the last line follows from (4.7). Multiplying equation (4.8) through by $a^2 \operatorname{sech}^2 ua$ then produces

$$\lambda a^2 \operatorname{sech}^2 ua\, f = -\frac{d^2}{du^2}f + a^4b^2\tanh^2 ua \operatorname{sech}^2 ua\, f.$$

Thus the spheroidal wave function, f is also a solution of the equation

$$-\frac{d^2}{du^2}f + \left[a^4b^2\tanh^2 ua \operatorname{sech}^2 ua - \lambda a^2 \operatorname{sech}^2 ua\right] f = 0,$$

as required. ■

4.3.1 Spheroidal Wave Functions as Eigenfunctions of the Operator S

In this section we show that the operator S as defined in Definition 2.5.0.14 has eigenfunctions that are spheroidal wave functions. We achieve this by showing that the eigenfunctions of S are the same as those of K_z defined in the previous section. In order to do this we will need to show that the operators S and K_z commute.

Proposition 4.3.1.1 *Let $x, t \in \mathbb{R}$ and let S and K_z for $z = x, t$ be operators as defined in Definition 2.5.0.14 and (4.6) respectively. Then $SK_x = K_t S$.*

Proof. First note that

$$K_t f = \left(a^2 - t^2\right) \frac{d^2}{dt^2} f - 2t \frac{d}{dt} f - b^2 t^2 f.$$

Then

$$\begin{aligned} S(K_t f)(t) &= \frac{1}{\pi} \int_{-a}^{a} [K_x f(x)] \frac{\sin b(t-x)}{t-x}\, dx \\ &= \frac{1}{\pi} \int_{-a}^{a} \left[\left(a^2 - x^2\right) \frac{d^2}{dx^2} f(x)\right] \frac{\sin b(t-x)}{t-x}\, dx - \frac{2}{\pi} \int_{-a}^{a} \left[x \frac{d}{dx} f(x)\right] \frac{\sin b(t-x)}{t-x}\, dx \\ &\quad -b^2 \frac{1}{\pi} \int_{-a}^{a} x^2 f(x) \frac{\sin b(t-x)}{t-x}\, dx. \end{aligned}$$

We want to perform integration by parts on the expression for $SK_t f$. In order to simplify the calculation, we first note that

$$\frac{\sin b(t-x)}{t-x} = \frac{1}{2} \int_{-b}^{b} e^{is(t-x)}\, ds. \tag{4.9}$$

Substituting equation (4.9) into the expression for $SK_t f$ we see that

$$\begin{aligned} &S(K_t f)(t) \\ &= \frac{1}{2\pi} \int_{-a}^{a} \left[\left(a^2 - x^2\right) \int_{-b}^{b} e^{is(t-x)}\, ds\right] \frac{d^2}{dx^2} f(x)\, dx - \frac{1}{\pi} \int_{-a}^{a} \left[x \int_{-b}^{b} e^{is(t-x)}\, ds\right] \frac{d}{dx} f(x)\, dx \\ &\quad - \frac{b^2}{2\pi} \int_{-a}^{a} x^2 f(x) \int_{-b}^{b} e^{is(t-x)}\, ds\, dx. \end{aligned} \tag{4.10}$$

We now perform integration by parts twice on the first integral of (4.10). So,

$$\int_{-a}^{a}\left[\left(a^2-x^2\right)\int_{-b}^{b}e^{is(t-x)}\,ds\right]\frac{d^2}{dx^2}f(x)\,dx$$
$$=\left[\left(\left(a^2-x^2\right)\int_{-b}^{b}e^{is(t-x)}\,ds\right)\frac{d}{dx}f(x)\right]_{x=-a}^{a}$$
$$-\int_{-a}^{a}\left[-2x\int_{-b}^{b}e^{is(t-x)}\,ds-i\left(a^2-x^2\right)\int_{-b}^{b}se^{is(t-x)}\,ds\right]\frac{d}{dx}f(x)\,dx$$
$$=-\left[\left(-2x\int_{-b}^{b}e^{is(t-x)}\,ds-i\left(a^2-x^2\right)\int_{-b}^{b}se^{is(t-x)}\,ds\right)f(x)\right]_{x=-a}^{a}$$
$$+\int_{-a}^{a}\left[-2\int_{-b}^{b}e^{is(t-x)}\,ds+4ix\int_{-b}^{b}se^{is(t-x)}\,ds-\left(a^2-x^2\right)\int_{-b}^{b}s^2e^{is(t-x)}\,ds\right]f(x)\,dx$$
$$=2a\left(\int_{-b}^{b}e^{is(t-a)}\,ds\right)f(a)+2a\left(\int_{-b}^{b}e^{is(t+a)}\,ds\right)f(-a)$$
$$+\int_{-a}^{a}\left[-2\int_{-b}^{b}e^{is(t-x)}\,ds+4ix\int_{-b}^{b}se^{is(t-x)}\,ds-\left(a^2-x^2\right)\int_{-b}^{b}s^2e^{is(t-x)}\,ds\right]f(x)\,dx.$$

Similarly, we perform one integration by parts on the second integral of (4.10), giving

$$\int_{-a}^{a}\left[x\int_{-b}^{b}e^{is(t-x)}\,ds\right]\frac{d}{dx}f(x)\,dx$$
$$=\left[\left(x\int_{-b}^{b}e^{is(t-x)}\,ds\right)f(x)\right]_{x=-a}^{a}-\int_{-a}^{a}\left[\int_{-b}^{b}e^{is(t-x)}\,ds-ix\int_{-b}^{b}se^{is(t-x)}\,ds\right]f(x)\,dx$$
$$=a\left(\int_{-b}^{b}e^{is(t-a)}\,ds\right)f(a)+a\left(\int_{-b}^{b}e^{is(t+a)}\,ds\right)f(-a)$$
$$-\int_{-a}^{a}\left[\int_{-b}^{b}e^{is(t-x)}\,ds-ix\int_{-b}^{b}se^{is(t-x)}\,ds\right]f(x)\,dx.$$

Hence,

$$S(K_tf)(t)=\frac{1}{\pi}\int_{-a}^{a}\left[ix\int_{-b}^{b}se^{is(t-x)}\,ds-\frac{1}{2}\left(a^2-x^2\right)\int_{-b}^{b}s^2e^{is(t-x)}\,ds-\frac{b^2}{2}x^2\int_{-b}^{b}e^{is(t-x)}\,ds\right]f(x)\,dx. \tag{4.11}$$

After relabelling by switching the roles of x and t, equation (4.11) then becomes

$$S(K_xf)(x)=\frac{1}{\pi}\int_{-a}^{a}\left[it\int_{-b}^{b}se^{is(x-t)}\,ds-\frac{1}{2}\left(a^2-t^2\right)\int_{-b}^{b}s^2e^{is(x-t)}\,ds-\frac{b^2}{2}t^2\int_{-b}^{b}e^{is(x-t)}\,ds\right]f(t)\,dt. \tag{4.12}$$

Now observe that by (4.9) we have

$$\frac{d}{dt}\left(\frac{\sin b(x-t)}{x-t}\right)=-\frac{i}{2}\int_{-b}^{b}se^{is(x-t)}\,ds;$$
$$\frac{d^2}{dt^2}\left(\frac{\sin b(x-t)}{x-t}\right)=-\frac{1}{2}\int_{-b}^{b}s^2e^{is(x-t)}\,ds.$$

Thus,

$$\begin{aligned} S(K_x f)(x) &= \frac{1}{\pi}\int_{-a}^{a}\left[-2t\frac{d}{dt}\left(\frac{\sin b(x-t)}{x-t}\right)+\left(a^2-t^2\right)\frac{d^2}{dt^2}\left(\frac{\sin b(x-t)}{x-t}\right)\right. \\ &\qquad \left. -b^2t^2\frac{\sin b(x-t)}{x-t}\right] f(t)\,dt \\ &= \frac{1}{\pi}\int_{-a}^{a}\left[K_t\frac{\sin b(x-t)}{x-t}\right] f(x)\,dx. \end{aligned}$$

Since $\frac{\sin b(x-t)}{x-t}$ is even it follows that $K_t\frac{\sin b(x-t)}{x-t} = K_t\frac{\sin b(t-x)}{t-x}$. Hence

$$\begin{aligned} & S(K_x f)(x) \\ &= \frac{1}{\pi}\int_{-a}^{a}\left[K_t\frac{\sin b(t-x)}{t-x}\right] f(x)\,dx \\ &= \frac{1}{\pi}\int_{-a}^{a}\left[\left(a^2-t^2\right)\frac{d^2}{dt^2}\frac{\sin b(t-x)}{t-x} - 2t\frac{d}{dt}\frac{\sin b(t-x)}{t-x} - b^2t^2\frac{\sin b(t-x)}{t-x}\right] f(x)\,dx \\ &= \left(a^2-t^2\right)\frac{d^2}{dt^2}(Sf)(t) - 2t\frac{d}{dt}(Sf)(t) - b^2t^2(Sf)(t) \\ &= K_t(Sf)(t) \end{aligned}$$

as required. ∎

An immediate consequence of Proposition 4.3.1.1 is that the eigenfunctions of S and K_t are the same. In Section 4.3 we saw that the eigenfunctions of the operator K_t are the spheroidal wave functions. Theorem 4.3.1.3 shows that S also admits spheroidal wave functions as its eigenfunctions. First we give a preliminary lemma that shows that zero is not an eigenvalue of the operator S.

Lemma 4.3.1.2 *Let S be the operator defined in Definition 2.5.0.14. Then zero is not an eigenvalue of S.*

Proof. Suppose that zero is an eigenvalue of S. Since $S = U^*U$ by Proposition 2.5.0.16 we have $U^*Uf = 0$ for some eigenfunction, $f \neq 0$. Now, $U^*Uf = 0$ implies $\langle U^*Uf, f\rangle = 0$, hence by Definition 2.1.0.7, $\|Uf\|^2 = 0$ and so $Uf = 0$. Recall Definition 2.5.0.8, since $U : L^2[-a,a] \to L^2[-b,b]$ it shows that $Uf = 0$ implies $Uf(t) = 0$ for $-b \le t \le b$. By Proposition 2.5.0.11, Uf is entire over the real line, hence has isolated zeros. We conclude that $Uf(t) = 0$ for all $t \in \mathbb{R}$ and $Uf(t) = \hat{f}(t)$. Now, using Proposition 2.4.1.3 we see that

$$\begin{aligned} f(x) &= \lim_{R\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-R}^{R}\hat{f}(t)e^{itx}\,dt \\ &= \lim_{R\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-R}^{R}Uf(t)e^{itx}\,dt \\ &= 0. \end{aligned}$$

Therefore, $Sf = 0$ implies that $f = 0$ everywhere, a contradiction. Hence zero cannot be an eigenvalue of S. ∎

Theorem 4.3.1.3 *Let S and K be as defined in Definition 2.5.0.14 and (4.6) respectively. Suppose that f is an eigenfunction of K associated with the eigenvalue λ. Furthermore, suppose that f is a spheroidal wave function of the first kind. Then f is also an eigenfunction of S.*

Proof. Let f be an eigenfunction of K corresponding to the eigenvalue λ, then $Kf = \lambda f$. Note that

$$SKf = \lambda Sf,$$

and since S and K commute by Proposition 4.3.1.1, we have

$$KSf = \lambda Sf$$

where Sf is twice continuously differentiable. By Lemma 4.3.1.2, zero is not an eigenvalue of S, hence $Sf \neq 0$. It follows that Sf is an eigenfunction of K corresponding to the eigenvalue λ. Therefore, we must have $Sf = g$ where g is some linear combination of spheroidal wave functions of the first and second kind. Now note that Sf is continuous by an application of the Dominated Convergence Theorem, hence Sf is finite at the points $z = \pm 1$. Therefore Sf must be a spheroidal wave function of the first kind. Since $Kf = \lambda f$ where f is the spheroidal wave function of the first kind corresponding to the eigenvalue λ, and Sf is also an eigenfunction of K corresponding to the eigenvalue λ, it follows that

$$Sf = \mu f$$

for some scalar, μ. Thus f is also an eigenfunction of S. ∎

4.4 The Spectrum of Hill's Equation

A spectrum, defined for an operator, contains the eigenvalues of the operator together with any limit points. In this section we seek to define the spectrum of Hill's equation by first writing (4.1) in operator form. We then introduce the monodromy operator, M_π and show how the spectrum of Hill's equation can be obtained from it. We finish by calculating the characteristic equation of the monodromy operator in preparation for Floquet's Theorem which we introduce in Section 4.4.2.

As previously stated, the definition of a spectrum is given in terms of operators. Therefore we first write (4.1) in operator form. The reader should also note that, unless otherwise specified, we take the space H_L to be as given in Definition 4.4.0.4.

Definition 4.4.0.4 *Let* $H_L = \{f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R})\}$. *Define* Hill's operator, $L : H_L \to L^2(\mathbb{R})$ *to be such that*

$$Lf(x) = -\frac{d^2}{dx^2}f(x) + q(x)f(x).$$

Remark 4.4.0.5 *With Hill's operator, L given by the above definition, Hill's equation, (4.1) becomes*

$$Lf(x) = \lambda f(x). \tag{4.13}$$

Having an operator form of Hill's equation, as in (4.13) allows us to define the spectrum of the operator, L. That is, we are able to define the spectrum of Hill's equation. We start this process by defining the spectrum of a general operator.

Definition 4.4.0.6 *Let H be a Hilbert space and $T : H \to H$ be an operator. The* spectrum *of T, denoted $\sigma(T)$ is defined to be*

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}.$$

It is clear from the definition of a spectrum that $\sigma(T)$ must contain all of the eigenvalues of T. If λ is an eigenvalue of T then $\lambda I - T$ fails to be one-to-one, hence is not bijective and therefore is not invertible. If an operator acts on a finite dimensional space then its spectrum consists precisely of its eigenvalues. If however, an operator acts on an infinite dimensional space then its spectrum may contain elements other than the eigenvalues. We summarise this in the following proposition.

Proposition 4.4.0.7 *Suppose that $T : H \to H$. If H is finite dimensional then*

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}.$$

If H is infinite dimensional then

$$\{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\} \subset \sigma(T).$$

In a finite dimensional Hilbert space, in order to find the spectrum of an operator, T we can consider its characteristic equation. The roots of the characteristic equation of T are precisely the eigenvalues of T.

Definition 4.4.0.8 *Let H be a finite dimensional Hilbert space and let $T : H \to H$ be an operator. We define the* characteristic equation *of T to be*

$$\det(\lambda I - T) = 0.$$

Remark 4.4.0.9 *When T is a trace class operator we can consider the equation*

$$\det(I - \lambda T) = 0.$$

The roots of this equation are the reciprocals of the eigenvalues of T.

In Theorem 4.4.0.19 we calculate the characteristic equation of the monodromy operator. It is this characteristic equation that is associated with Hill's equation. The proof of the theorem depends upon having Hill's equation in matrix form. We give a definition that introduces this matrix notation and then present some preliminary results.

Definition 4.4.0.10 *Let $H_L = \{f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R})\}$ and suppose that $f_1, f_2 \in H_L$ are the two fundamental solutions of Hill's equation. Set $g_i = f_i'$ for $i = 1, 2$ and let $\Psi = [\Psi_1, \Psi_2]$ where $\Psi_i = [f_i, g_i]^T$ for $i = 1, 2$. Also let*

$$D(x) = \begin{bmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{bmatrix}. \tag{4.14}$$

In matrix form, Hill's equation, (4.1) becomes

$$\Psi' = D\Psi. \tag{4.15}$$

Also, since f_1 and f_2 are fundamental solutions, the condition given by (4.3) is equivalent to $\Psi(0) = I$.

The following lemma shows that we can post multiply solutions to Hill's equation by scalar matrices to produce another solution.

Lemma 4.4.0.11 *Let Φ be a solution of Hill's equation so that $\Phi' = D\Phi$ where D is as given by (4.14). Suppose that W is a scalar valued matrix, then ΦW is also a solution of Hill's equation.*

Proof. The proof simply involves checking that ΦW satisfies (4.15). Let Φ be a solution of (4.15) and W a scalar valued matrix then

$$(\Phi W)' = \Phi' W + \Phi W'.$$

As W is a constant matrix its derivative with respect to x is zero. Also, since Φ is a solution of Hill's equation it satisfies $\Phi' = D\Phi$. Thus

$$(\Phi W)' = D\Phi W,$$

and so ΦW is also a solution of Hill's equation. ∎

We now begin to introduce the monodromy operator, M_π. In Definition 4.4.0.16 we take M_π to be a matrix similar to $\Psi(\pi)$ where Ψ denotes the fundamental solution matrix. The following Lemma is presented solely as a precursor to the proof of Proposition 4.4.0.14.

Lemma 4.4.0.12 *Let Ψ denote the 2×2 fundamental solution matrix. Also let Φ be any solution of $\Phi' = D\Phi$ where D is given by (4.14). Then Φ satisfies*

$$\Phi(\pi) = \Psi(\pi)\Phi(0).$$

Proof. Let Ψ be the 2×2 fundamental solution matrix, then Ψ satisfies Hill's equation, so $\Psi' = D\Psi$. Consider then $\Psi(x)\Phi(0)$. By Lemma 4.4.0.11, $\Psi(x)\Phi(0)$ is also a solution of Hill's equation, hence

$$[\Psi(x)\Phi(0)]' = D(x)\Psi(x)\Phi(0).$$

Also, since Ψ is a fundamental solution matrix it satisfies $\Psi(0) = I$ and so

$$\Psi(0)\Phi(0) = \Phi(0).$$

Therefore, at $x = 0$, $\Psi(x)\Phi(0)$ has derivative given by $D(0)\Psi(0)\Phi(0) = D(0)\Phi(0)$. That is, the derivatives of $\Psi(x)\Phi(0)$ and $\Phi(x)$, and the functions themselves are equal at the point $x = 0$. Hence, by uniqueness,

$$\Phi(x) = \Psi(x)\Phi(0).$$

So,

$$\Phi(\pi) = \Psi(\pi)\Phi(0)$$

as required. ∎

Recall that in Lemma 4.2.0.58 we saw that if $f(x)$ is a solution of Hill's equation then $f(x+\pi)$ is also a solution. The following lemma shows that we can write $f(x+\pi)$ as a linear combination of the fundamental solutions.

Lemma 4.4.0.13 *Let Ψ denote the 2×2 fundamental solution matrix then*

$$\Psi(x+\pi) = \Psi(x)\Psi(\pi).$$

Proof. Since Ψ is the 2×2 fundamental solution matrix we have

$$\Psi(x+\pi) = \begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix}(x+\pi).$$

Now, as $f_1(x)$ and $f_2(x)$ are solutions of Hill's equation, it follows from Lemma 4.2.0.58 that $f_1(x+\pi)$ and $f_2(x+\pi)$ are also solutions of Hill's equation. Therefore, we may write $f_1(x+\pi)$ and $f_2(x+\pi)$ as linear combinations of $f_1(x)$ and $f_2(x)$. Let

$$\begin{aligned} f_1(x+\pi) &= af_1(x) + bf_2(x), \\ f_2(x+\pi) &= cf_1(x) + df_2(x) \end{aligned}$$

for some constants a, b, c, d. Then

$$\begin{aligned} \Psi(x+\pi) &= \begin{bmatrix} af_1 + bf_2 & cf_1 + df_2 \\ af_1' + bf_2' & cf_1' + df_2' \end{bmatrix}(x) \\ &= \begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix}(x) \begin{bmatrix} a & c \\ b & d \end{bmatrix} \\ &= \Psi(x) \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \end{aligned} \tag{4.16}$$

Given that Ψ is a fundamental solution matrix and satisfies $\Psi(0) = I$, we use this to find a, b, c, d. So, taking $x = 0$ in (4.16) we have

$$\Psi(\pi) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Therefore,

$$\Psi(x+\pi) = \Psi(x)\Psi(\pi)$$

as required. ■

The following proposition is presented as motivation for the definition of the monodromy operator as appears in Definition 4.4.0.16.

Proposition 4.4.0.14 *Let Φ be any solution of $\Phi' = D\Phi$ such that $\Phi(0)$ is invertible. Then*

$$\Phi(x+\pi) = \Phi(x)\left[\Phi(0)^{-1}\Psi(\pi)\Phi(0)\right]$$

Proof. Let Ψ be the fundamental solution matrix and suppose that Φ is any solution of $\Phi' = D\Phi$ such that $\Phi(0)$ is invertible. Then by the proof of Lemma 4.4.0.12 we have $\Phi(x) = \Psi(x)\Phi(0)$ for all x. In particular,

$$\Phi(x+\pi) = \Psi(x+\pi)\Phi(0).$$

By Lemma 4.4.0.13 we have $\Psi(x+\pi) = \Psi(x)\Psi(\pi)$ and therefore

$$\Phi(x+\pi) = \Psi(x)\Psi(\pi)\Phi(0).$$

Finally, if $\Phi(x) = \Psi(x)\Phi(0)$ and $\Phi(0)$ is invertible then $\Psi(x) = \Phi(x)\Phi(0)^{-1}$. Hence,

$$\Phi(x+\pi) = \Phi(x)\Phi(0)^{-1}\Psi(\pi)\Phi(0).$$

■

Remark 4.4.0.15 *Note that the matrix $\Phi(0)^{-1}\Psi(\pi)\Phi(0)$ is similar to $\Psi(\pi)$.*

In the above proposition we saw that the matrix $\Psi(\pi)$ plays a special role. We saw that shifting any solution of Hill's equation by π results in post multiplication of the solution by $\Phi(0)^{-1}\Psi(\pi)\Phi(0)$. This observation prompts the following definition.

Definition 4.4.0.16 *Let $H_M = \{\Phi : \Phi' = D\Phi\}$ where D is as given by (4.14) and let Ψ denote the fundamental solution matrix. We define the* monodromy operator*, $M_\pi : H_M \to H_M$ to be such that*

$$\Phi(x) \mapsto \Phi(x+\pi)$$

where M_π is given by post multiplying by $\Phi(0)^{-1}\Psi(\pi)\Phi(0)$.

The monodromy operator, M_π is important since it is used to define Hill's discriminant. It is Hill's discriminant that gives the spectrum of Hill's equation. In Section 4.4.1 we will see how placing certain conditions upon Hill's discriminant allows us to calculate the periodic spectrum of Hill's equation. The relationship between Hill's discriminant and the monodromy operator is shown in the following definition.

Definition 4.4.0.17 *Let H_M be as given in Definition 4.4.0.16 and let $M_\pi : H_M \to H_M$ be the monodromy operator. Then* Hill's discriminant*, Δ is given by*

$$\Delta(\lambda) = \operatorname{tr} M_\pi.$$

Remark 4.4.0.18 *The above definition implies that Hill's discriminant takes the form*

$$\Delta(\lambda) = f_1(\pi) + f_2'(\pi),$$

where f_1 and f_2 are the fundamental solutions of Hill's equation.

As previously mentioned, Hill's discriminant gives the spectrum of Hill's equation. Therefore, by Remark 4.4.0.18, if we can find both fundamental solutions of Hill's equation we can calculate the spectrum.

The final task in this section is to calculate the characteristic equation of the monodromy operator. The roots of such a characteristic equation can be used to determine the type of solutions to Hill's equation that we can expect to find. This idea will be demonstrated in Proposition 4.4.2.2.

Theorem 4.4.0.19 *Let $H_M = \{\Psi : \Psi' = D\Psi\}$ where D is as given by (4.14) and let $M_\pi : H_M \to H_M$ be the monodromy operator. Then the characteristic equation of M_π is*

$$\mu^2 - (\operatorname{tr} M_\pi)\,\mu + \det M_\pi = 0$$

where $\det M_\pi = 1$.

Proof. The monodromy operator is represented by a 2×2 matrix, say

$$M_\pi = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $\operatorname{tr} M_\pi = a_{11} + a_{22}$ and $\det M_\pi = a_{11}a_{22} - a_{12}a_{21}$. The characteristic equation of M_π is given by

$$\begin{aligned} 0 &= \det[\mu I - M_\pi] \\ &= \det \begin{bmatrix} \mu - a_{11} & -a_{12} \\ -a_{21} & \mu - a_{22} \end{bmatrix} \\ &= (\mu - a_{11})(\mu - a_{22}) - a_{12}a_{21} \\ &= \mu^2 - (a_{11} + a_{22})\mu + a_{11}a_{22} - a_{12}a_{21} \\ &= \mu^2 - (\operatorname{tr} M_\pi)\,\mu + \det M_\pi. \end{aligned}$$

It remains to show that $\det M_\pi = 1$. Let Ψ denote the fundamental solution matrix then by Definition 4.4.0.16, $M_\pi = \Phi(0)^{-1}\Psi(\pi)\Phi(0)$ where Φ is a solution of Hill's equation. So,

$$\begin{aligned} \det M_\pi &= \det\left[\Phi(0)^{-1}\Psi(\pi)\Phi(0)\right] \\ &= \det \Psi(\pi). \end{aligned}$$

It therefore suffices to consider $\Psi(\pi)$ and show that $\det \Psi(\pi) = 1$. Let W denote the Wronskian of the fundamental solutions, f_1 and f_2. Then $W(x) = \det \Psi(x)$. Differentiating the Wronskian with respect to x we see that

$$\frac{d}{dx}W(x) = \det \begin{bmatrix} f_1' & f_2' \\ f_1' & f_2' \end{bmatrix}(x) + \det \begin{bmatrix} f_1 & f_2 \\ f_1'' & f_2'' \end{bmatrix}(x).$$

Clearly we have

$$\det \begin{bmatrix} f_1' & f_2' \\ f_1' & f_2' \end{bmatrix}(x) = 0.$$

Also, as f_1 and f_2 are solutions of Hill's equation we have, by (4.1)

$$f_i''(x) = [q(x) - \lambda]f_i(x)$$

for $i = 1, 2$. Thus,

$$\begin{aligned} \frac{d}{dx}W(x) &= \det \begin{bmatrix} f_1 & f_2 \\ (q-\lambda)f_1 & (q-\lambda)f_2 \end{bmatrix}(x) \\ &= 0. \end{aligned}$$

This implies that the Wronskian, $W(x)$ is constant for all x. In particular, $W(0) = W(\pi)$. Thus,

$$\begin{aligned} \det \Psi(\pi) &= W(\pi) \\ &= W(0) \\ &= \det \Psi(0). \end{aligned}$$

Since Ψ is a fundamental solution matrix, it follows from Definition 4.4.0.10 that $\Psi(0) = I$, hence $\det \Psi(\pi) = 1$. ■

In the case that the monodromy operator has distinct eigenvalues μ_1 and μ_2, M_π is diagonable and hence is similar to the matrix $\operatorname{diag}(\mu_1, \mu_2)$. Theorem 4.4.0.19 then shows that $1 = \det M_\pi = \mu_1\mu_2$. This observation provides motivation for the following definition.

Definition 4.4.0.20 *Let μ_1 and μ_2 be the roots of the characteristic equation of the monodromy operator, M_π. Then there exists some $\xi = a + bi \in \mathbb{C}$ such that $a, b \in [0, 2)$ and*

$$\begin{aligned} \mu_1 &= e^{i\xi\pi} \\ \mu_2 &= e^{-i\xi\pi}. \end{aligned}$$

We call ξ a characteristic exponent*.*

Remark 4.4.0.21 *It will sometimes be convenient to use the notation $\mu_i(\pi)$ for $i = 1, 2$.*

4.4.1 The Periodic Spectrum

In the case that Hill's equation has periodic solutions of period π or 2π, the associated eigenvalues lie in the periodic spectrum. This section defines the periodic spectrum of Hill's equation and shows how it can be calculated from Hill's discriminant. The remaining chapters of this thesis are dependent upon knowing the periodic spectrum of Hill's equation since we will sample from it. We show how the periodic spectrum can be split into two sets; the principal series relating to π-periodic solutions and the complementary series relating to 2π-periodic solutions. Further, we state a result known as the Oscillation Theorem that demonstrates the connection between the principal series and the complementary series.

Definition 4.4.1.1 *We define the* periodic spectrum *of Hill's equation to be,*

$$\sigma_p(L) = \{\lambda \in \mathbb{C} : \Delta^2(\lambda) - 4 = 0\}$$

where Δ is Hill's discriminant.

The periodic spectrum of Hill's equation is an infinite set as the following proposition shows.

Proposition 4.4.1.2 *Let $\Delta(\lambda)$ denote Hill's discriminant then the function*

$$\Delta^2(\lambda) - 4$$

has infinitely many zeros, $(\lambda_n)_{n=0}^{\infty}$. Further, if $\Delta^2(0) \neq 4$ then

$$\Delta^2(\lambda) - 4 = C \prod_{n=0}^{\infty} \left[1 - \frac{\lambda}{\lambda_n}\right].$$

Proof. Hill's discriminant, Δ has order $\frac{1}{2}$ by [35] (Theorem 2.2, page 20). The remainder of the statement now follows from Proposition 2.2.0.14. ■

Proposition 4.4.1.2 also shows us that if $0 \notin \sigma_p(L)$ then we can write the function $\Delta^2(\lambda) - 4$ as a convergent product.

Let $\sigma_p(L) = \{\lambda_n\}_{n \in \mathbb{N}_0}$ where the λ_n denote elements of the periodic spectrum. We can divide the set $\sigma_p(L)$ into two further sets that form sequences known as the principal and complementary series. The use of the term *series* is traditional in this context and does not imply summation.

Definition 4.4.1.3 *Suppose that λ belongs to the periodic spectrum. We say that λ lies in the* principal series *if it satisfies the equation*

$$\Delta(\lambda) - 2 = 0.$$

Similarly we say that λ lies in the complementary series *if it satisfies the equation,*

$$\Delta(\lambda) + 2 = 0.$$

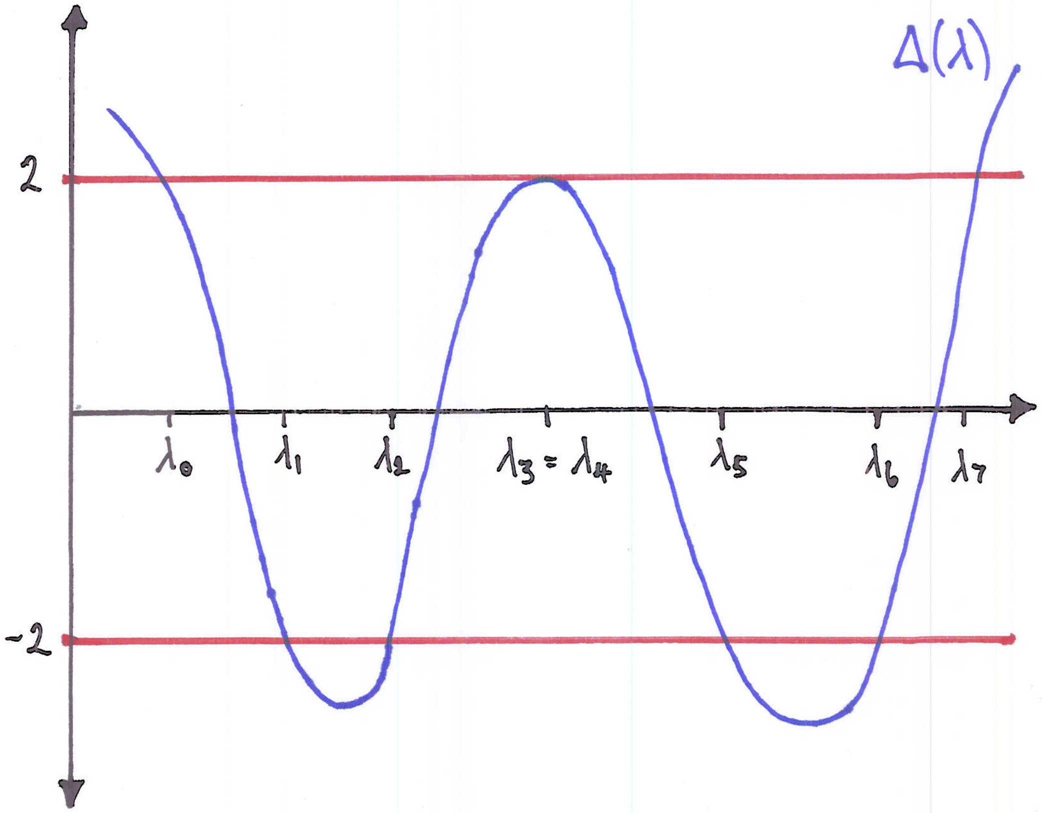
The following theorem is due to Liapounoff and Haupt and is known as the *Oscillation Theorem*. The theorem shows the relationship between the roots of $\Delta(\lambda) - 2 = 0$ and those of $\Delta(\lambda) + 2 = 0$. That is, it shows the relationship between eigenvalues in the principal series and those in the complementary series. Further, it shows what type of solutions to Hill's equation we can expect if a given eigenvalue is in either the principal or complementary series. The Oscillation Theorem, stated next, is presented in [35] (Theorem 2.1, page 11) along with a proof.

Proposition 4.4.1.4 *Hill's equation, (4.1) has a monotonically increasing infinite sequence of real numbers, $(\lambda_n)_{n=0}^{\infty}$ such that $\lambda_n \to \infty$ as $n \to \infty$ and the following inequalities hold,*

$$-\infty < \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \lambda_5 \leq \lambda_6 < \lambda_7 \leq \lambda_8 < \dots . \tag{4.17}$$

Furthermore, Hill's equation has a π-periodic solution if and only if λ_n belongs to the principal series, and a 2π-periodic solution if and only if λ_n belongs to the complementary series.

The sequence $(\lambda_n)_{n=0}^{\infty}$ is derived from the solutions to the equation $\Delta^2(\lambda) - 4 = 0$ and is monotonically increasing. By Definition 4.4.1.3 we see that each λ_n belongs to either the principal series or the complementary series. In [35] (Lemma 2.6, page 19) it was shown that λ_0 is a *simple root* of $\Delta^2(\lambda) - 4 = 0$ with $\Delta'(\lambda_0) < 0$. Further, λ_0 always lies in the principal series. The inequality $\lambda_0 < \lambda_1$ together with the fact that $\Delta'(\lambda_0) < 0$ then tells us that λ_1 lies in the complementary series. The next element in the sequence, λ_2 will again lie in the complementary series. If $\Delta'(\lambda_1) < 0$ then $\lambda_2 \neq \lambda_1$ and we say that both λ_1 and λ_2 are *simple roots*. Also, λ_1 and λ_2 will be separated by what we shall later refer to as an interval of instability. If $\Delta'(\lambda_1) = 0$ then $\lambda_2 = \lambda_1$ and we have a *double root*. This accounts for the inequality $\lambda_1 \leq \lambda_2$. The next inequality given in the Oscillation Theorem is $\lambda_2 < \lambda_3$ and this occurs since λ_3 lies in the principal series. Following this we have $\lambda_3 \leq \lambda_4$ denoting the fact that λ_3 and λ_4 can be either simple or double roots with λ_4 also in the principal series. This pattern of alternating between the principal series and complementary series repeats. The inequalities given in the Oscillation Theorem are structured such that for $n \in \mathbb{N}$, if λ_{2n-1} belongs to the principal series then λ_{2n} also belongs to the principal series. Similarly, if λ_{2n-1} belongs to the complementary series then λ_{2n} also belongs to the complementary series. As we have already briefly mentioned, the inequalities stated in (4.17) also define the intervals of *stability* and *instability*. These intervals are used to tell us whether or not our solutions to Hill's equation are bounded. Intervals of stability correspond to bounded solutions, whereas intervals of instability correspond to unbounded solutions. We will give more explanation of this concept in Section 4.4.2 but for now it suffices to note that an interval of stability is given by $\Delta^2(\lambda) - 4 < 0$ (if λ is a double root we also include the case $\Delta^2(\lambda) - 4 = 0$) and an interval of instability is anything outside of this. The following diagram plotting the function Δ makes the ideas just outlined more precise. Note that the scale on the y-axis refers only to the function Δ; the eigenvalues can take any real value.



$\Delta(\lambda)$
2
-2
λ_0
λ_1
λ_2
$\lambda_3 = \lambda_4$
λ_5
λ_6
λ_7

Remark 4.4.1.5 *From the diagram it is now easy to see that the principal series is given by*

$$\{\lambda_{4m}, \lambda_{4m+3} : m \in \mathbb{N}_0\}.$$

Similarly, the complementary series is given by

$$\{\lambda_{4m+1}, \lambda_{4m+2} : m \in \mathbb{N}_0\}.$$

The final proposition in this section explains the oscillatory nature of the function $\Delta(\lambda)$ that is evident in the diagram. It shows that we can express $\Delta(\lambda)$ in terms of the cosine function.

Proposition 4.4.1.6 *Let Δ denote Hill's discriminant. Then*

$$\Delta(\lambda) = 2\cos\xi\pi$$

where ξ is the characteristic exponent.

Proof. By Theorem 4.4.0.19, the monodromy operator has characteristic equation

$$0 = \mu^2 - \Delta(\lambda)\mu + 1. \tag{4.18}$$

Let μ_1 and μ_2 be the roots of the characteristic equation then

$$\begin{aligned} 0 &= (\mu - \mu_1)(\mu - \mu_2) \\ &= \mu^2 - (\mu_1 + \mu_2)\mu + \mu_1\mu_2. \end{aligned} \tag{4.19}$$

By comparing (4.18) with (4.19) we see that

$$\Delta(\lambda) = \mu_1 + \mu_2.$$

Finally, by Definition 4.4.0.20 we see that

$$\begin{aligned} \Delta(\lambda) &= e^{i\xi\pi} + e^{-i\xi\pi} \\ &= 2\cos\xi\pi \end{aligned}$$

as required. ■

4.4.2 Floquet's Theorem

The aim of this section is to state, without proof, Floquet's Theorem. Floquet's Theorem uses the roots of the characteristic equation of the monodromy operator to determine the nature of the solutions of Hill's equation. In particular, we see that when these roots are equal there exist periodic solutions to Hill's equation, of period π or 2π. As a consequence of Floquet's Theorem we are also able to determine when the solutions to Hill's equation are bounded.

Let the monodromy operator have eigenvalues μ_1 and μ_2. The following theorem gives a condition that Hill's discriminant must satisfy in order for μ_1 and μ_2 to be equal. It also shows that if $\mu_1 = \mu_2$ then λ belongs to the periodic spectrum.

Theorem 4.4.2.1 *Let M_π denote the monodromy operator with eigenvalues μ_1 and μ_2. Then $\mu_1 = \mu_2$ if and only if Hill's discriminant satisfies*

$$\Delta^2(\lambda) - 4 = 0.$$

Proof. By Theorem 4.4.0.19 and Definition 4.4.0.17, the characteristic equation of M_π is

$$\mu^2 - \Delta(\lambda)\mu + 1 = 0.$$

The roots of the characteristic equation of M_π are the eigenvalues of M_π by definition. We use the quadratic formula to find the roots μ_1 and μ_2. This gives

$$\mu = \frac{\Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 4}}{2},$$

from which it is clear that $\mu_1 = \mu_2$ if and only if $\Delta^2(\lambda) - 4 = 0$. ■

The following theorem is called *Floquet's Theorem* and it can be found with proof in [35] (page 4). Briefly, it tells us what type of solutions to Hill's equation we can expect to find.

Proposition 4.4.2.2 *Let $\mu_1(x) = e^{i\xi x}$ and $\mu_2(x) = e^{-i\xi x}$ where ξ is a characteristic exponent. Suppose that the characteristic equation of the monodromy operator, M_π has roots $\mu_1(\pi)$ and $\mu_2(\pi)$. If $\mu_1(\pi) \neq \mu_2(\pi)$ then Hill's equation has two linearly independent solutions, y_1 and y_2 such that*

$$\begin{aligned} y_1(x) &= \mu_1(x)p_1(x), \\ y_2(x) &= \mu_2(x)p_2(x) \end{aligned}$$

for π-periodic functions p_1, p_2. If $\mu_1(\pi) = \mu_2(\pi) = 1$ then Hill's equation has a non-trivial π-periodic solution, p_π. Suppose y is also a solution of Hill's equation and that y and p_π are linearly independent. Then

$$y(x + \pi) = y(x) + cp_\pi(x)$$

for some constant, c. If $\mu_1(\pi) = \mu_2(\pi) = -1$ then Hill's equation has a non-trivial 2π-periodic solution, $p_{2\pi}$. Suppose y is also a solution of Hill's equation and that y and $p_{2\pi}$ are linearly independent. Then

$$y(x + \pi) = -y(x) + cp_{2\pi}(x)$$

for some constant, c. If $c = 0$ then we have

$$\begin{aligned} &y_1(\pi) + y_2'(\pi) = \pm 2 \\ &y_1'(\pi) = 0 \\ &y_2(\pi) = 0. \end{aligned}$$

Remark 4.4.2.3 *We refer to the case in which $\mu_1(\pi) = \mu_2(\pi) = 1$ as the* periodic *case. The case in which $\mu_1(\pi) = \mu_2(\pi) = -1$ is referred to as the* antiperiodic *case.*

Floquet's Theorem tells us the nature of the solutions of Hill's equation based on the roots of the characteristic polynomial of the monodromy operator. It tells us that when those roots are equal and take the value 1, there exists a π-periodic solution to Hill's equation; when they take the value -1 there exists a 2π-periodic solution to Hill's equation. Notice also that when the roots are equal and $c = 0$ there exist two linearly independent periodic solutions. This corresponds to having a double root. Furthermore, when the roots are distinct there exist two linearly independent solutions that are products of a π-periodic function and some other function that is dependent upon the root.

The following corollary is a consequence of Floquet's theorem. It describes the conditions under which the solutions of Hill's equation are bounded.

Corollary 4.4.2.4 *Let ξ denote the characteristic exponent. If $\Delta^2(\lambda) - 4 < 0$ then ξ is real and all solutions of Hill's equation, (4.1) are bounded. If $\Delta^2(\lambda) - 4 > 0$ then all solutions of Hill's equation are unbounded. Finally, if $\Delta^2(\lambda) = 4$ then Hill's equation has a periodic solution of period π or 2π.*

Proof. We consider each case in turn. First suppose that $\Delta^2(\lambda) - 4 < 0$. By Definition 4.4.1.1 we are not in the periodic spectrum and so we must have $\mu_1 \neq \mu_2$, where μ_1 and μ_2 are the roots of the characteristic equation of the monodromy operator. Hence, by Floquet's Theorem 4.4.2.2 there exist two linearly independent solutions of Hill's equation,

$$\begin{aligned} y_1(x) &= \mu_1(x)p_1(x), \\ y_2(x) &= \mu_2(x)p_2(x). \end{aligned}$$

It follows that any solution of Hill's equation has the form

$$y(x) = \alpha y_1(x) + \beta y_2(x).$$

We show that y is bounded. Given $\Delta^2(\lambda) - 4 < 0$, or more simply, $-2 < \Delta(\lambda) < 2$, this is equivalent to $-1 < \cos \xi\pi < 1$ where ξ is a characteristic exponent by 4.4.1.6. Since $\xi = a + bi$ for $a, b \in [0, 2)$ it follows that we must have $a \neq 0$ and $b = 0$ so that $\xi \neq 0$ and real. Therefore,

$$\begin{aligned} |\mu_i(x)| &= \left|e^{\pm iax}\right| \\ &= 1 \end{aligned}$$

for $x \in \mathbb{R}$ and $i = 1, 2$. Hence,

$$\begin{aligned} |y(x)| &= |\alpha\mu_1(x)p_1(x) + \beta\mu_2(x)p_2(x)| \\ &\leq |\alpha|\,|p_1(x)| + |\beta|\,|p_2(x)|\,. \end{aligned}$$

By Definition 4.1.0.53, y_1 and y_2 are continuous. Since $\mu_1(x)$ and $\mu_2(x)$ are continuous it follows that the functions p_1 and p_2 must also be continuous. Further, by Floquet's Theorem 4.4.2.2 p_1 and p_2 are π-periodic and so are bounded on an interval of length π. It follows from the periodicity of p_1 and p_2 that they are both bounded on $\mathbb{R}$. Hence y is bounded.

Now suppose that $\Delta^2(\lambda) - 4 > 0$. Again, we are not in the periodic spectrum and so by Floquet's Theorem 4.4.2.2, any solution of Hill's equation has the form

$$y(x) = \alpha y_1(x) + \beta y_2(x)$$

where $y_1(x) = \mu_1(x)p_1(x)$ and $y_2(x) = \mu_2(x)p_2(x)$. Now, $\Delta^2(\lambda) - 4 > 0$ is equivalent to $|\cos \xi\pi| > 1$. Clearly this only happens when $\xi = a + bi$ for $b \neq 0$. Now,

$$\begin{aligned} |y(x)| &= |\alpha\mu_1(x)p_1(x) + \beta\mu_2(x)p_2(x)| \\ &\geq |\alpha|\,|\mu_1(x)|\,|p_1(x)| - |-\beta|\,|\mu_2(x)|\,|p_2(x)| \end{aligned}$$

where

$$\begin{aligned} |\mu_i(x)| &= \left|e^{\pm i(a+bi)x}\right| \\ &= e^{\pm(-b)x} \end{aligned}$$

for $x \in \mathbb{R}$ and $i = 1, 2$. Consider now the cases in which $x \to \pm\infty$. Suppose $x \to \infty$ then $|\mu_2(x)| = e^{bx} \to \infty$. If $x \to -\infty$ then $|\mu_1(x)| = e^{-bx} \to \infty$. In both cases the result is that y unbounded.

Finally, the existence of periodic solutions of period π or 2π when $\Delta^2(\lambda) = 4$ follows directly from Floquet's Theorem, 4.4.2.2. ■

The following definition provides some standard terminology.

