Problem-driven scenario generation: an analytical approach for stochastic programs with tail risk measure

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Abstract

Scenario generation is the construction of a discrete random vector to represent parameters of uncertain values in a stochastic program. Most approaches to scenario generation are *distribution-driven*, that is, they attempt to construct a random vector which captures well in a probabilistic sense the uncertainty. On the other hand, a *problem-driven* approach may be able to exploit the structure of a problem to provide a more concise representation of the uncertainty. There have been only a few problem-driven approaches proposed, and these have been heuristic in nature.

In this paper we propose what is, as far as we are aware, the first analytic approach to problem-driven scenario generation. This approach applies to stochastic programs with a tail risk measure, such as conditional value-at-risk. Since tail risk measures only depend on the upper tail of a distribution, standard methods of scenario generation, which typically spread there scenarios evenly across the support of the solution, struggle to adequately represent tail risk well.

1 Introduction

Stochastic programming is a tool for making decisions under uncertainty. Stochastic programs are used to model situations where an initial decision must be taken with some information unknown until after the decision has been made. In stochastic programming, uncertain parameters are modeled as random variables, and one attempts to minimize the expectation or risk measure of some loss function which depends on the initial decision. The power and flexibility of the stochastic programming approach comes at a price: stochastic programs are usually analytically intractable, and not susceptible to solution techniques for deterministic programs.

Typically, a stochastic program can only be solved when it is *scenario-based*, that is when the random vector for the problem has a finite discrete distribution. For example, stochastic linear programs become large-scale linear programs when the underlying random vector is discrete. In the stochastic programming literature, the mass points of this random vector are referred to as *scenarios*, the discrete distribution as the *scenario set* and the construction of this as *scenario generation*. Scenario generation can consist of discretizing a continuous probability distribution, or directly modeling the uncertain quantities as discrete random variables. The more scenarios in a set, the more computational power that is required to solve the problem. The key issue of scenario generation is therefore how to represent the uncertainty to ensure that the solution to the problem is reliable, while keeping the number of scenarios low so that the problem is computationally tractable. See [KW07] for methods of evaluating scenario generation methods and a discussion of what constitutes a reliable solution.

Minimizing the expectation of a loss function can be thought of as minimizing the long-term costs of a system. This is appropriate when the initial decision is going to be used again and again, and individual large losses do not matter in the short term. In other cases, the decision may be only used a

few times, and the occurrence of large losses may lead to bankruptcy. In this latter case, minimizing the expectation alone is not appropriate as this does not necessarily mitigate against the possibility of large losses. One possible action of recourse is to use a *risk measure* which penalises in some way the likelihood and severity of potential large losses. In these problems we try to find a decision which appropriately balances the expectation against the risk.

In this paper we are interested in problems which use *tail risk measures*. A precise definition of a tail-risk measure will be given in Section 2. But for now, one can think of a tail risk measure as a function of a random variable which only depends on the upper tail of its distribution function. Examples of tail risk measure include the Value-at-Risk [Jor96] and the Conditional Value-at-Risk [RU00], both of which are commonly used in financial contexts. Scenario generation is particularly challenging in the presence of a tail risk measure. Since these risk measures only depend on the upper tail tail of a distribution, standard methods for scenario generation, which spread their scenarios evenly across the support of the distribution, struggle to adequately represent the tail risk.

A common approach to scenario generation is to fit a statistical model to the uncertain problem parameters and then generate a random sample from this for the scenario set. This has desirable asymptotic properties [KR93, Sha03], but may require large sample sizes to ensure the reliability of the solutions it yields. This can be mitigated somewhat by using variance reduction techniques such as stratified sampling and importance sampling [LSW06]. Sampling also has the advantage that it can be used to construct confidence intervals on the true solution value [MMW99]. Another approach is to construct a scenario set whose distance from the true distribution, with respect to some probability metric, is small [Pfl01, HR09]. These approaches tend to yield better and much more stable solutions to stochastic programs than does sampling.

A characteristic of both of these approaches to scenario generation is that they are *distribution-based*; that is, they only aim to approximate a distribution and are divorced from the stochastic program for which they are producing scenarios. By exploiting the structure of a problem, it may be possible to find a more parsimonious representation of the uncertainty. Note that such a *problem-based* approach may not yield a discrete distribution which is close to the true distribution in a probabilistic sense; the aim is only to find a discrete distribution which yields a high quality solution to our problem.

A set of approaches which move away from the purely distribution-based paradigm of scenario generation are *constructive methods*. In these approaches, the modeler does not use a full probability distribution for the uncertain problem parameters, but specifies a set of target statistical properties they believe the are important for the underlying optimization problem, and generates a scenario set with these target properties. This approach was first proposed in [HW01], where it is postulated that the solution to a stochastic program will depend largely on a small set of statistical properties of the random variables, specific to that problem. That is, if we can generate a scenario set with the required properties, this should yield good solutions in our stochastic program even if the true distribution is significantly different. For example, it is known that for the classical Markowitz problem [Mar52] the first two moments of the return distributions determine exactly the solution. Constructive approaches have gained much popularity because they simplify the stochastic modeling of the uncertain parameters. Other constructive approaches can be found in [HKW03], and [KW11]. However, the major draw-back with constructive approaches is that it is not always clear which properties are important for a given problem.

There are only a few cases of problem-driven scenario generation in the literature and these are somewhat heuristic in nature. Constructive scenario generation methods can be considered problem-driven in the sense that a given stochastic program may only react to certain statistical properties. However, as we explained, an empirical investigation must be carried out to to identify which properties are important, and in reality these methods are often used as a convenient way of modeling the uncertain quantities. Other notable examples from the literature are the importance sampling approach in [DG90] which constructs a sampler from the loss function, and more recently, a scenario reduction approach [FR14] based on clustering scenarios which behave similarly with respect to the problem. The application in [ZW15] uses a problem-driven approach where the constructed scenario set is markedly different from the underlying distribution in the probabilistic sense.

In this paper, we present what is, as far as we are aware, the first truely analytical example of

problem-driven scenario generation. This approach is applicable to stochastic programs which use tail risk measures. We observe that the value of any tail risk measure depends only on scenarios confined to an area of the distribution that we call the *risk region*. This means that all scenarios that are not in the risk region can be aggregated into a single point. By concentrating scenarios in the risk region, we can calculate the value of a tail risk measure more accurately.

The idea that in stochastic programs with tail risk measures some scenarios do not contribute to the calculation of the tail-risk measure was also exploited in [GBM12]. However, they propose a solution algorithm rather than a method of scenario generation. Their approach is to iteratively solve the problem with a subset of scenarios, identify the scenarios which have loss in the tail, update their scenario set appropriately and resolve, until the true solution has been found. Their method has the benefit that it is not distribution dependent. On the other hand, their method works for only the β-CVaR risk measure, while our approach works in principle for any tail risk measure.

In general, finding a risk region is difficult as it is determined by both the problem and the distribution of the uncertain parameters. Therefore, we derive risk regions for two classes of problem as a proof-of-concept of our methodology. The first class of problems are those with monotonic loss functions which occur naturally in the context of network design. The second class are portfolio selection problems. For both cases, we implement our methodology and run numerical tests.

This paper is organized as follows: in Section 2 we define tail risk measures and their associated risk regions; in Section 3 we discuss how these risk regions can be exploited for the purposes of scenario generation and scenario reduction; in Section 4 we prove that our scenario generation method is consistent with sampling, in Sections 5 and 6 we derive risk regions for the two classes of problems described above; in Section 7 we present numerical tests; finally in Section 8 we summarize our results make some concluding remarks.

2 Tail risk measures and risk regions

In this section we define the core concepts related to our scenario generation methodology, and prove some results relating to these. Specifically, in Section 2.1 we formally define tail-risk measures of random variables and in Section 2.2 we define risk regions and present some key results related to these.

2.1 Tail risk of random variables

In our set-up we suppose we have some random variable representing an uncertain loss. For our purposes, we take a risk measure to be any function of a random variable[1]. The following formal definition is taken from [Tas02].

Definition 2.1 (Risk Measure)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and V be a non-empty set of $\mathcal{F}$-measurable real-valued random variables[2]. Then, a risk measure is some function $\rho : V \to \mathbb{R} \cup \{\infty\}$.

For a risk measure to be useful, it should in some way penalize potential large losses. For example, in the classical Markowitz problem [Mar52], one aims to minimize the variance of the return of a portfolio. By choosing a portfolio with a low variance, we reduce the probability of larges losses as a direct consequence of Chebyshev's inequality (see for instance [Bil95]). Various criteria for risk measures have been proposed; in [ADEH99] a *coherent risk measure* is defined to be a risk measure which satisfies axioms such as positive homogeneity and subadditivity; another perhaps desirable criterion for risk measures is that the risk measure is consistent with respect to first and second order stochastic dominance, see [OR02] for instance.

Besides not satisfying some of the above criteria, a major drawback with using variance as a measure is that it penalizes all large deviations from the mean, that is, it penalizes large profits as well as large losses. This motivates the idea of using risk measures which depend only on the upper tail of the loss

[1] The recent paper [RU13] which proposes a general framework for measures of risk and deviation, gives the following more specific characterization: a risk measure ρ should "model X as "adequately" $\leq C$ by the inequality $\rho(Z) \leq C$", where C is some loss one wishes not to exceed.

[2] We implicitly assume throughout that V is large enough to contain all constructed random variables

distribution. The upper tail is the restriction of the distribution function to outcomes with a loss greater than or equal to some quantile. We recall the definition of quantile function before stating the formal definition of tail risk measure.

Definition 2.2 (Quantile Function)**.** Suppose Z is a random variable with distribution function F_Z. Then the generalized inverse distribution function, or *quantile function* is defined as follows:

$$\begin{aligned} F_Z^{-1} : (0,1] &\to \mathbb{R} \cup \{\infty\} \\ \beta &\mapsto \inf\{x \in \mathbb{R} : F_Z(x) \geq \beta\}. \end{aligned}$$

Definition 2.3 (Tail Risk Measure)**.** Let $\rho_\beta : V \to \mathbb{R} \cup \{\infty\}$ be a risk measure as above, then ρ_β is a β-tail risk measure if $\rho_\beta(Z)$ depends only on the restriction of quantile function of Z above β, in the sense that if W and Z are random variables with $F_Z^{-1}|_{[\beta,1]} \equiv F_Z^{-1}|_{[\beta,1]}$ then $\rho_\beta(Z) = \rho_\beta(W)$.

To show that ρ_β is a β-tail risk measure, we must show that $\rho_\beta(Z)$ can be written as a function of the quantile function above or equal to β. Two very popular tail risk measures are the value-at-risk [Jor96] and the conditional value-at-risk [RU02]:

Example 2.4 (Value at risk)**.** Let Z be a random variable, and $0 < \beta < 1$. Then, the $\beta-$VaR for Z is defined to be the β-quantile of Z:

$$\beta\text{-VaR}(Z) := F_Z^{-1}(\beta).$$

Example 2.5 (Conditional value at risk)**.** Let Z be a random variable, and $0 < \beta < 1$. Then, the β-CVaR can be thought roughly as the conditional expectation of a random variable above its β-quantile. The following alternative characterization of β-CVaR [AT02] shows directly that it is a β-tail risk measure.

$$\beta\text{-CVaR}(Z) = \int_\beta^1 F_Z^{-1}(u)\ du.$$

The observation that we exploit for this work is that very different random variables will have the same β-tail risk measure as long as their β-tails are the same.

When showing that two distributions have the same β-tails, it is convenient to use distribution functions rather than quantile functions. An equivalent condition for showing that two random variables Z_1 and Z_2 have the same β-tail, that is $F_{Z_1}^{-1}(u) = F_{Z_2}^{-1}(u)$ for all $\beta \leq u \leq 1$, is the following:

$$F_{Z_1}^{-1}(\beta) = F_{Z_2}^{-1}(\beta) \text{ and } F_{Z_1}(z) = F_{Z_2}(z) \text{ for all } z \geq F_{Z_1}^{-1}(\beta). \tag{1}$$

2.2 Risk regions

In the optimization context we suppose that the loss depends on some decision $x \in \mathcal{X} \subseteq \mathbb{R}^k$ and the outcome of some latent random vector Y with support $\mathcal{Y} \subseteq \mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and which is independent of x. That is, we suppose our loss is determined by some function, $f : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}$, which we refer to as the *loss function*. For a given decision $x \in \mathcal{X}$, the random variable associated with the loss is thus $f(x, Y)$.

To avoid repeated use of cumbersome notation we introduce the following short-hand for distribution and quantile functions:

$$\begin{aligned} F_x(z) &:= F_{f(x,Y)}(z) = \mathbb{P}\left(f(x,Y) \leq z\right), \\ F_x^{-1}(\beta) &:= F_{f(x,Y)}^{-1}(\beta) = \inf\{z \in \mathbb{R} :\ F_x(z) \geq \beta\}. \end{aligned}$$

Since tail risk measures depend only on those outcomes which are in the β-tail, we aim to identify which outcomes lead to a loss in the β-tails for a feasible decision.

