IDEAL STRUCTURE OF THE ALGEBRA OF BOUNDED OPERATORS ACTING ON A BANACH SPACE

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In memoriam: Uffe Haagerup (1949–2015)

Abstract. We construct a Banach space Z such that the lattice of closed two-sided ideals of the Banach algebra $\mathscr{B}(Z)$ of bounded operators on Z is as follows:

$$\{0\} \subset \mathscr{K}(Z) \subset \mathscr{E}(Z) \begin{matrix} \subset \mathscr{M}_1 \subset \\ \subset \mathscr{M}_2 \subset \end{matrix} \mathscr{B}(Z).$$

We then determine which kinds of approximate identities (bounded/left/right), if any, each of the four non-trivial closed ideals of $\mathscr{B}(Z)$ contains, and we show that the maximal ideal $\mathscr{M}_1$ is generated as a left ideal by two operators, but not by a single operator, thus answering a question left open in our collaboration with Dales, Kochanek and Koszmider (*Studia Math.* 2013). In contrast, the other maximal ideal $\mathscr{M}_2$ is not finitely generated as a left ideal.

The Banach space Z is the direct sum of Argyros and Haydon's Banach space X_{AH} which has very few operators and a certain subspace Y of X_{AH}. The key property of Y is that every bounded operator from Y into X_{AH} is the sum of a scalar multiple of the inclusion map and a compact operator.

1. Introduction and statement of main results

A Banach space E has *very few operators* if E is infinite-dimensional and every bounded operator on E is the sum of a scalar multiple of the identity operator and a compact operator; that is, $\mathscr{B}(E) = \mathbb{K}I_E + \mathscr{K}(E)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ denotes the scalar field of E. Resolving a famous, long-standing open problem, Argyros and Haydon [2] established the existence of such Banach spaces by proving the following spectacular result.

Theorem 1.1 (Argyros and Haydon)**.** *There exists a Banach space X_{AH} such that:*
- (i) *X_{AH} has very few operators;*
- (ii) *X_{AH} has a shrinking Schauder basis;*
- (iii) *the dual space of X_{AH} is isomorphic to ℓ_1.*

The starting point of the present paper is the observation that X_{AH} contains a subspace Y which has certain special properties, as specified in following theorem; of these,

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property (iv) is by far the most important, and also the hardest to achieve. We are deeply grateful to Professor Argyros for having explained to us how to contruct such a subspace; details of its construction will be given in Section 2.

Theorem 1.2. *Argyros and Haydon's Banach space X_{AH} contains a closed, infinite-dimensional subspace Y which has the following four properties:*

(i) *Y is the closed linear span of a certain subsequence of the Schauder basis for X_{AH}, and hence Y has a shrinking Schauder basis;*
(ii) *Y has infinite codimension in X_{AH}, and hence Y is uncomplemented in X_{AH};*
(iii) *the dual space of Y is isomorphic to ℓ_1;*
(iv) *every bounded operator from Y into X_{AH} is the sum of a scalar multiple of the inclusion map $J\colon Y \to X_{\mathrm{AH}}$ and a compact operator.*

In the remainder of this paper, we shall consider the Banach space

$$Z = X_{\mathrm{AH}} \oplus Y, \tag{1.1}$$

where X_{AH} and Y are as in Theorems 1.1 and 1.2, respectively. For definiteness, we shall equip Z with the ℓ_∞-norm; that is, $\|(x,y)\| = \max\{\|x\|, \|y\|\}$ for $x \in X_{\mathrm{AH}}$ and $y \in Y$; all our results will, however, be of an isomorphic nature, so that any equivalent norm will do. Theorems 1.1(i) and 1.2(ii)+(iv) imply that every bounded operator T on Z has a unique representation as an operator-valued (2×2)-matrix of the form

$$T = \begin{pmatrix} \alpha_{1,1} I_{X_{\mathrm{AH}}} + K_{1,1} & \alpha_{1,2} J + K_{1,2} \\ K_{2,1} & \alpha_{2,2} I_Y + K_{2,2} \end{pmatrix}, \tag{1.2}$$

where $\alpha_{1,1}, \alpha_{1,2}$ and $\alpha_{2,2}$ are scalars, $I_{X_{\mathrm{AH}}}$ and I_Y denote the identity operators on X_{AH} and Y, respectively, $J\colon Y \to X_{\mathrm{AH}}$ is the inclusion map, and the operators $K_{1,1}\colon X_{\mathrm{AH}} \to X_{\mathrm{AH}}$, $K_{1,2}\colon Y \to X_{\mathrm{AH}}$, $K_{2,1}\colon X_{\mathrm{AH}} \to Y$ and $K_{2,2}\colon Y \to Y$ are compact.

Using this notation, we see that the sets

$$\mathscr{M}_1 = \{T \in \mathscr{B}(Z) : \alpha_{2,2} = 0\} \qquad \text{and} \qquad \mathscr{M}_2 = \{T \in \mathscr{B}(Z) : \alpha_{1,1} = 0\} \tag{1.3}$$

are maximal two-sided ideals of codimension one in $\mathscr{B}(Z)$. Our first main result gives a complete description of the lattice of closed two-sided ideals of $\mathscr{B}(Z)$. Its statement involves the following notion, which goes back to Kleinecke [14].

Definition 1.3. A bounded operator on a Banach space E is *inessential* if it belongs to the pre-image of the Jacobson radical of the Calkin algebra $\mathscr{B}(E)/\overline{\mathscr{F}(E)}$, where $\overline{\mathscr{F}(E)}$ denotes the norm-closure of the ideal of finite-rank operators on E.

We write $\mathscr{E}(E)$ for the set of inessential operators on the Banach space E. This is a closed two-sided ideal of $\mathscr{B}(E)$ which is proper if (and only if) E is infinite-dimensional.

Theorem 1.4. *The Banach algebra $\mathscr{B}(Z)$ of bounded operators on the Banach space Z defined by* (1.1) *contains exactly six closed two-sided ideals, namely*

$$
\begin{tikzcd}[arrows=dash]
 & \mathscr{B}(Z) \arrow[dl] \arrow[dr] & \\
\mathscr{M}_1 \arrow[dr] & & \mathscr{M}_2 \arrow[dl] \\
 & \mathscr{E}(Z) = \mathscr{M}_1 \cap \mathscr{M}_2 \arrow[d] & \\
 & \mathscr{K}(Z) \arrow[d] & \\
 & \{0\}, &
\end{tikzcd}
$$

where $\mathscr{M}_1$ and $\mathscr{M}_1$ are given by (1.3)*, and the lines denote proper inclusions, with the larger ideal at the top.*

We note that in the diagram, above, the smaller ideal has codimension one in the larger ideal in each of the inclusions, except the bottommost.

Remark 1.5. Not many infinite-dimensional Banach spaces E are known for which a complete classification of the closed two-sided ideals of $\mathscr{B}(E)$ exists. Indeed, to the best of our knowledge at the time of writing, the following list contains all such examples:

(i) the classical sequence spaces $E = \ell_p(\mathbb{I})$ for $1 \leqslant p < \infty$ and $E = c_0(\mathbb{I})$, where $\mathbb{I}$ is an arbitrary infinite index set; these results are due to Calkin [5] for countable $\mathbb{I}$ and $p = 2$; Gohberg–Markus–Feldman [10] for countable $\mathbb{I}$ and general p (including c_0); Gramsch [11] and Luft [21] for $p = 2$ and arbitrary $\mathbb{I}$; and Daws [9] in full generality;

(ii) the c_0-direct sum of the sequence of finite-dimensional Hilbert spaces of increasing dimension, that is, $E = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_{c_0}$, and its dual space $\big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_{\ell_1}$ (see [16] and [17], respectively);

(iii) $E = X_{\mathrm{AH}}$ by Theorem 1.1, above;

(iv) Tarbard's variants of the Argyros–Haydon space: for each $n \in \mathbb{N}$, there is a Banach space E such that E admits a strictly singular operator S which is nilpotent of order $n+1$, and every bounded operator on E has the form $\sum_{j=0}^{n} \alpha_j S^j + K$ for some scalars $\alpha_0, \ldots, \alpha_n$ and a compact operator K (see [24, Theorem 2.1]);

(v) $E = C(\Omega)$, where Ω is the Mrówka space constructed by Koszmider [15], assuming the Continuum Hypothesis (see [13, Theorem 5.5]; this result has also been obtained independently by Brooker (unpublished));

(vi) certain Banach spaces constructed by Motakis, Puglisi and Zisimopoulou [22]: for every countably infinite compact metric space Ω, there is a Banach space E such that the Banach algebra $\mathscr{B}(E)/\mathscr{K}(E)$ is isomorphic to the algebra $C(\Omega)$ of scalar-valued, continuous functions defined on Ω. (The classification of the closed two-sided ideals of $\mathscr{B}(E)$ is not stated explicitly in [22], but it is an easy consequence of [22, Theorem 5.1], together with the following two facts: (1) E is a $\mathscr{L}_\infty$-space, so it has the bounded approximation property, and therefore $\mathscr{K}(E)$ is the minimum non-zero closed two-sided ideal of $\mathscr{B}(E)$; (2) the closed ideals of the Banach algebra $C(\Omega)$ for a compact Hausdorff space Ω are precisely the zero sets of the closed subsets of Ω.)

