Higher Order Risk Vulnerability

James Huang*

Richard Stapleton†

October 22, 2015

*Lancaster University Management School, Lancaster, LA1 4YX, UK. Tel: +44 1524 593633. Fax: +44 1524 847321. Email: james.huang@lancaster.ac.uk.

†Manchester Business School, Crawford House, Oxford Road, Manchester, M13 9PL, UK.

Abstract

We add an independent unfair background risk to higher order risk-taking models in the current literature and examine its interaction with the main risk under consideration. Parallel to the well-known concept of risk vulnerability, which is defined by Gollier and Pratt (Gollier, C., Pratt, J. W.: Risk vulnerability and the tempering effect of background risk. Econometrica 64, 1109-1123 (1996)), an agent is said to have a type of higher order risk vulnerability if adding an independent unfair background risk to wealth raises his level of this type of higher order risk-aversion. We derive necessary and sufficient conditions for all types of higher order risk vulnerabilities and explain their behavioral implications. We find that as in the case of risk vulnerability, all familiar HARA utility functions have all types of higher order risk vulnerabilities except for a type of third order risk vulnerability corresponding to a downside risk aversion measure called the Schwarzian derivative.

Key words: background risk, downside risk aversion, downside risk vulnerability, higher order risk vulnerability.
JEL codes: D81.

1 Introduction

Recently, there has been an intense interest in the third or higher order risk attitudes in the literature. For example, we have seen a series of studies on the intensity of downside risk aversion (hereafter DRA), which lead to the establishment of five DRA measures.[1] Among the five DRA measures, the most well-known is the prudence measure defined as $P(x) = -u'''(x)/u''(x)$, where $u(x)$ is a Von Neumann-Morgenstern utility function. Kimball (1990) establishes this concept to explain precautionary savings. Chiu (2000), however, finds that prudence is linked to downside risk aversion.[2] Chiu (2005, 2010) further shows that prudence measures the intensity of preferences over a set of downside risk increases and explains its link with skewness preference.[3]

Keenan and Snow (2002, 2009, 2012) suggest another DRA measure, the Schwarzian derivative $S(x) = -R'(x) - 0.5R^2(x)$, where $R(x)$ is the Arrow-Pratt risk aversion measure, and characterize DRA by considering changes in risk that induce mean-and-variance-preserving downside risk increases in the utility distribution. A further measure for the intensity of DRA is proposed by Modica and Scarsini (2005) and Crainich and Eeckhoudt (2008). Both studies show that $D(x) = u'''(x)/u'(x)$ is linked to skewness preference.

Liu and Meyer (2012) propose another measure, $-R'(x)$, the negative slope of the Arrow-Pratt risk aversion measure along a similar line to Chiu (2005) and Keenan and Snow (2009), while Huang and Stapleton (2014) establish a fifth measure which is known as cautiousness $C(x) = (1/R(x))'$

[1] The concept of downside risk aversion is established by Menezes et al. (1980).

[2] Jindapon and Neilson (2007) and Keenan and Snow (2010) also explain how prudence is linked to DRA.

[3] Liu and Meyer (2012) also contribute to this result.

in the literature, using a simple portfolio problem with a risk-free bond, a stock, and an option. Huang (2012) identifies two sets of downside risk increases over which the intensity of an individual's preferences is measured by cautiousness and $D(x)$ respectively. He also explains the link between these downside risk aversion measures and skewness preference and shows the effect of downside risk aversion on option prices.

More work has been done on higher order risk attitudes. For example, Eeckhoudt and Schlesinger (2006) create a general framework based on the concept of risk apportionment for analyzing higher order risk attitudes. Eeckhoudt et al. (2009) extend the above analysis by apportioning risks via stochastic dominance. Denuit and Eeckhoudt (2010a) generalize Chiu's (2005) analysis of downside risk aversion to higher order cases and develop the nth order Arrow-Pratt risk aversion measure $(-u^{(n)}(x)/u^{(n-1)}(x))$.[4] They (2010b) also establish higher order Ross risk aversion measures $(-1)^{n-1}u^{(n)}(x)/u^{(1)}(y)$ and generalize the local DRA measure $D(x)$ to higher orders $(-1)^{n-1}u^{(n)}(x)/u^{(1)}(x)$. Jindapon and Neilson (2007) use a comparative statics approach to generalize Arrow-Pratt and Ross risk aversion measures to higher orders, while Liu and Meyer (2013) use the ratio of utility premiums to establish the (n/m)th order local Arrow-Pratt risk aversion measure $(-1)^{n-m}u^{(n)}(x)/u^{(m)}(x)$ and Ross risk aversion measure $(-1)^{n-m}u^{(n)}(x)/u^{(m)}(y)$.

All the third and higher order risk-taking models in the above studies deal with only one source of risk; however, as is well recognized, in the real world individuals may face multiple sources of risks. An important

[4]Denuit and Eeckhoudt (2010a) define the nth order Arrow-Pratt risk aversion measure as $-u^{(n+1)}(x)/u^{n}(x)$.

example of additional sources of risks is a decision maker's background risk.[5] As is explained by Gollier and Pratt (1996) in their seminal paper on risk vulnerability, even if risks are independent, they interact with each other, and "not taking these interactions into account can lead the theoretical model to dramatically misestimate optimal risk-taking."

In this paper we address this issue. We add an independent unfair background risk to the higher order risk-taking models in the current literature and examine its interaction with the main risk under consideration. Basically we ask the following question: what is the effect of an unfair background risk on the intensity of higher order risk attitudes towards another independent risk?

Gollier and Pratt (1996) argue that "conventional wisdom suggests that independent risks are substitutes for each other. In particular, adding an unfair background risk to wealth should increase risk aversion to other independent risks." This is equivalent to the condition that an undesirable risk is never made desirable by the presence of an independent unfair risk. They call this risk vulnerability. A similar argument may apply to higher order risk taking, and analogous to their concept of risk vulnerability, an agent is said to have a higher order risk vulnerability if adding an independent unfair background risk to wealth raises his level of a higher order risk aversion. We characterize this concept and derive necessary and sufficient conditions for higher order risk vulnerabilities.

Compared with risk vulnerability, higher order risk vulnerabilities are more complex. For example, in the case of the third order, we have five

[5] Different background risks discussed in the literature include labor income risk, housing risk, entrepreneurial risk, etc. See, for example, Campbell (2006) for a brief review of this literature.

types of downside risk vulnerabilities (hereafter DRV) corresponding to five different definitions of DRA measures respectively, while in the case of nth ($n > 3$) order, there are $(n-1)$ types of risk vulnerabilities corresponding to the $(n/1)$th, $(n/2)$th, ..., and $(n/(n-1))$th order Arrow-Pratt risk aversion measures respectively. We give a detailed analysis for each type of the third and higher order risk vulnerabilities.

The concept of higher order risk vulnerability we study in this paper is related to the concepts of standard prudence, proper prudence, and precautionary vulnerability, which Lajeri-Chaherli (2004) uses to explain the effect of background risk on precautionary savings. It is also related to Pratt and Zeckhauser's (1987) proper risk aversion, Kimball's (1993) standard risk aversion, and Franke et al.'s (2006) multiplicative risk vulnerability, which all explain the effect of background risk on risk aversion. The paper is also related to the work of Hara et al. (2011) who investigate the effect of background risk on cautiousness. Other related studies include Tsetlin and Winkler (2005) and Li (2011) who investigate the effect of background risk on risky choices and demand for risky assets, respectively.

The structure of the remaining paper is as follows. In Section 2, we characterize downside risk vulnerability. In Section 3, we characterize higher order risk vulnerability. In Section 4, we discuss an application of the results. The last section concludes the paper.

2 Downside Risk Vulnerability

We first consider the case of third order risk vulnerability or DRV. As we pointed out in the Introduction section, there are five types of DRV corresponding to five different DRA measures respectively. We will examine each type of DRV.

In this section, we assume that all utility functions are strictly increasing, strictly concave, and thrice continuously differentiable with a positive third derivative unless stated otherwise. Given a von Neumann-Morgenstern utility function $u(x)$, as was explained in the Introduction section, we have the following five alternative DRA measures: $P(x) = -\frac{u'''(x)}{u''(x)}$ (Kimball (1990) and Chiu (2005)), $S(x) = \frac{u'''(x)}{u'(x)} - \frac{3}{2}R^2(x)$ (Keenan and Snow (2002, 2009, 2012)), $-R'(x)$ (Liu and Meyer (2012)), $D(x) = \frac{u'''(x)}{u'(x)}$ (Modica and Scarsini (2005) and Crainich and Eeckhoudt (2008)), and $C(x) = (\frac{1}{R(x)})'$ (Huang and Stapleton (2014)), where $R(x)$ is the Arrow-Pratt measure of risk aversion. For convenience, sometimes we denote the five DRA measures by $\tau_1(x) = C(x), \tau_2 = D(x), \tau_3(x) = P(x), \tau_4(x) = -R'(x)$, and $\tau_5(x) = S(x)$, respectively. Detailed explanations of these five DRA measures are given in the studies specified above, and a comparison of these five DRA measures and a discussion of their relationships can be found in Huang (2012).