Definition 2.6 (Risk region)**.** For $0 < \beta < 1$ the β-risk region with respect to the decision $x \in \mathcal{X}$ is defined as follows:

$$\mathcal{R}_x(\beta) = \{y \in \mathbb{R}^d : F_x\left(f(x,y)\right) \geq \beta\},$$

or equivalently

$$\mathcal{R}_x(\beta) = \{y \in \mathbb{R}^d : f(x,y) \geq F_x^{-1}(\beta)\}. \tag{2}$$

The risk region with respect to the feasible region $\mathcal{X} \subset \mathbb{R}^k$ is defined to be:

$$\mathcal{R}_{\mathcal{X}}(\beta) = \bigcup_{x \in \mathcal{X}} \mathcal{R}_x(\beta). \tag{3}$$

The complement of this region is called the *non-risk region*. This can also be written

$$\mathcal{R}_{\mathcal{X}}(\beta)^c = \bigcap_{x \in \mathcal{X}} \mathcal{R}_x(\beta)^c. \tag{4}$$

The following basic properties of the risk region follow directly from the definition.

$$\text{(i) } 0 < \beta' < \beta < 1 \;\Rightarrow\; \mathcal{R}_{\mathcal{X}}(\beta) \subseteq \mathcal{R}_{\mathcal{X}}(\beta'); \tag{5}$$

$$\text{(ii) } \mathcal{X}' \subset \mathcal{X} \;\Rightarrow \mathcal{R}_{\mathcal{X}'}(\beta) \subseteq \mathcal{R}_{\mathcal{X}}(\beta); \tag{6}$$

$$\text{(iii) If } y \mapsto f(x,y) \text{ is upper semi-continuous then } \mathcal{R}_x(\beta) \text{ is closed and } \mathcal{R}_x(\beta)^c \text{ is open.} \tag{7}$$

We now state a technical property and prove that this ensures the distribution of the random vector in a given region completely determines the value of a tail risk measure. In essence, this condition ensures that there is enough mass in the set to ensure that the β-quantile does not depend on the probability distribution outside of it.

Definition 2.7 (Aggregation condition)**.** Suppose that $\mathcal{R}_{\mathcal{X}}(\beta) \subseteq \mathcal{R} \subset \mathbb{R}^d$ and that for all $x \in \mathcal{X}$, $\mathcal{R}$ satisfies the following condition:

$$\mathbb{P}\left(Y \in \{y : z' < f(x,y) \leq F_x^{-1}(\beta)\} \cap \mathcal{R}\right) > 0 \qquad \forall\, z' < F_x^{-1}(\beta). \tag{8}$$

Then $\mathcal{R}$ is said to satisfy the *β-aggregation condition*.

The motivation for the term *aggregation condition* comes from Theorem 2.8 which follows. This result ensures that if a set satisfies the aggregation condition then we can transform the probability distribution of Y so that all the mass in the complement of this set can be aggregated into a single point without affecting the value of the tail risk measure. This property is particularly relevant to scenario generation as if we have such a set, then all scenarios which it does not contain can be aggregated, reducing the size of the stochastic program.

Theorem 2.8. Suppose that $\mathcal{R}_{\mathcal{X}}(\beta) \subseteq \mathcal{R} \subset \mathbb{R}^d$ satisfies the β-aggregation condition and that $\tilde{Y}$ is a random vector for which

$$\mathbb{P}(Y \in \mathcal{A}) = \mathbb{P}\left(\tilde{Y} \in \mathcal{A}\right) \qquad \text{for any measurable } \mathcal{A} \subseteq \mathcal{R}. \tag{9}$$

Then for any tail risk measure ρ_β we have $\rho_\beta(f(x,Y)) = \rho_\beta\left(f(x,\tilde{Y})\right)$ for all $x \in \mathcal{X}$.

Proof. Fix $x \in \mathcal{X}$. To show that $\rho_\beta(f(x,Y)) = \rho_\beta\left(f(x,\tilde{Y})\right)$ we must show that the β-quantile and the β-tail distributions of $f(x,Y)$ and $f(x,\tilde{Y})$ are the same. The following two conditions are necessary and sufficient for this to occur:

$$F_x(z) = F_{f(x,\tilde{Y})}(z) \;\; \forall\, z \geq F_x^{-1}(\beta) \text{ and } F_{f(x,\tilde{Y})}(z) < \beta \;\; \forall\, z < F_x^{-1}(\beta).$$

In the first case suppose that $z' \geq F_x^{-1}(\beta)$. Note that as a direct consequence of (9) we have

$$\mathbb{P}(Y \in \mathcal{B}) = \mathbb{P}\left(\tilde{Y} \in \mathcal{B}\right) \qquad \text{for any } \mathcal{B} \supseteq \mathcal{R}^c. \tag{10}$$

Now,

$$\begin{aligned}
F_{f(x,\tilde{Y})}(z') &= \mathbb{P}\left(\tilde{Y} \in \{y:\ f(x,y) \le z'\}\right) \\
&= \mathbb{P}\left(\tilde{Y} \in \underbrace{\mathcal{R}^c \cap \{y:\ f(x,y) \le z'\}}_{=\mathcal{R}^c}\right) + \mathbb{P}\left(\tilde{Y} \in \underbrace{\mathcal{R} \cap \{y:\ f(x,Y) \le z'\}}_{\subset \mathcal{R}}\right) \\
&= \mathbb{P}\left(Y \in \mathcal{R}^c\right) + \mathbb{P}\left(Y \in \mathcal{R} \cap \{y : f(x,y) \le z'\}\right) \qquad \text{by (9) and (10)} \\
&= \mathbb{P}\left(Y \in \{y : f(x,y) \le z'\}\right) = F_x(z') \qquad \text{as required.}
\end{aligned}$$

Now suppose $z' < F_x^{-1}(\beta)$. There are two subcases; in the first instance we assume $\mathbb{P}\left(f(x,Y) = F_x^{-1}(\beta)\right) > 0$, then we have:

$$\begin{aligned}
F_{f(x,\tilde{Y})}(z') &\le \mathbb{P}\left(f(x,\tilde{Y}) < F_x^{-1}(\beta)\right) \\
&= \mathbb{P}\left(f(x,Y) < F_x^{-1}(\beta)\right) < \beta \qquad \text{as required.}
\end{aligned}$$

In the case where $\mathbb{P}\left(f(x,Y) = F_x^{-1}(\beta)\right) = 0$ we have:

$$\begin{aligned}
F_{f(x,\tilde{Y})}(z') &= \mathbb{P}\left(\tilde{Y} \in \{y : f(x,y) \le z'\}\right) \le \mathbb{P}\left(\tilde{Y} \in \mathcal{R}^c \cup \{y : f(x,y) \le z'\}\right) \\
&= \mathbb{P}\left(\tilde{Y} \in \underbrace{\{y : f(x,y) \le F_x^{-1}(\beta)\}}_{\supseteq \mathcal{R}^c}\right) - \mathbb{P}\left(\tilde{Y} \in \underbrace{\mathcal{R} \cap \{y : z' < f(x,y) \le F_x^{-1}(\beta)\}}_{\subseteq \mathcal{R}}\right) \\
&= \mathbb{P}\left(Y \in \{y : f(x,y) \le F_x^{-1}(\beta)\}\right) - \mathbb{P}\left(Y \in \mathcal{R} \cap \{y : z' < f(x,y) \le F_x^{-1}(\beta)\}\right) \qquad \text{by (9) and (10)} \\
&< \mathbb{P}\left(Y \in \{y : f(x,y) \le F_x^{-1}(\beta)\}\right) \qquad \text{by (8)} \\
&= \beta \qquad \text{since } \mathbb{P}\left(f(x,Y) = F_x^{-1}(\beta)\right) > 0
\end{aligned}$$

as required. □

The β-aggregation condition is difficult to verify directly. The following shows that it immediately holds for $\mathcal{R}_{\mathcal{X}}(\beta')$ when $\beta' < \beta$.

Proposition 2.9. Suppose $\beta' < \beta$. Then, $\mathcal{R}_{\mathcal{X}}(\beta')$ satisfies the β-aggregation condition. That is for all $x \in \mathcal{X}$

$$\mathbb{P}\left(Y \in \{y : z' \le f(x,y) \le F_x^{-1}(\beta)\} \cap \mathcal{R}_{\mathcal{X}}(\beta')\right) > 0 \qquad \forall\ z' < F_x^{-1}(\beta).$$

Proof. Fix $x \in \mathcal{X}$. We prove this for two cases. In the first case, we assume that $F_x^{-1}(\beta') = F_x^{-1}(\beta)$. In this case, the distribution function F_x has a discontinuity at $z = F_x^{-1}(\beta)$, that is $\mathbb{P}(f(x,Y) = z) > 0$. Therefore, for $z' < z$ we have

$$\mathbb{P}\left(Y \in \{y : z' \le f(x,y) \le F_x^{-1}(\beta)\} \cap \mathcal{R}_{\mathcal{X}}(\beta')\right) \ge \mathbb{P}(f(x,Y) = z) > 0.$$

We the second case we assume that $F_x^{-1}(\beta') < F_x^{-1}(\beta)$. In this case for all $F_x^{-1}(\beta') < z' < F_x^{-1}(\beta)$, we have $\{y : z' < f(x,y) \le F_x^{-1}(\beta)\} \subset \mathcal{R}_{\mathcal{X}}(\beta')$ and so

$$\mathbb{P}\left(Y \in \{y : z' \le f(x,y) \le F_x^{-1}(\beta)\} \cap \mathcal{R}_{\mathcal{X}}(\beta')\right) = \mathbb{P}\left(z' \le f(x,Y) \le F_x^{-1}(\beta)\right) > 0.$$

□

For convenience, we now drop β from our notation and terminology. Thus, we refer to the β-risk region and β-aggregation condition as simply the risk region and aggregation condition respectively, and write $\mathcal{R}_{\mathcal{X}}(\beta)$ as $\mathcal{R}_{\mathcal{X}}$.

All sets satisfying the aggregation condition must contain the risk region, however, the aggregation condition does not necessarily hold for the risk region itself. It is guaranteed to hold if Y has a discrete

distribution, since in this case for all $x \in \mathcal{X}$ and $z' < F_x^{-1}(\beta)$ we have:

$$\mathbb{P}\left(Y \in \{y : z' < f(x,y) \leq F_x^{-1}(\beta)\} \cap \mathcal{R}_{\mathcal{X}}\right) \geq \mathbb{P}\left(f(x,Y) = F_x^{-1}(\beta)\right) > 0.$$

In the non-discrete case we must impose extra conditions on the problem to avoid some degenerate cases. Recall that $\mathcal{Y}$ denotes the support of the random vector Y.

Proposition 2.10. Suppose that Y is continuous and the following conditions hold:

(i) $\operatorname{int}(\mathcal{Y}) \cap \operatorname{int}(\mathcal{R}_{\mathcal{X}})$ is connected

(ii) $y \mapsto f(x,y)$ is continuous for all $x \in \mathcal{X}$

(iii) For each $x \in \mathcal{X}$ there exists $x' \in \mathcal{X}$ such that

$$\operatorname{int}(\mathcal{Y}) \cap \operatorname{int}(\mathcal{R}_x \cap \mathcal{R}_{x'}) \neq \emptyset \text{ and } \operatorname{int}(\mathcal{Y}) \cap \operatorname{int}(\mathcal{R}_{x'} \setminus \mathcal{R}_x) \neq \emptyset \tag{11}$$

Then the risk region $\mathcal{R}_{\mathcal{X}}$ satisfies the aggregation condition.

Proof. Fix $x \in \mathcal{X}$ and $z' < F_x^{-1}(\beta)$. Pick $x' \in \mathcal{X}$ such that (11) holds. Also, let $y_0 \in \operatorname{int}(\mathcal{Y}) \cap \operatorname{int}(\mathcal{R}_{x'} \setminus \mathcal{R}_x)$ and $y_1 \in \operatorname{int}(\mathcal{Y}) \cap \operatorname{int}(\mathcal{R}_x \cap \mathcal{R}_{x'})$. Since $\operatorname{int}(\mathcal{Y}) \cap \operatorname{int}(\mathcal{R}_{\mathcal{X}})$ is connected there exists continuous path from y_0 to y_1. That is, there exists $\gamma : [0,1] \to \operatorname{int}(\mathcal{Y}) \cap \operatorname{int}(\mathcal{R}_{\mathcal{X}})$ such that $\gamma(0) = y_0$ and $\gamma(1) = y_1$. Now, $f(x,y_0) < F_x^{-1}(\beta)$ and $f(x,y_1) \geq F_x^{-1}(\beta)$ and so given that $t \mapsto f(x,\gamma(t))$ is continuous there must exist $0 < t < 1$ such that $z' < f(x,\gamma(t)) < F_x^{-1}(\beta)$. That is, $\operatorname{int}(\mathcal{Y}) \cap \operatorname{int}(\mathcal{R}_{\mathcal{X}}) \cap \{y : z' < f(x,y) < F_x^{-1}(\beta)\}$ is non-empty. This is a non-empty open set contained in the support of Y and so has positive probability, hence the aggregation condition holds. □

The following Proposition gives a condition under which the non-risk region is convex.

Proposition 2.11. Suppose that for each $x \in \mathcal{X}$ the function $y \mapsto f(x,y)$ is convex. Then the non-risk region $\mathcal{R}_{\mathcal{X}}^c$ is convex.