Definition 4.4.2.5 *If $\lambda \in \mathbb{R}$ is such that $\Delta^2(\lambda) - 4 < 0$, or $\Delta^2(\lambda) - 4 = 0$ and λ is a double root, then we say that λ belongs to an interval of* stability*. If $\Delta^2(\lambda) - 4 > 0$, or $\Delta^2(\lambda) - 4 = 0$ and λ is a simple root then we say that λ belongs to an interval of* instability*.*

We see from Corollary 4.4.2.4 and Definition 4.4.2.5 that the intervals of stability relate to bounded solutions of Hill's equation, whereas the intervals of instability relate to unbounded solutions.

4.4.3 The Bloch Spectrum

Traditionally, the Bloch spectrum is used by physicists and is defined to be the set of eigenvalues for which all solutions of an equation are bounded on the real line. This is an analytic approach. Here we define the Bloch spectrum in terms of Hill's discriminant, Δ which is derived from the monodromy operator, a geometrical quantity.

Definition 4.4.3.1 *Let L denote Hill's operator. We define the* Bloch spectrum *of Hill's equation to be*

$$\sigma_B(L) = \{\lambda \in \mathbb{R} : \Delta^2(\lambda) - 4 < 0\} \cup \{\lambda \in \mathbb{R} : \Delta^2(\lambda) - 4 = 0 \ \textit{and} \ \lambda \ \textit{is a double root}\}.$$

The Bloch spectrum is therefore the set of eigenvalues that belong to intervals of stability. That is, $\lambda \in \sigma_B(L)$ if every solution of

$$-f''(x) + q(x)f(x) = \lambda f(x)$$

is bounded. We show this in the following lemma.

Lemma 4.4.3.2 *Suppose that λ belongs to the Bloch spectrum, $\sigma_B(L)$. Let f_λ denote a solution to Hill's equation corresponding to the eigenvalue, λ. Then f_λ is bounded.*

Proof. Let $\lambda \in \sigma_B(L)$ then either $\Delta^2(\lambda) - 4 < 0$ or $\Delta^2(\lambda) - 4 = 0$ and λ is a double root. We denote by f_λ the eigenfunction associated with the eigenvalue, λ. If $\Delta^2(\lambda) - 4 < 0$ then by Corollary 4.4.2.4, f_λ is a bounded solution of Hill's equation. If $\Delta^2(\lambda) - 4 = 0$ then again by Corollary 4.4.2.4, Hill's equation has a continuous, periodic solution, f_λ. Since f_λ is continuous and periodic it is therefore bounded, completing the result. ■

The following proposition shows that the Bloch spectrum of Hill's equation is indeed a subset of the spectrum of Hill's equation, for if $\lambda I - L$ is not invertible then $\lambda \in \sigma(L)$ by Definition 4.4.0.6.

Proposition 4.4.3.3 *Let L denote Hill's operator. Suppose that λ lies in the Bloch spectrum, $\sigma_B(L)$. Then $\lambda I - L$ is not invertible.*

Proof. Let $\lambda \in \sigma_B(L)$ then by Lemma 4.4.3.2, there exists a bounded function, f_λ associated with λ and such that

$$-f_\lambda''(x) + q(x) f_\lambda = \lambda f_\lambda.$$

Suppose that $f_\lambda \in L^2(\mathbb{R})$ then by Definition 4.4.0.4,

$$L f_\lambda = \lambda f_\lambda$$

where L denotes Hill's operator. Therefore, λ is an eigenvalue with eigenfunction f_λ. By Proposition 4.4.0.7, λ belongs to the spectrum of L, hence, by Definition 4.4.0.6, $\lambda I - L$ is not invertible.

Now suppose that $\lambda \in \sigma_B(L)$ and $f_\lambda \notin L^2(\mathbb{R})$. We construct an approximate eigenfunction so that $\lambda I - L$ is not invertible. Take $T > 0$ large and let $\varphi_T : \mathbb{R} \to [0,1]$ be even, infinitely differentiable and such that

$$\varphi_T(x) = \begin{cases} 0 & \text{for} \quad x < -T-1, \\ 1 & \text{for} \quad -T < x < T, \\ 0 & \text{for} \quad x > T+1. \end{cases}$$

We take φ_T to be decreasing on $(0,\infty)$ and note that the families $\{\varphi_T''\}_{T>0}$ and $\{\varphi_T'\}_{T>0}$ are uniformly bounded on $\mathbb{R}$. Also,

$$\varphi_T f_\lambda(x) = \begin{cases} 0 & \text{for} \quad x < -T-1, \\ f_\lambda(x) & \text{for} \quad -T < x < T, \\ 0 & \text{for} \quad x > T+1. \end{cases}$$

Since f_λ satisfies Hill's equation it follows that $\varphi_T f_\lambda$ satisfies Hill's equation on the set

$$S = (-\infty, -T-1) \cup (-T, T) \cup (T+1, \infty).$$

Note that for $x \in (-\infty, -T-1) \cup (T+1, \infty)$ we have the trivial solution. Therefore, writing Hill's equation in terms of Hill's operator we see that the equation

$$(\lambda I - L)(\varphi_T f_\lambda) = 0 \tag{4.20}$$

holds on the set S. Now, since f_λ is differentiable on $\mathbb{R}$ it follows that it is continuous on $\mathbb{R}$. Thus $\varphi_T f_\lambda \in L^2(\mathbb{R})$ since it is continuous and has support on $(-T, T)$. It then follows from (4.20) that

$$\|(\lambda I - L)(\varphi_T f_\lambda)\|_{L^2(\mathbb{R})} = \|(\lambda I - L)(\varphi_T f_\lambda)\|_{L^2(\mathbb{R} \backslash S)}.$$

On the set $\mathbb{R} \backslash S$ we have

$$\begin{aligned} (\lambda I - L)(\varphi_T f_\lambda)(x) &= \lambda (\varphi_T f_\lambda)(x) + (\varphi_T f_\lambda)''(x) - q(x)(\varphi_T f_\lambda)(x) \\ &= (\varphi_T'' f_\lambda)(x) + 2(\varphi_T' f_\lambda')(x) + (\varphi_T f_\lambda'')(x) + [\lambda - q(x)](\varphi_T f_\lambda)(x). \end{aligned}$$

It therefore follows from the triangle inequality that

$$\begin{aligned} &\left\|\left(\lambda I-L\right)\left(\varphi_T f_\lambda\right)\right\|_{L^2(\mathbb{R})} \\ &\quad \le \left\|\varphi_T'' f_\lambda\right\|_{L^2(\mathbb{R}\setminus S)} + 2\left\|\varphi_T' f_\lambda'\right\|_{L^2(\mathbb{R}\setminus S)} + \left\|\varphi_T f_\lambda''\right\|_{L^2(\mathbb{R}\setminus S)} + \left\|\lambda-q\right\|_{L^\infty(\mathbb{R}\setminus S)}\left\|\varphi_T f_\lambda\right\|_{L^2(\mathbb{R}\setminus S)}. \end{aligned}$$

We show that each term in the above inequality is bounded. First note that for any $x \in \mathbb{R}$ we have $|\varphi_T| \le 1$, hence

$$\begin{aligned} \left\|\varphi_T\right\|^2_{L^2(\mathbb{R}\setminus S)} &= \int_{-T-1}^{-T} \left|\varphi_T(x)\right|^2\, dx + \int_{T}^{T+1} \left|\varphi_T(x)\right|^2\, dx \\ &\le \int_{-T-1}^{-T} 1\, dx + \int_{T}^{T+1} 1\, dx \\ &= 2. \end{aligned}$$

Also, since φ_T' and φ_T'' are uniformly bounded, we can choose constants $C', C'' > 0$ such that $|\varphi_T'| \le C'$ and $|\varphi_T''| \le C''$. By the same reasoning used to calculate $\left\|\varphi_T\right\|^2_{L^2(\mathbb{R}\setminus S)}$, we see that

$$\begin{aligned} \left\|\varphi_T'\right\|^2_{L^2(\mathbb{R}\setminus S)} &\le 2(C')^2, \\ \left\|\varphi_T''\right\|^2_{L^2(\mathbb{R}\setminus S)} &\le 2(C'')^2. \end{aligned}$$

We also note that since q is continuous and periodic it is bounded and so we may take

$$|\lambda - q(x)| \le Q \tag{4.21}$$

for all $x \in \mathbb{R}$ and some constant $Q > 0$. Furthermore, since $\lambda \in \sigma_B(L)$, f_λ is bounded, say

$$|f_\lambda(x)| \le M \tag{4.22}$$

for $x \in \mathbb{R}$ and some constant $M > 0$. Thus

$$\begin{aligned} \left\|f_\lambda\right\|^2_{L^2(\mathbb{R}\setminus S)} &= \int_{-T-1}^{-T} \left|f_\lambda(x)\right|^2\, dx + \int_{T}^{T+1} \left|f_\lambda(x)\right|^2\, dx \\ &\le \int_{-T-1}^{-T} M^2\, dx + \int_{T}^{T+1} M^2\, dx \\ &= 2M^2. \end{aligned}$$

Also, since f_λ satisfies Hill's equation we may write,

$$f_\lambda'' = [q(x) - \lambda]\, f_\lambda$$

and so by (4.21) and (4.22) we have

$$|f_\lambda''(x)| \le QM$$

for all $x \in \mathbb{R}$. Thus, as above

$$\begin{aligned} \left\|f_\lambda''\right\|^2_{L^2(\mathbb{R}\setminus S)} &= \int_{-T-1}^{-T} \left|f_\lambda''(x)\right|^2\, dx + \int_{T}^{T+1} \left|f_\lambda''(x)\right|^2\, dx \\ &\le \int_{-T-1}^{-T} (QM)^2\, dx + \int_{T}^{T+1} (QM)^2\, dx \\ &= 2(QM)^2. \end{aligned}$$

In order to show that f_λ' is bounded we split into two cases. First suppose that $\Delta^2(\lambda) - 4 < 0$ then by Proposition 4.4.2.2,

$$f_\lambda(x) = e^{\pm i\xi x} p(x)$$

where p is a π-periodic function. Now,

$$\begin{aligned} |f_\lambda'(x)| &= \left| \pm i\xi e^{\pm i\xi x} p(x) + e^{\pm i\xi x} p'(x) \right| \\ &\leq \xi \left| e^{\pm i\xi x} \right| |p(x)| + \left| e^{\pm i\xi x} \right| |p'(x)| . \end{aligned}$$

Since f_λ and the exponential function are continuous, it follows that p must also be continuous. Since p is a continuous, periodic function, it follows that p is bounded on $\mathbb{R}$. Also, since f_λ is a solution of Hill's equation, it must be continuously differentiable, hence p' is continuous and periodic and therefore bounded on $\mathbb{R}$. We thus take $|p(x)| \leq P$ and $|p'(x)| \leq P'$ for all $x \in \mathbb{R}$ and constants $P, P' > 0$. Furthermore, since $\Delta^2(\lambda) - 4 < 0$ we have $\xi \in \mathbb{R}$ by Corollary 4.4.2.4 and so $\left| e^{\pm i\xi x} \right| \leq 1$. Thus,

$$|f_\lambda'(x)| \leq \xi P + P'.$$

Now suppose that $\Delta^2(\lambda) - 4 = 0$ and λ is a double root. Then f_λ is a bounded, periodic, continuously differentiable solution of Hill's equation. It therefore follows that f_λ' is continuous and periodic, hence bounded and so

$$|f_\lambda'(x)| \leq D$$

for all $x \in \mathbb{R}$ and some constant $D > 0$. Let $M' = \max\{\xi P + P', D\}$ then in both cases we have

$$\begin{aligned} \|f_\lambda'\|^2_{L^2(\mathbb{R}\setminus S)} &= \int_{-T-1}^{-T} |f_\lambda'(x)|^2 \, dx + \int_{T}^{T+1} |f_\lambda'(x)|^2 \, dx \\ &\leq \int_{-T-1}^{-T} (M')^2 \, dx + \int_{T}^{T+1} (M')^2 \, dx \\ &= 2(M')^2. \end{aligned}$$

It now follows that

$$\begin{aligned} \|(\lambda I - L)(\varphi_T f_\lambda)\|_{L^2(\mathbb{R})} &\leq 2C''M + 4C'M' + 2QM + 2QM \\ &= 2\left[C''M + 2C'M' + 2QM\right]. \end{aligned}$$

Let $C = 2\left[C''M + 2C'M' + 2QM\right]$ then we have

$$\|(\lambda I - L)(\varphi_T f_\lambda)\|_{L^2(\mathbb{R})} \leq C$$

where $C > 0$ is some constant, independent of T. Finally we calculate $\|\varphi_T f_\lambda\|_{L^2(\mathbb{R})}$. First note that

$$\begin{aligned} \|\varphi_T f_\lambda\|^2_{L^2(\mathbb{R})} &= \|\varphi_T f_\lambda\|^2_{L^2(S)} + \|\varphi_T f_\lambda\|^2_{L^2(\mathbb{R}\setminus S)} \\ &= \|f_\lambda\|^2_{L^2(-T,T)} + \|\varphi_T f_\lambda\|^2_{L^2(\mathbb{R}\setminus S)} . \end{aligned}$$

We have assumed that $f_\lambda \notin L^2(\mathbb{R})$ and so we may take

$$\|f_\lambda\|_{L^2(-T,T)} \to \infty$$

as $T \to \infty$. Hence, $\|\varphi_T f_\lambda\|_{L^2(\mathbb{R})} \to \infty$ as $T \to \infty$. It follows that

$$\lim_{T\to\infty} \frac{\|(\lambda I - L)(\varphi_T f_\lambda)\|_{L^2(\mathbb{R})}}{\|\varphi_T f_\lambda\|_{L^2(\mathbb{R})}} = 0,$$

so we can choose C_T such that

$$\|(\lambda I - L)\,(\varphi_T f_\lambda)\|_{L^2(\mathbb{R})} \leq C_T\,\|\varphi_T f_\lambda\|_{L^2(\mathbb{R})}$$

where $C_T \to 0$ as $T \to \infty$. Hence as $T \to \infty$, $\|\lambda I - L\|_{L^2(\mathbb{R})} \to 0$ and so we cannot invert $\lambda I - L$. ■

4.5 Characterisation of the Periodic Spectrum in Terms of Hill's Determinants

The purpose of this section is to find criteria, not dependent upon solving Hill's equation, for an eigenvalue to be in the periodic spectrum. By following the approach of Hill in [28], we write the solutions and the potential of Hill's equation in terms of Fourier series from which we can construct another matrix representation of Hill's equation. Hill then continued to calculate various determinants associated with the resulting matrix. In a similar fashion we also construct determinants whose roots are elements of the periodic spectrum. We do this by constructing separate determinants for the principal series and the complementary series. Our method of construction has the advantage of using holomorphic functions instead of meromorphic functions as seen in [35] (page 32). We then proceed to extend Hill's method by writing the resulting determinants in terms of linear systems. This is done via a convolution operation.

Let q be the potential of Hill's equation, (4.1). Throughout this section we take q to be as in the following definition.

Definition 4.5.0.4 *Let $q \in L^2[0, \pi]$ be π-periodic and twice continuously differentiable. Suppose that q has Fourier series given by*

$$q(t) = \sum_{j=-\infty}^{\infty} \theta_j e^{2ijt}$$

for Fourier coefficients, $\theta_j \in \mathbb{C}$.

Lemma 4.5.0.5 *Let q be as in Definition 4.5.0.4, then q satisfies*

$$\begin{aligned} \|q\|^2_{L^2[0,\pi]} &= \sum_{j=-\infty}^{\infty} |\theta_j|^2 \\ &< \infty. \end{aligned}$$

Furthermore,

$$\|q'\|^2_{L^2[0,\pi]} = 4\sum j^2\,|\theta_j|^2.$$

Proof. First note that since $q \in L^2[0,\pi]$, it has finite norm. Given $\theta_j \in \mathbb{C}$ we have,

$$\begin{aligned}
\|q\|^2_{L^2[0,\pi]} &= \frac{1}{\pi}\int_0^\pi |q(t)|^2 \, dt \\
&= \frac{1}{\pi}\int_0^\pi \left| \sum_{j=-\infty}^{\infty} \theta_j e^{2ijt} \right|^2 dt \\
&= \frac{1}{\pi}\int_0^\pi \left(\sum_{j=-\infty}^{\infty} \theta_j e^{2ijt} \right) \left(\sum_{k=-\infty}^{\infty} \overline{\theta_k} e^{-2ikt} \right) dt \\
&= \frac{1}{\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \theta_j \overline{\theta_k} \int_0^\pi e^{2i(j-k)t} \, dt.
\end{aligned}$$

Note that

$$\int_0^\pi e^{2i(j-k)t} \, dt = \begin{cases} \pi & \text{for } \ k = j, \\ 0 & \text{for } \ k \neq j, \end{cases}$$

therefore,

$$\|q\|^2_{L^2[0,\pi]} = \sum_{j=-\infty}^{\infty} |\theta_j|^2$$

as required.

Next we calculate $\|q'\|^2_{L^2[0,\pi]}$. First we differentiate q term-by-term giving

$$q'(t) = \sum_{j=-\infty}^{\infty} 2ij\theta_j e^{2ijt}.$$

Now, as before with $\theta_j \in \mathbb{C}$ we have

$$\begin{aligned}
\|q'\|^2_{L^2[0,\pi]} &= \frac{1}{\pi}\int_0^\pi |q'(t)|^2 \, dt \\
&= \frac{1}{\pi}\int_0^\pi \left| \sum_{j=-\infty}^{\infty} 2ij\theta_j e^{2ijt} \right|^2 dt \\
&= \frac{1}{\pi}\int_0^\pi \left(\sum_{j=-\infty}^{\infty} 2ij\theta_j e^{2ijt} \right) \left(\sum_{k=-\infty}^{\infty} -2ik\overline{\theta_k} e^{-2ikt} \right) dt \\
&= \frac{1}{\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 4jk\theta_j \overline{\theta_k} \int_0^\pi e^{2i(j-k)t} \, dt \\
&= \sum_{j=-\infty}^{\infty} 4j^2 |\theta_j|^2
\end{aligned}$$

as required. ■

The following proposition is the starting point that Hill used in his paper [28]. It provides equations which, once solved, give solutions of Hill's equation whose corresponding eigenvalues lie in the periodic spectrum.

Proposition 4.5.0.6 *Let λ be an eigenvalue of Hill's equation. Then λ belongs to the principal series if and only if there exists a sequence $(b_j)_{j\in\mathbb{Z}} \in \ell^2$ satisfying*

$$b_j + \frac{\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k}{4j^2 - 1} = \frac{(\lambda - 1)b_j}{4j^2 - 1}$$

for all $j \in \mathbb{Z}$. Similarly, λ belongs to the complementary series if and only if there exists a sequence $(b_j)_{j\in\mathbb{Z}} \in \ell^2$ satisfying

$$b_j + \frac{\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k}{(1+2j)^2} = \frac{\lambda b_j}{(1+2j)^2}$$

for all $j \in \mathbb{Z}$.

Proof. By the Oscillation Theorem 4.4.1.4, λ belongs to the principal series if and only if there exists a periodic solution, f, to Hill's equation of period π. Since f is periodic it therefore has a Fourier series,

$$f(t) = \sum_{j=-\infty}^{\infty} b_j e^{2ijt}.$$

Hill's equation, (4.1) then becomes

$$\begin{aligned} \lambda \sum_{j=-\infty}^{\infty} b_j e^{2ijt} &= -\sum_{j=-\infty}^{\infty} 4i^2 j^2 b_j e^{2ijt} + \left(\sum_{k=-\infty}^{\infty} \theta_k e^{2ikt}\right)\left(\sum_{j=-\infty}^{\infty} b_j e^{2ijt}\right) \\ &= 4\sum_{j=-\infty}^{\infty} j^2 b_j e^{2ijt} + \sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k\right) e^{2ijt}. \end{aligned} \quad (4.23)$$

If we compare the coefficients of e^{2ijt} in (4.23) we see that

$$\lambda b_j = 4j^2 b_j + \sum_{k=-\infty}^{\infty} \theta_{j-k} b_k. \quad (4.24)$$

We would like to divide (4.24) by $4j^2$, however this leads to division by zero in the case that $j = 0$. In order to avoid dividing by zero we first subtract a b_j from either side of (4.24) and then divide by $4j^2 - 1$. We obtain,

$$b_j + \frac{\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k}{4j^2 - 1} = \frac{(\lambda - 1) b_j}{4j^2 - 1}$$

as required.

The case in which λ belongs to the complementary series is shown similarly. First note that by the Oscillation Theorem 4.4.1.4, λ lies in the complementary series if and only if there exists a 2π-periodic solution to Hill's equation. Let f denote such a solution, then f has Fourier series

$$f(t) = \sum_{j=-\infty}^{\infty} b_j e^{i(1+2j)t}.$$

Substituting this into Hill's equation then gives

$$\lambda \sum_{j=-\infty}^{\infty} b_j e^{i(1+2j)t} = \sum_{j=-\infty}^{\infty} (1+2j)^2 b_j e^{i(1+2j)t} + \sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k\right) e^{i(1+2j)t}. \quad (4.25)$$

Comparing the coefficients of $e^{i(1+2j)t}$ in (4.25) and dividing by $(1+2j)^2$ we obtain

$$b_j + \frac{\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k}{(1+2j)^2} = \frac{\lambda b_j}{(1+2j)^2}$$

completing the proof. ■

The following lemma is used to show that the matrices

$$\left[\frac{\theta_{j-k}}{4j^2-1}\right] - \text{diag}\left[\frac{\lambda-1}{4j^2-1}\right] \quad \text{and} \quad \left[\frac{\theta_{j-k}}{(1+2j)^2}\right] - \text{diag}\left[\frac{\lambda}{(1+2j)^2}\right]$$

appearing in Proposition 4.5.0.8, are trace class. Once we know that these matrices are trace class we are then able to use the Fredholm and Carleman determinants interchangeably (see Corollary 2.3.0.29).

Lemma 4.5.0.7 *The matrices*

$$\text{diag}\left[\frac{\lambda-1}{4j^2-1}\right]_{j=-\infty}^{\infty}, \left[\frac{\theta_{j-k}}{4j^2-1}\right]_{j,k=-\infty}^{\infty}, \text{diag}\left[\frac{\lambda}{(1+2j)^2}\right]_{j=-\infty}^{\infty} \quad \textit{and} \quad \left[\frac{\theta_{j-k}}{(1+2j)^2}\right]_{j,k=-\infty}^{\infty}$$

are trace class. Hence,

$$\left[\frac{\theta_{j-k}}{4j^2-1}\right]_{j,k=-\infty}^{\infty} - \text{diag}\left[\frac{\lambda-1}{4j^2-1}\right]_{j=-\infty}^{\infty},$$
$$\left[\frac{\theta_{j-k}}{(1+2j)^2}\right]_{j,k=-\infty}^{\infty} - \text{diag}\left[\frac{\lambda}{(1+2j)^2}\right]_{j=-\infty}^{\infty}$$

are trace class.

Proof. Let $A = \text{diag}\left[\frac{\lambda-1}{4j^2-1}\right]_{j=-\infty}^{\infty}$ and let $B = \left[\frac{\theta_{j-k}}{4j^2-1}\right]_{j,k=-\infty}^{\infty}$. We use Lemma 2.1.2.4 to show that both A and B are trace class. To see that the matrix A is trace class we note that

$$|\lambda - 1| \sum_{j=-\infty}^{\infty} \frac{1}{|4j^2-1|} < \infty$$

by comparison with $\sum_{j=-\infty}^{\infty} \frac{1}{j^2}$. Now consider the matrix B. We have

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{|\theta_{j-k}|}{|4j^2-1|} = \sum_{j=-\infty}^{\infty} \frac{1}{|4j^2-1|} \sum_{k=-\infty}^{\infty} |\theta_{j-k}|.$$

Now, $\sum_{k=-\infty}^{\infty} |\theta_{j-k}|$ is convergent by Lemma 4.5.0.5 and $\sum_{j=-\infty}^{\infty} \frac{1}{|4j^2-1|}$ is convergent by comparison with $\sum_{j=-\infty}^{\infty} \frac{1}{j^2}$. Hence $\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{|\theta_{j-k}|}{|4j^2-1|}$ and so B is indeed trace class.

A similar argument can be used to show that the matrices $\text{diag}\left[\frac{\lambda}{(1+2j)^2}\right]_{j=-\infty}^{\infty}$ and $\left[\frac{\theta_{j-k}}{(1+2j)^2}\right]_{j,k=-\infty}^{\infty}$ are trace class. It now follows from Proposition 2.1.2.2 that the sum of trace class matrices is also trace class, thus

$$\left[\frac{\theta_{j-k}}{4j^2-1}\right]_{j,k=-\infty}^{\infty} - \text{diag}\left[\frac{\lambda-1}{4j^2-1}\right]_{j=-\infty}^{\infty},$$
$$\left[\frac{\theta_{j-k}}{(1+2j)^2}\right]_{j,k=-\infty}^{\infty} - \text{diag}\left[\frac{\lambda}{(1+2j)^2}\right]_{j=-\infty}^{\infty}$$

are also trace class. ■

The following proposition gives several criteria for an eigenvalue of Hill's equation to lie in the periodic spectrum. Note that the conditions do not require that we solve Hill's equation, they only require that we know the Fourier coefficients of the potential.

Proposition 4.5.0.8 *Let λ be an eigenvalue of Hill's equation. Then λ lies in the principal series if and only if the Fredholm determinant*

$$\det\left(I - \text{diag}\left[\frac{\lambda-1}{4j^2-1}\right] + \left[\frac{\theta_{j-k}}{4j^2-1}\right]\right) = 0.$$

Equivalently, λ lies in the principal series if and only if the Carleman determinant

$$\det_2\left(I - \text{diag}\left[\frac{\lambda - 1}{4j^2 - 1}\right] + \left[\frac{\theta_{j-k}}{4j^2 - 1}\right]\right) = 0.$$

Also, λ lies in the complementary series if and only if the Fredholm determinant

$$\det\left(I - \text{diag}\left[\frac{\lambda}{(1+2j)^2}\right] + \left[\frac{\theta_{j-k}}{(1+2j)^2}\right]\right) = 0.$$

Equivalently, λ lies in the complementary series if and only if the Carleman determinant

$$\det_2\left(I - \text{diag}\left[\frac{\lambda}{(1+2j)^2}\right] + \left[\frac{\theta_{j-k}}{(1+2j)^2}\right]\right) = 0.$$

Proof. Let λ be an eigenvalue of Hill's equation then, by Proposition 4.5.0.6, λ belongs to the principal series if and only if there exists $(b_j)_{j\in\mathbb{Z}} \in \ell^2$ such that the equation

$$b_j + \frac{\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k}{4j^2 - 1} = \frac{(\lambda - 1)b_j}{4j^2 - 1} \tag{4.26}$$

holds for all $j \in \mathbb{Z}$. Let $B = [\ldots, b_{-1}, b_0, b_1, \ldots]^T \neq 0$ then, in matrix form, (4.26) becomes

$$\left(I - \text{diag}\left[\frac{\lambda - 1}{4j^2 - 1}\right] + \left[\frac{\theta_{j-k}}{4j^2 - 1}\right]\right) B = 0.$$

Now, since $B \neq 0$ we must have $\left(I - \text{diag}\left[\frac{\lambda-1}{4j^2-1}\right] + \left[\frac{\theta_{j-k}}{4j^2-1}\right]\right)$ not invertible, hence

$$\det\left(I - \text{diag}\left[\frac{\lambda - 1}{4j^2 - 1}\right] + \left[\frac{\theta_{j-k}}{4j^2 - 1}\right]\right) = 0.$$

Finally, by Lemma 4.5.0.7, the matrix $\left[\frac{\theta_{j-k}}{4j^2-1}\right] - \text{diag}\left[\frac{\lambda-1}{4j^2-1}\right]$ is trace class and so it then follows from Corollary 2.3.0.29 that

$$\det_2\left(I - \text{diag}\left[\frac{\lambda - 1}{4j^2 - 1}\right] + \left[\frac{\theta_{j-k}}{4j^2 - 1}\right]\right) = 0.$$

Similarly, using Proposition 4.5.0.6, λ belongs to the complementary series if and only if there exists $(b_j)_{j\in\mathbb{Z}} \in \ell^2$ such that the equation

$$b_j + \frac{\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k}{(1+2j)^2} = \frac{\lambda b_j}{(1+2j)^2}$$

holds for all $j \in \mathbb{Z}$. In matrix form this becomes

$$\left(I - \text{diag}\left[\frac{\lambda}{(1+2j)^2} + \left[\frac{\theta_{j-k}}{(1+2j)^2}\right]\right]\right) B = 0$$

where $B = [\ldots, b_{-1}, b_0, b_1, \ldots]^T \neq 0$. Again, since $B \neq 0$ it follows that

$$\det\left(I - \text{diag}\left[\frac{\lambda}{(1+2j)^2} + \left[\frac{\theta_{j-k}}{(1+2j)^2}\right]\right]\right) = 0.$$

By Lemma 4.5.0.7, the matrix $\left[\frac{\theta_{j-k}}{(1+2j)^2}\right] - \text{diag}\left[\frac{\lambda}{(1+2j)^2}\right]$ is trace class, hence by Corollary 2.3.0.29,

$$\det_2\left(I - \text{diag}\left[\frac{\lambda}{(1+2j)^2} + \left[\frac{\theta_{j-k}}{(1+2j)^2}\right]\right]\right) = 0.$$

■

Remark 4.5.0.9 *In [35], Magnus and Winkler consider*

$$\det\left(I + \left[\frac{\theta_{j-k}}{\lambda - (\xi + 2j)^2}\right]\right)$$

where ξ is a characteristic exponent. This determinant is meromorphic with respect to λ and has poles when $\lambda = (\xi + 2j)^2$.

The following proposition shows how the determinant conditions found in Proposition 4.5.0.8 can be simplified in the case of zero potential. We see that when $q = 0$, the periodic spectrum is given by $\{n^2 : n \in \mathbb{N}_0\}$.

Proposition 4.5.0.10 *Let q be the potential of Hill's equation. Given $q = 0$, λ belongs to the principal series if and only if*

$$\lambda \prod_{j=1}^{\infty} \left(1 - \frac{\lambda - 1}{4j^2 - 1}\right)^2 = 0.$$

Similarly, λ belongs to the complementary series if and only if

$$\prod_{j=0}^{\infty} \left[1 - \frac{\lambda}{(1 + 2j)^2}\right]^2 = 0.$$

Further, the periodic spectrum is given by $\{n^2 : n \in \mathbb{Z}\}$.

Proof. Suppose that Hill's equation has zero potential, then $\theta_j = 0$ for all $j \in \mathbb{Z}$. Therefore, by Proposition 4.5.0.8, λ lies in the principal series if and only if λ is a root of

$$\begin{aligned} 0 &= \det\left(I - \operatorname{diag}\left[\frac{\lambda - 1}{4j^2 - 1}\right]\right) \\ &= \prod_{j=-\infty}^{\infty} \left(1 - \frac{\lambda - 1}{4j^2 - 1}\right) \\ &= \lambda \prod_{j=1}^{\infty} \left(1 - \frac{\lambda - 1}{4j^2 - 1}\right)^2. \end{aligned} \tag{4.27}$$

Clearly (4.27) has roots when $\lambda = 4j^2$, thus the principal series is given by

$$\{4j^2 : j \in \mathbb{Z}\}. \tag{4.28}$$

Similarly, if $q = 0$, it follows from Proposition 4.5.0.8 that λ lies in the complementary series if and only if λ is a root of

$$\begin{aligned} 0 &= \det\left(I - \operatorname{diag}\left[\frac{\lambda}{(1 + 2j)^2}\right]\right) \\ &= \prod_{j=-\infty}^{\infty} \left[1 - \frac{\lambda}{(1 + 2j)^2}\right] \\ &= \prod_{j=0}^{\infty} \left[1 - \frac{\lambda}{(1 + 2j)^2}\right]^2. \end{aligned} \tag{4.29}$$

Clearly (4.29) has roots when $\lambda = (1 + 2j)^2$ thus giving the complementary series

$$\{(1 + 2j)^2 : j \in \mathbb{Z}\}. \tag{4.30}$$

We note that the products $\prod_{j=1}^{\infty}\left(1-\frac{\lambda-1}{4j^2-1}\right)^2$ and $\prod_{j=0}^{\infty}\left[1-\frac{\lambda}{(1+2j)^2}\right]^2$ are convergent and therefore not always zero. First, $\prod_{j=1}^{\infty}\left(1-\frac{\lambda-1}{4j^2-1}\right)^2$ is convergent if $\sum_{j=1}^{\infty}\left|\frac{1-\lambda}{4j^2-1}\right|$ is convergent. Clearly this holds by comparison with $\sum_{j=1}^{\infty}\frac{1}{j^2}$. Hence $\prod_{j=1}^{\infty}\left(1-\frac{\lambda-1}{4j^2-1}\right)^2$ is convergent. Since $\prod_{j=1}^{\infty}\left(1-\frac{\lambda-1}{4j^2-1}\right)^2$ is convergent, it follows that the zeros of $\lambda\prod_{j=1}^{\infty}\left(1-\frac{\lambda-1}{4j^2-1}\right)^2$ are exactly those as given by $\lambda=4j^2$. A similar argument shows that $\prod_{j=0}^{\infty}\left[1-\frac{\lambda}{(1+2j)^2}\right]^2$ is convergent and thus has zeros that are exactly those as given by $\lambda=(1+2j)^2$.

Finally we recall that the periodic spectrum is given by the union of the principal series and the complementary series, thus by (4.28) and (4.30), the periodic spectrum is given by

$$\{4j^2 : j\in\mathbb{Z}\}\cup\{(1+2j)^2 : j\in\mathbb{Z}\}=\{n^2 : n\in\mathbb{Z}\}.$$

■

The following example shows the reader why functions of order $\frac{1}{2}$ have zeros that grow like j^2.

Example 4.5.0.11

The function $\frac{\sin\sqrt{z}}{\sqrt{z}}$ has zeros when $z=\pi^2 j^2$.

In order to show that this statement is true we use the following formula that was proved by Euler:

$$\sin(\pi z)=\pi z\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right).$$

Write $\sqrt{z}=\frac{\pi\sqrt{z}}{\pi}$, then by the above formula we have

$$\begin{aligned}\sin\sqrt{z} &= \sin\left[\pi\left(\frac{\sqrt{z}}{\pi}\right)\right]\\ &= \pi\left(\frac{\sqrt{z}}{\pi}\right)\prod_{j=1}^{\infty}\left(1-\frac{\left(\frac{\sqrt{z}}{\pi}\right)^2}{j^2}\right)\\ &= \sqrt{z}\prod_{j=1}^{\infty}\left(1-\frac{z}{\pi^2 j^2}\right).\end{aligned}$$

The result follows immediately.

The remainder of this section shows how we can extend Hill's method to include determinant conditions that involve the use of linear systems. We shall introduce linear systems via the convolution operation. The idea is to write Hill's equation in terms of convolutions and then, using a known linear system, we can rewrite the equation of convolutions in terms of operators that are specified by the linear system. The result is a condition for λ being in the periodic spectrum based on linear systems.

Definition 4.5.0.12 *Let ρ_p be the continuous, π-periodic function defined by*

$$\rho_p(t)=\sum_{j=-\infty}^{\infty}\frac{e^{2ijt}}{4j^2-1}.$$

Similarly, let ρ_c be the continuous, 2π-periodic function defined by

$$\rho_c(t) = \sum_{j=-\infty}^{\infty} \frac{e^{i(1+2j)t}}{(1+2j)^2}.$$

Remark 4.5.0.13 *Note that the functions ρ_p and ρ_c are even since*

$$\begin{aligned} \rho_p(t) &= -1 + \sum_{j=1}^{\infty} \frac{e^{2ijt} + e^{-2ijt}}{4j^2 - 1}, \\ \rho_c(t) &= \sum_{j=0}^{\infty} \frac{e^{i(1+2j)t} + e^{-i(1+2j)t}}{(1+2j)^2}. \end{aligned}$$

Definition 4.5.0.14 *For α continuous, we define the* convolution *operation for the function ρ_p to be*

$$[\rho_p * \alpha](t) = \frac{1}{\pi} \int_0^{\pi} \rho_p(t-s)\alpha(s)\, ds.$$

Likewise, for α continuous, we define the convolution operation for the function ρ_c to be

$$[\rho_c * \alpha](t) = \frac{1}{2\pi} \int_0^{2\pi} \rho_c(t-s)\alpha(s)\, ds.$$

The following proposition gives conditions based on the convolution operation, for an eigenvalue to belong to the periodic spectrum.

Proposition 4.5.0.15 *Let λ be an eigenvalue of Hill's equation. Then λ lies in the principal series if and only if λ satisfies*

$$f + \rho_p * (qf) = (\lambda - 1)(\rho_p * f), \tag{4.31}$$

for some $f \in L^2[0, \pi]$. Similarly, λ lies in the complementary series if and only if λ satisfies

$$f + \rho_c * (qf) = \lambda(\rho_c * f) \tag{4.32}$$

for some $f \in L^2[0, 2\pi]$.

Proof. By Proposition 4.5.0.6, λ belongs to the principal series if and only if

$$b_j + \frac{\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k}{4j^2 - 1} = \frac{(\lambda - 1) b_j}{4j^2 - 1} \tag{4.33}$$

for all $j \in \mathbb{Z}$. The proof therefore reduces to showing the equivalence of (4.31) and (4.33). We begin by evaluating the convolutions separately. Recall that $\rho_p(t) = \sum_{j=-\infty}^{\infty} \frac{e^{2ijt}}{4j^2-1}$ and $f(t) = \sum_{j=-\infty}^{\infty} b_j e^{2ijt}$, thus

$$\begin{aligned} [\rho_p * f](t) &= \frac{1}{\pi} \int_0^{\pi} \rho_p(t-s) f(s)\, ds \\ &= \frac{1}{\pi} \int_0^{\pi} \left(\sum_{j=-\infty}^{\infty} \frac{e^{2ij(t-s)}}{4j^2 - 1} \right) \left(\sum_{k=-\infty}^{\infty} b_k e^{2iks} \right) ds \\ &= \frac{1}{\pi} \int_0^{\pi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{b_k}{4j^2 - 1} e^{2ij(t-s)+2iks}\, ds. \end{aligned}$$

Note that $e^{2ij(t-s)+2iks} = e^{2ijt}e^{2i(k-j)s}$, therefore

$$\begin{aligned}
[\rho_p * f](t) &= \frac{1}{\pi}\int_0^\pi \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{b_k e^{2ijt}}{4j^2-1} e^{2i(k-j)s}\,ds \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{b_k e^{2ijt}}{4j^2-1}\int_0^\pi \frac{e^{2i(k-j)s}}{\pi}\,ds.
\end{aligned}$$

Consider the sum

$$\sum_{k=-\infty}^{\infty} \frac{b_k e^{2ijt}}{4j^2-1}\int_0^\pi \frac{e^{2i(k-j)s}}{\pi}\,ds,$$

we have

$$\int_0^\pi \frac{e^{2i(k-j)s}}{\pi}\,ds = \begin{cases} 0 & \text{when } k \neq j, \\ 1 & \text{when } k = j, \end{cases}$$

and so

$$\sum_{k=-\infty}^{\infty} \frac{b_k e^{2ijt}}{4j^2-1}\int_0^\pi \frac{e^{2i(k-j)s}}{\pi}\,ds = \frac{b_j e^{2ijt}}{4j^2-1}.$$

Hence,

$$[\rho_p * f](t) = \sum_{j=-\infty}^{\infty} \frac{b_j}{4j^2-1} e^{2ijt}.$$

Similarly,

$$\begin{aligned}
[\rho_p * (qf)](t) &= \frac{1}{\pi}\int_0^\pi \rho_p(t-s)(qf)(s)\,ds \\
&= \frac{1}{\pi}\int_0^\pi \left(\sum_{j=-\infty}^{\infty} \frac{e^{2ij(t-s)}}{4j^2-1}\right)\left(\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \theta_{k-l} b_l e^{2iks}\right) ds \\
&= \frac{1}{\pi}\int_0^\pi \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{\sum_{l=-\infty}^{\infty} \theta_{k-l} b_l e^{2ijt}}{4j^2-1} e^{2i(k-j)s}\,ds \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{\sum_{l=-\infty}^{\infty} \theta_{k-l} b_l e^{2ijt}}{4j^2-1}\int_0^\pi \frac{e^{2i(k-j)s}}{\pi}\,ds \\
&= \sum_{j=-\infty}^{\infty} \frac{\sum_{l=-\infty}^{\infty} \theta_{j-l} b_l}{4j^2-1} e^{2ijt}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&f(t) + [\rho_p * (qf)](t) - (\lambda-1)(\rho_p * f)(t) \\
&\quad = \sum_{j=-\infty}^{\infty} b_j e^{2ijt} + \sum_{j=-\infty}^{\infty} \frac{\sum_{l=-\infty}^{\infty} \theta_{j-l} b_l}{4j^2-1} e^{2ijt} - (\lambda-1)\sum_{j=-\infty}^{\infty} \frac{b_j}{4j^2-1} e^{2ijt}
\end{aligned}$$

and the equivalence now follows from Proposition 4.5.0.6.

Now suppose that λ lies in the complementary series. By Proposition 4.5.0.6, λ belongs to the complementary series if and only if

$$b_j + \frac{\sum_{k=-\infty}^{\infty} \theta_{j-k} b_k}{(1+2j)^2} = \frac{\lambda b_j}{(1+2j)^2} \tag{4.34}$$

for all $j \in \mathbb{Z}$. Again the proof entails showing the equivalence of (4.32) and (4.34). As before, we evaluate the convolutions separately. Recall that since λ lies in the complementary series, $f(t) = \sum_{j=-\infty}^{\infty} b_j e^{i(1+2j)t}$. Now,

$$\begin{aligned}
[\rho_c * f](t) &= \frac{1}{2\pi}\int_0^{2\pi} \rho_c(t-s)f(s)\,ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{j=-\infty}^{\infty} \frac{e^{i(1+2j)(t-s)}}{(1+2j)^2}\right)\left(\sum_{k=-\infty}^{\infty} b_k e^{i(1+2k)s}\right) ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{b_k e^{i(1+2j)t}}{(1+2j)^2} e^{2i(k-j)s}\,ds \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{b_k e^{i(1+2j)t}}{(1+2j)^2}\int_0^{2\pi} \frac{e^{2i(k-j)s}}{2\pi}\,ds.
\end{aligned}$$

Since

$$\int_0^{2\pi} \frac{e^{2i(k-j)s}}{2\pi}\,ds = \begin{cases} 0 & \text{when } k \neq j, \\ 1 & \text{when } k = j, \end{cases}$$

it follows that

$$[\rho_c * f](t) = \sum_{j=-\infty}^{\infty} \frac{b_j}{(1+2j)^2} e^{i(1+2j)t}.$$

Similarly,

$$\begin{aligned}
[\rho_c * (qf)](t) &= \frac{1}{2\pi}\int_0^{2\pi} \rho_c(t-s)(qf)(s)\,ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{j=-\infty}^{\infty} \frac{e^{i(1+2j)(t-s)}}{(1+2j)^2}\right)\left(\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \theta_{k-l} b_l e^{i(1+2k)s}\right) ds \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{\sum_{l=-\infty}^{\infty} \theta_{k-l} b_l e^{i(1+2j)t}}{(1+2j)^2}\int_0^{2\pi} \frac{e^{2i(k-j)s}}{2\pi}\,ds \\
&= \sum_{j=-\infty}^{\infty} \frac{\sum_{l=-\infty}^{\infty} \theta_{j-l} b_l}{(1+2j)^2} e^{i(1+2j)t}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&f(t) + [\rho_c * (qf)](t) - \lambda(\rho_c * f)(t) \\
&= \sum_{j=-\infty}^{\infty} b_j e^{i(1+2j)t} + \sum_{j=-\infty}^{\infty} \frac{\sum_{l=-\infty}^{\infty} \theta_{j-l} b_l}{(1+2j)^2} e^{i(1+2j)t} - \lambda \sum_{j=-\infty}^{\infty} \frac{b_j}{(1+2j)^2} e^{i(1+2j)t}
\end{aligned}$$

and the equivalence now follows from Proposition 4.5.0.6. ■

We now have two forms of Hill's equation in terms of convolutions that will determine whether or not an eigenvalue lies in the periodic spectrum. We use these equations to give conditions involving linear systems. First, we write the convolutions as integrals as in the next lemma.

Lemma 4.5.0.16 *Let λ be an eigenvalue of Hill's equation. Then λ lies in the principal series if and only if λ satisfies the equation*

$$f(s) + \frac{1}{\pi}\int_0^{\pi} \rho_p(s-t)[q(t) + 1 - \lambda] f(t)\,dt = 0 \tag{4.35}$$

for some $f \in L^2[0,\pi]$. *Likewise,* λ *lies in the complementary series if and only if* λ *satisfies the equation*

$$f(s) + \frac{1}{2\pi}\int_0^{2\pi} \rho_c(s-t)[q(t)-\lambda]f(t)\,dt = 0 \tag{4.36}$$

for some $f \in L^2[0,2\pi]$.

Proof. By Proposition 4.5.0.15, λ lies in the principal series if and only if the equation

$$f(s) + [\rho_p * (qf)](s) - (\lambda - 1)[\rho_p * f](s) = 0 \tag{4.37}$$

holds for some $f \in L^2[0,\pi]$. We use Definition 4.5.0.14 to show the equivalence of (4.35) and (4.37). Thus we have,

$$\begin{aligned} [\rho_p * (qf)](s) &= \frac{1}{\pi}\int_0^{\pi} \rho_p(s-t)(qf)(t)\,dt, \\ [\rho_p * f](s) &= \frac{1}{\pi}\int_0^{\pi} \rho_p(s-t)f(t)\,dt. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= f(s) + [\rho_p * (qf)](s) - (\lambda-1)[\rho_p * f](s) \\ &= f(s) + \frac{1}{\pi}\int_0^{\pi} \rho_p(s-t)(qf)(t)\,dt - (\lambda-1)\frac{1}{\pi}\int_0^{\pi} \rho_p(s-t)f(t)\,dt \\ &= f(s) + \frac{1}{\pi}\int_0^{\pi} \rho_p(s-t)[q(t)+1-\lambda]f(t)\,dt \end{aligned}$$

as required.