Now consider the situation where there is an independent background risk $\tilde{\epsilon}$ to the wealth of an agent $u(x)$. Denote his derived utility function by $\hat{u}(x)$, i.e., $\hat{u}(x) = Eu(x + \tilde{\epsilon})$. We now give the formal definition of DRV:

Definition 1 *An agent $u(x)$ is said to have DRV of the ith type if $\tau_i(x)$ is increased by any independent unfair background risk, $i = 1, ..., 5$.*

According to this definition, there are five types of DRV corresponding to the five DRA measures respectively. We will explain behavioral implications of these five types of DRV later in Subsection 2.5.

2.1 A Necessary and Sufficient Condition

In this section we derive a necessary and sufficient condition for each type of DRV. When there is background risk $\tilde{\epsilon}$ to the wealth of an agent $u(x)$, we denote the type i DRA measure of his derived utility function $\hat{u}(x)$ by $\hat{\tau}_i(x)$, i.e., $\hat{\tau}_1(x) = \hat{C}(x) = (\frac{1}{\hat{R}(x)})'$, $\hat{\tau}_2(x) = \hat{D}(x) = \frac{Eu'''(x+\tilde{\epsilon})}{Eu'(x+\tilde{\epsilon})}$, $\hat{\tau}_3(x) = \hat{P}(x) = -\frac{Eu'''(x+\tilde{\epsilon})}{Eu''(x+\tilde{\epsilon})}$, $\hat{\tau}_4(x) = -\hat{R}'(x)$, and $\hat{\tau}_5(x) = \hat{S}(x) = \frac{Eu'''(x+\tilde{\epsilon})}{Eu'(x+\tilde{\epsilon})} - \frac{3}{2}\hat{R}^2(x)$, where $\hat{R}(x) = -\frac{Eu''(x+\tilde{\epsilon})}{Eu'(x+\tilde{\epsilon})}$. We present the following result.

Theorem 1 *An agent $u(x)$ has type i DRV to zero-mean [unfair] background risks if and only if $\tau_i(x)$ is increased by all independent binary zero-mean background risks, i.e., the ith of the following five inequalities is true, [and $\tau_i(x)$ is monotone decreasing], $i = 1, ..., 5$.*

$$-\frac{\partial\psi(w,x,y)}{\partial w} \geq C(w)\psi^2(w,x,y), \ \forall\ x>0, y>0, w, \tag{1}$$

$$u'(y)(D(y) - D(x)) \geq u'(x)D'(x)(y-x), \ \forall\ x, y, \tag{2}$$

$$-u''(y)(P(y) - P(x)) \geq -u''(x)P'(x)(y-x), \ \forall\ x, y, \tag{3}$$

$$-\frac{\partial\psi(w,x,y)}{\partial w} \geq -R'(w), \ \forall\ x>0, y>0, w, \tag{4}$$

$$-\frac{\partial\psi(w,x,y)}{\partial w} - 0.5\psi^2(w,x,y) \geq S(w), \ \forall\ x>0, y>0, w, \tag{5}$$

where $\psi(w,x,y) = \frac{xu''(w-y)+yu''(w+x)}{xu'(w-y)+yu'(w+x)}$.

Proof: See Appendix A.

The above result presents a necessary and sufficient condition for each type of DRV. We may note that the conditions for type 2 and type 3 DRVs are bivariate while those for type 1, type 4, and type 5 DRVs are trivariate. In the latter three cases, for technical reasons we cannot reduce the trivariate conditions to bivariate forms as we have done for the other two cases;[6]

[6]The main technical reason is that in the cases of type 2 and type 3 DRVs, $\tau_i(x)$ is a ratio of two linear functions of the derivatives of a utility function, thus characterizing

however, we will be able to derive bivariate sufficient conditions for these three cases later in Theorem 3, which are close in spirit to the above bivariate conditions for the other two cases.

2.2 Univariate Necessary Conditions

In the last subsection, we have derived a necessary and sufficient condition for each of the five types of DRV; however, these conditions are inconvenient as they are multivariate. In this subsection we search for univariate necessary conditions for the five types of DRV using the case of small background risks. We have the following result.

Theorem 2 *Assume that $u(x)$ is six times continuously differentiable. $\tau_i(x)$ is increased by all small independent zero-mean [unfair] background risks only if for all x the ith of the following five inequalities holds [and $\tau_i(x)$ is monotone decreasing], $i = 1, ..., 5$.*

$$C''(x) - 2(1 + 2C(x))C'(x)R(x) + 2C^2(x)(1 + C(x))R^2(x) \geq 0, \quad (6)$$

$$D''(x) - 2R(x)D'(x) \geq 0, \quad (7)$$

$$P''(x) - 2P(x)P'(x) \geq 0, \quad (8)$$

$$-R'''(x) + 2R(x)R''(x) + 2R'^2(x) \geq 0, \quad (9)$$

$$S''(x) - 2R(x)S'(x) + 3(S(x) + 0.5R^2(x))^2 \geq 0. \quad (10)$$

Proof: See Appendix B.

The above result presents a univariate necessary condition for each type of DRV. The first part ($i = 1$) of the theorem is derived by Hara et al. (2011) under the assumption that utility functions are real analytic. We relax this

DRV in these two cases is equivalent to solving a linear-fractional programming problem, while in the other three cases, it is not.

assumption to the condition that utility functions are six times continuously differentiable.

2.3 Univariate Sufficient Conditions

As we pointed out above, the necessary and sufficient conditions for the five types of DRV presented in Theorem 1 are inconvenient. In this subsection we derive more convenient sufficient conditions. We first present the following result.

Theorem 3 *Given $i \in \{1, ..., 5\}$, if for all x and y, $h_i(y)(\tau_i(y) - \tau_i(x)) \geq (y - x)h_i(x)\tau_i'(x)$, where $h_1(x) = \frac{(u''(x))^2}{u'(x)}$, $h_2(x) = h_4(x) = h_5(x) = u'(x)$, $h_3(x) = -u''(x)$, [and $\tau_i(x)$ is monotone decreasing] then, $\tau_i(x)$ is increased by all independent zero-mean [unfair] background risks, and the converse is true for $i = 2, 3$.*

Proof: See Appendix C.

The above result gives a more convenient sufficient condition for DRV of types 1, 4, and 5 than Theorem 1. Nevertheless, the main conditions in the above theorem are still complicated and inconvenient as they are bivariate, i.e., they involve computations of functions at two levels of wealth x and y. Similar to Gollier and Pratt's characterization of risk vulnerability, in the remainder of this subsection we derive some univariate conditions. We have the following result.

Theorem 4 *For $i = 1$, assume $C(x) \geq -0.5$. If for all x, $\tau_i''(x) \geq \zeta_i(x)\tau_i'(x)$, where $\zeta_1(x) = 2P(x) - R(x)$, $\zeta_2(x) = \zeta_4(x) = \zeta_5(x) = R(x)$, $\zeta_3(x) = P(x)$, (and $\tau_i(x)$ is monotone decreasing) then, $\tau_i(x)$ is increased by all independent zero-mean (unfair) background risks, $i = 1, ..., 5$.*

Proof: See Appendix D.

The above theorem gives a univariate sufficient condition for each type of DRV, which is similar in spirit to the univariate sufficient condition for risk vulnerability in Proposition 3 of Gollier and Pratt (1996). We attribute the third part ($i = 3$) of the above theorem and Corollary 1 below to Lajeri-Chaherli (2004) who uses these two results to characterize precautionary vulnerability in the context of precautionary savings.

As an immediate consequence of the above theorem, we have the following corollary.

Corollary 1 *For $i = 1$, assume $C(x) \geq -0.5$. If $\tau_i(x)$ is decreasing and convex then it is increased by all independent unfair background risks, $i = 1, ..., 5$.*

We attribute the result where $i = 1$ to Hara et al. (2011) who show that if cautiousness is positive, decreasing, and convex, then it is increased by all unfair background risks. Here we have relaxed the requirement of positive cautiousness to the condition that $C(x) \geq -0.5$. As we mentioned earlier the third part of the result ($i = 3$) is due to Lajeri-Chaherli (2004).