Proof. For $x \in \mathcal{X}$, if $y \mapsto f(x,y)$ is convex then the set $\mathcal{R}_x^c = \{y \in \mathbb{R}^d : f(x,y) < F_x^{-1}(\beta)\}$ must be convex. The arbitrary intersection of convex sets is convex, hence $\mathcal{R}_{\mathcal{X}}^c = \bigcap_{x \in \mathcal{X}} \mathcal{R}_x^c$ is convex. □

This convexity condition is held by a large class of stochastic programs, for instance, all two-stage linear recourse problems with fixed recourse will have this property (see, for instance,[BL97]).

The random vector in the following definition plays a special role in our theory.

Definition 2.12 (Aggregated random vector)**.** For some set $\mathcal{R} \subset \mathbb{R}^d$ satisfying the aggregation condition, the *aggregated random vector* is defined as follows:

$$\psi_{\mathcal{R}}(Y) := \begin{cases} Y & \text{if } Y \in \mathcal{R}, \\ \mathbb{E}\left[\, Y | Y \in \mathcal{R}^c \,\right] & \text{otherwise.} \end{cases}$$

If we have $\mathbb{E}\left[Y | Y \in \mathcal{R}^c\right] \in \mathcal{R}^c$ then Theorem 2.8 guarantees that $\rho_\beta\left(f\left(x, \psi_{\mathcal{R}}(Y)\right)\right) = \rho_\beta\left(f\left(x,Y\right)\right)$ for all $x \in \mathcal{X}$. For example, the conditions of Proposition 2.11 will guarantee this. As well as preserving the value of the tail risk measure, the function $\psi_{\mathcal{R}}$ will preserve the expectation for affine loss functions.

Corollary 2.13. Suppose for each $x \in \mathcal{X}$ the function $y \mapsto f(x,y)$ is affine and for a set $\mathcal{R} \subset \mathbb{R}^d$ satisfying the aggregation condition we have that $\mathbb{E}\left[Y | Y \in \mathcal{R}^c\right] \in \mathcal{R}^c$.Then,

$$\rho_\beta\left(f\left(x, \psi_{\mathcal{R}}(Y)\right)\right) = \rho_\beta\left(f\left(x,Y\right)\right) \text{ and } \mathbb{E}\left[f\left(x, \psi_{\mathcal{R}^c}(Y)\right)\right] = \mathbb{E}\left[f(x,Y)\right] \text{ for all } x \in \mathcal{X}.$$

Proof. The equality of the tail-risk measures follows immediately from Theorem 2.8. For the expectation function we have

$$\begin{aligned} \mathbb{E}\left[\psi_{\mathcal{R}}(Y)\right] &= \mathbb{P}\left(Y \in \mathcal{R}\right) \mathbb{E}\left[\psi_{\mathcal{R}}(Y) | Y \in \mathcal{R}\right] + \mathbb{P}\left(Y \in \mathcal{R}^c\right) \mathbb{E}\left[\psi_{\mathcal{R}}(Y) | Y \in \mathcal{R}^c\right] \\ &= \mathbb{P}\left(Y \in \mathcal{R}\right) \mathbb{E}\left[Y | Y \in \mathcal{R}\right] + \mathbb{P}\left(Y \in \mathcal{R}^c\right) \mathbb{E}\left[Y | Y \in \mathcal{R}^c\right] = \mathbb{E}\left[Y\right]. \end{aligned}$$

Since $y \mapsto f(x, y)$ is affine this means that

$$\mathbb{E}\left[f(x, \psi_{\mathcal{R}}(Y))\right] = f(x, \mathbb{E}\left[\psi_{\mathcal{R}}(Y)\right]) = f(x, \mathbb{E}\left[Y\right]) = \mathbb{E}\left[f(x, Y)\right].$$

□

3 Scenario generation

In the previous section, we showed that under mild conditions the value of a tail risk measure only depends on the distribution of outcomes in the risk region. In this section we demonstrate how this feature may be exploited for the purposes of scenario generation and scenario reduction.

We assume throughout this section that our scenario sets are constructed from some underlying probabilistic model from which we can draw independent identically distributed samples. We also assume we have a set $\mathcal{R} \subset \mathbb{R}^d$ which satisfies the aggregation condition and for which we can easily test membership. We refer to $\mathcal{R}$ as a *a conservative risk region*, and its complement $\mathcal{R}^c$ as the conservative non-risk region or *aggregation region*. Note that we reserve the symbol $\mathcal{R}_{\mathcal{X}}$ to denote the *exact* risk region as defined in (3). Our general approach is to prioritize the construction of scenarios in the approximate risk region.

In Section 3.1 we present and analyse two concrete approaches: aggregation sampling and aggregation reduction. In Section 3.2 we briefly discuss alternative ways of exploiting risk regions for scenario generation.

3.1 Aggregation sampling and reduction

In *aggregation sampling* the user specifies a number of *risk scenarios*, that is, the number of scenarios to represent the approximate risk region. The algorithm then draws samples from the distribution, storing those samples which lie in the approximate risk region and aggregating those in the aggregation region into a single point. In particular, the samples in the aggregation region are aggregated into their mean. The algorithm terminates when the specified number of risk scenarios has been reached. This is detailed in Algorithm 1. In *aggregation reduction* one draws a fixed number of samples from the distribution and then aggregates all those in the aggregation region.

input : $\mathcal{R} \subset \mathbb{R}^d$ set satisfying aggregation condition, $N_{\mathcal{R}}$ number of required risk scenarios
output: $\{(y_s, p_s)\}_{s=1}^{N_{\mathcal{R}}+1}$ scenario set
$n_{\mathcal{R}^c} \leftarrow 0$, $n_{\mathcal{R}} \leftarrow 0$, $y_{\mathcal{R}^c} = \mathbf{0}$;
while $n_{\mathcal{R}} < N_{\mathcal{R}}$ **do**
  Sample new point y;
  if $y \in \mathcal{R}$ **then**
    $n_{\mathcal{R}} \leftarrow n_{\mathcal{R}} + 1$; $y_{n_{\mathcal{R}}} \leftarrow y$;
  end
  else
    $n_{\mathcal{R}^c} \leftarrow n_{\mathcal{R}^c} + 1$; $y_{\mathcal{R}^c} \leftarrow \frac{1}{n_{\mathcal{R}^c}+1}\left(n_{\mathcal{R}^c} y_{\mathcal{R}^c} + y\right)$
  end
end
foreach i *in* $1, \ldots, N_{\mathcal{R}}$ **do** $p_i \leftarrow \frac{1}{(n_{\mathcal{R}^c}+N_{\mathcal{R}})}$;
if $n_{\mathcal{R}^c} > 0$ **then**
  $p_{n_{\mathcal{R}^c}+1} \leftarrow \frac{n_{\mathcal{R}^c}}{n_{\mathcal{R}^c}+\mathcal{N}_{\mathcal{R}}}$;
end
else
  Sample new point y;
  $n_{\mathcal{R}^c} \leftarrow 1$; $y_{N_{\mathcal{R}}+1} \leftarrow y$;
end
$p_{N_{\mathcal{R}}+1} \leftarrow \frac{n_{\mathcal{R}^c}}{n_{\mathcal{R}^c}+N_{\mathcal{R}}}$

Algorithm 1: Aggregation sampling

Aggregation sampling and aggregation reduction can be thought of as equivalent to sampling from the aggregated random vector from Definition 2.12 for large sample sizes. Aggregation sampling and aggregation reduction are thus consistent with sampling only if $\mathcal{R}$ satisfies the aggregation condition and $\mathbb{E}\left[Y|Y \in \mathcal{R}^c\right] \in \mathcal{R}^c$. The precise conditions required for consistency will be stated and proved in Section 4.

We now study the performance of our methodology. Let q be the probability of the aggregation region, and n the desired number of risk scenarios. Let $N(n)$ denote the *effective sample size* for aggregation sampling, that is, the number of samples drawn until the algorithm terminates[3]. The aggregation sampling algorithm can be viewed as a sequence of Bernoulli trials where a trial is a success if the corresponding sample lies in the aggregation region, and which terminates once we have reached n failures, that is, once we have sampled n scenarios from the risk region. We can therefore write down the distribution of $N(n)$:

$$N(n) \sim n + \mathcal{NB}(n, q),$$

where $\mathcal{NB}(N, q)$ denotes a *negative binomial* random variable whose probability mass function is as follows:

$$\binom{k+n-1}{k}(1-q)^n q^k, \qquad k \geq 0.$$

The expected effective sample size of aggregation sampling is thus:

$$\mathbb{E}\left[N(n)\right] = n + n\frac{q}{1-q} \tag{12}$$

Let $R(n)$ denote the number of scenarios which are aggregated in the aggregation reduction method. Aggregation reduction can similarly be viewed as a sequence of n Bernoulli trials, where success and failure are defined in the same way as described above. The number of aggregated scenarios in aggregation reduction is therefore distributed as follows:

$$R(n) \sim \mathcal{B}(n, q)$$

where $\mathcal{B}(n, q)$ denotes a binomial random variable and so we have

$$\mathbb{E}\left[R(n)\right] = nq. \tag{13}$$

From (12) and (13) we can see that for both aggregation sampling and aggregation reduction the effectiveness of the method improves as the probability of the aggregation region q increases. In particular, given the properties of risk regions in (5) and (6), we can expect the performance of our methods to improve as the level of the tail risk measure increases and as the problem becomes more constrained.

3.2 Alternative approaches

The above algorithms and analyses assume that the samples of Y were identically, indepenedently distributed. However, in principle the algorithms will work for any unbiased sequence of samples. This opens up the possibility of enhancing the scenario aggregation and reduction algorithms by using them in conjuction with variance reduction techniques such as importance sampling, or antithetic sampling [Hig98][4]. The formulae (12) and (13) will still hold, but q will the probability of a *sample* occuring in the aggregation region rather than the actual probability of the aggregation region itself.

The above algorithms can also be generalized in how they represent the non-risk region. Because aggregation sampling and aggregating reduction only represent the non-risk region with a single scenario, they do not in general preserve the overall expectation of the loss function, or any other statistics of the loss function except for the value of a tail risk measure. These algorithms should therefore generally only be used for problems which only involve tail risk measures. However, if the loss function is affine (in the

[3]For simplicity of exposition we discount the event that the while loop of the algorithm terminates with $n_{\mathcal{R}^c} = 0$ which occurs with probability q^n

[4]Batch sampling methods such as stratified sampling will not work with aggregation sampling which requires samples to be drawn sequentially.

sense of Corollary 2.13), then collapsing all points in the non-risk region to the conditional expectation preserves the overall expectation.

If expectation or any other statistic of the cost function is used in the optimization problem then one could represent the non-risk region region with many scenarios. For example, instead of aggregating all scenarios in the non-risk region into a single point we could apply a clustering algorithm to them such as k-means. The ideal allocation of points between the risk and non-risk regions will be problem dependent and is beyond the scope of this paper.

4 Consistency of aggregation sampling

The reason that aggregation sampling and aggregation reduction work is that, for large sample sizes, they are equivalent to sampling from the aggregated random vector, and if the aggregation condition holds then the aggregated random vector yields the same optimization problem as the original random vector. We only prove consistency for aggregation sampling and not aggregation reduction as the proofs are very similar. Essentially, the only difference is that aggregation sampling has the additional complication of terminating after a random number of samples.

We suppose in this section that we have a sequence of independently identically distributed (i.i.d.) random vectors $Y_1, Y_2, \ldots$ with the same distribution as Y, and which are defined on the product probability space Ω^∞.

4.1 Uniform convergence of empirical β-quantiles

The i.i.d. sequence of random vectors $Y_1, Y_2, \ldots$ can be used to estimate the distribution and quantile functions of Y. We introduce the additional short-hand for the empirical distribution and quantile functions:

$$F_{n,x}(z) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{f(x,Y_i)\le z\}} \quad \text{and} \quad F^{-1}_{n,x}(\beta) := \inf\{z \in \mathbb{R} : \ F_{n,x}(z) \ge \beta\}.$$

Note that these are random-valued functions on the probability space Ω^∞. It is immediate from the strong law of large numbers that for all $\bar{x} \in \mathbb{R}^d$ and $z \in \mathbb{R}$, we have $F_{n,\bar{x}}(z) \overset{\text{w.p.1}}{\to} F_{\bar{x}}(z)$ as $n \to \infty$. In addition, if $F_{\bar{x}}$ is strictly increasing at $z = F^{-1}_{\bar{x}}$ then we also have $F^{-1}_{n,\bar{x}}(\beta) \overset{\text{w.p.1}}{\to} F^{-1}_{\bar{x}}(\beta)$ as $n \to \infty$; see for instance [Ser80][Chapter 2]. The following result extends this pointwise convergence to a convergence which is uniform with respect to $x \in \mathcal{X}$.

Theorem 4.1. Suppose the following hold:

(i) For each $x \in \mathcal{X}$, F_x is strictly increasing and continuous in some neighborhood of $F^{-1}_x(\beta)$

(ii) For all $\bar{x} \in \mathcal{X}$ the mapping $x \mapsto f(x, Y)$ is continuous at $\bar{x}$ with probability 1.