Now let λ lie in the complementary series. By Proposition 4.5.0.15

$$f(s) + [\rho_c * (qf)](s) - \lambda[\rho_c * f](s) = 0 \tag{4.38}$$

for some $f \in L^2[0,2\pi]$. Again we must show the equivalence of (4.36) and (4.38). Note that

$$\begin{aligned} [\rho_c * (qf)](s) &= \frac{1}{2\pi}\int_0^{2\pi} \rho_c(s-t)(qf)(t)\,dt, \\ [\rho_c * f](s) &= \frac{1}{2\pi}\int_0^{2\pi} \rho_c(s-t)f(t)\,dt, \end{aligned}$$

and so

$$\begin{aligned} 0 &= f(s) + [\rho_c * (qf)](s) - \lambda[\rho_c * f](s) \\ &= f(s) + \frac{1}{2\pi}\int_0^{2\pi} \rho_c(s-t)(qf)(t)\,dt - \lambda\frac{1}{2\pi}\int_0^{2\pi} \rho_c(s-t)f(t)\,dt \\ &= f(s) + \frac{1}{2\pi}\int_0^{2\pi} \rho_c(s-t)[q(t)-\lambda]f(t)\,dt \end{aligned}$$

completing the proof. ∎

The convolution equations give several different operations which we can separately express in terms of a linear system. We therefore define two linear systems, one for the principal series and one for the complementary series, and write equations (4.31) and (4.32) in terms of these linear systems. First we define the system that we shall use when working with the principal series.

Definition 4.5.0.17 *Let $\mathbb{C}$ be the input space and output space and let $L^2[0,\pi]$ be the state space. Take $-A$ be the generator of the strongly continuous semigroup $\{T_t\}_{t\geq 0}$ where $T_t = e^{-tA}$. We introduce a new system, $(-A, B_p, C, M_p)$ given by the linear system $(-A, B_p, C)$ together with an additional operator, M_p. Thus we define the operators*

$$\begin{aligned} T_t &: L^2[0,\pi] \to L^2[0,\pi] \\ B_p &: \mathbb{C} \to L^2[0,\pi] \\ C &: \mathcal{D}_{[0,\pi]}(A) \to \mathbb{C} \\ M_p &: L^2[0,\pi] \to L^2[0,\pi] \end{aligned}$$

to be such that

$$\begin{aligned} T_t f(x) &= f(x+t) \\ B_p b &= \rho_p(x) b \\ Cf(x) &= f(0) \\ M_p f(x) &= [q(x) + 1 - \lambda] f(x). \end{aligned}$$

Likewise, we define a system for use with the complementary series.

Definition 4.5.0.18 *Let $\mathbb{C}$ be the input space and output space and let $L^2[0,2\pi]$ be the state space. Take $-A$ be the generator of the strongly continuous semigroup $\{T_t\}_{t\geq 0}$ where $T_t = e^{-tA}$. We introduce a new system, $(-A, B_c, C, M_c)$ given by the linear system $(-A, B_c, C)$ together with an additional operator, M_c. Thus we define the operators*

$$\begin{aligned} T_t &: L^2[0,2\pi] \to L^2[0,2\pi] \\ B_c &: \mathbb{C} \to L^2[0,2\pi] \\ C &: \mathcal{D}_{[0,2\pi]}(A) \to \mathbb{C} \\ M_c &: L^2[0,2\pi] \to L^2[0,2\pi] \end{aligned}$$

to be such that

$$\begin{aligned} T_t f(x) &= f(x+t) \\ B_c b &= \rho_c(x) b \\ Cf(x) &= f(0) \\ M_c f(x) &= [q(x) - \lambda] f(x). \end{aligned}$$

In the following definition we define the operators whose Carleman determinants have zeros belonging to either the principal series or the complementary series.

Definition 4.5.0.19 *Given the system, $(-A, B_p, C, M_p)$, define the operator $R_p : \mathcal{D}_{[0,\pi]}(A) \to L^2[0,\pi]$ to be*

$$R_p = \frac{1}{\pi} \int_0^{\pi} e^{-tA} B_p C e^{tA} M_p \, dt.$$

Similarly, given the system, $(-A, B_c, C, M_c)$, define the operator $R_c : \mathcal{D}_{[0,2\pi]}(A) \to L^2[0,2\pi]$ to be

$$R_c = \frac{1}{2\pi} \int_0^{2\pi} e^{-tA} B_c C e^{tA} M_c \, dt.$$

Note the similarity between the operators R_p and R_c defined above and the operator R_x that appeared in Chapter 3.

Defining the systems $(-A, B_p, C, M_p)$ and $(-A, B_c, C, M_c)$ enables us to write $R_p f$ and $R_c f$ as integrals. Since R_p and R_c are operators defined by linear systems, having $R_p f$ and $R_c f$ in the form of an integral provides the link between linear systems and the convolution equations (4.31) and (4.32). The following theorem shows that the operators R_p and R_c can be used to give suitable conditions which, when satisfied, tells us that an eigenvalue of Hill's equation lies in the periodic spectrum.

Theorem 4.5.0.20 *Let the operators R_p and R_c be as defined by Definition 4.5.0.19 and let λ be an eigenvalue of Hill's equation. Then λ belongs to the principal series if and only if there exists a non-zero $f \in L^2[0, \pi]$ such that*

$$[I + R_p]f(x) = 0,$$

where

$$R_p f(x) = \frac{1}{\pi} \int_0^\pi \rho_p(x - t)[q(t) + 1 - \lambda]f(t)\, dt.$$

Also, λ belongs to the complementary series if and only if there exists a non-zero $f \in L^2[0, 2\pi]$ such that

$$[I + R_c]f(x) = 0,$$

where

$$R_c f(x) = \frac{1}{2\pi} \int_0^{2\pi} \rho_c(x - t)[q(t) - \lambda]f(t)\, dt.$$

Proof. Our first task is to write $R_p f$ as an integral. Let x be the active variable and let $R_p = \frac{1}{\pi}\int_0^\pi e^{-tA} B_p C e^{tA} M_p\, dt$. We evaluate $R_p f(x)$ by applying each operator in turn. Thus

$$\begin{aligned}
R_p f(x) &= \frac{1}{\pi}\int_0^\pi e^{-tA} B_p C e^{tA} M_p f(x)\, dt \\
&= \frac{1}{\pi}\int_0^\pi e^{-tA} B_p C e^{tA} [q(x) + 1 - \lambda] f(x)\, dt \\
&= \frac{1}{\pi}\int_0^\pi e^{-tA} B_p C [q(x - t) + 1 - \lambda] f(x - t)\, dt \\
&= \frac{1}{\pi}\int_0^\pi e^{-tA} B_p [q(-t) + 1 - \lambda] f(-t)\, dt \\
&= \frac{1}{\pi}\int_0^\pi e^{-tA} \rho_p(x) [q(-t) + 1 - \lambda] f(-t)\, dt \\
&= \frac{1}{\pi}\int_0^\pi \rho_p(x + t) [q(-t) + 1 - \lambda] f(-t)\, dt.
\end{aligned}$$

We can further simplify $R_p f$ by noting that ρ_p, q and f are π-periodic. Hence,

$$R_p f(x) = \frac{1}{\pi} \int_0^\pi \rho_p(x - t)[q(t) + 1 - \lambda]f(t)\, dt.$$

Now, by Lemma 4.5.0.16, λ lies in the principal series if and only if

$$\begin{aligned}
0 &= f(x) + \frac{1}{\pi}\int_0^\pi \rho_p(x - t)[q(t) + 1 - \lambda] f(t)\, dt \\
&= f(x) + R_p f(x)
\end{aligned}$$

completing the first part of the proof.

Following the same method we write $R_c f$ as an integral. Let x be the active variable and let $R_c = \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} B_c C e^{tA} M_c \, dt$. We evaluate $R_c f(x)$ by applying each operator in turn. Thus

$$\begin{aligned}
R_c f(x) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} B_c C e^{tA} M_c f(x)\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} B_c C e^{tA} [q(x) - \lambda] f(x)\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} B_c C [q(x-t) - \lambda] f(x-t)\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} B_c [q(-t) - \lambda] f(-t)\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} \rho_c(x) [q(-t) - \lambda] f(-t)\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \rho_c(x+t) [q(-t) - \lambda] f(-t)\, dt.
\end{aligned}$$

As before, we can further simplify $R_c f$. Note that ρ_c and f are 2π-periodic and q is π-periodic we have

$$R_c f(x) = \frac{1}{2\pi}\int_0^{2\pi} \rho_c(x-t)[q(t) - \lambda] f(t)\, dt.$$

Now, by Lemma 4.5.0.16, λ lies in the complementary series if and only if

$$\begin{aligned}
0 &= f(x) + \frac{1}{2\pi}\int_0^{2\pi} \rho_c(x-t)[q(t) - \lambda] f(t)\, dt \\
&= f(x) + R_c f(x)
\end{aligned}$$

as required. ■

Remark 4.5.0.21 *Theorem 4.5.0.20 shows the dependence of R_p and R_c upon λ.*

We have therefore found conditions that tell us whether or not an eigenvalue lies in the periodic spectrum without having to solve Hill's equation. However, depending on the potential, q, the conditions given in Theorem 4.5.0.20 may not be so easy to calculate since the integrals $R_p f(x)$ and $R_c f(x)$ may be rather complicated. We therefore seek to simplify these conditions by first writing R_p and R_c as the product of two operators.

Definition 4.5.0.22 *Let * denote the adjoint. Define the operators $P_p : L^2[0,\pi] \to \mathcal{D}_{[0,\pi]}(A)$ and $P_c : L^2[0,2\pi] \to \mathcal{D}_{[0,2\pi]}(A)$ by*

$$\begin{aligned}
P_p f &= \frac{1}{\pi}\int_0^{\pi} M_p^* e^{tA^*} C^* f(t)\, dt, \\
P_c f &= \frac{1}{2\pi}\int_0^{2\pi} M_c^* e^{tA^*} C^* f(t)\, dt.
\end{aligned}$$

Also, define the operators $Q_p : L^2[0,\pi] \to L^2[0,\pi]$ and $Q_c : L^2[0,2\pi] \to L^2[0,2\pi]$ by

$$\begin{aligned}
Q_p f &= \frac{1}{\pi}\int_0^{\pi} e^{-tA} B_p f(t)\, dt, \\
Q_c f &= \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} B_c f(t)\, dt.
\end{aligned}$$

Proposition 4.5.0.23 *Let P_p, P_c, Q_p and Q_c be as in Definition 4.5.0.22. Then*

$$\begin{aligned} R_p &= Q_p P_p^*, \\ P_p^* Q_p &= \frac{1}{\pi} C e^{sA} M_p \int_0^{\pi} e^{-tA} B_p \, dt, \end{aligned}$$

where $P_p^* : \mathcal{D}_{[0,\pi]}(A) \to L^2[0,\pi]$ is given by

$$P_p^* f = C e^{tA} M_p f.$$

Also,

$$\begin{aligned} R_c &= Q_c P_c^*, \\ P_c^* Q_c &= \frac{1}{2\pi} C e^{sA} M_c \int_0^{2\pi} e^{-tA} B_c \, dt, \end{aligned}$$

where $P_c^* : \mathcal{D}_{[0,2\pi]}(A) \to L^2[0,2\pi]$ is given by

$$P_c^* f = C e^{tA} M_c f.$$

Proof. We first calculate P_p^*. By Definition 2.1.0.7 we know that if $T : H_1 \to H_2$ is an operator then T and its adjoint, T^* must satisfy the equation

$$\langle Tf, g \rangle_{H_1} = \langle f, T^* g \rangle_{H_2}. \tag{4.39}$$

Now, given $P_p f = \frac{1}{\pi} \int_0^{\pi} M_p^* e^{tA^*} C^* f(t) \, dt$ we have

$$\begin{aligned} \langle f, P_p^* g \rangle &= \langle P_p f, g \rangle \\ &= \left\langle \frac{1}{\pi} \int_0^{\pi} M_p^* e^{tA^*} C^* f(t) \, dt, g \right\rangle \\ &= \frac{1}{\pi} \int_0^{\pi} \langle M_p^* e^{tA^*} C^* f(t), g \rangle \, dt. \end{aligned}$$

By (4.39) we can write

$$\langle M_p^* e^{tA^*} C^* f(t), g \rangle = \langle f(t), C e^{tA} M_p g \rangle,$$

where $Ce^{tA} M_p : \mathcal{D}_{[0,\pi]}(A) \to \mathbb{C}$. Since both $f(t)$ and $Ce^{tA} M_p g$ are scalar-valued, it follows that

$$\langle f, P_p^* g \rangle = \frac{1}{\pi} \int_0^{\pi} f(t) \overline{C e^{tA} M_p g} \, dt.$$

Hence, $P_p^* g = C e^{tA} M_p g$. Now observe that

$$Q_p P_p^* = \frac{1}{\pi} \int_0^{\pi} e^{-tA} B_p C e^{tA} M_p \, dt$$

which agrees with Definition 4.5.0.19, thus $Q_p P_p^* = R_p$. Similarly,

$$P_p^* Q_p = C e^{sA} M_p \frac{1}{\pi} \int_0^{\pi} e^{-tA} B_p \, dt$$

as required.

In a similar fashion, we calculate P_c^*. Given $P_c f = \frac{1}{2\pi}\int_0^{2\pi} M_c^* e^{tA^*} C^* f(t)\, dt$ we again have

$$\begin{aligned}
\langle f, P_c^* g\rangle &= \langle P_c f, g\rangle \\
&= \left\langle \frac{1}{2\pi}\int_0^{2\pi} M_c^* e^{tA^*} C^* f(t)\, dt, g\right\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle M_c^* e^{tA^*} C^* f(t), g\rangle\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{Ce^{tA} M_c g}\, dt.
\end{aligned}$$

Hence, $P_c^* g = Ce^{tA} M_c g$. Finally we observe that

$$Q_c P_c^* = \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} B_c C e^{tA} M_c\, dt$$

which agrees with Definition 4.5.0.19, thus $Q_c P_c^* = R_c$. Similarly,

$$P_c^* Q_c = Ce^{sA} M_c \frac{1}{2\pi}\int_0^{2\pi} e^{-tA} B_c\, dt$$

completing the proof. ■

We would like to find determinant expressions for R_p and R_c which we will then simplify by relating them to $P_p^* Q_p$ and $P_c^* Q_c$ respectively. However, we first need to be sure that such determinants exist. This is the purpose of the following two lemmas.

Lemma 4.5.0.24 *The operators $R_p = Q_p P_p^*$, $R_c = Q_c P_c^*$, $P_p^* Q_p$ and $P_c^* Q_c$ are Hilbert–Schmidt.*

Proof. We use Proposition 2.1.1.9 to show first that the operators $Q_p P_p^*$ and $P_p^* Q_p$ are Hilbert–Schmidt. In order to do this we use the system $(-A, B_p, C, M_p)$ to write the operator Q_p in integral form. Let x be the active variable then,

$$\begin{aligned}
Q_p f &= \frac{1}{\pi}\int_0^{\pi} e^{-tA} B_p f(t)\, dt \\
&= \frac{1}{\pi}\int_0^{\pi} e^{-tA} \rho_p(x) f(t)\, dt \\
&= \frac{1}{\pi}\int_0^{\pi} \rho_p(x+t) f(t)\, dt.
\end{aligned}$$

As Q_p is now in the form of an integral operator with kernel $\rho_p(x+t)$, we can apply Proposition 2.1.1.6 to show that Q_p is Hilbert–Schmidt. Thus,

$$\begin{aligned}
\int_0^{\pi}\int_0^{\pi} |\rho_p(x+t)|^2\, dt\, dx &= \int_0^{\pi}\int_0^{\pi} \left| \sum_{j=-\infty}^{\infty} \frac{e^{2ij(x+t)}}{4j^2-1} \right|^2 dt\, dx \\
&\leq \int_0^{\pi}\int_0^{\pi} \left[\sum_{j=-\infty}^{\infty} \frac{1}{|4j^2-1|} \right]^2 dt\, dx.
\end{aligned}$$

By comparison with $\sum_{j=-\infty}^{\infty} \frac{1}{j^2}$ we find that $\sum_{j=-\infty}^{\infty} \frac{1}{|4j^2-1|}$ is convergent and so it follows that $\int_0^{\pi}\int_0^{\pi} |\rho_p(x+t)|^2\, dt\, dx$ is finite. Hence Q_p is Hilbert–Schmidt. Next we show that P_p^* is bounded. In Lemma 2.6.0.32 we saw that C is bounded and in Theorem 2.6.0.25 we saw that e^{tA} is bounded. Therefore, both operators have an upper bound given by their operator norm.

Also, the operator M_p is bounded. To see this first note that by Definition 4.5.0.4, q is twice continuously differentiable and π-periodic, therefore q is bounded on $[0, \pi]$. Let $|q| \leq C'$ for some constant C'. Then, using the triangle inequality

$$\begin{aligned}
\|M_p f\|^2_{L^2[0,\pi]} &= \int_0^\pi |[q(x) + 1 - \lambda]f(x)|^2 \, dx \\
&\leq \int_0^\pi (|q(x)| + |1 - \lambda|)^2 \, |f(x)|^2 \, dx \\
&\leq (C' + |1 - \lambda|)^2 \int_0^\pi |f(x)|^2 \, dx \\
&= (C' + |1 - \lambda|)^2 \, \|f\|^2_{L^2[0,\pi]} \, .
\end{aligned}$$

Hence M_p is bounded. It now follows that

$$\begin{aligned}
\left\|P_p^* f\right\|_{L^2[0,\pi]} &= \left\|Ce^{tA} M_p f\right\|_{L^2[0,\pi]} \\
&\leq \|C\|_{\text{op}} \left\|e^{tA}\right\|_{\text{op}} \|M_p\|_{\text{op}} \|f\|_{L^2[0,\pi]}
\end{aligned}$$

and so P_p^* is bounded. Hence, by Proposition 2.1.1.9, both $Q_p P_p^*$ and $P_p^* Q_p$ are Hilbert–Schmidt.

In the same manner we show that $Q_c P_c^*$ and $P_c^* Q_c$ are Hilbert–Schmidt. Given the linear system $(-A, B_c, C, M_c)$ we write Q_c as an integral operator. Let x be the active variable, then

$$\begin{aligned}
Q_c f &= \frac{1}{2\pi} \int_0^{2\pi} e^{-tA} B_c f(t) \, dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \rho_c(x + t) f(t) \, dt
\end{aligned}$$

and Q_c has kernel $\rho_c(x + t)$. Now,

$$\begin{aligned}
\int_0^{2\pi} \int_0^{2\pi} |\rho_c(x + t)|^2 \, dt \, dx &= \int_0^{2\pi} \int_0^{2\pi} \left| \sum_{j=-\infty}^{\infty} \frac{e^{i(1+2j)(x+t)}}{(1 + 2j)^2} \right|^2 dt \, dx \\
&\leq \int_0^{2\pi} \int_0^{2\pi} \left[\sum_{j=-\infty}^{\infty} \frac{1}{(1 + 2j)^2} \right]^2 dt \, dx \\
&< \infty
\end{aligned}$$

since $\sum_{j=-\infty}^{\infty} \frac{1}{(1+2j)^2}$ is convergent by comparison with $\sum_{j=-\infty}^{\infty} \frac{1}{j^2}$. Thus Q_c is Hilbert–Schmidt. It remains to show that P_c^* is bounded. First note that since q is twice continuously differentiable and π-periodic, it is bounded on $[0, \pi]$ and hence on $[0, 2\pi]$. Let $|q| \leq C'$ for some constant C'. Again, by the triangle inequality

$$\begin{aligned}
\|M_c f\|^2_{L^2[0,2\pi]} &= \int_0^{2\pi} |[q(x) - \lambda]f(x)|^2 \, dx \\
&\leq (C' + |\lambda|)^2 \, \|f\|^2_{L^2[0,2\pi]} \, ,
\end{aligned}$$

thus M_c is bounded. So it follows that

$$\begin{aligned}
\|P_c^* f\|_{L^2[0,2\pi]} &= \left\|Ce^{tA} M_c f\right\|_{L^2[0,2\pi]} \\
&\leq \|C\|_{\text{op}} \left\|e^{tA}\right\|_{\text{op}} \|M_c\|_{\text{op}} \|f\|_{L^2[0,2\pi]}
\end{aligned}$$

hence P_c^* is bounded. Finally, by Proposition 2.1.1.9, $Q_c P_c^*$ and $P_c^* Q_c$ are Hilbert–Schmidt. ■

Since the operators R_p and R_c are Hilbert–Schmidt, we now know that the Carleman determinants, $\det_2(I + R_p)$ and $\det_2(I + R_c)$ exist. The following theorem provides alternative formulae for $\det_2(I+R_p)$ and $\det_2(I+R_c)$ that are somewhat easier to evaluate. It also provides further conditions that determine if an eigenvalue belongs to the periodic spectrum.

Theorem 4.5.0.25 *Let P_p, P_c, Q_p and Q_c be as given in Definition 4.5.0.22 and R_p and R_c as in Definition 4.5.0.19. Then*

$$\begin{aligned} \det_2(I + R_p) &= \det_2(I + P_p^* Q_p), \\ \det_2(I + R_c) &= \det_2(I + P_c^* Q_c). \end{aligned}$$

Furthermore, if λ is an eigenvalue of Hill's equation then λ belongs to the principal series if and only if $\det_2(I + R_p) = 0$. Likewise, λ belongs to the complementary series if and only if $\det_2(I + R_c) = 0$.

Proof. By Proposition 4.5.0.23 we have $R_p = Q_p P_p^*$ and since R_p is Hilbert–Schmidt,

$$\det_2(I + R_p) = \det_2(I + Q_p P_p^*).$$

It follows from Sylvester's Determinant Theorem 2.3.0.30 that

$$\det_2(I + Q_p P_p^*) = \det_2(I + P_p^* Q_p),$$

proving the first part of the result for the principal series. Similarly for the complementary series, we find that since $R_c = Q_c P_c^*$ is Hilbert–Schmidt,

$$\det_2(I + R_c) = \det_2(I + Q_c P_c^*).$$

Sylvester's Determinant Theorem 2.3.0.30 now gives

$$\det_2(I + Q_c P_c^*) = \det_2(I + P_c^* Q_c).$$

The second part of the theorem follows immediately from Theorem 4.5.0.20. ■

In the following corollary we use the systems $(-A, B_p, C, M_p)$ and $(-A, B_c, C, M_c)$ to write $P_p^* Q_p f$ and $P_c^* Q_c f$, respectively, as integrals. This should make it clear that the conditions found in Theorem 4.5.0.25 are much easier to evaluate than the conditions found in Theorem 4.5.0.20.

Proposition 4.5.0.26 *Let P_p, P_c, Q_p and Q_c be as defined in Definition 4.5.0.22 and let R_p and R_c be as defined in Definition 4.5.0.19. Then*

$$\begin{aligned} P_p^* Q_p f &= \frac{1}{\pi}[q(-s) + 1 - \lambda] \int_0^{\pi} \rho_p(s-t) f(t)\, dt, \\ P_c^* Q_c f &= \frac{1}{2\pi}[q(-s) - \lambda] \int_0^{2\pi} \rho_c(s-t) f(t)\, dt. \end{aligned}$$

Proof. Let x be the active variable. In the proof of Lemma 4.5.0.24 we saw that

$$Q_p f = \frac{1}{\pi} \int_0^{\pi} \rho_p(x+t) f(t)\, dt.$$

We continue from this point, thus

$$\begin{aligned}
P_p^* Q_p f &= P_p^* \frac{1}{\pi}\int_0^{\pi} \rho_p(x+t) f(t)\, dt \\
&= \frac{1}{\pi} C e^{sA} M_p \int_0^{\pi} \rho_p(x+t) f(t)\, dt \\
&= \frac{1}{\pi} C e^{sA} [q(x)+1-\lambda] \int_0^{\pi} \rho_p(x+t) f(t)\, dt \\
&= \frac{1}{\pi} C [q(x-s)+1-\lambda] \int_0^{\pi} \rho_p(x-s+t) f(t)\, dt \\
&= \frac{1}{\pi} [q(-s)+1-\lambda] \int_0^{\pi} \rho_p(t-s) f(t)\, dt.
\end{aligned}$$

By Remark 4.5.0.13, ρ_p is even and so $\rho_p(t-s) = \rho_p(s-t)$. The first part of the result now follows.

Similarly, given x is the active variable we have

$$Q_c f = \frac{1}{2\pi}\int_0^{2\pi} \rho_c(x+t) f(t)\, dt$$

by the proof of Lemma 4.5.0.24. Thus,

$$\begin{aligned}
P_c^* Q_c f &= P_c^* \frac{1}{2\pi}\int_0^{2\pi} \rho_c(x+t) f(t)\, dt \\
&= \frac{1}{2\pi} C e^{sA} M_c \int_0^{2\pi} \rho_c(x+t) f(t)\, dt \\
&= \frac{1}{2\pi} C e^{sA} [q(x)-\lambda] \int_0^{2\pi} \rho_c(x+t) f(t)\, dt \\
&= \frac{1}{2\pi} C [q(x-s)-\lambda] \int_0^{2\pi} \rho_c(x-s+t) f(t)\, dt \\
&= \frac{1}{2\pi} [q(-s)-\lambda] \int_0^{2\pi} \rho_c(t-s) f(t)\, dt.
\end{aligned}$$

The result now follows from Remark 4.5.0.13 since ρ_c is even. ■

From the previous proposition it can be seen that it is easier to evaluate $P_p^* Q_p f$ and $P_c^* Q_c f$ since in both cases, the potential lies outside the integral and the integrand contains only two functions. This is in contrast to $R_p f$ and $R_c f$ whose integrands contain three functions, one of which is the potential.

Definition 4.5.0.27 *Let M_0 and M_1 be multiplication operators such that*

$$\begin{aligned}
M_0 &= q(x) \\
M_1 &= q(x)+1.
\end{aligned}$$

In the case of the principal series we constructed the multiplication operator, $M_p = q(x) + 1 - \lambda$. Using Definition 4.5.0.27 we can write $M_p = M_1 - \lambda I$. Similarly, in the case of the complementary series we have the multiplication operator, $M_c = q(x) - \lambda$ which we may write as $M_c = M_0 - \lambda I$. These observations allow us to create some further identities based on $\det_2(I + R_p)$ and $\det_2(I + R_c)$.

Definition 4.5.0.28 *Let * denote the adjoint. Define the operators $P_1 : L^2[0,\pi] \to \mathcal{D}_{[0,\pi]}(A)$ and $S_p : L^2[0,\pi] \to \mathcal{D}_{[0,\pi]}(A)$ by*

$$\begin{aligned} P_1 &= \frac{1}{\pi}\int_0^\pi M_1^* e^{tA^*} C^*\, dt; \\ S_p &= \frac{1}{\pi}\int_0^\pi e^{tA^*} C^*\, dt. \end{aligned}$$

Similarly, define the operators $P_0 : L^2[0,2\pi] \to \mathcal{D}_{[0,2\pi]}(A)$ and $S_c : L^2[0,2\pi] \to \mathcal{D}_{[0,2\pi]}(A)$ by

$$\begin{aligned} P_0 &= \frac{1}{2\pi}\int_0^{2\pi} M_0^* e^{tA^*} C^*\, dt; \\ S_c &= \frac{1}{2\pi}\int_0^{2\pi} e^{tA^*} C^*\, dt. \end{aligned}$$

Lemma 4.5.0.29 *Let P_1, P_0, S_p and S_c be as given in Definition 4.5.0.28. Then*

$$\begin{aligned} P_p &= P_1 - \bar{\lambda} S_p; \\ P_p^* &= P_1^* - \lambda S_p^* \end{aligned}$$

where $P_1^* = Ce^{tA}M_1$ and $S_p^* = Ce^{tA}$. Similarly,

$$\begin{aligned} P_c &= P_0 - \bar{\lambda} S_c; \\ P_c^* &= P_0^* - \lambda S_c^* \end{aligned}$$

where $P_0^* = Ce^{tA}M_0$ and $S_c^* = Ce^{tA}$.

Proof. By Definition 4.5.0.22 we have

$$P_p = \frac{1}{\pi}\int_0^\pi M_p^* e^{tA^*} C^*\, dt,$$

where $M_p = M_1 - \lambda I$. We calculate the adjoint of M_p using Proposition 2.1.0.8, thus

$$\begin{aligned} M_p^* &= [M_1 - \lambda I]^* \\ &= M_1^* - \bar{\lambda} I. \end{aligned}$$

It follows that

$$\begin{aligned} P_p &= \frac{1}{\pi}\int_0^\pi M_1^* e^{tA^*} C^*\, dt - \bar{\lambda}\frac{1}{\pi}\int_0^\pi e^{tA^*} C^*\, dt \\ &= P_1 - \bar{\lambda} S_p \end{aligned}$$

as required. Applying Proposition 2.1.0.8 again to P_p now gives

$$\begin{aligned} P_p^* &= [P_1 - \bar{\lambda} S_p]^* \\ &= P_1^* - \lambda S_p^*. \end{aligned}$$

We calculate the adjoint's of P_1 and S_p using Definition 2.1.0.7. Thus,

$$\begin{aligned} \langle f, P_1^* g\rangle &= \langle P_1 f, g\rangle \\ &= \left\langle \frac{1}{\pi}\int_0^\pi M_1^* e^{tA^*} C^* f(t)\, dt, g \right\rangle \\ &= \frac{1}{\pi}\int_0^\pi \langle M_1^* e^{tA^*} C^* f(t), g\rangle\, dt. \end{aligned}$$

Using Definition 2.1.0.7 again we can write

$$\langle M_1^* e^{tA^*} C^* f(t), g\rangle = \langle f(t), Ce^{tA} M_1 g\rangle,$$

where $Ce^{tA}M_1 : \mathcal{D}_{[0,\pi]}(A) \to \mathbb{C}$. Since both $f(t)$ and $Ce^{tA}M_1 g$ are scalar-valued, it follows that

$$\langle f, P_1^* g\rangle = \frac{1}{\pi}\int_0^{\pi} f(t)\overline{Ce^{tA}M_1 g}\, dt.$$

Hence $P_1^* g = Ce^{tA}M_1 g$ as required. Similarly, given $S_p f = \frac{1}{\pi}\int_0^{\pi} e^{tA^*}C^* f(t)\, dt$ we have

$$\begin{aligned}
\langle f, S_p^* g\rangle &= \langle S_p f, g\rangle \\
&= \left\langle \frac{1}{\pi}\int_0^{\pi} e^{tA^*}C^* f(t)\, dt, g\right\rangle \\
&= \frac{1}{\pi}\int_0^{\pi} \langle e^{tA^*}C^* f(t), g\rangle\, dt \\
&= \frac{1}{\pi}\int_0^{\pi} \langle f(t), Ce^{tA} g\rangle\, dt \\
&= \frac{1}{\pi}\int_0^{\pi} f(t)\overline{Ce^{tA} g}\, dt.
\end{aligned}$$

Hence $S_p^* g = Ce^{tA} g$ as required.

Likewise, by Definition 4.5.0.22

$$P_c = \frac{1}{2\pi}\int_0^{2\pi} M_c^* e^{tA^*} C^*\, dt,$$

where $M_c = M_0 - \lambda I$. We therefore have

$$M_c^* = M_0^* - \bar{\lambda} I$$

and so

$$\begin{aligned}
P_c &= \frac{1}{2\pi}\int_0^{2\pi} M_0^* e^{tA^*} C^*\, dt - \bar{\lambda}\frac{1}{2\pi}\int_0^{2\pi} e^{tA^*} C^*\, dt \\
&= P_0 - \bar{\lambda} S_c.
\end{aligned}$$

Furthermore,

$$P_c^* = P_0^* - \lambda S_c^*$$

where the adjoint's of P_0 and S_c are calculated using Definition 2.1.0.7. First we have

$$\begin{aligned}
\langle f, P_0^* g\rangle &= \langle P_0 f, g\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle M_0^* e^{tA^*} C^* f(t), g\rangle\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{Ce^{tA} M_0 g}\, dt,
\end{aligned}$$

giving $P_0^* g = Ce^{tA} M_0 g$ as required. Lastly, as before, given $S_c f = \frac{1}{2\pi}\int_0^{2\pi} e^{tA^*} C^* f(t)\, dt$ we have

$$\begin{aligned}
\langle f, S_c^* g\rangle &= \langle S_c f, g\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{Ce^{tA} g}\, dt.
\end{aligned}$$

Hence, $S_c^* g = Ce^{tA} g$ as required. ∎

Remark 4.5.0.30 *Note that if $\lambda = 0$ then $M_p = M_1$. Further, if $q = 0$ then M_1 denotes multiplication by 1, hence, $P_p = S_p$.*

The final theorem of this section provides yet more conditions involving determinants for which an eigenvalue of Hill's equation will belong to the periodic spectrum.

Theorem 4.5.0.31 *Let R_p and R_c be as given in Definition 4.5.0.19 and let Q_p and Q_c be as defined in Definition 4.5.0.22. Also let P_1^*, P_0^*, S_p^* and S_c^* be as defined in Lemma 4.5.0.29. Then*

$$\begin{aligned} R_p &= Q_p\left(P_1^* - \lambda S_p^*\right); \\ R_c &= Q_c\left(P_0^* - \lambda S_c^*\right). \end{aligned}$$

Furthermore, an eigenvalue, λ of Hill's equation belongs to the principal series if and only if

$$\det{}_2\left(I + P_1^* Q_p - \lambda S_p^* Q_p\right) = 0.$$

Similarly, λ belongs to the complementary series if and only if

$$\det{}_2\left(I + P_0^* Q_c - \lambda S_c^* Q_c\right) = 0.$$

Proof. By Proposition 4.5.0.23 we have $R_p = Q_p P_p^*$ and $R_c = Q_c P_c^*$. Also, by Lemma 4.5.0.29, $P_p^* = P_1^* - \lambda S_p^*$ and $P_c^* = P_0^* - \lambda S_c^*$. Therefore,

$$\begin{aligned} R_p &= Q_p\left(P_1^* - \lambda S_p^*\right); \\ R_c &= Q_c\left(P_0^* - \lambda S_c^*\right) \end{aligned}$$

giving the first part of the result.

To see the second part, note that by Theorem 4.5.0.25, λ belongs to the principal series if and only if $\det_2(I + R_p) = 0$. So, by the first part of this theorem and Sylvester's Determinant Theorem 2.3.0.30, λ lies in the principal series if and only if

$$\begin{aligned} 0 &= \det{}_2\left[I + Q_p\left(P_1^* - \lambda S_p^*\right)\right] \\ &= \det{}_2\left(I + P_1^* Q_p - \lambda S_p^* Q_p\right). \end{aligned}$$

The same argument shows that λ belongs to the complementary series if and only if

$$\begin{aligned} 0 &= \det{}_2\left(I + R_c\right) \\ &= \det{}_2\left[I + Q_c\left(P_0^* - \lambda S_c^*\right)\right] \\ &= \det{}_2\left(I + P_0^* Q_c - \lambda S_c^* Q_c\right). \end{aligned}$$

■

4.6 The Construction of the Potential

The inverse spectral problem is concerned with reconstructing the potential given that the spectrum is known. In this section we consider a problem similar to that of the inverse spectral problem in that we ultimately want to reconstruct the potential. We shall look at how to construct the potential given that a linear system, $(-A, B, C)$ and the scattering function, ϕ are known. By Corollary 3.3.0.48, we know that given a linear system, $(-A, B_r, C)$ and scattering function, $\phi(x) = C\left(e^{-xA} + e^{xA}\right) B_r$, if

$$T(x,y) = -C\left(e^{-xA} + e^{xA}\right)\left[I + \mu R_x\right]^{-1}\left(e^{-yA} + e^{yA}\right) B_r$$

then T satisfies the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) T(x,y) = q(x) T(x,y)$$

where $q(x) = 4\mu \frac{d}{dx} T(x,x)$. This information provides a way in which we can calculate the potential. If the linear system, $(-A, B_r, C)$ and the scattering function, ϕ are known then we can calculate $T(x,y)$ and hence $q(x)$.

The calculation of the potential, q depends on calculating the operator T, given a specific linear system, $(-A, B_r, C)$. However, the calculation of $T(x,y)$ depends on R_x, therefore we first evaluate R_x.

For the calculations that follow, we ask the reader to recall that when using the linear system $(-A, B_r, C)$ as defined by Example 2.6, the scattering function, ϕ satisfies the relation

$$\phi(x) = \psi(x) + \psi(-x)$$

where $\psi \in \mathcal{D}_{[a,b]}(A)$ is absolutely continuous. This formula was given in Proposition 3.1.0.39. It should now be clear where the function ψ, that appears in the following calculations, comes from.

Proposition 4.6.0.32 *Let $(-A, B_r, C)$ be the linear system defined in Example 2.6 and let ϕ be the scattering function in Definition 3.1.0.37. Then, the operator R_x defined by Definition 3.3.0.44 has the form*

$$R_x f(t) = 2\int_{-x}^{x} \left[\psi(t+z) + \psi(t-z)\right] f(z)\, dz.$$

Proof. Let ϕ be the scattering function defined by Definition 3.1.0.37 and let

$$R_x = \int_{-x}^{x} \left(e^{-zA} + e^{zA}\right) B_r C \left(e^{-zA} + e^{zA}\right)\, dz.$$

We apply the linear system, $(-A, B_r, C)$ to a function, f. Let t be the variable then

$$\begin{aligned}
R_x f(t) &= \int_{-x}^{x} \left(e^{-zA} + e^{zA}\right) B_r C \left(e^{-zA} + e^{zA}\right) f(t)\, dz \\
&= \int_{-x}^{x} \left(e^{-zA} + e^{zA}\right) B_r C[f(t+z) + f(t-z)]\, dz \\
&= \int_{-x}^{x} \left(e^{-zA} + e^{zA}\right) B_r[f(z) + f(-z)]\, dz \\
&= \int_{-x}^{x} \left(e^{-zA} + e^{zA}\right) \psi(t)[f(z) + f(-z)]\, dz \\
&= \int_{-x}^{x} [\psi(t+z) + \psi(t-z)][f(z) + f(-z)]\, dz.
\end{aligned}$$

Finally we note that the above simplifies to give

$$R_x f(t) = 2\int_{-x}^{x} \left[\psi(t+z) + \psi(t-z)\right] f(z)\, dz$$

as required. ■

Proposition 4.6.0.33 *Suppose that $x \in [-2\pi, 2\pi]$. The operator $R_x : L^2[-2\pi, 2\pi] \to L^2[-2\pi, 2\pi]$ is Hilbert–Schmidt and therefore has a Carleman determinant.*

Proof. We use Definition 2.1.0.6 to rewrite R_x as an integral operator with a kernel. It is then straightforward to apply Proposition 2.1.1.6 to show that R_x is Hilbert–Schmidt. Now,

$$\begin{aligned} R_x f(t) &= 2\int_{-x}^{x} [\psi(t+z) + \psi(t-z)]\, f(z)\, dz \\ &= 2\int_{-2\pi}^{2\pi} \mathbb{I}_{(-x,x)}(z)\, [\psi(t+z) + \psi(t-z)]\, f(z)\, dz, \end{aligned}$$

so by Definition 2.1.0.6, R_x has kernel $k(t,z) = 2\mathbb{I}_{(-x,x)}(z)[\psi(t+z) + \psi(t-z)]$. Thus, by Proposition 2.1.1.6,

$$\begin{aligned} \int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi} |k(t,z)|^2\, dz\, dt &= \int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi} \left|2\mathbb{I}_{(-x,x)}(z)[\psi(t+z) + \psi(t-z)]\right|^2\, dz\, dt \\ &= 4\int_{-2\pi}^{2\pi}\int_{-x}^{x} |\psi(t+z) + \psi(t-z)|^2\, dz\, dt. \end{aligned}$$

Observe that

$$\begin{aligned} |\psi(t+z) + \psi(t-z)|^2 &= |\psi(t+z)|^2 + 2\mathrm{Re}\left(\psi(t+z)\overline{\psi(t-z)}\right) + |\psi(t-z)|^2 \\ &\le |\psi(t+z)|^2 + 2\,|\psi(t+z)|\,|\psi(t-z)| + |\psi(t-z)|^2 \\ &= (|\psi(t+z)| + |\psi(t-z)|)^2 \\ &\le 4\left(|\psi(t+z)|^2 + |\psi(t-z)|^2\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi} |k(t,z)|^2\, dz\, dt &\le 16\int_{-2\pi}^{2\pi}\int_{-x}^{x} \left(|\psi(t+z)|^2 + |\psi(t-z)|^2\right) dz\, dt \\ &\le 32\,\|\psi\|^2_{L^2(-4\pi,4\pi)} \int_{-2\pi}^{2\pi} 1\, dt \\ &= 128\pi\,\|\psi\|^2_{L^2(-4\pi,4\pi)} \\ &< \infty, \end{aligned}$$

hence R_x is Hilbert–Schmidt. By Definition 2.3.0.27 R_x has a Carleman determinant. ■

So far we have calculated $R_x f$ but the operator T involves the term $[I + \mu R_x]^{-1}$. We therefore proceed to find $[I + \mu R_x]^{-1} f$. Note that if we expand $[I + \mu R_x]^{-1}$ we have

$$[I + \mu R_x]^{-1} = I - \mu R_x + \mu^2 R_x^2 - \mu^3 R_x^3 + \dots. \tag{4.40}$$

This suggests that we should calculate R_x^n for $n = 1, 2, \dots$.

Theorem 4.6.0.34 *Let $(-A, B_r, C)$ be the linear system given in Example 2.6 and let ϕ be the scattering function defined in Definition 3.1.0.37. Let R_x be as in Definition 3.3.0.44 then,*

$$\begin{aligned} R_x^n f(t) = {} & 2^n \int_{-x}^{x}\cdots\int_{-x}^{x} [\psi(t+x_1) + \psi(t-x_1)] \left(\prod_{j=1}^{n-1} [\psi(x_j + x_{j+1}) + \psi(x_j - x_{j+1})]\right) \\ & f(x_n)\, dx_1 \dots dx_n. \end{aligned}$$

Furthermore,

$$\begin{aligned}[I+\mu R_x]^{-1}f(t) &= f(t) \\ &\quad +\sum_{n=1}^{\infty}(-2\mu)^n \int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(t+x_1)+\psi(t-x_1)] \\ &\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right) f(x_n)\,dx_1\dots dx_n\end{aligned}$$

defines a convergent series for $|\mu|\,\|R_x\|<1$.

Proof. Given, the linear system $(-A,B_r,C)$ and scattering function, ϕ, it follows from Proposition 4.6.0.32 that

$$R_xf(t)=2\int_{-x}^{x}[\psi(t+z)+\psi(t-z)]\,f(z)\,dz.$$

Therefore,

$$\begin{aligned}R_x^2f(t) &= 2\int_{-x}^{x}[\psi(t+z)+\psi(t-z)]\,R_xf(z)\,dz \\ &= 2^2\int_{-x}^{x}[\psi(t+z)+\psi(t-z)]\int_{-x}^{x}[\psi(z+w)+\psi(z-w)]f(w)\,dw\,dz \\ &= 2^2\int_{-x}^{x}\int_{-x}^{x}[\psi(t+z)+\psi(t-z)][\psi(z+w)+\psi(z-w)]f(w)\,dw\,dz.\end{aligned}$$

It is then easily seen that

$$\begin{aligned}R_x^nf(t) &= 2^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(t+x_1)+\psi(t-x_1)]\left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right) \\ &\quad f(x_n)\,dx_1\dots dx_n.\end{aligned}$$

Substituting this information into (4.40) we obtain

$$\begin{aligned}[I+\mu R_x]^{-1}f(t) &= f(t)+\sum_{n=1}^{\infty}(-\mu)^nR_x^nf(t) \\ &= f(t) \\ &\quad +\sum_{n=1}^{\infty}(-2\mu)^n \int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(t+x_1)+\psi(t-x_1)] \\ &\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right) f(x_n)\,dx_1\dots dx_n\end{aligned}$$

as required. ■

We can now provide a formula for the function $T(x,y)$ in terms of a linear system.

