Supplementary Material to *Modelling across extremal dependence classes*

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Derivations of ray dependence functions ($\lambda > 0$ and $\lambda < 0$) and spectral density ($\lambda > 0$)

Derivation of $d(q)$ for $\lambda > 0$

This follows simply by noting that Proposition 6 gives that marginal quantile functions are

$$q_A(tx) = (tx)^\lambda l_A(tx), \quad q_B(ty) = (ty)^\lambda l_B(ty),$$

for $tx, ty \geq 1$ so that using the same dominated convergence arguments as in $\lim_{t\to\infty} \theta(t)$ given in the proof of Proposition 1,

$$\lim_{t\to\infty} t\mathsf{P}\{A > q_A(tx), B > q_B(ty)\} = \lambda^{-1/\lambda} \int_0^1 \min\left\{\frac{\tau(v)^{-1/\lambda}}{\mu_1 x}, \frac{\tau(1-v)^{-1/\lambda}}{\mu_2 y}\right\} \, \mathrm{d}F_V(v). \tag{1}$$

Therefore $\mathsf{P}\{A > q_A(tq), B > q_B(t(1-q))\}/\mathsf{P}\{A > q_A(t), B > q_B(t)\}$ converges to $q^{-1/2}(1-q)^{-1/2}d(q)$ with d the form claimed in Remark 1.

Derivation of h for $\lambda > 0$

To derive h, consider (1), with $\mathrm{d}F_V(v) = f_V(v)\,\mathrm{d}v$. This expression can be set equal to

$$\int_0^1 2\min\left(\frac{w^*}{x}, \frac{1-w^*}{y}\right) h(w^*)\,\mathrm{d}w^* = \int_0^{\frac{x}{x+y}} \frac{2w^*}{x} h(w^*)\,\mathrm{d}w^* + \int_{\frac{x}{x+y}}^1 \frac{2(1-w^*)}{x} h(w^*)\,\mathrm{d}w^*.$$

By differentiating under the integral sign, we have

$$\frac{\partial^2}{\partial x \partial y}\left\{\int_0^{\frac{x}{x+y}} \frac{2w^*}{x} h(w^*)\,\mathrm{d}w^* + \int_{\frac{x}{x+y}}^1 \frac{2(1-w^*)}{y} h(w^*)\,\mathrm{d}w^*\right\} = \frac{2}{(x+y)^3} h\left(\frac{x}{x+y}\right),$$

so that h is recovered upon setting $x = w, y = 1-w$, and dividing by two. Thus we begin with

$$\begin{aligned}\lambda^{-1/\lambda}\int_0^1 \min\left\{\frac{\tau(v)^{-1/\lambda}}{\mu_1 x}, \frac{\tau(1-v)^{-1/\lambda}}{\mu_2 y}\right\} f_V(v)\,\mathrm{d}v &= \lambda^{-1/\lambda}\int_0^{r(x,y)} \frac{\tau(v)^{-1/\lambda}}{\mu_1 x} f_V(v)\,\mathrm{d}v \\ &\quad + \lambda^{-1/\lambda}\int_{r(x,y)}^1 \frac{\tau(1-v)^{-1/\lambda}}{\mu_2 y} f_V(v)\,\mathrm{d}v,\end{aligned}$$

with $r(x,y) = \frac{(x\mu_1)^\lambda}{(x\mu_1)^\lambda + (y\mu_2)^\lambda}$. Differentiating with respect to x yields

$$\begin{aligned}\lambda^{-1/\lambda}\Bigg\{\int_0^{r(x,y)} -\frac{\tau(v)^{-1/\lambda}}{\mu_1 x^2} f_V(v)\,\mathrm{d}v + \frac{\tau\{r(x,y)\}^{-1/\lambda}}{\mu_1 x} f_V\{r(x,y)\}\frac{\partial}{\partial x} r(x,y) \\ - \frac{\tau\{1-r(x,y)\}^{-1/\lambda}}{\mu_2 y} f_V\{r(x,y)\}\frac{\partial}{\partial x} r(x,y)\Bigg\} = \int_0^{r(x,y)} -\frac{\tau(v)^{-1/\lambda}}{\mu_1 x^2} f_V(v)\,\mathrm{d}v,\end{aligned}$$

whilst differentiating what remains with respect to y gives

$$-\lambda^{-1/\lambda}\frac{\tau\{r(x,y)\}^{-1/\lambda}}{\mu_1 x^2} f_V\{r(x,y)\}\frac{\partial}{\partial y}r(x,y).$$