(iii) $\mathcal{X} \subset \mathbb{R}^k$ is compact

then $F^{-1}_{n,x}(\beta) \to F^{-1}_x(\beta)$ uniformly on $\mathcal{X}$ with probability 1.

The proof of this result relies on various continuity properties of the distribution and quantile functions which are provided in Appendix A. Some elements of the proof below have been adapted from [SDR09, Theorem 7.48], a result which concerns the uniform convergence of expectation functions.

Proof. Fix $\epsilon_0 > 0$ and $\bar{x} \in \mathcal{X}$. Since $F_{\bar{x}}$ is continuous in a neighborhood of $F^{-1}_{\bar{x}}(\beta)$, there exists $0 < \epsilon < \epsilon_0$ such $F_{\bar{x}}$ is continuous at $F^{-1}_{\bar{x}}(\beta) \pm \epsilon$. Since $F_{\bar{x}}$ is strictly increasing at $F^{-1}_{\bar{x}}(\beta)$,

$$\delta := \min\{\beta - F_{\bar{x}}\left(F^{-1}_{\bar{x}}(\beta) - \epsilon\right), \ F_{\bar{x}}\left(F^{-1}_{\bar{x}}(\beta) + \epsilon\right) - \beta\} > 0.$$

By Corollary A.2 the mapping $x \mapsto F_x\left(F^{-1}_{\bar{x}}(\beta) - \epsilon\right)$ is continuous at $\bar{x}$ with probability 1. Applying Lemma A.4, there exists a neighborhood W of $\bar{x}$ such that with probability 1, for n large enough

$$\sup_{x \in W \cap \mathcal{X}} \left| F_{n,x}(F^{-1}_{\bar{x}}(\beta) - \epsilon) - F_{n,\bar{x}}(F^{-1}_{\bar{x}}(\beta) - \epsilon) \ \right| < \frac{\delta}{2}.$$

In addition, by the strong law of large numbers, with probability 1, for n large enough

$$\left|F_{n,\bar{x}}\left(F_{\bar{x}}^{-1}(\beta)-\epsilon\right)-F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta)-\epsilon\right)\right|<\frac{\delta}{2}. \tag{14}$$

Thus, for all $x \in W \cap \mathcal{X}$ we have that

$$\left|F_{n,x}\left(F_{\bar{x}}^{-1}(\beta)-\epsilon\right)-F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta)-\epsilon\right)\right|<\delta.$$

Similarly, we we can choose W so that we also have

$$\left|F_{n,x}\left(F_{\bar{x}}^{-1}(\beta)+\epsilon\right)-F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta)+\epsilon\right)\right|<\delta.$$

and so

$$F_{n,x}\left(F_{\bar{x}}^{-1}(\beta)-\epsilon\right)<\beta<F_{n,x}\left(F_{\bar{x}}^{-1}(\beta)+\epsilon\right).$$

Hence, we have that with probability 1, for n large enough

$$\sup_{x\in W\cap\mathcal{X}}\left|F_{n,x}^{-1}(\beta)-F_{\bar{x}}^{-1}(\beta)\right|\leq\epsilon<\epsilon_0. \tag{15}$$

Also, by Proposition A.3 the function $x \mapsto F_x^{-1}(\beta)$ is continuous and so the neighborhood can also be chosen so that

$$\sup_{x\in W\cap\mathcal{X}}\left|F_{\bar{x}}^{-1}(\beta)-F_{x}^{-1}(\beta)\right|<\epsilon_0, \tag{16}$$

and so combining (15) and (16) we have

$$\sup_{x\in W\cap\mathcal{X}}\left|F_{n,x}^{-1}(\beta)-F_{x}^{-1}(\beta)\right|<2\epsilon_0.$$

Finally, since $\mathcal{X}$ is compact, there exists a finite number of points $x_1, \ldots, x_m \in \mathcal{X}$ with corresponding neighborhoods $W_1, \ldots, W_m$ covering $\mathcal{X}$, such that with probability 1, for n large enough the following holds:

$$\sup_{x\in W_j\cap\mathcal{X}}\left|F_{n,x}^{-1}(\beta)-F_{x}^{-1}(\beta)\right|<2\epsilon_0 \qquad \text{for } i=1,\ldots,m$$

that is, with probability 1, for n large enough

$$\sup_{x\in\mathcal{X}}\left|F_{n,x}^{-1}(\beta)-F_{x}^{-1}(\beta)\right|<2\epsilon_0.$$

□

In the next subsection this result will be used to show that any point in the interior of the non-risk region will, with probability 1, be in the non-risk region of the sampled scenario set for a large enough sample size.

4.2 Equivalence of aggregation sampling with sampling from aggregated random vector

The main obstacle in showing that aggregation sampling is equivalent to sampling from the aggregated random vector is to show that the aggregated scenario in the non-risk region converges almost surely to the conditional expectation of the non-risk region as the number of specified risk scenarios tends to infinity. Recall from Section 3 that $N(n)$ denotes the effective sample size in aggregation sampling when we require n risk scenarios and is distributed as $n + \mathcal{NB}(n, q)$ where q is the probability of the non-risk region. The purpose of the next Lemma is to show that as $n \to \infty$ the number of samples drawn from the non-risk region almost surely tends to infinity.

Lemma 4.2. Suppose $M(n) \sim \mathcal{NB}(n, p)$ where $0 < p < 1$. Then with probability 1 we have that $\lim_{n\to\infty} M(n) = \infty$.

Proof. First note that,

$$\{\lim_{n\to\infty} M(n) = \infty\}^c = \bigcup_{k\in\mathbb{N}} \left(\bigcap_{n\in\mathbb{N}} \bigcup_{t>n} \{M(t) > k\}^c \right) = \bigcup_{k\in\mathbb{N}} \limsup_{n\to\infty} \{M(n) \leq k\}.$$

Hence, to show that $\mathbb{P}(\{\lim_{n\to\infty} M(n) = \infty\}) = 1$ it is enough to show for each $k \in \mathbb{N}$ we have that

$$\mathbb{P}\left(\limsup_{n\to\infty} \{M(n) \leq k\} \right) = 0. \tag{17}$$

Now, fix $k \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we have that

$$\mathbb{P}(M(n) = k) = \binom{k+n-1}{k}(1-p)^n\ p^k,$$

and in particular,

$$\mathbb{P}(M(n+1) = k) = \binom{k+n}{k}(1-p)^{n+1}p^k = \frac{k+n}{n}(1-p)\ \mathbb{P}(M(n) = k).$$

For large enough n we have that $\frac{k+n}{n}(1-p) < 1$, hence $\sum_{n=1}^{\infty} \mathbb{P}(M(n) = k) < +\infty$ and so

$$\sum_{n=1}^{\infty} \mathbb{P}(M(n) \leq k) = \sum_{n=1}^{\infty}\sum_{j=1}^{k} \mathbb{P}(M(n) = j) = \sum_{j=1}^{k}\sum_{n=1}^{\infty} \mathbb{P}(M(n) = j) < \infty.$$

The result (17) now holds by the first Borel-Cantelli Lemma [Bil95, Section 4]. □

The next Corollary shows that the strong law of large numbers still applies for the conditional expectation of the non-risk region in aggregation sampling despite the sample size being a random quantity.

Corollary 4.3. Suppose $\mathbb{E}[|Y|] < +\infty$ and $\mathbb{P}(Y \in \mathcal{R}^c) > 0$, then

$$\frac{1}{N(n)-n} \sum_{i\in 1\ldots,N(n):\ Y_i\in\mathcal{R}^c} Y_i \to \mathbb{E}[\ Y|Y \in \mathcal{R}^c\ ] \text{ with probability 1 as } n \to \infty$$

This theorem could be proved by viewing the random variable $\sum_{i\in 1\ldots,N(n):\ Y_i\in\mathcal{R}^c} Y_i \to \mathbb{E}[\ Y|Y \in \mathcal{R}^c\ ]$ as part of an appropriately defined renewal-reward process, and then using standard asymptotic results which apply to these; see [GS01, Chapter 10]. To keep this paper self-contained, we provide an elementary proof.

Proof. Define the following measurable subsets of Ω^∞:

$$\Omega_1 = \{\omega \in \Omega : \lim_{n\to\infty} N(n)(\omega) - n = \infty\},$$
$$\Omega_2 = \{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}} Y_i(\omega) = \mathbb{E}\left[\mathbb{1}_{\{Y\in\mathcal{R}^c\}} Y\right]\},$$
$$\Omega_3 = \{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}} = \mathbb{P}(Y \in \mathcal{R}^c)\}.$$

By the strong law of large numbers Ω_2 and Ω_3 have probability one. Since $N(n) - n \sim \mathcal{NB}(n, q)$, where $q = \mathbb{P}(Y \in \mathcal{R}^c)$, Ω_1 has probability 1 by Lemma 4.2. Therefore, $\Omega_1 \cap \Omega_2 \cap \Omega_3$ has probability 1 and so it is enough to show that for any $\omega \in \Omega_1 \cap \Omega_2 \cap \Omega_3$ we have that

$$\frac{1}{N(n)(\omega)-n} \sum_{i\in 1\ldots,N(n):\ Y_i(\omega)\in\mathcal{R}^c} Y_i(\omega) \to \mathbb{E}[\ Y|Y \in \mathcal{R}^c\ ] \text{ as } n \to \infty.$$

Let $\omega \in \Omega_1 \cap \Omega_2 \cap \Omega_3$. Since $\omega \in \Omega_2 \cap \Omega_3$, we have that as $m \to \infty$:

$$\frac{1}{\frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}}} \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}} Y_i \to \frac{1}{\mathbb{P}\left(Y \in \mathcal{R}^c\right)} \mathbb{E}\left[\mathbb{1}_{\{Y\in\mathcal{R}^c\}} Y\right] = \mathbb{E}\left[Y | Y \in \mathcal{R}^c\right].$$

Now, fix $\epsilon > 0$. Then there exists $N_1(\omega) \in \mathbb{N}$ such

$$m > N_1(\omega) \implies \left| \frac{1}{\frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}}} \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}} Y_i - \mathbb{E}\left[Y | Y \in \mathcal{R}^c\right] \right| < \epsilon.$$

Since $\omega \in \Omega_1$ there exists $N_2(\omega)$ such that

$$n > N_2(\omega) \implies N(n)(\omega) > N_1(\omega).$$

Noting that

$$\begin{aligned}
&\frac{1}{\frac{1}{N(n)(\omega)}\sum_{i=1}^{N(n)(\omega)} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}}} \frac{1}{N(n)(\omega)} \sum_{i=1}^{N(n)(\omega)} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}} Y_i(\omega) \\
&\qquad = \frac{1}{\frac{N(n)(\omega)-n}{N(n)(\omega)}} \frac{1}{N(n)(\omega)} \sum_{i=1}^{N(n)(\omega)} \mathbb{1}_{\{Y_i(\omega)\in\mathcal{R}^c\}} Y_i(\omega) \\
&\qquad = \frac{1}{N(n)(\omega) - n} \sum_{i:Y_i(\omega)\in\mathcal{R}^c} Y_i,
\end{aligned}$$

we have that

$$n > N_2 \implies \left| \frac{1}{N(n)(\omega) - n} \sum_{i:Y_i(\omega)\in\mathcal{R}^c} Y_i(\omega) - \mathbb{E}\left[Y | Y \in \mathcal{R}^c\right] \right| < \epsilon$$

and so $\frac{1}{N(n)(\omega)-n} \sum_{i:Y_i(\omega)\in\mathcal{R}^c} Y_i(\omega) \to \mathbb{E}\left[Y | Y \in \mathcal{R}^c\right]$ as $n \to \infty$. □

To show that aggregation sampling yields solutions consistent with the underlying random vector Y, we show that with probability 1, for n large enough, it is equivalent to sampling from the aggregated random vector $\psi_{\mathcal{R}}(Y)$, as defined in Definition 2.12. If the region $\mathcal{R}$ satisfies the aggregation condition, and $\mathbb{E}\left[Y | Y \in \mathcal{R}^c\right] \in \mathcal{R}^c_{\mathcal{X}}$, Theorem 2.8 tells us that $\rho_\beta\left(f(x, \psi_{\mathcal{R}}(Y))\right) = \rho_\beta\left(f(x, Y)\right)$ for all $x \in \mathcal{X}$. Hence, if sampling is consistent for the risk measure ρ_β, then aggregation sampling also consistent.

Denote by $\tilde{F}_{n,x}, \tilde{F}^{-1}_{n,x}$, the empirical distribution, and quantile functions respectively and by $\tilde{\rho}_{n,\beta}(x)$ the value of the tail-risk measure for the decision $x \in \mathcal{X}$ for the sample from the aggregated random vector: $\psi_{\mathcal{R}}(Y_1), \ldots, \psi_{\mathcal{R}}(Y_n)$. Similarly, denote by $\hat{F}_{n,x}, \hat{F}^{-1}_{n,x}$, and $\hat{\rho}_{n,\beta}$ the analogous functions for the scenario set constructed by aggregation sampling with n risk scenarios. Note that these latter functions will depend on the sample $Y_1, \ldots, Y_{N(n)}$. Note also that like $F_{n,x}$ and $F^{-1}_{n,x}$, all these functions are random and defined on the same sample space Ω^∞.