Lemma 4.6.0.35 *Let* $(-A,B_r,C)$ *be the linear system as stated in Example 2.6 and let* ϕ *be the scattering function defined in Definition 3.1.0.37. If*

$$T(x,y)=-C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}\left(e^{-yA}+e^{yA}\right)B_r$$

then

$$
\begin{aligned}
T(x,y) &= -[\psi(x+y)+\psi(x-y)+\psi(-x+y)+\psi(-x-y)] \\
&\quad -\sum_{n=1}^{\infty}(-2\mu)^n \int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\
&\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+y)+\psi(x_n-y)]\,dx_1\ldots dx_n.
\end{aligned}
$$

Proof. Let $(-A, B_r, C)$ be the linear system as stated in Example 2.6 and let ϕ be the scattering function defined in Definition 3.1.0.37. Given $T(x,y)$, we apply the linear system $(-A, B_r, C)$ directly to obtain a formula for $T(x,y)$. Let t be the variable and b a scalar, then

$$
\begin{aligned}
T(x,y)b &= -C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}\left(e^{-yA}+e^{yA}\right)B_r b \\
&= -C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}\left(e^{-yA}+e^{yA}\right)\psi(t)b \\
&= -C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}[\psi(t+y)+\psi(t-y)]b.
\end{aligned}
$$

We use Lemma 4.6.0.34 to evaluate $[I+\mu R_x]^{-1}[\psi(t+y)+\psi(t-y)]b$, thus,

$$
\begin{aligned}
&T(x,y)b \\
&\quad = -C\left(e^{-xA}+e^{xA}\right)\{[\psi(t+y)+\psi(t-y)]b \\
&\qquad +\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(t+x_1)+\psi(t-x_1)]\left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right) \\
&\qquad [\psi(x_n+y)+\psi(x_n-y)]b\,dx_1\ldots dx_n\} \\
&\quad = -C\{[\psi(t+x+y)+\psi(t+x-y)]b \\
&\qquad +\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(t+x+x_1)+\psi(t+x-x_1)] \\
&\qquad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+y)+\psi(x_n-y)]b\,dx_1\ldots dx_n \\
&\qquad +[\psi(t-x+y)+\psi(t-x-y)]b+\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(t-x+x_1)+\psi(t-x-x_1)] \\
&\qquad \left.\left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+y)+\psi(x_n-y)]b\,dx_1\ldots dx_n\right\}.
\end{aligned}
$$

Before continuing with our calculation, we simplify the above line. This gives

$$
\begin{aligned}
&T(x,y)b \\
&\quad = -C\{[\psi(t+x+y)+\psi(t+x-y)+\psi(t-x+y)+\psi(t-x-y)]b \\
&\qquad +\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(t+x+x_1)+\psi(t+x-x_1)+\psi(t-x+x_1)+\psi(t-x-x_1)] \\
&\qquad \left.\left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+y)+\psi(x_n-y)]b\,dx_1\ldots dx_n\right\}.
\end{aligned}
$$

Finally, applying the operator C completes the calculation. We thus have,

$$\begin{aligned} T(x,y)b &= -[\psi(x+y)+\psi(x-y)+\psi(-x+y)+\psi(-x-y)]b \\ &\quad -\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\ &\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+y)+\psi(x_n-y)]b\,dx_1\ldots dx_n. \end{aligned}$$

The result now follows. ■

We now have all the ingredients necessary to calculate the potential, q. The following theorem shows how to calculate the potential if the linear system, $(-A,B_r,C)$ is known.

Theorem 4.6.0.36 *Let $(-A,B_r,C)$ be the linear system specified by Example 2.6 and let ϕ be the scattering function given in Definition 3.1.0.37. Also let*

$$T(x,y) = -C\left(e^{-xA}+e^{xA}\right)[I+\mu R_x]^{-1}\left(e^{-yA}+e^{yA}\right)B_r$$

where R_x is as defined by Definition 3.3.0.44. Then the potential, q has the form,

$$\begin{aligned} q(x) &= -8[\psi'(2x)-\psi'(-2x)] \\ &\quad -4\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi'(x+x_1)+\psi'(x-x_1)-\psi'(-x+x_1)-\psi'(-x-x_1)] \\ &\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+x)+\psi(x_n-x)]\,dx_1\ldots dx_n \\ &\quad -4\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\ &\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi'(x_n+x)-\psi'(x_n-x)]\,dx_1\ldots dx_n \\ &\quad -4\sum_{n=1}^{\infty}(-2\mu)^n\sum_{k=1}^{n}\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\ &\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+x)+\psi(x_n-x)] \\ &\quad dx_1\ldots dx_{k-1}\,dx_{k+1}\ldots dx_n|_{x_k=x} \\ &\quad -4\sum_{n=1}^{\infty}(-2\mu)^n\sum_{k=1}^{n}\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\ &\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+x)+\psi(x_n-x)] \\ &\quad dx_1\ldots dx_{k-1}\,dx_{k+1}\ldots dx_n|_{x_k=-x}. \end{aligned}$$

Proof. Given scattering function $\phi(x)=C\left(e^{-xA}+e^{xA}\right)B_r$ and operator

$$R_x = \int_{-x}^{x}\left(e^{-zA}+e^{zA}\right)B_rC\left(e^{-zA}+e^{zA}\right)\,dz,$$

the function $T(x,y)$ satisfies the Gelfand–Levitan integral equation (3.3) by Theorem 3.3.0.47. Therefore, by Corollary 3.3.0.48 we know that $q(x)=4\frac{d}{dx}T(x,x)$ and so we begin by calculating

$T(x,x)$. By Proposition 4.6.0.35,

$$\begin{aligned}
T(x,x) &= -[\psi(2x)+2\psi(0)+\psi(-2x)] \\
&\quad -\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\
&\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+x)+\psi(x_n-x)]\,dx_1\ldots dx_n.
\end{aligned}$$

The next step is to differentiate $T(x,x)$ which we do term-by-term. The fact that $T(x,x)$ is sufficiently well behaved to allow term-by-term differentiation follows from [1] (Theorem 13-14, page 403). Since the expression for $T(x,x)$ is somewhat complicated, we first describe the process of differentiation. Differentiating the first term of $T(x,x)$ is obvious. The remaining part of $T(x,x)$ involves a sum of integrals where each integral contains a product of terms. We proceed to differentiate the sum term-by-term, differentiating each integrand using the product rule; first differentiating the term $\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)$ and then differentiating the term $\psi(x_n+x)+\psi(x_n-x)$, finally addressing the limits. Therefore,

$$\begin{aligned}
q(x) &= 4\frac{d}{dx}T(x,x) \\
&= -8[\psi'(2x)-\psi'(-2x)] \\
&\quad -4\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi'(x+x_1)+\psi'(x-x_1)-\psi'(-x+x_1)-\psi'(-x-x_1)] \\
&\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+x)+\psi(x_n-x)]\,dx_1\ldots dx_n \\
&\quad -4\sum_{n=1}^{\infty}(-2\mu)^n\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\
&\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi'(x_n+x)-\psi'(x_n-x)]\,dx_1\ldots dx_n \\
&\quad -4\sum_{n=1}^{\infty}(-2\mu)^n\sum_{k=1}^{n}\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\
&\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+x)+\psi(x_n-x)] \\
&\quad dx_1\ldots dx_{k-1}\,dx_{k+1}\ldots dx_n|_{x_k=x} \\
&\quad -4\sum_{n=1}^{\infty}(-2\mu)^n\sum_{k=1}^{n}\int_{-x}^{x}\cdots\int_{-x}^{x}[\psi(x+x_1)+\psi(x-x_1)+\psi(-x+x_1)+\psi(-x-x_1)] \\
&\quad \left(\prod_{j=1}^{n-1}[\psi(x_j+x_{j+1})+\psi(x_j-x_{j+1})]\right)[\psi(x_n+x)+\psi(x_n-x)] \\
&\quad dx_1\ldots dx_{k-1}\,dx_{k+1}\ldots dx_n|_{x_k=-x}.
\end{aligned}$$

■

4.6.1 Finding $(I + R_x)^{-1}$

In this section we propose an alternative method to find $(I + \mu R_x)^{-1}$. By Proposition 4.6.0.32 we know that

$$(I + \mu R_x)f(t) = f(t) + 2\mu \int_{-x}^{x} [\psi(t+z) + \psi(t-z)]f(z)\,dz.$$

Let

$$h(t) = f(t) + 2\mu \int_{-x}^{x} [\psi(t+z) + \psi(t-z)]f(z)\,dz, \tag{4.41}$$

then $(I + \mu R_x)f(t) = h(t)$ and so $f(t) = (I + \mu R_x)^{-1}h(t)$. This shows another way in which we can calculate $(I + \mu R_x)^{-1}$.

The inspiration for the following proposition comes from the work done with the operator, T. Note that the Gelfand–Levitan integral equation that appears in the following proposition has the same form as that in Definition 3.2.0.40. Also, in Proposition 4.2.0.60 we saw that if T satisfies a Gelfand–Levitan integral equation then $\cos x\sqrt{\lambda} + 2\int_0^x T(x,y)\cos y\sqrt{\lambda}\,dy$ is a solution of Hill's equation. This observation provides the motivation for defining the function f in the following proposition.

Proposition 4.6.1.1 *Let Ψ satisfy the Gelfand–Levitan integral equation*

$$\Psi(z,t) + \psi(t+z) + \psi(t-z) + 2\mu \int_{-x}^{x} \Psi(z,w)[\psi(t+w) + \psi(t-w)]\,dw = 0, \tag{4.42}$$

and let

$$f(t) = h(t) + 2\mu \int_{-x}^{x} \Psi(z,t)h(z)\,dz.$$

Then f satisfies equation (4.41).

Proof. Suppose that Ψ satisfies (4.42). Multiply both sides of (4.42) by $2\mu h(z)$ and then integrate with respect to z over the interval $[-x, x]$. We obtain

$$\begin{aligned} 0 &= 2\mu \int_{-x}^{x} \Psi(z,t)h(z)\,dz + 2\mu \int_{-x}^{x} [\psi(t+z) + \psi(t-z)]h(z)\,dz \\ &\quad + 4\mu^2 \int_{-x}^{x}\int_{-x}^{x} \Psi(z,w)[\psi(t+w) + \psi(t-w)]h(z)\,dw\,dz. \end{aligned}$$

Note that we can rewite $\int_{-x}^{x}\int_{-x}^{x} \Psi(z,w)[\psi(t+w) + \psi(t-w)]h(z)\,dw\,dz$ by first changing the order of integration and then relabelling (using the same variables), thus

$$\begin{aligned} &\int_{-x}^{x}\int_{-x}^{x} \Psi(z,w)[\psi(t+w) + \psi(t-w)]h(z)\,dw\,dz \\ &= \int_{-x}^{x}\int_{-x}^{x} \Psi(z,w)[\psi(t+w) + \psi(t-w)]h(z)\,dz\,dw \\ &= \int_{-x}^{x}\int_{-x}^{x} \Psi(w,z)[\psi(t+z) + \psi(t-z)]h(w)\,dw\,dz. \end{aligned}$$

So,

$$\begin{aligned}
0 &= 2\mu\int_{-x}^{x}\Psi(z,t)h(z)\,dz + 2\mu\int_{-x}^{x}[\psi(t+z)+\psi(t-z)]h(z)\,dz \\
&\quad +4\mu^2\int_{-x}^{x}\int_{-x}^{x}\Psi(w,z)[\psi(t+z)+\psi(t-z)]h(w)\,dw\,dz \\
&= 2\mu\int_{-x}^{x}\Psi(z,t)h(z)\,dz \\
&\quad +2\mu\int_{-x}^{x}\left\{[\psi(t+z)+\psi(t-z)]h(z)+2\mu\int_{-x}^{x}\Psi(w,z)[\psi(t+z)+\psi(t-z)]h(w)\,dw\right\}\,dz \\
&= 2\mu\int_{-x}^{x}\Psi(z,t)h(z)\,dz \\
&\quad +2\mu\int_{-x}^{x}[\psi(t+z)+\psi(t-z)]\left\{h(z)+2\mu\int_{-x}^{x}\Psi(w,z)h(w)\,dw\right\}\,dz. \qquad (4.43)
\end{aligned}$$

Given $f(t) = h(t) + 2\mu\int_{-x}^{x}\Psi(z,t)h(z)\,dz$ we see that (4.43) becomes

$$0 = f(t) - h(t) + 2\mu\int_{-x}^{x}[\psi(t+z)+\psi(t-z)]f(z)\,dz,$$

that is, f satisfies equation 4.41 as required. ■

Chapter 5

Sampling Sequences and their Applications

In this chapter we look at sampling sequences related to Paley–Wiener spaces and some of their applications. The periodic spectrum of Hill's equation is a sequence, $(\lambda_n)_{n=0}^{\infty}$ where λ_n is of order n^2 in size. To analyse the behaviour in detail, we first take square roots and introduce t_n where t_n is of order n. Then we compare the sequence $(t_n)_{n\in\mathbb{Z}}$ in detail with $(n)_{n\in\mathbb{Z}}$ and show that $(t_n)_{n\in\mathbb{Z}}$ is a sampling sequence. First we see how the sequence $(t_n)_{n\in\mathbb{Z}}$ can be used to reconstruct a function given that the function is known at the sampling points, t_n. This then leads us to consider the question of bases. The sequence $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ gives an orthonormal basis for $L^2[-\pi,\pi]$, so it is therefore natural to consider if the sequence $\left\{e^{it_nx}\right\}_{n\in\mathbb{Z}}$ also gives a basis. Usually this will not be an orthonormal basis, but we can formulate conditions under which it is a frame or a Riesz basis. Since the space $L^2[-\pi,\pi]$ and the Paley–Wiener space, $PW(\pi)$ are naturally isomorphic via the Fourier transform, we often express the results of this chapter in terms of Paley–Wiener space.

One of the main new results of this thesis is that the sampling sequence, $(t_n)_{n\in\mathbb{Z}}$ is associated with a Carleman determinant which depends in a Lipschitz continuous way on $(t_n)_{n\in\mathbb{Z}}$. This is a crucial technical point that Blower, Brett and Doust present in their paper [7].

5.1 A Sampling Sequence Derived from the Spectrum of Hill's Equation

The aim of this section is to construct a sampling sequence from the periodic spectrum of Hill's equation. We define the notion of a sampling sequence and pick points from the periodic spectrum of Hill's equation that will be used to form a sequence, $(t_n)_{n\in\mathbb{Z}}$. Using Borg's estimates we are able to show that $(t_n)_{n\in\mathbb{Z}}$ is indeed a sampling sequence.

Definition 5.1.0.2 *Let $(t_n)_{n\in\mathbb{Z}}$ be a sequence satisfying the following conditions,*
i) $t_n < t_{n+1}$ for all n;
ii) $t_n \to \pm\infty$ as $n \to \pm\infty$;

iii) there exists some constant $\delta > 0$ such that $t_{n+1} - t_n \leq \delta$ for all n.
Then we say that $(t_n)_{n\in\mathbb{Z}}$ is a sampling sequence*.*

Remark 5.1.0.3 *Conditions (i)-(iii) in Definition 5.1.0.2 are known as the* sampling conditions*.*

We now use the periodic spectrum of Hill's equation to construct a sequence of sampling points, $(t_n)_{n\in\mathbb{Z}}$. Let $(\lambda_n)_{n=0}^{\infty}$ denote the periodic spectrum of Hill's equation. We begin the construction of our sampling sequence by removing the points λ_{2m} for $m \in \mathbb{N}_0$. We therefore consider only the sequence $(\lambda_{2n+1})_{n\in\mathbb{N}_0}$. This is equivalent to choosing the left-hand endpoint of each interval of instability that occurs after λ_0. Where the interval of instability disappears we take the left-hand endpoint of the double root. Note that λ_1 is in the complementary series, while λ_3 is in the principal series. The λ_{2n+1}'s then continue to alternate between the complementary series and the principal series. We summarise this in the following remark.

Remark 5.1.0.4 *If $n \in \mathbb{N}_0$ is odd then λ_{2n+1} belongs to the principal series. If $n \in \mathbb{N}_0$ is even then λ_{2n+1} belongs to the complementary series.*

We are now ready to define our sampling sequence.

Definition 5.1.0.5 *Let $(\lambda_n)_{n=0}^{\infty}$ denote the periodic spectrum of Hill's equation. Define the sequence, $(t_n)_{n\in\mathbb{Z}}$ as follows,*

$$t_n = \begin{cases} -\sqrt{\lambda_{-(2n+1)}} & \text{for } n \leq -1 \\ 0 & \text{for } n = 0 \\ \sqrt{\lambda_{2n-1}} & \text{for } n \geq 1. \end{cases} \tag{5.1}$$

Remark 5.1.0.6 *Note that $t_{-n} = -t_n$.*

Having constructed a sequence, we check that it is indeed a sampling sequence. In order to do this we first need to introduce some estimates that will enable us to check that the sampling conditions from Definition 5.1.0.2 hold. Proposition 5.1.0.7 presents *Borg's estimates* and can be found in [35] (Theorem 2.11, page 39). First we ask the reader to recall Remark 4.4.1.5 as this should help with the understanding of the following estimates.

Proposition 5.1.0.7 *Let q be the potential in Hill's equation and suppose that q is π-periodic and satisfies*

$$\int_0^{\pi} q(x)\, dx = 0.$$

Also let

$$\frac{1}{\pi}\int_0^{\pi} |q(x)|\, dx = C$$

for some constant, C. Given λ_{4n-1} and λ_{4n} belong to the principal series then, for any $n \in \mathbb{N}_0$ such that $n > \frac{C}{2}$ we have

$$\begin{aligned} \left|\sqrt{\lambda_{4n-1}} - 2n\right| &\leq \frac{C}{4n}, \\ \left|\sqrt{\lambda_{4n}} - 2n\right| &\leq \frac{C}{4n}. \end{aligned}$$

Given λ_{4n-3} and λ_{4n-2} belong to the complementary series then, for any $n \in \mathbb{N}_0$ such that $n > \frac{C}{2}$ we have,

$$\begin{aligned} \left|\sqrt{\lambda_{4n-3}} - (2n-1)\right| &\leq \frac{C}{4n-2}, \\ \left|\sqrt{\lambda_{4n-2}} - (2n-1)\right| &\leq \frac{C}{4n-2}. \end{aligned}$$

Borg's estimates are used to show how close the square root of an element in the periodic spectrum is to its nearest integer. Estimates are given for all elements in the periodic spectrum. However, for our work it is not necessary to have a complete set of estimates since we are only working with the set $\{\lambda_{2n+1}\}_{n\in\mathbb{N}_0}$. We therefore reformulate Proposition 5.1.0.7 to provide suitable estimates for use with our sampling sequence, $(t_n)_{n\in\mathbb{Z}}$.

Corollary 5.1.0.8 *Let q be the potential in Hill's equation and suppose that q is π-periodic and satisfies*

$$\int_0^\pi q(x)\,dx = 0.$$

Further, suppose that

$$\frac{1}{\pi}\int_0^\pi |q(x)|\,dx < \frac{1}{2}.$$

Then for all $n \in \mathbb{N}$,

$$\begin{aligned} \left|\sqrt{\lambda_{4n-1}} - 2n\right| &< \frac{1}{8n}, \\ \left|\sqrt{\lambda_{4n-3}} - (2n-1)\right| &< \frac{1}{8n-4}. \end{aligned}$$

Remark 5.1.0.9 *Note that in both cases, the estimates in Corollary 5.1.0.8 have the form,*

$$\left|\sqrt{\lambda_{2m-1}} - m\right| < \frac{1}{4m}$$

for all $m \in \mathbb{N}$.

Proof. Both estimates follow directly from Proposition 5.1.0.7. ■

We can now show that the sequence, $(t_n)_{n\in\mathbb{Z}}$ is indeed a sampling sequence.

Theorem 5.1.0.10 *Suppose that Hill's equation has π-periodic potential, q such that*

$$\int_0^\pi q(x)\,dx = 0,$$

and

$$\frac{1}{\pi}\int_0^\pi |q(x)|\,dx < \frac{1}{2}.$$

Then the sequence $(t_n)_{n\in\mathbb{Z}}$ as defined in Definition 5.1.0.5 is real and satisfies

$$|t_n - n| < \frac{1}{4}.$$

Furthermore, $(t_n)_{n\in\mathbb{Z}}$ is a sampling sequence such that $t_{n+1} - t_n < \frac{3}{2}$.

Proof. We first show that $|t_n - n| < \frac{1}{4}$. Note that since $t_{-n} = -t_n$ by Remark 5.1.0.6, it follows that

$$\begin{aligned} |t_{-n} - (-n)| &= |-t_n + n| \\ &= |t_n - n|. \end{aligned}$$

It therefore suffices to consider the case $n \geq 1$ since the case $n = 0$ is trivial. Now, by Definition 5.1.0.5, $t_n = \sqrt{\lambda_{2n-1}}$ for $n \geq 1$, hence, by Remark 5.1.0.9,

$$\begin{aligned} |t_n - n| &= \left|\sqrt{\lambda_{2n-1}} - n\right| \\ &< \frac{1}{4n}. \end{aligned}$$

The result now follows since $n \geq 1$.

Next we show that $t_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$. The sequence $(t_n)_{n\in\mathbb{Z}}$ will be real if the λ_{2n-1} are real and positive for all $n \in \mathbb{N}$. By the Oscillation Theorem 4.4.1.4 the λ_{2n-1} are all real. Also, since the λ_n satisfy the inequalities, (4.17) and are therefore monotonically increasing, it suffices to show that $\lambda_1 \geq 0$ in order to prove that the λ_{2n-1} are positive. We prove the positivity of λ_1 by contradiction. Suppose that $\lambda_1 < 0$ then $\lambda_1 = -|\lambda_1|$ where $|\lambda_1| > 0$. Taking the square root of both sides gives

$$\sqrt{\lambda_1} = \pm i\sqrt{|\lambda_1|}.$$

Next we subtract 1 from both sides of the above equation and then take the modulus, finishing by squaring both sides. This gives,

$$\begin{aligned} \left|\sqrt{\lambda_1} - 1\right|^2 &= \left|\pm i\sqrt{|\lambda_1|} - 1\right|^2 \\ &= |\lambda_1| + 1 \\ &> 1, \end{aligned}$$

which implies that $\left|\sqrt{\lambda_1} - 1\right| > 1$. However, since $|t_n - n| < \frac{1}{4}$ for all $n \in \mathbb{Z}$,

$$\begin{aligned} \left|\sqrt{\lambda_1} - 1\right| &= |t_1 - 1| \\ &< \frac{1}{4}. \end{aligned}$$

This is a contradiction and so we conclude that $\lambda_1 \geq 0$. Therefore, $\lambda_n \geq 0$ for all $n \in \mathbb{N}$ and so we have proved that the sequence $(t_n)_{n\in\mathbb{Z}}$ is indeed real.

It remains to prove that $(t_n)_{n\in\mathbb{Z}}$ is a sampling sequence. First we note that since the λ_{2n-1} satisfy the inequalities (4.17), then $t_n < t_{n+1}$ for all n. Also, by the Oscillation Theorem 4.4.1.4, the λ_{2n-1} form a monotonically increasing sequence tending to infinity. Thus, $t_n \to \infty$ as $n \to \infty$ and $t_{-n} \to -\infty$ as $n \to -\infty$. The final sampling condition requires that the distance between any two adjacent sampling points is strictly less than $\frac{3}{2}$. Given $|t_n - n| < \frac{1}{4}$ for all $n \in \mathbb{Z}$, it is easily seen that

$$\begin{aligned} |t_{n+1} - t_n| &= |[t_{n+1} - (n+1)] - [t_n - n] + 1| \\ &\leq |t_{n+1} - (n+1)| + |t_n - n| + 1 \\ &< \frac{3}{2} \end{aligned}$$

as required. This completes the proof. ■

5.2 Sampling Theorems

This section aims to reconstruct a function given that the value of the function is known at a set of sampling points. Whittaker, Kotel'nikov and Shannon have shown that a function can be reconstructed if its value is known at points that are equally spaced along the real line. The crucial point here is that the samples are equally spaced and so sampling occurs at a constant rate. In what follows we discover what happens when the samples do not occur at a constant rate, but instead are shifted within a small region around an integer.

Before stating the Whittaker–Kotel'nikov–Shannon Sampling Theorem we introduce a new function known as the sinc function that will appear in an application of the theorem.

Definition 5.2.0.11 *Define the* sinc function *to be*

$$\operatorname{sinc} x = \begin{cases} \frac{\sin \pi x}{\pi x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

The following lemma provides some obvious but useful facts concerning the sinc function.

Lemma 5.2.0.12 *Let $m, n \in \mathbb{Z}$. Then*

$$\operatorname{sinc}(m-n) = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$

Proof. Let $m, n \in \mathbb{Z}$ and suppose that $n \neq m$. Then,

$$\begin{aligned} \operatorname{sinc}(m-n) &= \frac{\sin \pi(m-n)}{\pi(m-n)} \\ &= 0 \end{aligned}$$

since $\sin k\pi = 0$ for $k \in \mathbb{Z}$. The case $n = m$ follows directly from Definition 5.2.0.11. ∎

The following theorem is known as the *Whittaker–Kotel'nikov–Shannon Sampling Theorem* and appears in [41] (Theorem 7.2.2, page 209).

Proposition 5.2.0.13 *For $f \in PW(b)$ we can recover f from the samples $\left\{ f\left(\frac{n\pi}{b}\right)\right\}_{n\in\mathbb{Z}}$ using the convergent (in the $L^2(\mathbb{R})$ norm) formula*

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{b}\right) \frac{\sin b\left(t - \frac{n\pi}{b}\right)}{b\left(t - \frac{n\pi}{b}\right)}.$$

Example 5.2.0.14

Let $f \in PW(\pi)$ then by Proposition 5.2.0.13,

$$\begin{aligned} f(t) &= \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \\ &= \sum_{n\in\mathbb{Z}} f(n) \operatorname{sinc}(t-n). \end{aligned}$$

This shows that $\{\operatorname{sinc}(t-n)\}_{n\in\mathbb{Z}}$ is a spanning sequence for $PW(\pi)$.

In the case that we are working in the space $PW(\pi)$, the Whittaker–Kotel'nikov–Shannon Sampling Theorem 5.2.0.13 tells us that every $f \in PW(\pi)$ can be written as a linear combination of the elements $\{\operatorname{sinc}(t-n)\}_{n\in\mathbb{Z}}$. In fact, $\{\operatorname{sinc}(t-n)\}_{n\in\mathbb{Z}}$ is actually an orthonormal basis as justified by the following proposition.

Proposition 5.2.0.15 *The set* $\{\operatorname{sinc}(t-n)\}_{n\in\mathbb{Z}}$ *forms an orthonormal basis for* $PW(\pi)$.

Proof. Recall Proposition 2.4.3.4, we saw that $k_t(x) = \frac{\sin b(t-x)}{\pi(t-x)}$ is a reproducing kernel for $PW(b)$. Therefore, $k_x(t) = \operatorname{sinc}(t-x)$ is a reproducing kernel for $PW(\pi)$. Let $f \in PW(\pi)$ then in particular,

$$f(n) = \langle f, \operatorname{sinc}(\cdot - n)\rangle.$$

Thus, by the Whittaker–Kotel'nikov–Shannon Sampling Theorem 5.2.0.13, if $f \in PW(\pi)$ then

$$\begin{aligned} f(t) &= \sum_{n\in\mathbb{Z}} f(n)\operatorname{sinc}(t-n) \\ &= \sum_{n\in\mathbb{Z}} \langle f, \operatorname{sinc}(\cdot - n)\rangle \operatorname{sinc}(t-n). \end{aligned}$$

Furthermore, $\langle \operatorname{sinc}(\cdot - n), \operatorname{sinc}(\cdot - m)\rangle = \operatorname{sinc}(m-n)$, again since $\operatorname{sinc}(t-x)$ is a reproducing kernel for $PW(\pi)$. It now follows from Lemma 5.2.0.12 that for $m, n \in \mathbb{Z}$,

$$\langle \operatorname{sinc}(\cdot - n), \operatorname{sinc}(\cdot - m)\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

This completes the proof that $\{\operatorname{sinc}(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $PW(\pi)$. ∎

We now ask if we can reconstruct a function given that the sampling does not occur at a constant rate. It turns out that we can gain some insight into this if we use a sampling sequence whose rate of sampling is bounded. The following result can be found, with proof, in [41] (Corollary 7.3.7, page 222).

Lemma 5.2.0.16 *Let* (t_n) *be a sampling sequence such that* $t_{n+1} - t_n < \frac{\pi}{b}$ *for all* n*. Then there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1 \|f\|^2 \leq \sum_{n\in\mathbb{Z}} |f(t_n)|^2 (t_{n+1} - t_{n-1}) \leq C_2 \|f\|^2$$

for all $f \in PW(b)$.

Theorem 5.2.0.17 *Suppose that Hill's equation has* π*-periodic potential,* q *such that*

$$\int_0^\pi q(x)\,dx = 0,$$

and

$$\frac{1}{\pi}\int_0^\pi |q(x)|\,dx < \frac{1}{2}.$$

Let $(t_n)_{n\in\mathbb{Z}}$ *be the sampling sequence defined by Definition 5.1.0.5. Then there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1 \|f\|^2 \leq \sum_{n\in\mathbb{Z}} |f(t_n)|^2 \leq C_2 \|f\|^2$$

for all $f \in PW(2)$.

Proof. By Theorem 5.1.0.10, $(t_n)_{n\in\mathbb{Z}}$ is a sampling sequence such that $t_{n+1} - t_n < \frac{3}{2} < \frac{\pi}{2}$. By Lemma 5.2.0.16, it follows that for all $f \in PW(2)$,

$$A \| f \|^2 \leq \sum_{n\in\mathbb{Z}} | f(t_n)|^2 (t_{n+1} - t_{n-1}) \leq B \| f \|^2 \tag{5.2}$$

for some constants $A, B > 0$. By Theorem 5.1.0.10 we have $|t_n - n| < \frac{1}{4}$ for all n. From this estimate it follows that

$$\begin{aligned} t_{n+1} - t_{n-1} &\geq \left(n + 1 - \frac{1}{4}\right) - \left(n - 1 + \frac{1}{4}\right) \\ &= \frac{3}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned} t_{n+1} - t_{n-1} &\leq \left(n + 1 + \frac{1}{4}\right) - \left(n - 1 - \frac{1}{4}\right) \\ &= \frac{5}{2}. \end{aligned}$$

Since $\frac{3}{2} \leq t_{n+1} - t_{n-1} \leq \frac{5}{2}$ we can simplify (5.2), thus

$$C_1 \| f \|^2 \leq \sum_{n\in\mathbb{Z}} | f(t_n)|^2 \leq C_2 \| f \|^2$$

for some constants $C_1, C_2 > 0$, proving the result. ∎

5.3 Frames and Riesz Bases

In this section we introduce the concepts of frames and Riesz bases with the aim being to use our sampling sequence, $(t_n)_{n\in\mathbb{Z}}$ to construct a Riesz basis. Analogous to the example of $\{e^{inx}\}_{n\in\mathbb{Z}}$ being an orthonormal basis for $L^2[-\pi, \pi]$, we see that $\{e^{it_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi, \pi]$. We prove this using Borg's estimates from Section 5.1 and Kadec's Quarter Theorem. Further information on the topics contained within this section can be found in [9], [27] and [40] (Section 4.1, page 300).

Definition 5.3.0.18 *Let H be a Hilbert space. A set $\{f_n\}_{n\in\mathbb{Z}} \in H$ is a* frame *if there exist constants $C_1, C_2 > 0$ such that,*

$$C_1 \| f \|^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \leq C_2 \| f \|^2$$

for all $f \in H$.

The following proposition gives an example of a frame for a specific Paley–Wiener space. Note that the frame is constructed using the sampling sequence, $(t_n)_{n\in\mathbb{Z}}$ found in Section 5.1.

Proposition 5.3.0.19 *Let*

$$k_t(x) = \frac{\sin 2(t - x)}{\pi(t - x)}$$

be the reproducing kernel for the space, $PW(2)$. Further, let $(t_n)_{n\in\mathbb{Z}}$ be the sampling sequence defined in Definition 5.1.0.5. Then,

$$k_{t_n}(x) = \frac{\sin 2(t_n - x)}{\pi(t_n - x)}$$

is a frame for $PW(2)$.

Proof. Let $(t_n)_{n\in\mathbb{Z}}$ be the sampling sequence as given in Definition 5.1.0.5. Then, by Theorem 5.2.0.17, there exist constants $C_1, C_2 > 0$ such that for all $f \in PW(2)$,

$$C_1 \|f\|^2 \leq \sum_{n\in\mathbb{Z}} |f(t_n)|^2 \leq C_2 \|f\|^2 . \tag{5.3}$$

By Proposition 2.4.3.4 we know that

$$k_t(x) = \frac{\sin 2(t-x)}{\pi(t-x)}$$

is a reproducing kernel for the space $PW(2)$. Therefore, for all $f \in PW(2)$, k_t satisfies $\langle f, k_t\rangle = f(t)$. In particular, $f(t_n) = \langle f, k_{t_n}\rangle$. It now follows from (5.3) that

$$C_1 \|f\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, k_{t_n}\rangle|^2 \leq C_2 \|f\|^2 .$$

Hence, by Definition 5.3.0.18, $\{k_{t_n}\}_{n\in\mathbb{Z}}$ is a frame for $PW(2)$. ■

In linear algebra we have the notion of a spanning sequence which we can refine to a basis. If we liken a frame to a spanning sequence then we can ask, is it possible to refine a frame to a basis? It turns out that in some, but not all cases we can refine a frame to a particular type of basis known as a Riesz basis. Again, as in linear algebra, just as a basis is still a spanning sequence, every Riesz basis is a frame. We use the definition of a Riesz basis as given in [9].

Definition 5.3.0.20 *Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for a Hilbert space, H. Suppose that $F : H \to H$ is a bounded bijective operator and let $r_n = Fe_n$. Then $\{r_n\}_{n\in\mathbb{Z}}$ is a* Riesz basis *for H.*

In the following proposition we present a criterion for a set, $\{r_n\}_{n\in\mathbb{Z}}$ to be a Riesz basis. Note how the condition is similar to that given for a frame. The result with its proof can be found in [41] (Proposition 2.5.7, page 73).

Proposition 5.3.0.21 *The set $\{r_n\}_{n\in\mathbb{Z}}$ is a Riesz basis if and only if there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \left\| \sum_{n\in\mathbb{Z}} a_n r_n \right\|^2 \leq C_2 \sum_{n\in\mathbb{Z}} |a_n|^2,$$

for all square summable sequences, (a_n).

In practise, Riesz bases are often hard to find. An easier route to finding a Riesz basis of the form $\{e^{is_n x}\}_{n\in\mathbb{Z}}$ is by applying *Kadec's Quarter Theorem.* We state Kadec's theorem in Proposition 5.3.0.22, without proof and note that the value of $\frac{1}{4}$ is best possible. More information regarding conditions for $\{e^{is_n x}\}_{n\in\mathbb{Z}}$ to be a Riesz basis can be found in [27] (page 78).

Proposition 5.3.0.22 *Let $(s_n)_{n\in\mathbb{Z}}$ be a real sequence and suppose that*

$$|s_n - n| < \frac{1}{4}.$$

Then the set $\{e^{is_n x}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi, \pi]$.

We can apply Kadec's Quarter Theorem to construct a Riesz basis from our sampling sequence, $(t_n)_{n\in\mathbb{Z}}$.

Theorem 5.3.0.23 *Suppose that Hill's equation has π-periodic potential, q such that*

$$\int_0^\pi q(x)\,dx = 0,$$

and

$$\frac{1}{\pi}\int_0^\pi |q(x)|\,dx < \frac{1}{2}.$$

Let $(t_n)_{n\in\mathbb{Z}}$ be the sampling sequence defined by Definition 5.1.0.5. Then $\{e^{it_nx}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi,\pi]$.

Proof. Let $(t_n)_{n\in\mathbb{Z}}$ be the sampling sequence as in Definition 5.1.0.5. By Theorem 5.1.0.10, $(t_n)_{n\in\mathbb{Z}}$ is a real sequence and

$$|t_n - n| < \frac{1}{4}$$

for all $n \in \mathbb{Z}$. It now follows from Kadec's Quarter Theorem 5.3.0.22 that the set $\{e^{it_nx}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi,\pi]$. ■

5.3.1 Dual Riesz Bases

To each Riesz basis, $\{r_n\}$ there corresponds a sequence, $\{r_n^*\}$ such that $\{r_n^*\}$ is also a Riesz basis. Further, $\{r_n\}$ and $\{r_n^*\}$ are biorthogonal. Here we define the sequence $\{r_n^*\}$, known as the dual Riesz basis and then, using a specific example, we look at how we may construct a dual Riesz basis. The idea behind this section is to construct a Riesz basis for $L^2[0,\pi]$ from the sampling sequence, $(t_n)_{n\in\mathbb{Z}}$ and then use the linear system, $(-A,B,C)$ to calculate the dual Riesz basis.

We begin by defining biorthogonal sequences and then we introduce the notion of a dual Riesz basis.

Definition 5.3.1.1 *Let $\{e_n\}_{n\in\mathbb{Z}}$ and $\{f_n\}_{n\in\mathbb{Z}}$ be sequences in H. We say that $\{e_n\}_{n\in\mathbb{Z}}$ and $\{f_n\}_{n\in\mathbb{Z}}$ are* biorthogonal *if,*

$$\langle e_n, f_m\rangle_H = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases}$$

Definition 5.3.1.2 *Let $\{r_n\}_{n\in\mathbb{Z}}$ be a Riesz basis such that*

$$Fe_n = r_n$$

for some orthonormal basis, $\{e_n\}_{n\in\mathbb{Z}}$ and some bounded, bijective operator, F. We define the dual Riesz basis *of $\{r_n\}_{n\in\mathbb{Z}}$ to be $\{r_n^*\}_{n\in\mathbb{Z}}$ where*

$$r_n^* = \left(F^{-1}\right)^* e_n,$$

and $\left(F^{-1}\right)^*$ is the adjoint of the operator F^{-1}.

Remark 5.3.1.3 *The dual Riesz basis is also referred to as the* biorthogonal basis.

Lemma 5.3.1.4 *Let $\{r_n\}_{n\in\mathbb{Z}}$ be a Riesz basis in H and let $\{r_n^*\}_{n\in\mathbb{Z}}$ be its dual Riesz basis. Then $\{r_n\}_{n\in\mathbb{Z}}$ and $\{r_n^*\}_{n\in\mathbb{Z}}$ are biorthogonal.*

Proof. Let $\{r_n\}_{n\in\mathbb{Z}}$ be a Riesz basis in H and suppose that $r_n = Fe_n$ for some bounded, bijective operator, F and an orthonormal basis, $\{e_n\}_{n\in\mathbb{Z}}$. Also suppose that $\{r_n^*\}_{n\in\mathbb{Z}}$ is the dual Riesz basis corresponding to $\{r_n\}_{n\in\mathbb{Z}}$ and that $r_n^* = \left(F^{-1}\right)^* e_n$. Using Definition 2.1.0.7 we see that

$$\begin{aligned}\langle r_n, r_m^*\rangle &= \langle Fe_n, \left(F^{-1}\right)^* e_m\rangle \\ &= \langle F^{-1}Fe_n, e_m\rangle \\ &= \langle e_n, e_m\rangle \\ &= \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases}\end{aligned}$$

This shows that a Riesz basis and its dual are indeed biorthogonal. ■

It is natural to ask whether the dual Riesz basis is in fact a Riesz basis. It turns out that this is the case. Further, we can use a Riesz basis for a space, H together with its dual Riesz basis to write an element of H as a linear combination of the Riesz basis. The following proposition summarises these ideas. It can be found, with proof, in [9] (Theorem 3.6.3, page 64).

Proposition 5.3.1.5 *Let $\{r_n\}_{n\in\mathbb{Z}}$ be a Riesz basis for a Hilbert space, H and let $\{r_n^*\}_{n\in\mathbb{Z}}$ be its dual Riesz basis. Then $\{r_n^*\}_{n\in\mathbb{Z}}$ is the unique sequence such that*

$$f = \sum_{n\in\mathbb{Z}} \langle f, r_n^*\rangle r_n$$

for all $f \in H$. Further, $\{r_n^*\}_{n\in\mathbb{Z}}$ is a Riesz basis.

Remark 5.3.1.6 *It can be seen from the Banach Isomorphism Theorem that F^{-1} is bounded since F is a bounded, bijective operator. This shows that $\{r_n^*\}_{n\in\mathbb{Z}}$ is a Riesz basis.*

We are aiming to calculate the dual Riesz basis for a particular Riesz basis. In the next proposition we give the form of the Riesz basis that we will work with. Again, it is derived from the sampling sequence, $(t_n)_{n\in\mathbb{Z}}$. Since we are changing the Riesz basis we also need to change the space in which we are working. First we state a lemma that will make clear the new space in which we will work, as well as providing some insight into how to prove that a given sequence is a Riesz basis for our space.

Lemma 5.3.1.7 *The space of functions,*

$$\left\{f \in L^2[-\pi,\pi] : f(-x) = f(x)\right\}$$

is canonically, unitarily equivalent to the space $L^2[0,\pi]$.

Proof. First we note that $f \in L^2[-\pi,\pi]$ if and only if

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty.$$

This condition is equivalent to

$$\int_0^{\pi} |f(-x)|^2 \, dx + \int_0^{\pi} |f(x)|^2 \, dx < \infty.$$

Now, given $f(-x) = f(x)$, we see that $f \in \{f \in L^2[-\pi, \pi] : f(-x) = f(x)\}$ if and only if

$$\int_0^{\pi} |f(x)|^2 \, dx < \infty,$$

that is, $f \in L^2[0, \pi]$. ■

The previous lemma shows that we can prove a Riesz basis exists for $L^2[0, \pi]$ by proving that it exists for the even functions in $L^2[-\pi, \pi]$.

Theorem 5.3.1.8 *Suppose that Hill's equation has π-periodic potential, q such that*

$$\int_0^{\pi} q(x) \, dx = 0,$$

and

$$\frac{1}{\pi} \int_0^{\pi} |q(x)| \, dx < \frac{1}{2}.$$

Let $(t_n)_{n \in \mathbb{Z}}$ be the sampling sequence defined by Definition 5.1.0.5. Then $\{\cos t_n x\}_{n \in \mathbb{N}}$ forms a Riesz basis for $L^2[0, \pi]$.

Proof. We use Lemma 5.3.1.7 and begin by considering the space, $L^2[-\pi, \pi]$. By Theorem 5.3.0.23, the set $\{e^{it_n x}\}_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2[-\pi, \pi]$, where $(t_n)_{n \in \mathbb{Z}}$ is the sampling sequence of Definition 5.1.0.5. Further, by Proposition 5.3.1.5, for $f \in L^2[-\pi, \pi]$ we have

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{it_n x}.$$

Now suppose that f is an even function so that $0 = f(x) - f(-x)$ for all x. Then

$$\begin{aligned} 0 &= \sum_{n \in \mathbb{Z}} a_n \left(e^{it_n x} - e^{-it_n x} \right) \\ &= 2i \sum_{n \in \mathbb{Z}} a_n \sin t_n x \\ &= 2i \left[\sum_{n=-\infty}^{-1} a_n \sin t_n x + a_0 \sin t_0 x + \sum_{n=1}^{\infty} a_n \sin t_n x \right] \\ &= 2i \left[\sum_{n=1}^{\infty} a_{-n} \sin t_{-n} x + a_0 \sin t_0 x + \sum_{n=1}^{\infty} a_n \sin t_n x \right]. \end{aligned}$$

By Remark 5.1.0.6 we have $t_{-n} = -t_n$ and so

$$0 = 2i \left[a_0 \sin t_0 x + \sum_{n=1}^{\infty} (a_n - a_{-n}) \sin t_n x \right].$$

Since the above must hold for all x, we conclude that $f \in L^2[-\pi, \pi]$ is even if and only if $a_0 = 0$ and $a_{-n} = a_n$. Therefore, an even function, $f \in L^2[-\pi, \pi]$ has the form

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n \left(e^{-it_n x} + e^{it_n x} \right) \\ &= 2 \sum_{n=1}^{\infty} a_n \cos t_n x. \end{aligned} \tag{5.4}$$

This shows that $\{\cos t_n x\}_{n\in\mathbb{N}}$ is a basis for the even functions in $L^2[-\pi,\pi]$. By Lemma 5.3.1.7, $\{\cos t_n x\}_{n\in\mathbb{N}}$ is therefore a basis for $L^2[0,\pi]$. To see that $\{\cos t_n x\}_{n\in\mathbb{N}}$ is a Riesz basis we observe the following: given $\{e^{it_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$ we have, for $f(x)=\sum_{n\in\mathbb{Z}} a_n e^{it_n x}\in L^2[-\pi,\pi]$,

$$C_1\sum_{n\in\mathbb{Z}}|a_n|^2\le\|f\|^2\le C_2\sum_{n\in\mathbb{Z}}|a_n|^2$$

by Proposition 5.3.0.21. The above inequality holds for all f so in particular it holds for even f. Thus, for $f\in L^2[-\pi,\pi]$ even such that f takes the form of equation (5.4), we have

$$C_1\sum_{n\in\mathbb{Z}}|a_n|^2\le\left\|\sum_{n=1}^{\infty}a_n\cos t_n x\right\|^2\le C_2\sum_{n\in\mathbb{Z}}|a_n|^2$$

for some constants, $C_1, C_2>0$. Therefore $\{\cos t_n x\}_{n\in\mathbb{N}}$ is a Riesz basis by Proposition 5.3.0.21. ■

For the remainder of this section we use Hill's equation in the form

$$-f_n''+qf_n=t_n^2 f_n$$

where q is a constant potential and the t_n are as described in Definition 5.1.0.5. Let $(\lambda_n)_{n=0}^{\infty}$ denote the elements of the periodic spectrum then, since q is constant, it follows from [35] (Theorem 7.12, page 112) that each λ_n for $n\ge 1$ is a double root. That is, when q is a constant, with the exception of $(-\infty,\lambda_0)$, there are no intervals of instability. Thus every solution of Hill's equation corresponding to an eigenvalue in the interval (λ_0,∞) is bounded. Recall that by Proposition 4.2.0.60, a solution of Hill's equation, f_n has the form,

$$f_n(x)=\cos t_n x+2\int_0^x T(x,y)\cos t_n y\,dy.$$

Lemma 5.3.1.9 *Suppose that Hill's equation has potential $q(x)=0$ for all x. Furthermore, suppose that $T(0,0)=0$. The set $\{f_n\}_{n\in\mathbb{Z}}$ where*

$$f_n(x)=\cos t_n x+2\int_0^x T(x,y)\cos t_n y\,dy.$$

forms an orthonormal basis in $L^2[0,\pi]$.

Proof. Suppose that $T(0,0)=0$ then by Theorem 4.2.0.59, f_n is a fundamental solution satisfying the boundary conditions

$$f_n(0)=1\quad\text{and}\quad f_n'(0)=0.$$

Also, since $q(x)=0$ for all x, it follows from [35] (Theorem 7.12, page 112) that the sequence, $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of double roots. Thus, by Floquet's Theorem 4.4.2.2, we also have the boundary condition

$$f_n'(\pi)=0.$$

It follows from Proposition 4.1.0.57 that Hill's equation together with the solutions, $\{f_n\}_{n\in\mathbb{Z}}$ forms a regular Sturm–Liouville system. Therefore, $\{f_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis in $L^2[0,\pi]$ by the Sturm–Liouville Theorem, see [55] (Theorem 11.1, page 131). ■

We now show how the linear system, $(-A, B, C)$ can be used to find the dual Riesz basis of $\{\cos t_n x\}_{n\in\mathbb{N}}$.