In each of the cases (i)–(v), above, the lattice of closed two-sided ideals of $\mathscr{B}(E)$ is linearly ordered, whereas in case (vi), it is infinite. Hence the Banach space Z given by (1.1) appears to be the first Banach space E for which we have a complete classification of the lattice of closed two-sided ideals of the Banach algebra $\mathscr{B}(E)$, and this lattice is finite, but it is not linearly ordered.

Note added in proof. We shall here describe another family of Banach spaces E such that the lattice of closed two-sided ideals of $\mathscr{B}(E)$ is finite and not linearly ordered. For each $n \in \mathbb{N}$, we apply [2, Theorem 10.4] to obtain Banach spaces $X_1, \ldots, X_n$, each having very few operators, each having a Schauder basis, and such that every bounded operator from X_j to X_k is compact whenever $j, k \in \{1, \ldots, n\}$ are distinct. Take $m_1, \ldots, m_n \in \mathbb{N}$, and set $E = X_1^{m_1} \oplus \cdots \oplus X_n^{m_n}$. Then $\mathscr{K}(E)$ is the smallest non-zero closed two-sided ideal of $\mathscr{B}(E)$, and we have

$$\mathscr{B}(E)/\mathscr{K}(E) \cong M_{m_1}(\mathbb{K}) \oplus \cdots \oplus M_{m_n}(\mathbb{K}),$$

where $M_m(\mathbb{K})$ denotes the algebra of scalar-valued $(m \times m)$-matrices. By Wedderburn's structure theorem (see, *e.g.*, [7, Theorem 1.5.9]), this shows that every finite-dimensional, semi-simple complex algebra can arise as the Calkin algebra of a Banach space. Moreover, we note that the choice $m_1 = \cdots = m_n = 1$ gives a counterpart of the result of Motakis, Puglisi and Zisimopoulou that we described in (vi), above, in the case where the underlying space Ω is finite.

Returning to the general case where $m_1, \ldots, m_n \in \mathbb{N}$ are arbitrary, we may consider each bounded operator T on E as an operator-valued $(n \times n)$-matrix $(T_{j,k})_{j,k=1}^n$, where we have $T_{j,k} \in \mathscr{B}(X_k^{m_k}, X_j^{m_j})$ for each j, k. Since $M_m(\mathbb{K})$ is simple for each $m \in \mathbb{N}$, the map

$$N \mapsto \bigl\{(T_{j,k})_{j,k=1}^n : T_{j,j} \in \mathscr{K}(X_j^{m_j}) \ (j \notin N)\bigr\}$$

is an order isomorphism of the power set of $\{1, \ldots, n\}$ onto the lattice of non-zero closed two-sided ideals of $\mathscr{B}(E)$. Hence the lattice of closed two-sided ideals of $\mathscr{B}(E)$ has $2^n + 1$ elements, and it is not linearly ordered for $n \geqslant 2$.

Let us finally remark that the ideal lattices obtained in this way are different from that of $\mathscr{B}(Z)$ that we described in Theorem 1.4, above; for instance, none of these lattices has precisely six elements.

After seeing Argyros and Haydon's main results as they were stated in Theorem 1.1, above, Dales observed that they imply that the Banach algebra $\mathscr{B}(X_{\mathrm{AH}})$ is amenable [2, Proposition 10.6], thus disproving a long-standing conjecture of B. E. Johnson. In contrast, we note that $\mathscr{B}(Z)$ does not share this property.

Proposition 1.6. *The Banach algebra $\mathscr{B}(Z)$ is not amenable.*

The study of amenability is intimately related to the existence of approximate identities, as explained in [7, Section 2.9], for instance. Our second main result, which will be proved in Section 4, describes what kinds of approximate identities, if any, can be found in each of the four non-trivial closed two-sided ideals of $\mathscr{B}(Z)$. Before we state this result formally, let us introduce the relevant terminology.

Definition 1.7. A net $(e_j)_{j\in\mathbb{J}}$ in a Banach algebra $\mathscr{A}$ is a *left approximate identity* if the net $(e_ja)_{j\in\mathbb{J}}$ converges to a for each $a \in \mathscr{A}$, and a *right approximate identity* if the net $(ae_j)_{j\in\mathbb{J}}$ converges to a for each $a \in \mathscr{A}$. If in addition $\sup_{j\in\mathbb{J}} \|e_j\| < \infty$, then $(e_j)_{j\in\mathbb{J}}$ is a *bounded* left or right approximate identity. A *bounded two-sided approximate identity* is a net which is simultaneously a bounded left and right approximate identity.

Theorem 1.8.
(i) *The ideal $\mathscr{M}_1$ has a bounded left approximate identity, but it has no right approximate identity.*
(ii) *The ideal $\mathscr{M}_2$ has a bounded right approximate identity, but it has no left approximate identity.*
(iii) *The ideal $\mathscr{E}(Z) = \mathscr{M}_1 \cap \mathscr{M}_2$ has no left or right approximate identity.*
(iv) *The ideal $\mathscr{K}(Z)$ has a bounded two-sided approximate identity.*

Our third and final main result uses the Banach space Z to answer two questions that were left open in [8] regarding the maximal left ideals of the Banach algebra $\mathscr{B}(E)$ for an infinite-dimensional Banach space E. To set the stage for this result, we require some background information from [8], beginning with the easy observation that, for each non-zero element x of E, the set

$$\mathscr{M\!L}_x = \{T \in \mathscr{B}(E) : Tx = 0\} \tag{1.4}$$

is a maximal left ideal of $\mathscr{B}(E)$, and it is generated as a left ideal by a single operator, namely any projection $P \in \mathscr{B}(E)$ with $\ker P = \mathbb{K}x$. The maximal left ideals of the form (1.4) were termed *fixed* in [8], inspired by the analogous terminology for ultrafilters, and the following question was studied extensively:

> *Is every finitely-generated, maximal left ideal of the Banach algebra $\mathscr{B}(E)$ necessarily fixed?*

Indeed, a positive answer to this question was established for many Banach spaces E, but, somewhat surprisingly, it was also shown that the answer is not always positive: for $E = X_{\mathrm{AH}} \oplus \ell_\infty$, the Banach algebra $\mathscr{B}(E)$ contains a non-fixed, singly-generated, maximal left ideal of codimension one, namely

$$\mathscr{K}_1 = \left\{ \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \in \mathscr{B}(E) : T_{1,1} \text{ is compact} \right\}.$$

The Banach space $E = X_{\mathrm{AH}} \oplus \ell_\infty$ is evidently not separable. We shall show in Section 5 that a similar example exists based on the separable Banach space Z given by (1.1). This example will also enable us to answer another question implicitly left open in [8] (see [8, Proposition 2.2] and the remark following it) because the non-fixed, finitely-generated maximal left ideal of $\mathscr{B}(Z)$ that we identify is not generated by one, but by two operators. More precisely, our result is as follows.

Theorem 1.9. *The ideals $\mathscr{M}_1$ and $\mathscr{M}_2$ are the only non-fixed, maximal left ideals of $\mathscr{B}(Z)$, and*

(i) *$\mathscr{M}_1$ is generated as a left ideal by the two operators*

$$\begin{pmatrix} I_{X_{\mathrm{AH}}} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}, \tag{1.5}$$

but $\mathscr{M}_1$ is not generated as a left ideal by a single bounded operator on Z;

(ii) *$\mathscr{M}_2$ is not finitely generated as a left ideal.*

Remark 1.10. In a post on *Mathematics Stack Exchange*, Petry [23] asked whether there is a one-sided version of the Nakayama lemma, in the following specific sense: let R be a unital non-commutative ring, and let L be a finitely-generated left ideal of R such that $L = L \cdot L$ (that is, each element of L can be written as the sum of products of elements of L). Must L be generated (as a left ideal) by a single idempotent element?