Substituting in τ and noting that

$$\frac{\partial}{\partial y}r(x,y) = \frac{\partial}{\partial y}\frac{(x\mu_1)^\lambda}{(x\mu_1)^\lambda + (y\mu_2)^\lambda} = -\lambda\frac{x^\lambda y^{\lambda-1}\mu_1^\lambda\mu_2^\lambda}{\{(x\mu_1)^\lambda + (y\mu_2)^\lambda\}^2}$$

gives

$$\frac{x^{\lambda-1}y^{\lambda-1}\mu_1^\lambda\mu_2^\lambda}{\|(x\mu_1)^\lambda,(y\mu_2)^\lambda\|_m^{1/\lambda}\{(x\mu_1)^\lambda + (y\mu_2)^\lambda\}^2} f_V\left\{\frac{(x\mu_1)^\lambda}{(y\mu_1)^\lambda + (y\mu_2)^\lambda}\right\},$$

so that substituting $x = w, y = 1-w$ and dividing by two yields

$$h(w) = \frac{\lambda^{1-1/\lambda}}{2}\frac{w^{\lambda-1}(1-w)^{\lambda-1}\mu_1^\lambda\mu_2^\lambda}{\|(w\mu_1)^\lambda,((1-w)\mu_2)^\lambda\|_m^{1/\lambda}\{(w\mu_1)^\lambda + ((1-w)\mu_2)^\lambda\}^2} f_V\left\{\frac{(w\mu_1)^\lambda}{(w\mu_1)^\lambda + ((1-w)\mu_2)^\lambda}\right\},$$

which is denoted $h(\cdot;\lambda,f_V)$ in Remark 1.

Derivation of $d(q)$ for $\lambda < 0$

This follows firstly by noting that Proposition 9 gives that marginal quantile functions are

$$q_A(tx) = \Lambda - (tx)^\lambda l_A(tx), \quad q_B(ty) = \Lambda - (ty)^\lambda l_B(ty),$$

for $tx, ty \geq 1$. The ray dependence function can be found by following the proof of Proposition 4 through with these $q_A(tx)$ and $q_B(ty)$, which reveals that

$$\lim_{t\to\infty} t^{1-\lambda}\mathsf{P}\{A > q_A(tx), B > q_B(ty)\} = \frac{F_V'(1/2)}{4}\left\{\min(xm_+, ym_-)^\lambda - \frac{1+\lambda}{1-\lambda}\max(xm_+, ym_-)^\lambda\right\}\max(xm_+, ym_-)^{-1}.$$

Therefore $\mathsf{P}\{A > q_A(tq), B > q_B(t(1-q))\}/\mathsf{P}\{A > q_A(t), B > q_B(t)\}$ converges to $q^{-\frac{1-\lambda}{2}}(1-q)^{-\frac{1-\lambda}{2}}d(q)$ with d the form claimed in Remark 2.