Theorem 5.3.1.10 *Suppose that Hill's equation has potential $q(x) = 0$ for all x and that $T(0,0) = 0$. Let $(t_n)_{n\in\mathbb{Z}}$ be the sampling sequence defined in Definition 5.1.0.5 and let*

$$Vg(x) = 2\int_0^x T(x,y)g(y)\,dy.$$

Then the Riesz basis $\{r_n\}_{n\in\mathbb{N}}$ where $r_n(x) = \cos t_n x$ has dual Riesz basis $\{r_n^*\}_{n\in\mathbb{N}}$ where

$$r_n^* = (I + V^*)(I + V)r_n.$$

Proof. Let t_n belong to the sampling sequence described in Definition 5.1.0.5 and suppose that f_n is the solution of Hill's equation corresponding to the eigenvalue t_n^2, where

$$f_n(x) = \cos t_n x + 2\int_0^x T(x,y)\cos t_n y\,dy.$$

By Lemma 5.3.1.9, $\{f_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2[0,\pi]$ and by Theorem 5.3.1.8, $\{\cos t_n x\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[0,\pi]$. Let $r_n(x) = \cos t_n x$, then by Definition 5.3.0.20 there exists some bounded, bijective operator, F such that

$$Ff_n = r_n.$$

This implies that

$$f_n = F^{-1}r_n, \tag{5.5}$$

that is

$$F^{-1}\cos t_n x = \cos t_n x + 2\int_0^x T(x,y)\cos t_n y\,dy. \tag{5.6}$$

Now let V be the operator

$$Vg(x) = 2\int_0^x T(x,y)g(y)\,dy.$$

It follows from (5.6) that $F^{-1} = I + V$. In order to calculate the dual Riesz basis, Definition 5.3.1.2 requires that we calculate $\left(F^{-1}\right)^*$. Thus,

$$\begin{aligned}\left(F^{-1}\right)^* &= (I+V)^* \\ &= I + V^*.\end{aligned}$$

Hence, by Definition 5.3.1.2, the dual Riesz basis is given by

$$\begin{aligned} r_n^* &= \left(F^{-1}\right)^* f_n \\ &= (I+V^*)f_n.\end{aligned}$$

Now, using the fact that $F^{-1} = I + V$ it follows from (5.5) that

$$\begin{aligned} f_n &= F^{-1}r_n \\ &= (I+V)r_n.\end{aligned}$$

Hence,

$$r_n^* = (I+V^*)(I+V)r_n$$

as required. ■

Remark 5.3.1.11 *In [38] (Section 2, page 220), McKean and van Moerbeke consider the auxiliary spectrum and show that the corresponding solutions of Hill's equation satisfy boundary conditions at both* 0 *and* π*. The auxiliary spectrum is obtained by taking points that lie in intervals of instability and in the case that this interval disappears, we take the double root. A sampling sequence* $(s_n)_{n\in\mathbb{Z}}$ *can then be constructed from the auxiliary spectrum in exactly the same way as we created* $(t_n)_{n\in\mathbb{Z}}$ *from the periodic spectrum. It then follows that* $\{\cos s_n x\}_{n\in\mathbb{N}}$ *will also be a Riesz basis. Furthermore, the solutions,* g_n *of Hill's equation corresponding to the sampling points* s_n *will satisfy the boundary conditions* $g_n(0) = 0 = g_n(\pi)$*, and hence will form a Sturm–Liouville system. It follows then that the set* $\{g_n\}_{n\in\mathbb{Z}}$ *is an orthonormal basis. We can then follow exactly the same method as demonstrated in Theorem 5.3.1.10 to derive the dual Riesz basis for* $\{\cos s_n x\}_{n\in\mathbb{N}}$*. This shows that we can calculate the dual Riesz basis using linear systems, without the assumption of having zero potential.*

Suppose that the function, $T(x,y)$ can be written in terms of the linear system, $(-A, B, C)$. Theorem 5.3.1.10 therefore shows that we can write the operator V in terms of $(-A, B, C)$ and hence calculate the dual Riesz basis corresponding to $\{\cos t_n x\}_{n\in\mathbb{N}}$. The following corollary makes precise the form of the dual Riesz basis.

Corollary 5.3.1.12 *Suppose that Hill's equation has potential* $q(x) = 0$ *for all* x *and that* $T(0,0) = 0$*. Let* $\{r_n^*\}_{n\in\mathbb{N}}$ *be the dual Riesz basis found in Theorem 5.3.1.10. Then*

$$\begin{aligned} r_n^*(x) &= \cos t_n x + 2\int_0^x T(x,y)\cos t_n y\,dy + 2\int_x^\pi \overline{T(y,x)}\cos t_n y\,dy \\ &\quad +4\int_0^\pi \cos t_n y \int_{\max\{x,y\}}^\pi \overline{T(z,x)}T(z,y)\,dz\,dy. \end{aligned}$$

Proof. From Theorem 5.3.1.10 we know that

$$\begin{aligned} r_n^*(x) &= (I+V^*)(I+V)\cos t_n x \\ &= (I+V+V^*+V^*V)\cos t_n x \end{aligned} \tag{5.7}$$

where $Vg(x) = 2\int_0^x T(x,y)g(y)\,dy$. We proceed by calculating V^*. By Definition 2.1.0.7 we have

$$\begin{aligned} \langle g, V^*h\rangle_{L^2[-\pi,\pi]} &= \langle Vg, h\rangle_{L^2[-\pi,\pi]} \\ &= \int_{-\pi}^\pi Vg\,\overline{h}\,dx \\ &= \int_{-\pi}^\pi 2\int_0^x T(x,y)g(y)\,dy\,\overline{h(x)}\,dx \\ &= 2\int_{-\pi}^\pi\int_0^x T(x,y)g(y)\overline{h(x)}\,dy\,dx. \end{aligned}$$

Now, reversing the order of integration we obtain

$$\begin{aligned} \langle g, V^*h\rangle_{L^2[-\pi,\pi]} &= 2\int_{-\pi}^\pi\int_y^\pi T(x,y)g(y)\overline{h(x)}\,dx\,dy \\ &= 2\int_{-\pi}^\pi g(y)\overline{\int_y^\pi \overline{T(x,y)}h(x)\,dx}\,dy. \end{aligned}$$

Therefore,

$$V^*h(y) = 2\int_y^\pi \overline{T(x,y)}h(x)\,dx.$$

It follows from (5.7) that the dual Riesz basis, $\{r_n^*\}_{n\in\mathbb{Z}}$ is given by

$$\begin{aligned} r_n^*(x) &= \cos t_n x + 2\int_0^x T(x,y)\cos t_n y\,dy + 2\int_x^\pi \overline{T(z,x)}\cos t_n z\,dz \\ &\quad +2\int_x^\pi \overline{T(z,x)}2\int_0^z T(z,y)\cos t_n y\,dy\,dz \\ &= \cos t_n x + 2\int_0^x T(x,y)\cos t_n y\,dy + 2\int_x^\pi \overline{T(y,x)}\cos t_n y\,dy \\ &\quad +4\int_x^\pi\int_0^z \overline{T(z,x)}T(z,y)\cos t_n y\,dy\,dz \\ &= \cos t_n x + 2\int_0^x T(x,y)\cos t_n y\,dy + 2\int_x^\pi \overline{T(y,x)}\cos t_n y\,dy \\ &\quad +4\int_0^\pi\int_{\max\{x,y\}}^\pi \overline{T(z,x)}T(z,y)\cos t_n y\,dz\,dy \\ &= \cos t_n x + 2\int_0^x T(x,y)\cos t_n y\,dy + 2\int_x^\pi \overline{T(y,x)}\cos t_n y\,dy \\ &\quad +4\int_0^\pi \cos t_n y\int_{\max\{x,y\}}^\pi \overline{T(z,x)}T(z,y)\,dz\,dy. \end{aligned}$$

■

5.4 Gram Matrices

Here we define the notion of a Gram matrix for a given sequence and look at a way to calculate the determinant of such a matrix. We see that by definition, the Gram matrix of an orthonormal sequence is the identity matrix. Therefore, by calculating Gram matrices we are able to compare sequences with orthonormal sequences. We use the sampling sequence, $(t_n)_{n\in\mathbb{Z}}$ obtained from the periodic spectrum of Hill's equation to create a sequence of reproducing kernels for the space $PW(\pi)$. It is this sequence of reproducing kernels that we calculate the Gram matrix and its determinant for. It will be seen that the Gram matrix for the sequence of reproducing kernels in $PW(\pi)$ is equivalent to the Gram matrix of the Riesz basis $\left\{e^{it_n x}\right\}_{n\in\mathbb{Z}}$ in $L^2[-\pi,\pi]$.

Definition 5.4.0.13 *Let $\{x_n\}$ be a sequence in an inner product space. The* Gram matrix *for the sequence $\{x_n\}$ is given by*

$$G = [\langle x_n, x_m\rangle]_{n,m}\,.$$

Remark 5.4.0.14 *If the sequence $\{x_n\}$ is orthonormal then the Gram matrix is the identity matrix.*

Recall from Proposition 2.4.3.4 that the function

$$k_t(x) = \frac{\sin b(t-x)}{\pi(t-x)}$$

is a reproducing kernel for $PW(b)$. Therefore, by Definition 5.2.0.11, $k_t(x) = \operatorname{sinc}(t-x)$ is a reproducing kernel for $PW(\pi)$.

Proposition 5.4.0.15 *Let $(t_n)_{n\in\mathbb{Z}}$ be the sampling sequence given in Definition 5.1.0.5. Then the sequence of reproducing kernels, $\{\operatorname{sinc}(t_n-x)\}_{n\in\mathbb{Z}}$ has Gram matrix*

$$G = [\operatorname{sinc}(t_n - t_m)]_{n,m\in\mathbb{Z}}\,.$$

Proof. By Definition 5.4.0.13, the Gram matrix of the sequence, $\{\operatorname{sinc}(t_n - x)\}_{n\in\mathbb{Z}}$ has entries given by

$$\langle \operatorname{sinc}(t_n - x), \operatorname{sinc}(t_m - x)\rangle.$$

Since $k_{t_n} = \operatorname{sinc}(t_n - x)$ is a reproducing kernel for $PW(\pi)$, it follows from Definition 2.4.3.3 that

$$\langle \operatorname{sinc}(t_n - x), \operatorname{sinc}(t_m - x)\rangle = \operatorname{sinc}(t_n - t_m).$$

Hence the result. ∎

The following lemma provides an alternative way of expressing the Gram matrix of $\{\operatorname{sinc}(t_n - x)\}_{n\in\mathbb{Z}}$. It allows us to calculate the determinant of the Gram matrix via Andréief's Identity 2.3.0.31. Furthermore, it also shows that the Gram matrix of $\{\operatorname{sinc}(t_n - x)\}_{n\in\mathbb{Z}}$ is equal to the Gram matrix of $\{e^{it_n x}\}_{n\in\mathbb{Z}}$ since

$$\left\langle e^{it_n x}, e^{it_m x}\right\rangle_{L^2[-\pi,\pi]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(t_n - t_m)x}\, dx.$$

Lemma 5.4.0.16 *Let $(t_n)_{n\in\mathbb{Z}}$ be the sampling sequence given in Definition 5.1.0.5. Also, let $\{\operatorname{sinc}(t_n - x)\}_{n\in\mathbb{Z}}$ be a sequence of reproducing kernels in $PW(\pi)$ with Gram matrix, G. Then,*

$$G = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(t_n - t_m)x}\, dx\right]_{n,m\in\mathbb{Z}}.$$

Proof. By Proposition 5.4.0.15, the sequence, $\{\operatorname{sinc}(t_n - x)\}_{n\in\mathbb{Z}}$ has Gram matrix

$$G = [\operatorname{sinc}(t_n - t_m)]_{n,m\in\mathbb{Z}}.$$

Now,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(t_n - t_m)x}\, dx &= \frac{1}{2\pi}\left[\frac{e^{i(t_n - t_m)x}}{i(t_n - t_m)}\right]_{x=-\pi}^{\pi} \\
&= \frac{e^{i\pi(t_n - t_m)} - e^{-i\pi(t_n - t_m)}}{2\pi i(t_n - t_m)} \\
&= \frac{\sin\pi(t_n - t_m)}{\pi(t_n - t_m)} \\
&= \operatorname{sinc}(t_n - t_m),
\end{aligned}$$

giving the result. ∎

Given a sampling sequence, $(t_n)_{n\in\mathbb{Z}}$ obtained from the periodic spectrum of Hill's equation, we can create a sequence of reproducing kernels, $\{\operatorname{sinc}(t_n - x)\}_{n\in\mathbb{Z}}$ for the space $PW(\pi)$. In the event that the sequence of reproducing kernels is finite, the following theorem shows how we can calculate the determinant of the corresponding Gram matrix.

Theorem 5.4.0.17 *Suppose that $j, k \in \{1, \ldots, n\}$ where $n < \infty$ and let $\{\operatorname{sinc}(t_k - x)\}_{k=1}^{n}$ be a sequence of reproducing kernels in $PW(\pi)$. Let G_n denote the Gram matrix of the sequence $\{\operatorname{sinc}(t_k - x)\}_{k=1}^{n}$, then G_n has determinant given by*

$$\det G_n = \frac{1}{n!}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \det\left[e^{it_j x_k}\right]_{j,k=1}^{n} \det\left[e^{-it_l x_k}\right]_{l,k=1}^{n} \frac{dx_1}{2\pi}\cdots\frac{dx_n}{2\pi}.$$

Proof. It follows from Lemma 5.4.0.16 that the determinant of the Gram matrix corresponding to the sequence $\{\operatorname{sinc}(t_k - x)\}_{k=1}^n$ is

$$\det G_n = \det \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(t_j - t_k)x} \, dx \right]_{j,k=1}^n .$$

Hence, by Andréief's Identity 2.3.0.31,

$$\det G_n = \frac{1}{n!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \det \left[e^{it_j x_k} \right]_{j,k=1}^n \det \left[e^{-it_l x_k} \right]_{l,k=1}^n \frac{dx_1}{2\pi} \cdots \frac{dx_n}{2\pi},$$

as required. ∎

Since the Gram matrix for the Riesz basis $\{e^{it_n x}\}_{n \in \mathbb{Z}}$ is equal to the Gram matrix for the reproducing kernels $\{\operatorname{sinc}(t_n - x)\}_{n \in \mathbb{Z}}$, Theorem 5.4.0.17 also holds for $\{e^{it_n x}\}_{n \in \mathbb{Z}}$.

5.4.1 The Operator $I + \Phi_n$

Let $n \in \mathbb{N}$ and recall that $\{e^{inx}\}_{n \in \mathbb{Z}}$ is an orthonormal basis and $\{e^{it_n x}\}_{n \in \mathbb{Z}}$ is a Riesz basis for the space $L^2[-\pi, \pi]$. We introduce the operator $I + \Phi_n$ to allow us to compare $\{e^{it_n x}\}_{n \in \mathbb{Z}}$ with an orthonormal sequence. In this section we define an operator $I + \Phi_n$ acting on the orthonormal basis $\{e^{inx}\}_{n \in \mathbb{Z}}$ such that $e^{ijx} \mapsto e^{it_j x}$ for $|j| \leq n$ and $e^{ijx} \mapsto e^{ijx}$ otherwise. We then seek to calculate the matrix associated with the operator $I + \Phi_n$ so that we may ultimately calculate its determinant. If $\{e^{it_n x}\}_{n \in \mathbb{Z}}$ is orthonormal then the matrix associated with $I + \Phi_n$ will be the identity matrix and it will have determinant equal to 1.

Definition 5.4.1.1 *Define the operator $I + \Phi_n : L^2[-\pi, \pi] \to L^2[-\pi, \pi]$ to be such that*

$$(I + \Phi_n) e^{ijx} = \begin{cases} e^{it_j x} & \text{for } |j| \leq n, \\ e^{ijx} & \text{for } |j| > n. \end{cases}$$

Remark 5.4.1.2 *We note that when $t_j = j$ for all j then $\Phi_n = 0$.*

We wish to construct the matrix of the operator $I + \Phi_n$ with respect to the basis $\{e^{inx}\}_{n \in \mathbb{Z}}$. Note that by Definition 2.3.0.20, the matrix of $I + \Phi_n$ with respect to the basis $\{e^{inx}\}_{n \in \mathbb{Z}}$ is given by

$$\left[\left\langle (I + \Phi_n) e^{ikx}, e^{ijx} \right\rangle_{L^2[-\pi, \pi]} \right]_{j,k} .$$

Proposition 5.4.1.3 *The operator $I + \Phi_n$ acting on $L^2[-\pi, \pi]$ has block matrix*

$$\left[\left\langle (I + \Phi_n) e^{ikx}, e^{ijx} \right\rangle_{L^2[-\pi, \pi]} \right]_{j,k} = \begin{bmatrix} I & [\operatorname{sinc}(t_k - j)]_{j < -n, |k| \leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k - j)]_{|j|, |k| \leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k - j)]_{j > n, |k| \leq n} & I \end{bmatrix}$$

with respect to the basis $\{e^{ijx}\}_{j \in \mathbb{Z}}$.

Proof. We must calculate the inner products, $\left\langle (I + \Phi_n) e^{ikx}, e^{ijx} \right\rangle_{L^2[-\pi, \pi]}$ for all j and k. From Definition 5.4.1.1 we have,

$$\left\langle (I + \Phi_n) e^{ikx}, e^{ijx} \right\rangle_{L^2[-\pi, \pi]} = \begin{cases} \left\langle e^{it_k x}, e^{ijx} \right\rangle_{L^2[-\pi, \pi]} & \text{for } |k| \leq n, \\ \left\langle e^{ikx}, e^{ijx} \right\rangle_{L^2[-\pi, \pi]} & \text{for } |k| > n. \end{cases}$$

Since $\{e^{ijx}\}_{j\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi,\pi]$, we know that

$$\langle e^{ikx}, e^{ijx}\rangle_{L^2[-\pi,\pi]} = \delta_{kj} = \begin{cases} 0 & \text{for } j \neq k, \\ 1 & \text{for } j = k. \end{cases}$$

Also,

$$\begin{aligned} \langle e^{it_k x}, e^{ijx}\rangle_{L^2[-\pi,\pi]} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(t_k-j)x}\,dx \\ &= \frac{e^{i\pi(t_k-j)} - e^{-i\pi(t_k-j)}}{2\pi i(t_k-j)} \\ &= \operatorname{sinc}(t_k - j). \end{aligned}$$

Therefore,

$$\langle (I+\Phi_n)e^{ikx}, e^{ijx}\rangle_{L^2[-\pi,\pi]} = \begin{cases} \operatorname{sinc}(t_k-j) & \text{for } |k| \leq n, \\ \delta_{kj} & \text{for } |k| > n. \end{cases}$$

It follows that $I+\Phi_n$ has matrix

$$\left[\langle (I+\Phi_n)e^{ikx}, e^{ijx}\rangle_{L^2[-\pi,\pi]}\right]_{j,k} = \begin{bmatrix} I & [\operatorname{sinc}(t_k-j)]_{j<-n,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k-j)]_{|j|,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k-j)]_{j>n,|k|\leq n} & I \end{bmatrix},$$

where j denotes the row and k the column. ■

Note the shape of the block matrix in Proposition 5.4.1.3. The central element is a finite, square $(2n+1)\times(2n+1)$ matrix. Above and below the central element are infinite matrices with $2n+1$ columns and an infinite number of rows. We progress to calculate the determinant of the operator $I+\Phi_n$.

Proposition 5.4.1.4 *The operator Φ_n is trace class. Furthermore, $I+\Phi_n$ has determinant given by*

$$\det(I+\Phi_n) = \frac{1}{(2n+1)!}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \det[e^{it_k x_l}]_{k,l=-n}^{n} \det[e^{-ijx_l}]_{j,l=-n}^{n} \frac{dx_{-n}}{2\pi}\cdots\frac{dx_n}{2\pi}.$$

Proof. From Proposition 5.4.1.3 we know that $I+\Phi_n$ has matrix

$$\begin{aligned} \left[\langle (I+\Phi_n)e^{ikx}, e^{ijx}\rangle_{L^2[-\pi,\pi]}\right]_{j,k} &= \begin{bmatrix} I & [\operatorname{sinc}(t_k-j)]_{j<-n,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k-j)]_{|j|,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k-j)]_{j>n,|k|\leq n} & I \end{bmatrix} \\ &= I + \begin{bmatrix} 0 & [\operatorname{sinc}(t_k-j)]_{j<-n,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k-j)-\delta_{jk}]_{|j|,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k-j)]_{j>n,|k|\leq n} & 0 \end{bmatrix}. \end{aligned}$$

Therefore the matrix associated with Φ_n is

$$\begin{bmatrix} 0 & [\operatorname{sinc}(t_k-j)]_{j<-n,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k-j)-\delta_{jk}]_{|j|,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k-j)]_{j>n,|k|\leq n} & 0 \end{bmatrix}. \tag{5.8}$$

Now, the rank of Φ_n is equal to the column rank of the matrix given by (5.8). Hence, rank $\Phi_n \leq 2n+1$ which means that Φ_n has finite rank. By [51] (Corollary 2.3, page 17), every finite rank operator is a trace class operator. So it follows that Φ_n is trace class and thus $\det(I+\Phi_n)$ is defined. We proceed to calculate this determinant. Let

$$\begin{aligned} A &= [\operatorname{sinc}(t_k - j)]_{j<-n, |k|\leq n}, \\ B &= [\operatorname{sinc}(t_k - j)]_{|j|,|k|\leq n}, \\ C &= [\operatorname{sinc}(t_k - j)]_{j>n, |k|\leq n}. \end{aligned}$$

Then

$$\begin{aligned} \begin{bmatrix} I & [\operatorname{sinc}(t_k - j)]_{j<-n,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k - j)]_{|j|,|k|\leq n} & 0 \\ 0 & [\operatorname{sinc}(t_k - j)]_{j>n,|k|\leq n} & I \end{bmatrix} &= \begin{bmatrix} I & A & 0 \\ 0 & B & 0 \\ 0 & C & I \end{bmatrix} \qquad (5.9) \\ &= \begin{bmatrix} I & 0 & 0 \\ 0 & B & 0 \\ 0 & C & I \end{bmatrix} \begin{bmatrix} I & A & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & C & I \end{bmatrix} \begin{bmatrix} I & A & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \qquad (5.10) \end{aligned}$$

Taking the determinant of both sides of (5.10) and noting that each matrix is triangular, we obtain

$$\begin{aligned} \det(I+\Phi_n) &= \det \begin{bmatrix} I & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{bmatrix} \det \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & C & I \end{bmatrix} \det \begin{bmatrix} I & A & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \det B \\ &= \det[\operatorname{sinc}(t_k - j)]_{j,k=-n}^{n}. \end{aligned}$$

Given that

$$\operatorname{sinc}(t_k - j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(t_k - j)x}\, dx$$

it follows that

$$\det(I+\Phi_n) = \det \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(t_k - j)x}\, dx \right]_{j,k=-n}^{n}.$$

Finally, applying Andréief's Identity 2.3.0.31 produces the desired result,

$$\det(I+\Phi_n) = \frac{1}{(2n+1)!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \det[e^{it_k x_l}]_{k,l=-n}^{n} \det[e^{-ijx_l}]_{j,l=-n}^{n} \frac{dx_{-n}}{2\pi} \cdots \frac{dx_n}{2\pi}.$$

■

Note the similarity between the determinant given in Proposition 5.4.1.4 and the determinant of the Gram matrix appearing in Theorem 5.4.0.17. We return to look at determinants of the type given in Proposition 5.4.1.4, in Chapter 6.

5.5 Lipschitz Dependence of the Sampling Sequence on the Potential

This section covers one of the main new results of this thesis, it can also be found in [7]. Let $\Phi_t : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ be the linear operator defined by $\Phi_t \sum a_n e^{inx} = \sum a_n e^{it_n x}$ where $\{e^{it_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$. Note that by Proposition 5.3.0.22, $\{e^{it_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis if $|t_n - n| < \frac{1}{4}$. Also, since $\{e^{it_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis it follows that Φ_t is well defined and bounded. Now, $\Phi_t : e^{inx} \mapsto e^{it_n x}$, therefore the Gram matrix associated with the operator Φ is equal to the Gram matrix given by the sequence $\left(e^{it_n x}\right)_{n\in\mathbb{Z}}$. In this section we see that the Gram matrix of $\left(e^{it_n x}\right)_{n\in\mathbb{Z}}$ is a Lipschitz function of the sequence $(t_n)_{n\in\mathbb{Z}}$. The results of this section have been shown to hold for any sequence $(t_n)_{n\in\mathbb{Z}}$ such that $\{e^{it_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis. Hence all results found here will certainly hold for the sampling sequence derived from the periodic spectrum of Hill's equation as defined in Definition 5.1.0.5. The structure of the periodic spectrum of Hill's equation as a set of points is therefore linked to the sequence $\left(e^{it_n x}\right)_{n\in\mathbb{Z}}$ in Hilbert space.

Definition 5.5.0.5 *Let H_1 and H_2 be Hilbert spaces. We say that $\Phi : H_1 \to H_2$ is* Lipschitz *with constant L if there exists $L > 0$ such that*

$$\|\Phi(x) - \Phi(y)\|_{H_2} \leq L \|x - y\|_{H_1}$$

for all $x, y \in H_1$.

Theorem 5.5.0.6 *Suppose that $t_n \in \mathbb{R}$ for all n and that $\{e^{it_n x}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi,\pi]$. Set $t = (t_n)$ and suppose that $(t_n - n) \in \ell^2$. Let $\Phi_t : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ be the linear operator*

$$\Phi_t \sum a_n e^{inx} = \sum a_n e^{it_n x}.$$

Then,
1) $\Phi_t - I$ is Hilbert–Schmidt;
2) the map $(t_n - n) \mapsto \Phi_t - I$ is Lipschitz $\ell^2 \to \mathrm{HS}$.

Proof. Given $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi,\pi]$, let $\Phi_t\left(e^{inx}\right) = e^{it_n x}$. Then

$$\begin{aligned}
\|\Phi_t - I\|_{HS}^2 &= \sum_n \left\|(\Phi_t - I)(e^{inx})\right\|^2_{L^2[-\pi,\pi]} \\
&= \sum_n \left\|e^{it_n x} - e^{inx}\right\|^2_{L^2[-\pi,\pi]} \\
&= \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|e^{it_n x} - e^{inx}\right|^2 dx \\
&= \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|e^{i(t_n-n)x} - 1\right|^2 dx.
\end{aligned}$$

Using the fact that $|z|^2 = z\bar{z}$ for complex z we have

$$\begin{aligned}
\left|e^{i(t_n-n)x} - 1\right|^2 &= \left[e^{i(t_n-n)x} - 1\right]\left[e^{-i(t_n-n)x} - 1\right] \\
&= 2 - e^{i(t_n-n)x} - e^{-i(t_n-n)x} \\
&= 2 - 2\cos(t_n - n)x.
\end{aligned}$$

The double angle formulae then gives

$$2 - 2\cos(t_n - n)x = 4\sin^2 \frac{1}{2}(t_n - n)x,$$

thus,

$$\|\Phi_t - I\|_{HS}^2 = \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sin^2 \frac{1}{2}(t_n - n)x\,dx.$$

Given $|\sin x| \leq x$ for all $x > 0$ we have the following inequality,

$$\sin^2 \frac{1}{2}(t_n - n)x \leq \frac{1}{4}|t_n - n|^2 x^2.$$

Hence,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sin^2 \frac{1}{2}(t_n - n)x\,dx &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |t_n - n|^2 x^2\,dx \\
&= \frac{1}{2\pi}|t_n - n|^2 \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}|t_n - n|^2 \left[\frac{2\pi^3}{3}\right] \\
&= \frac{\pi^2}{3}|t_n - n|^2.
\end{aligned}$$

Therefore

$$\|\Phi_t - I\|_{HS}^2 \leq \frac{\pi^2}{3}\sum_n |t_n - n|^2.$$

Since $(t_n - n) \in \ell^2$ it follows that

$$\|(t_n) - (n)\|_{\ell^2}^2 = \sum_n |t_n - n|^2$$

is convergent, hence

$$\|\Phi_t - I\|_{HS}^2 \leq \frac{\pi^2}{3}\|(t_n) - (n)\|_{\ell^2}^2$$

proving that $\Phi_t - I$ is Hilbert–Schmidt.

We now prove, using the same method, that the map $(t_n - n) \mapsto \Phi_t - I$ is Lipschitz. First note that $(\Phi_t - I) - (\Phi_s - I) = \Phi_t - \Phi_s$ and so the proof reduces to showing that Φ_t satisfies the Lipschitz condition. By the first part of the theorem, $\Phi_t - I$ is Hilbert–Schmidt thus the map $(t_n - n) \mapsto \Phi_t - I$ takes the space ℓ^2 to HS. Now,

$$\begin{aligned}
\|\Phi_t - \Phi_s\|_{HS}^2 &= \sum_n \left\|(\Phi_t - \Phi_s)(e^{inx})\right\|_{L^2[-\pi,\pi]}^2 \\
&= \sum_n \left\|e^{it_n x} - e^{is_n x}\right\|_{L^2[-\pi,\pi]}^2 \\
&= \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|e^{it_n x} - e^{is_n x}\right|^2 dx \\
&= \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|e^{i(t_n - s_n)x} - 1\right|^2 dx.
\end{aligned}$$

Again we use $|z|^2 = z\bar{z}$ followed by the double angle formulae to simplify the above giving,

$$\begin{aligned}
\|\Phi_t - \Phi_s\|_{HS}^2 &= \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} 2 - 2\cos(t_n - s_n)x\,dx \\
&= \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sin^2 \frac{1}{2}(t_n - s_n)x\,dx.
\end{aligned}$$

Since $|\sin x| \le x$ for all $x > 0$ it follows that

$$\sin^2 \frac{1}{2}(t_n - s_n)x \le \frac{1}{4}|t_n - s_n|^2 x^2,$$

hence

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sin^2 \frac{1}{2}t_n - s_n)x\,dx &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |t_n - s_n|^2 x^2\,dx \\ &= \frac{1}{2\pi}|t_n - s_n|^2 \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}|t_n - s_n|^2 \left[\frac{2\pi^3}{3}\right] \\ &= \frac{\pi^2}{3}|t_n - s_n|^2 .\end{aligned}$$

Therefore

$$\begin{aligned}\|\Phi_t - \Phi_s\|_{HS}^2 &\le \sum_n \frac{\pi^2}{3}|t_n - s_n|^2 \\ &= \frac{\pi^2}{3}\|(t_n) - (s_n)\|_{\ell^2}^2 ,\end{aligned}$$

and so the map $(t_n - n) \mapsto \Phi_t - I$ is Lipschitz $\ell^2 \to \text{HS}$. ■

The following corollary shows that the determinant of the Gram matrix associated with the sequence $\left\{e^{it_n x}\right\}_{n\in\mathbb{Z}}$ is a Lipschitz continuous function of $(t_n)_{n\in\mathbb{Z}}$.

Corollary 5.5.0.7 *The map* $t \mapsto \det_2 \Phi_t$ *is continuous* $\ell^2 \to \mathbb{C}$. *Further, when* $t = (n)$, $\det_2 \Phi_n = 1$.

Proof. Let T be a Hilbert–Schmidt operator then the map $T \mapsto \det_2(I + T)$ is continuous by [15] (Lemma 22(b), page 1106). By Theorem 5.5.0.6, $\Phi_t - I$ is Hilbert–Schmidt and so

$$\Phi_t - I \mapsto \det_2 \Phi_t$$

is continuous. Finally, we note that when $t = (n)$ we have $\Phi_n = I$ and so $\det_2 \Phi_n = 1$. ■

Chapter 6

Determinants Associated with an Integral of Ramanujan

The integral $I_a(t)$ arises in several contexts, examples being sampling theory and orthogonal polynomials. The main aim of this chapter is to systematically analyse determinants with entries $I_a(t_j - k)$ where

$$I_a(t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\, dx,$$

is an integral associated with Ramanujan. Ramanujan is credited with proving an identity for $I_a(t)$ in terms of the Gamma function. Of particular interest is the case in which $a \in \mathbb{N}$ and $t_j = j$ since then $I_a(t_j - k)$ can be given a factorial expression, via the Gamma function, which leads to a Toeplitz type matrix.

We also show that $\{I_a\}$ gives a basis for the even functions in the Paley–Wiener space, $PW\left(\frac{\pi}{2}\right)$. Further, this basis is related to the reproducing kernel for the same Paley–Wiener space.

We finish this chapter with a short section on Chebyshev polynomials and show how the function I_a can be used to evaluate some integrals involving Chebyshev polynomials.

6.1 Unitary Groups and the Weyl Denominator Formula

This brief section is intended to introduce the Weyl Denominator Formula as defined for the unitary group. We note the formula in order to make comparisons with calculations carried out within this chapter.

We begin by defining the unitary group.

Definition 6.1.0.8 *Given $n \in \mathbb{N}$, we denote by $GL(n, \mathbb{C})$ the set of all invertible $n \times n$ complex matrices. We call $GL(n, \mathbb{C})$ the* general linear group*. The group operation of $GL(n, \mathbb{C})$ is matrix multiplication. The* unitary group*, $U(n, \mathbb{C})$ is defined to be*

$$U(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : A^* A = I\}.$$

Also, we define the special unitary group, $SU(n,\mathbb{C})$ *to be*

$$SU(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) : A^*A = I, \det A = 1\}.$$

The following proposition shows that a unitary matrix has complex eigenvalues that lie on the unit circle. Further, the sum of the arguments of the eigenvalues of a special unitary matrix is a multiple of 2π.

Proposition 6.1.0.9 *Suppose that $A \in U(n,\mathbb{C})$. Then A has eigenvalues, $\left\{e^{i\theta_1},\ldots,e^{i\theta_n}\right\}$. Further, if $A \in SU(n,\mathbb{C})$ with eigenvalues $\left\{e^{i\theta_1},\ldots,e^{i\theta_n}\right\}$, then*

$$\sum_{j=1}^{n} \theta_j = 2\pi k$$

for $k \in \mathbb{Z}$.

Proof. Let $A \in U(n,\mathbb{C})$ and suppose that λ is an eigenvalue of A. Then $Ax = \lambda x$ where x is the eigenvector corresponding to λ. Now,

$$\begin{aligned} |\lambda|^2 \langle x,x\rangle &= \langle \lambda x, \lambda x\rangle \\ &= \langle Ax, Ax\rangle \\ &= \langle A^*Ax, x\rangle \end{aligned}$$

where the last line follows from Definition 2.1.0.7. Since $A \in U(n,\mathbb{C})$, $A^*A = I$ and so

$$|\lambda|^2 \langle x,x\rangle = \langle x,x\rangle.$$

Therefore we must have $|\lambda| = 1$. That is, the eigenvalues lie on the unit circle.

Now let $A \in SU(n,\mathbb{C})$ and suppose that A has eigenvalues $\left\{e^{i\theta_1},\ldots,e^{i\theta_n}\right\}$. Then A is similar to the matrix, $D = \text{diag}\left(e^{i\theta_1},\ldots,e^{i\theta_n}\right)$. Also, since $A \in SU(n,\mathbb{C})$, $\det D = \det A = 1$. Therefore,

$$\begin{aligned} 1 &= \det \text{diag}\left(e^{i\theta_1},\ldots,e^{i\theta_n}\right) \\ &= \prod_{j=1}^{n} e^{i\theta_j} \\ &= e^{i\sum_{j=1}^{n}\theta_j}. \end{aligned} \tag{6.1}$$

It is now easily seen that in order for (6.1) to hold we must have $\sum_{j=1}^{n}\theta_j = 2\pi k$ for some $k \in \mathbb{Z}$. ■

Proposition 6.1.0.9 provides some insight into the motivation behind the Weyl Denominator Formula 6.1.0.10. Indeed the formula is defined for the unitary group which, as we have just seen has complex eigenvalues that lie on the unit circle. In the definition that follows we view $\left\{e^{i\theta_j}\right\}$ as eigenvalues of a unitary matrix.

Definition 6.1.0.10 *The* Weyl denominator *for the group $SU(n,\mathbb{C})$ is given by the product*

$$\prod_{j<k}\left[e^{2\pi i(\theta_j-\theta_k)} - 1\right].$$

6.2 The Integral I_a

We introduce the function, I_a, an integral associated with Ramanujan. In Section 6.4 we will see that Ramanujan evaluated the integral I_a using Gamma functions. Ramanujan's formula will be used in Section 6.5 where we evaluate determinants of I_a at points, t_n. Also in this section we show that, under certain conditions, the function I_a lies in a Paley–Wiener space.

Definition 6.2.0.11 *Let $a \in \mathbb{R}$. For $t \in \mathbb{R}$ define the function I_a to be*

$$I_a(t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\, dx.$$

In the following lemma we show that I_a is an even function. This fact will be used in various proofs throughout the remaining sections.

Lemma 6.2.0.12 *Let I_a be as defined in Definition 6.2.0.11. Then I_a is an even function.*

Proof. We follow the standard method of proving that a function is even by showing $I_a(t) = I_a(-t)$. Thus,

$$\begin{aligned} I_a(-t) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{-itx}\, dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos(-x)]^{a-2} e^{it(-x)}\, dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{-itx}\, dx \\ &= I_a(t) \end{aligned}$$

as required. ∎

Under certain conditions, the function I_a is a Paley–Wiener function. The following proposition shows that for $a > \frac{3}{2}$, I_a belongs to a Paley–Wiener space.

Proposition 6.2.0.13 *Let I_a be as given in Definition 6.2.0.11. If $a > \frac{3}{2}$ then $I_a \in PW\left(\frac{\pi}{2}\right)$.*

Proof. First note that $I_a(t)$ takes the form specified by the converse of the Paley–Wiener Theorem 2.4.3.7. We show that $(\cos x)^{a-2} \in L^2\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for $a > \frac{3}{2}$ by showing that the inequality,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|(\cos x)^{a-2}\right|^2 dx < \infty$$

is satisfied. We note that, on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have $|\cos x| = \cos x$ since $0 \leq \cos x \leq 1$. Therefore, we need to show that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{2a-4}\, dx < \infty. \tag{6.2}$$

For $a \geq 2$, (6.2) is easily satisfied since

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{2a-4}\, dx &\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, dx \\ &= \pi \end{aligned}$$

which is clearly finite. Now take $\frac{3}{2} < a < 2$. This means that $2a-4$ is negative. Write $2a-4 = -(4-2a)$ where $4-2a$ is positive, then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{2a-4}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(\cos x)^{4-2a}}\, dx.$$

Next we make the substitution $x = t - \frac{\pi}{2}$. This yields

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{2a-4}\, dx &= \int_0^{\pi} \frac{1}{\left(\cos\left(t-\frac{\pi}{2}\right)\right)^{4-2a}}\, dt \\ &= \int_0^{\pi} \frac{1}{(\sin t)^{4-2a}}\, dt \end{aligned}$$

where, in the last line we have used the fact that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$. From the symmetry of the graph $\frac{1}{\sin t}$ we have

$$\int_0^{\pi} \frac{1}{(\sin t)^{4-2a}}\, dt = 2\int_0^{\frac{\pi}{2}} \frac{1}{(\sin t)^{4-2a}}\, dt.$$

To finish the calculation we find a bound on $\frac{1}{(\sin t)^{4-2a}}$. We achieve this by showing that the function $\frac{\sin t}{t}$ is decreasing on the interval $\left[0, \frac{\pi}{2}\right]$ and then use this fact to find a lower bound for $\frac{\sin t}{t}$, thus giving an upper bound for $\frac{1}{\sin t}$. Now, $\frac{\sin t}{t}$ is decreasing for $0 \leq t \leq \frac{\pi}{2}$ since

$$\begin{aligned} \frac{d}{dt}\left(\frac{\sin t}{t}\right) &= \frac{t\cos t - \sin t}{t^2} \\ &= \frac{\cos t(t - \tan t)}{t^2} \\ &\leq 0. \end{aligned}$$

In the above the last line follows as on the interval $\left[0, \frac{\pi}{2}\right]$, we have t^2 non-negative, $0 \leq \cos t \leq 1$ and $t - \tan t \leq 0$. To see that $t - \tan t$ is negative we note that it is decreasing on the interval $\left[0, \frac{\pi}{2}\right]$ since

$$\begin{aligned} \frac{d}{dt}(t - \tan t) &= 1 - \frac{1}{\cos^2 t} \\ &\leq 0 \end{aligned}$$

because $0 \leq \cos t \leq 1$. Therefore, $t - \tan t$ attains its highest bound on the interval $\left[0, \frac{\pi}{2}\right]$ at the point 0 and so $t - \tan t \leq 0$. This shows that the function $\frac{\sin t}{t}$ is decreasing on $\left[0, \frac{\pi}{2}\right]$, meaning that $\frac{\sin t}{t}$ has a lower bound at the point $\frac{\pi}{2}$. Thus,

$$\begin{aligned} \frac{\sin t}{t} &\geq \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} \\ &= \frac{2}{\pi}. \end{aligned}$$

The desired bound for the function $\frac{1}{(\sin t)^{4-2a}}$ is therefore

$$\frac{1}{(\sin t)^{4-2a}} \leq \left(\frac{\pi}{2t}\right)^{4-2a}.$$

Continuing now to show that $(\cos x)^{a-2} \in L^2\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we see that we have

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{2a-4}\, dx &= 2\int_0^{\frac{\pi}{2}} \frac{1}{(\sin t)^{4-2a}}\, dt \\
&\leq 2\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2t}\right)^{4-2a} dt \\
&= \frac{\pi^{4-2a}}{2^{3-2a}}\left[\frac{1}{(2a-3)t^{3-2a}}\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi^{4-2a}}{2^{3-2a}}\left[\frac{2^{3-2a}}{(2a-3)\pi^{3-2a}}\right] \\
&= \frac{\pi}{2a-3} \\
&< \infty
\end{aligned}$$

as required. Hence, for $a > \frac{3}{2}$, $(\cos x)^{a-2} \in L^2\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. It then follows from the converse of the Paley–Wiener Theorem that $I_a(t) \in L^2(\mathbb{R})$ and is entire.

It remains to show that $\hat{I}_a \in L^2\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that we can write

$$\begin{aligned}
I_a(t) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\, dx \\
&= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \sqrt{2\pi}\,\mathbb{I}_{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}(x)(\cos x)^{a-2} e^{itx}\, dx.
\end{aligned}$$

This shows that I_a is the inverse Fourier transform of $\sqrt{2\pi}\,\mathbb{I}_{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}(x)(\cos x)^{a-2}$. It follows then that

$$\hat{I}_a(x) = \sqrt{2\pi}\,\mathbb{I}_{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}(x)(\cos x)^{a-2}$$

from which it is easily seen (by the first calculation of this proof) that $\hat{I}_a \in L^2\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. ■

6.3 Using I_a to Find a Basis for $PW\left(\frac{\pi}{2}\right)$

In the case that $a \in \mathbb{N}$, the formula for I_a can be greatly simplified. In fact, when $a \in \mathbb{N}$, I_a can be written as a sum of sinc functions. We use this observation to ultimately show that the set $\{I_{2b}\}_{b\in\mathbb{N}}$ gives a basis for the even functions in the space $PW\left(\frac{\pi}{2}\right)$. On the way to proving this we note that the sinc functions arising as reproducing kernels of the space $PW\left(\frac{\pi}{2}\right)$ can be used to construct an orthonormal basis for $PW\left(\frac{\pi}{2}\right)$.

The first task is to write I_a in terms of sinc functions. The following proposition shows that when $a \in \mathbb{N}$, the function I_a can be expressed as a sum of sinc functions.

Proposition 6.3.0.14 *Suppose that $a \in \mathbb{N}$. Then,*

$$I_a(t) = \frac{\pi}{2^{a-2}} \sum_{j=0}^{a-2} \binom{a-2}{j} \operatorname{sinc} \frac{1}{2}[t + a - 2(j+1)].$$

Proof. Let $a \in \mathbb{N}$, then

$$\begin{aligned}
I_a(t) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\, dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{e^{ix}+e^{-ix}}{2}\right]^{a-2} e^{itx}\, dx.
\end{aligned}$$

We use the binomial theorem to expand $(\cos x)^{a-2}$ as follows

$$\begin{aligned}\left[\frac{e^{ix}+e^{-ix}}{2}\right]^{a-2} &= \frac{1}{2^{a-2}}\sum_{j=0}^{a-2}\binom{a-2}{j}e^{i(a-j-2)x}e^{-ijx}\\ &= \frac{1}{2^{a-2}}\sum_{j=0}^{a-2}\binom{a-2}{j}e^{i(a-2j-2)x}.\end{aligned}$$

Therefore,

$$\begin{aligned}I_a(t) &= \frac{1}{2^{a-2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\sum_{j=0}^{a-2}\binom{a-2}{j}e^{i(a-2j-2)x}\right]e^{itx}\,dx\\ &= \frac{1}{2^{a-2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_{j=0}^{a-2}\binom{a-2}{j}e^{i(t+a-2j-2)x}\,dx\\ &= \frac{1}{2^{a-2}}\sum_{j=0}^{a-2}\binom{a-2}{j}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{i(t+a-2j-2)x}\,dx.\end{aligned}$$

Now,

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{i(t+a-2j-2)x}\,dx &= \left[\frac{e^{i(t+a-2j-2)x}}{i(t+a-2j-2)}\right]_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= \frac{e^{i\frac{\pi}{2}(t+a-2j-2)}-e^{-i\frac{\pi}{2}(t+a-2j-2)}}{i(t+a-2j-2)}\\ &= \frac{2\sin\frac{\pi}{2}(t+a-2j-2)}{t+a-2j-2}\\ &= \pi\operatorname{sinc}\frac{1}{2}(t+a-2j-2).\end{aligned}$$

Hence,

$$I_a(t) = \frac{\pi}{2^{a-2}}\sum_{j=0}^{a-2}\binom{a-2}{j}\operatorname{sinc}\frac{1}{2}[t+a-2(j+1)]$$

as required. ∎

Remark 6.3.0.15 *Notice that if a is even so that $a = 2b$ for some $b \in \mathbb{N}$ then*

$$I_{2b}(t) = \frac{\pi}{2^{2b-2}}\sum_{j=0}^{2b-2}\binom{2b-2}{j}\operatorname{sinc}\frac{1}{2}[t+2(b-j-1)].$$

Similarly if $a = 2b+1$ for some $b \in \mathbb{N}$ so that a is odd then

$$I_{2b+1}(t) = \frac{\pi}{2^{2b-1}}\sum_{j=0}^{2b-1}\binom{2b-1}{j}\operatorname{sinc}\frac{1}{2}[t+2(b-j)-1].$$

The remainder of this section is devoted to finding a basis based on the functions I_a, for the even functions in the space $PW\left(\frac{\pi}{2}\right)$. We first seek to show that the reproducing kernels for $PW\left(\frac{\pi}{2}\right)$ form an orthonormal basis for the space.