In reply, Schwiebert outlined an example which shows that the answer is in general negative. We observe that our results provide another such example. Indeed, let $R = \mathscr{B}(Z)$, and let $L = \mathscr{M}_1$. Theorem 1.9(i) shows that L is finitely generated, but not by a single element (idempotent or not), while Theorem 1.8(i) in tandem with Cohen's Factorization Theorem (see, *e.g.,* [7, Corollary 2.9.25]) implies that each element of L can be written as the product of two elements of L. Being a Banach algebra, this example has a very different flavour from Schwiebert's, which is based on an algebra over a finite field constructed by Andruszkiewicz and Puczyłowski [1].

2. The construction of the subspace Y and the proof of Theorem 1.2

Schauder decompositions. Let E be a Banach space. A sequence $(F_j)_{j\in\mathbb{N}}$ of non-zero subspaces of E is a *Schauder decomposition* for E if, for each $x \in E$, there is a unique sequence $(x_j)_{j\in\mathbb{N}}$, where $x_j \in F_j$ for each $j \in \mathbb{N}$, such that the series $\sum_{j=1}^{\infty} x_j$ is norm-convergent with sum x. In this case, for each $n \in \mathbb{N}$, we can define a projection $P_n \in \mathscr{B}(E)$ by $P_n x = \sum_{j=1}^{n} x_j$; this is the n^{th} *canonical projection* associated with the decomposition. The number $\sup_{n\in\mathbb{N}} \|P_n\|$ turns out to be finite; this is the *decomposition constant.*

A Schauder decomposition $(F_j)_{j\in\mathbb{N}}$ for E is:

- *shrinking* if $\|x^* - P_n^* x^*\| \to 0$ for each $x^* \in E^*$;
- *finite-dimensional* (or an *FDD* for short) if $\dim F_j < \infty$ for each $j \in \mathbb{N}$. (Note: the case where each F_j is one-dimensional, say $F_j = \mathbb{K} b_j$ ($j \in \mathbb{N}$), corresponds to $(b_j)_{j\in\mathbb{N}}$ being a Schauder basis for E.)

We shall require the following elementary observation concerning compact operators into or out of a Banach space with an FDD. It goes back to at least [3, Remark, p. 14] in the case of a single Banach space with a Schauder basis. For completeness, we outline a proof.

Lemma 2.1. *Let D and E be Banach spaces, where E has an FDD, and denote by $(P_n)_{n\in\mathbb{N}}$ the canonical projections associated with this FDD.*

(i) *For each bounded operator $S\colon D\to E$, the following two conditions are equivalent:*
  (a) *S is compact;*
  (b) *$\|S-P_nS\|\to 0$ as $n\to\infty$.*

(ii) *Suppose that the FDD for E is shrinking. Then, for each bounded operator $T\colon E\to D$, the following three conditions are equivalent:*
  (a) *T is compact;*
  (b) *$\|T-TP_n\|\to 0$ as $n\to\infty$;*
  (c) *$\|Tx_j\|\to 0$ as $j\to\infty$ for every bounded block sequence $(x_j)_{j\in\mathbb{N}}$ with respect to the FDD for E.*

Proof. Let $C=\sup_{n\in\mathbb{N}}\|P_n\|<\infty$ be the decomposition constant.

(i). The implication (b)$\Rightarrow$(a) is clear because P_n has finite-dimensional image for each $n\in\mathbb{N}$. Conversely, suppose contrapositively that, for some $\varepsilon>0$ and each $m\in\mathbb{N}$, there is an integer $n\geqslant m$ such that $\|(I_E-P_n)S\|>\varepsilon$. By recursion, we can choose a sequence $(x_j)_{j\in\mathbb{N}}$ of unit vectors in D and a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ of natural numbers such that $\|(I_E-P_{k_j})Sx_j\|>\varepsilon$ and $\|(I_E-P_m)Sx_j\|<\varepsilon/2$ whenever $j,m\in\mathbb{N}$ and $m\geqslant k_{j+1}$. This implies that

$$(C+1)\|Sx_{i+j}-Sx_i\|\geqslant\|(I_E-P_{k_{i+j}})Sx_{i+j}\|-\|(I_E-P_{k_{i+j}})Sx_i\|>\frac{\varepsilon}{2}\qquad(i,j\in\mathbb{N}),$$

which shows that no subsequence of $(Sx_i)_{i\in\mathbb{N}}$ is Cauchy, and therefore the operator S is not compact.

(ii). The equivalence of conditions (a) and (b) follows by dualizing (i) and using Schauder's theorem together with the fact that $(P_n^*)_{n\in\mathbb{N}}$ are the canonical projections associated with an FDD for the dual space E^* of E.

The implication (b)$\Rightarrow$(c) is easy because, for every block sequence $(x_j)_{j\in\mathbb{N}}$ in E and each $n\in\mathbb{N}$, we can find $j_0\in\mathbb{N}$ such that $P_nx_j=0$ whenever $j\geqslant j_0$.

(c)$\Rightarrow$(b). Suppose contrapositively that, for some $\varepsilon>0$ and each $m\in\mathbb{N}$, there is an integer $k\geqslant m$ such that $\|T(I_E-P_k)\|>\varepsilon$. Then we can find a unit vector $w\in E$ and a further integer $j>k$ such that $\|T(P_j-P_k)w\|>\varepsilon$, and hence we can recursively choose integers $1\leqslant k_1<j_1\leqslant k_2<j_2\leqslant\cdots$ and unit vectors $w_1,w_2,\ldots\in E$ such that $\|T(P_{j_i}-P_{k_i})w_i\|>\varepsilon$ for each $i\in\mathbb{N}$. This implies that $(x_i)_{i\in\mathbb{N}}:=((P_{j_i}-P_{k_i})w_i)_{i\in\mathbb{N}}$ is a $2C$-bounded block sequence for which (c) fails. □

The Bourgain–Delbaen construction. Argyros and Haydon used the *Bourgain–Delbaen construction* [4] to define their Banach space X_{AH}. We shall now summarize those parts of this method that are required for our present purposes. We follow the notation and terminology used in [2] as far as possible, with the notable exception that our focus is on both real and complex scalars, whereas [2] considered real scalars only. For this reason, it is convenient to introduce a single symbol for the following countable, dense subfield of the scalar field $\mathbb{K}$ that will play the role of the rationals in the real case:

$$\mathbb{L} = \begin{cases} \mathbb{Q} & \text{for } \mathbb{K} = \mathbb{R} \\ \mathbb{Q} + \mathrm{i}\mathbb{Q} & \text{for } \mathbb{K} = \mathbb{C}. \end{cases} \tag{2.1}$$

For a (non-empty, countable) set Γ, we consider the Banach spaces

$$\ell_\infty(\Gamma) = \Big\{ x \colon \Gamma \to \mathbb{K} : \sup_{\gamma\in\Gamma} |x(\gamma)| < \infty \Big\} \quad \text{and} \quad \ell_1(\Gamma) = \Big\{ x^* \colon \Gamma \to \mathbb{K} : \sum_{\gamma\in\Gamma} |x^*(\gamma)| < \infty \Big\},$$

and identify $\ell_\infty(\Gamma)$ with the dual space of $\ell_1(\Gamma)$ via the duality bracket

$$\langle x^*, x\rangle = \sum_{\gamma\in\Gamma} x^*(\gamma)x(\gamma) \qquad (x^* \in \ell_1(\Gamma),\ x \in \ell_\infty(\Gamma)).$$

We write e_γ and e_γ^* for the elements of $\ell_\infty(\Gamma)$ and $\ell_1(\Gamma)$, respectively, given by

$$e_\gamma(\gamma) = 1 = e_\gamma^*(\gamma) \qquad \text{and} \qquad e_\gamma(\eta) = 0 = e_\gamma^*(\eta) \qquad (\eta \in \Gamma \setminus \{\gamma\}).$$

Let $p = 1$ or $p = \infty$. Then $\operatorname{supp} x$ denotes the *support* of an element $x \in \ell_p(\Gamma)$. Given a non-empty subset Δ of Γ, we identify $\ell_p(\Delta)$ with the subspace $\{x \in \ell_p(\Gamma) : \operatorname{supp} x \subseteq \Delta\}$ of $\ell_p(\Gamma)$.