Additional figures from Section 5

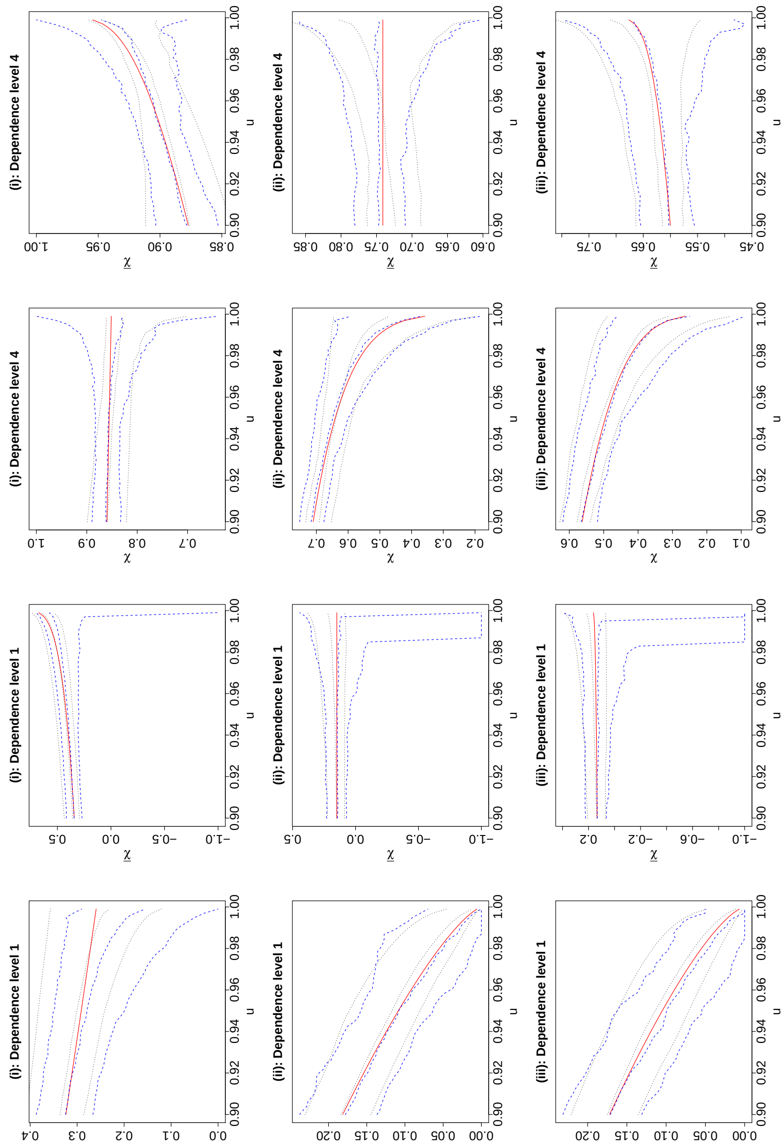

Figure 1: Estimates of $\chi(u)$ (left) and $\bar{\chi}(u)$ (right) for dependence levels 1 and 4 of dependence structures (i)–(iii) using the new model (dotted lines) and the Heffernan–Tawn model (dashed lines). The three lines represent pointwise means and upper 95% and lower 5% quantiles of the 100 repetitions. Red solid line: true value for the copula. The dependence structures and levels are given as the figure title.

Additional proofs from Appendix A

Proof of Lemma 1. The expression $s \mapsto s\phi^{-1/\beta}(s)$ defines a strictly increasing continuous map $[s_0, \infty) \to [1, \infty)$ which is regularly varying with index 1 (note that $\phi^{-1/\beta}$ is slowly varying). Let $\sigma : [1, \infty) \to [s_0, \infty)$ denote the corresponding inverse, which is also regularly varying with index 1, and set $u(t) = t^{-\beta}\sigma^{\beta}(t)$ for all $t \geq 1$; it follows that u is continuous and slowly varying. Setting $s = \sigma(t) = tu^{1/\beta}(t)$ we then get

$$t = s\phi^{-1/\beta}(s) = tu^{1/\beta}(t)\, \phi^{-1/\beta}\{tu^{1/\beta}(t)\} \implies u(t) = \phi\{tu^{1/\beta}(t)\} = \phi(s).$$

The final part of the result follows (note that $tu^{1/\beta}(t) \to \infty$ as $t \to \infty$ since u is slowly varying). □

Proof of Proposition 6. We have

$$\phi(s) := s^{\beta}\mathsf{P}(A > s^{\lambda\beta}) = \int_0^1 \{s^{-\lambda\beta} + \lambda\tau(v)\}_+^{-1/\lambda}\, \mathrm{d}F_V(v).$$

As s increases from 0 to ∞, $s^{-\lambda\beta} + \lambda\tau(v)$ decreases monotonically to $\lambda\tau(v) \geq \lambda$; hence $\{s^{-\lambda\beta} + \lambda\tau(v)\}_+^{-1/\lambda}$ increases monotonically to $\{\lambda\tau(v)\}^{-1/\lambda} \leq \lambda^{-1/\lambda}$. Dominated convergence then gives

$$\lim_{s\to\infty} \phi(s) = \int_0^1 \{\lambda\tau(v)\}^{-1/\lambda}\, \mathrm{d}F_V(v) = \mu_1.$$

Since this limit is non-zero it follows that ϕ is slowly varying. The result for $q_A(t^{\beta})$ now follows from Lemma 1 (with $l_A = u^{\lambda}$). The $q_B(t^{\gamma})$ case is similar. □

Proof of Lemma 2. For each $\delta > 0$ set $J_\delta = \{v \in [0, 1] : a(v) \leq \alpha + \delta\}$.