The condition of decreasing and convex DRA measures appears quite reasonable as we show in the next subsection that most familiar HARA utility functions satisfy this condition.

2.4 The Case of HARA Utility Functions

The case of HARA utility functions deserves some special attention as they are most frequently used in economics and finance. Some simple calculations show that for all HARA utility functions with a marginal utility function $u'(x) = (x + a)^{-\gamma}$, where $\gamma > 0$, $R(x)$, $D(x)$, $P(x)$, and $-R'(x)$ are all decreasing and convex, and $C(x) \equiv C = \frac{1}{\gamma}$ is constant; thus all these HARA utility functions satisfy the sufficient conditions for the first four types of DRV in Corollary 1.

The Schwarzian derivative, however, is an exception. For these HARA utility functions, we have $R(x) = \frac{\gamma}{x+a}$ and $S(x) = -R'(x) - 0.5R^2(x) = \frac{\gamma}{(x+a)^2} - \frac{0.5\gamma^2}{(x+a)^2} = \frac{\gamma(2-\gamma)}{2(x+a)^2}$. It follows that $S(x)$ is decreasing and convex if and only if $\gamma \in (0, 2)$, i.e., its cautiousness $C > 0.5$. Thus these HARA utility functions satisfy the sufficient condition for type 5 DRV in Corollary 1 only if $C \geq 0.5$.

In fact, when $C < 0.5$, i.e., $\gamma > 2$, we have $S'(x) > 0$, which violates the necessary condition for type 5 DRV to unfair risks. Thus when $C < 0.5$, the HARA utility functions do not have type 5 DRV to unfair risks. Moreover, we can verify that when $C < 0.5$, the necessary and sufficient condition for type 5 DRV to zero-mean risks in Theorem 1 is violated, thus these HARA utility functions do not have type 5 DRV to zero-mean risks either.[7]

2.5 Behavioral Implications

In this subsection, we explain behavioral implications of the five types of DRV. We first present the following lemma.

Lemma 1 *Given two CDFs $F(x)$ and $G(x)$ whose supports are both contained in $[a, b]$, assume that $G^{-1}(F(x))$ is concave and that $u(x)$ is indifferent between $F(x)$ and $G(x)$. If $\int_a^b \eta_i(x) \int_a^x u'(z)(G(z) - F(z))dzdx = 0$, where $\eta_1(x) = R^2(x)$, $\eta_2(x) = \frac{1}{u'^2(x)}$, $\eta_3(x) = \frac{R(x)}{u'(x)}$, $\eta_4(x) = 1$, and $\eta_5(x) = u'(x)$, then, $F(x)$ is preferred to $G(x)$ by any agent $v(x)$ who has a greater DRA measure of the ith type than $u(x)$, $i = 1, ..., 5$.*

[7] The verification is omitted for brevity but is available on request.

The lemma is part of Theorem 2 in Huang (2012), and its proof can be found there.

As is explained by Chiu (2005) and Huang (2012), according to Van Zwet (1964), a convex (concave) transformation of a random variable results in a strong increase (decrease) in skewness. This implies that the set of risk changes which satisfy the conditions (i) $G^{-1}(F(x))$ is concave, (ii) $u(x)$ is indifferent between $F(x)$ and $G(x)$, and (iii) $\int_a^b \eta_i(x) \int_a^x u'(z)(G(z) - F(z))dzdx = 0$ is a set of strong decreases in skewness. Thus the above lemma shows that the five sets of strong skewness decreases are indifferent for $u(x)$ but are unfavored by those who have a greater DRA measure of the ith type, $i = 1, ..., 5$, respectively.

This result also gives a glimpse into the relationship between the five DRA measures: they all explain the intensity of an agent's preference for strong increases in skewness, but the sets of strong increases in skewness over which the intensity of preference they explain are different from each other.

As an immediate consequence of the above lemma, we have the following result.

Proposition 1 *Let $i \in \{1, ..., 5\}$. Given two CDFs $F(x)$ and $G(x)$ whose supports are both contained in $[a, b]$, assume that $G^{-1}(F(x))$ is concave, an agent $u(x)$ is indifferent between $F(x)$ and $G(x)$, and $\int_a^b \eta_i(x) \int_a^x u'(z)(G(z) - F(z))dzdx = 0$, where $\eta_i(x)$ is defined in Lemma 1. If the agent has type i DRV, in the presence of an independent unfair background risk, he prefers $F(x)$ to $G(x)$.*

This result shows that if an agent has type i DRV, the five sets of strong skewness decreases which are indifferent for him become unfavored by him after an independent unfair background risk is added to his wealth. In other words, if he has type i DRV, an independent unfair background risk will increase the intensity of his preference for skewness.

3 Higher Order Risk Vulnerability

3.1 The Definition

As we explained in the introduction section, we have higher order risk aversion measures $R_{n/m}(x) = (-1)^{n-m}u^{(n)}(x)/u^{(m)}(x)$, where $1 \leq m < n$. In this section, we assume that $(-1)^i u^{(i)}(x) < 0$, $i = 1, ..., n$.[8] We now give the following definition.

Definition 2 *An agent is said to have (n/m)th order risk vulnerability if any independent unfair background risk $\tilde{\epsilon}$ makes him behave in a more (n/m)th order risk-averse way, i.e., $R_{n/m}(x)$ is increased by $\tilde{\epsilon}$.*

The above definition nests risk vulnerability as a special case where $n = 2, m = 1$. Also, the second and third notions of downside risk vulnerability discussed earlier are special cases of (n/m)th order risk vulnerability where $n = 3, m = 1$ and $n = 3, m = 2$ respectively.

3.2 Necessary and Sufficient Conditions

We present the following result:

Theorem 5 *The following statements are true.*

[8] If a utility function exhibits nth order strictly risk aversion for every n, it is said to have mixed risk aversion by Caballe and Pomansky (1996). They point out that most utility functions used in examples have mixed risk aversion.

1. *An agent* $u(x)$ *is* (n/m)*th order risk vulnerable if and only if* $R_{n/m}(x)$ *is decreasing and* $(-1)^{m-1}\xi(w,x) \geq 0$ *for all* w *and* x*, where* $\xi(w,x)$ *is defined by*

$$\xi(w,x) = u^{(m)}(x)(R_{n/m}(x) - R_{n/m}(w)) - u^{(m)}(w)R'_{n/m}(w)(x-w).$$

2. *An agent* $u(x)$ *is* (n/m)*th order risk vulnerable only if* $R_{n/m}(x)$ *is decreasing and* $R''_{n/m}(x) \geq 2R'_{n/m}(x)R_{(m+1)/m}(x)$ *for every* x*.*

3. *An agent* $u(x)$ *is* (n/m)*th order risk vulnerable if* $R_{n/m}(x)$ *is decreasing and* $R''_{n/m}(x) \geq R'_{n/m}(x)R_{(m+1)/m}(x)$ *for every* x*.*

4. *An agent* $u(x)$ *is* $((n+1)/n)$*th order risk vulnerable if both* $R_{(n+1)/n}(x)$ *and* $R_{(n+2)/(n+1)}(x)$ *are decreasing.*

Proof: see Appendix E

The above result gives a bivariate necessary and sufficient condition, a univariate necessary condition, and a univariate sufficient condition for the (n/m)th order risk vulnerability. When $n = 2, m = 1$, from the three statements in the above result we obtain Gollier and Pratt's (1996) Proposition 2, the condition (13), and Proposition 3, respectively.

It is not difficult to verify that any HARA class utility function with a marginal utility function $(x+a)^{-\gamma}$, where $\gamma > 0$, has decreasing and convex (n/m)th order risk aversion, which implies that it satisfies the third condition in the above theorem; thus all such HARA utility functions have (n/m)th order risk vulnerability.

3.3 Behavioral Implications

There are alternative ways to give behavioral implications of higher order risk vulnerabilities: For example, we may use the results on $R_{n/(n-1)}(x)$ in

Denuit and Eeckhoudt's (2010a) and Jindapon and Neilson (2007) to give two behavioral implications of the $(n/(n-1))$th order risk vulnerability. We may also use Denuit and Eeckhoudt's (2010b) result on $(-1)^{n-1}u^{(n)}(x)/u^{(1)}(x)$ to give a behavioral implication of the $(n/1)$th order risk vulnerability.