Proposition 6.3.0.16 *Let $k_{2n}(t) = \frac{1}{2}\operatorname{sinc}\frac{1}{2}(t-2n)$ be a reproducing kernel for $PW(\frac{\pi}{2})$. Then the set $\left\{\sqrt{2}k_{2n}\right\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $PW(\frac{\pi}{2})$.*

Proof. Let $f \in PW(\frac{\pi}{2})$ then by the Whittaker–Kotel'nikov–Shannon Sampling Theorem 5.2.0.13,

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f(2n) \frac{\sin \frac{\pi}{2}(t-2n)}{\frac{\pi}{2}(t-2n)} \\ &= \sum_{n=-\infty}^{\infty} f(2n) \operatorname{sinc} \frac{1}{2}(t-2n). \end{aligned}$$

By Proposition 2.4.3.4, the space $PW(\frac{\pi}{2})$ has reproducing kernel

$$k_s(t) = \frac{1}{2} \operatorname{sinc} \frac{1}{2}(t-s),$$

thus

$$\operatorname{sinc} \frac{1}{2}(t-2n) = 2k_{2n}(t).$$

Further, as k_{2n} is a reproducing kernel we have

$$f(2n) = \langle f, k_{2n} \rangle.$$

Therefore,

$$\begin{aligned} f(t) &= 2 \sum_{n=-\infty}^{\infty} \langle f, k_{2n} \rangle k_{2n}(t) \\ &= \sum_{n=-\infty}^{\infty} \langle f, \sqrt{2}k_{2n} \rangle \sqrt{2}k_{2n}(t). \end{aligned}$$

Finally we note that

$$\begin{aligned} \langle \sqrt{2}k_{2m}, \sqrt{2}k_{2n} \rangle &= 2k_{2m}(2n) \\ &= \operatorname{sinc}(n-m). \end{aligned}$$

Thus, by Lemma 5.2.0.12

$$\langle \sqrt{2}k_{2m}, \sqrt{2}k_{2n} \rangle = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$

It now follows that $\left\{\sqrt{2}k_{2n}\right\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $PW\left(\frac{\pi}{2}\right)$. ■

Remark 6.3.0.17 *Notice that the* sinc *term in I_a for $a = 2b$ takes the form*

$$\operatorname{sinc} \frac{1}{2}[t + 2(b-j-1)] = 2k_{2n}(t)$$

where $n = -(b-j-1)$ is an integer. Similarly, when $a = 2b+1$ we have

$$\operatorname{sinc} \frac{1}{2}[t + 2(b-j) - 1] = 2k_{1-2(b-j)}(t).$$

In Proposition 6.3.0.14 we saw that the function I_a could be expressed in terms of sinc functions. Remark 6.3.0.17 shows that we can express these sinc functions in terms of the reproducing kernel for $PW\left(\frac{\pi}{2}\right)$. That is, we can use the orthonormal basis, $\left\{\frac{\sqrt{2}}{2} \operatorname{sinc} \frac{1}{2}(t-2n)\right\}_{n\in\mathbb{Z}}$ to construct I_a. It is therefore natural to ask whether $\{I_a\}_{a\in\mathbb{Z}}$, or at least $\{I_{2b}\}_{b\in\mathbb{Z}}$, forms a basis for $PW\left(\frac{\pi}{2}\right)$. We proceed to answer this question.

Lemma 6.3.0.18 *Let a be even so that $a = 2b$ for some $b \in \mathbb{N}$. Then I_{2b} is an even function and*

$$I_{2b}(t) = \frac{\pi}{2^{2b-1}} \sum_{j=0}^{2b-2} \binom{2b-2}{j} \left\{ \operatorname{sinc} \frac{1}{2}[t + 2(b-j-1)] + \operatorname{sinc} \frac{1}{2}[t - 2(b-j-1)] \right\}.$$

Proof. Let $b \in \mathbb{N}$ and set $a = 2b$. By Proposition 6.3.0.14 we have

$$I_{2b}(t) = \frac{\pi}{2^{2b-2}} \sum_{j=0}^{2b-2} \binom{2b-2}{j} \operatorname{sinc} \frac{1}{2}[t + 2(b-j-1)]. \tag{6.3}$$

If we run the summation 'backwards', that is, sum from the $(2b-2)^{\text{nd}}$ term to the zeroth term then I_{2b} becomes

$$\begin{aligned} I_{2b}(t) &= \frac{\pi}{2^{2b-2}} \sum_{j=0}^{2b-2} \binom{2b-2}{2b-j-2} \operatorname{sinc} \frac{1}{2}\{t + 2[b - (2b-j-2) - 1]\} \\ &= \frac{\pi}{2^{2b-2}} \sum_{j=0}^{2b-2} \binom{2b-2}{j} \operatorname{sinc} \frac{1}{2}[t - 2(b-j-1)]. \end{aligned} \tag{6.4}$$

Adding (6.3) and (6.4) produces

$$2I_{2b}(t) = \frac{\pi}{2^{2b-2}} \sum_{j=0}^{2b-2} \binom{2b-2}{j} \left\{ \operatorname{sinc} \frac{1}{2}[t + 2(b-j-1)] + \operatorname{sinc} \frac{1}{2}[t - 2(b-j-1)] \right\}$$

as required.

Finally we note that I_{2b} is even by Lemma 6.2.0.12. ■

The following proposition gives a basis for the even functions in $PW\left(\frac{\pi}{2}\right)$. Again, the basis is constructed using reproducing kernels for $PW\left(\frac{\pi}{2}\right)$.

Proposition 6.3.0.19 *The set*

$$B_E = \left\{ \frac{\sqrt{2}}{2} \operatorname{sinc} \frac{1}{2} t \right\} \cup \left\{ \frac{1}{2} \left[\operatorname{sinc} \frac{1}{2}(t + 2n) + \operatorname{sinc} \frac{1}{2}(t - 2n) \right] \right\}_{n=1}^{\infty}$$

forms an orthonormal basis for the even functions in the space $PW\left(\frac{\pi}{2}\right)$.

Proof. By Proposition 6.3.0.16, we know that $\left\{ \frac{\sqrt{2}}{2} \operatorname{sinc} \frac{1}{2}(t - 2n) \right\}_{n \in \mathbb{Z}}$ forms an orthonormal basis for $PW\left(\frac{\pi}{2}\right)$ and so, for $f \in PW\left(\frac{\pi}{2}\right)$, we may write

$$f(t) = \frac{\sqrt{2}}{2} \sum_{n=-\infty}^{\infty} a_n \operatorname{sinc} \frac{1}{2}(t - 2n) \tag{6.5}$$

where $a_n = \left\langle f, \frac{\sqrt{2}}{2} \operatorname{sinc} \frac{1}{2}(s - 2n) \right\rangle$ for $n \in \mathbb{Z}$. Note that we can reformulate (6.5) so that the summation runs over $\mathbb{N}_0$ as follows,

$$f(t) = \frac{\sqrt{2}}{2} a_0 \operatorname{sinc} \frac{1}{2} t + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left[a_n \operatorname{sinc} \frac{1}{2}(t - 2n) + a_{-n} \operatorname{sinc} \frac{1}{2}(t + 2n) \right].$$

Further, if f is even then f must satisfy $f(t) = f(-t)$. So, $f \in PW\left(\frac{\pi}{2}\right)$ is even if and only if

$$\begin{aligned} 0 &= f(t) - f(-t) \\ &= \frac{\sqrt{2}}{2} a_0 \operatorname{sinc} \frac{1}{2} t + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left[a_n \operatorname{sinc} \frac{1}{2}(t - 2n) + a_{-n} \operatorname{sinc} \frac{1}{2}(t + 2n) \right] - \frac{\sqrt{2}}{2} a_0 \operatorname{sinc} \frac{1}{2}(-t) \\ &\quad - \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left[a_n \operatorname{sinc} \frac{1}{2}(-t - 2n) + a_{-n} \operatorname{sinc} \frac{1}{2}(-t + 2n) \right]. \end{aligned}$$

Since the sinc function is even, we may simplify the above to obtain

$$\begin{aligned}0 &= \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\left[a_n \operatorname{sinc}\frac{1}{2}(t-2n) + a_{-n}\operatorname{sinc}\frac{1}{2}(t+2n)\right] \\ &\quad -\frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\left[a_n \operatorname{sinc}\frac{1}{2}(t+2n) + a_{-n}\operatorname{sinc}\frac{1}{2}(t-2n)\right] \\ &= \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}(a_n - a_{-n})\left[\operatorname{sinc}\frac{1}{2}(t-2n) - \operatorname{sinc}\frac{1}{2}(t+2n)\right]. \qquad (6.6)\end{aligned}$$

Now, if (6.6) is true for all t then we must have $a_n - a_{-n} = 0$. It follows that $f \in PW\left(\frac{\pi}{2}\right)$ is even if and only if $a_n = a_{-n}$. Hence for even $f \in PW\left(\frac{\pi}{2}\right)$ we have

$$\begin{aligned}f(t) &= \frac{\sqrt{2}}{2}a_0 \operatorname{sinc}\frac{1}{2}t + \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\left[a_n \operatorname{sinc}\frac{1}{2}(t-2n) + a_{-n}\operatorname{sinc}\frac{1}{2}(t+2n)\right] \\ &= \frac{\sqrt{2}}{2}a_0 \operatorname{sinc}\frac{1}{2}t + \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}a_n\left[\operatorname{sinc}\frac{1}{2}(t-2n) + \operatorname{sinc}\frac{1}{2}(t+2n)\right].\end{aligned}$$

It is now clear that the set $B_E = \left\{\frac{\sqrt{2}}{2}\operatorname{sinc}\frac{1}{2}t\right\} \cup \left\{\frac{1}{2}\left[\operatorname{sinc}\frac{1}{2}(t+2n) + \operatorname{sinc}\frac{1}{2}(t-2n)\right]\right\}_{n=1}^{\infty}$ does indeed form a basis for the even functions in the space $PW\left(\frac{\pi}{2}\right)$. It remains to show that the basis B_E is orthonormal. Observe that since $a_n = a_{-n}$, we have

$$\begin{aligned}2a_n &= a_n + a_{-n} \\ &= \left\langle f, \frac{\sqrt{2}}{2}\operatorname{sinc}\frac{1}{2}(s-2n)\right\rangle + \left\langle f, \frac{\sqrt{2}}{2}\operatorname{sinc}\frac{1}{2}(s+2n)\right\rangle \\ &= \left\langle f, \frac{\sqrt{2}}{2}\left[\operatorname{sinc}\frac{1}{2}(s-2n) + \operatorname{sinc}\frac{1}{2}(s+2n)\right]\right\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}f(t) &= \frac{\sqrt{2}}{2}\left\langle f, \frac{\sqrt{2}}{2}\operatorname{sinc}\frac{1}{2}s\right\rangle \operatorname{sinc}\frac{1}{2}t \\ &\quad + \frac{\sqrt{2}}{4}\sum_{n=1}^{\infty}\left\langle f, \frac{\sqrt{2}}{2}\left[\operatorname{sinc}\frac{1}{2}(s-2n) + \operatorname{sinc}\frac{1}{2}(s+2n)\right]\right\rangle \left[\operatorname{sinc}\frac{1}{2}(t-2n)\right. \\ &\quad \left. + \operatorname{sinc}\frac{1}{2}(t+2n)\right] \\ &= \frac{1}{2}\left\langle f, \operatorname{sinc}\frac{1}{2}s\right\rangle \operatorname{sinc}\frac{1}{2}t \\ &\quad + \frac{1}{4}\sum_{n=1}^{\infty}\left\langle f, \operatorname{sinc}\frac{1}{2}(s-2n) + \operatorname{sinc}\frac{1}{2}(s+2n)\right\rangle \left[\operatorname{sinc}\frac{1}{2}(t-2n) + \operatorname{sinc}\frac{1}{2}(t+2n)\right],\end{aligned}$$

showing that for $f \in PW\left(\frac{\pi}{2}\right)$ even, f can be wriiten as a linear combination of the elements of B_E. We conclude by showing that the elements of B_E are orthonormal. First recall that $k_{2n}(t) = \frac{1}{2}\operatorname{sinc}\frac{1}{2}(t-2n)$ is a reproducing kernel for $PW\left(\frac{\pi}{2}\right)$. In order to simplify the notation and exploit the properties of a reproducing kernel, we rewrite the set B_E in terms of the reproducing kernel. Thus

$$B_E = \left\{\sqrt{2}k_0\right\} \cup \{k_{-2n} + k_{2n}\}_{n=1}^{\infty}$$

where

$$\begin{aligned} k_0(t) &= \frac{1}{2}\operatorname{sinc}\frac{1}{2}t, \\ k_{-2n}(t)+k_{2n}(t) &= \frac{1}{2}\left[\operatorname{sinc}\frac{1}{2}(t+2n)+\operatorname{sinc}\frac{1}{2}(t-2n)\right]. \end{aligned}$$

We now check that the elements of B_E are orthogonal. Let $n \in \mathbb{N}$ then, given k_{2n} is a reproducing kernel,

$$\begin{aligned} \langle k_{-2n}+k_{2n}, \sqrt{2}k_0\rangle &= \sqrt{2}\langle k_{-2n}, k_0\rangle + \sqrt{2}\langle k_{2n}, \sqrt{2}k_0\rangle \\ &= \sqrt{2}k_{-2n}(0)+\sqrt{2}k_{2n}(0). \end{aligned}$$

By Lemma 5.2.0.12 we have $k_{-2n}(0) = 0 = k_{2n}(0)$, hence $\langle k_{-2n}+k_{2n}, \sqrt{2}k_0\rangle = 0$. Now let $m \in \mathbb{N}$ and suppose that $m \neq n$. Again, since k is a reproducing kernel we have

$$\begin{aligned} \langle k_{-2m}+k_{2m}, k_{-2n}+k_{2n}\rangle &= \langle k_{-2m}, k_{-2n}\rangle + \langle k_{-2m}, k_{2n}\rangle + \langle k_{2m}, k_{-2n}\rangle + \langle k_{2m}, k_{2n}\rangle \\ &= k_{-2m}(-2n)+k_{-2m}(2n)+k_{2m}(-2n)+k_{2m}(2n). \end{aligned}$$

Lemma 5.2.0.12 now gives $\langle k_{-2m}(t)+k_{2m}(t), k_{-2n}(t)+k_{2n}(t)\rangle = 0$ for $m \neq n$. We have thus shown that the set B_E is orthogonal and it remains to show that the elements are normalised. Now, again using the fact that k is a reproducing kernel we see that

$$\begin{aligned} \langle \sqrt{2}k_0, \sqrt{2}k_0\rangle &= 2\langle k_0, k_0\rangle \\ &= 2k_0(0) \\ &= \operatorname{sinc} 0. \end{aligned}$$

It then follows from Lemma 5.2.0.12 that $\langle \sqrt{2}k_0, \sqrt{2}k_0\rangle = 1$. Similarly, using Lemma 5.2.0.12,

$$\begin{aligned} \langle k_{-2n}+k_{2n}, k_{-2n}+k_{2n}\rangle &= \langle k_{-2n}, k_{-2n}\rangle + \langle k_{-2n}, k_{2n}\rangle + \langle k_{2n}, k_{-2n}\rangle + \langle k_{2n}, k_{2n}\rangle \\ &= k_{-2n}(-2n)+k_{-2n}(2n)+k_{2n}(-2n)+k_{2n}(2n) \\ &= 1. \end{aligned}$$

Thus B_E is indeed an orthonormal basis for the even functions in the space $PW\left(\frac{\pi}{2}\right)$. ■

As a point of interest we show that we can also find a basis for the odd functions in $PW\left(\frac{\pi}{2}\right)$.

Proposition 6.3.0.20 *The set*

$$B_O = \left\{\frac{1}{2}\left[\operatorname{sinc}\frac{1}{2}(t-2n)-\operatorname{sinc}\frac{1}{2}(t+2n)\right]\right\}_{n=1}^{\infty}$$

forms an orthonormal basis for the odd functions in the space $PW\left(\frac{\pi}{2}\right)$.

Proof. Again, by Proposition 6.3.0.16, $\left\{\frac{\sqrt{2}}{2}\operatorname{sinc}\frac{1}{2}(t-2n)\right\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $PW\left(\frac{\pi}{2}\right)$. For $f \in PW\left(\frac{\pi}{2}\right)$ we have

$$\begin{aligned} f(t) &= \frac{\sqrt{2}}{2}\sum_{n=-\infty}^{\infty} a_n \operatorname{sinc}\frac{1}{2}(t-2n) \\ &= \frac{\sqrt{2}}{2}a_0 \operatorname{sinc}\frac{1}{2}t + \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\left[a_n \operatorname{sinc}\frac{1}{2}(t-2n)+a_{-n}\operatorname{sinc}\frac{1}{2}(t+2n)\right] \end{aligned}$$

where $a_n = \left\langle f, \frac{\sqrt{2}}{2} \operatorname{sinc} \frac{1}{2}(s-2n) \right\rangle$ for $n \in \mathbb{N}_0$. Further, if f is odd then f must satisfy $f(t) = -f(-t)$. Therefore, $f \in PW\left(\frac{\pi}{2}\right)$ is odd if and only if

$$\begin{aligned} 0 &= f(t) + f(-t) \\ &= \frac{\sqrt{2}}{2} a_0 \operatorname{sinc} \frac{1}{2} t + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left[a_n \operatorname{sinc} \frac{1}{2}(t-2n) + a_{-n} \operatorname{sinc} \frac{1}{2}(t+2n) \right] + \frac{\sqrt{2}}{2} a_0 \operatorname{sinc} \frac{1}{2}(-t) \\ &\quad + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left[a_n \operatorname{sinc} \frac{1}{2}(-t-2n) + a_{-n} \operatorname{sinc} \frac{1}{2}(-t+2n) \right]. \end{aligned}$$

Since the sinc function is even, we may simplify the above to obtain

$$\begin{aligned} 0 &= \sqrt{2} a_0 \operatorname{sinc} \frac{1}{2} t + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left[a_n \operatorname{sinc} \frac{1}{2}(t-2n) + a_{-n} \operatorname{sinc} \frac{1}{2}(t+2n) \right] \\ &\quad + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left[a_n \operatorname{sinc} \frac{1}{2}(t+2n) + a_{-n} \operatorname{sinc} \frac{1}{2}(t-2n) \right] \\ &= \sqrt{2} a_0 \operatorname{sinc} \frac{1}{2} t + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} (a_{-n} + a_n) \left[\operatorname{sinc} \frac{1}{2}(t-2n) + \operatorname{sinc} \frac{1}{2}(t+2n) \right]. \end{aligned} \tag{6.7}$$

Notice that if (6.7) is true for all t then we must have $a_0 = 0$ and $a_{-n} + a_n = 0$. It follows that $f \in PW\left(\frac{\pi}{2}\right)$ is odd if and only if $a_0 = 0$ and $a_{-n} = -a_n$. Hence for odd $f \in PW\left(\frac{\pi}{2}\right)$ we have

$$\begin{aligned} f(t) &= \frac{\sqrt{2}}{2} a_0 \operatorname{sinc} \frac{1}{2} t + \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left[a_n \operatorname{sinc} \frac{1}{2}(t-2n) + a_{-n} \operatorname{sinc} \frac{1}{2}(t+2n) \right] \\ &= \frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} a_n \left[\operatorname{sinc} \frac{1}{2}(t-2n) - \operatorname{sinc} \frac{1}{2}(t+2n) \right]. \end{aligned}$$

Again we have shown that the set $B_O = \left\{ \frac{1}{2} \left[\operatorname{sinc} \frac{1}{2}(t-2n) - \operatorname{sinc} \frac{1}{2}(t+2n) \right] \right\}_{n=1}^{\infty}$ forms a basis for the odd functions in $PW\left(\frac{\pi}{2}\right)$. We finish by showing that B_O is an orthonormal basis. Note that since $a_{-n} = -a_n$, we have

$$\begin{aligned} 2a_n &= a_n - a_{-n} \\ &= \left\langle f, \frac{\sqrt{2}}{2} \operatorname{sinc} \frac{1}{2}(s-2n) \right\rangle - \left\langle f, \frac{\sqrt{2}}{2} \operatorname{sinc} \frac{1}{2}(s+2n) \right\rangle \\ &= \left\langle f, \frac{\sqrt{2}}{2} \left[\operatorname{sinc} \frac{1}{2}(s-2n) - \operatorname{sinc} \frac{1}{2}(s+2n) \right] \right\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} f(t) &= \frac{\sqrt{2}}{4} \sum_{n=1}^{\infty} \left\langle f, \frac{\sqrt{2}}{2} \left[\operatorname{sinc} \frac{1}{2}(s-2n) - \operatorname{sinc} \frac{1}{2}(s+2n) \right] \right\rangle \left[\operatorname{sinc} \frac{1}{2}(t-2n) - \operatorname{sinc} \frac{1}{2}(t+2n) \right] \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left\langle f, \operatorname{sinc} \frac{1}{2}(s-2n) - \operatorname{sinc} \frac{1}{2}(s+2n) \right\rangle \left[\operatorname{sinc} \frac{1}{2}(t-2n) - \operatorname{sinc} \frac{1}{2}(t+2n) \right] \end{aligned}$$

showing that the set B_O is a spanning sequence for the odd functions in the space $PW\left(\frac{\pi}{2}\right)$. We conclude by showing that the elements of B_O are orthonormal. Recall that $k_{2n}(t) = \frac{1}{2} \operatorname{sinc} \frac{1}{2}(t-2n)$ is a reproducing kernel for $PW\left(\frac{\pi}{2}\right)$ and rewrite the set B_O in terms of this reproducing kernel. Thus

$$B_O = \{k_{2n} - k_{-2n}\}_{n=1}^{\infty}.$$

We now check that the elements of B_O are orthonormal. Let $n, m \in \mathbb{N}$ and suppose that $m \neq n$ then, given k is a reproducing kernel,

$$\begin{aligned}\langle k_{2m} - k_{-2m}, k_{2n} - k_{-2n}\rangle &= \langle k_{2m}, k_{2n}\rangle - \langle k_{2m}, k_{-2n}\rangle - \langle k_{-2m}, k_{2n}\rangle + \langle k_{-2m}, k_{-2n}\rangle \\ &= k_{2m}(2n) - k_{2m}(-2n) - k_{-2m}(2n) + k_{-2m}(-2n).\end{aligned}$$

It now follows from Lemma 5.2.0.12 that $\langle k_{2m} - k_{-2m}, k_{2n} - k_{-2n}\rangle = 0$ for $m \neq n$. Finally, and again using Lemma 5.2.0.12,

$$\begin{aligned}\langle k_{2n} - k_{-2n}, k_{2n} - k_{-2n}\rangle &= \langle k_{2n}, k_{2n}\rangle - \langle k_{2n}, k_{-2n}\rangle - \langle k_{-2n}, k_{2n}\rangle + \langle k_{-2n}, k_{-2n}\rangle \\ &= k_{2n}(2n) - k_{2n}(-2n) - k_{-2n}(2n) + k_{-2n}(-2n) \\ &= 1.\end{aligned}$$

Thus B_O is indeed an orthonormal basis for the odd functions in the space $PW\left(\frac{\pi}{2}\right)$. ■

When considering operators we are able to construct the matrix of an operator with respect to a given basis. In the following theorem we construct the matrix of I_{2b} with respect to the basis B_E. That is, we construct a matrix, T whose n^{th} column is given by the coefficients of I_{2n} when expressed as a linear combination of the basis, B_E. Furthermore, we show that the functions $\{I_{2b}\}_{b\in\mathbb{N}}$ form a basis for the even Paley–Wiener functions over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Theorem 6.3.0.21 *Let X denote the matrix of I_{2b} with respect to the basis B_E. Then the b^{th} column of X is given by*

$$\frac{\pi}{2^{2b-3}}\begin{bmatrix}\frac{\sqrt{2}}{2}\binom{2b-2}{b-1} & \binom{2b-2}{b-2} & \cdots & \binom{2b-2}{1} & \binom{2b-2}{0} & 0 & 0 & \cdots\end{bmatrix}^T.$$

The matrix X is an upper triangular matrix with strictly positive entries on and above the leading diagonal and takes the form

$$X = \pi\begin{bmatrix}\sqrt{2} & \frac{1}{2}\binom{2}{1}\left(\frac{\sqrt{2}}{2}\right) & \frac{1}{2}\binom{4}{2}\left(\frac{\sqrt{2}}{2^3}\right) & \frac{1}{2}\binom{6}{3}\left(\frac{\sqrt{2}}{2^5}\right) & & \cdots \\ 0 & \binom{2}{0}\left(\frac{1}{2}\right) & \binom{4}{1}\left(\frac{1}{2^3}\right) & \binom{6}{2}\left(\frac{1}{2^5}\right) & & \cdots \\ 0 & 0 & \binom{4}{0}\left(\frac{1}{2^3}\right) & \binom{6}{1}\left(\frac{1}{2^5}\right) & & \cdots \\ 0 & 0 & 0 & \binom{6}{0}\left(\frac{1}{2^5}\right) & & \\ 0 & 0 & 0 & 0 & \ddots & \\ & & & & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots\end{bmatrix}.$$

Furthermore, the set $\{I_{2b}\}_{b\in\mathbb{N}}$ gives a basis for the even functions in the space $PW\left(\frac{\pi}{2}\right)$.

Proof. We first express I_{2b} as a linear combination of the basis, B_E. The coefficients of I_{2b} with respect to B_E then form the b^{th} column of the required matrix. By Lemma 6.3.0.18,

$$\begin{aligned}
I_{2b}(t) &= \frac{\pi}{2^{2b-1}} \sum_{j=0}^{2b-2} \binom{2b-2}{j} \left\{ \operatorname{sinc} \frac{1}{2}[t+2(b-j-1)] + \operatorname{sinc} \frac{1}{2}[t-2(b-j-1)] \right\} \\
&= \frac{\pi}{2^{2b-1}} \left[\binom{2b-2}{0} \left\{ \operatorname{sinc} \frac{1}{2}[t+2(b-1)] + \operatorname{sinc} \frac{1}{2}[t-2(b-1)] \right\} \right. \\
&\quad + \binom{2b-2}{1} \left\{ \operatorname{sinc} \frac{1}{2}[t+2(b-2)] + \operatorname{sinc} \frac{1}{2}[t-2(b-2)] \right\} + \ldots \\
&\quad + \binom{2b-2}{b-2} \left\{ \operatorname{sinc} \frac{1}{2}(t+2) + \operatorname{sinc} \frac{1}{2}(t-2) \right\} \\
&\quad + \binom{2b-2}{b-1} \left\{ \operatorname{sinc} \frac{1}{2}t + \operatorname{sinc} \frac{1}{2}t \right\} + \binom{2b-2}{b} \left\{ \operatorname{sinc} \frac{1}{2}(t-2) + \operatorname{sinc} \frac{1}{2}(t+2) \right\} \\
&\quad + \cdots + \binom{2b-2}{2b-3} \left\{ \operatorname{sinc} \frac{1}{2}[t-2(b-2)] + \operatorname{sinc} \frac{1}{2}[t+2(b-2)] \right\} \\
&\quad \left. + \binom{2b-2}{2b-2} \left\{ \operatorname{sinc} \frac{1}{2}[t-2(b-1)] + \operatorname{sinc} \frac{1}{2}[t+2(b-1)] \right\} \right].
\end{aligned}$$

Upon expanding the above summation we note that we can pair off the terms, thus

$$\begin{aligned}
I_{2b}(t) &= \frac{\pi}{2^{2b-1}} \left[2\binom{2b-2}{b-1} \operatorname{sinc} \frac{1}{2}t + \left\{ \binom{2b-2}{b-2} + \binom{2b-2}{b} \right\} \left\{ \operatorname{sinc} \frac{1}{2}(t-2) + \operatorname{sinc} \frac{1}{2}(t+2) \right\} \right. \\
&\quad + \cdots + \left\{ \binom{2b-2}{1} + \binom{2b-2}{2b-3} \right\} \left\{ \operatorname{sinc} \frac{1}{2}[t-2(b-2)] + \operatorname{sinc} \frac{1}{2}[t+2(b-2)] \right\} \\
&\quad \left. + \left\{ \binom{2b-2}{0} + \binom{2b-2}{2b-2} \right\} \left\{ \operatorname{sinc} \frac{1}{2}[t-2(b-1)] + \operatorname{sinc} \frac{1}{2}[t+2(b-1)] \right\} \right].
\end{aligned}$$

Since

$$\binom{n}{r} + \binom{n}{n-r} = 2\binom{n}{r},$$

it follows that

$$\begin{aligned}
I_{2b}(t) &= \frac{\pi}{2^{2b-1}} \left[2\binom{2b-2}{b-1} \operatorname{sinc} \frac{1}{2}t + 2\binom{2b-2}{b-2} \left\{ \operatorname{sinc} \frac{1}{2}(t-2) + \operatorname{sinc} \frac{1}{2}(t+2) \right\} + \ldots \right. \\
&\quad + 2\binom{2b-2}{1} \left\{ \operatorname{sinc} \frac{1}{2}[t-2(b-2)] + \operatorname{sinc} \frac{1}{2}[t+2(b-2)] \right\} \\
&\quad \left. + 2\binom{2b-2}{0} \left\{ \operatorname{sinc} \frac{1}{2}[t-2(b-1)] + \operatorname{sinc} \frac{1}{2}[t+2(b-1)] \right\} \right].
\end{aligned}$$

Finally we adjust each term so that it corresponds to an element of B_E. We therefore have

$$\begin{aligned}
I_{2b}(t) &= \frac{\pi}{2^{2b-1}} \left[2\sqrt{2}\binom{2b-2}{b-1} \left[\frac{\sqrt{2}}{2} \operatorname{sinc} \frac{1}{2}t \right] + 4\binom{2b-2}{b-2} \left\{ \frac{1}{2} \left[\operatorname{sinc} \frac{1}{2}(t-2) + \operatorname{sinc} \frac{1}{2}(t+2) \right] \right\} \right. \\
&\quad + \cdots + 4\binom{2b-2}{1} \left\{ \frac{1}{2} \left[\operatorname{sinc} \frac{1}{2}[t-2(b-2)] + \operatorname{sinc} \frac{1}{2}[t+2(b-2)] \right] \right\} \\
&\quad \left. + 4\binom{2b-2}{0} \left\{ \frac{1}{2} \left[\operatorname{sinc} \frac{1}{2}[t-2(b-1)] + \operatorname{sinc} \frac{1}{2}[t+2(b-1)] \right] \right\} \right].
\end{aligned}$$

Hence the b^{th} column of the matrix of I_{2b} with respect to the basis B_E is

$$\frac{\pi}{2^{2b-3}} \begin{bmatrix} \frac{\sqrt{2}}{2}\binom{2b-2}{b-1} & \binom{2b-2}{b-2} & \ldots & \binom{2b-2}{1} & \binom{2b-2}{0} & 0 & 0 & \ldots \end{bmatrix}^T.$$

Substituting in values for b then produces the matrix

$$X = \pi \begin{bmatrix} \sqrt{2} & \frac{1}{2}\binom{2}{1}\left(\frac{\sqrt{2}}{2}\right) & \frac{1}{2}\binom{4}{2}\left(\frac{\sqrt{2}}{2^3}\right) & \frac{1}{2}\binom{6}{3}\left(\frac{\sqrt{2}}{2^5}\right) & & \cdots \\ 0 & \binom{2}{0}\left(\frac{1}{2}\right) & \binom{4}{1}\left(\frac{1}{2^3}\right) & \binom{6}{2}\left(\frac{1}{2^5}\right) & & \cdots \\ 0 & 0 & \binom{4}{0}\left(\frac{1}{2^3}\right) & \binom{6}{1}\left(\frac{1}{2^5}\right) & & \cdots \\ 0 & 0 & 0 & \binom{6}{0}\left(\frac{1}{2^5}\right) & & \\ 0 & 0 & 0 & 0 & \ddots & \\ & & & & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{bmatrix}.$$

In order to see that the set $\{I_{2b}\}_{b\in\mathbb{N}}$ forms a basis for the even functions in $PW\left(\frac{\pi}{2}\right)$, we simply need to show that each element of B_E can be expressed as a linear combination of the elements $\{I_{2b}\}_{b\in\mathbb{N}}$. Let X^T denote the transpose of the matrix X. Notice that X is upper triangular with non-zero, positive entries on the leading diagonal. Each upper left $n \times n$ block has non-zero determinant and is therefore invertible. It follows that X and consequently X^T are invertible. Let Z denote the column vector

$$Z = \begin{bmatrix} I_2 & I_4 & I_6 & \ldots \end{bmatrix}^T$$

and let Y denote the column vector

$$Y = \left[\frac{\sqrt{2}}{2}\operatorname{sinc}\frac{1}{2}t \quad \frac{1}{2}\left[\operatorname{sinc}\frac{1}{2}(t+2) + \operatorname{sinc}\frac{1}{2}(t-2)\right] \quad \frac{1}{2}\left[\operatorname{sinc}\frac{1}{2}(t+4) + \operatorname{sinc}\frac{1}{2}(t-4)\right] \quad \ldots\right].$$

By the first part of the theorem we have

$$Z = X^T Y.$$

The invertibility of X^T then allows us to write

$$Y = \left(X^T\right)^{-1} Z,$$

thus expressing the basis elements, B_E in terms of the functions, $\{I_{2b}\}_{b\in\mathbb{N}}$. This completes the proof. ■

6.4 A Formula by Ramanujan: The Integral I_a in Terms of the Gamma Function

In this section we introduce the Gamma function, Γ along with some relations satisfied by Γ. We then state and prove a result known to Ramanujan that expresses the integral I_a as defined in Definition 6.2.0.11 in terms of the Gamma function. Further, we see that I_a arises as the characteristic function of a probability density function. Also, in Section 6.3 we found expressions for I_a given that $a \in \mathbb{Z}$, we continue with this here. Noting that when $n \in \mathbb{N}$ we can express the Gamma function in terms of factorials, we use this idea to further simplify I_a.

We begin with a brief introduction to the Gamma function. A detailed construction of the definition of the Gamma function can be found in [31] (Section 8.8, page 229).

Definition 6.4.0.22 *For $t \in \mathbb{C}$ such that* $\mathrm{Re}(t) > 0$ *we define the* Gamma function *to be,*

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx.$$

Remark 6.4.0.23 *The Gamma function as defined in Definition 6.4.0.22 is holomorphic on the half-plane* $\mathrm{Re}(t) > 0$. *See [31] (page 230).*

In the next set of definitions we introduce some well-known identities that will be used throughout subsequent calculations.

Definition 6.4.0.24 *For* $\mathrm{Re}(t) > 0$, *the Gamma function satisfies the identity,*

$$\Gamma(t) = (t-1)\Gamma(t-1). \tag{6.8}$$

Further, if $t = n \in \mathbb{N}$ then

$$\Gamma(n) = (n-1)!. \tag{6.9}$$

Remark 6.4.0.25 *In [31] (page 230), it can be seen that the identity, (6.8) can be used to extend the definition of the Gamma function to include those values of t for which* $\mathrm{Re}(t) \leq 0$. *In fact, the identity*

$$\Gamma(t) = \frac{\Gamma(t+n)}{t(t+1)\dots(t+n-1)} \tag{6.10}$$

holds for $\mathrm{Re}(t) > -n$. *Furthermore, by Remark 6.4.0.23,* $\Gamma(t+n)$ *is holomorphic when* $\mathrm{Re}(t) > -n$. *It now follows from (6.10) that $\Gamma(t)$ is a meromorphic function with poles at the points $t = 0, -1, \dots, -n+1$. This shows that $\frac{1}{\Gamma(t)}$ is an entire function.*

Definition 6.4.0.26 *The Gamma function satisfies the relation*

$$\frac{1}{\Gamma(t)\Gamma(1-t)} = \frac{\sin \pi t}{\pi} = t \operatorname{sinc} t. \tag{6.11}$$

The final identity that we will make use of in this section is the *Legendre duplication formula.* This is stated in the following proposition. A proof of the Legendre duplication formula can be found in [54] (Section 1.86, page 55).

Proposition 6.4.0.27 *For any $z \in \mathbb{C}$ we have*

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z + \frac{1}{2}\right).$$

The next proposition that we will give is due to Ramanujan. The proof depends upon integration around a suitable contour in the complex plane and the use of Beta functions to evaluate the integrals. We fist define a Beta function and give a useful relation between the Beta function and the Gamma function.

Definition 6.4.0.28 *The* Beta function *is defined by*

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx$$

and it satisfies the relation

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

We now present Ramanujan's formula for the integral, I_a. Here we state the result and give a full proof for completeness. The idea behind the method for the proof has been taken from [53] (Section 7.6, page 186).

Proposition 6.4.0.29 *Let I_a be as defined in Definition 6.2.0.11. Then for $a > 1$,*

$$I_a(t) = \frac{\pi\Gamma(a-1)}{2^{a-2}\Gamma\left(\frac{a+t}{2}\right)\Gamma\left(\frac{a-t}{2}\right)}.$$

Proof. We prove that the identity holds for $1 < a < 2$ and $t + a - 2 > 0$, then use an analytic continuation argument to show that the identity holds for $a > 1$ and $t \in \mathbb{R}$. Let $0 < \delta < 1$. Define a contour, γ as follows:

$$\gamma = \gamma_1 \oplus [i, \delta i] \oplus \gamma_2 \oplus [-\delta i, -i]$$

where γ_1 is the unit semicircle in the right half plane and γ_2 denotes the semicircle centre 0, radius δ, traced clockwise in the right half plane. Consider the function

$$f(z) = \frac{1}{i}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^{t-1}.$$

Now, f is holomorphic on and inside the contour γ, therefore, by Cauchy's theorem,

$$\begin{aligned} 0 &= \int_\gamma \frac{1}{i}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^{t-1}\,dz \\ &= \int_{\gamma_1} + \int_{[i,\delta i]} + \int_{\gamma_2} + \int_{[-\delta i, -i]} \left\{\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{1}{iz}\right\} dz. \end{aligned} \tag{6.12}$$

We want to write the integral in polar form so we make the substitution $z = re^{ix}$, where $r > 0$ denotes the radius and $-\pi \leq x \leq \pi$ denotes the angle between z and the real axis. Now, γ_1 is the unit semicircle in the right half plane, therefore we take $r = 1$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The first integral then becomes

$$\begin{aligned} \int_{\gamma_1}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{dz}{iz} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^{a-2}\left(e^{ix}\right)^t \frac{ie^{ix}}{ie^{ix}}\,dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\,dx. \end{aligned}$$

Similarly, for the third integral we have $r = \delta$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ for γ_2 which is traced in a clockwise direction. Thus,

$$\begin{aligned} \int_{\gamma_2}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{dz}{iz} &= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\delta e^{ix}+\frac{1}{\delta e^{ix}}}{2}\right)^{a-2}\left(\delta e^{ix}\right)^t \frac{i\delta e^{ix}}{i\left(\delta e^{ix}\right)}\,dx \\ &= -\frac{1}{2^{a-2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\delta e^{ix}+\frac{1}{\delta e^{ix}}\right)^{a-2}\left(\delta e^{ix}\right)^t\,dx. \end{aligned}$$

We treat the second and fourth integrals a little differently. Considering the second integral we see that

$$\begin{aligned} \int_{[i,\delta i]}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{dz}{iz} &= \int_1^\delta\left(\frac{iy+\frac{1}{iy}}{2}\right)^{a-2}(iy)^t \frac{i}{i^2 y}\,dy \\ &= \frac{1}{2^{a-2}}\int_\delta^1\left(iy+\frac{1}{iy}\right)^{a-2}(iy)^t \frac{i}{y}\,dy \\ &= \frac{1}{2^{a-2}}\int_\delta^1\left(ye^{i\frac{\pi}{2}}+\frac{1}{ye^{i\frac{\pi}{2}}}\right)^{a-2}\left(ye^{i\frac{\pi}{2}}\right)^t \frac{e^{i\frac{\pi}{2}}}{y}\,dy. \end{aligned}$$

We can remove a factor of $e^{i\frac{\pi}{2}}$ from $ye^{i\frac{\pi}{2}} + \frac{1}{ye^{i\frac{\pi}{2}}}$, giving

$$\begin{aligned}\left(ye^{i\frac{\pi}{2}} + \frac{1}{ye^{i\frac{\pi}{2}}}\right)^{a-2} &= \left[e^{i\frac{\pi}{2}}\left(y + \frac{1}{ye^{i\pi}}\right)\right]^{a-2} \\ &= e^{i\frac{(a-2)\pi}{2}}\left(y + \frac{1}{ye^{i\pi}}\right)^{a-2}.\end{aligned}$$

Noting that $e^{i\pi} = -1$ we therefore have

$$\begin{aligned}\int_{[i,\delta i]}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{dz}{iz} &= \frac{1}{2^{a-2}}\int_\delta^1 e^{i\frac{(a-2)\pi}{2}}\left(y + \frac{1}{ye^{i\pi}}\right)^{a-2} y^{t-1}e^{i\frac{(t+1)\pi}{2}}\,dy \\ &= -\frac{e^{i\frac{(t+a+1)\pi}{2}}}{2^{a-2}}\int_\delta^1 \left(y + \frac{e^{-i\pi}}{y}\right)^{a-2} y^{t-1}\,dy.\end{aligned}$$

Our aim is to write $\int_{[i,\delta i]}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{dz}{iz}$ as a Beta integral as this will allow us to evaluate the integral in terms of Gamma functions; we come to this later. For now we note that

$$\begin{aligned}\left(y + \frac{e^{-i\pi}}{y}\right)^{a-2} y^{t-1} &= \left(y^2 + e^{-i\pi}\right)^{a-2} y^{t-a+1} \\ &= \left[e^{-i\pi}\left(1-y^2\right)\right]^{a-2} y^{t-a+1} \\ &= e^{i(2-a)\pi}\left(1-y^2\right)^{a-2} y^{t-a+1} \\ &= e^{-ia\pi}\left(1-y^2\right)^{a-2} y^{t-a+1}\end{aligned}$$

where the last line follows since $e^{2\pi i} = 1$. Hence

$$\begin{aligned}\int_{[i,\delta i]}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{dz}{iz} &= -\frac{e^{i\frac{(t+a+1)\pi}{2}}}{2^{a-2}}\int_\delta^1 e^{-ia\pi}\left(1-y^2\right)^{a-2} y^{t-a+1}\,dy \\ &= -\frac{e^{i\frac{(t-a+1)\pi}{2}}}{2^{a-2}}\int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\,dy.\end{aligned}$$

We use the same approach to evaluate the fourth integral. First,

$$\begin{aligned}\int_{[-\delta i,-i]}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{dz}{iz} &= \int_\delta^1\left(\frac{-iy + \frac{1}{-iy}}{2}\right)^{a-2} (-iy)^t \frac{(-i)}{(-i^2)y}\,dy \\ &= \frac{1}{2^{a-2}}\int_\delta^1\left(-iy + \frac{1}{-iy}\right)^{a-2} (-iy)^t \frac{(-i)}{y}\,dy \\ &= \frac{1}{2^{a-2}}\int_\delta^1\left(ye^{-i\frac{\pi}{2}} + \frac{1}{ye^{-i\frac{\pi}{2}}}\right)^{a-2} \left(ye^{-i\frac{\pi}{2}}\right)^t \frac{e^{-i\frac{\pi}{2}}}{y}\,dy.\end{aligned}$$

Next remove a factor of $e^{-i\frac{\pi}{2}}$ from $ye^{-i\frac{\pi}{2}} + \frac{1}{ye^{-i\frac{\pi}{2}}}$, giving

$$\begin{aligned}\left(ye^{-i\frac{\pi}{2}} + \frac{1}{ye^{-i\frac{\pi}{2}}}\right)^{a-2} &= \left[e^{-i\frac{\pi}{2}}\left(y + \frac{1}{ye^{-i\pi}}\right)\right]^{a-2} \\ &= e^{-i\frac{(a-2)\pi}{2}}\left(y + \frac{e^{i\pi}}{y}\right)^{a-2}.\end{aligned}$$

Using this together with the fact that $e^{i\pi} = -1$ we have

$$\begin{aligned}\int_{[-\delta i,-i]}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t \frac{dz}{iz} &= \frac{1}{2^{a-2}}\int_\delta^1 e^{-i\frac{(a-2)\pi}{2}}\left(y + \frac{e^{i\pi}}{y}\right)^{a-2} y^{t-1}e^{-i\frac{(t+1)\pi}{2}}\,dy \\ &= -\frac{e^{-i\frac{(t+a+1)\pi}{2}}}{2^{a-2}}\int_\delta^1 \left(y + \frac{e^{i\pi}}{y}\right)^{a-2} y^{t-1}\,dy.\end{aligned}$$

