LAMPLIGHTER GROUPS AND VON NEUMANN'S CONTINUOUS REGULAR RING

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Abstract. Let Γ be a discrete group. Following Linnell and Schick one can define a continuous ring $c(\Gamma)$ associated with Γ. They proved that if the Atiyah Conjecture holds for a torsion-free group Γ, then $c(\Gamma)$ is a skew field. Also, if Γ has torsion and the Strong Atiyah Conjecture holds for Γ, then $c(\Gamma)$ is a matrix ring over a skew field. The simplest example when the Strong Atiyah Conjecture fails is the lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$. It is known that $\mathbb{C}(\mathbb{Z}_2 \wr \mathbb{Z})$ does not even have a classical ring of quotients. Our main result is that if H is amenable, then $c(\mathbb{Z}_2 \wr H)$ is isomorphic to a continuous ring constructed by John von Neumann in the $1930's$.

Keywords. continuous rings, von Neumann algebras, the algebra of affiliated operators, lamplighter group

1. Introduction

Let us consider $\mathrm{Mat}_{k\times k}(\mathbb{C})$ the algebra of k by k matrices over the complex field. This ring is a unital $*$-algebra with respect to the conjugate transposes. For each element $A \in \mathrm{Mat}_{k\times k}(\mathbb{C})$ one can define A^* satisfying the following properties.

- $(\lambda A)^* = \overline{\lambda} A^*$
- $(A+B)^* = A^* + B^*$
- $(AB)^* = B^* A^*$
- $0^* = 0$, $1^* = 1$

Also, each element has a normalized rank $\mathrm{rk}(A) = \mathrm{Rank}(A)/k$ with the following properties.

- $\mathrm{rk}(0) = 0, \mathrm{rk}(1) = 1,$
- $\mathrm{rk}(A+B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$
- $\mathrm{rk}(AB) \leq \min\{\mathrm{rk}(A), \mathrm{rk}(B)\}$
- $\mathrm{rk}(A^*) = \mathrm{rk}(A)$
- If e and f are orthogonal idempotents then $\mathrm{rk}(e+f) = \mathrm{rk}(e) + \mathrm{rk}(f)$.

The ring $\mathrm{Mat}_{k\times k}(\mathbb{C})$ has an algebraic property namely, von Neumann called regularity: Any principal left-(or right) ideal can be generated by an idempotent. Furthermore, among these generating idempotents there is a unique projection (that is $\mathrm{Mat}_{k\times k}(\mathbb{C})$ is a $*$-regular ring). In a von Neumann regular ring any non-zerodivisor is necessarily invertible. One can also observe that the algebra of matrices is proper, that is $\sum_{i=1}^n a_i a_i^* = 0$ implies that all the matrices a_i are zero. One should note that if R is a $*$-regular ring with a rank

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function, then the rank extends to $\mathrm{Mat}_{k\times k}(R)$ [6], where the extended rank has the same property as rk except that the rank of the identity is k.
One can immediately see that the rank function defines a metric $d(A,B) := \mathrm{rk}(A-B)$ on any algebra with a rank, and the matrix algebra is complete with respect to this metric. These complete $*$-regular algebras are called continuous $*$-algebras (see [5] for an extensive study of continuous rings). Note that for the matrix algebras the possible values of the rank functions are $0, 1/k, 2/k, \dots, 1$. John von Neumann observed that there are some interesting examples of infinite dimensional continuous $*$-algebras, where the rank function can take any real values in between 0 and 1. His first example was purely algebraic.

Example 1. Let us consider the following sequence of diagonal embeddings.

$$\mathbb{C} \to \mathrm{Mat}_{2\times 2}(\mathbb{C}) \to \mathrm{Mat}_{4\times 4}(\mathbb{C}) \to \mathrm{Mat}_{8\times 8}(\mathbb{C}) \to \dots$$

One can observe that all the embeddings are preserving the rank and the $*$-operation. Hence the direct limit $\varinjlim \mathrm{Mat}_{2^k\times 2^k}(\mathbb{C})$ is a $*$-regular ring with a proper rank function. The addition, multiplication, the $*$-operation and the rank function can be extended to the metric completion $\mathcal{M}$ of the direct limit ring. The resulting algebra $\mathcal{M}$ is a simple, proper, continuous $*$-algebra, where the rank function can take all the values on the unit interval.

Example 2. Consider a finite, tracial von Neumann algebra $\mathcal{N}$ with trace function $\mathrm{tr}_\mathcal{N}$. Then $\mathcal{N}$ is a $*$-algebra equipped with a rank function. If P is a projection, then $\mathrm{rk}_\mathcal{N}(P) = \mathrm{tr}_\mathcal{N}(P)$. For a general element $A \in \mathcal{N}$, $\mathrm{rk}_\mathcal{N}(A) = 1 - \lim_{t\to\infty} \int_0^t \mathrm{tr}_\mathcal{N}(E_\lambda) d\lambda$, where $\int_0^\infty E_\lambda\, d\lambda$ is the spectral decomposition of A^*A. In general, $\mathcal{N}$ is not regular, but it has the Ore property with respect to its zero divisors. The Ore localization of $\mathcal{N}$ with respect to its non-zerodivisors is called the algebra of affiliated operators and denoted by $U(\mathcal{N})$. These algebras are also proper continuous $*$-algebras [1]. The rank of an element $A \in U(\mathcal{N})$ is given by the trace of the projection generating the principal ideal $U(\mathcal{N})A$. It is important to note, that $U(\mathcal{N})$ is the rank completion of $\mathcal{N}$ (Lemma 2.2 ([12]).
Linnell and Schick observed [9] that if X is a subset of a proper $*$-regular algebra R, then there exists a smallest $*$-regular subalgebra containing X, the $*$-regular closure. Now let Γ be a countable group and $\mathbb{C}\Gamma$ be its complex group algebra. Then one can consider the natural embedding of the group algebra to its group von Neumann algebra $\mathbb{C}\Gamma \to \mathcal{N}\Gamma$. Let $U(\Gamma)$ denote the Ore localization of $\mathcal{N}(\Gamma)$ and the embedding $\mathbb{C}\Gamma \to U(\Gamma)$. Since $U(\Gamma)$ is a proper $*$-regular ring, one can consider the smallest $*$-algebra $\mathcal{A}(\Gamma)$ in $U(\Gamma)$ containing $\mathbb{C}(\Gamma)$. Let $c(\Gamma)$ be the completion of the algebra $\mathcal{A}$ above. It is a continuous $*$-algebra [5]. Of course, if the rank function has only finitely many values in $\mathcal{A}$, then $c(\Gamma)$ equals to $\mathcal{A}(\Gamma)$. Note that if $\mathbb{C}\Gamma$ is embedded into a continuous $*$-algebra T, then one can still define $c_T(\Gamma)$ as the smallest continuous ring containing $\mathbb{C}\Gamma$. In [3] we proved that if Γ is amenable, $c(\Gamma) = c_T(\Gamma)$ for any embedding $\mathbb{C}\Gamma \to T$ associated to sofic representations of Γ, hence $c(\Gamma)$ can be viewed as a canonical object. Linnell and Schick calculated the algebra $c(\Gamma)$ for several groups, where the rank function has only finitely many values on $\mathcal{A}$. They proved the following results:

- If Γ is torsion-free and the Atiyah Conjecture holds for Γ, then $c(\Gamma)$ is a skew-field. This is the case, when Γ is amenable and $\mathbb{C}\Gamma$ is a domain. Then $c(\Gamma)$ is the Ore localization of $\mathbb{C}\Gamma$. If Γ is the free group of k generators, then $c(\Gamma)$ is the Cohen-Amitsur free skew field of k generators. The Atiyah Conjecture for a torsion-free group means that the rank of an element in $\mathrm{Mat}_{k\times k}(\mathbb{C}\Gamma) \subset \mathrm{Mat}_{k\times k}(U(\mathcal{N}(\Gamma)))$ is an integer.
- If the orders of the finite subgroups of Γ are bounded and the Strong Atiyah Conjecture holds for Γ, then $c(\Gamma)$ is a finite dimensional matrix ring over some skew field. In this case the Strong Atiyah Conjecture means that the ranks of an element in $\mathrm{Mat}_{k\times k}(\mathbb{C}\Gamma) \subset \mathrm{Mat}_{k\times k}(U(\mathcal{N}(\Gamma)))$ is in the abelian group $\frac{1}{\mathrm{lcm}(\Gamma)}\mathbb{Z}$, where $\mathrm{lcm}(\Gamma)$ indicates the least common multiple of the orders of the finite subgroups of Γ.

The lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$ has finite subgroups of arbitrarily large orders. Also, although Γ is amenable, $\mathbb{C}\Gamma$ does not satisfy the Ore condition with respect to its non-zerodivisors [8]. In other words, it has no classical ring of quotients. The goal of this paper is to calculate $c(\mathbb{Z}_2 \wr \mathbb{Z})$ and even $c(\mathbb{Z}_2 \wr H)$, where H is a countably infinite amenable group.

Theorem 1. *If H is a countably infinite amenable group, then $c(\mathbb{Z}_2 \wr H)$ is the simple continuous ring $\mathcal{M}$ of von Neumann.*

2. Crossed Product Algebras

In this section we recall the notion of crossed product algebras and the group-measure space construction of Murray and von Neumann. Let $\mathcal{A}$ be a unital, commutative $*$-algebra and $\phi : \Gamma \to \mathrm{Aut}(\mathcal{A})$ be a representation of the countable group Γ by $*$-automorphisms. The associated crossed product algebra $\mathcal{A} \rtimes \Gamma$ is defined the following way. The elements of $\mathcal{A} \rtimes \Gamma$ are the finite formal sums

$$\sum_{\gamma\in\Gamma} a_\gamma \cdot \gamma \,,$$

where $a_\gamma \in \mathcal{A}$. The multiplicative structure is given by

$$\delta \cdot a_\gamma = \phi(\delta)(a_\gamma) \cdot \delta \,.$$

The $*$-structure is defined by $\gamma^* = \gamma^{-1}$ and $(\gamma \cdot a)^* = a^* \cdot \gamma^{-1}$. Note that

$$(\delta \cdot a_\gamma)^* = (\phi(\delta)a_\gamma \cdot \delta)^* = \delta^* \cdot \phi(\delta)a^*_\gamma = \phi(\delta^{-1})\phi(\delta)a^*_\gamma \cdot \delta^{-1} = a^*_\gamma \cdot \delta^* \,.$$

Now let (X, μ) be a probability measure space and $\tau : \Gamma \curvearrowright X$ be a measure preserving action of a countable group Γ on X. Then we have a $*$-representation $\hat{\tau}$ of Γ in $\mathrm{Aut}(L^\infty(X, \mu))$, where $L^\infty(X, \mu)$ is the commutative $*$-algebra of bounded measurable functions on X (modulo zero measure perturbations).

$$\hat{\tau}(\gamma)(f)(x) = f(\tau(\gamma^{-1})(x)) \,.$$

Let $\mathcal{H} = l^2(\Gamma, L^2(X,\mu))$ be the Hilbert-space of $L^2(X,\mu)$-valued functions on Γ. That is, each element of $\mathcal{H}$ can be written in the form of

$$\sum_{\gamma\in\Gamma} b_\gamma \cdot \gamma\,,$$

where $\sum_{\gamma\in\Gamma} \|b_\gamma\|^2 < \infty$. Then we have a representation L of $L^\infty(X,\mu)) \rtimes \Gamma$ on $l^2(\Gamma, L^2(X,\mu))$ by

$$L(\sum_{\gamma\in\Gamma} a_\gamma \cdot \gamma)(\sum_{\delta\in\Gamma} b_\delta \cdot \delta) = \sum_{\delta\in\Gamma}\left(\sum_{\gamma\in\Gamma} a_\gamma(\hat{\tau}(\gamma)(\beta_\delta))\cdot \gamma\delta\right)\,.$$