The Bourgain–Delbaen construction, as Argyros and Haydon present it, begins with the singleton set $\Delta_1 = \{1\}$ and the functional $c_1^* = 0$. A sequence $(\Delta_n)_{n\in\mathbb{N}}$ of non-empty, finite, disjoint sets is then defined recursively, together with functionals $c_\gamma^* \in \operatorname{span}\{e_\eta^* : \eta \in \Gamma_n\}$ for each $n \in \mathbb{N}$ and $\gamma \in \Delta_{n+1}$, where $\Gamma_n := \bigcup_{j=1}^n \Delta_j$, in such a way that the sequence

$$(d_\gamma^*)_{\gamma\in\Gamma} := (e_\gamma^* - c_\gamma^*)_{\gamma\in\Gamma}$$

is a Schauder basis for the Banach space $\ell_1(\Gamma)$, where $\Gamma := \bigcup_{j\in\mathbb{N}} \Delta_j$, endowed with the lexicographic order induced by $\Delta_1, \Delta_2, \ldots$ (The finite sets $\Delta_1, \Delta_2, \ldots$ are *a priori* unordered; they can each be given an arbitrary linear order to ensure that Γ is linearly ordered.) In particular, the finite-dimensional subspaces $\operatorname{span}\{d_\gamma^* : \gamma \in \Delta_n\}$ $(n \in \mathbb{N})$ form an FDD for $\ell_1(\Gamma)$. We write $P_{(0,n]}^*$ for the n^{th} canonical projection on $\ell_1(\Gamma)$ associated with this decomposition; that is, $P_{(0,n]}^*$ is given by $P_{(0,n]}^* d_\gamma^* = d_\gamma^*$ if $\gamma \in \Gamma_n$ and $P_{(0,n]}^* d_\gamma^* = 0$ otherwise. For later reference, we note that the image of $P_{(0,n]}^*$ is given by

$$\operatorname{span}\{d_\gamma^* : \gamma \in \Gamma_n\} = \operatorname{span}\{e_\gamma^* : \gamma \in \Gamma_n\} = \ell_1(\Gamma_n). \tag{2.2}$$

Let $(d_\gamma)_{\gamma\in\Gamma}$ be the sequence of coordinate functionals in $\ell_1(\Gamma)^* = \ell_\infty(\Gamma)$ associated with the Schauder basis $(d_\gamma^*)_{\gamma\in\Gamma}$ for $\ell_1(\Gamma)$. The *Bourgain–Delbaen space* $X(\Gamma)$ determined by the set Γ is now defined as the closed subspace of $\ell_\infty(\Gamma)$ spanned by $\{d_\gamma : \gamma \in \Gamma\}$, so that, by definition, $(d_\gamma)_{\gamma\in\Gamma}$ is a Schauder basis for $X(\Gamma)$. Denote by $P_{(0,n]}$ the adjoint of the projection $P_{(0,n]}^*$ for each $n \in \mathbb{N}$. Since the image of $P_{(0,n]}$ is equal to $\operatorname{span}\{d_\gamma : \gamma \in \Gamma_n\}$, we may consider $P_{(0,n]}$ as an operator into $X(\Gamma)$. We observe that the subspaces

$$M_n := \operatorname{span}\{d_\gamma : \gamma \in \Delta_n\} \qquad (n \in \mathbb{N})$$

form an FDD for $X(\Gamma)$, and $(P_{(0,n]}|X(\Gamma))_{n\in\mathbb{N}}$ are the associated projections.

Let $n \in \mathbb{N}$. By (2.2), we may regard $P^*_{(0,\,n]}$ as a surjection onto $\ell_1(\Gamma_n)$. The adjoint of this operator, which we shall denote by $i_n\colon \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma)$, plays an important role in the study of Bourgain–Delbaen spaces. It is an extension operator, in the sense that $i_n(x)(\gamma) = (x)(\gamma)$ for each $x \in \ell_\infty(\Gamma_n)$ and $\gamma \in \Gamma_n$, and it satisfies

$$\|x\|_\infty \leqslant \|i_n(x)\|_\infty \leqslant M\|x\|_\infty \qquad (x \in \ell_\infty(\Gamma_n)), \tag{2.3}$$

where M is the basis constant of $(d^*_\gamma)_{\gamma\in\Gamma}$. We can describe i_n explicitly by the formula $i_n = P_{(0,\,n]}|_{\ell_\infty(\Gamma_n)}$. In particular, its image is spanned by $\{d_\gamma : \gamma \in \Gamma_n\}$, and so we may regard i_n as an operator from $\ell_\infty(\Gamma_n)$ into $X(\Gamma)$.

Let $x \in \operatorname{span}\{d_\gamma : \gamma \in \Gamma\}$. By the *range* of x, we understand the smallest interval $\mathbb{I}$ of $\mathbb{N}$ such that $x \in \operatorname{span}\bigl\{d_\gamma : \gamma \in \bigcup_{i\in\mathbb{I}} \Delta_i\bigr\}$. We write $\operatorname{ran} x$ for the range of x. Suppose that $\operatorname{ran} x \subseteq (p, q]$ for some non-negative integers $p < q$. Then, as observed in [2, p. 12], the element $u := x|_{\Gamma_q} \in \ell_\infty(\Gamma_q)$ satisfies

$$x = i_q(u) \qquad \text{and} \qquad \operatorname{supp} u \subseteq \Gamma_q \setminus \Gamma_p. \tag{2.4}$$

Suppose that $x \neq 0$, and set $m = \max\operatorname{ran} x \in \mathbb{N}$. Then we define the *local support* of x by

$$\operatorname{locsupp} x := \operatorname{supp}(x|_{\Gamma_m}) = \{\gamma \in \Gamma_m : x(\gamma) \neq 0\}.$$

Further, suppose that $x = i_n(w)$ for some $n \in \mathbb{N}$ and $w \in \ell_\infty(\Gamma_n)$. Then we have $n \geqslant m$ because $i_n[\ell_\infty(\Gamma_n)] = \operatorname{span}\{d_\gamma : \gamma \in \Gamma_n\}$, and hence

$$x(\gamma) = \langle e^*_\gamma, i_n(w)\rangle = \langle P^*_{(0,n]} e^*_\gamma, w\rangle = w(\gamma) \qquad (\gamma \in \Gamma_m), \tag{2.5}$$

which proves that $\operatorname{locsupp} x = (\operatorname{supp} w) \cap \Gamma_m$.

We reserve the term 'block sequence' for a block sequence with respect to the FDD $(M_n)_{\in\mathbb{N}}$, in the following precise sense. Let $\mathbb{I}$ be a non-empty (finite or infinite) interval of $\mathbb{N}$. A *block sequence* indexed by $\mathbb{I}$ is a sequence $(x_i)_{i\in\mathbb{I}}$ in $X(\Gamma) \setminus \{0\}$ such that $x_i \in \operatorname{span}\{d_\gamma : \gamma \in \Gamma\}$ for each $i \in \mathbb{I}$ and $\max\operatorname{ran} x_{i-1} < \min\operatorname{ran} x_i$ whenever $i \neq \min\mathbb{I}$.

The set Γ^{AH}. Argyros and Haydon's Banach space X_{AH} is the Bourgain–Delbaen space $X(\Gamma^{\mathrm{AH}})$ determined by a very clever choice of $\Gamma^{\mathrm{AH}} := \bigcup_{j\in\mathbb{N}} \Delta^{\mathrm{AH}}_j$ that we shall now attempt to describe, following [2, Section 4]. The first step is to fix two fast-increasing sequences $(m_j)_{j\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ of natural numbers which satisfy the following conditions (see [2, Assumption 2.3]):

- $m_1 \geqslant 4$ and $n_1 \geqslant m_1^2$;
- $m_{j+1} \geqslant m_j^2$ and $n_{j+1} \geqslant m_{j+1}^2 (4n_j)^{\log_2 m_{j+1}}$ for each $j \in \mathbb{N}$.

The recursive definition of the sets $(\Delta^{\mathrm{AH}}_n)_{n\in\mathbb{N}}$ and the associated functionals $(c^*_\gamma)_{\gamma\in\Gamma^{\mathrm{AH}}}$ requires that several other objects are defined simultaneously, as part of the same recursion. Indeed, we shall also choose a strictly increasing sequence $(N_n)_{n\in\mathbb{N}_0}$ of integers and construct four maps called 'rank', 'age', σ and 'weight'. Each of these maps will be defined on the set Γ^{AH}. The first three will take their values in $\mathbb{N}$, while 'weight' maps into the set $\{1/m_j : j \in \mathbb{N}\}$. The map σ must be injective and satisfy $\sigma(\gamma) > \operatorname{rank}\gamma$ for each $\gamma \in \Gamma^{\mathrm{AH}}$.