Eventually we will write $\int_{[-\delta i,-i]}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t\,\frac{dz}{iz}$ as a Beta integral so for now we note that

$$\begin{aligned}\left(y+\frac{e^{i\pi}}{y}\right)^{a-2} y^{t-1} &= \left(\frac{y^2+e^{i\pi}}{y}\right)^{a-2} y^{t-1}\\ &= \left[e^{i\pi}\left(\frac{1-y^2}{y}\right)\right]^{a-2} y^{t-1}\\ &= e^{i(a-2)\pi}\left(1-y^2\right)^{a-2} y^{t-a+1}\\ &= e^{ia\pi}\left(1-y^2\right)^{a-2} y^{t-a+1}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{[-\delta i,-i]}\left(\frac{z+z^{-1}}{2}\right)^{a-2} z^t\,\frac{dz}{iz} &= -\frac{e^{-i\frac{(t+a+1)\pi}{2}}}{2^{a-2}}\int_\delta^1 e^{ia\pi}\left(1-y^2\right)^{a-2} y^{t-a+1}\,dy\\ &= -\frac{e^{-i\frac{(t-a+1)\pi}{2}}}{2^{a-2}}\int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\,dy.\end{aligned}$$

Putting this information together we see that equation (6.12) now becomes

$$\begin{aligned}0 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\,dx - \frac{e^{i\frac{(t-a+1)\pi}{2}}}{2^{a-2}}\int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\,dy\\ &\quad -\frac{1}{2^{a-2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\delta e^{ix}+\frac{1}{\delta e^{ix}}\right)^{a-2}\left(\delta e^{ix}\right)^t\,dx - \frac{e^{-i\frac{(t-a+1)\pi}{2}}}{2^{a-2}}\int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\,dy.\end{aligned}$$

Rearranging and simplifying the above gives

$$\begin{aligned}I_a(t) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\,dx\\ &= \left(\frac{e^{i\frac{(t-a+1)\pi}{2}}+e^{-i\frac{(t-a+1)\pi}{2}}}{2^{a-2}}\right)\int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\,dy\\ &\quad +\frac{1}{2^{a-2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\delta e^{ix}+\frac{1}{\delta e^{ix}}\right)^{a-2}\left(\delta e^{ix}\right)^t\,dx\\ &= \frac{2\cos\frac{\pi}{2}(t-a+1)}{2^{a-2}}\int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\,dy + \frac{1}{2^{a-2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\delta e^{ix}+\frac{1}{\delta e^{ix}}\right)^{a-2}\left(\delta e^{ix}\right)^t\,dx.\end{aligned}$$

Note that

$$\begin{aligned}\cos\frac{\pi}{2}(t-a+1) &= \sin\left[\frac{\pi}{2}(t-a)+\pi\right]\\ &= \sin\frac{\pi}{2}(t-a)\cos\pi\\ &= -\sin\frac{\pi}{2}(t-a)\end{aligned}$$

using the double angle formulae. Also, as sin is an odd function we have $-\sin\frac{\pi}{2}(t-a) = \sin\frac{\pi}{2}(a-t)$. Definition 6.4.0.26 then allows us to write $\sin\frac{\pi}{2}(a-t)$ in terms of Gamma functions, thus

$$\begin{aligned}\cos\frac{\pi}{2}(t-a+1) &= \frac{\pi\sin\frac{\pi}{2}(a-t)}{\pi}\\ &= \frac{\pi}{\Gamma\left(\frac{a-t}{2}\right)\Gamma\left(1-\frac{a-t}{2}\right)}.\end{aligned}$$

It follows that

$$\begin{aligned}I_a(t) &= \frac{2\pi}{2^{a-2}\Gamma\left(\frac{a-t}{2}\right)\Gamma\left(1-\frac{a-t}{2}\right)}\int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\,dy\\ &\quad +\frac{1}{2^{a-2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\delta e^{ix}+\frac{1}{\delta e^{ix}}\right)^{a-2}\left(\delta e^{ix}\right)^t\,dx. \qquad (6.13)\end{aligned}$$

We finish the calculation by taking the limit as $\delta \to 0^+$ of (6.13). First, recall that we wanted to write $\int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\, dy$ as a Beta integral; we proceed to do this via a substitution. Let $u = y^2$ then

$$\begin{aligned} \int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\, dy &= \int_{\delta^2}^1 (1-u)^{a-2} \left(\sqrt{u}\right)^{t-a+1} \frac{du}{2\sqrt{u}} \\ &= \frac{1}{2}\int_{\delta^2}^1 (1-u)^{a-2} \left(\sqrt{u}\right)^{t-a} du \\ &\to \frac{1}{2} B\left(\frac{t-a}{2}+1, a-1\right) \end{aligned}$$

as $\delta \to 0^+$. By the relation stated in Definition 6.4.0.28 it follows that

$$\lim_{\delta\to 0^+} \int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\, dy = \frac{\Gamma\left(\frac{t-a+2}{2}\right)\Gamma(a-1)}{2\Gamma\left(\frac{a+t}{2}\right)}.$$

Finally we consider $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\delta e^{ix} + \frac{1}{\delta e^{ix}}\right)^{a-2} \left(\delta e^{ix}\right)^t dx$. Let $1 < a < 2$, then

$$\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\delta e^{ix} + \frac{1}{\delta e^{ix}}\right)^{a-2} \left(\delta e^{ix}\right)^t dx \right| \le \delta^t \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \delta e^{ix} + \frac{1}{\delta e^{ix}} \right|^{a-2} dx.$$

We note that

$$\begin{aligned} \left| \delta e^{ix} + \frac{1}{\delta e^{ix}} \right|^2 &= \left(\delta e^{ix} + \frac{1}{\delta e^{ix}}\right)\left(\delta e^{-ix} + \frac{1}{\delta e^{-ix}}\right) \\ &= \delta^2 + e^{2ix} + e^{-2ix} + \frac{1}{\delta^2} \\ &= \delta^2 + 2\cos 2x + \frac{1}{\delta^2} \\ &\le \delta^2 + 2 + \frac{1}{\delta^2}. \end{aligned}$$

Since $\delta > 0$ and real, it follows that

$$\left| \delta e^{ix} + \frac{1}{\delta e^{ix}} \right|^2 \le \left(\delta + \frac{1}{\delta}\right)^2.$$

Thus,

$$\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\delta e^{ix} + \frac{1}{\delta e^{ix}}\right)^{a-2} \left(\delta e^{ix}\right)^t dx \right| \le \delta^t \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\delta + \frac{1}{\delta}\right)^{a-2} dx.$$

Now observe that

$$\begin{aligned} \delta^t \left(\delta + \frac{1}{\delta}\right)^{a-2} &= \delta^{t-a+2}\left(\delta^2+1\right)^{a-2} \\ &\to 0 \end{aligned}$$

as $\delta \to 0^+$ whenever $t-a+2 > 0$. Since $1 < a < 2$ we have $t-a+2 > 0$ whenever $t > 0$. Thus for $t > 0$

$$\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\delta e^{ix} + \frac{1}{\delta e^{ix}}\right)^{a-2} \left(\delta e^{ix}\right)^t dx \right| \to 0$$

as $\delta \to 0^+$. We conclude that for $1 < a < 2$ and $t > 0$,

$$\begin{aligned} I_a(t) &= \lim_{\delta \to 0^+} \left\{ \frac{2\pi}{2^{a-2}\Gamma\left(\frac{a-t}{2}\right)\Gamma\left(1-\frac{a-t}{2}\right)} \int_\delta^1 \left(1-y^2\right)^{a-2} y^{t-a+1}\, dy \right. \\ &\quad \left. + \frac{1}{2^{a-2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\delta e^{ix} + \frac{1}{\delta e^{ix}}\right)^{a-2} \left(\delta e^{ix}\right)^t dx \right\} \\ &= \frac{\pi\Gamma\left(\frac{t-a+2}{2}\right)\Gamma(a-1)}{2^{a-2}\Gamma\left(\frac{a-t}{2}\right)\Gamma\left(\frac{t-a+2}{2}\right)\Gamma\left(\frac{a+t}{2}\right)} + 0 \\ &= \frac{\pi\Gamma(a-1)}{2^{a-2}\Gamma\left(\frac{a-t}{2}\right)\Gamma\left(\frac{a+t}{2}\right)}. \end{aligned}$$

Since I_a is an even function it is clear that the above identity also holds for $t < 0$. It remains to show that the identity holds when $t = 0$. Let $t = 0$ then by [26] (3.621(5) and 8.384(1)),

$$\begin{aligned} I_a(0) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2}\, dx \\ &= 2\int_0^{\frac{\pi}{2}} (\sin x)^0 (\cos x)^{a-2}\, dx \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a-1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}. \end{aligned}$$

Note that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ then by the Legendre duplication formula 6.4.0.27 we have

$$\Gamma(a-1) = \frac{2^{a-2}}{\sqrt{\pi}}\Gamma\left(\frac{a-1}{2}\right)\Gamma\left(\frac{a}{2}\right).$$

Therefore

$$I_a(0) = \frac{\pi\Gamma(a-1)}{2^{a-2}\Gamma\left(\frac{a}{2}\right)^2}$$

as expected.

To complete the proof and show that the identity holds for $a > 1$ we first note that the map $a \mapsto I_a(t)$ is holomorphic on $\mathrm{Re}\,(a) > 1$. Also, the map

$$a \mapsto \frac{\pi\Gamma(a-1)}{2^{a-2}\Gamma\left(\frac{a+t}{2}\right)\Gamma\left(\frac{a-t}{2}\right)}$$

is holomorphic on $\mathrm{Re}\,(a) > 1$ since $\frac{1}{\Gamma\left(\frac{a+t}{2}\right)\Gamma\left(\frac{a-t}{2}\right)}$ is entire and $\Gamma(a-1)$ is holomorphic on $\mathrm{Re}\,(a) > 1$. Since

$$I_a(t) = \frac{\pi\Gamma(a-1)}{2^{a-2}\Gamma\left(\frac{a-t}{2}\right)\Gamma\left(\frac{a+t}{2}\right)} \tag{6.14}$$

for $1 < a < 2$ and $t \in \mathbb{R}$, it follows from the Identity Theorem [43] (Section 15.8, page 180) that (6.14) holds for $a > 1$ and $t \in \mathbb{R}$. ■

In the proof of the following proposition we see that the integral, I_a arises as the characteristic function of a probability density function. Further, we can evaluate the characteristic function using Proposition 6.4.0.29.

Proposition 6.4.0.30 *Let $a > 1$. For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the function*

$$p_a(x) = \frac{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)}{\pi\Gamma\left(\frac{a-1}{2}\right)}(\cos x)^{a-2}$$

is a probability density function with characteristic function

$$\rho(t) = \frac{\Gamma\left(\frac{a}{2}\right)^2}{\Gamma\left(\frac{a+t}{2}\right)\Gamma\left(\frac{a-t}{2}\right)}.$$

Proof. Let $a > 1$. The fact that p_a is a probability density function is easily seen for if

$$I_a(0) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2}\, dx,$$

then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p_a(x)\, dx = \frac{\Gamma\left(\frac{a}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a-1}{2}\right)} I_a(0).$$

By Proposition 6.4.0.29,

$$I_a(0) = \frac{\pi\Gamma(a-1)}{2^{a-2}\Gamma\left(\frac{a}{2}\right)^2}.$$

We use the Legendre duplication formula 6.4.0.27 to write

$$\Gamma(a-1) = \frac{2^{a-2}}{\sqrt{\pi}}\Gamma\left(\frac{a-1}{2}\right)\Gamma\left(\frac{a}{2}\right).$$

So,

$$I_a(0) = \frac{\sqrt{\pi}\Gamma\left(\frac{a-1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}$$

and it is now easily seen that p_a is a probability density function.

To calculate the characteristic function, ρ of p_a, first note that by definition we have

$$\begin{aligned}\rho(t) &= \int_{-\infty}^{\infty} p_a(x)e^{itx}\, dx \\ &= \frac{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)}{\pi\Gamma\left(\frac{a-1}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\, dx.\end{aligned}$$

Note that $I_a(t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{itx}\, dx$, thus, by Proposition 6.4.0.29 we have

$$\rho(t) = \frac{\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)\Gamma(a-1)}{2^{a-2}\Gamma\left(\frac{a-1}{2}\right)\Gamma\left(\frac{a+t}{2}\right)\Gamma\left(\frac{a-t}{2}\right)}.$$

Finally, we apply the Legendre duplication formula 6.4.0.27 to obtain

$$\Gamma(a-1) = \frac{2^{a-2}}{\sqrt{\pi}}\Gamma\left(\frac{a-1}{2}\right)\Gamma\left(\frac{a}{2}\right),$$

giving

$$\begin{aligned}\rho(t) &= \frac{2^{a-2}\sqrt{\pi}\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a-1}{2}\right)\Gamma\left(\frac{a}{2}\right)}{2^{a-2}\sqrt{\pi}\Gamma\left(\frac{a-1}{2}\right)\Gamma\left(\frac{a+t}{2}\right)\Gamma\left(\frac{a-t}{2}\right)} \\ &= \frac{\Gamma\left(\frac{a}{2}\right)^2}{\Gamma\left(\frac{a+t}{2}\right)\Gamma\left(\frac{a-t}{2}\right)}.\end{aligned}$$

■

In the case that $a \in \mathbb{N}$ we can use Proposition 6.4.0.29 to evaluate $I_a(t)$ and produce some useful relations. To do this, we must consider the cases of a odd and even separately. Following this we see that we can use our relations to evaluate $I_a(t)$ when a is any integer. Before considering a to be a general integer we first look at the case that $a = 2$ as in that case, the function I_a is simplified greatly.

Lemma 6.4.0.31 *Suppose that $a = 2$ then*

$$I_2(t_j - k) = \pi \operatorname{sinc} \frac{1}{2}(t_j - k).$$

Furthermore,

$$I_2(t_j - k) = \frac{\pi}{\Gamma\left(1 + \frac{t_j - k}{2}\right)\Gamma\left(1 - \frac{t_j - k}{2}\right)}.$$

Proof. Let $a = 2$ then by Definition 6.2.0.11 it is clear that

$$I_2(t_j - k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(t_j - k)x}\, dx.$$

Calculating this integral directly we obtain

$$\begin{aligned}
I_2(t_j - k) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(t_j - k)x}\, dx \\
&= \left[\frac{e^{i(t_j - k)x}}{i(t_j - k)}\right]_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \left[\frac{e^{i\frac{\pi}{2}(t_j - k)} - e^{-i\frac{\pi}{2}(t_j - k)}}{i(t_j - k)}\right] \\
&= \frac{2\sin\frac{\pi}{2}(t_j - k)}{(t_j - k)} \\
&= \pi \operatorname{sinc} \frac{1}{2}(t_j - k).
\end{aligned}$$

This completes the first part of the lemma.

To find $I_2(t_j - k)$ in terms of Gamma functions, we use Proposition 6.4.0.29. Thus,

$$I_2(t_j - k) = \frac{\pi\Gamma(1)}{\Gamma\left(1 + \frac{t_j - k}{2}\right)\Gamma\left(1 - \frac{t_j - k}{2}\right)}.$$

By Definition 6.4.0.24, $\Gamma(1) = 1$ and $\Gamma\left(1 + \frac{t_j - k}{2}\right) = \left(\frac{t_j - k}{2}\right)\Gamma\left(\frac{t_j - k}{2}\right)$ and so

$$I_2(t_j - k) = \frac{\pi}{\left(\frac{t_j - k}{2}\right)\Gamma\left(\frac{t_j - k}{2}\right)\Gamma\left(1 - \frac{t_j - k}{2}\right)}.$$

Note that this value does agree with that found in the first part of the lemma, for by Definition 6.4.0.26,

$$\frac{1}{\Gamma\left(\frac{t_j - k}{2}\right)\Gamma\left(1 - \frac{t_j - k}{2}\right)} = \left(\frac{t_j - k}{2}\right) \operatorname{sinc} \frac{1}{2}(t_j - k),$$

hence

$$\begin{aligned}
I_2(t_j - k) &= \frac{\pi\left(\frac{t_j - k}{2}\right) \operatorname{sinc} \frac{1}{2}(t_j - k)}{\frac{t_j - k}{2}} \\
&= \pi \operatorname{sinc} \frac{1}{2}(t_j - k).
\end{aligned}$$

■

Next we turn our attention to the case that $a \in \mathbb{Z}$. As in Lemma 6.4.0.31, we use Definition 6.4.0.24 and 6.4.0.26 to simplify the function I_a. The proofs of the following two lemmas contain calculations that are elementary and standard. However, we give them in detail so as to make clear the precise features of the formulae. We begin by considering the case that $a = 2n$ for some $n \in \mathbb{N}$, showing how to evaluate $I_{2n}(t)$.

Lemma 6.4.0.32 *Let $\mathbb{C}(t)$ be the field of rational functions in the variable t. Adjoin the transcendental function $\sin \frac{\pi}{2} t$ to form the algebra $\mathbb{C}(t)\left[\sin \frac{\pi}{2} t\right]$. Then for $n \in \mathbb{Z}$,*

$$I_{2n}(t) = \frac{\pi (2n-2)! \operatorname{sinc} \frac{1}{2} t}{2^{2n-2} \prod_{j=1}^{n-1} \left[(n-j)^2 + \left(\frac{t}{2}\right)^2 \right]}$$

so $I_{2n}(t) \in \mathbb{C}(t)\left[\sin \frac{\pi}{2} t\right]$.

Proof. Let $a = 2n$ for some $n \in \mathbb{N}$ then

$$\begin{aligned} I_{2n}(t) &= \frac{\pi \Gamma(2n-1)}{2^{2n-2} \Gamma\left(\frac{2n+t}{2}\right) \Gamma\left(\frac{2n-t}{2}\right)} \\ &= \frac{\pi \Gamma(2n-1)}{2^{2n-2} \Gamma\left(n + \frac{t}{2}\right) \Gamma\left(n - \frac{t}{2}\right)}. \end{aligned}$$

By equation (6.9) we have $\Gamma(2n-1) = (2n-2)!$. Furthermore, we can repeatedly apply (6.8) to obtain

$$\begin{aligned} \Gamma\left(n + \frac{t}{2}\right) &= \left(n - 1 + \frac{t}{2}\right) \Gamma\left(n - 1 + \frac{t}{2}\right) \\ &= \left(n - 1 + \frac{t}{2}\right)\left(n - 2 + \frac{t}{2}\right) \Gamma\left(n - 2 + \frac{t}{2}\right) \\ &= \ldots \\ &= \left(n - 1 + \frac{t}{2}\right)\left(n - 2 + \frac{t}{2}\right) \ldots \left(\frac{t}{2}\right) \Gamma\left(\frac{t}{2}\right) \\ &= \left[\prod_{j=1}^{n} \left(n - j + \frac{t}{2}\right)\right] \Gamma\left(\frac{t}{2}\right). \end{aligned}$$

Similarly we find that

$$\begin{aligned} \Gamma\left(n - \frac{t}{2}\right) &= \left(n - 1 - \frac{t}{2}\right) \Gamma\left(n - 1 - \frac{t}{2}\right) \\ &= \left(n - 1 - \frac{t}{2}\right)\left(n - 2 - \frac{t}{2}\right) \Gamma\left(n - 2 - \frac{t}{2}\right) \\ &= \ldots \\ &= \left(n - 1 - \frac{t}{2}\right)\left(n - 2 - \frac{t}{2}\right) \ldots \left(1 - \frac{t}{2}\right) \Gamma\left(1 - \frac{t}{2}\right) \\ &= \left[\prod_{j=1}^{n-1} \left(n - j - \frac{t}{2}\right)\right] \Gamma\left(1 - \frac{t}{2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} I_{2n}(t) &= \frac{\pi (2n-2)!}{2^{2n-2} \left[\prod_{j=1}^{n} \left(n - j + \frac{t}{2}\right)\right] \left[\prod_{k=1}^{n-1} \left(n - k - \frac{t}{2}\right)\right] \Gamma\left(\frac{t}{2}\right) \Gamma\left(1 - \frac{t}{2}\right)} \\ &= \frac{\pi (2n-2)!}{2^{2n-2} \frac{t}{2} \prod_{j=1}^{n-1} \left[(n-j)^2 + \left(\frac{t}{2}\right)^2\right] \Gamma\left(\frac{t}{2}\right) \Gamma\left(1 - \frac{t}{2}\right)}. \end{aligned}$$

Finally, equation (6.11) allows us to write

$$\frac{1}{\Gamma\left(\frac{t}{2}\right)\Gamma\left(1-\frac{t}{2}\right)} = \frac{t}{2}\operatorname{sinc}\frac{1}{2}t.$$

Hence,

$$\begin{aligned} I_{2n}(t) &= \frac{\pi(2n-2)!\frac{t}{2}\operatorname{sinc}\frac{1}{2}t}{2^{2n-2}\frac{t}{2}\prod_{j=1}^{n-1}\left[(n-j)^2+\left(\frac{t}{2}\right)^2\right]} \\ &= \frac{\pi(2n-2)!\operatorname{sinc}\frac{1}{2}t}{2^{2n-2}\prod_{j=1}^{n-1}\left[(n-j)^2+\left(\frac{t}{2}\right)^2\right]}. \end{aligned}$$

To finish, we note that $I_{2n}(t)$ is a rational function of t multiplied by $\sin\frac{\pi}{2}t$. Therefore $I_{2n}(t) \in \mathbb{C}(t)\left[\sin\frac{\pi}{2}t\right]$. ■

In the next lemma we consider the case that $a = 2n+1$ for $n \in \mathbb{Z}$. Our calculations provide a relation between $I_{2n+1}(t)$ and $I_{2n}(t)$. We also include a relation for $I_{2n+2}(t)$ and $I_{2n}(t)$.

Lemma 6.4.0.33 *Let $\mathbb{C}(t)$ be the field of rational functions in the variable t. Adjoin the transcendental function $\cos\frac{\pi}{2}t$ to form the algebra $\mathbb{C}(t)\left[\cos\frac{\pi}{2}t\right]$. Then, for $n \in \mathbb{N}$ the following relations hold:*

$$\begin{aligned} I_{2n+1}(t) &= \frac{2\pi(2n-1)!(2n-2)!\operatorname{sinc} t}{\prod_{j=1}^{2n-1}\left[(2n-j)^2-t^2\right]}\frac{1}{I_{2n}(t)}; \\ I_{2n+2}(t) &= \frac{2n(2n-1)}{(2n+t)(2n-t)}I_{2n}(t), \end{aligned}$$

where $I_{2n}(t)$ has a rational coefficient in the formula for $I_{2n+2}(t)$. Furthermore, $I_{2n+1}(t) \in \mathbb{C}(t)\left[\cos\frac{\pi}{2}t\right]$.

Proof. Let $a = 2n+1$ then

$$\begin{aligned} I_{2n+1}(t) &= \frac{\pi\Gamma(2n)}{2^{2n-1}\Gamma\left(\frac{2n+1+t}{2}\right)\Gamma\left(\frac{2n+1-t}{2}\right)} \\ &= \frac{\pi\Gamma(2n)}{2^{2n-1}\Gamma\left(\frac{2n+t}{2}+\frac{1}{2}\right)\Gamma\left(\frac{2n-t}{2}+\frac{1}{2}\right)}. \end{aligned}$$

We use the Legendre duplication formula 6.4.0.27 to write

$$\begin{aligned} \Gamma\left(\frac{2n+t}{2}+\frac{1}{2}\right) &= \frac{\sqrt{\pi}\Gamma(2n+t)}{2^{2n+t-1}\Gamma\left(\frac{2n+t}{2}\right)}; \\ \Gamma\left(\frac{2n-t}{2}+\frac{1}{2}\right) &= \frac{\sqrt{\pi}\Gamma(2n-t)}{2^{2n-t-1}\Gamma\left(\frac{2n-t}{2}\right)}. \end{aligned}$$

Thus

$$\begin{aligned} I_{2n+1}(t) &= \frac{2^{4n-2}\pi\Gamma(2n)\Gamma\left(\frac{2n+t}{2}\right)\Gamma\left(\frac{2n-t}{2}\right)}{2^{2n-1}\pi\Gamma(2n+t)\Gamma(2n-t)} \\ &= \frac{2^{2n-1}\Gamma(2n)\Gamma\left(\frac{2n+t}{2}\right)\Gamma\left(\frac{2n-t}{2}\right)}{\Gamma(2n+t)\Gamma(2n-t)} \\ &= \frac{2^{2n-1}\Gamma(2n)\Gamma\left(n+\frac{t}{2}\right)\Gamma\left(n-\frac{t}{2}\right)}{\Gamma(2n+t)\Gamma(2n-t)}. \end{aligned}$$

We recognise $\Gamma\left(n+\frac{t}{2}\right)\Gamma\left(n-\frac{t}{2}\right)$ since this appeared in Lemma 6.4.0.32. The lemma therefore allows us to write

$$\Gamma\left(n+\frac{t}{2}\right)\Gamma\left(n-\frac{t}{2}\right) = \frac{\pi\Gamma(2n-1)}{2^{2n-2}I_{2n}(t)},$$

hence,

$$\begin{aligned} I_{2n+1}(t) &= \frac{2^{2n-1}\pi\Gamma(2n)\Gamma(2n-1)}{2^{2n-2}\Gamma(2n+t)\Gamma(2n-t)I_{2n}(t)} \\ &= \frac{2\pi\Gamma(2n)\Gamma(2n-1)}{\Gamma(2n+t)\Gamma(2n-t)I_{2n}(t)}. \end{aligned}$$

We use the same technique as in the proof of Lemma 6.4.0.32 to deduce that

$$\begin{aligned} \Gamma(2n+t) &= \left[\prod_{j=1}^{2n}(2n-j+t)\right]\Gamma(t); \\ \Gamma(2n-t) &= \left[\prod_{j=1}^{2n-1}(2n-j-t)\right]\Gamma(1-t). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{\Gamma(2n+t)\Gamma(2n-t)} &= \frac{1}{\left[\prod_{j=1}^{2n}(2n-j+t)\right]\left[\prod_{j=1}^{2n-1}(2n-j-t)\right]\Gamma(t)\Gamma(1-t)} \\ &= \frac{1}{t\left\{\prod_{j=1}^{2n-1}[(2n-j)^2-t^2]\right\}\Gamma(t)\Gamma(1-t)} \\ &= \frac{\operatorname{sinc} t}{\prod_{j=1}^{2n-1}[(2n-j)^2-t^2]} \end{aligned}$$

where the last line follows from (6.11). So we have

$$\begin{aligned} I_{2n+1}(t) &= \frac{2\pi\Gamma(2n)\Gamma(2n-1)\operatorname{sinc} t}{\left\{\prod_{j=1}^{2n-1}[(2n-j)^2-t^2]\right\}I_{2n}(t)} \\ &= \frac{2\pi(2n-1)!(2n-2)!\operatorname{sinc} t}{\left\{\prod_{j=1}^{2n-1}[(2n-j)^2-t^2]\right\}I_{2n}(t)}. \end{aligned}$$

Notice that $I_{2n+1}(t)I_{2n}(t) \in \mathbb{C}(t)[\sin\pi t]$ and $I_{2n}(t) \in \mathbb{C}(t)\left[\sin\frac{\pi}{2}t\right]$, therefore, $I_{2n+1}(t)$ is a rational function multiplied by $\frac{\sin\pi t}{\sin\frac{\pi}{2}t}$. From the double angle formulae we note that

$$\sin\pi t = 2\sin\frac{\pi}{2}t\cos\frac{\pi}{2}t$$

and so it follows that $I_{2n+1}(t) \in \mathbb{C}(t)\left[\cos\frac{\pi}{2}t\right]$.

Now let $a = 2n+2$ then, using (6.8) we have

$$\begin{aligned} I_{2n+2}(t) &= \frac{\pi\Gamma(2n+1)}{2^{2n}\Gamma\left(\frac{2n+2+t}{2}\right)\Gamma\left(\frac{2n+2-t}{2}\right)} \\ &= \frac{\pi\Gamma(2n+1)}{2^{2n}\Gamma\left(n+1+\frac{t}{2}\right)\Gamma\left(n+1-\frac{t}{2}\right)} \\ &= \frac{\pi\Gamma(2n+1)}{2^{2n}\left(n+\frac{t}{2}\right)\left(n-\frac{t}{2}\right)\Gamma\left(n+\frac{t}{2}\right)\Gamma\left(n-\frac{t}{2}\right)}. \end{aligned}$$

Again, we recognise $\Gamma\left(n+\frac{t}{2}\right)\Gamma\left(n-\frac{t}{2}\right)$ and so following Lemma 6.4.0.32 we obtain

$$\Gamma\left(n+\frac{t}{2}\right)\Gamma\left(n-\frac{t}{2}\right) = \frac{\pi\Gamma(2n-1)}{2^{2n-2}I_{2n}(t)}.$$

Hence,

$$\begin{aligned} I_{2n+2}(t) &= \frac{2^{2n-2}\pi\Gamma(2n+1)I_{2n}(t)}{2^{2n}\pi\left(n+\frac{t}{2}\right)\left(n-\frac{t}{2}\right)\Gamma(2n-1)} \\ &= \frac{\Gamma(2n+1)I_{2n}(t)}{2^{2}\left(n+\frac{t}{2}\right)\left(n-\frac{t}{2}\right)\Gamma(2n-1)} \\ &= \frac{(2n)!I_{2n}(t)}{2^{2}\left(n+\frac{t}{2}\right)\left(n-\frac{t}{2}\right)(2n-2)!} \\ &= \frac{2n(2n-1)}{(2n+t)(2n-t)}I_{2n}(t). \end{aligned}$$

■

Remark 6.4.0.34 *The relations stated in the previous lemmas allow us to work out $I_a(t)$ for $a \in \{2, 3, \dots\}$. However, we can also run these relations backwards to obtain $I_a(t)$ for $a \in \{\dots, -1, 0, 1\}$. These relations can therefore be used to extend Ramanujan's formula to cover the cases when a is an integer.*

6.5 Determinants Associated with Ramanujan's Integral, I_a

We have thus arrived at the main section of this chapter. The previous sections have been designed so that they lead up to the calculation of determinants associated with the function I_a. Given a sequence, $(t_j)_{j=-n}^{n}$ we calculate the determinant of the matrix $[I_a(t_j - k)]_{j,k=-n}^{n}$ using Andréief's Identity 2.3.0.31. In the case that $t_j = j$ we can use Ramanujan's formula 6.4.0.29 to write the $I_a(t_j - k)$ in terms of factorials. This allows us to give the matrix form of $[I_a(t_j - k)]_{j,k=-n}^{n}$. In fact we see that $[I_a(j - k)]_{j,k=-n}^{n}$ is a Toeplitz matrix. In previous sections we found different formulae for I_a depending on whether a was odd or even. It should come as no surprise then that the matrix $[I_a(j - k)]_{j,k=-n}^{n}$ differs with a odd or even, although in both cases it takes the Toeplitz form.

Throughout this section we give results for a sequence, $(t_n)_{n\in\mathbb{Z}}$. Although the results stated hold for a general sequence, the reader should note that in particular they hold for the sampling sequence found in Chapter 5. Therefore, we can evaluate determinants with entries based on Ramanujan's integral, I_a at sampling points corresponding to the periodic spectrum of Hill's equation.

The following lemma shows the relationship between two points on the unit circle. We use it to explain the terms appearing in the formulae stated in Theorem 6.5.0.36.

Lemma 6.5.0.35 *Let z_1 and z_2 be points on the unit circle, with angles x_1 and x_2 respectively, in relation to the real axis. The distance between the points z_1 and z_2 is given by,*

$$|z_1 - z_2| = 2\left|\sin\frac{1}{2}(x_1 - x_2)\right|.$$

Proof. Let z_1 and z_2 be points on the unit circle. Then, in polar form $z_1 = e^{ix_1}$ and $z_2 = e^{ix_2}$.

Let $|z_1 - z_2|$ denote the distance between the points z_1 and z_2 then,

$$\begin{aligned} |z_1 - z_2|^2 &= (z_1 - z_2)\overline{(z_1 - z_2)} \\ &= \left(e^{ix_1} - e^{ix_2}\right)\left(e^{-ix_1} - e^{-ix_2}\right) \\ &= 2 - e^{i(x_1 - x_2)} - e^{-i(x_1 - x_2)} \\ &= 2\left[1 - \cos(x_1 - x_2)\right]. \end{aligned}$$

Using the double angle formulae to evaluate $1 - \cos(x_1 - x_2)$ we therefore obtain

$$|z_1 - z_2|^2 = 4\sin^2 \frac{1}{2}(x_1 - x_2).$$

Taking the square root of both sides now produces the desired result. ■

In the following theorem we regard x_j as points distributed on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ according to the probability density function p_a described in Proposition 6.4.0.30. The reader should also note the interaction term

$$\prod_{-n \le j < k \le n} 4\sin^2 \frac{1}{2}(x_k - x_j) \tag{6.15}$$

arising from the (Vandermonde) product, $\prod_{-n \le j < k \le n} \left|e^{ix_j} - e^{ix_k}\right|^2$. Lemma 6.5.0.35 shows that (6.15) is the product of squared distances between points on the unit circle. Note also that the product (6.15) is analogous to the Weyl denominator formula. By Definition 6.1.0.10, for $-n \le j, k \le n$, the modulus of the Weyl denominator is given by

$$\begin{aligned} \prod_{-n \le j < k \le n} \left|e^{2\pi i(x_j - x_k)} - 1\right| &= \prod_{-n \le j < k \le n} \left|e^{2\pi i x_j} - e^{2\pi i x_k}\right| \\ &= \prod_{-n \le j < k \le n} 2\left|\sin \pi(x_j - x_k)\right| \end{aligned}$$

where the last line follows from Lemma 6.5.0.35.

Theorem 6.5.0.36 *Let $n \in \mathbb{N}$ then*

$$\det\left[\frac{1}{\pi} I_a(t_j - k)\right]_{j,k=-n}^{n}$$
$$= \frac{1}{(2n+1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \det\left[e^{it_j x_l}\right]_{j,l=-n}^{n} \det\left[e^{-ikx_l}\right]_{l,k=-n}^{n} \left[\prod_{j=-n}^{n} (\cos x_j)^{a-2}\right] \frac{dx_{-n}}{\pi} \cdots \frac{dx_n}{\pi}. \tag{6.16}$$

Suppose further that $t_j = j$ for $j \in \mathbb{Z}$ then

$$\det\left[\frac{1}{\pi} I_a(j - k)\right]_{j,k=-n}^{n}$$
$$= \frac{1}{(2n+1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\prod_{-n \le j < k \le n} 4\sin^2 \frac{1}{2}(x_k - x_j)\right] \left[\prod_{j=-n}^{n} (\cos x_j)^{a-2}\right] \frac{dx_{-n}}{\pi} \cdots \frac{dx_n}{\pi}.$$

Proof. From Definition 6.2.0.11 we have

$$\begin{aligned} \det\left[\frac{1}{\pi} I_a(t_j - k)\right]_{j,k=-n}^{n} &= \det\left[\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^{a-2} e^{i(t_j - k)x}\, dx\right]_{j,k=-n}^{n} \\ &= \det\left[\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{it_j x}(\cos x)^{a-2} e^{-ikx}\, dx\right]_{j,k=-n}^{n}. \end{aligned}$$

Applying Andréief's Identity, 2.3.0.31 to the above then yields

$$\det\left[\frac{1}{\pi}I_a(t_j-k)\right]_{j,k=-n}^{n}$$
$$=\frac{1}{(2n+1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\det\left[e^{it_jx_l}\right]_{j,l=-n}^{n}\det\left[(\cos x_l)^{a-2}e^{-ikx_l}\right]_{l,k=-n}^{n}\frac{dx_{-n}}{\pi}\cdots\frac{dx_n}{\pi}.$$

We use elementary row operations to simplify $\left[(\cos x_l)^{a-2}e^{-ikx_l}\right]_{l,k=-n}^{n}$. First note that every element in row l of the matrix $\left[(\cos x_l)^{a-2}e^{-ikx_l}\right]_{l,k=-n}^{n}$ contains a factor of $(\cos x_l)^{a-2}$. We can remove each of these factors and place it into an elementary matrix that premultiplies the matrix $\left[e^{-ikx_l}\right]_{l,k=-n}^{n}$. The resulting elementary matrix will be diagonal with the $(j,j)^{\text{th}}$ entry being $(\cos x_j)^{a-2}$. Thus, we have

$$\left[(\cos x_l)^{a-2}e^{-ikx_l}\right]_{l,k=-n}^{n}=\operatorname{diag}\left[(\cos x_l)^{a-2}\right]_{l=-n}^{n}\left[e^{-ikx_l}\right]_{l,k=-n}^{n},$$

from which we obtain

$$\det\left[(\cos x_l)^{a-2}e^{-ikx_l}\right]_{l,k=-n}^{n}=\left[\prod_{l=-n}^{n}(\cos x_l)^{a-2}\right]\det\left[e^{-ikx_l}\right]_{l,k=-n}^{n}.$$

Hence,

$$\det\left[\frac{1}{\pi}I_a(t_j-k)\right]_{j,k=-n}^{n}$$
$$=\frac{1}{(2n+1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\det\left[e^{it_jx_l}\right]_{j,l=-n}^{n}\det\left[e^{-ikx_l}\right]_{l,k=-n}^{n}\left[\prod_{j=-n}^{n}(\cos x_j)^{a-2}\right]\frac{dx_{-n}}{\pi}\cdots\frac{dx_n}{\pi}.$$

This completes the proof of the first part of the result.

For the second part of the theorem, suppose that $t_j=j$ for $j\in\mathbb{Z}$. Then, from (6.16) we have

$$\det\left[\frac{1}{\pi}I_a(t_j-k)\right]_{j,k=-n}^{n}$$
$$=\frac{1}{(2n+1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\det\left[e^{ijx_l}\right]_{j,l=-n}^{n}\det\left[e^{-ikx_l}\right]_{l,k=-n}^{n}\left[\prod_{j=-n}^{n}(\cos x_j)^{a-2}\right]\frac{dx_{-n}}{\pi}\cdots\frac{dx_n}{\pi}.$$

In order to evaluate the determinants of $\left[e^{ijx_l}\right]_{j,l=-n}^{n}$ and $\left[e^{-ikx_l}\right]_{l,k=-n}^{n}$ we first write them as Vandermonde matrices. Notice that

$$\left[e^{ijx_l}\right]_{j,l=-n}^{n}=\begin{bmatrix}\left(e^{ix_{-n}}\right)^{-n} & \cdots & \left(e^{ix_0}\right)^{-n} & \cdots & \left(e^{ix_n}\right)^{-n}\\ \vdots & & \vdots & & \vdots\\ 1 & \cdots & 1 & \cdots & 1\\ \vdots & & \vdots & & \vdots\\ \left(e^{ix_{-n}}\right)^{n} & \cdots & \left(e^{ix_0}\right)^{n} & \cdots & \left(e^{ix_n}\right)^{n}\end{bmatrix},$$

so, if we remove a factor of $\left(e^{ix_l}\right)^{-n}$ from the l^{th} column for each $l\in\{-n,\ldots,n\}$, then the resulting matrix will be a Vandermonde matrix. The matrix $\left[e^{ijx_l}\right]_{j,l=-n}^{n}$ is therefore equivalent to post multiplying a Vandermonde matrix by a diagonal elementary matrix. Thus,

$$\left[e^{ijx_l}\right]_{j,l=-n}^{n}=\left[e^{i(j-1)x_l}\right]_{1\le j\le 2n+1,-n\le l\le n}\operatorname{diag}\left[e^{-inx_l}\right]_{l=-n}^{n}$$

Now, $\left[e^{i(j-1)x_l}\right]_{1\leq j\leq 2n+1, -n\leq l\leq n}$ is a Vandermonde matrix but before we can evaluate its determinant we need to relabel the x_l so that $1 \leq l \leq 2n+1$, that is, k and l run over the same indices. It is easy to see that the map $l \mapsto l+n+1$ produces the desired index. It follows that

$$\left[e^{ijx_l}\right]_{j,l=-n}^{n} = \left[e^{i(j-1)x_{l-n-1}}\right]_{j,l=1}^{2n+1} \operatorname{diag}\left[e^{-inx_l}\right]_{l=-n}^{n}.$$

Taking the determinant and using Definition 2.3.0.17 produces

$$\begin{aligned}
\det\left[e^{ijx_l}\right]_{j,l=-n}^{n} &= \det\left(\left[e^{i(j-1)x_{l-n-1}}\right]_{j,l=1}^{2n+1} \operatorname{diag}\left[e^{-inx_l}\right]_{l=-n}^{n}\right) \\
&= \det\left[e^{i(j-1)x_{l-n-1}}\right]_{j,l=1}^{2n+1} \det\left(\operatorname{diag}\left[e^{-inx_l}\right]_{l=-n}^{n}\right) \\
&= \left[\prod_{1\leq j<l\leq 2n+1}\left(e^{ix_{l-n-1}} - e^{ix_{j-n-1}}\right)\right]\left[\prod_{l=-n}^{n} e^{-inx_l}\right] \\
&= \left[\prod_{-n\leq j<l\leq n}\left(e^{ix_l} - e^{ix_j}\right)\right]\left[\prod_{l=-n}^{n} e^{-inx_l}\right].
\end{aligned}$$

Similarly we find that

$$\det\left[e^{-ikx_l}\right]_{l,k=-n}^{n} = \left[\prod_{l=-n}^{n} e^{inx_l}\right]\left[\prod_{-n\leq l<k\leq n}\left(e^{-ix_k} - e^{-ix_l}\right)\right].$$

Therefore,

$$\begin{aligned}
&\det\left[\frac{1}{\pi}I_a(t_j-k)\right]_{j,k=-n}^{n} \\
&= \frac{1}{(2n+1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\prod_{-n\leq j<l\leq n}\left(e^{ix_l}-e^{ix_j}\right)\right]\left[\prod_{-n\leq l<k\leq n}\left(e^{-ix_k}-e^{-ix_l}\right)\right] \\
&\quad \left[\prod_{j=-n}^{n}(\cos x_j)^{a-2}\right]\frac{dx_{-n}}{\pi}\cdots\frac{dx_n}{\pi} \\
&= \frac{1}{(2n+1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\prod_{-n\leq j<k\leq n}\left|e^{ix_k}-e^{ix_j}\right|^2\right]\left[\prod_{j=-n}^{n}(\cos x_j)^{a-2}\right]\frac{dx_{-n}}{\pi}\cdots\frac{dx_n}{\pi}.
\end{aligned}$$

Note that $\left|e^{ix_k} - e^{ix_j}\right|$ is the distance between two points on the unit circle. It follows that $\prod_{-n\leq j<k\leq n}\left|e^{ix_k} - e^{ix_j}\right|^2$ is a product of squared distances thus, by Lemma 6.5.0.35 we have,

$$\prod_{-n\leq j<k\leq n}\left|e^{ix_k}-e^{ix_j}\right|^2 = \prod_{-n\leq j<k\leq n} 4\sin^2\frac{1}{2}(x_k - x_j).$$

Hence,

$$\begin{aligned}
&\det\left[\frac{1}{\pi}I_a(j-k)\right]_{j,k=-n}^{n} \\
&= \frac{1}{(2n+1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\prod_{-n\leq j<k\leq n}4\sin^2\frac{1}{2}(x_k-x_j)\right]\left[\prod_{j=-n}^{n}(\cos x_j)^{a-2}\right]\frac{dx_{-n}}{\pi}\cdots\frac{dx_n}{\pi}
\end{aligned}$$

as required. ■

We want to make the determinants given in Theorem 6.5.0.36 more explicit. Under certain circumstances this can be done. We look closely at cases when $a \in \mathbb{N}$ and $t_j \in \mathbb{Z}$ and evaluate the resulting determinants. Before doing this we look at the case $a = 2$ and $t_j \in \mathbb{R}$ as this produces a simple result.

Proposition 6.5.0.37 *Let $a = 2$ then the determinant of the matrix $\left[\frac{1}{\pi}I_2(t_j - k)\right]_{j,k=-n}^{n}$ reduces to*

$$\det\left[\frac{1}{\pi}I_2(t_j - k)\right]_{j,k=-n}^{n} = \det\left[\operatorname{sinc}\frac{1}{2}(t_j - k)\right]_{j,k=-n}^{n}.$$

Proof. Let $a = 2$ then by Lemma 6.4.0.31

$$\frac{1}{\pi}I_2(t_j - k) = \operatorname{sinc}\frac{1}{2}(t_j - k).$$

Hence,

$$\det\left[\frac{1}{\pi}I_2(t_j - k)\right]_{j,k=-n}^{n} = \det\left[\operatorname{sinc}\frac{1}{2}(t_j - k)\right]_{j,k=-n}^{n}$$

as required. ■

The following corollary is placed here merely to check the constants of Theorem 6.5.0.36. Its proof makes use of the preceding proposition.

Corollary 6.5.0.38 *Suppose that $n = 1$ and $a = 2$. Let $t_j = j$ then the determinant of the matrix*

$$\left[\frac{1}{\pi}I_2(j - k)\right]_{j,k=-1}^{1}$$

satisfies Theorem 6.5.0.36.