Theorem 4.4. Suppose the following conditions hold:

(i) $(x, y) \mapsto f(x, y)$ is continuous on $\mathcal{X} \times \mathbb{R}^d$

(ii) For each $x \in \mathcal{X}$, F_x is strictly increasing and continuous in some neighborhood of $F_x^{-1}(\beta)$

(iii) $\mathbb{E}\left[\, Y | Y \in \mathcal{R}^c \,\right] \in \operatorname{int}\left(\mathcal{R}^c_{\mathcal{X}}\right)$

(iv) $\mathcal{X}$ is compact.

Then, with probability 1, for n large enough $\tilde{\rho}_{n,\beta} \equiv \hat{\rho}_{N(n),\beta}$.

Proof. Note that if

$$z > \max\left\{ f\left(x, \frac{1}{N(n)(\omega) - n} \sum_{i\in 1\ldots, N(n)(\omega):\ Y_i(\omega)\in\mathcal{R}^c} Y_i(\omega) \right), f\left(x, \mathbb{E}\left[\, Y | Y \in \mathcal{R}^c \,\right]\right) \right\}$$

then

$$\begin{aligned}\hat{F}_{n,x}(z)(\omega) &= \frac{N(n)(\omega)-n}{N(n)(\omega)} \\ &\quad + \frac{1}{N(n)(\omega)}|\{1 \le i \le N(n)(\omega) \mid f(x, Y_i(\omega)) \le z \text{ and } Y_i(\omega) \in \mathcal{R}\}| \\ &= \tilde{F}_{N(n),x}(z)(\omega).\end{aligned}$$

So if we have

$$\hat{F}^{-1}_{n,x}(\beta)(\omega) > \max\left\{ f\left(x, \frac{1}{N(n)(\omega)-n} \sum_{i\in 1\ldots,N(n)(\omega):\ Y_i(\omega)\in\mathcal{R}^c} Y_i(\omega)\right), f\left(x, \mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\,\right]\right)\right\} \tag{18}$$

then this implies that $\hat{F}^{-1}_{n,x}(u)(\omega) = \tilde{F}^{-1}_{N(n),x}(u)(\omega)$ for all $u \ge \beta$, which in turn implies $\hat{\rho}_{\beta,n}(x)(\omega) = \tilde{\rho}_{\beta,N(n)}(x)(\omega)$. Hence, it is enough to show that with probability 1, for sufficiently large n, the inequality (18) holds for all $x \in \mathcal{X}$.

Since $\mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\,\right] \in \operatorname{int}\left(\mathcal{R}^c_{\mathcal{X}}\right)$ we have that $f(x, \mathbb{E}\left[Y|Y\in\mathcal{R}^c\right]) < F_x^{-1}(\beta)$ for all $x \in \mathcal{X}$ and since $\mathcal{X}$ is compact there exists $\delta > 0$ such that

$$\sup_{x\in\mathcal{X}} \left(F_x^{-1}(\beta) - f\left(x, \mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\,\right]\right)\right) > \delta. \tag{19}$$

The continuity of $f(x,y)$ and again the compactness of $\mathcal{X}$ implies that there exists $\gamma > 0$ such that

$$|y - \mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\,\right]| < \gamma \implies \sup_{x\in\mathcal{X}} |f(x,y) - f(x, \mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\right]) \,| < \frac{\delta}{2}$$

Thus, by Corollary 4.3, with probability 1, for n large enough

$$\left| f\left(x, \frac{1}{N(n)-n} \sum_{i\in 1\ldots,N(n):\ Y_i\in\mathcal{R}^c} Y_i\right) - f\left(x, \mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\right]\right) \right| < \frac{\delta}{2}. \tag{20}$$

Also, by Theorem 4.1, given $N(n) > n$, for n large enough

$$\sup_{x\in\mathcal{X}} \left|F_x^{-1}(\beta) - \tilde{F}^{-1}_{N(n),x}(\beta)\right| < \frac{\delta}{2}, \tag{21}$$

which implies for all $x \in \mathcal{X}$

$$\begin{aligned}\tilde{F}^{-1}_{N(n),x}(\beta) - f&\left(x, \frac{1}{N(n)-n} \sum_{i\in 1\ldots,N(n):\ Y_i\in\mathcal{R}^c} Y_i\right) \\ &\ge \left(F_x^{-1}(\beta) - \frac{\delta}{2}\right) - \left(f\left(x, \mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\,\right]\right) + \frac{\delta}{2}\right) \qquad \text{by (20) and (21)} \\ &= \underbrace{\left(F_x^{-1}(\beta) - f\left(x, \mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\,\right]\right)\right)}_{>\delta \text{ by (19)}} - \delta > 0.\end{aligned}$$

Similarly with probability 1 for n large enough we have $\tilde{F}^{-1}_{N(n),x}(\beta) > f\left(x, \mathbb{E}\left[\, Y|Y\in\mathcal{R}^c\,\right]\right)$ for all $x \in \mathcal{X}$. Therefore the inequality (18) holds with probability 1 for sufficiently large n as required. □

5 A conservative risk region for monotonic loss functions

In order to use risk regions for scenario generation, we need to have characterisation which conveniently allows us to test whether or not a point belongs to it. In general this is a difficult as the risk region depends on the loss function, the distribution and the problem constraints. Therefore, as a proof-of-

concept, in the following two sections we derive risk regions for two classes of problems. In this section we propose a conservative risk region for problems which have monotonic loss functions.

Definition 5.1 (Monotonic loss function)**.** We call a loss function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ monotonic increasing if for all $x \in \mathcal{X}$ we have $y < \tilde{y} \Rightarrow f(x, y) < f(x, \tilde{y})$ where the inequality holds element-wise. Similarly, we say it is monotonic decreasing if for all $x \in \mathcal{X}$ we have $y < \tilde{y} \Rightarrow f(x, y) > f(x, \tilde{y})$.

Without loss of generality we shall conduct our analysis for loss functions which are monotonic increasing and simply refer to these as monotonic. Monotonic loss functions occur naturally in stochastic linear programming. Recall, that in stochastic linear programs, the loss function is decomposed into a deterministic first-stage cost, and a second stage recourse cost which depends on the first-stage decision and the outcome of the random vector. The general form of this recourse function is as follows:

$$Q(x, Y(\omega)) = \min\{q(\omega)^T z : \ W(\omega)z + T(\omega)x = h(\omega)^T, \ z \geq 0\} \tag{22}$$

where $Y(\omega) = (q(\omega), W(\omega), T(\omega), h(\omega))$. The recourse function is said to be *relatively complete* if for all $x \in \mathcal{X}$ there is a feasible solution z to the minimization in (22). The following result gives an example of a class of recourse functions in which are monotonic.

Proposition 5.2. Suppose $\mathcal{X} \subset \mathbb{R}^n_+$ and we have a relatively complete recourse function defined as follows:

$$Q(x, y) = \min\{q^T z : Wz \geq y, \ Tz \leq Vx, \ z \geq 0\} \tag{23}$$

where $q \in \mathbb{R}^p$ has strictly positive elements, and the matrices $W \in \mathbb{R}^{d \times p}$, $T \in \mathbb{R}^{r \times p}$, and $V \in \mathbb{R}^{r \times n}$ all have non-negative elements. Then, Q is a monotonic loss function.

Proof. Fix $x \in \mathcal{X}$. By taking the dual of the linear program (23), we can rewrite the recourse function as follows:

$$Q(x, y) = \max\{y^T \pi - x^T V^T \nu : W^T \pi - T^T \nu \leq q, \ \pi, \nu \geq 0\}. \tag{24}$$

Note that since all elements of q are strictly positive and $x^T V^T \nu \geq 0$, an optimal value for the dual variable π to the maximization in (24) will have at least one strictly positive element. Now, suppose we have two vectors $\bar{y}, \tilde{y} \in \mathbb{R}^d$ such that $\bar{y} < \tilde{y}$, and and that $(\bar{\pi}, \bar{\nu})$ is an optimal solution to the dual problem corresponding to $y = \bar{y}$. Then, $(\bar{\pi}, \bar{\nu})$ is also a feasible solution to the dual problem corresponding to $y = \tilde{y}$. Hence,

$$\begin{aligned} Q(x, \bar{y}) &= \bar{y}^T \bar{\pi} - x^T V^T \bar{\nu} \\ &< \tilde{y}^T \bar{\pi} - x^T V^T \bar{\nu} \\ &\leq Q(x, \tilde{y}). \end{aligned}$$

□

This problem class arises in network design. In this context the first-stage decision x may represent how much capacity to give nodes or arcs in a transportation network. They may also represent binary decisions on whether these nodes or arcs can be used at all. The second stage decision z then may represent the flow of the commodity through the network, and the random vector Y the demand which must be met.

For a problem with a monotonic loss function, the following result defines a conservative risk region.

Theorem 5.3. Suppose the loss function $f : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}$ is monotonic, then the following set is a conservative risk region:

$$\mathcal{R} = \{y \in \mathbb{R}^d : \mathbb{P}(Y > y) \leq 1 - \beta\}. \tag{25}$$

Proof. To show that $\mathcal{R}$ is a risk region, we show that for all $y \in \mathcal{R}^C$ and $x \in \mathcal{X}$ we have have that

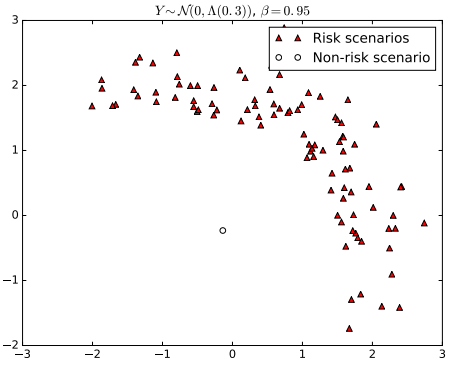
$F_x(f(x,y)) < \beta$. Fix $y \in \mathcal{R}^C$ and $x \in \mathcal{X}$, then

$$\begin{aligned} F_x(f(x,y)) = \mathbb{P}\left(f(x,Y) \leq f(x,y)\right) &= 1 - \mathbb{P}\left(f(x,Y) > f(x,y)\right) \\ &\leq 1 - \mathbb{P}\left(Y > y\right) \qquad \text{by monotonicity} \\ &< 1 - (1-\beta) = \beta. \end{aligned}$$

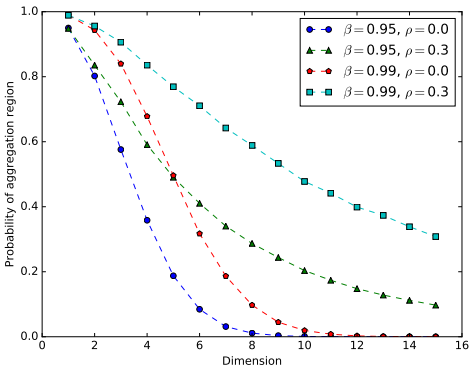
□

The above theorem says that a point lies in the β-risk region if the survivor function at that point is less than or equal to $(1-\beta)$. In Figure 1a is plotted a scenario set constructed by aggregation sampling for this risk region where Y follows a multivariate Normal distribution.

Recall that the efficacy of our scenario generation algorithms depends on the probability of the aggregation region. In Figure 1b we have plotted these probabilities for a range of dimensions for multivariate Normal distributions with zero mean and covariance matrix $\Lambda(\rho) \in \mathbb{R}^{d\times d}$ where $\Lambda(\rho)_{ij} = \rho$ for $i \neq j$ and $\rho > 0$, and $\Lambda(\rho)_{ij} = 1$. In this case the probability of the aggregation region does not have a convenient analytic expression, and so these calculations have been done via Monte Carlo simulation. The results show that these probabilities tend to zero as the dimension of the distribution increases. In the case where the marginal distributions are independent ($\rho = 0$), this probability is already negligible for $d = 10$ for both $\beta = 0.95$ and $\beta = 0.99$. However, in the presence of modest positive correlations ($\rho = 0.3$) this convergence is much slow, meaning our scenario generation method would still be useful in higher dimensions.



(a) Scenario set constructed by aggregation sampling



(b) Probability of non-risk region

Figure 1: Non-risk region monotonic loss function and multivariate Normal distribution

6 An exact risk region for the portfolio selection problem

In this section, we characterize exactly the risk region of the portfolio selection problem when the asset returns are elliptically distributed. Because this problem, under some conditions, can be solved exactly through various methods, we use it to test the performance of our scenario generation methodology in Section 7.

In the portfolio selection problem, one aims to choose a portfolio of financial assets with uncertain returns. For $i = 1, \ldots, d$, let x_i denote the amount to invest in asset i, and Y_i the random return of asset i. The loss function in this problem is the negative total return, that is $f(x,Y) = \sum_{i=1}^{d} -x_i Y_i = -x^T Y$. The set $\mathcal{X} \in \mathbb{R}^d$ of feasible portfolios may encompass constraints like no short-selling ($x \geq 0$), total investment ($\sum_{i=1}^{d} x_i = 1$) and quotas on certain stocks or combinations of stocks ($x \leq c$).