As we have already mentioned, the recursion begins with the set $\Delta_1^{\mathrm{AH}} = \{1\}$ and the functional $c_1^* = 0$. We set $N_0 = 0$ and define

$$\operatorname{rank}\gamma = \operatorname{age}\gamma = 1, \qquad \sigma(\gamma) = 2 \qquad \text{and} \qquad \operatorname{weight}\gamma = \frac{1}{m_1} \qquad (\gamma = 1 \in \Delta_1^{\mathrm{AH}}).$$

Now assume recursively that, for some $n \in \mathbb{N}$, we have defined the sets $\Delta_1^{\mathrm{AH}}, \ldots, \Delta_n^{\mathrm{AH}}$ and the functionals c_γ^* for $\gamma \in \Gamma_n^{\mathrm{AH}}$ (where $\Gamma_n^{\mathrm{AH}} := \bigcup_{j=1}^n \Delta_j^{\mathrm{AH}}$ by convention, as above), as well as the integers $N_0 < N_1 < \cdots < N_{n-1}$ and the maps $\operatorname{rank}, \operatorname{age}, \sigma \colon \Gamma_n^{\mathrm{AH}} \to \mathbb{N}$ and $\operatorname{weight} \colon \Gamma_n^{\mathrm{AH}} \to \{1/m_j : j \in \mathbb{N}\}$, where σ is injective and satisfies $\sigma(\gamma) > \operatorname{rank}\gamma$ for each $\gamma \in \Gamma_n^{\mathrm{AH}}$. Choose $N_n > N_{n-1}$ such that the set

$$B_{p,n} := \Biggl\{ \sum_{\eta \in \Gamma_n^{\mathrm{AH}} \setminus \Gamma_p^{\mathrm{AH}}} a_\eta e_\eta^* : a_\eta \in \mathbb{L},\ \sum_\eta |a_\eta| \leqslant 1 \text{ and the denominator of } a_\eta \text{ divides } N_n! \text{ for each } \eta \in \Gamma_n^{\mathrm{AH}} \setminus \Gamma_p^{\mathrm{AH}} \Biggr\}$$

is a 2^{-n}-net in the unit ball of $\ell_1(\Gamma_n^{\mathrm{AH}} \setminus \Gamma_p^{\mathrm{AH}})$ for each $p \in \{0, 1, \ldots, n-1\}$, where we have introduced $\Gamma_0^{\mathrm{AH}} := \emptyset$ for convenience. (When talking about 'the denominator' of an element a_η of $\mathbb{L}$ in the complex case, we suppose that a_η has been written in the form $a_\eta = (j + k\mathrm{i})/m$ for some $j, k \in \mathbb{Z}$ and $m \in \mathbb{N}$.) We admit into $\Delta_{n+1}^{\mathrm{AH}}$ elements γ of two types:

(i) Elements of *type 1* are triples of the form

$$\gamma = \Bigl(n+1, \frac{1}{m_j}, b^*\Bigr),$$

where $b^* \in B_{0,n}$ and $j \in \{1, \ldots, n+1\}$. If j is even, then we admit each γ of this form into $\Delta_{n+1}^{\mathrm{AH}}$, whereas if j is odd, we admit γ into $\Delta_{n+1}^{\mathrm{AH}}$ if and only if $b^* = e_\eta^*$, where $\eta \in \Gamma_n^{\mathrm{AH}}$ has weight $1/m_{4i-2}$ for some $i \in \mathbb{N}$, and this weight satisfies $1/m_{4i-2} < 1/n_j^2$. In both cases we define

$$c_\gamma^* = \frac{b^*}{m_j}, \qquad \operatorname{rank}\gamma = n+1, \qquad \operatorname{weight}\gamma = \frac{1}{m_j} \qquad \text{and} \qquad \operatorname{age}\gamma = 1.$$

(ii) Elements of *type 2* are quadruples of the form

$$\gamma = \Bigl(n+1, \xi, \frac{1}{m_j}, b^*\Bigr),$$

where $j \in \{1, \ldots, n+1\}$, $\xi \in \Delta_p^{\mathrm{AH}}$ for some $p \in \{1, \ldots, n-1\}$, $\operatorname{weight}\xi = 1/m_j$, $\operatorname{age}\xi < n_j$ and $b^* \in B_{p,n}$. Again, if j is even, then we admit each γ of this form into $\Delta_{n+1}^{\mathrm{AH}}$, whereas if j is odd, we admit γ into $\Delta_{n+1}^{\mathrm{AH}}$ if and only if $b^* = e_\eta^*$, where $\eta \in \Gamma_n^{\mathrm{AH}} \setminus \Gamma_p^{\mathrm{AH}}$ has weight $1/m_{4\sigma(\xi)}$. In both cases we define

$$c_\gamma^* = e_\xi^* + \frac{b^* - P_{(0,p]}^* b^*}{m_j}, \quad \operatorname{rank}\gamma = n+1, \quad \operatorname{weight}\gamma = \frac{1}{m_j}, \quad \operatorname{age}\gamma = 1 + \operatorname{age}\xi.$$

It remains to extend the definition of σ to $\Delta_{n+1}^{\mathrm{AH}}$. Set $m = \max \sigma[\Gamma_n^{\mathrm{AH}}]$. Then $m > n$, and we may therefore define $\sigma(\gamma)$ for $\gamma \in \Delta_{n+1}^{\mathrm{AH}}$ by assigning to it any value in $\mathbb{N} \cap (m, \infty)$ that we wish, as long as we choose distinct values for distinct elements of $\Delta_{n+1}^{\mathrm{AH}}$. This completes the recursive construction and hence the definition of Argyros and Haydon's Banach space X_{AH}.

Remark 2.2. For later reference, we record the following two facts.

(i) As noted in [2, p. 17], the basis constant M of $(d_\gamma^*)_{\gamma \in \Gamma^{\mathrm{AH}}}$ is at most 2.

(ii) The Schauder basis $(d_\gamma)_{\gamma \in \Gamma^{\mathrm{AH}}}$ of X_{AH} is shrinking, so that $(d_\gamma^*)_{\gamma \in \Gamma^{\mathrm{AH}}}$ forms a Schauder basis for the dual space X_{AH}^*, and therefore $X_{\mathrm{AH}}^* \cong \ell_1(\Gamma^{\mathrm{AH}})$. Indeed, the proof of [2, Proposition 5.12] shows that the FDD $(M_n)_{n \in \mathbb{N}}$ for X_{AH} is shrinking, and hence the conclusion follows from the elementary general fact that if a Schauder basis has a finite-dimensional blocking which is shrinking, then the basis is itself shrinking.

We are now ready to define the subspace Y of X_{AH} that will have the properties stated in Theorem 1.2.

Definition 2.3. We begin by recursively defining a sequence $(\Delta_n')_{n \geqslant 2}$ of non-empty, proper subsets of $(\Delta_n^{\mathrm{AH}})_{n \geqslant 2}$.

To start the recursion, we choose an element β_0 in Δ_2^{AH} and set $\Delta_2' = \{\beta_0\}$. This is certainly a non-empty subset of Δ_2^{AH}. It is also proper because Δ_2^{AH} contains at least two distinct elements, namely $(2, 1/m_2, \pm e_1^*)$.

Now let $n \geqslant 2$, and assume recursively that we have defined non-empty, proper subsets $\Delta_2', \dots, \Delta_n'$ of $\Delta_2^{\mathrm{AH}}, \dots, \Delta_n^{\mathrm{AH}}$, respectively. Set $\Gamma_n' = \bigcup_{j=2}^n \Delta_j'$, and define

$$\Delta_{n+1}' = \{\gamma \in \Delta_{n+1}^{\mathrm{AH}} : c_\gamma^*(\eta) \neq 0 \text{ for some } \eta \in \Gamma_n'\}. \tag{2.6}$$

Then Δ_{n+1}' is non-empty because it contains the element $(n+1, 1/m_2, e_{\beta_0}^*)$. To see that Δ_{n+1}' is a proper subset of $\Delta_{n+1}^{\mathrm{AH}}$, choose $\zeta \in \Delta_2^{\mathrm{AH}} \setminus \Delta_2'$. Then we have

$$\gamma := \Big(n+1, \frac{1}{m_2}, e_\zeta^*\Big) \in \Delta_{n+1}^{\mathrm{AH}},$$

and $c_\gamma^*(\eta) = e_\zeta^*(\eta)/m_2 = 0$ for each $\eta \in \Gamma^{\mathrm{AH}} \setminus \{\zeta\} \supseteq \Gamma_n'$, so that $\gamma \notin \Delta_{n+1}'$. This completes the recursion.

Set $\Gamma' = \bigcup_{n=2}^\infty \Delta_n'$, and define Y to be the closed subspace of X_{AH} spanned by the basic sequence $(d_\gamma)_{\gamma \in \Gamma'}$.

The definition of Y shows immediately that Y is infinite-dimensional and has infinite codimension in X_{AH} (because the sets Γ' and $\Gamma^{\mathrm{AH}} \setminus \Gamma'$ are infinite), and that $(d_\gamma)_{\gamma \in \Gamma'}$ is a Schauder basis for Y. This basis is shrinking because it is a subsequence of the shrinking basis $(d_\gamma)_{\gamma \in \Gamma^{\mathrm{AH}}}$ for X_{AH}. Thus clauses (i) and (ii) of Theorem 1.2 are satisfied. To establish the other two clauses, we require some further observations concerning Γ' and Y.