Note that $L(\sum_{\gamma\in\Gamma} a_\gamma \cdot \gamma)$ is always a bounded operator. A trace is given on $L^\infty(X,\mu)) \rtimes \Gamma$ by

$$\mathrm{Tr}(S) = \int_X a_1(x)d\mu(x)\,.$$

The weak operator closure of $L(L^\infty_c(X,\mu)) \rtimes \Gamma)$ in $B\left(l^2(\Gamma, L^2(X,\mu))\right)$ is the von Neumann algebra $\mathcal{N}(\tau)$ associated to the action. Here $L^\infty_c(X,\mu)$ denotes the subspace of functions in $L^\infty(X,\mu)$ having only countable many values.
Note that one can extend Tr to $\mathrm{Tr}_{\mathcal{N}(\tau)}$ on the von Neumann algebra to make it a tracial von Neumann algebra.
We will denote by $c(\tau)$ the smallest continuous algebra in $U(\mathcal{N}(\tau))$ containing $L^\infty_c(X,\mu)\rtimes\Gamma$. One should note that the weak closure of $L^\infty_c(X,\mu) \rtimes \Gamma$ in $B\left(l^2(\Gamma, L^2(X,\mu))\right)$ is the same as the weak closure of $L^\infty(X,\mu) \rtimes \Gamma$. Hence our definition for the von Neumann algebra of an action coincides with the classical definition. On the other hand, $c(L^\infty_c(X,\mu) \rtimes \Gamma)$ is smaller than $c(L^\infty(X,\mu) \rtimes \Gamma)$.

3. The Bernoulli Algebra

Let H be a countable group. Consider the Bernoulli shift space $B_H := \prod_{h\in H}\{0,1\}$ with the usual product measure ν_H. The probability measure preserving action $\tau_H : H \curvearrowright (B_H, \nu_H)$ is defined by

$$\tau_H(\delta)(x)(h) = x(\delta^{-1}h)\,,$$

where $x \in B_H$, $\delta, h \in H$. Let $\mathcal{A}_H$ be the commutative $*$-algebra of functions that depend only on finitely many coordinates of the shift space. It is well-known that the Rademacher functions $\{R_S\}_{S\subset H,\,|S|<\infty}$ form a basis in $\mathcal{A}_H$, where

$$R_S(x) = \prod_{\delta\in S} exp(i\pi x(\delta))\,.$$

The Rademacher functions with respect to the pointwise multiplication form an Abelian group isomorphic to $\oplus_{h\in H} Z_2$ the Pontrjagin dual of the compact group B_H satisfying

- $R_S R_{S'} = R_{S\triangle S'}$
- $\int_{B_H} R_S\, d\nu = 0$, if $|S| > 0$
- $R_\emptyset = 1$.

The group H acts on $\mathcal{A}_H$ by

$$\hat{\tau}_H(\delta)(f)(x) = f\left(\tau_H(\delta^{-1})(x)\right) .$$

Hence,

$$\hat{\tau}_H(\delta)R_S = R_{\delta S} .$$

Therefore, the elements of $\mathcal{A}_H \rtimes H$ can be uniquely written as in the form of the finite sums

$$\sum_\delta \sum_S c_{\delta,S} R_S \cdot \delta ,$$

where $\delta \cdot R_S = R_{\delta S} \cdot \delta$.
Now let us turn our attention to the group algebra $\mathbb{C}(\mathbb{Z}_2 \wr H)$. For $\delta \in H$, let t_δ be the generator in $\sum_{h \in H} Z_2$ belonging to the δ-component. Any element of $\mathbb{C}(\mathbb{Z}_2 \wr H)$ can be written in a unique way as a finite sum

$$\sum_\delta \sum_S c_{\delta,S} t_S \cdot \delta ,$$

where $t_S = \prod_{s \in S} t_s$, $\delta \cdot t_S = t_{\delta S}$, $t_S t_{S'} = t_{S \triangle S'}$. Also note that

$$\mathrm{Tr}(\sum_\delta \sum_S c_{\delta,S} t_S \cdot \delta) = c_{1,\emptyset} .$$

Hence we have the following proposition.

Proposition 3.1. *There exists a trace preserving $*$-isomorphism $\kappa : \mathbb{C}(\mathbb{Z}_2 \wr H) \to \mathcal{A}_H \rtimes H$ such that*

$$\kappa(\sum_\delta \sum_S c_{\delta,S} t_S \cdot \delta) = \sum_\delta \sum_S c_{\delta,S} R_S \cdot \delta .$$

Recall that if $A \subset \mathcal{N}_1$, $B \subset \mathcal{N}_1$ are weakly dense *-subalgebras in finite tracial von Neumann algebras $\mathcal{N}_1$ and $\mathcal{N}_2$ and $\kappa : A \to B$ is a trace preserving $*$-homomorphism, then κ extends to a trace preserving isomorphism between the von Neumann algebras themselves (see e.g. [7] Corollary 7.1.9.). Therefore, $\kappa : \mathbb{C}(\mathbb{Z}_2 \wr H) \to \mathcal{A}_H \rtimes H$ extends to a trace (and hence rank) preserving isomorphism between the von Neumann algebras $\mathcal{N}(\mathbb{Z}_2 \wr H)$ and $\mathcal{N}(\tau_H)$.

Proposition 3.2. *For any countable group H,*

$$c(\mathbb{Z}_2 \wr H) \cong c(\tau_H) .$$

Proof. The rank preserving isomorphism $\kappa : \mathcal{N}(\mathbb{Z}_2 \wr H) \to \mathcal{N}(\tau_H)$ extends to a rank preserving isomorphism between the rank completions, that is, the algebras of affiliated operators. It is enough to prove that the rank closure of $\mathcal{A}_H \rtimes H$ is $L^\infty_c(B_H, \nu_H) \rtimes H$.