For a given portfolio $x \in \mathcal{X}$, the corresponding risk region is the half-space of points where loss is greater than or equal to the β-quantile $\mathcal{R}_x = \{y \in \mathbb{R}^d : \ -x^T y \geq F_x^{-1}(\beta)\}$. The following corollary gives sufficient conditions for the risk region to satisfy the aggregation condition for continuous distributions.

Corollary 6.1. Suppose that $\mathcal{Y} = \mathbb{R}^d$ and there exist $x_1, x_2 \in \mathcal{X}$ which are linearly independent. Then, for any $\mathcal{R} \supseteq \mathcal{R}_{\mathcal{X}}$, $\mathcal{R}$ satisfies the aggregation condition. Moreover, if Y is continuous and $\mathcal{X}$ is compact, then aggregation sampling with respect to $\mathcal{R}$ is consistent in the sense of Theorem 4.4.

Proof. For the first part of this result, it is enough to show that $\mathcal{R}_{\mathcal{X}}$ satisfies the aggregation condition. We prove this by showing that all the conditions of Proposition 2.10 hold. Note that $x \mapsto -x^T y$ is continuous so condition (ii) holds immediately.

For all $x \in \mathcal{X}$ the interior of the corresponding risk region and non-risk region are open half-spaces:

$$\operatorname{int}(\mathcal{R}_x) = \{y \in \mathbb{R}^d : -x^T y > F_x^{-1}(\beta)\} \quad \text{and} \quad \operatorname{int}(\mathcal{R}_x^c) = \{y \in \mathbb{R}^d : -x^T y < F_x^{-1}(\beta)\}.$$

Fix $\bar{x} \in \mathcal{X}$. Then either $\bar{x}$ is linearly independent to x_1 or it is linearly independent to x_2. Assume it is linearly independent to x_1. Now, $\operatorname{int}(\mathcal{R}_{\bar{x}})$ and $\operatorname{int}(\mathcal{R}_{x_1})$ are non-parallel half-spaces and so both $\operatorname{int}(\mathcal{R}_{\bar{x}} \cap \mathcal{R}_{x_1})$ and $\operatorname{int}(\mathcal{R}_{x_1} \setminus \mathcal{R}_{\bar{x}}) = \operatorname{int}(\mathcal{R}_{x_1}) \cap \operatorname{int}(\mathcal{R}_{\bar{x}}^c)$ are non-empty so condition (iii) is satisfied.

Since $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$ are non-parallel half-spaces, their union $\mathcal{R}_{x_1} \cup \mathcal{R}_{x_2}$ is connected. Similarly, for any $x \in \mathcal{X}$, we must have $\mathcal{R}_x$ being non-parallel with either $\mathcal{R}_{x_1}$ or $\mathcal{R}_{x_2}$ and so $\mathcal{R}_x \cup \mathcal{R}_{x_1} \cup \mathcal{R}_{x_2}$ must also be connected. Hence, $\mathcal{R}_{\mathcal{X}} = \bigcup_{x \in \mathcal{X}} (\mathcal{R}_x \cup \mathcal{R}_{x_1} \cup \mathcal{R}_{x_2})$ is connected so condition (i) is also satisfied.

It now remains to show that aggregation sampling is consistent in the sense of Theorem 4.4. Conditions (i) and (iv) of this theorem hold trivially. By Proposition 2.11 $\mathcal{R}_{\mathcal{X}}$ is convex and condition (iii) holds. Since Y is continuous and has support $\mathcal{Y} = \mathbb{R}^d$, Y has a density f such that $f(y) > 0$ for all $y \in \mathbb{R}^d$. Hence, for all $x \in \mathcal{X}$, the function F_x is continuous and increasing everywhere so condition (ii) holds. □

By exploiting the structure of a parametric distribution, it may be possible to characterize its associated risk region in a more convenient manner. We now do this for elliptically distributed returns. Elliptical distributions are a general class of distributions which include among others the multivariate Normal and multivariate t-distributions. See [FKN89] for a full overview of the subject.

Definition 6.2 (Spherical and Elliptical Distributions)**.** Let X be a random vector in $\mathbb{R}^d$, then X is said to be *spherical* if its distribution is invariant under orthonormal transformations; that is, if

$$X \sim UX \qquad \text{for all } U \in \mathbb{R}^{d \times d} \text{ orthonormal.}$$

Let Y be a random vector in $\mathbb{R}^d$, then Y is said to be *elliptical* if it can be written $Y = PX + \mu$ where $P \in \mathbb{R}^{d \times d}$ is non-singular, $\mu \in \mathbb{R}^d$, and X is random vector with spherical distribution. We will denote this $Y \sim \text{Elliptical}(X, P, \mu)$.

We will assume throughout that $Y \sim \text{Elliptical}(X, P, \mu)$, is continuous and $\mathcal{Y} = \mathbb{R}^d$ so that we can apply Corollary 6.1. An important property of elliptical distributions is that for any $x \in \mathbb{R}^d$ we can characterize exactly the distribution of $x^T Y$:

$$x^T Y \sim \|Px\| X_1 + x^T \mu, \tag{26}$$

where X_1 is the first component of the random vector X, and $\|\cdot\|$ denotes the standard Euclidean norm. This property allows us to solve some portfolio selection problems for elliptical distributions where the risk measure is positive homogeneous and translation invariant via quadratic programming or interior point algorithms. Such risk measures include the β-VaR, β-CVaR and all coherent risk measures [ADEH99]. For more details see [KBF07]. By (26) the β-quantile of the loss of a portfolio is as follows:

$$F_x^{-1}(\beta) = \|Px\| F_{X_1}^{-1}(\beta) - x^T \mu.$$

Therefore, using (4), the exact non-risk region for $Y \sim \text{Elliptical}(X, P, \mu)$, is as follows:

$$\{y \in \mathbb{R}^d : -x^T y \leq \|Px\| F_{X_1}^{-1}(\beta) - x^T \mu \qquad \forall x \in \mathcal{X}\}. \tag{27}$$

This characterization is not practical for testing whether or not a point belongs to the risk region, which is required for our scenario generation algorithms. However, a more convenient form is available in the

case where $\mathcal{X} \subset \mathbb{R}^d$ is convex. Before stating the result, we recall the required concept of a projection onto a convex set.

Definition 6.3 (Projection)**.** Let $C \subset \mathbb{R}^d$ be a closed convex set. Then for any point $y \in \mathbb{R}^d$, we define the projection of y onto C to be the unique point $p_C(y) \in C$ such that $\inf_{x \in C} \|x - y\| = \|p_C(y) - y\|$.

By a slight abuse of notation, for a set $\mathcal{A} \subset \mathbb{R}^d$ and a matrix $T \in \mathbb{R}^{d \times d}$, we write $T(\mathcal{A}) := \{Ty : y \in \mathcal{A}\}$. Finally, recall that the conic hull of a set $\mathcal{A} \subset \mathbb{R}^d$, which we denote $\text{conic}(\mathcal{A})$, is the smallest convex cone containing $\mathcal{A}$.

Theorem 6.4. Suppose $\mathcal{X} \subset \mathbb{R}^d$ is a convex set, then the exact non-risk region in (27) can be written as follows:

$$\mathcal{R}^c_{\mathcal{X}} = P^T \left(\{\tilde{y} : \|p_{K'}(\tilde{y} - \mu)\| \leq F^{-1}_{X_1}(\beta)\} \right) \tag{28}$$

where $K' = PK$ and $K = \text{conic}(\mathcal{X})$.

Proof. The result follows directly from Corollary B.5 in Appendix B by taking $\alpha = F^{-1}_{X_1}(\beta)$. □

In general, to test whether a point belongs to the risk region we need to calculate the projections of points onto a cone. For details on how this is done see [Ujv07, FTW15]. However, if we take $\mathcal{X} = \mathbb{R}^d$, then the set (28) can be easily shown to simplify to an ellipsoid:

$$\mathcal{R}^c_{\mathbb{R}^d} = \{y \in \mathbb{R}^d : (y - \mu)^T \Sigma^{-1} (y - \mu) \leq F^{-1}_{X_1}(\beta)^2\}. \tag{29}$$

where $\Sigma = P^T P$. Note that by (6) we have $\mathcal{R}_{\mathcal{X}} \subset \mathcal{R}_{\mathbb{R}^d}$ and so $\mathcal{R}_{\mathbb{R}^d}$ is always a risk region for this problem.

As discussed in Section 3 on scenario generation, the greater the probability of the non-risk region, the greater the benefit of our methodology over regular sampling. To gauge the utility of our methodology we calculate the probability of the region (29) for the Normal distribution. If $Y \sim \mathcal{N}(\mu, \Sigma)$ this can be calculated exactly:

$$\begin{aligned} \mathbb{P}\left((Y - \mu)^T \Sigma^{-1} (Y - \mu) \leq \Phi^{-1}(\beta)^2 \right) &= \mathbb{P}\left(X^T X \leq \Phi^{-1}(\beta)^2 \right) \qquad \text{where } Y = PX + \mu \\ &= \mathbb{P}\left(\chi^2_d \leq \Phi^{-1}(\beta)^2 \right), \end{aligned} \tag{30}$$

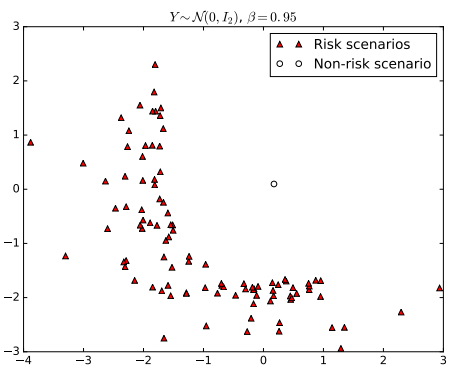
where Φ is the distribution function of the standard Normal distribution. That is, the probability of the non-risk region is invariant to the mean and covariance and can be calculated from a χ^2_d distribution function.

Suppose now that $\mathcal{X} = \{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1,\ x \geq 0\}$, that is, we forbid short-selling. The probability of the risk region is now no longer invariant to the covariance of the risk region and so for simplicity we assume that $Y \sim \mathcal{N}(0, I_d)$ where I_d denotes the $d \times d$ identity matrix. In this case we have $K = \mathbb{R}^d_+$, $K' = \mathbb{R}^d_+$, and $p_{K'}(y) = y_+$ where $y_+ = \max(0, y)$. Hence, the non-risk region can be written as:

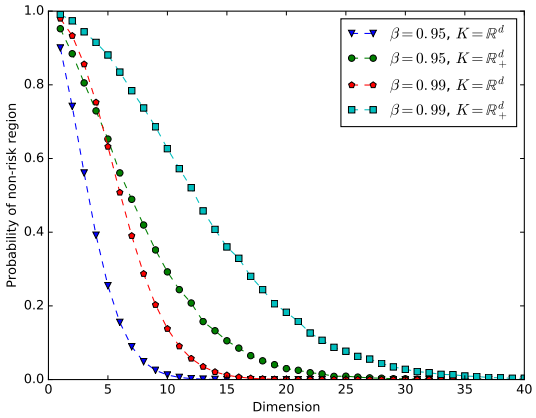
$$\mathcal{R}^c_{\mathcal{X}} = \{y \in \mathbb{R}^d : \|y_+\| \leq \Phi^{-1}(\beta)\}. \tag{31}$$

A scenario set constructed via aggregation sampling with the non-risk region (31) is shown in Figure 2a. In Figure 2 we have plotted the probability of the non-risk region for $Y \sim \mathcal{N}(0, I_d)$ for a range of dimensions, for the cases where $K = \mathbb{R}^d$ and $K = \mathbb{R}^d_+$, and $\beta = 0.95$ and $\beta = 0.99$. For the cases where $K = \mathbb{R}^d$, we can calculate the probabilities directly from the formula (30). For the cases where $K = \mathbb{R}^d_+$, the probability of the set (31) cannot be calculated analytically and so we again resort to Monte Carlo simulation.

The plot again shows that the probability of the non-risk region converges to zero as we increase the dimension of the problem. For the cases where $K = \mathbb{R}^d$, the convergence is so quick that even for relatively small dimensions the probability of the non-risk region is tiny, rendering the benefit of our methodology negligible. For $K = \mathbb{R}^d_+$ the probability of the non-risk region is higher and decays at a much slower rate as the dimension increases. This underlines the importance of making use of our constraints for finding a risk region.



(a) Scenario set constructed by aggregation sampling



(b) Probability of non-risk region

Figure 2: Non-risk region for multivariate Normal random vector

7 Numerical tests

In this section, we test the performance of the aggregation sampling algorithm from Section 3 for the risk regions presented in Sections 5 and 6 against basic sampling. To allow us to directly compare the performance for risk regions of monotonic loss functions and the exact risk regions for the portfolio selection problem, we use a portfolio selection problem with constraints to preclude short-selling. In this case, the loss function is monotonic decreasing.

7.1 Experimental Set-up

For our numerical tests we use the following problem:

$$\begin{aligned} \underset{x \geq 0}{\text{minimize}} \ & \beta\text{-CVaR}(-x^T Y) \\ \text{subject to } & \mathbb{E}\left[x^T Y\right] \geq t. \end{aligned} \tag{P}$$

We will in particular assume that the asset returns follow a Normal distribution, that is $Y \sim \mathcal{N}(\mu, \Sigma)$. We fit these distributions from monthly return data from between January 2007 and December 2016 for randomly selected companies in the FTSE 100 index. We use the target expected return $t = 0.01$ which is feasible for the constructed problems.