Lemma 2.4. *Let $\gamma \in \Gamma^{\mathrm{AH}}$. Then:*

(i) $\gamma \in \Gamma' \setminus \{\beta_0\}$ *if and only if* $c_\gamma^*|_{\Gamma'} \neq 0$;

(ii) $\gamma \in \Gamma'$ *if and only if* $d_\gamma^*|_{\Gamma'} \neq 0$.

Proof. Set $n = (\operatorname{rank}\gamma) - 1 \in \mathbb{N}_0$, so that $\gamma \in \Delta^{\mathrm{AH}}_{n+1}$.

(i). This is (almost) immediate from the definition of Γ'. Indeed, if $\gamma \in \Gamma' \setminus \{\beta_0\}$, then we have $n \geqslant 2$ and $c^*_\gamma(\eta) \neq 0$ for some $\eta \in \Gamma'_n$ by (2.6), so that $c^*_\gamma|_{\Gamma'} \neq 0$.

Conversely, suppose that $c^*_\gamma(\eta) \neq 0$ for some $\eta \in \Gamma'$. Then $\operatorname{rank}\eta \leqslant n$ because $\operatorname{supp} c^*_\gamma \subseteq \Gamma_n$. Hence $\eta \in \Gamma'_n$, and therefore $\gamma \in \Delta'_{n+1}$ by (2.6). We cannot have $\gamma = \beta_0$ because $\operatorname{rank}\beta_0 = 2$, so that $\operatorname{supp} c^*_{\beta_0} \subseteq \Gamma_1 = \{1\}$, which is disjoint from Γ'.

(ii). Recall that $d^*_\gamma = e^*_\gamma - c^*_\gamma$.

Suppose first that $\gamma \in \Gamma'$. Then $d^*_\gamma(\gamma) = 1$ because $c^*_\gamma(\gamma) = 0$, and so $d^*_\gamma|_{\Gamma'} \neq 0$.

Conversely, suppose that $d^*_\gamma(\eta) \neq 0$ for some $\eta \in \Gamma'$. If $\gamma = \eta$, then $\gamma \in \Gamma'$. Otherwise $e^*_\gamma(\eta) = 0$, so that $c^*_\gamma(\eta) \neq 0$, and the conclusion follows from (i). □

Lemma 2.5. *Let $\gamma \in \Gamma'$. Then $d_\gamma|_{\Gamma^{\mathrm{AH}} \setminus \Gamma'} = 0$.*

Proof. We shall prove the result inductively by showing that $d_\gamma(\eta) = 0$ for each $m \in \mathbb{N}$ and $\eta \in \Delta^{\mathrm{AH}}_m \setminus \Delta'_m$. To begin the induction, we observe that this is true whenever $m \leqslant \operatorname{rank}\gamma$ because $\operatorname{supp} d_\gamma \subseteq \{\gamma\} \cup (\Gamma^{\mathrm{AH}} \setminus \Gamma^{\mathrm{AH}}_{\operatorname{rank}\gamma})$ (see [2, p. 12]).

Now let $m \geqslant \operatorname{rank}\gamma$ and $\eta \in \Delta^{\mathrm{AH}}_{m+1} \setminus \Delta'_{m+1}$, and assume inductively that $d_\gamma(\xi) = 0$ for each $\xi \in \Gamma^{\mathrm{AH}}_m \setminus \Gamma'_m$. By Lemma 2.4(i), we have $c^*_\eta|_{\Gamma'} = 0$ and thus

$$c^*_\eta = \sum_{\xi \in \Gamma^{\mathrm{AH}}_m \setminus \Gamma'_m} c^*_\eta(\xi) e^*_\xi.$$

This implies that

$$d_\gamma(\eta) = \langle d_\gamma, d^*_\eta + c^*_\eta \rangle = 0 + \sum_{\xi \in \Gamma^{\mathrm{AH}}_m \setminus \Gamma'_m} c^*_\eta(\xi) d_\gamma(\xi) = 0$$

by the induction hypothesis, and hence the induction continues. □

To state the following two results concisely, we set $\Gamma'_0 = \Gamma'_1 = \emptyset$.

Corollary 2.6. *Let $y \in Y$. Then:*

(i) $\operatorname{supp} y \subseteq \Gamma'$.

(ii) *Suppose that* $\operatorname{ran} y \subseteq (p, q]$ *for some non-negative integers $p < q$. Then $y = i_q(y|_{\Gamma^{\mathrm{AH}}_q})$ and* $\operatorname{supp}(y|_{\Gamma^{\mathrm{AH}}_q}) \subseteq \Gamma'_q \setminus \Gamma'_p$.

Proof. (i). By the definition of Y, it suffices to show that $\operatorname{supp} d_\gamma \subseteq \Gamma'$ for each $\gamma \in \Gamma'$, that is, $d_\gamma(\eta) = 0$ for each $\eta \in \Gamma^{\mathrm{AH}} \setminus \Gamma'$, which is true by Lemma 2.5.

(ii). This follows by combining (i) with (2.4). □

Corollary 2.7. *Let $p < q$ be natural numbers. Then*

$$i_q[\ell_\infty(\Gamma'_q \setminus \Gamma'_p)] = \operatorname{span}\{d_\gamma : \gamma \in \Gamma'_q \setminus \Gamma'_p\}.$$

Proof. Set $F = \operatorname{span}\{d_\gamma : \gamma \in \Gamma'_q \setminus \Gamma'_p\}$. Corollary 2.6(ii) implies that $F \subseteq i_q[\ell_\infty(\Gamma'_q \setminus \Gamma'_p)]$, so that

$$|\Gamma'_q \setminus \Gamma'_p| = \dim F \leqslant \dim i_q[\ell_\infty(\Gamma'_q \setminus \Gamma'_p)] \leqslant \dim \ell_\infty(\Gamma'_q \setminus \Gamma'_p) = |\Gamma'_q \setminus \Gamma'_p| < \infty.$$

Hence $i_q[\ell_\infty(\Gamma'_q \setminus \Gamma'_p)]$ has the same finite dimension as its subspace F, so they are equal. □

Proof of Theorem 1.2(iii). Since Y has a shrinking basis, its dual is separable, so by a result of Lewis and Stegall (see [18, the second corollary of Theorem 2]), it suffices to show that Y is a $\mathscr{L}_\infty$-space. This follows from an argument similar to [2, Proposition 3.2]. Indeed, $(\operatorname{span}\{d_\gamma : \gamma \in \Gamma'_q\})_{q=2}^\infty$ is an increasing sequence of subspaces of Y whose union is dense in Y, and these subspaces are uniformly isomorphic to the finite-dimensional ℓ_∞-spaces of the corresponding dimensions by (2.3) and Corollary 2.7 (applied with $p = 1$). □

Clause (iv) of Theorem 1.2 is, not surprisingly, significantly harder to prove than clauses (i)–(iii). We shall follow closely Argyros and Haydon's proof of [2, Theorem 7.4], which shows that all bounded operators on X_{AH} have the form scalar-plus-compact. 'Rapidly increasing sequences' play a central role in this proof; their definition is as follows.

Definition 2.8. A *rapidly increasing sequence* (or *RIS* for short) in X_{AH} is a block sequence $(x_i)_{i\in\mathbb{I}}$ indexed by a non-empty (finite or infinite) interval $\mathbb{I}$ of $\mathbb{N}$ such that there are a constant $C > 0$ and a strictly increasing sequence $(j_i)_{i\in\mathbb{I}}$ of natural numbers satisfying

(i) $\|x_i\|_\infty \leqslant C$ for each $i \in \mathbb{I}$;
(ii) $\max\operatorname{ran} x_{i-1} < j_i$ for each $i \in \mathbb{I} \setminus \{\min \mathbb{I}\}$;
(iii) $|x_i(\gamma)| \leqslant C/m_k$ for each $i \in \mathbb{I}$ and each $\gamma \in \Gamma^{\mathrm{AH}}$ with $\operatorname{weight}\gamma = 1/m_k$ for some $k \in \mathbb{N} \cap [1, j_i)$.

If we need to specify the constant C in this definition, we refer to a *C-RIS*.

We say that a RIS $(x_i)_{i\in\mathbb{I}}$ is *semi-normalized* if $\inf_{i\in\mathbb{I}} \|x_i\|_\infty > 0$. (Note that condition (i), above, ensures that $\sup_{i\in\mathbb{I}} \|x_i\|_\infty < \infty$.)

Let W be a subset of X_{AH}. By a *RIS in W*, we mean a sequence $(x_i)_{i\in\mathbb{I}}$ that is a RIS in the above sense and satisfies $x_i \in W$ for each $i \in \mathbb{I}$.