It is also possible to give behavioral interpretations using Eeckhoudt et al.'s (2009) concept of risk apportionment via stochastic dominance, however, in the following discussion, for convenience we use Liu and Meyer's (2013) result on $R_{n/m}(x)$ to explain the behavioral implications of the (n/m)th order risk vulnerability. Let $F(x)$ and $G(x)$ be two CDFs on $[a,b]$. Let $F_1(x) = F(x)$ and $F_i(x) = \int_a^x F_{i-1}(y)dy$. According to Ekern's (1980) definition, $G(x)$ is said to have more nth degree risk on $[a,b]$ than $F(x)$ if $G_i(b) = F_i(b)$, $i = 1, ..., n$, and $\forall x \in [a,b]$, $G_n(x) \geq F_n(x)$.

As before, given an independent background risk $\tilde{\epsilon}$, we use $\hat{u}(x)$ to denote $Eu(x+\tilde{\epsilon})$. Also, let $\hat{R}_{n/m}(x)$ denote $(-1)^{n-m}\hat{u}^{(n)}(x)/\hat{u}^{(m)}(x)$. We have the following result.

Proposition 2 *Assume $u(x)$ has the (n/m)th order risk vulnerability. Given an independent unfair background risk $\tilde{\epsilon}$, if $\hat{R}_{n/m}(x) \neq R_{n/m}(x)$ then there exists $\delta > 0$, such that*

$$\frac{\int_{x-\delta}^{x+\delta} \hat{u}d(F-G)}{\int_{x-\delta}^{x+\delta} \hat{u}d(F-H)} > \frac{\int_{x-\delta}^{x+\delta} ud(F-G)}{\int_{x-\delta}^{x+\delta} ud(F-H)}$$

for all $F(y), G(y)$, and $H(y)$ on $[x-\delta, x+\delta]$ such that $G(y)$ has more nth degree risk than $F(y)$ and $H(y)$ has more mth degree risk than $F(y)$.

Proof: As $u(x)$ has the (n/m)th order risk vulnerability, if $\hat{R}_{n/m}(x) \neq R_{n/m}(x)$ then $\hat{R}_{n/m}(x) > R_{n/m}(x)$. Then the result immediately follows from Liu and Meyer's (2013) Theorem 3.[9] Q.E.D.

[9] Liu and Meyer's (2013) Theorem 3 states that if $u(x)$ has strictly greater (n/m)th or-

Using Liu and Meyer's (2013) explanation, $\hat{R}_{n/m}(x) > R_{n/m}(x)$ implies that, in the presence of an independent unfair background risk $\tilde{\epsilon}$, "...restricted to a sufficiently small neighborhood about x, agent u is willing to pay more in terms of an mth degree risk increase to avoid any nth-degree risk increase than is agent v."

4 An Application to Precautionary Savings

In this section we give an example of applications of the results in this paper to economic problems. Start with a two-period consumption-saving model which is widely used in the literature. Consider an agent whose first-period and second-period utility functions are u and v respectively. Assume that u and v are both strictly increasing and strictly concave. Also assume that u is continuously differentiable and v is $(n+1)$ times continuously differentiable. With wealth w and a risk $\tilde{x}$, which has a CDF $F(x)$ defined on a real interval $[a, b]$, he has the following maximization problem:

$$\max_s U(s; F) = u(w - s) + \int_a^b v(s + x)dF(x), \tag{11}$$

where s is his saving for the second period.

Suppose that just before he finalizes his consumption-saving plan, he made a financial or non-financial deal which results in some certain change in his wealth and an additional unfair risk $\tilde{\epsilon}$ independent from the original risk $\tilde{x}$. With this change, his consumption-saving problem becomes:

$$\max_s \hat{U}(s; F) = u(\hat{w} - s) + \int_a^b E_{\tilde{\epsilon}} v(s + \tilde{\epsilon} + x)dF(x), \tag{12}$$

der risk aversion than $v(x)$ then there exists $\delta > 0$, such that $\int_{x-\delta}^{x+\delta} ud(F-G)/\int_{x-\delta}^{x+\delta} ud(F-H) > \int_{x-\delta}^{x+\delta} vd(F-G)/\int_{x-\delta}^{x+\delta} vd(F-H)$ for all $F(y), G(y)$, and $H(y)$ on $[x-\delta, x+\delta]$ such that $G(y)$ and $H(y)$ have more nth and mth degree risk than $F(y)$ respectively.

where $\hat{w}$ is his certain amount of wealth after the deal. Suppose that the deal is carefully considered such that it does not affect his optimal consumption-saving plan, i.e., the above two problems have the same optimal solution s^*.

4.1 Mean-Utility-Preserving Risk Increases

Now consider the class of changes in the agent's original risk $\tilde{x}$ which are called mean-utility-preserving risk increases for utility function $-v'$ with wealth s^* defined by Diamond and Stiglitz (1974).[10] In this case, since such risk increases do not change the expectation of $v'(s^* + \tilde{x})$, they do not change the optimal solution to the agent's consumption-saving problem (11). However, what is the effect of such risk increases on problem (12)?

Let P_v and T_v denote the absolute prudence and temperance of v.[11] Assume that P_v is decreasing and either T_v is decreasing or $P_v \geq P_v' P_v$. Then according to Theorem 5, the absolute risk aversion of $-v'$, which is equal to the absolute prudence of v, is increased by the unfair risk $\tilde{\epsilon}$. In this case, it is well known $-E_{\tilde{x}} E_{\tilde{\epsilon}} v'(s^* + \tilde{\epsilon} + \tilde{x})$ is reduced by such mean-utility-preserving risk increases.[12] It follows that $\hat{U}'(s^*; G) \geq \hat{U}'(s^*; F)$, which implies an increase in the optimal saving. Hence, while these risk changes do not affect the optimal saving in problem (11), they raise the optimal saving in problem (12).

[10] According to Diamond and Stiglitz (1974), a change in risk $F(x) \rightarrow G(x)$ on $[a, b]$ is a mean-utility-preserving risk increase for a utility function u with wealth w_0 if $\int_a^y u'(w_0 + x)G(x)dx \geq \int_a^b u'(w_0 + x)F(x)dx$, $\forall x \in [a, b]$, with the equality holding at $y = b$.

[11] The absolute temperance is defined by Eeckhoudt et al. (1996) and Gollier and Pratt (1996) as $-v''''/v'''$.

[12] See, for example, Theorem 3 in Diamond and Stiglitz (1974).

4.2 Mean-Utility-Preserving Higher Degree Risk Increases

We first present the following definition of mean-utility-preserving nth degree risk increases given by Denuit and Eeckhoudt (2010a):

Definition 3 *Assume that u is $(n-1)$th degree risk averse, i.e., $(-1)^n u^{(n-1)} > 0$. A risk change $F(x) \to G(x)$ on $[a, b]$ is said to be a mean-utility-preserving nth degree risk increases for u with wealth w_0 if it preserves the first $n-2$ moments and the mean utility and satisfies the condition that $G_n(b) < F_n(b)$ and there exists z such that $G_n(x) \geq F_n(x)$, $\forall x \leq z$ and $G_{n-1}(x) \leq F_{n-1}(x)$, $\forall x \geq z$.*

Consider the class of mean-utility-preserving nth degree increase in risk for $-v'$ with wealth s^*. In this case, since such risk increases do not change the expectation of $v'(s^* + \tilde{x})$, they do not change the optimal solution to the agent's consumption-saving problem (11). However, what is the effect of such risk increases on problem (12)?

Let $R^v_{(n+1)/n}$ denote the $((n+1)/n)$th order absolute risk aversion of v, i.e., $R^v_{(n+1)/n} = -\frac{v^{(n+1)}}{v^{(n)}}$. Assume that $R^v_{(n+1)/n}$ is decreasing and either $R^v_{(n+2)/(n+1)}$ is decreasing or $R^{v''}_{(n+1)/n} \geq R^{v'}_{(n+1)/n} R^v_{(n+1)/n}$. Then according to Theorem 5, any unfair risk $\tilde{\epsilon}$ raises the $((n+1)/n)$th order absolute risk aversion of v.

According to the Proposition 1 in Denuit and Eeckhoudt (2010a), since $-\hat{v}'(x) = -E_{\tilde{\epsilon}} v'(x + \tilde{\epsilon})$ has greater nth order absolute risk aversion than $-v'$, the above class of risk increases will reduce $-E_{\tilde{x}} E_{\tilde{\epsilon}} v'(s^* + \tilde{\epsilon} + \tilde{x})$. It follows that $\hat{U}'(s)$ is increased by these risk increases, which implies an increase in the optimal saving. Hence, while these mean-utility-preserving nth degree risk increases do not affect the optimal saving in problem (11), they raise the optimal saving in problem (12).