Lemma 3.1. *Let $f \in L^\infty_c(B_H, \nu_H)$. Then $rk_{\mathcal{N}(\tau_H)}(f) = \nu_H(supp(f))$.*

Proof. By definition,

$$\mathrm{rk}_{\mathcal{N}(\tau_H)}(f) = 1 - \lim_{\lambda\to 0} \mathrm{tr}_{\mathcal{N}(\tau_H)} E_\lambda\,,$$

where E_λ is the spectral projection of f^*f corresponding to λ.

$$\mathrm{tr}_{\mathcal{N}(\tau_H)} E_\lambda = \nu_H(\{x \mid |f^2(x)| \leq \lambda\})\,.$$

Hence, $\mathrm{rk}_{\mathcal{N}(\tau_H)}(f) = 1 - \nu_H(\{x \mid f^2(x) = 0\}) = \nu_H(\mathrm{supp}(f))\,.$ □

Let $\{m_n\}_{n=1}^\infty \subset \mathcal{A}_H, m_n \overset{\mathrm{rk}}{\to} m \in L_c^\infty(B_H,\nu_H)$. Then $m_n\cdot\gamma \overset{\mathrm{rk}}{\to} m\cdot\gamma$. Therefore our proposition follows from the lemma below.

Lemma 3.2. *$\mathcal{A}_H$ is dense in $L_c^\infty(B_H,\nu_H)$ with respect to the rank metric.*

Proof. By Lemma 3.1, $L^\infty_{fin}(B_H,\nu_H)$ is dense in $L_c^\infty(B_H,\nu_H)$, where $L^\infty_{fin}(B_H,\nu_H)$ is the $*$-algebra of functions taking only finitely many values. Recall that $V\subset B_H$ is a basic set if $1_V \in \mathcal{A}_H$. It is well-known that any measurable set in B_H can be approximated by basic sets, that is for any $U\subset B_H$, there exists a sequence of basic sets $\{V_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \nu_H(V_n \triangle U) = 0\,. \tag{1}$$

By (1) and Lemma 3.1

$$\lim_{n\to\infty} \mathrm{rk}_{\mathcal{N}(\tau_n)}(1_{V_n} - 1_U) = 0\,.$$

Let $f = \sum_{m=1}^l c_m 1_{U_m}$, where U_m are disjoint measurable sets. Let $\lim_{n\to\infty}\nu_H(V_n^m\triangle U_m) = 0$, where $\{V_n^m\}_{n=1}^\infty$ are basic sets. Then

$$\lim_{n\to\infty} \mathrm{rk}_{\mathcal{N}(\tau_n)}(\sum_{m=1}^l c_m 1_{V_n^m} - f) = 0\,.$$

Therefore, $\mathcal{A}_H$ is dense in $L^\infty_{fin}(B_H,\nu_H)$. □

4. The Odometer Algebra

The Odometer Algebra is constructed via the odometer action using the algebraic crossed product construction. Let us consider the compact group of 2-adic integers $\hat{\mathbb{Z}}_{(2)}$. Recall that $\hat{\mathbb{Z}}_{(2)}$ is the completion of the integers with respect to the dyadic metric

$$d_{(2)}(n,m) = 2^{-k}\,,$$

where k is the power of two in the prime factor decomposition of $|m-n|$. The group $\hat{\mathbb{Z}}_{(2)}$ can be identified with the compact group of one way infinite sequences with respect to the binary addition.

The Haar-measure μ_{haar} on $\hat{\mathbb{Z}}_{(2)}$ is defined by $\mu_{\mathrm{haar}}(U_n^l) = 1/2^n$, where $0\leq l\leq 2^n-1$ and U_n^l is the clopen subset of elements in $\hat{\mathbb{Z}}_{(2)}$ having residue l modulo 2^n. Let T be the addition map $x\to x+1$ in $\hat{\mathbb{Z}}_{(2)}$. The map T defines an action $\rho:\mathbb{Z}\curvearrowright(\hat{\mathbb{Z}}_{(2)},\mu_{\mathrm{haar}})$ The dynamical system $(T,\hat{\mathbb{Z}}_{(2)},\mu_{\mathrm{haar}})$ is called the odometer action. As in Section 3, we consider the $*$-subalgebra of function $\mathcal{A}_M$ in $L^\infty(\hat{\mathbb{Z}}_{(2)}, \mu_{\text{haar}})$ that depend only on finitely many coordinates of $\hat{\mathbb{Z}}_{(2)}$. We consider a basis for $\mathcal{A}_M$. For $n \geq 0$ and $0 \leq l \leq 2^n - 1$ let

$$F_n^l(x) = exp\left(\frac{2\pi i x (mod\, 2^n)}{2^n} l\right) .$$

Notice that $F_{n+1}^{2l} = F_n^l$. Then the functions $\{F_n^l\}_{n,l|(l,n)=1}$ form the Prüfer 2-group

$$\mathbb{Z}_{(2)} = \mathbb{Z}_1 \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \dots$$

with respect to the pointwise multiplication. The discrete group $\mathbb{Z}_{(2)}$ is the Pontrjagin dual of the compact Abelian group $\hat{\mathbb{Z}}_{(2)}$. The element F_n^1 is the generator of the cyclic subgroup $\mathbb{Z}_{2^n}$. Note that

$$\int_{\hat{\mathbb{Z}}_{(2)}} F_n^l \, d\mu_{\text{haar}} = 0$$

except if $l = 0, n = 0$, when $F_n^l \equiv 1$. Observe that if $k \in \mathbb{Z}$ then

$$\rho(k) F_n^l = F_n^{l+k(mod\, 2^n)} \tag{2}$$

since $F_n^l(x-k) = F_n^{l+k(mod\, 2^n)}(x)$. Hence we have the following lemma.

Lemma 4.1. *The elements of $\mathcal{A}_M \rtimes \mathbb{Z}$ can be uniquely written as finite sums in the form*

$$\sum_k \sum_{n \geq 0} \sum_{l|(l,n)=1} c_{n,l,k} F_n^l \cdot k \, ,$$

where $k \cdot F_n^l = F_n^{l+k(mod\, 2^n)}$ and $F_0^0 = 1$.