Our first aim is to establish the following variant of [2, Proposition 5.11] for bounded operators defined on the subspace Y of X_{AH}.

Proposition 2.9. *Let T be a bounded operator from Y into a Banach space. Then the following three conditions are equivalent:*

(a) *every RIS $(x_i)_{i\in\mathbb{N}}$ in Y has a subsequence $(x'_i)_{i\in\mathbb{N}}$ such that $\|Tx'_i\| \to 0$ as $i \to \infty$;*
(b) *every bounded block sequence $(x_i)_{i\in\mathbb{N}}$ in Y has a subsequence $(x'_i)_{i\in\mathbb{N}}$ such that $\|Tx'_i\| \to 0$ as $i \to \infty$;*
(c) *the operator T is compact.*

As in [2], the proof of this result relies heavily on the following two notions.

Definition 2.10. A block sequence $(x_i)_{i\in\mathbb{N}}$ in $X_{\mathrm{AH}} \setminus \{0\}$ has:

- *bounded local weight* if $\inf\bigl\{\operatorname{weight}\gamma : \gamma \in \bigcup_{i\in\mathbb{N}} \operatorname{locsupp} x_i\bigr\} > 0$;
- *rapidly decreasing local weight* if, for each $i \in \mathbb{N}$ and $\gamma \in \operatorname{locsupp} x_{i+1}$, we have $\operatorname{weight}\gamma < 1/m_{q_i}$, where $q_i := \max\operatorname{ran} x_i$.

Proposition 2.11 ([2, Proposition 5.10])**.** *Let $(x_i)_{i\in\mathbb{N}}$ be a bounded block sequence in $X_{\mathrm{AH}} \setminus \{0\}$, and suppose that $(x_i)_{i\in\mathbb{N}}$ has either bounded local weight or rapidly decreasing local weight. Then $(x_i)_{i\in\mathbb{N}}$ is a RIS.*

Proof of Proposition 2.9. The implication (b)$\Rightarrow$(a) is obvious.

(b)$\Leftrightarrow$(c). Each subsequence of a bounded block sequence is evidently itself a bounded block sequence. Hence condition (b) is equivalent to the formally stronger statement that $\|Tx_i\| \to 0$ as $i \to \infty$ for every bounded block sequence $(x_i)_{i\in\mathbb{N}}$ in Y, and this latter statement is in turn equivalent to condition (c) by Lemma 2.1(ii), which applies because the basis $(d_\gamma)_{\gamma\in\Gamma'}$ for Y is shrinking.

It remains to prove that (a)$\Rightarrow$(b), which we shall accomplish by adapting the proof of [2, Proposition 5.11]. We begin by observing that since each subsequence of a RIS is a RIS, condition (a) is equivalent to the formally stronger statement that $\|Tx_i\| \to 0$ as $i \to \infty$ for every RIS $(x_i)_{i\in\mathbb{N}}$ in Y. Suppose that this statement holds true, let $(x_j)_{j\in\mathbb{N}}$ be a bounded block sequence in Y, and choose integers $0 = q_0 < q_1 < q_2 < \cdots$ such that $\operatorname{ran} x_j \subseteq (q_{j-1}, q_j]$ for each $j \in \mathbb{N}$. Fix $j, k \in \mathbb{N}$, set $u_j = x_j|_{\Gamma^{\mathrm{AH}}_{q_j}}$ and, for each $\gamma \in \Gamma^{\mathrm{AH}}_{q_j}$, define

$$v_j^k(\gamma) = \begin{cases} u_j(\gamma) & \text{if weight}\,\gamma \geqslant 1/m_k \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad w_j^k(\gamma) = \begin{cases} u_j(\gamma) & \text{if weight}\,\gamma < 1/m_k \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $u_j = v_j^k + w_j^k$, $\|v_j^k\|_\infty \vee \|w_j^k\|_\infty = \|u_j\|_\infty \leqslant \|x_j\|_\infty$ and

$$\operatorname{supp} v_j^k \cup \operatorname{supp} w_j^k = \operatorname{supp} u_j \subseteq \Gamma'_{q_j} \setminus \Gamma'_{q_{j-1}}$$

by Corollary 2.6(ii). Hence $y_j^k := i_{q_j}(v_j^k)$ and $z_j^k := i_{q_j}(w_j^k)$ satisfy $y_j^k + z_j^k = i_{q_j}(u_j) = x_j$, they both belong to $\operatorname{span}\{d_\gamma : \gamma \in \Gamma'_{q_j} \setminus \Gamma'_{q_{j-1}}\}$ by Corollary 2.7, and their norms are at most $2\|x_j\|_\infty$ by (2.3) and Remark 2.2(i). Thus $(y_j^k)_{j\in\mathbb{N}}$ and $(z_j^k)_{j\in\mathbb{N}}$ are bounded block sequences in Y. Using (2.5), we obtain

$$\operatorname{locsupp} y_j^k \subseteq \operatorname{supp} v_j^k = \Big\{\gamma \in \operatorname{supp} u_j : \text{weight}\,\gamma \geqslant \frac{1}{m_k}\Big\},$$

so that $(y_j^k)_{j\in\mathbb{N}}$ has bounded local weight, and it is therefore a RIS by Proposition 2.11. Hence the assumption implies that $\|Ty_j^k\| \to 0$ as $j \to \infty$, so that we can recursively choose integers $1 < j_1 < j_2 < \cdots$ such that $\|Ty_{j_k}^k\| \to 0$ as $k \to \infty$. Set $k_1 = 1$ and, recursively, define $k_{p+1} = q_{j_{k_p}}$ for $p \in \mathbb{N}$. Then $(z_{j_{k_p}}^{k_p})_{p\in\mathbb{N}}$ is a bounded block sequence with rapidly decreasing local weight, so it is a RIS by Proposition 2.11, and hence $\|Tz_{j_{k_p}}^{k_p}\| \to 0$ as $p \to \infty$. It now follows that $x'_p := x_{j_{k_p}} = y_{j_{k_p}}^{k_p} + z_{j_{k_p}}^{k_p}$ $(p \in \mathbb{N})$ is a subsequence of $(x_j)_{j\in\mathbb{N}}$ such that $\|Tx'_p\| \to 0$ as $p \to \infty$. $\square$

We shall next establish a lemma which generalizes [2, Lemma 7.2 and Proposition 7.3]. While we shall require it only for $\Upsilon = \Gamma'$, we have chosen to state it in greater generality to highlight that, unlike Proposition 2.9, it does not depend on any special properties of the set Γ'. The statement of this lemma involves three further notions. First, for a subspace W of X_{AH}, we denote by $W \cap \mathbb{L}^{\Gamma^{\mathrm{AH}}}$ the set of $w \in W$ such that $w(\gamma) \in \mathbb{L}$ for each $\gamma \in \Gamma^{\mathrm{AH}}$, where we recall that $\mathbb{L}$ is the subfield of the scalar field given by (2.1). Second, for natural numbers $p < q$, we write $P^*_{(p,q]}$ for the operator $P^*_{(0,q]} - P^*_{(0,p]}$ and denote by $P_{(p,q]}$ its adjoint. Third, we require the following piece of terminology, which originates from [2, Definition 6.1].

Definition 2.12. Let $C > 0$ and $j \in \mathbb{N}$. A $(C, j, 0)$-*exact pair* is a pair $(z, \eta) \in X_{\mathrm{AH}} \times \Gamma^{\mathrm{AH}}$ that satisfies:

- $|\langle d_\xi^*, z\rangle| \leqslant C/m_j$ for each $\xi \in \Gamma^{\mathrm{AH}}$, weight $\eta = 1/m_j$, $\|z\|_\infty \leqslant C$ and $z(\eta) = 0$;
- $|z(\xi)| \leqslant C/m_{i\wedge j}$ for each $i \in \mathbb{N} \setminus \{j\}$ and each $\xi \in \Gamma^{\mathrm{AH}}$ with weight $\xi = 1/m_i$.