5 Conclusion

In this paper we have added an independent unfair background risk to the higher order risk-taking models in the literature and examined its interaction with the main risk under consideration. An agent is said to have a higher order risk vulnerability if adding an independent unfair background risk to wealth raises his level of a higher order risk-aversion. We have presented analytical necessary and sufficient conditions for this concept.

Compared with risk vulnerability, the case of higher order risk vulnerability is more complex. For example, in the case of the third order, corresponding to the five definitions of DRA measures respectively, there are five types of third order risk vulnerabilities or DRVs, while in the case of the nth ($n > 3$) order, there are $(n-1)$ types of risk vulnerabilities corresponding to the $(n/1)$th, $(n/2)$th, ..., and $(n/(n-1))$th orders of Arrow-Pratt risk aversion measure respectively. We have given a detailed analysis for each type of the third and higher order risk vulnerabilities.

As in the case of risk vulnerability, all familiar DARA utility functions — in fact, all HARA (CARA, CRRA,...) functions — have all types of higher order risk vulnerability except for the type of DRV corresponding to the Schwarzian derivative. These HARA utility functions will have this type of DRV if and only if we additionally require that cautiousness is larger than 0.5.

Appendix A Proof of Theorem 1

To prove the theorem, we need the following lemma. Let Ω_0 be the set of all zero-mean binary random variables. Let Ω be any set of zero-mean random variables which contains Ω_0. Let $f(x)$, $g(x)$, and $h(x)$ be three functions such that $\forall\ \tilde{\epsilon} \in \Omega$, $Ef(\tilde{\epsilon}) \geq 0$, $Eg(\tilde{\epsilon}) \geq 0$, and $Eh(\tilde{\epsilon}) \geq 0$.

Lemma 2 *$\forall\ \tilde{\epsilon} \in \Omega$, $Eg(\tilde{\epsilon})Eh(\tilde{\epsilon}) \geq [Ef(\tilde{\epsilon})]^2$ if and only if the inequality is true for all $\tilde{\epsilon} \in \Omega_0$.*

Proof: We first prove the following statement: $\forall\ \tilde{\epsilon} \in \Omega$, $Eg(\tilde{\epsilon})Eh(\tilde{\epsilon}) \geq [Ef(\tilde{\epsilon})]^2$, if and only if $\forall\ \tilde{\epsilon}_1 \in \Omega$ and $\forall\ \tilde{\epsilon}_2 \in \Omega$, $Eg(\tilde{\epsilon}_1)Eh(\tilde{\epsilon}_2)+Eg(\tilde{\epsilon}_2)Eh(\tilde{\epsilon}_1) \geq 2Ef(\tilde{\epsilon}_1)Ef(\tilde{\epsilon}_2)$. This is proved as follows. To prove the "if" part, we let $\tilde{\epsilon}_1 = \tilde{\epsilon}_2$ in the latter inequality, while to prove the "only if" part, we need only note that

$$Eg(\tilde{\epsilon}_1)Eh(\tilde{\epsilon}_2) + Eg(\tilde{\epsilon}_2)Eh(\tilde{\epsilon}_1) \geq 2\sqrt{Eg(\tilde{\epsilon}_1)Eh(\tilde{\epsilon}_1)Eh(\tilde{\epsilon}_2)Eg(\tilde{\epsilon}_2)}.$$

On the other hand, if we fix either of the two random variables, the expression $Eg(\tilde{\epsilon}_1)Eh(\tilde{\epsilon}_2)+Eg(\tilde{\epsilon}_2)Eh(\tilde{\epsilon}_1)-2Ef(\tilde{\epsilon}_1)Ef(\tilde{\epsilon}_2)$ is linear in the distribution of the other (in the probabilities of its possible values). This implies that $\forall\ \tilde{\epsilon}_1 \in \Omega$ and $\forall\ \tilde{\epsilon}_2 \in \Omega$, $Eg(\tilde{\epsilon}_1)Eh(\tilde{\epsilon}_2)+Eg(\tilde{\epsilon}_2)Eh(\tilde{\epsilon}_1) \geq 2Ef(\tilde{\epsilon}_1)Ef(\tilde{\epsilon}_2)$ if and only if $\forall\ \tilde{\epsilon}_1 \in \Omega_0$ and $\forall\ \tilde{\epsilon}_2 \in \Omega_0$, the inequality is true. Now applying the preceding statement for the case where $\Omega = \Omega_0$, we immediately obtain the lemma. Q.E.D.

With the help of Lemma 2, we are now ready to prove the theorem. The result where $i = 2$ and 3 is a special case of Statement 1 of Theorem 5 which is proved in the last appendix, thus we need only prove the result for the other three cases. Note that as an unfair risk can be decomposed into a certain reduction in wealth and a zero-mean risk, a necessary [sufficient] condition for all independent zero-mean background risks to increase $\tau_i(x)$ combined with the condition that $\tau_i(x)$ is monotone decreasing is a necessary [sufficient] condition for all independent unfair background risks to increase $\tau_i(x)$. Thus we need only prove the result for the case of zero-mean background risks.

To prove the result for $\tau_1(x) = C(x)$, note that the inequality $\hat{C}(x) \geq C(x)$ is equivalent to

$$\frac{u'(x)u'''(x)}{u''^2(x)}[Eu''(x+\tilde{\epsilon})]^2 \leq Eu'(x+\tilde{\epsilon})Eu'''(x+\tilde{\epsilon}). \tag{13}$$

Thus the problem is to characterize the utility function $u(x)$ which satisfies the following condition

$$E\tilde{\epsilon} = 0 \Rightarrow \frac{u'(x)u'''(x)}{u''^2(x)}[Eu''(x+\tilde{\epsilon})]^2 \leq Eu'(x+\tilde{\epsilon})Eu'''(x+\tilde{\epsilon}).$$

It is straightforward to see that Lemma 2 is applicable to this case. From this lemma it is clear that the inequality $\hat{C}(x) \geq C(x)$ is true for all zero-mean risks if and only if it is true for all zero-mean binary risks, which is equivalent to Inequality (1) as $C(x) = (\frac{1}{R(x)})'$. This proves the result for $\tau_1(x)$.

To prove the result for $\tau_4(x) = -R'(x)$, we have

$$-\hat{R}'(x) = \frac{Eu'''(x+\tilde{\epsilon})Eu'(x+\tilde{\epsilon}) - [Eu''(x+\tilde{\epsilon})]^2}{[Eu'(x+\tilde{\epsilon})]^2}.$$

Thus the inequality $-\hat{R}'(x) \geq -R'(x)$ is equivalent to

$$E[u'''(x+\tilde{\epsilon}) + R'(x)u'(x+\tilde{\epsilon})]Eu'(x+\tilde{\epsilon}) \geq [Eu''(x+\tilde{\epsilon})]^2.$$

It follows that the problem is to characterize the utility function $u(x)$ which satisfies the following condition

$$E\tilde{\epsilon} = 0 \Rightarrow [Eu''(x+\tilde{\epsilon})]^2 \leq E[u'''(x+\tilde{\epsilon}) + R'(x)u'(x+\tilde{\epsilon})]Eu'(x+\tilde{\epsilon}).$$

If for some zero-mean $\tilde{\epsilon}$, $E[u'''(x+\tilde{\epsilon}) + R'(x)u'(x+\tilde{\epsilon})] < 0$, then there must exist a zero-mean binary $\tilde{\epsilon}$ such that $E[u'''(x+\tilde{\epsilon})+R'(x)u'(x+\tilde{\epsilon})] < 0$. Thus we need only consider the case where for all zero-mean risks, $E[u'''(x+\tilde{\epsilon}) + R'(x)u'(x+\tilde{\epsilon})] \geq 0$. Now Lemma 2 is applicable to this case, and the inequality $-\hat{R}'(x) \geq -R'(x)$ is true for all zero-mean risks if and only if it is true for all binary zero-mean risks, which is equivalent to Inequality (4). This proves the result for $\tau_4(x)$.

To prove the result for $\tau_5(x) = S(x)$, we have

$$\hat{S}(x) = \frac{Eu'''(x+\tilde{\epsilon})}{Eu'(x+\tilde{\epsilon})} - \frac{3}{2}[\frac{Eu''(x+\tilde{\epsilon})}{Eu'(x+\tilde{\epsilon})}]^2.$$

Thus the inequality $\hat{S}(x) \geq S(x)$ is equivalent to

$$E[u'''(x+\tilde{\epsilon}) - S(x)u'(x+\tilde{\epsilon})]Eu'(x+\tilde{\epsilon}) \geq \frac{3}{2}[Eu''(x+\tilde{\epsilon})]^2. \tag{14}$$

It follows that the problem is to characterize the utility function $u(x)$ which satisfies the following condition

$$E\tilde{\epsilon} = 0 \Rightarrow \frac{3}{2}[Eu''(x+\tilde{\epsilon})]^2 \leq E[u'''(x+\tilde{\epsilon}) - S(x)u'(x+\tilde{\epsilon})]Eu'(x+\tilde{\epsilon}).$$

Similar to the case of $\tau_4(x)$, we need only consider the case where for all zero-mean risks, $E[u'''(x+\tilde{\epsilon}) - S(x)u'(x+\tilde{\epsilon})] \geq 0$. Again, Lemma 2 is applicable to this case, and Inequality (14) is true for all zero-mean risks if and only if it is true for all binary zero-mean risks, which is equivalent to Inequality (5) as $S(x) = -R'(x) - 0.5R^2(x)$. This proves the result for $\tau_5(x)$. Q.E.D.