5. Periodic operators

Definition 5.1. *A function $\mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is a periodic operator if there exists some $n \geq 1$ such that*

- *$A(x, y) = 0$, if $|x - y| > 2^n$*
- *$A(x, y) = A(x + 2^n, y + 2^n)$.*

Observe that the periodic operators form a $*$-algebra, where

- $(A + B)(x, y) = A(x, y) + B(x, y)$
- $AB(x, y) = \sum_{z \in \mathbb{Z}} A(x, z) B(z, y)$
- $A^*(x, y) = A(y, x)$

Proposition 5.1. *The algebra of periodic operators $\mathcal{P}$ is $*$-isomorphic to a dense subalgebra of $\mathcal{M}$.*

Proof. We call $A \in \mathcal{P}$ an element of type-n if

- $A(x, y) = A(x + 2^n, y + 2^n)$
- $A(x, y) = 0$ if $0 \leq x \leq 2^n - 1$, $y > 2^n - 1$
- $A(x, y) = 0$ if $0 \leq x \leq 2^n - 1$, $y < 0$.

Clearly, the elements of type-n form an algebra $\mathcal{P}_n$ isomorphic to $\mathrm{Mat}_{2^n\times 2^n}(\mathbb{C})$ and $\mathcal{P}_n\to\mathcal{P}_{n+1}$ is the diagonal embedding. Hence, we can identify the algebra of finite type elements $\mathcal{P}_f=\cup_{n=1}^\infty\mathcal{P}_n$ with $\varinjlim \mathrm{Mat}_{2^n\times 2^n}(\mathbb{C})$.
For $A\in\mathcal{P}$, if $n\geq 1$ is large enough, let $A_n\in\mathcal{P}_n$ be defined the following way.

- $A_n(x,y)=A(x,y)$ if $2^nl\leq x,y\leq 2^nl+2^n-1$ for some $l\in\mathbb{Z}$.
- Otherwise, $A(x,y)=0$.

Lemma 5.1. **(i):** *$\{A_n\}_{n=1}^\infty$ is a Cauchy-sequence in $\mathcal{M}$.*
(ii): $(A+B)_n=A_n+B_n$.
(iii): $rk_{\mathcal{M}}(A_n^*-(A^*)_n)=0$.
(iv): $rk_{\mathcal{M}}((AB_n)-A_nB_n)=0,$.
(v): $\lim_{n\to\infty}A_n=0$ *if and only if* $A=0$.

Proof. First observe that for any $Q\in\mathcal{P}_n$

$$\mathrm{rk}_{\mathcal{M}}(Q)\leq\frac{|\{0\leq x\leq 2^n-1\mid \exists\, 0\leq y\leq 2^n-1 \text{ such that } A_n(x,y)\neq 0.\}|}{2^n}$$

Suppose that $A(x,y)=A(x+2^k,y+2^k)$ and $k<n<m$. Then

$$|\{0\leq x\leq 2^n-1\mid A_n(x,y)\neq A_m(x,y)\,\text{for some } 0\leq y\leq 2^n-1\}|\leq 2^k2^{m-n}\,.$$

Hence by the previous observation, $\{A_n\}_{n=1}^\infty$ is a Cauchy-sequence. Note that (iii) and (iv) can be proved similarly, the proof of (ii) is straightforward. In order to prove (v) let us suppose that $A(x,y)=0$ whenever $|x-y|\geq 2^k$. Let $n>k$ and $0\leq y\leq 2^k-1$ such that $A(x,y)\neq 0$ for some $-2^k\leq x\leq 2^k-1$. Therefore $\mathrm{rk}_{\mathcal{M}}A_n\geq\frac{2^{n-k-1}}{2^n}$. Thus (v) follows. □
Let us define $\phi:\mathcal{P}\to\mathcal{M}$ by $\phi(A)=\lim_{n\to\infty}A_n$. By the previous lemma, ϕ is an injective $*$-homomorphism. □

Definition 5.2. *A periodic operator A is diagonal if $A(x,y)=0$, whenever $x\neq y$. The diagonal operators form the Abelian $*$-algebra $\mathcal{D}\subset\mathcal{P}$.*

Lemma 5.2. *We have the isomorphism $\mathcal{D}\cong\mathbb{C}(\mathbb{Z}_{(2)})$, where $\mathbb{Z}_{(2)}$ is the Prüfer 2-group.*

Proof. For $n\geq 1$ and $0\leq l\leq 2^n-1$ let $E_n^l\in\mathcal{D}$ be defined by

$$E_n^l(x,x):=exp\left(\frac{2\pi ix(mod\,2^n)}{2^n}l\right)\,.$$

It is easy to see that $E_{n+1}^{2l}=E_n^l$ and the multiplicative group generated by E_n^1 is isomorphic to $\mathbb{Z}_{2^n}$. Observe that the set $\{E_n^l\}_{n,l,(l,n)=1}$ form a basis in the space of n-type diagonal operators. Therefore, $\mathcal{D}\cong\cup_{n=1}^\infty\mathbb{C}(Z_{2^n})=\mathbb{C}(\mathbb{Z}_{(2)})$. □
Let $J\in\mathcal{P}$ be the following element.

- $J(x,y)=1$, if $y=x+1$.
- Otherwise, $J(x,y)=0$.

Then

$$J\cdot E_n^l=E_n^{l+1(mod\,2^n)}\,. \tag{3}$$

Also, any periodic operator A can be written in a unique way as a finite sum

$$\sum_{k\in\mathbb{Z}} D_k \cdot J^k ,$$

where D_k is a diagonal operator in the form

$$D_k = \sum_{n=0}^{\infty} \sum_{l|(l,n)=1} c_{l,n,k} E_n^l .$$

Thus, by (2) and (3), we have the following corollary.

Corollary 5.1. *The map $\psi : \mathcal{P} \to \mathcal{A}_M \rtimes \mathbb{Z}$ defined by*

$$\psi(\sum_k \sum_{n\geq 0} \sum_{l|(l,n)=1} c_{l,n,k} E_n^l \cdot k) = \sum_k \sum_{n\geq 0} \sum_{l|(l,n)=1} c_{l,n,k} F_n^l \cdot k$$

is a $*$-isomorphism of algebras.