Claim 1: there exists $S_{1,\delta}$ such that $|a(v) - \alpha| \leq \delta$ when $s \geq S_{1,\delta}$ and $v \in I_s \cap J_\delta$. The continuity of a implies $U := \{v \in [0, 1] : a(v) > \alpha - \delta\}$ is an open neighbourhood of $I \cap J_\delta \neq \emptyset$. Since $I_s \to I$ as $s \to \infty$ it follows that $I_s \cap J_\delta \subseteq U$ for all sufficiently large s.

Claim 2: there exists $S_{2,\delta}$ and $C_\delta > 0$ such that $\int_{I_s \cap J_{\delta/4}} dF_V(v) \geq C_\delta$ for all $s \geq S_{2,\delta}$. Choose $\tilde{v} \in I$ and $\delta_0 > 0$ so that $a(\tilde{v}) = \alpha$ and $J' := [\tilde{v} - \delta_0, \tilde{v} + \delta_0] \subseteq J_{\delta/4}$. Then $I \cap J'$ is an interval of length at least $\delta_1 = \min(\delta_0, |I|) > 0$ (recall that I is an interval). Since I_s is an interval converging to I it follows that, for all sufficiently large s, $I_s \cap J'$ is an interval of length at least $\delta_1/2$, which is contained in $I_s \cap J_{\delta/4}$. We can then let C_δ be the infimum of $\int_K \mathrm{d}F_V(v)$, taken over all intervals $K \subseteq [0, 1]$ of length at least $\delta_1/2$; this quantity is positive by Assumption 1.

Setting

$$\phi_\delta(s) = \int_{I_s \cap J_\delta} u^{-a(v)}(s)\, \mathrm{d}F_V(v) \quad \text{and} \quad \psi_\delta(s) = \int_{I_s \setminus J_\delta} u^{-a(v)}(s)\, \mathrm{d}F_V(v)$$

we clearly have

$$\phi(s) = \phi_\delta(s) + \psi_\delta(s). \tag{2}$$

Claim 3: there exists $S_{3,\delta}$ such that

$$1 \leq \frac{\phi(s)}{\phi_\delta(s)} \leq 1 + C_\delta^{-1} s^{-\rho\delta/4} \quad \textit{for } s \geq S_{3,\delta}. \tag{3}$$

Set $\sigma = \rho\delta/\{4(\alpha + \delta)\} \in (0, \rho/4]$. Since u is regularly varying with index ρ there exists $S'_{3,\delta} \geq 1$ such that

$$s^{\rho-\sigma} \leq u(s) \leq s^{\rho+\sigma} \quad \text{for } s \geq S'_{3,\delta}.$$

If $v \in J_{\delta/4}$ then $a(v) \leq \alpha + \delta/4$ so

$$a(v)(\rho + \sigma) \leq \alpha\rho + \sigma(\alpha + \delta/4) + \rho\delta/4 \leq \alpha\rho + \sigma(\alpha + \delta) + \rho\delta/4 = \alpha\rho + \rho\delta/2$$

so, for any $s \geq S'_{3,\delta}$,

$$u^{-a(v)}(s) \geq s^{-a(v)(\rho+\sigma)} \geq s^{-\alpha\rho-\rho\delta/2}.$$

When $s \geq \max\{S_{2,\delta}, S'_{3,\delta}\}$, Claim 2 then leads to

$$\phi_\delta(s) \geq \phi_{\delta/4}(s) = \int_{I_s \cap J_{\delta/4}} u^{-a(v)}(s)\, \mathrm{d}F_V(v) \geq s^{-\alpha\rho-\rho\delta/2} \int_{I_s \cap J_{\delta/4}} \mathrm{d}F_V(v) \geq C_\delta s^{-\alpha\rho-\rho\delta/2}.$$