Appendix B Proof of Theorem 2

As was explained at the beginning of the proof of Theorem 1, we need only prove the result for the case of zero-mean background risks. The result where $i = 1$ follows from Theorem 1 in Hara et al. (2011), and its proof can be found there. The result where $i = 2$ and 3 is a special case of Statement 2 of Theorem 5 which is proved in the last appendix, thus we need only prove the result for the other two cases.

To prove the result where $i = 4$, we have $\hat{R}(x) = E(\frac{u'}{Eu'}R)$.[13] Differentiating the equation with respect to x, we obtain

$$\hat{R}'(x) = E[(\frac{u'}{Eu'})'R] + E(\frac{u'}{Eu'}R'). \quad (15)$$

As $E(\frac{u'}{Eu'})' = 0$, we have $E[(\frac{u'}{Eu'})'R] = E[(\frac{u'}{Eu'})'(R - \hat{R}(x))]$. Moreover, as $(\frac{u'}{Eu'})' = \frac{u'}{Eu'}(\hat{R}(x) - R)$, we obtain

$$E[(\frac{u'}{Eu'})'R] = -E[\frac{u'}{Eu'}(\hat{R}(x) - R)^2]. \quad (16)$$

This and Equation (15) lead to

$$-\hat{R}'(x)Eu' = -E(u'R') + E[u'(\hat{R}(x) - R)^2]. \quad (17)$$

Adding $R'(x)Eu'$ to both sides, we obtain

$$(-\hat{R}'(x) + R'(x))Eu' = R'(x)Eu' - E(u'R') + E[u'(\hat{R}(x) - R)^2]. \quad (18)$$

Meanwhile, we have $(Eu')^2E(u'(\hat{R}(x) - R)^2) = E[u'(-Eu'' - REu')^2] = E[u'(R'(x)u'(x)\epsilon + O(\epsilon^2))^2] = R'^2(x)u'^3(x)\sigma_\epsilon^2 + O(\sigma_\epsilon^3)$. This implies that

$$E(u'(\hat{R}(x) - R)^2) = u'(x)R'^2(x)\sigma_\epsilon^2 + O(\sigma_\epsilon^3). \quad (19)$$

We also have

$$R'(x)Eu' - E(u'R') = 0.5[R'(x)u'''(x) - (u'(x)R'(x))'']\sigma_\epsilon^2 + O(\sigma_\epsilon^3).$$

[13] For brevity, in the rest of the proof, we omit the argument $(x + \tilde{\epsilon})$ of the functions under the expectation operator.

The last two equation, together with Equation (18), imply that $(-\hat{R}'(x) + R'(x))Eu'$ is equal to

$$0.5(R'(x)u'''(x) - (u'(x)R'(x))'')\sigma_\epsilon^2 + u'(x)R'^2(x)\sigma_\epsilon^2 + O(\sigma_\epsilon^3).$$

Thus for arbitrarily small σ_ϵ, $-\hat{R}'(x) \geq -R'(x)$ only if $R'(x)u'''(x) + 2u'(x) R'^2(x) - (u'(x)R'(x))'' \geq 0$, which is equivalent to $-R'''(x) + 2R(x)R''(x) + 2R'^2(x) \geq 0$.

To prove the result where $i = 5$, we have

$$\hat{S}(x) = -\hat{R}'(x) - 0.5\hat{R}^2(x) = -[E(\frac{u'}{Eu'}R)]' - 0.5[E(\frac{u'}{Eu'}R)]^2,$$

where again for brevity, we have omitted the argument $(x+\tilde{\epsilon})$ of the functions under the expectation operators.

Hence we obtain

$$\hat{S}(x) = -E[(\frac{u'}{Eu'})'R] + E[\frac{u'}{Eu'}(-R' - 0.5R^2)] + 0.5E(\frac{u'}{Eu'}R^2) - 0.5(E\frac{u'}{Eu'}R)^2.$$

In the meantime, from Equation (16), we have $-E[(\frac{u'}{Eu'})'R] = E[\frac{u'}{Eu'}(\hat{R}(x) - R)^2]$, while $E(\frac{u'}{Eu'}R^2) - (E\frac{u'}{Eu'}R)^2 = E[\frac{u'}{Eu'}(\hat{R}(x) - R)^2]$. Hence from the preceding equation, we have

$$\hat{S}(x) = E[\frac{u'}{Eu'}S] + \frac{3}{2}E[\frac{u'}{Eu'}(\hat{R}(x) - R)^2]. \tag{20}$$

Rewrite it as

$$(\hat{S}(x) - S(x))Eu' = -S(x)Eu' + E(u'S) + \frac{3}{2}E[u'(\hat{R}(x) - R)^2].$$

As we have $Eu' = u'(x) + u'''(x)\sigma_\epsilon^2 + O(\sigma_\epsilon^3)$, $E(u'S) = u'(x)S(x) + (u'(x)S(x))'' \sigma_\epsilon^2 + O(\sigma_\epsilon^3)$, and Equation (19), it follows that $(\hat{S}(x) - S(x))Eu'$ is equal to

$$[-0.5S(x)u'''(x) + 0.5(u'(x)S(x))'' + \frac{3}{2}u'(x)R'^2(x)]\sigma_\epsilon^2 + O(\sigma_\epsilon^3).$$

Thus for arbitrarily small σ_ϵ, $\hat{S}(x) \geq S(x)$, only if $-S(x)u'''(x)+(u'(x)S(x))'' +3u'(x)R'^2(x) \geq 0$. As $S(x) = -R'(x) - 0.5R^2(x)$, it can be rewritten as $S(x)(u'(x)R(x))' - (u'(x)R(x)S(x))' + (u'(x)S'(x))' + 3u'(x)(S(x) +0.5R^2(x))^2 \geq 0$. Simplifying it, we obtain $-2R(x)S'(x)+S''(x)+3(S(x)+ 0.5R^2(x))^2 \geq 0$. Q.E.D.

Appendix C Proof of Theorem 3

The result where $i = 2$ and 3 is a special case of Statement 1 of Theorem 5 which is proved in the last appendix, thus we need only prove the result for the other three cases. As was explained at the beginning of the proof of Theorem 1 in Appendix A, we need only prove the result for the case of zero-mean risks. We first prove the following lemma.

Lemma 3 *For $i \in \{1, 4, 5\}$, $\tau_i(x)$ is increased by a zero-mean background risk $\tilde{\epsilon}$ if*

$$E(\tau_i(x+\tilde{\epsilon})h_i(x+\tilde{\epsilon})) \geq \tau_i(x)Eh_i(x+\tilde{\epsilon}), \tag{21}$$

where $h_i(x)$ is defined in Theorem 3.