This problem has been constructed so that we can easily calculate exactly the optimality gap of any candidate portfolio $x \geq 0$. The following formula is easily verified by recalling that for continuous probability distributions, the β-CVaR is just the conditional expectation of the random variable above the β-quantile (see [RU00] for instance):

$$\beta\text{-CVaR}(-x^T Y) = (1-\beta)\mu^T x + \sqrt{x^T \Sigma x} \int_{\Phi^{-1}(\beta)}^{\infty} z \, d\Phi(z) \tag{32}$$

where Φ denotes the distribution function of the standard Normal distribution. The problem (P) can therefore be solved exactly using an interior point algorithm.

We are interested in the quality and stability of the solutions that are yielded by our method as compared to sampling. To this end, in each experiment, for a range of scenario set sizes, we construct 50 scenario sets using sampling and aggregation sampling, solve the resulting problems, and calculate the optimality gaps for the solutions that these yield. If any constructed scenario set has an expected return which makes the problem infeasible, we discard it and sample a new set.

7.2 Results

In Figure 3 are presented the results of these stability tests for two different problems. In the first problem we have $d = 5$ and $\beta = 0.95$. In the second problem we have $d = 10$ and $\beta = 0.99$. For each scenario set size we have plotted the mean optimality gap with a 95% error bar. In the legend of each plot we have also given the estimated probability of the aggregation regions, q.

In both cases, both aggregation sampling methods outperform basic sampling for all scenario set sizes in terms of mean and standard error of the calculated optimality gaps. This is because for aggregation sampling we are effectively sampling more scenarios compared with basic sampling. Aggregation sampling with the conservative and exact non-risk regions are on a par in terms of mean optimality gap, but use of the exact risk region yields a smaller standard error. The improved performance of aggregation sampling with the exact non-risk region can be expected given that its probability is greater than that of the conservative risk region which gives a greated effective sample size.

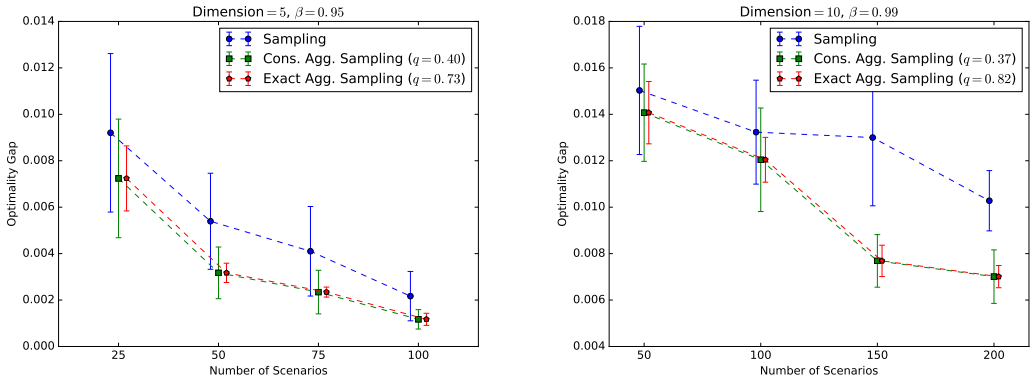


Figure 3: Error bar plot of optimality gaps yielded by sampling and aggregation sampling

8 Conclusions

In this paper we have demonstrated that for stochastic programs which use a tail risk measure, a significant portion of the support of the random variables in the problem do not participate in the calculation of that tail risk measure, whatever feasible decision is used. As a consequence, for scenario-based problems, if we concentrate our scenarios in the region of the distribution which is important to the problem, the risk region, we can represent the uncertainty in our problem in a more parsimonious way, thus reducing the computational burden of solving it.

We have proposed and analyzed two specific methods of scenario generation using risk regions: aggregation sampling and aggregation reduction. Both of these methods were shown to be more effective as the probability of the non-risk region increases: in essence the higher this probability the more redundancy there is in the original distribution. Therefore, our methodology becomes more valuable as as our problem becomes more constrained, and as the level of the tail-risk increases, since these changes cause the probability of the non-risk region to decrease.

The application of our methodology relies on having a convenient characterisation of a risk region. For portfolio selection problems we derived the exact risk region when returns have an elliptical distribution. However, a characterisation of the exact risk region will generally not be possible. Nevertheless, it is suffiecient to have a conservative risk region. For stochastic programs with monotonic loss functions, a wide class which includes some network design problems, we were able to derive such a region.

The effectiveness of our methodology depends on the probability of the aggregation region, that is the exact or conservative non-risk region used in our scenario generation algorithms. We observed that for both the stochastic programs with monotonic loss function and portfolio selection problems that this probability tends to zero as the dimension of the random vector in the problem increases. However, in some circumstances this effect is mitigated. In the case of monotonic loss functions we observed that

small positive correlations slowed down this convergence, and in the portfolio selection problem, the addition of positivity constraints had a similar effect.

We tested the performance of our aggregation sampling algorithm for portfolio selection problems using both the exact non-risk region and the conservative risk region for monotonic loss functions. This demonstrated a significant improvement on the performance of basic sampling, particularly when an exact non-risk region was used.

The methodology has much potential. For some small to moderately-sized network design problems this methodology could yield much better solutions. In particular the methodology is agnostic to the presence of integer variables, and so this methodology could be used to solve difficult integer programs.

In the paper [FTW15] we demonstrate that our methodology may be applied to more difficult and realistic portfolio selection problems such as those involving integer variables, and for which the asset returns are no longer elliptically distributed. In the same paper we also discuss some of the technical issues involved in applying the method, such as finding the conic hull of the feasible region, and methods of projecting points onto this. We also investigate the use of artificial constraints as a way of making our methodology more effective.

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A Continuity of Distribution and Quantile Functions

Throughout we use the following set-up: $\mathcal{X} \subset \mathbb{R}^k$ a decision space, Y a random vector with support $\mathcal{Y} \subset \mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, and a cost function $f : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}$. The quantity is $f(x, Y)$ is assumed to be measurable for all $x \in \mathcal{X}$. In this appendix we prove a series of technical results related to the continuity of the distribution and quantile functions for $f(x, Y)$. These are required for the proofs in Section 4.

The following elementary result concerns the continuity of an expectation function.

Proposition A.1. Suppose for $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, and a given $\bar{x} \in \mathcal{X}$ the following holds:

(i) $x \mapsto g(x, Y)$ is continuous at $\bar{x}$ with probability 1.

(ii) There exists a neighborhood W of $\bar{x}$ and integrable $h : \mathcal{Y} \to \mathbb{R}$ such that for all $x \in W$ we have $g(x, Y) \leq h(Y)$ with probability 1.

Then, $x \mapsto \mathbb{E}\left[g(x, Y)\right]$ is continuous at $\bar{x}$.

Proof. Let $(x_k)_{k=1}^{\infty}$ be some sequence in $\mathcal{X}$ such that $x_k \to \bar{x}$ as $k \to \infty$. Without loss of generality $x_k \in W$ for all $k \in \mathbb{N}$. By assumption (i), almost surely we have $g(x_k, Y) \to f(\bar{x}, Y)$ as $k \to \infty$. Using assumption (ii) we can apply the Lebesgue theorem of dominated convergence so that:

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}\left[g(x_k, Y)\right] &= \mathbb{E}\left[\lim_{k\to\infty} g(x_k, Y)\right] \\ &= \mathbb{E}\left[g(\bar{x}, Y)\right]\end{aligned}$$

and hence $x \mapsto \mathbb{E}\left[g(x, Y)\right]$ is continuous at $\bar{x}$. □

The continuity of the distribution function immediately follows from the above proposition.

Corollary A.2. Suppose for a given $\bar{x} \in \mathcal{X}$ that $x \mapsto f(x, Y)$ is continuous with probability 1 at $\bar{x}$, and for $z \in \mathbb{R}$ the distribution function $F_{\bar{x}}$ is continuous at z. Then, $x \mapsto F_x(z)$ is continuous at $\bar{x}$.

Proof. Let $g(x, Y) = \mathbb{1}_{\{f(x,Y)\leq z\}}$ so that $F_x(z) = \mathbb{E}\left[g(x, Y)\right]$. The function $g(x, Y)$ is clearly dominated by the integrable function $h(Y) = 1$. It is therefore enough to show that $x \mapsto g(x, Y)$ is almost surely continuous at $\bar{x}$ as the result will then follow from Proposition A.1.

Since $F_{\bar{x}}$ is continuous at z, we must have $\mathbb{P}\left(f(\bar{x}, Y) = z\right) = 0$. Almost surely, we have that for $\omega \in \Omega$ that $x \mapsto f(x, Y(\omega))$ is continuous at $\bar{x}$. Let's first assume that $f(\bar{x}, Y(\omega)) > z$. In this case, there exist some neighborhood V of $\bar{x}$ such that $x \in V \Rightarrow f(x, Y(\omega)) > z$, which in turn implies $|g(x, Y) - g(\bar{x}, Y)| = 0$. Hence $x \mapsto g(x, Y(\omega))$ is continuous at $\bar{x}$. The same argument holds if $f(\bar{x}, Y(\omega)) < z$. Hence, with probability 1, $x \mapsto g(x, Y)$ is continuous at $\bar{x}$. □

Continuity of the quantile function follows from the continuity of the distribution function but requires that the distribution function is strictly increasing at the required quantile.

Proposition A.3. Suppose for some $\bar{x} \in \mathcal{X}$, and $z = F_{\bar{x}}^{-1}(\beta)$ that the conditions of Corollary A.2 hold, and in addition that $F_{\bar{x}}$ is strictly increasing at $F_{\bar{x}}^{-1}(\beta)$, that is for all $\epsilon > 0$

$$F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta) - \epsilon\right) < \beta < F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta) + \epsilon\right).$$

Then $x \mapsto F_x^{-1}(\beta)$ is continuous at $\bar{x}$.

Proof. Assume $x \mapsto F_x^{-1}(\beta)$ is not continuous at $\bar{x}$. This means there exists $\epsilon > 0$ such that for all neighborhoods W of $\bar{x}$

$$\text{there exists } x' \in W \text{ such that } \left|F_{\bar{x}}^{-1}(\beta) - F_{x'}^{-1}(\beta)\right| > \epsilon.$$

Now set,

$$\gamma := \min\{\beta - F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta) - \epsilon\right), F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta) + \epsilon\right) - \beta\} > 0 \qquad \text{since } F_{\bar{x}} \text{ strictly increasing at } F_{\bar{x}}^{-1}(\beta).$$

By the continuity of $x \mapsto F_x\left(F_{\bar{x}}^{-1}(\beta)\right)$ at $\bar{x}$ there exists W a neighborhood of $\bar{x}$, such that:

$$x \in W \Longrightarrow \left|F_x\left(F_{\bar{x}}^{-1}(\beta)\right) - F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta)\right)\right| < \gamma. \tag{33}$$

But for the x' identified above That is,

$$F_{x'}^{-1}(\beta) < F_{\bar{x}}^{-1}(\beta) - \epsilon \qquad \text{or} \qquad F_{x'}^{-1}(\beta) > F_{\bar{x}}^{-1}(\beta) + \epsilon$$

and so given that $F_{\bar{x}}$ is non-decreasing, and by the definition of γ we must have:

$$\left|F_{\bar{x}}\left(F_{\bar{x}}^{-1}(\beta)\right) - F_{\bar{x}}\left(F_{x'}^{-1}(\beta)\right)\right| \geq \gamma$$

which contradicts (33). □

Recall, that for a sequence of i.i.d. random vectors $Y_1, Y_2, \ldots$ with the same distribution as Y, we define the sampled distribution function as follows:

$$F_{n,x}(z) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{f(x,Y_i)\leq z\}}.$$

The final result concerns the continuity of the sampled distribution function.