6. Lück's Approximation Theorem revisited

The goal of this section is to prove the following proposition.

Proposition 6.1. *We have $c(\rho) \cong \mathcal{M}$ where ρ is the odometer action.*

Proof. Let us define the linear map $t : \mathcal{P} \to \mathbb{C}$ by

$$t(A) := \frac{\sum_{i=0}^{2^n-1} A(i,i)}{2^n} ,$$

where $A \in \mathcal{P}$ and $A(x + 2^n, y + 2^n)$ for all $x, y \in \mathbb{Z}$.

Lemma 6.1. *$Tr_{\mathcal{N}(\rho)}(\psi(A)) = t(A)$, where ψ is the $*$-isomorphism of Corollary 5.1.*

Proof. Recall that $\mathrm{Tr}_{\mathcal{N}(\rho)}(F_n^l) = 0$, except, when $l = 0, n = 0, F_n^l = 1$. If $n \neq 0$ and $l \neq 0$, then $t(E_n^l)$ is the sum of all k-th roots of unity for a certain k, hence $t(E_n^l) = 0$. Also, $t(1) = 1$. Thus, the lemma follows. □

It is enough to prove that

$$\mathrm{rk}_{\mathcal{M}}(A) = \mathrm{rk}_{\mathcal{N}(\rho)}(\psi(A)) \tag{4}$$

Indeed by (4), ψ is a rank-preserving $*$-isomorphism between $\mathcal{P}$ and $\mathcal{A}_M \rtimes \mathbb{Z}$. Hence the isomorphism ψ extends to a metric isomorphism

$$\hat{\psi} : \overline{\mathcal{P}} \to \overline{\mathcal{A}_M \rtimes \mathbb{Z}} ,$$

where $\overline{\mathcal{P}}$ is the closure of $\mathcal{P}$ in $\mathcal{M}$ and $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$ is the closure of $\mathcal{A}_M \rtimes \mathbb{Z}$ in $U(\mathcal{N}(\rho))$. Since $\mathcal{P}$ is dense in $\mathcal{M}$, $\overline{\mathcal{P}} \cong \mathcal{M}$. Also, $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$ is a $*$-subalgebra of $U(\mathcal{N}(\rho))$, since the $*$-ring operations are continuous with respect to the rank metric. Therefore $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$ is a continuous algebra isomorphic to $\mathcal{M}$. Observe that the rank closure $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$ is isomorphic to the rank closure of $L_c^\infty(\hat{\mathbb{Z}}_{(2)}, \mu_{\text{haar}}) \rtimes \mathbb{Z}$ by the argument of Lemma 3.2. Therefore, $c(\rho) \cong \mathcal{M}$. Thus from now on, our only goal is to prove (4).

Lemma 6.2. *Let $A \in \mathcal{P}$ and $A_n \in Mat_{2^n\times 2^n}(\mathbb{C})$ as in Section 5. Then the matrices $\{A_n\}_{n=1}^\infty$ have uniformly bounded norms.*

Proof. Let M, N be chosen in such a way that

- $|A_n(x,y)| \leq M$ for any $x, y \in \mathbb{Z}, n \geq 1$.
- $|A_n(x,y)| = 0$ if $|x-y| \geq \frac{N}{2}$.

Now let $v = (v(1), v(2), \dots, v(2^n)) \in \mathbb{C}^{2^n}$, $\|v\|^2 = 1$. Then

$$\|A_n v\|^2 = \sum_{x=1}^{2^n} |\sum_{y\,||x-y|<N/2} A_n(x,y)v(y)|^2 \leq M^2 \sum_{x=1}^{2^n} |\sum_{y\,||x-y|<N/2} v(y)|^2 \leq$$

$$\leq M^2 N \sum_{x=1}^{2^n} \sum_{y\,||x-y|<N/2} |v(y)|^2 \leq M^2 \sum_{y=1}^{2^n} N|v(y)|^2 = M^2N^2 .$$

Therefore, for any $n \geq 1$, $\|A_n\| \leq MN$. □

Lemma 6.3. *Let $A \in \mathcal{P}$. Then for any $k \geq 1$*

$$\lim_{k\to\infty} t((A_n^*A_n)^k) = t((A^*A)^k) = Tr_{\mathcal{N}(\rho)}(\psi(A^*A)^k)) .$$

Proof. Let $m \geq 1, l \geq 1, q \geq 1$ be integers such that

- $A(x,y) = A(x+2^m, y+2^m)$ for any $x, y \in \mathbb{Z}$.
- $A(x,y) = 0$, if $|x-y| \geq l$.
- $|(A^*A)^k(x,x)| \leq q$ and $|(A_n^*A_n)^k(x,x)| \leq q$ for any $x \in \mathbb{Z}$.

By definition,

$$t((A_n^*A_n)^k) = \frac{\sum_{x=1}^{2^n}(A_n^*A_n)^k(x,x)}{2^n}$$

$$t((A^*A)^k) = \frac{\sum_{x=1}^{2^n}(A^*A)^k(x,x)}{2^n} .$$

Observe that if $2lk < x, 2^n - 2lk$, then

$$(A^*A)^k(x,x) = (A_n^*A_n)(x,x) .$$

Hence,

$$|t((A^*A)^k) - t((A_n^*A_n)^k)| \leq \frac{4klq}{2^n} .$$

Thus our lemma follows. □

Now, we follow the idea of Lück [10]. Let μ be the spectral measure of $\psi(A) \in \mathcal{N}(\rho)$. That is

$$\mathrm{Tr}_{\mathcal{N}(\rho)} f(A^*A) = \int_0^K f(x)\, d\mu(x) ,$$

for all $f \in C[0,K]$, where $K > 0$ is chosen in such a way that $\mathrm{Spec}\,\psi(A^*A) \subset [0,K]$ and $\|A_n^*A_n\| \leq K$ for all $n \geq 1$. Also, let μ_n be the spectral measure of $A_n^*A_n$, that is,