Proof: The results where $i = 4$ and 5 immediately follow from Equations (17) and (20), respectively, thus we need only prove the case where $i = 1$. To prove this case, we have

$$\hat{C}(x) + 1 = \frac{Eu'(x+\tilde{\epsilon})Eu'''(x+\tilde{\epsilon})}{(Eu''(x+\tilde{\epsilon}))^2}.$$

Some simple calculations lead to

$$\hat{C}(x) + 1 = \frac{Eu'(x+\tilde{\epsilon})}{(Eu''(x+\tilde{\epsilon}))^2}E[(C(x+\tilde{\epsilon})+1)\frac{(u''(x+\tilde{\epsilon}))^2}{u'(x+\tilde{\epsilon})}]. \tag{22}$$

Meanwhile, from the Cauchy-Schwarz inequality ($E\tilde{\epsilon}_1^2 E\tilde{\epsilon}_2^2 \geq [E(\tilde{\epsilon}_1\tilde{\epsilon}_2)]^2$), we have

$$Eu'(x+\tilde{\epsilon})E\frac{(u''(x+\tilde{\epsilon}))^2}{u'(x+\tilde{\epsilon})} \geq (Eu''(x+\tilde{\epsilon}))^2.$$

Rewrite it as

$$\frac{Eu'(x+\tilde{\epsilon})}{(Eu''(x+\tilde{\epsilon}))^2} \geq \frac{1}{E\frac{(u''(x+\tilde{\epsilon}))^2}{u'(x+\tilde{\epsilon})}}.$$

This, together with Equation (22), implies that

$$\begin{aligned}\hat{C}(x)+1 &\geq \frac{1}{E\frac{(u''(x+\tilde{\epsilon}))^2}{u'(x+\tilde{\epsilon})}}E[(C(x+\tilde{\epsilon})+1)\frac{(u''(x+\tilde{\epsilon}))^2}{u'(x+\tilde{\epsilon})}] \\ &= \frac{1}{E\frac{(u''(x+\tilde{\epsilon}))^2}{u'(x+\tilde{\epsilon})}}E[C(x+\tilde{\epsilon})\frac{(u''(x+\tilde{\epsilon}))^2}{u'(x+\tilde{\epsilon})}]+1.\end{aligned}$$

Hence $\hat{C}(x) \geq C(x)$ if $E[C(x+\tilde{\epsilon})h_1(x+\tilde{\epsilon})] \geq C(x)Eh_1(x+\tilde{\epsilon})$, where $h_1(x) = \frac{(u''(x))^2}{u'(x)}$. Q.E.D.

We now use the lemma to prove the theorem. Applying the lemma, we need only prove that the inequality in Theorem 3 is necessary and sufficient for Inequality (21). This is shown as follows. It is well known that Inequality (21) is true for all zero-mean $\tilde{\epsilon}$ if and only if it is true for all zero-mean binary $\tilde{\epsilon}$, i.e., $p\tau_i(x+\epsilon_1)h_i(x+\epsilon_1)+(1-p)\tau_i(x+\epsilon_2)h_i(x+\epsilon_2) \geq \tau_i(x)[ph_i(x+\epsilon_1)+(1-p)h_i(x+\epsilon_2)]$, for all $p \in (0,1)$, ϵ_1 and ϵ_2 satisfying $p\epsilon_1+(1-p)\epsilon_2 = 0$. After the elimination of p, this is equivalent to the condition that there exists a scalar $m(x)$ such that

$$h_i(x+\epsilon_1)\frac{\tau_i(x+\epsilon_1)-\tau_i(x)}{\epsilon_1} \geq m(x) \geq h_i(x+\epsilon_2)\frac{\tau_i(x+\epsilon_2)-\tau_i(x)}{\epsilon_2},$$

for any $\epsilon_1 > 0 > \epsilon_2$. By symmetry, the only candidate for $m(x)$ is $h_i(x)\tau_i'(x)$. The above condition is thus equivalent to $h_i(y)(\tau_i(y)-\tau_i(x))-h_i(x)\tau_i'(x)(y-x) \geq 0$. Q.E.D.

Appendix D Proof of Theorem 4

As was explained at the beginning of the proof of Theorem 1 in Appendix A, we need only prove the result for the case of zero-mean risks. We first prove the following lemma. The proof uses a technique that is very close to the one used by Gollier and Pratt (1996) in the proof of their Proposition 3.

Lemma 4 *Assume that for a given* w*,* $g(w) > 0$*, and for all* x*,* $f(x) > 0$*,* $f'(x) \leq 0$*, and* $(\frac{g(x)}{f(x)})' \leq 0$*. If for all* x*,* $(\frac{g(x)}{f(x)})'' \geq -(\frac{g(x)}{f(x)})'\frac{f'(x)}{f(x)}$ *then, for any unfair* $\tilde{\epsilon}$*,* $\frac{Eg(w+\tilde{\epsilon})}{Ef(w+\tilde{\epsilon})} \geq \frac{g(w)}{f(w)}$.

Proof: Note that as for all x, $f(x) > 0$ and $(\frac{g(x)}{f(x)})'' \geq -(\frac{g(x)}{f(x)})'\frac{f'(x)}{f(x)}$, we have $[(\frac{g(x)}{f(x)})'f(x)]' = (\frac{g(x)}{f(x)})''f(x) + (\frac{g(x)}{f(x)})'f'(x) \geq 0$, i.e., $(\frac{g(x)}{f(x)})'f(x) = g'(x) - \frac{g(x)f'(x)}{f(x)}$ is increasing. Thus as $g(w) > 0$ and for all x, $f'(x) \leq 0$, $\frac{g(x)}{f(x)}$ is decreasing, and $g'(x) - \frac{g(x)f'(x)}{f(x)}$ is increasing, y has the same sign as

$$\frac{1}{g(w)}[(g'(w+y) - \frac{g(w+y)f'(w+y)}{f(w+y)}) - (g'(w) - \frac{g(w)f'(w)}{f(w)})] + \frac{f'(w+y)}{g(w)}(\frac{g(w+y)}{f(w+y)} - \frac{g(w)}{f(w)}),$$

which is equal to

$$\frac{g'(w+y)}{g(w)} - \frac{f'(w+y)}{f(w)} - (\frac{g'(w)}{g(w)} - \frac{f'(w)}{f(w)}).$$

Therefore, since the direction of integration cancels out the sign, we have

$$\int_0^{\epsilon}[\frac{g'(w+y)}{g(w)} - \frac{f'(w+y)}{f(w)} - (\frac{g'(w)}{g(w)} - \frac{f'(w)}{f(w)})]dy \geq 0.$$

Since this is true for any ϵ, taking the expectation over a random variable $\tilde{\epsilon}$ yields

$$\frac{Eg(w+\tilde{\epsilon}) - g(w)}{g(w)} - \frac{Ef(w+\tilde{\epsilon}) - f(w)}{f(w)} \geq (\frac{g'(w)}{g(w)} - \frac{f'(w)}{f(w)})E\tilde{\epsilon}.$$

As $f(w)$ and $g(w)$ are both strictly positive and $\frac{g(w)}{f(w)}$ is decreasing, the right hand side is non-negative, and it follows that $\frac{Eg(w+\tilde{\epsilon})}{Ef(w+\tilde{\epsilon})} \geq \frac{g(w)}{f(w)}$. Q.E.D.

We now use the lemma to prove Theorem 4. Let $f(x) = h_i(x)$ and $g(x) = (\tau_i(x) + \alpha_0)h_i(x)$, where $h_i(x)$ is defined in Theorem 3 and $\alpha_0 > 0$ is arbitrarily large. Note that for $i = 1, ..., 5$, for all x, $f(x) = h_i(x) > 0$, and for any given w, as $\alpha_0 > 0$ is arbitrarily large, $g(w) = (\tau_i(w)+\alpha_0)h_i(w) > 0$. Also note that for $i = 1$, $C(x) \geq -0.5$ implies $h_1'(x) = -h_1(x)R(x)(2C(x)+1) \leq 0$, for $i = 3$, $h_i'(x) = -u'''(x) \leq 0$, and for $i = 2,\ 4,\ 5$, $h_i'(x) = u''(x) \leq 0$. Moreover, as $\tau_i(x)$ is decreasing, $\frac{g(x)}{f(x)} = \tau_i(x) + \alpha_0$ is also decreasing. Furthermore, we have $-(\ln h_i(x))' = \zeta_i(x)$, where $\zeta_i(x)$ is defined in the theorem, and it follows from the given condition in the theorem that $(\frac{g(x)}{f(x)})'' + (\frac{g(x)}{f(x)})'\frac{f'(x)}{f(x)} = \tau_i''(x) - \tau_i'(x)\zeta_i(x) \geq 0$. Thus $f(x)$ and $g(x)$ satisfy all the conditions in the lemma. Now applying the lemma, we obtain $\frac{Eg(w+\tilde{\epsilon})}{Ef(w+\tilde{\epsilon})} \geq \frac{g(w)}{f(w)}$ $\forall\tilde{\epsilon}$. This implies that $\forall\tilde{\epsilon}$, $\frac{E[(\tau_i(w+\tilde{\epsilon})+\alpha_0)h_i(w+\tilde{\epsilon})]}{Eh_i(w+\tilde{\epsilon})} \geq \frac{(\tau_i(w)+\alpha_0)h_i(w)}{h_i(w)}$, i.e., $E[\tau_i(w+\tilde{\epsilon})h_i(w+\tilde{\epsilon})] \geq \tau_i(w)Eh_i(w+\tilde{\epsilon})$. Applying Lemma 3 of the last appendix, we immediately obtain the conclusion in the theorem. Q.E.D.