On the other hand, if $v \notin J_\delta$ then $a(v) \geq \alpha + \delta$ so

$$a(v)(\rho - \sigma) \geq (\alpha+\delta)(\rho-\sigma) = \alpha\rho - \sigma(\alpha+\delta) + \rho\delta = \alpha\rho + 3\rho\delta/4,$$

and thus, for any $s \geq S'_{3,\delta}$,

$$u^{-a(v)}(s) \leq s^{-a(v)(\rho-\sigma)} \leq s^{-\alpha\rho-3\rho\delta/4}.$$

When $s \geq S'_{3,\delta}$ it follows that

$$\psi_\delta(s) = \int_{I_s \setminus J_\delta} u^{-a(v)}(s)\, \mathrm{d}F_V(v) \leq s^{-\alpha\rho-3\rho\delta/4} \int_{I_s \setminus J_\delta} \mathrm{d}F_V(v) \leq s^{-\alpha\rho-3\rho\delta/4}.$$

When $s \geq \max(S_{2,\delta}, S'_{3,\delta})$ our estimates for $\phi_\delta(s)$ and $\psi_\delta(s)$ can be combined with (2) to give (3).

Let $l \geq 1$ and $\epsilon > 0$. Choose $\delta \in (0,1]$ so that $(1+\delta)^{\alpha+\delta} l^{\rho\delta} \leq 1+\epsilon$. Since u is regularly varying with index ρ we can find $S_{4,\delta}$ such that

$$(1+\delta)^{-1} l^\rho \leq \frac{u(ls)}{u(s)} \leq (1+\delta) l^\rho \quad \text{for } s \geq S_{4,\delta}.$$

If $v \in I_s \cap J_\delta$ and $s \geq \max\{S_{1,\delta}, S_{4,\delta}\}$, Claim 1 gives $\alpha - \delta \leq a(v) \leq \alpha + \delta$ and so

$$\begin{aligned}(1+\epsilon)^{-1} l^{-\alpha\rho} &\leq (1+\delta)^{-(\alpha+\delta)} l^{-(\alpha+\delta)\rho} \leq (1+\delta)^{-a(v)} l^{-a(v)\rho} \\ &\leq \frac{u^{-a(v)}(ls)}{u^{-a(v)}(s)} \leq (1+\delta)^{a(v)} l^{-a(v)\rho} \leq (1+\delta)^{\alpha+\delta} l^{-(\alpha-\delta)\rho} \leq (1+\epsilon) l^{-\alpha\rho}.\end{aligned}$$

Integration then gives

$$\frac{\phi_\delta(ls)}{\phi_\delta(s)} \in [(1+\epsilon)^{-1} l^{-\alpha\rho}, (1+\epsilon) l^{-\alpha\rho}]. \tag{4}$$

Choose $S \geq \max\{S_{1,\delta}, \ldots, S_{4,\delta}\}$ so that $S^{-\rho\delta/4} \leq C_\delta \epsilon$. Now

$$\frac{\phi(ls)}{\phi(s)} = \frac{\phi(ls)}{\phi_\delta(ls)} \frac{\phi_\delta(ls)}{\phi_\delta(s)} \frac{\phi_\delta(s)}{\phi(s)}.$$

For $s \geq S$ the middle term on the right hand side belongs to $[(1+\epsilon)^{-1} l^{-\alpha\rho}, (1+\epsilon) l^{-\alpha\rho}]$ by (4), while the first and third terms belong to $[1, 1+\epsilon]$ and $[(1+\epsilon)^{-1}, 1]$ respectively by (3) (note that, $l \geq 1$ so $ls \geq s \geq S$). Thus $\phi(ls)/\phi(s) \in [(1+\epsilon)^{-2} l^{-\alpha\rho}, (1+\epsilon)^2 l^{-\alpha\rho}]$ for any $s \geq S$. Since $\epsilon > 0$ was arbitrary it follows that $\phi(ls)/\phi(s) \to l^{-\alpha\rho}$ as $s \to \infty$; hence ϕ is regularly varying with index $-\alpha\rho$. ☐

Proof of Proposition 7. For $s \geq 1$, using (A.1),

$$\phi(s) := s^\beta \mathsf{P}(A > \beta \log s) = s^\beta \int_0^1 e^{-\beta\tau(v)\log s}\, \mathrm{d}F_V(v) = \int_0^1 s^{-\beta\{\tau(v)-1\}}\, \mathrm{d}F_V(v).$$