$$t(f(A_n^*A_n)) = \int_0^K f(x)\, d\mu_n(x) ,$$

or all $f \in C[0,K]$. As in [10], we can see that the measures $\{\mu_n\}_{n=1}^\infty$ converge weakly to μ. Indeed by Lemma 6.3,

$$\lim_{n\to\infty} t(P(A_n^* A_n)) = \mathrm{Tr}_{\mathcal{N}(\rho)} P(A^*A)$$

for any real polynomial P, therefore

$$\lim_{n\to\infty} t(f(A_n^* A_n)) = \mathrm{Tr}_{\mathcal{N}(\rho)} f(A^*A)$$

for all $f \in C[0,K]$.
Since $\mathrm{rk}_{\mathcal{M}}(A_n) = \mathrm{rk}_{\mathcal{M}}(A_n^* A_n)$ and $\mathrm{rk}_{\mathcal{N}(\rho)}(\psi(A)) = \mathrm{rk}_{\mathcal{N}(\rho)})\psi(A^*A))$, in order to prove (4) it is enough to see that

$$\lim_{n\to\infty} \mathrm{rk}_{\mathcal{M}}(A_n^* A_n) = \mathrm{rk}_{\mathcal{N}(\rho)}(\psi(A^*A))\,.$$

Observe that $\mathrm{rk}_{\mathcal{M}}(A_n^* A_n) = 1 - \mu_n(0)$ and

$$\mathrm{rk}_{\mathcal{N}(\rho)}(\psi(A^*A)) = 1 - \lim_{\lambda\to 0} \mathrm{Tr}_{\mathcal{N}(\rho)} E_\lambda = \mu(0)\,.$$

Hence, our proposition follows from the lemma below (an analogue of Lück's Approximation Theorem).

Lemma 6.4. $\lim_{n\to\infty} \mu_n(0) = \mu(0)\,.$

Proof. Let $F_n(\lambda) = \int_0^\lambda \mu_n(t)\,dt$ and $F(\lambda) = \int_0^\lambda \mu(t)\,dt$ be the distribution functions of our spectral measures. Since $\{\mu_n\}_{n=1}^\infty$ weakly converges to the measure μ, it is enough to show that $\{F_n\}_{n=1}^\infty$ converges uniformly. Let $n \le m$ and $D_m^n : \mathrm{Mat}_{2^n\times 2^n}(\mathbb{C}) \to \mathrm{Mat}_{2^m\times 2^m}(\mathbb{C})$ be the diagonal operator. Let $\varepsilon > 0$. By Lemma 5.1, if n, m are large enough,

$$\mathrm{Rank}(D_m^n(A_n) - A_m) \le \varepsilon 2^m\,.$$

Hence, by Lemma 3.5 [2],

$$\|F_n - F_m\|_\infty \le \varepsilon\,.$$

Therefore, $\{F_n\}_{n=1}^\infty$ converges uniformly. ☐

7. Orbit Equivalence

First let us recall the notion of orbit equivalence. Let $\tau_1 : \Gamma_1 \curvearrowright (X,\mu)$ resp. $\tau_2 : \Gamma_2 \curvearrowright (Y,\nu)$ be essentially free probability measure preserving actions of the countably infinite groups Γ_1 resp. Γ_2. The two actions are called orbit equivalent if there exists a measure preserving bijection $\Psi : (X,\mu) \to (Y,\nu)$ such that for almost all $x \in X$ and $\gamma \in \Gamma_1$ there exists $\gamma_x \in \Gamma_2$ such that

$$\tau_2(\gamma_x)(\Psi(x)) = \Psi(\tau_1(\gamma)(x))\,.$$

Feldman and Moore [4] proved that if τ_1 and τ_2 are orbit equivalent then $\mathcal{N}(\tau_1) \cong \mathcal{N}(\tau_2)\,.$ The goal of this section is to prove the following proposition.

Proposition 7.1. *If τ_1 and τ_2 are orbit equivalent actions, then $c(\tau_1) \cong c(\tau_2)$.*

Our Theorem 1 follows from the proposition. Indeed, by Proposition 3.2 and Proposition 6.1

$$\mathcal{M} \cong c(\rho) \quad \text{and} \quad c(\mathbb{Z}_2 \wr H) \cong c(\tau_H)\,.$$

By the famous theorem of Ornstein and Weiss [11], the odometer action and the Bernoulli shift action of a countably infinite amenable group are orbit equivalent. Hence $\mathcal{M} \cong c(\mathbb{Z}_2 \wr H)$. □

Proof. We build the proof of our proposition on the original proof of Feldman and Moore. Let $\gamma \in \Gamma_1$, $\delta \in \Gamma_2$. Let

$$M(\delta,\gamma) = \{y \in Y \ \mid \tau_2(\delta)(y) = \Psi(\tau_1(\gamma)\Psi^{-1}(y))\}$$

$$N(\gamma,\delta) = \{x \in X \ \mid \tau_1(\gamma)(x) = \Psi^{-1}(\tau_2(\delta)\Psi(x))\}.$$

Observe that $\Psi(N(\delta,\gamma)) = M(\gamma,\delta)$. Following Feldman and Moore ([4], Proposition 2.1) for any $\gamma \in \Gamma_1$, $\delta \in \Gamma_2$

$$\kappa(\gamma) = \sum_{h\in\Gamma_2} h \cdot 1_{M(h,\gamma)}$$

and

$$\lambda(\delta) = \sum_{g\in\Gamma_1} g \cdot 1_{N(g,\delta)}$$

are well-defined. That is, $\sum_{n=1}^k h_n \cdot 1_{M(h_n,\gamma)}$ converges weakly to $\kappa(\gamma) \in \mathcal{N}(\tau_2)$ as $k \to \infty$ and $\sum_{n=1}^k g_n \cdot 1_{N(g_n,\delta)}$ converges weakly to $\lambda(\delta) \in \mathcal{N}(\tau_1)$ as $k \to \infty$, where $\{\gamma_n\}_{n=1}^\infty$ resp. $\{\delta_n\}_{n=1}^\infty$ are enumerations of the elements of Γ_1 resp. Γ_2.
Furthermore, one can extend κ resp. λ to maps

$$\kappa' : L^\infty((X,\mu) \rtimes \Gamma_1) \to \mathcal{N}(\tau_2)$$

resp.