Appendix E Proof of Theorem 5

As was explained at the beginning of the proof of Theorem 1 in Appendix A, we need only prove the result for the case of zero-mean risks. We first prove Statement 1. The inequality $(-1)^{n-m}\frac{Eu^{(n)}(x+\tilde{\epsilon})}{Eu^{(m)}(x+\tilde{\epsilon})} \geq R_{n/m}(x)$ is equivalent to

$$(-1)^{n-1}Eu^{(n)}(x+\tilde{\epsilon}) \geq (-1)^{m-1}R_{n/m}(x)Eu^{(m)}(x+\tilde{\epsilon}).$$

It is well known that the above inequality is true for all zero-mean $\tilde{\epsilon}$ if and only if it is true for all zero-mean binary $\tilde{\epsilon}$.[14] Thus the inequality is true for

[14]See, for example, Gollier and Pratt's (1996) argument in the proof of their Proposition 2.

all zero-mean $\tilde{\epsilon}$ if and only if for all $x_2 < x < x_1$,

$$[(-1)^{n-1}u^{(n)}(x_1) - (-1)^{m-1}R_{n/m}(x)u^{(m)}(x_1)](x - x_2)$$
$$+[(-1)^{n-1}u^{(n)}(x_2) - (-1)^{m-1}R_{n/m}(x)u^{(m)}(x_2)](x_1 - x) \geq 0.$$

This is equivalent to the condition that there exists a scaler $\kappa(x)$ such that

$$(-1)^{m-1}u^{(m)}(x_1)\frac{R_{n/m}(x_1) - R_{n/m}(x)}{x_1 - x} \geq \kappa(x)$$
$$\geq (-1)^{m-1}u^{(m)}(x_2)\frac{R_{n/m}(x) - R_{n/m}(x_2)}{x - x_2}.$$

The only candidate for $\kappa(x)$ is $u^{(m)}(x)R'_{n/m}(x)$, and it follows that the above condition is equivalent to $(-1)^{m-1}\xi(w, x) \geq 0$, where $\xi(w, x) = u^{(m)}(x)(R_{n/m}(x) - R_{n/m}(w)) - u^{(m)}(w)R'_{n/m}(w)(x - w)$. This proves the first statement.

To prove the second statement, observe that $\xi(w, w) = 0$ and

$$\frac{\partial \xi(w, x)}{\partial x} = u^{(m+1)}(x)(R_{n/m}(x) - R_{n/m}(w)) + u^{(m)}(x)R'_{n/m}(x) - u^{(m)}(w)R'_{n/m}(w),$$

which implies $\frac{\partial \xi(w,x)}{\partial x}|_{x=w} = 0$. It follows that to have $(-1)^{m-1}\xi(w, x) \geq 0$ $\forall w, x$, it is necessary that $(-1)^{m-1}\frac{\partial^2 \xi(w,x)}{\partial x^2}|_{x=w} \geq 0 \;\; \forall w$, which is equivalent to $R''_{n/m}(w) \geq 2R_{(m+1)/m}(w)R'_{n/m}(w) \;\; \forall w$.

To prove the third statement, let $f(x) = (-1)^{m-1}u^{(m)}(x)$ and $g(x) = (-1)^{n-1}u^{(n)}(x)$. Applying Lemma 4 in the last appendix, we immediately conclude that the third statement is true.

The proof of the fourth statement follows from the proof of Proposition 1 in Kimball (1993) and from the fact that $R_{(n+1)/n}$ is isomorphic to risk aversion as applied to $(-1)^{n-1}u^{(n-1)}$. Q.E.D.

REFERENCES

1. Caballe, J., Pomansky, A.: Mixed risk aversion. J Econ Theory 71, 485-513(1996).

2. Campbell, J. Y.: Household finance. J Financ 61, 1553-1604(2006).

3. Chiu, W. H.: On the propensity to self-protect. J Risk Insur 67, 555-578(2000).

4. Chiu, W. H.: Skewness preference, risk aversion, and the precedence relations on stochastic changes. Manag Sci 12, 1816-1828(2005).

5. Chiu, W. H.: Skewness preference, risk taking and expected utility maximisation. Geneva Risk Ins Rev 35, 108-129(2010).

6. Chiu, W. H., Eeckhoudt, L., Rey, B.: On relative and partial risk attitudes: theory and implications. Econ Theory 50, 151-167(2012).

7. Crainich, D., Eeckhoudt, L.: On the intensity of downside risk aversion. J Risk Uncertainty 36, 267-276(2008).

8. Denuit, M., Eeckhoudt, L.: A general index of absolute risk attitude. Manag Sci 56, 712-715(2010a).

9. Denuit, M., Eeckhoudt, L.: Stronger measures of higher-order risk attitudes. J Econ Theory 145 2027-2036(2010b).

10. Diamond, P., Stiglitz, J.: Increases in risk and in risk aversion. J Econ Theory 8, 337-360(1974).

11. Eeckhoudt, L., Gollier, C., Schlesinger, H.: Changes in background risks and risk taking behavior. Econometrica 64, 683-689(1996).

12. Eeckhoudt, L., Gollier, C.: The impact of prudence on optimal prevention. Econ Theory 26, 989-994 (2005)

13. Eeckhoudt L., Schlesinger, H.: Putting risk in its proper place. Am Econ Rev 96, 280-289, 2006.

14. Eeckhoudt, L., Schlesinger, H., Tsetlin, I.: Apportioning of risks via stochastic dominance. J Econ Theory 144, 994-1003(2009).

15. Franke, G., Schlesinger, H., Stapleton, R.C.: Multiplicative background risk, Manag Sci 52, 146-153(2006).

16. Franke, G., Stapleton, R.C., Subrahmanyam, M.G.: Background risk and the demand for state-contingent claims. Econ Theory 23, 321-335(2004).

17. Gollier, C., Pratt, J. W.: Risk vulnerability and the tempering effect of background risk. Econometrica 64, 1109-1123(1996).

18. Hara, C., Huang, J., Kuzmics, C.: Effects of background risks on cautiousness with an application to a portfolio choice problem. J Econ Theory 146, 346-358(2011).

19. Huang, J.: On downside risk aversion measures. SSRN working paper (2012). Available at http://ssrn.com/abstract=2192665 or http://dx.doi.org/10.2139/ssrn.2192665.

20. Huang, J., Stapleton, R.: Cautiousness, skewness preference, and the demand for options. Rev Financ 18, 2375-2395 (2014).

21. Jindapon, P., Neilson, W.: Higher-order generalizations of ArrowPratt and Ross risk aversion: a comparative statics approach. J Econ Theory

136, 719-728(2007).

22. Keenan, D., Snow, A.: Greater downside risk aversion. J Risk Uncertainty 24, 267-277(2002).

23. Keenan, D., Snow, A.: Greater downside risk aversion in the large. J Econ Theory 144, 1092-1101(2009).

24. Keenan, D., Snow, A.: Greater prudence and greater downside risk aversion. J Econ Theory 145, 2018-2026(2010).

25. Keenan, D., Snow, A.: The Schwarzian derivative as a ranking of downside risk aversion. J Risk Uncertainty 44, 149-160(2012).

26. Kimball, M. S.: Precautionary saving in the small and in the large. Econometrica 58, 53-73(1990).

27. Kimball, M. S.: Standard risk aversion. Econometrica 61, 589-611(1993).

28. Lajeri-Chaherli, F. Proper prudence, standard prudence and precautionary vulnerability. Econ Lett 82, 29-34(2004).

29. Li, J.: The demand for a risky asset in the presence of a background risk. J Econ Theory 146, 372-391(2011).

30. Liu, L., Meyer, J.: Decreasing absolute risk aversion, prudence and increased downside risk aversion. J Risk Uncertainty 44, 243-260(2012).

31. Liu, L., Meyer, J.: Substituting one risk increase for another: a method for measuring risk aversion. J Econ Theory 148, 2706-2718(2013).

32. Menezes, C., Geiss, C., Tressler, J.: Increasing downside risk. Am Econ Rev 70, 921-932(1980).

33. Modica, S., Scarsini, M.: A note on comparative downside risk aversion. J Econ Theory 122, 267-271(2005).

34. Pratt, J.W.: Risk aversion in the small and in the large. Econometrica 32 122-136(1964).

35. Pratt, J.W.: Aversion to one risk in the presence of others. J Risk Uncertainty 1, 395-413(1988).

36. Pratt, J.W., Zeckhauser, R.J.: Proper risk aversion. Econometrica 55, 143-154(1987).

37. Quiggin, J.: Background risk in generalized expected utility theory. Econ Theory 22, 607-611(2003).

38. Tsetlin, I., Winkler, R.L.: Risky choices and correlated background risk. Manag Sci 51, 1336-1345(2005).