Proof. Set $n = 1$ and $a = 2$ and let $t_j = j$. By Proposition 6.5.0.37 we have,

$$\begin{aligned}
\det\left[\frac{1}{\pi}I_2(j - k)\right]_{j,k=-1}^{1} &= \det\left[\operatorname{sinc}\frac{1}{2}(j - k)\right]_{j,k=-1}^{1} \\
&= \det\begin{bmatrix} \operatorname{sinc}(0) & \operatorname{sinc}\left(-\frac{1}{2}\right) & \operatorname{sinc}(-1) \\ \operatorname{sinc}\left(\frac{1}{2}\right) & \operatorname{sinc}(0) & \operatorname{sinc}\left(-\frac{1}{2}\right) \\ \operatorname{sinc}(1) & \operatorname{sinc}\left(\frac{1}{2}\right) & \operatorname{sinc}(0) \end{bmatrix} \\
&= \det\begin{bmatrix} \operatorname{sinc}(0) & \operatorname{sinc}\left(\frac{1}{2}\right) & \operatorname{sinc}(1) \\ \operatorname{sinc}\left(\frac{1}{2}\right) & \operatorname{sinc}(0) & \operatorname{sinc}\left(\frac{1}{2}\right) \\ \operatorname{sinc}(1) & \operatorname{sinc}\left(\frac{1}{2}\right) & \operatorname{sinc}(0) \end{bmatrix},
\end{aligned}$$

where the last line holds as the sinc function is an even function. Given that $\operatorname{sinc}(0) = 1$, $\operatorname{sinc}(1) = 0$ and $\operatorname{sinc}\left(\frac{1}{2}\right) = \frac{2}{\pi}$, it follows that

$$\begin{aligned}
\det\left[\frac{1}{\pi}I_2(j - k)\right]_{j,k=-1}^{1} &= \det\begin{bmatrix} 1 & \frac{2}{\pi} & 0 \\ \frac{2}{\pi} & 1 & \frac{2}{\pi} \\ 0 & \frac{2}{\pi} & 1 \end{bmatrix} \\
&= \det\begin{bmatrix} 1 & \frac{2}{\pi} \\ \frac{2}{\pi} & 1 \end{bmatrix} - \frac{2}{\pi}\det\begin{bmatrix} \frac{2}{\pi} & \frac{2}{\pi} \\ 0 & 1 \end{bmatrix} \\
&= 1 - \left(\frac{2}{\pi}\right)^2 - \left(\frac{2}{\pi}\right)^2 \\
&= 1 - \frac{8}{\pi^2}.
\end{aligned}$$

We now check the value of $\det\left[\frac{1}{\pi}I_2(j-k)\right]_{j,k=-1}^{1}$ as stated by Theorem 6.5.0.36. We use the full matrix notation to make our calculations clear, thus

$$\begin{aligned}
&\det\left[\frac{1}{\pi}I_2(j-k)\right]_{j,k=-1}^{1}\\
&=\frac{1}{3!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\det\left[e^{ijx_l}\right]_{j,l=-1}^{1}\det\left[e^{-ikx_l}\right]_{k,l=-1}^{1}\frac{dx_{-1}}{\pi}\frac{dx_0}{\pi}\frac{dx_1}{\pi}\\
&=\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\det\left(\left[e^{ijx_l}\right]_{j,l=-1}^{1}\cdot\left[e^{-ikx_l}\right]_{k,l=-1}^{1}\right)dx_{-1}\,dx_0\,dx_1\\
&=\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\det\left(\begin{bmatrix}e^{-ix_{-1}} & e^{-ix_0} & e^{-ix_1}\\ 1 & 1 & 1\\ e^{ix_{-1}} & e^{ix_0} & e^{ix_1}\end{bmatrix}\begin{bmatrix}e^{ix_{-1}} & e^{ix_0} & e^{ix_1}\\ 1 & 1 & 1\\ e^{-ix_{-1}} & e^{-ix_0} & e^{-ix_1}\end{bmatrix}\right)dx_{-1}\,dx_0\,dx_1.
\end{aligned}$$

Note that the two matrices in the above expression are row equivalent and so

$$\begin{aligned}
&\det\left[\frac{1}{\pi}I_2(j-k)\right]_{j,k=-1}^{1}\\
&=\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\det\left(\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}e^{ix_{-1}} & e^{ix_0} & e^{ix_1}\\ 1 & 1 & 1\\ e^{-ix_{-1}} & e^{-ix_0} & e^{-ix_1}\end{bmatrix}^2\right)dx_{-1}\,dx_0\,dx_1\\
&=-\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\det\begin{bmatrix}e^{ix_{-1}} & e^{ix_0} & e^{ix_1}\\ 1 & 1 & 1\\ e^{-ix_{-1}} & e^{-ix_0} & e^{-ix_1}\end{bmatrix}^2 dx_{-1}\,dx_0\,dx_1\\
&=-\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\det\begin{bmatrix}e^{ix_{-1}} & e^{ix_0} & e^{ix_1}\\ 1 & 1 & 1\\ e^{-ix_{-1}} & e^{-ix_0} & e^{-ix_1}\end{bmatrix}\right)^2 dx_{-1}\,dx_0\,dx_1.
\end{aligned}$$

Expanding the determinant gives

$$\begin{aligned}
&\det\begin{bmatrix}e^{ix_{-1}} & e^{ix_0} & e^{ix_1}\\ 1 & 1 & 1\\ e^{-ix_{-1}} & e^{-ix_0} & e^{-ix_1}\end{bmatrix}\\
&=-\det\begin{bmatrix}e^{ix_0} & e^{ix_1}\\ e^{-ix_0} & e^{-ix_1}\end{bmatrix}+\det\begin{bmatrix}e^{ix_{-1}} & e^{ix_1}\\ e^{-ix_{-1}} & e^{-ix_1}\end{bmatrix}-\det\begin{bmatrix}e^{ix_{-1}} & e^{ix_0}\\ e^{-ix_{-1}} & e^{-ix_0}\end{bmatrix}\\
&=-e^{i(x_0-x_1)}+e^{-i(x_0-x_1)}+e^{i(x_{-1}-x_1)}-e^{-i(x_{-1}-x_1)}-e^{i(x_{-1}-x_0)}+e^{-i(x_{-1}-x_0)}\\
&=-2i\sin(x_0-x_1)+2i\sin(x_{-1}-x_1)-2i\sin(x_{-1}-x_0).
\end{aligned}$$

A further expansion of the terms inside the brackets then produces

$$
\begin{aligned}
&\det\left[\frac{1}{\pi}I_2(j-k)\right]_{j,k=-1}^{1}\\
&= -\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{-2i\sin(x_0-x_1)+2i\sin(x_{-1}-x_1)-2i\sin(x_{-1}-x_0)\}^2\,dx_{-1}\,dx_0\,dx_1\\
&= -\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{-4\sin^2(x_0-x_1)+8\sin(x_{-1}-x_1)\sin(x_0-x_1)-4\sin^2(x_{-1}-x_1)\\
&\qquad -8\sin(x_{-1}-x_0)\sin(x_0-x_1)+8\sin(x_{-1}-x_0)\sin(x_{-1}-x_1)-4\sin^2(x_{-1}-x_0)\}\\
&\qquad dx_{-1}\,dx_0\,dx_1.
\end{aligned}
$$

We then use the double angle formulae to further simplify the integrand so that we may easily evaluate it. Hence,

$$
\begin{aligned}
&\det\left[\frac{1}{\pi}I_2(j-k)\right]_{j,k=-1}^{1}\\
&= -\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{-2+2\cos 2(x_0-x_1)+4\cos(x_{-1}-x_0)-4\cos(x_{-1}+x_0-2x_1)-2\\
&\qquad +2\cos 2(x_{-1}-x_1)-4\cos(x_{-1}-2x_0+x_1)+4\cos(x_{-1}-x_1)+4\cos(x_1-x_0)\\
&\qquad -4\cos(2x_{-1}-x_0-x_1)-2+2\cos 2(x_{-1}-x_0)\}\,dx_{-1}\,dx_0\,dx_1\\
&= -\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{-6+4\cos(x_0-x_1)+2\cos 2(x_0-x_1)+4\cos(x_{-1}-x_0)\\
&\qquad +4\cos(x_{-1}-x_1)+2\cos 2(x_{-1}-x_0)+2\cos 2(x_{-1}-x_1)-4\cos(x_{-1}+x_0-2x_1)\\
&\qquad -4\cos(x_{-1}-2x_0+x_1)-4\cos(2x_{-1}-x_0-x_1)\}\,dx_{-1}\,dx_0\,dx_1.
\end{aligned}
$$

We evaluate the inner integral as follows,

$$
\begin{aligned}
I_{-1} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{-6+4\cos(x_0-x_1)+2\cos 2(x_0-x_1)+4\cos(x_{-1}-x_0)+4\cos(x_{-1}-x_1)\\
&\qquad +2\cos 2(x_{-1}-x_0)+2\cos 2(x_{-1}-x_1)-4\cos(x_{-1}+x_0-2x_1)\\
&\qquad -4\cos(x_{-1}-2x_0+x_1)-4\cos(2x_{-1}-x_0-x_1)\}\,dx_{-1}\\
&= [\{-6+4\cos(x_0-x_1)+2\cos 2(x_0-x_1)\}x_{-1}+4\sin(x_{-1}-x_0)+4\sin(x_{-1}-x_1)\\
&\qquad +\sin 2(x_{-1}-x_0)+\sin 2(x_{-1}-x_1)-4\sin(x_{-1}+x_0-2x_1)-4\sin(x_{-1}-2x_0+x_1)\\
&\qquad -2\sin(2x_{-1}-x_0-x_1)]_{x_{-1}=-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
&= \pi\left[-6+4\cos(x_0-x_1)+2\cos 2(x_0-x_1)\right]+4\sin\left(\frac{\pi}{2}-x_0\right)-4\sin\left(-\frac{\pi}{2}-x_0\right)\\
&\qquad +4\sin\left(\frac{\pi}{2}-x_1\right)-4\sin\left(-\frac{\pi}{2}-x_1\right)+\sin(\pi-2x_0)-\sin(-\pi-2x_0)+\sin(\pi-2x_1)\\
&\qquad -\sin(-\pi-2x_1)-4\sin\left(\frac{\pi}{2}+x_0-2x_1\right)+4\sin\left(-\frac{\pi}{2}+x_0-2x_1\right)\\
&\qquad -4\sin\left(\frac{\pi}{2}-2x_0+x_1\right)+4\sin\left(-\frac{\pi}{2}-2x_0+x_1\right)-2\sin(\pi-x_0-x_1)\\
&\qquad +2\sin(-\pi-x_0-x_1).
\end{aligned}
$$

Using the double angle formulae we can simplify the integral above to obtain,

$$
\begin{aligned}
I_{-1} &= \pi\left[-6+4\cos(x_0-x_1)+2\cos 2(x_0-x_1)\right]+8\cos(x_0)+8\cos(x_1)-8\cos(x_0-2x_1)\\
&\qquad -8\cos(2x_0-x_1).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\det\left[\frac{1}{\pi}I_2(j-k)\right]_{j,k=-1}^{1}\\
&= -\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{-6\pi+8\cos(x_1)+8\cos(x_0)+4\pi\cos(x_0-x_1)-8\cos(x_0-2x_1)\\
&\quad -8\cos(2x_0-x_1)+2\pi\cos 2(x_0-x_1)\}\;dx_0\,dx_1.
\end{aligned}$$

Again, evaluating the inner integral, i.e. the integral with respect to x_0, we see that

$$\begin{aligned}
I_0 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{-6\pi+8\cos(x_1)+8\cos(x_0)+4\pi\cos(x_0-x_1)-8\cos(x_0-2x_1)-8\cos(2x_0-x_1)\\
&\quad +2\pi\cos 2(x_0-x_1)\}\;dx_0\\
&= [\{-6\pi+8\cos(x_1)\}\,x_0+8\sin(x_0)+4\pi\sin(x_0-x_1)-8\sin(x_0-2x_1)-4\sin(2x_0-x_1)\\
&\quad +\pi\sin 2(x_0-x_1)]_{x_0=-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
&= \pi\left[-6\pi+8\cos(x_1)\right]+8\sin\left(\frac{\pi}{2}\right)-8\sin\left(-\frac{\pi}{2}\right)+4\pi\sin\left(\frac{\pi}{2}-x_1\right)-4\pi\sin\left(-\frac{\pi}{2}-x_1\right)\\
&\quad -8\sin\left(\frac{\pi}{2}-2x_1\right)+8\sin\left(-\frac{\pi}{2}-2x_1\right)-4\sin(\pi-x_1)+4\sin(-\pi-x_1)\\
&\quad +\pi\sin(\pi-2x_1)-\pi\sin(-\pi-2x_1)\\
&= \pi\left[-6\pi+8\cos(x_1)\right]+16+8\pi\cos(x_1)-16\cos(2x_1).
\end{aligned}$$

Finally, we evaluate the remaining integral. Thus,

$$\begin{aligned}
&\det\left[\frac{1}{\pi}I_2(j-k)\right]_{j,k=-1}^{1}\\
&= -\frac{1}{6\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{16-6\pi^2+16\pi\cos(x_1)-16\cos(2x_1)\right\}\;dx_1\\
&= -\frac{1}{6\pi^3}\left[\left(16-6\pi^2\right)x_1+16\pi\sin(x_1)-8\sin(2x_1)\right]_{x_1=-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
&= -\frac{1}{6\pi^3}\left[\pi\left(16-6\pi^2\right)+16\pi\sin\left(\frac{\pi}{2}\right)-16\pi\sin\left(-\frac{\pi}{2}\right)-8\sin(\pi)+8\sin(-\pi)\right]\\
&= -\frac{1}{6\pi^3}\left[48\pi-6\pi^3\right]\\
&= 1-\frac{8}{\pi^2}
\end{aligned}$$

as required. ■

Theorem 6.5.0.36 focuses on evaluating the integral expression of I_a. However, we can also use Ramanujan's formula 6.4.0.29 to evaluate $\det\left[I_a(t_j-k)\right]_{j,k=-n}^{n}$. Recall that for $n \in \mathbb{N}$, $\Gamma(n)=(n-1)!$. We investigate the cases where $\frac{1}{2}(a+t_j-k)$ and $\frac{1}{2}(a-t_j+k)$ are positive integers to see how the general formula for $I_a(t_j-k)$ reduces to an expression involving factorials. Firstly, we must set $t_j=j$ where $j\in\mathbb{Z}$. Now, $\frac{1}{2}(a+j-k)$ and $\frac{1}{2}(a-j+k)$ are positive integers if $a+j-k$ and $a-j+k$ are even numbers. This can occur in two ways. Firstly, if a is even then $j-k$ must also be even. Secondly, if a is odd then $j-k$ must also be odd. We consider cases with $a\in\mathbb{N}$ and take $n\in\mathbb{N}$ throughout. The following lemma show what happens to the function I_a when a is even.

Lemma 6.5.0.39 *Suppose that $a = 2b$ for some $b \in \mathbb{N}$. If $j-k$ is even so that $\frac{j-k}{2} \in \mathbb{Z}$ then*

$$I_{2b}(j-k) = \frac{\pi(2b-2)!}{2^{2b-2}\left(b+\frac{j-k}{2}-1\right)!\left(b-\frac{j-k}{2}-1\right)!}$$

for $j-k \leq a-2$. Otherwise, $I_{2b}(j-k) = 0$.

Proof. Let $a = 2b$ for some $b \in \mathbb{N}$ and suppose that 2 divides $j-k$. Then, following on from Ramanujan's formula 6.4.0.29, for $2b > 1$ we have

$$I_{2b}(j-k) = \frac{\pi\Gamma(2b-1)}{2^{2b-2}\Gamma\left(\frac{2b+j-k}{2}\right)\Gamma\left(\frac{2b-j+k}{2}\right)}.$$

Note that $2b$ is an even integer that is strictly bigger than 1, therefore $2b \geq 2$. Clearly we can write $\Gamma(2b-1) = (2b-2)!$. In order to write $\Gamma\left(\frac{2b+j-k}{2}\right)$ as a factorial, a necessary condition is that

$$\frac{2b+j-k}{2} \geq 1.$$

This is clearly true in the case that $j-k$ is positive. Similarly, if we assume that $j-k$ is positive then in order to write $\Gamma\left(\frac{2b-j+k}{2}\right)$ as a factorial we must have

$$\frac{2b-j+k}{2} \geq 1.$$

This is equivalent to the condition $j-k \leq 2b-2$. It suffices to show that the factorials exist in the case $j-k$ positive because I_{2b} is an even function. Hence, for $j-k \leq a-2$

$$I_{2b}(j-k) = \frac{\pi(2b-2)!}{2^{2b-2}(b+\frac{j-k}{2}-1)!(b-\frac{j-k}{2}-1)!}.$$

Note that for $j-k > 2b-2$, $\frac{2b-j+k}{2}$ will be a negative integer. Now, the Gamma function has poles at the negative integers, therefore, $\frac{1}{\Gamma}$ has roots at the negative integers. It follows that

$$\frac{1}{\Gamma\left(\frac{2b-j+k}{2}\right)} = 0,$$

hence $I_{2b}(j-k) = 0$ for $j-k > 2b-2$. ■

We continue with an analogous lemma for the case a odd.

Lemma 6.5.0.40 *Suppose that a is an odd integer so that $a = 2b+1$ for some $b \in \mathbb{N}$. If $j-k$ is odd then*

$$I_{2b+1}(j-k) = \frac{\pi(2b-1)!}{2^{2b-1}\left(b+\frac{j-k-1}{2}\right)!\left(b-\frac{j-k-1}{2}-1\right)!}$$

for $j-k \leq a-2$. Otherwise, $I_{2b+1}(j-k) = 0$.

Proof. Let $a = 2b+1$ for $b \in \mathbb{N}$. Suppose that $j-k$ is odd, then $j-k \pm 1$ is even, hence $\frac{j-k\pm1}{2} \in \mathbb{Z}$. By Ramanujan's formula 6.4.0.29,

$$\begin{aligned} I_{2b+1}(j-k) &= \frac{\pi\Gamma(2b)}{2^{2b-1}\Gamma\left(\frac{2b+1+j-k}{2}\right)\Gamma\left(\frac{2b+1-j+k}{2}\right)} \\ &= \frac{\pi\Gamma(2b)}{2^{2b-1}\Gamma\left(b+\frac{j-k+1}{2}\right)\Gamma\left(b-\frac{j-k-1}{2}\right)}. \end{aligned}$$

Since $b \in \mathbb{N}$, $2b-1 \geq 1$ and so $\Gamma(2b) = (2b-1)!$ is well defined. We wish to write $\Gamma\left(b + \frac{j-k+1}{2}\right)$ and $\Gamma\left(b - \frac{j-k-1}{2}\right)$ as factorial expressions. As in the previous lemma, it suffices to show that this can be done in the case $j-k$ positive as I_{2b+1} is an even function. Thus take $j-k \geq 0$ then, since $b \in \mathbb{N}$ we certainly have

$$b + \frac{j-k+1}{2} \geq 1.$$

Also,

$$b - \frac{j-k-1}{2} \geq 1$$

if and only if $j-k \leq 2b-1$, that is $j-k \leq a-2$. Hence, for $j-k \leq a-2$ we have

$$\begin{aligned} I_{2b+1}(j-k) &= \frac{\pi(2b-1)!}{2^{2b-1}\left(b+\frac{j-k+1}{2}-1\right)!\left(b-\frac{j-k-1}{2}-1\right)!} \\ &= \frac{\pi(2b-1)!}{2^{2b-1}\left(b+\frac{j-k-1}{2}\right)!\left(b-\frac{j-k-1}{2}-1\right)!} \end{aligned}$$

as required. Finally we note that if $j-k > 2b-1$ then $b - \frac{j-k-1}{2} < 1$ and so $\frac{1}{\Gamma(z)}$ has a root at $z = b - \frac{j-k-1}{2}$, completing the proof. ■

Since $a > 1$ can take any value and $j, k \in \mathbb{Z}$, the case in which $a+j-k$ and $a-j+k$ are odd integers can also occur. Now, in order for this to happen, if a is even then $j-k$ must be odd. Similarly, if a is odd then $j-k$ must be even. We summarise what happens in these cases in the following two lemmas.

Lemma 6.5.0.41 *Suppose that a is an even integer so that $a = 2b$ for some $b \in \mathbb{N}$. Let $j-k$ be odd then*

$$I_{2b}(j-k) = \frac{2^{2b}\left(b-\frac{j-k}{2}-1\right)(2b-2)!\left(b+\frac{j-k-1}{2}-1\right)!\left(b-\frac{j-k-1}{2}-1\right)!}{(2b+j-k-2)!(2b-j+k)!}$$

for $j-k \leq a-1$. Otherwise, $I_{2b}(j-k) = 0$.

Proof. Let $a = 2b$ for some $b \in \mathbb{N}$. Suppose for simplicity that $j-k = 2p+1$ for some $p \in \mathbb{N}$. Note that we may assume $p \in \mathbb{N}$ since I_{2b} is an even function. We have

$$I_{2b}(j-k) = \frac{\pi\Gamma(2b-1)}{2^{2b-2}\Gamma\left(b+p+\frac{1}{2}\right)\Gamma\left(b-p-\frac{1}{2}\right)}.$$

Now, using the Legendre duplication formula 6.4.0.27 we see that

$$\Gamma\left(b+p+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2b+2p)}{2^{2b+2p-1}\Gamma(b+p)}.$$

Similarly,

$$\begin{aligned} \Gamma\left(b-p-\frac{1}{2}\right) &= \frac{\Gamma\left(b-p+\frac{1}{2}\right)}{b-p-\frac{1}{2}} \\ &= \frac{\sqrt{\pi}\Gamma(2b-2p)}{2^{2b-2p-1}(b-p-\frac{1}{2})\Gamma(b-p)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
I_{2b}(j-k) &= \frac{\pi\Gamma(2b-1)}{2^{2b-2}\left(\frac{\sqrt{\pi}\Gamma(2b+2p)}{2^{2b+2p-1}\Gamma(b+p)}\right)\left(\frac{\sqrt{\pi}\Gamma(2b-2p)}{2^{2b-2p-1}(b-p-\frac{1}{2})\Gamma(b-p)}\right)} \\
&= \frac{2^{4b-2}\pi\left(b-p-\frac{1}{2}\right)\Gamma(2b-1)\Gamma(b+p)\Gamma(b-p)}{2^{2b-2}\pi\Gamma(2b+2p)\Gamma(2b-2p)} \\
&= \frac{2^{2b}\left(b-p-\frac{1}{2}\right)\Gamma(2b-1)\Gamma(b+p)\Gamma(b-p)}{\Gamma(2b+2p)\Gamma(2b-2p)}. \qquad (6.17)
\end{aligned}$$

Now, since $b \in \mathbb{N}$ it is clear that $2b-2 \geq 0$, hence $\Gamma(2b-1) = (2b-2)!$ is well defined. We consider if the other terms involving Γ can be easily converted into factorial expressions. As $b, p \in \mathbb{N}$ it is also apparent that $\Gamma(b+p)$ and $\Gamma(2b+2p)$ can be expressed as factorials. It therefore remains to check that $\Gamma(b-p)$ and $\Gamma(2b-2p)$ have factorial expressions. Now, $\Gamma(b-p)$ can be written as a factorial if $b-p \geq 1$. This is equivalent to the condition $2p+1 \leq 2b-1$, or $j-k \leq a-1$. Similarly, we see that $2b-2p \geq 1$ if and only if $2p+1 \leq 2b$ and so $\Gamma(2b-2p)$ can be written as a factorial if $j-k \leq a$. Hence,

$$I_{2b}(j-k) = \frac{2^{2b}\left(b-p-\frac{1}{2}\right)(2b-2)!(b+p-1)!(b-p-1)!}{(2b+2p-1)!(2b-2p-1)!}$$

for $j-k \leq a-1$. Returning to our original notation involving j and k we have, for $j-k \leq a-1$,

$$\begin{aligned}
I_{2b}(j-k) &= \frac{2^{2b}\left(b-\frac{j-k-1}{2}-\frac{1}{2}\right)(2b-2)!\left(b+\frac{j-k-1}{2}-1\right)!\left(b-\frac{j-k-1}{2}-1\right)!}{(2b+j-k-1-1)!(2b-j+k+1-1)!} \\
&= \frac{2^{2b}\left(b-\frac{j-k}{2}-1\right)(2b-2)!\left(b+\frac{j-k-1}{2}-1\right)!\left(b-\frac{j-k-1}{2}-1\right)!}{(2b+j-k-2)!(2b-j+k)!}.
\end{aligned}$$

Finally, using (6.17), we note that if $j-k = 2p+1 > a$ then $2b-2p < 1$ and so $\frac{1}{\Gamma(z)}$ has a zero at $z = 2b-2p$. Therefore, $I_{2b}(j-k) = 0$ for $j-k > a$. In the case that $j-k = a = 2b$ we observe that

$$\begin{aligned}
b-p-\frac{1}{2} &= b-\frac{j-k-1}{2}-\frac{1}{2} \\
&= b-\frac{2b-1}{2}-\frac{1}{2} \\
&= 0.
\end{aligned}$$

Hence, $I_{2b}(j-k) = 0$ for $j-k \geq a$. ■

Lemma 6.5.0.42 *Suppose that a is an odd integer so that $a = 2b+1$ for some $b \in \mathbb{N}$. If $j-k$ is even then*

$$I_{2b+1}(j-k) = \frac{2^{2b-1}(2b-1)!\left(b+\frac{j-k}{2}-1\right)!\left(b-\frac{j-k}{2}-1\right)!}{(2b+j-k-1)!(2b-j+k-1)!}$$

for $j-k \leq a-3$. Otherwise, $I_{2b+1}(j-k) = 0$.

Proof. Let $a = 2b+1$ for some $b \in \mathbb{N}$ and suppose for simplicity that $j-k = 2p$ for some $p \in \mathbb{N}$. Then

$$I_{2b+1}(j-k) = \frac{\pi\Gamma(2b)}{2^{2b-1}\Gamma\left(b+p+\frac{1}{2}\right)\Gamma\left(b-p+\frac{1}{2}\right)}.$$

Using the Legendre duplication formula 6.4.0.27 we see that

$$\Gamma\left(b+p+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2b+2p)}{2^{2b+2p-1}\Gamma(b+p)}$$

and

$$\Gamma\left(b-p+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2b-2p)}{2^{2b-2p-1}\Gamma(b-p)}.$$

Therefore,

$$\begin{aligned} I_{2b+1}(j-k) &= \frac{\pi\Gamma(2b)}{2^{2b-1}\left(\frac{\sqrt{\pi}\Gamma(2b+2p)}{2^{2b+2p-1}\Gamma(b+p)}\right)\left(\frac{\sqrt{\pi}\Gamma(2b-2p)}{2^{2b-2p-1}\Gamma(b-p)}\right)} \\ &= \frac{2^{4b-2}\Gamma(2b)\Gamma(b+p)\Gamma(b-p)}{2^{2b-1}\Gamma(2b+2p)\Gamma(2b-2p)}. \end{aligned} \tag{6.18}$$

Now, $a = 2b+1 > 1$ so clearly $2b+1 \geq 3$, that is $b \geq 1$. Using this, together with the fact that $p \in \mathbb{N}$ so $p \geq 1$, it is obvious that $\Gamma(2b)$, $\Gamma(b+p)$ and $\Gamma(2b+2p)$ can be written as factorials. As before, we check the conditions under which $b-p \geq 1$ and $2b-2p \geq 1$. First, $b-p \geq 1$ if and only if $2p \leq 2b-2$. This is equivalent to $j-k \leq a-3$. Similarly, $2b-2p \geq 1$ if and only if $2p \leq 2b-1$, which is equivalent to $j-k \leq a-2$. Thus,

$$I_{2b+1}(j-k) = \frac{2^{2b-1}(2b-1)!(b+p-1)!(b-p-1)!}{(2b+2p-1)!(2b-2p-1)!}$$

for $j-k \leq a-3$. Again, returning to the original notation yields

$$I_{2b+1}(j-k) = \frac{2^{2b-1}(2b-1)!\left(b+\frac{j-k}{2}-1\right)!\left(b-\frac{j-k}{2}-1\right)!}{(2b+j-k-1)!(2b-j+k-1)!},$$

for $j-k \leq a-3$. Finally, from (6.18) we see that for $j-k > a-2$ we have $I_{2b+1}(j-k) = 0$ since $\frac{1}{\Gamma(2b-2p)}$ will have a zero, completing the proof. ■

With these lemmas in place, we are now ready to construct the matrices $[I_a(j-k)]_{j,k=-n}^{n}$, in the cases of a odd and a even. We show the results in separate theorems.

Theorem 6.5.0.43 *Let $n \in \mathbb{N}$. Suppose that a is even so that a has the form $a = 2b$ for some $b \in \mathbb{N}$. Then $[I_a(j-k)]_{j,k=-n}^{n}$ is a real symmetric Toeplitz matrix with entries in $\mathbb{Q}[\pi]$. Furthermore, let $I = I_a$ then $[I_a(j-k)]_{j,k=-n}^{n}$ has the form*

$$\left[\begin{array}{ccccccccc} I(0) & I(1) & I(2) & & I(2b-2) & I(2b-1) & 0 & & 0 \\ I(1) & I(0) & I(1) & & & I(2b-2) & I(2b-1) & \ddots & \\ I(2) & I(1) & I(0) & \ddots & & & I(2b-2) & \ddots & 0 \\ & & \ddots & \ddots & & & & \ddots & I(2b-1) \\ & & & & & & & & I(2b-2) \\ & & & \ddots & \ddots & \ddots & & & \\ I(2b-2) & & & & & & & & \\ I(2b-1) & I(2b-2) & & & & \ddots & \ddots & & \\ 0 & I(2b-1) & I(2b-2) & & & \ddots & I(0) & I(1) & I(2) \\ & \ddots & \ddots & \ddots & & & I(1) & I(0) & I(1) \\ 0 & & 0 & I(2b-1) & I(2b-2) & & I(2) & I(1) & I(0) \end{array}\right],$$

where

$$I_{2b}(j-k)=\frac{\pi(2b-2)!}{2^{2b-2}\left(b+\frac{j-k}{2}-1\right)!\left(b-\frac{j-k}{2}-1\right)!}$$

if $j-k\le 2b-2$ *is even and*

$$I_{2b}(j-k)=\frac{2^{2b}\left(b-\frac{j-k}{2}-1\right)(2b-2)!\left(b+\frac{j-k-1}{2}-1\right)!\left(b-\frac{j-k-1}{2}-1\right)!}{(2b+j-k-2)!(2b-j+k)!}$$

if $j-k\le 2b-1$ *is odd.*

Remark 6.5.0.44 *Note that for* $j-k$ *even,* $I_{2b}(j-k)$ *is a rational multiple of* π *whereas for* $j-k$ *odd,* $I_{2b}(j-k)$ *is rational.*

Proof. Let $n\in\mathbb{N}$ and suppose that $a=2b$ for some $b\in\mathbb{N}$. It is then clear that

$$\begin{aligned}[I_{2b}(j-k)]_{j,k=-n}^{n} &= \begin{bmatrix} I_{2b}(0) & I_{2b}(-1) & I_{2b}(-2) & & I_{2b}(-2n)\\ I_{2b}(1) & I_{2b}(0) & I_{2b}(-1) & \ddots & \\ I_{2b}(2) & I_{2b}(1) & I_{2b}(0) & \ddots & \ddots\\ & \ddots & \ddots & \ddots & \ddots\\ I_{2b}(2n) & & \ddots & \ddots & I_{2b}(0)\end{bmatrix}\\ &= \begin{bmatrix} I_{2b}(0) & I_{2b}(1) & I_{2b}(2) & & I_{2b}(2n)\\ I_{2b}(1) & I_{2b}(0) & I_{2b}(1) & \ddots & \\ I_{2b}(2) & I_{2b}(1) & I_{2b}(0) & \ddots & \ddots\\ & \ddots & \ddots & \ddots & \ddots\\ I_{2b}(2n) & & \ddots & \ddots & I_{2b}(0)\end{bmatrix}\end{aligned}$$

since the function I_a is even. The desired entries for the matrix follow from Lemmas 6.5.0.39 and 6.5.0.41, completing the proof. ■

Note the banded nature of the matrix in Theorem 6.5.0.43. The following example gives an application of Theorem 6.5.0.43 and helps the reader to see the shape of the matrix $[I_{2b}(j-k)]_{j,k=-n}^{n}$.

Example 6.5.0.45

Set $n=1$ so that we are working with a 3×3 matrix. Let $a=4$ then

$$[I_4(j-k)]_{j,k=-1}^{1} = \begin{bmatrix} I_4(0) & I_4(1) & I_4(2)\\ I_4(1) & I_4(0) & I_4(1)\\ I_4(2) & I_4(1) & I_4(0)\end{bmatrix}.$$

We use the theorem to evaluate each entry in turn. In the style of Theorem 6.5.0.43, we have $a=2(2)$ giving $b=2$. First we look at the cases of $j-k$ even. When $j-k=0$ we have

$$\begin{aligned}I_4(0) &= \frac{\pi(4-2)!}{2^{4-2}\left(2+\frac{0}{2}-1\right)!\left(2-\frac{0}{2}-1\right)!}\\ &= \frac{\pi}{2}.\end{aligned}$$

Similarly, when $j-k=2$ we have

$$\begin{aligned} I_4(2) &= \frac{\pi(4-2)!}{2^{4-2}\left(2+\frac{2}{2}-1\right)!\left(2-\frac{2}{2}-1\right)!} \\ &= \frac{2\pi}{22^2} \\ &= \frac{\pi}{4}. \end{aligned}$$

Now, when $j-k$ is odd, that is $j-k=1$ we have

$$\begin{aligned} I_4(1) &= \frac{2^4\left(2-\frac{1}{2}-1\right)(4-2)!\left(2+\frac{1-1}{2}-1\right)!\left(2-\frac{1-1}{2}-1\right)!}{(4+1-2)!(4-1)!} \\ &= \frac{2^4\left(\frac{1}{2}\right)(2)}{(3!)^2} \\ &= \frac{4}{9}. \end{aligned}$$

Hence,

$$[I_4(j-k)]_{j,k=-1}^{1} = \begin{bmatrix} \frac{\pi}{2} & \frac{4}{9} & \frac{\pi}{4} \\ \frac{4}{9} & \frac{\pi}{2} & \frac{4}{9} \\ \frac{\pi}{4} & \frac{4}{9} & \frac{\pi}{2} \end{bmatrix}.$$

The following theorem shows what happens to $[I_a(j-k)]_{j,k=-n}^{n}$ in the case that a is odd. It is analogous to Theorem 6.5.0.43. Again, note the banded nature of the matrix.

Theorem 6.5.0.46 *Let $n \in \mathbb{N}$. Suppose that a is odd so that $a = 2b+1$ for some $b \in \mathbb{N}$. Then $[I_a(j-k)]_{j,k=-n}^{n}$ is a real symmetric Toeplitz matrix with entries in $\mathbb{Q}[\pi]$. Furthermore, let $I = I_a$ then the matrix $[I_a(j-k)]_{j,k=-n}^{n}$ has the form*

$$\begin{bmatrix} I(0) & I(1) & I(2) & & I(2b-2) & I(2b-1) & 0 & & 0 \\ I(1) & I(0) & I(1) & & & I(2b-2) & I(2b-1) & \ddots & \\ I(2) & I(1) & I(0) & \ddots & & & I(2b-2) & \ddots & 0 \\ & & \ddots & \ddots & & & & & I(2b-1) \\ & & & & & & & & I(2b-2) \\ & & & \ddots & \ddots & \ddots & & & \\ I(2b-2) & & & & & & & & \\ I(2b-1) & I(2b-2) & & & & \ddots & \ddots & & \\ 0 & I(2b-1) & I(2b-2) & & & \ddots & I(0) & I(1) & I(2) \\ & \ddots & \ddots & \ddots & & & I(1) & I(0) & I(1) \\ 0 & & 0 & I(2b-1) & I(2b-2) & & I(2) & I(1) & I(0) \end{bmatrix},$$

where

$$I_{2b+1}(j-k) = \frac{2^{2b-1}(2b-1)!\left(b+\frac{j-k}{2}-1\right)!\left(b-\frac{j-k}{2}-1\right)!}{(2b+j-k-1)!(2b-j+k-1)!}$$

if $j-k$ is even and

$$I_{2b+1}(j-k) = \frac{\pi(2b-1)!}{2^{2b-1}\left(b+\frac{j-k-1}{2}\right)!\left(b-\frac{j-k-1}{2}-1\right)!}$$

if $j-k$ is odd.

Remark 6.5.0.47 *Note that for $j-k$ even, $I_{2b+1}(j-k)$ is rational whereas for $j-k$ odd, $I_{2b+1}(j-k)$ is a rational multiple of π.*

Proof. Let $n \in \mathbb{N}$ and suppose that $a = 2b+1$ for some $b \in \mathbb{N}$. As I_{2b+1} is an even function we clearly have the symmetric matrix

$$[I_{2b+1}(j-k)]_{j,k=-n}^{n} = \begin{bmatrix} I_{2b+1}(0) & I_{2b+1}(1) & I_{2b+1}(2) & & I_{2b+1}(2n) \\ I_{2b+1}(1) & I_{2b+1}(0) & I_{2b+1}(1) & \ddots & \\ I_{2b+1}(2) & I_{2b+1}(1) & I_{2b+1}(0) & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ I_{2b+1}(2n) & & \ddots & \ddots & I_{2b+1}(0) \end{bmatrix}.$$

The entries for the matrix follows from Lemmas 6.5.0.40 and 6.5.0.42, completing the proof. ■

We close this section with an example to illustrate Theorem 6.5.0.46.

Example 6.5.0.48

Again let $n = 1$ so that we are working with a 3×3 matrix. Set $a = 5$ then

$$[I_5(j-k)]_{j,k=-1}^{1} = \begin{bmatrix} I_5(0) & I_5(1) & I_5(2) \\ I_5(1) & I_5(0) & I_5(1) \\ I_5(2) & I_5(1) & I_5(0) \end{bmatrix}.$$

Now, using Theorem 6.5.0.46 we evaluate $I_5(j-k)$ in the cases that $j-k$ is odd and even. If $a = 2b+1$ we take $b = 2$. Taking $j-k = 0$ we have

$$\begin{aligned} I_5(0) &= \frac{2^{4-1}(4-1)!\left(2+\frac{0}{2}-1\right)!\left(2-\frac{0}{2}-1\right)!}{(4-1)!(4-1)!} \\ &= \frac{2^3 3!}{(3!)^2} \\ &= \frac{4}{3}. \end{aligned}$$

Similarly, taking $j-k = 2$ we see that

$$\begin{aligned} I_5(2) &= \frac{2^{4-1}(4-1)!\left(2+\frac{2}{2}-1\right)!\left(2-\frac{2}{2}-1\right)!}{(4+2-1)!(4-2-1)!} \\ &= \frac{2^3 3!(2)}{5!} \\ &= \frac{4}{5} \end{aligned}$$

Finally, letting $j-k = 1$ we have

$$\begin{aligned} I_5(1) &= \frac{\pi(4-1)!}{2^{4-1}\left(2+\frac{1-1}{2}\right)!\left(2-\frac{1-1}{2}-1\right)!} \\ &= \frac{\pi 3!}{2^3(2)} \\ &= \frac{3\pi}{8}. \end{aligned}$$

Hence,

$$[I_5(j-k)]_{j,k=-1}^{1} = \begin{bmatrix} \frac{4}{3} & \frac{3\pi}{8} & \frac{4}{5} \\ \frac{3\pi}{8} & \frac{4}{3} & \frac{3\pi}{8} \\ \frac{4}{5} & \frac{3\pi}{8} & \frac{4}{3} \end{bmatrix}.$$

6.6 The Evaluation of I_a Using Jacobi Polynomials

In this section we show that the function I_a can be used to give a value for various integrals that arise in the theory of Jacobi polynomials. Specifically we are able to show that I_a represents the integral of a Chebyshev polynomial whose weight function corresponds to the weight function of the Jacobi polynomials. We refer the reader to [52] for background information on the Chebyshev and Jacobi polynomials, both of which are orthogonal polynomials.

We begin by defining orthogonal polynomials and then we define the Jacobi polynomials. We adopt this approach since the Jacobi polynomials are defined with a weight function. Therefore, upon seeing that I_a is the integral of a Chebyshev polynomial with a weight function, the reader will understand where this weight function comes from.

Definition 6.6.0.49 *Let $w(x) \geq 0$ be a* weight *such that*

$$0 < \int_a^b w(x)dx < \infty.$$

Suppose that

$$\int_a^b |x|^n \, w(x) \, dx < \infty$$

for all $n \geq 0$. If we orthogonalise the set $\{1, x, \ldots, x^n, \ldots\}$ we obtain a set of polynomials

$$\{p_0(x), p_1(x), \ldots, p_n(x), \ldots\}$$

uniquely determined by the following conditions:
i) $p_n(x)$ has degree n with the coefficient of x^n being positive;
ii) the system, $\{p_n(x)\}_{n=0}^{\infty}$ is orthonormal so that $\int_a^b p_r(x)p_s(x)\, dm(x) = \delta_{rs}$.
We call $\{p_n(x)\}_{n=0}^{\infty}$ the orthogonal polynomials.

As previously mentioned, the Jacobi polynomials are orthogonal polynomials. We define the Jacobi polynomials next and use them to derive the Chebyshev polynomials.

Definition 6.6.0.50 *Suppose that $\alpha, \beta > -1$. Let $\{p_n\}_{n=0}^{\infty}$ be orthogonal polynomials associated with the weight*

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

on the interval $[-1, 1]$. We define the Jacobi polynomials, *$P_n^{(\alpha,\beta)}(x)$ to be such that*

$$P_n^{(\alpha,\beta)}(x) \propto p_n(x)$$

with normalisation given by $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$.

Definition 6.6.0.51 *Let $P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$ be the Jacobi polynomial with corresponding weight function $\left(1-x^2\right)^{-\frac{1}{2}}$. Then*

$$P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) = \frac{1.3...(2n-1)}{2.4...2n} T_n(x)$$

where $T_n(x)$ is a Chebyshev polynomial of the first kind. *If $x = \cos\theta$ then*

$$T_n(x) = \cos n\theta.$$

In the following two propositions we demonstrate how we can use I_a to evaluate two different functions involving Chebyshev polynomials.

Proposition 6.6.0.52 *Let T_k be a Chebyshev polynomial whose order, k is even. Then*

$$I_a(k) = \frac{1}{\left(\sqrt{2}\right)^{a-2}} \int_{-1}^{1} \frac{\left(\sqrt{1+x}\right)^{a-2}}{\sqrt{1-x^2}} T_k\left(\sqrt{\frac{1+x}{2}}\right) dx$$

where $T_k\left(\sqrt{\frac{1+x}{2}}\right)$ is a Chebyshev polynomial of order $\frac{k}{2}$.

Proof. First we observe that

$$\begin{aligned} I_a(k) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{a-2} e^{ikt}\, dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{a-2} \cos kt\, dt + i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{a-2} \sin kt\, dt. \end{aligned}$$

Since $(\cos t)^{a-2} \sin xt$ is odd we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{a-2} \sin kt\, dt = 0,$$

therefore

$$I_a(k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{a-2} \cos kt\, dt.$$

Also, as $(\cos t)^{a-2} \cos xt$ is even we may write

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{a-2} \cos kt\, dt = 2 \int_{0}^{\frac{\pi}{2}} (\cos t)^{a-2} \cos kt\, dt.$$

Noting that $\cos kt = T_k(\cos t)$ it follows that

$$I_a(k) = 2 \int_{0}^{\frac{\pi}{2}} (\cos t)^{a-2} T_k(\cos t)\, dt.$$

We now make the substitution $x = \cos 2t$. Using the double angle formulae we see that

$$\cos 2t = 2\cos^2 t - 1,$$

which rearranges to give

$$\begin{aligned} \cos t &= \sqrt{\frac{\cos 2t + 1}{2}} \\ &= \sqrt{\frac{x+1}{2}}. \end{aligned}$$

We also note that $dx = -2 \sin 2t\, dt$. Furthermore, using $\sin^2\theta + \cos^2\theta = 1$ we see that

$$\begin{aligned} 1 - x^2 &= 1 - \cos^2 2t \\ &= \sin^2 2t, \end{aligned}$$

giving

$$\sin 2t = \sqrt{1-x^2}.$$

Thus, with a change of limits, $\left[0, \frac{\pi}{2}\right] \mapsto [1, -1]$ we arrive at

$$\begin{aligned} I_a(k) &= 2\int_1^{-1} \left(\sqrt{\frac{x+1}{2}}\right)^{a-2} T_k\left(\sqrt{\frac{x+1}{2}}\right) \frac{(-1)}{2\sqrt{1-x^2}}\, dx \\ &= \frac{1}{\left(\sqrt{2}\right)^{a-2}} \int_{-1}^{1} \frac{\left(\sqrt{x+1}\right)^{a-2}}{\sqrt{1-x^2}} T_k\left(\sqrt{\frac{x+1}{2}}\right) dx \end{aligned}$$

as required. ■

Next we present a more general version of the above result in the sense that it holds for a Chebyshev polynomial of any order. The proof follows the same method as Proposition 6.6.0.52. The only alteration is a different choice of substitution.

Proposition 6.6.0.53 *Let T_k denote a Chebyshev polynomial of order k then*

$$I_a(k) = 2\int_0^1 \frac{x^{a-2}}{\sqrt{1-x^2}} T_k(x)\, dx.$$

Proof. As in the proof of Proposition 6.6.0.52 we may write

$$I_a(k) = 2\int_0^{\frac{\pi}{2}} (\cos t)^{a-2} T_k(\cos t)\, dt.$$

However, this time we make the substitution $x = \cos t$. We note that $dx = -\sin t\, dt$ and use $\sin^2\theta + \cos^2\theta = 1$ to obtain

$$\begin{aligned} \sin t &= \sqrt{1-\cos^2 t} \\ &= \sqrt{1-x^2}. \end{aligned}$$

A final note of the change of limits tells us that $\left[0, \frac{\pi}{2}\right] \mapsto [1, 0]$, hence

$$\begin{aligned} I_a(k) &= 2\int_1^0 x^{a-2} T_k(x) \frac{(-1)}{\sqrt{1-x^2}}\, dx \\ &= 2\int_0^1 \frac{x^{a-2}}{\sqrt{1-x^2}} T_k(x)\, dx \end{aligned}$$

as required. ■

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