$$\lambda' : L^\infty((Y,\nu) \rtimes \Gamma_2) \to \mathcal{N}(\tau_1)$$

by

$$\kappa'(\sum_{\gamma\in\Gamma_1} a_\gamma \cdot \gamma) = \sum_{\gamma\in\Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \kappa(\gamma) = \sum_{\gamma\in\Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \sum_{n=1}^{\infty} h_n \cdot 1_{M(h_n,\gamma)}$$

and

$$\lambda'(\sum_{\delta\in\Gamma_2} b_\delta \cdot \delta) = \sum_{\delta\in\Gamma_2} (b_\delta \circ \Psi) \cdot \lambda(\delta) = \sum_{\delta\in\Gamma_2} (b_\delta \circ \Psi) \cdot \sum_{n=1}^{\infty} g_n \cdot 1_{N(g_n,\delta)}\,.$$

The maps κ' resp. λ' are injective trace-preserving $*$-homomorphisms with weakly dense ranges. Hence they extend to isomorphisms of von Neumann algebras

$$\hat{\kappa} : \mathcal{N}(\tau_1) \to \mathcal{N}(\tau_2), \hat{\lambda} : \mathcal{N}(\tau_2) \to \mathcal{N}(\tau_1)\,,$$

where $\hat{\kappa}$ and $\hat{\lambda}$ are, in fact, the inverses of each other.

Lemma 7.1.

$$\lim_{k\to\infty} rk_{\mathcal{N}(\tau_2)}\left(\sum_{\gamma\in\Gamma_1}(a_\gamma\circ\Psi^{-1})\cdot\sum_{n=1}^{k} h_n\cdot 1_{M(h_n,\gamma)} - \hat{\kappa}(\sum_{\gamma\in\Gamma_1} a_\gamma\cdot\gamma)\right)=0\,. \tag{5}$$

$$\lim_{k\to\infty} rk_{\mathcal{N}(\tau_1)}\left(\sum_{\delta\in\Gamma_2}(b_\delta\circ\Psi)\cdot\sum_{n=1}^{k} g_n\cdot 1_{N(g_n,\delta)} - \hat{\lambda}(\sum_{\delta\in\Gamma_2} b_\delta\cdot\delta)\right)=0\,. \tag{6}$$

Proof. By definition, the disjoint union $\cup_{n=1}^\infty M(h_n,\gamma)$ equals to Y (modulo a set of measure zero). We need to show that if $\{\sum_{n=1}^k T_n\cdot 1_{M(h_n,\gamma)}\}_{k=1}^\infty$ weakly converges to an element $S\in\mathcal{N}(\tau_2)$, then $\{\sum_{n=1}^k T_n\cdot 1_{M(h_n,\gamma)}\}_{k=1}^\infty$ converges to S in the rank metric as well, where $T_n\in L^\infty_c(Y,\nu)\rtimes\Gamma_2$. Let $P_k=\sum_{n=1}^k 1_{M(h_n,\gamma)}\in l^2(\Gamma,L^2(Y,\nu))$. We denote by $\hat{P}_k$ the element $\sum_{n=1}^k 1_{M(h_n,\gamma)}$ in $L^\infty_c(Y,\nu)\rtimes\Gamma_2$. By definition, if $L(A)(P_k)=0$ then $A\hat{P}_k=0$. Now, by weak convergence,

$$L(S)(P_k)=\lim_{l\to\infty}\sum_{n=1}^{l} T_n\cdot 1_{M(h_n,\gamma)}(P_k)\,.$$

That is,

$$L(S-\sum_{n=1}^{k} T_n\cdot 1_{M(h_n,\gamma)})(P_k)=0\,.$$

Therefore,

$$(S-\sum_{n=1}^{k} T_n\cdot 1_{M(h_n,\gamma)})\hat{P}_k=0\,.$$

Thus,

$$(S-\sum_{n=1}^{k} T_n\cdot 1_{M(h_n,\gamma)})=(S-\sum_{n=1}^{k} T_n\cdot 1_{M(h_n,\gamma)})(1-\hat{P}_k)\,.$$

By Lemma 3.1, $\mathrm{rk}_{\mathcal{N}(\tau_2)}(1-\hat{P}_k)=1-\sum_{n=1}^k\nu(M(h_n,\gamma))$, hence

$$\lim_{k\to\infty}\mathrm{rk}_{\mathcal{N}(\tau_2)}(S-\sum_{n=1}^{k} T_n\cdot 1_{M(h_n,\gamma)})=0\,.\qquad\square$$

Now let us turn back to the proof of our proposition. By (5), $\hat{\kappa}$ maps the algebra $L^\infty_c(X,\mu)\rtimes\Gamma_1$ into the rank closure of $L^\infty_c(Y,\nu)\rtimes\Gamma_2$. Since $\hat{\kappa}$ preserves the rank, $\hat{\kappa}$ maps the rank closure of $L^\infty_c(X,\mu)\rtimes\Gamma_1$ into the rank closure of $L^\infty_c(Y,\nu)\rtimes\Gamma_2$. Similarly, $\hat{\lambda}$ maps the rank closure of $L^\infty_c(Y,\nu)\rtimes\Gamma_2$ into the rank closure of $L^\infty_c(X,\mu)\rtimes\Gamma_1$. That is, $\hat{\kappa}$ provides an isomorphism between the rank closures of $L^\infty_c(X,\mu)\rtimes\Gamma_1$ and $L^\infty_c(Y,\nu)\rtimes\Gamma_2$. Therefore, the smallest continuous ring containing $L^\infty_c(X,\mu)\rtimes\Gamma_1$ in $U(\mathcal{N}(\tau_1))$ is mapped to the smallest continuous ring containing $L^\infty_c(Y,\nu)\rtimes\Gamma_2$ in $U(\mathcal{N}(\tau_2))$. $\square$

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