Now $\beta\{\tau(v) - 1\} \geq 0$ with equality iff $v \in \Omega_0$. Dominated convergence then gives

$$\lim_{s\to\infty} \phi(s) = \int_0^1 \lim_{s\to\infty} s^{-\beta\{\tau(v)-1\}}\, \mathrm{d}F_V(v) = \int_{\Omega_0} \mathrm{d}F_V(v) = m_+.$$

By Lemma 2 we know that ϕ is slowly varying. The result for $q_A(t^\beta)$ now follows from Lemma 1 (with $l_A = u$). The $q_B(t^\gamma)$ case is similar. ☐

Proof of Proposition 8. From (A.2a) and Proposition 7 we have

$$\mathsf{P}\{A > q_A(t^\beta),\, B > q_B(t^\gamma)\} = \int_0^1 \min\bigl[e^{-\tau(v)\log\{t^\beta l_A(t)\}},\, e^{-\tau(1-v)\log\{t^\gamma l_B(t)\}}\bigr]\,\mathrm{d}F_V(v).$$

By Proposition 3 we then get $\theta(t) = \int_0^1 g_v(t)\,\mathrm{d}F_V(v)$ where

$$g_v(t) = t^{\widehat{\nu}} \min\bigl\{t^{-\beta\tau(v)} l_A^{-\tau(v)}(t),\, t^{-\gamma\tau(1-v)} l_B^{-\tau(1-v)}(t)\bigr\}.$$

Now $\tau \geq 1$ so $l_A^{-\tau(v)}(t), l_B^{-\tau(1-v)}(t) \leq C = \max\{m_+^{-1}, m_-^{-1}\}$ using Proposition 7. Furthermore $\widehat{\nu} \leq \max\{\beta\tau(v),\, \gamma\tau(1-v)\}$ (by definition) leading to $g_v(t) \leq C$ for all v and $t \geq 1$. If $v \notin \Omega$ then $\widehat{\nu} < \max\{\beta\tau(v),\, \gamma\tau(1-v)\}$ so $g_v(t) \to 0$ as $t \to \infty$. In particular, if $\omega \in [1-v', v']$ it follows that $\Omega = \{\omega\}$ and hence $g_v(t) \to 0$ as $t \to \infty$ whenever $v \neq \omega$; dominated convergence then gives $\lim_{t\to\infty} \theta(t) = 0$. □

Proof of Proposition 9. Set $S_0 = \Lambda^{1/(\lambda\beta)}$. For $s \geq S_0$ we get

$$\begin{aligned}
\phi(s) := s^\beta \mathsf{P}\bigl(A > \Lambda - s^{\lambda\beta}\bigr) &= \int_0^1 \bigl[s^{-\lambda\beta}\bigl\{1 - \lambda(1/\lambda + s^{\lambda\beta})\tau(v)\bigr\}\bigr]_+^{-1/\lambda}\,\mathrm{d}F_V(v) \\
&= \int_0^1 \bigl[(s^{-\lambda\beta} + \lambda)\{1 - \tau(v)\} - \lambda\bigr]_+^{-1/\lambda}\,\mathrm{d}F_V(v),
\end{aligned} \tag{5}$$

using (A.1). For $s \geq S_0$ we have $(s^{-\lambda\beta} + \lambda)\{1 - \tau(v)\} \leq 0$ (recall that $\tau(v) \geq 1$) so the integrand in (5) is bounded above by $(-\lambda)^{-1/\lambda}$. Also note that $s^{-\lambda\beta} \to +\infty$ as $s \to \infty$, so

$$\lim_{s\to\infty} \bigl[(s^{-\lambda\beta} + \lambda)\{1 - \tau(v)\} - \lambda\bigr]_+ = \begin{cases} 0 & \text{if } \tau(v) > 1, \\ -\lambda & \text{if } \tau(v) = 1. \end{cases}$$

As $\{v : \tau(v) = 1\} = \Omega_0$, dominated convergence now gives

$$\lim_{s\to\infty} \phi(s) = \int_{\Omega_0} (-\lambda)^{-1/\lambda}\,\mathrm{d}F_V(v) = (-\lambda)^{-1/\lambda} m_+.$$

Since this limit is non-zero it follows that ϕ is slowly varying. The result for $q_A(t^\beta)$ now follows from Lemma 1 (with $l_A = u^\lambda$). The $q_B(t^\gamma)$ case is similar. □