Lemma A.4. Suppose for $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, and $\bar{x} \in \mathcal{X}$ the conditions from Proposition A.1 hold. Then for all $\epsilon > 0$ there exists a neighborhood W, of $\bar{x}$, such that with probability 1

$$\limsup_{n\to\infty} \sup_{x\in W\cap\mathcal{X}} \left|\frac{1}{n}\sum_{i=1}^{n} g(x,Y_i) - \frac{1}{n}\sum_{i=1}^{n} g(\bar{x},Y_i)\right| < \epsilon$$

In particular, if $x \mapsto f(x,Y)$ is continuous at $\bar{x}$ with probability 1 and $F_{\bar{x}}$ is continuous at $z \in \mathbb{R}$ then for all $\epsilon > 0$ there exists a neighborhood W, of $\bar{x}$ such that with probability 1

$$\limsup_{n\to\infty} \sup_{x\in W\cap\mathcal{X}} |F_{n,x}(z) - F_{n,\bar{x}}(z)\ | < \epsilon. \tag{34}$$

Proof. Fix $\bar{x} \in \mathcal{X}$, and $\epsilon > 0$. Let $(\gamma_k)_{k=1}^{\infty}$ be any sequence of positive numbers converging to zero and define

$$V_k := \{x \in \mathcal{X} : \|x - \bar{x}\| \leq \gamma_k\},$$
$$\delta_k(Y) := \sup_{x\in V_k} |g(x,Y) - g(\bar{x},Y)\ |.$$

Note first that the quantity $\delta_k(Y)$ is Lebesgue measurable (see [SDR09, Theorem 7.37] for instance). By assumption (1) the mapping $x \mapsto g(x,Y)$ is continuous at $\bar{x}$ with probability 1, hence $\delta_k(Y) \to 0$ almost surely as $k \to \infty$. Now, since $|g(x,Y)| \leq h(Y)$ we must have $|\delta_k(Y)| \leq 2h(Y)$, therefore, by the Lebesgue dominated convergence theorem, we have that

$$\lim_{k\to\infty} \mathbb{E}\left[\delta_k(Y)\right] = \mathbb{E}\left[\lim_{k\to\infty} \delta_k(Y)\right] = 0. \tag{35}$$

Note also that

$$\sup_{x\in V_k}\left|\frac{1}{n}\sum_{i=1}^{n}g(x,Y_i)-\frac{1}{n}\sum_{i=1}^{n}g(\bar{x},Y_i)\right|\leq\frac{1}{n}\sum_{i=1}^{n}\sup_{x\in V_k}|g(x,Y_i)-g(\bar{x},Y_i)|$$

and so

$$\sup_{x\in V_k}\left|\frac{1}{n}\sum_{i=1}^{n}g(x,Y_i)-\frac{1}{n}\sum_{i=1}^{n}g(\bar{x},Y_i)\right|\leq\frac{1}{n}\sum_{i=1}^{n}\delta_k(Y_i).$$

Since the sequence of random vectors $Y_1, Y_2, \ldots$ is i.i.d. we have by the strong law of large numbers that the right-hand side of (36) converges with probability 1 to $\mathbb{E}\left[\delta_k(Y)\right]$ as $n\to\infty$. Hence, with probability 1

$$\limsup_{n\to\infty}\sup_{x\in V_k}\left|\frac{1}{n}\sum_{i=1}^{n}g(x,Y_i)-\frac{1}{n}\sum_{i=1}^{n}g(\bar{x},Y_i)\right|\leq\mathbb{E}\left[\delta_k(Y)\right]. \tag{36}$$

By (35) we can pick $k\in\mathbb{N}$ such that $\mathbb{E}\left[\delta_k(Y)\right]<\epsilon$ and so setting $W=V_k$ we have by (36) with probability 1

$$\limsup_{n\to\infty}\sup_{x\in W\cap\mathcal{X}}\left|\frac{1}{n}\sum_{i=1}^{n}g(x,Y_i)-\frac{1}{n}\sum_{i=1}^{n}g(\bar{x},Y_i)\right|<\epsilon.$$

The result (34) follows immediately as the special case $g(x,Y)=\mathbb{1}_{\{f(x,Y)\leq z\}}$. □

B Convex cone results

The results in this appendix relate to the characterization of the non-risk region for the portfolio selection problem with elliptically distributed returns. This first result allows for an exact characterization of this region for the unconstrained portfolio selection problem.

Proposition B.1. Suppose $\alpha>0, \mu\in\mathbb{R}^d$ and $P\in\mathbb{R}^{d\times d}$. Then, for all $y\in\mathbb{R}^d$:

$$\left(y^T-\mu\right)\Sigma^{-1}\left(y-\mu\right)\leq\alpha^2\Longleftrightarrow x^T\left(y-\mu\right)\leq\|Px\|\ \alpha\qquad\forall x\in\mathbb{R}^d, \tag{37}$$

where $\Sigma=P^TP$.

Proof. Assume without loss of generality that $\mu=0$. So we have to prove:

$$y^T\Sigma^{-1}y\leq\alpha^2\Longleftrightarrow x^Ty\leq\sqrt{x^T\Sigma x}\ \alpha\ \forall x\in\mathbb{R}^d. \tag{38}$$

We first prove the forward implication. We do this by proving the converse, that is, we suppose for some $y\in\mathbb{R}^d$ that there exists $\tilde{x}\in\mathbb{R}^d$ such that $\tilde{x}^Ty>\|P\tilde{x}\|\ \alpha$. First, set $y_0=\frac{\Sigma\tilde{x}\alpha}{\|P\tilde{x}\|}=\frac{\Sigma\tilde{x}\alpha}{\sqrt{\tilde{x}^T\Sigma\tilde{x}}}$. Now,

$$y_0\Sigma^{-1}y_0=\frac{\tilde{x}^T\Sigma^T\Sigma^{-1}\Sigma\tilde{x}\alpha^2}{\tilde{x}^T\Sigma\tilde{x}}=\alpha^2\ \text{ and }\ \tilde{x}^Ty_0=\tilde{x}^T\frac{\Sigma\tilde{x}\alpha}{\sqrt{\tilde{x}^T\Sigma\tilde{x}}}=\sqrt{\tilde{x}^T\Sigma\tilde{x}}\ \alpha=\|P\tilde{x}\|\ \alpha.$$

That is, y_0 satisfies the inequalities of this Proposition with equality. Note that we also have,

$$(y-y_0)^T\Sigma^{-1}(y-y_0)>0\qquad\text{since }\Sigma^{-1}\text{ is positive definite.}$$

Expanding and rearranging this expression we have,

$$\begin{aligned}
&y^T\Sigma^{-1}y-2y_0^T\Sigma^{-1}y+y_0\Sigma^{-1}\tilde{y_0}>0\\
\Leftrightarrow\quad&y^T\Sigma^{-1}y-2\frac{\alpha}{\sqrt{\tilde{x}^T\Sigma\tilde{x}}}\tilde{x}^Ty+\alpha^2>0\\
\Rightarrow\quad&y\Sigma^{-1}y>\alpha^2\qquad\text{since }\tilde{x}^Ty>\ \|Px\|\ \alpha.
\end{aligned}$$

We now prove the backwards implication. We again do this by proving the converse, in this case,

that if $y^T\Sigma^{-1}y > \alpha^2$ then there exists $\tilde{x} \in \mathbb{R}^d \setminus \{0\}$ such that $\tilde{x}^T y > \sqrt{x^T\Sigma x}\alpha$. Let $\tilde{x} = \Sigma^{-1}y$, then,

$$\tilde{x}^T y = y^T\Sigma^{-1}y = \underbrace{\sqrt{y^T\Sigma^{-1}y}}_{=\sqrt{\tilde{x}^T\Sigma\tilde{x}}}\underbrace{\sqrt{y^T\Sigma^{-1}y}}_{>\alpha} > \sqrt{\tilde{x}^T\Sigma\tilde{x}}\ \alpha = \|Px\|\ \alpha.$$

□

The following two propositions give properties about projections onto convex cones which are required in the proof of the main results of this appendix.

Proposition B.2. Suppose $K \subset \mathbb{R}^d$ is a convex cone, then, for all $y \in \mathbb{R}^d$:

$$p_K(y)^T\left(y - p_K(y)\right) = 0$$

Proof. First note that we must have $p_K(y)^T y \geq 0$. If this is not the the case then

$$\|y - p_K(y)\|^2 = \|p_K(y)\|^2 - 2p_K(y)^T y + \|y\|^2 > \|y\|^2 = \|y - 0\|^2$$

which contradicts the definition of $p_K(y)$ since $0 \in K$. Now assume that $p_K(y)^T\left(y - p_K(y)\right) \neq 0$, and set $\tilde{x} = \frac{p_K(y)^T y}{\|p_K(y)\|^2}p_K(y) \in K$. Now,

$$p_K(y)^T(\tilde{x} - y) = p_K^T y - p_K^T y = 0.$$

By assumption $p_k^T y \neq \|p_K(y))\|^2$, and so $\tilde{x} \neq p_K(y)$, hence

$$\begin{aligned}\|p_K(y) - y\|^2 &= \|(p_K(y) - \tilde{x}) + (\tilde{x} - y)\|^2 \\ &= \|(p_K(y) - \tilde{x})\|^2 - 2\underbrace{(p_K(y) - \tilde{x})^T(\tilde{x} - y)}_{=0} + \|(\tilde{x} - y)\|^2 > \|(\tilde{x} - y)\|^2\end{aligned}$$

which, again, contradictions the definition of $p_K(y)$ since $\tilde{x} \in K$. □

Proposition B.3. Suppose $K \subset \mathbb{R}^d$ be a convex cone and $x \in K$. Then for any $y \in \mathbb{R}^d$

$$x^T y \leq x^T p_K(y).$$

Proof. The result holds trivially if $y \in K$ so we assume $y \notin K$. Assume there exists $\tilde{x} \in K$ such that $\tilde{x}^T y > \tilde{x}^T p_K(y)$. For all $0 \leq \lambda \leq 1$ we have $\lambda x + (1-\lambda)p_K(y) \in K$. Now,

$$\begin{aligned}\|(\lambda\tilde{x} + (1-\lambda)p_K(y)) - y\|^2 - \|y - p_K(y)\|^2 &= \lambda^2\|\tilde{x} - p_K(y)\|^2 + 2\lambda(\tilde{x} - p_K(y))^T(p_K(y) - y) \\ &= \lambda^2\|\tilde{x} - p_K(y)\|^2 - 2\lambda\underbrace{\tilde{x}^T(y - p_K(y))}_{>0 \text{ by assumption}}.\end{aligned}$$

That is, for $0 < \lambda < \frac{\tilde{x}^T(y - p_K(y))}{2\|p_K(y) - \tilde{x}\|}$ we have $\|\lambda\tilde{x} + (1-\lambda)p_K(y) - y\| < \|y - p_K(y)\|$ which contradicts the definition of $p_K(y)$. □

The next two results generalize Proposition B.1 to the case where $x \in \mathbb{R}^d$ is restricted to a convex cone. The first describes the region in the case where $P = I$, and the second generalizes the result to any non-singular matrix. In particular, it is Corollary B.5 that allows us to characterize the non-risk region of portfolio selection problem for a convex feasible region.

Theorem B.4. Suppose $\mathcal{X} \subset \mathbb{R}^d$ is convex, and let $\mathcal{A} := \{y : x^T y \leq \|x\|\ \alpha\ \forall x \in \mathcal{X}\}$ and $\mathcal{B} := \{y : \|p_K(y)\| \leq \alpha\}$ where $K = \text{conic}\,(\mathcal{X})$. Then, $\mathcal{A} = \mathcal{B}$.

Proof. ($\mathcal{B} \subseteq \mathcal{A}$)

Suppose $y \in \mathcal{B}$ and let $x \in \mathcal{X}$, then $x \in K$ and so

$$\begin{aligned} x^T y &\leq x^T p_K(y) \qquad \text{by Proposition B.3} \\ &\leq \|x\| \ \|p_K(y)\| \qquad \text{by the Cauchy-Schwartz inequality} \\ &\leq \|x\| \ \alpha \qquad \text{since } y \in \mathcal{B}. \end{aligned}$$

Hence $y \in \mathcal{A}$.
$(\mathcal{A} \subseteq \mathcal{B})$
Suppose $y \notin \mathcal{B}$ and set $x = p_K(y) \in K$. Now,

$$\begin{aligned} x^T y &= p_K(y)^T y \\ &= p_K(y)^T p_K(y) + p_K(y)^T (y - p_K(y)) \\ &= p_K(y)^T p_K(y) \qquad \text{by Proposition B.2} \\ &\geq \|x\| \ \alpha \qquad \text{since } y \notin \mathcal{B}. \end{aligned}$$

Since $\mathcal{X}$ is convex we have $x = \lambda \bar{x}$ for some $\bar{x} \in \mathcal{X}$ and so we must also have $\bar{x}^T y > \|\bar{x}\| \alpha$, hence $y \notin \mathcal{A}$. □

Corollary B.5. Suppose K is a convex cone, and $P \in \mathbb{R}^{d \times d}$ is a non-singular matrix. Let, $\mathcal{A} := \{y : x^T y \leq \|Px\| \ \alpha \ \forall x \in K\}$ and $\mathcal{B} := P^T \left(\{\tilde{y} : \|p_{K'}(\tilde{y})\| \leq \alpha\}\right)$ where $K' = PK$. Then, $\mathcal{A} = \mathcal{B}$.

Proof.

$$\begin{aligned} \mathcal{B} &= P^T \left(\{\tilde{y} : \|p_{K'}(\tilde{y})\| \leq \alpha\}\right) \\ &= P^T \left(\{\tilde{y} : \tilde{x}^T \tilde{y} \leq \|\tilde{x}\| \ \alpha \ \forall \tilde{x} \in K'\}\right) \qquad \text{by Theorem B.4} \\ &= \{y : \tilde{x}^T (P^T)^{-1} y \leq \sqrt{\tilde{x}^T \tilde{x}} \alpha \ \forall \tilde{x} \in K'\} \\ &= \{y : x^T P^T (P^T)^{-1} y \leq \|Px\| \ \alpha \ \forall x \in K\} \\ &= \{y : x^T y \leq \|Px\| \ \alpha \ \forall x \in K\} = \mathcal{A} \end{aligned}$$

□