Lemma 2.13. *Let $C > 0$, let $W = \overline{\operatorname{span}}\{d_\gamma : \gamma \in \Upsilon\}$ for some non-empty subset Υ of Γ^{AH}, and let $T \colon W \to X_{\mathrm{AH}}$ be a bounded operator.*

(i) *Let $(x_i)_{i\in\mathbb{I}}$ be a C-RIS in W, where $\mathbb{I}$ is a non-empty interval of $\mathbb{N}$. Then, for each $\varepsilon > 0$, there is a $(C + \varepsilon)$-RIS $(y_i)_{i\in\mathbb{I}}$ in $W \cap \mathbb{L}^{\Gamma^{\mathrm{AH}}}$ such that $\|x_i - y_i\|_\infty \leqslant \varepsilon$ for each $i \in \mathbb{I}$.*

(ii) *Suppose that $\operatorname{dist}(Tx_i, \mathbb{K}x_i) \to 0$ as $i \to \infty$ for every RIS $(x_i)_{i\in\mathbb{N}}$ in $W \cap \mathbb{L}^{\Gamma^{\mathrm{AH}}}$. Then $\operatorname{dist}(Tx_i, \mathbb{K}x_i) \to 0$ as $i \to \infty$ for every RIS $(x_i)_{i\in\mathbb{N}}$ in W.*

(iii) *Let $\delta > 0$, and let $(x_i)_{i\in\mathbb{N}}$ be a C-RIS in $W \cap \mathbb{L}^{\Gamma^{\mathrm{AH}}}$ such that $\operatorname{dist}(Tx_i, \mathbb{K}x_i) > \delta$ for each $i \in \mathbb{N}$. Then, for each $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$, there are $z \in \operatorname{span}\{x_i : i \in \mathbb{N}\} \subseteq W$, $q \in \mathbb{N} \cap (p, \infty)$ and $\eta \in \Delta_q^{\mathrm{AH}}$ such that the following five conditions are satisfied:*
   (1) $\operatorname{ran} z \subseteq (p, q)$;
   (2) (z, η) *is a* $(16C, 2j, 0)$*-exact pair;*
   (3) $\operatorname{Re}(Tz)(\eta) > 7\delta/16$;
   (4) $\|(I_{X_{\mathrm{AH}}} - P_{(p,q]})Tz\|_\infty < \delta/m_{2j}$;
   (5) $\operatorname{Re}\langle Tz, P_{(p,q]}^* e_\eta^*\rangle > 3\delta/8$.

(iv) *For every RIS $(x_i)_{i\in\mathbb{N}}$ in W, $\operatorname{dist}(Tx_i, \mathbb{K}x_i) \to 0$ as $i \to \infty$.*

Proof. Clauses (i) and (ii) are both proved by standard approximation arguments. We omit the details.

(iii). Since $(x_i)_{i\in\mathbb{N}}$ is a bounded block sequence with respect to the shrinking basis $(d_\gamma)_{\gamma\in\Upsilon}$ for W, it is weakly null in W. Being bounded, the operator T is automatically weakly continuous, so that $(Tx_i)_{i\in\mathbb{N}}$ is weakly null in X_{AH}. Now the remainder of the proof of [2, Lemma 7.2] carries over verbatim. (Note the need for the real part in conditions (3) and (5); this is due to the fact that we consider complex as well as real scalars.)

(iv). Assume towards a contradiction that there is a RIS $(x_i)_{i\in\mathbb{N}}$ in W such that $\operatorname{dist}(Tx_i, \mathbb{K}x_i) \not\to 0$ as $i \to \infty$. By (ii), we may suppose that $x_i \in W \cap \mathbb{L}^{\Gamma^{\mathrm{AH}}}$ for each $i \in \mathbb{N}$. We may now proceed exactly as in the proof of [2, Proposition 7.3] to reach a contradiction, using (iii) instead of [2, Lemma 7.2] and noting that the element

$$z = \frac{1}{n_{2j_0-1}} \sum_{i=1}^{n_{2j_0-1}} z_i$$

defined in [2, p. 34] belongs to W, so that we may apply the operator T to it. □

Finally, we can prove clause (iv) of Theorem 1.2.

Proof of Theorem 1.2(iv). Lemma 2.13(iv) shows that, for each RIS $(x_i)_{i\in\mathbb{N}}$ in Y, there is a scalar sequence $(\lambda_i)_{i\in\mathbb{N}}$ such that $\|Tx_i - \lambda_i x_i\|_\infty \to 0$ as $i \to \infty$. Suppose that $(x_i)_{i\in\mathbb{N}}$ is semi-normalized. Then, arguing as in the proof of [2, Theorem 7.4], we deduce that $(\lambda_i)_{i\in\mathbb{N}}$ is convergent and that the limit is independent of the choice of $(\lambda_i)_{i\in\mathbb{N}}$ and $(x_i)_{i\in\mathbb{N}}$; that is, we have a scalar λ such that

$$\|Tx_i - \lambda x_i\|_\infty \leqslant \|Tx_i - \lambda_i x_i\|_\infty + |\lambda - \lambda_i|\,\|x_i\|_\infty \to 0 \quad \text{as} \quad i \to \infty \tag{2.7}$$

for every semi-normalized RIS $(x_i)_{i\in\mathbb{N}}$ in Y.

We shall now complete the proof by showing that the operator $T - \lambda J$ is compact. By Proposition 2.9, we must show that every RIS $(x_i)_{i\in\mathbb{N}}$ in Y has a subsequence $(x'_i)_{i\in\mathbb{N}}$ such that $\|Tx'_i - \lambda x'_i\|_\infty \to 0$ as $i \to \infty$. If $(x_i)_{i\in\mathbb{N}}$ is semi-normalized, then this follows from (2.7) (and there is no need to pass to a subsequence). Otherwise $(x_i)_{i\in\mathbb{N}}$ has a subsequence $(x'_i)_{i\in\mathbb{N}}$ which is norm-null, in which case the conclusion is obvious (because the operator $T - \lambda J$ is bounded). □

3. The lattice of closed two-sided ideals of $\mathscr{B}(Z)$: the proofs of Theorem 1.4 and Proposition 1.6

Denote by $\mathscr{T}_2$ the algebra of upper triangular (2×2)-matrices over $\mathbb{K}$. Since every bounded operator on Z has a unique matrix representation of the form (1.2), we can define unital algebra homomorphisms by

$$\varphi\colon\ \begin{pmatrix} \alpha_{1,1}I_{X_{\mathrm{AH}}} + K_{1,1} & \alpha_{1,2}J + K_{1,2} \\ K_{2,1} & \alpha_{2,2}I_Y + K_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ 0 & \alpha_{2,2} \end{pmatrix}, \quad \mathscr{B}(Z) \to \mathscr{T}_2, \tag{3.1}$$

and

$$\psi\colon\ \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ 0 & \alpha_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_{1,1}I_{X_{\mathrm{AH}}} & \alpha_{1,2}J \\ 0 & \alpha_{2,2}I_Y \end{pmatrix}, \quad \mathscr{T}_2 \to \mathscr{B}(Z). \tag{3.2}$$

Clearly $\ker\varphi = \mathscr{K}(Z)$, and the composition $\varphi\circ\psi$ is equal to the identity operator on $\mathscr{T}_2$, so that we have a split-exact sequence

$$\{0\} \longrightarrow \mathscr{K}(Z) \xrightarrow{\ \iota\ } \mathscr{B}(Z) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathscr{T}_2 \longrightarrow \{0\},$$

where $\iota\colon \mathscr{K}(Z) \to \mathscr{B}(Z)$ is the inclusion map.

Proof of Theorem 1.4. For each two-sided ideal $\mathscr{I}$ of $\mathscr{T}_2$, the pre-image $\varphi^{-1}[\mathscr{I}]$ under φ is a two-sided ideal of $\mathscr{B}(Z)$. The identity $\varphi^{-1}[\mathscr{I}] = \psi[\mathscr{I}] + \mathscr{K}(Z)$ shows that this ideal is closed (as the sum of a finite-dimensional subspace and a closed subspace), and the map $\mathscr{I} \mapsto \varphi^{-1}[\mathscr{I}]$ is an order isomorphism of the lattice of two-sided ideals of $\mathscr{T}_2$ onto the lattice of closed two-sided ideals of $\mathscr{B}(Z)$ that contain $\mathscr{K}(Z)$. Since X_{AH} and Y both have Schauder bases, $\mathscr{K}(Z)$ is the minimum non-zero closed two-sided ideal of $\mathscr{B}(Z)$. Hence the conclusion follows from the standard elementary fact that the lattice of two-sided ideals of $\mathscr{T}_2$ is given by

$$\begin{tikzcd}[column sep=small]
 & \mathscr{T}_2 \arrow[dl, dash] \arrow[dr, dash] & \\
\left\{\begin{pmatrix}\alpha_{1,1} & \alpha_{1,2}\\ 0 & 0\end{pmatrix} : \alpha_{1,1},\, \alpha_{1,2} \in \mathbb{K}\right\} = \mathscr{R}_1 \arrow[dr, dash] & & \mathscr{C}_2 = \left\{\begin{pmatrix}0 & \alpha_{1,2}\\ 0 & \alpha_{2,2}\end{pmatrix} : \alpha_{1,2},\, \alpha_{2,2} \in \mathbb{K}\right\} \arrow[dl, dash] \\
 & \operatorname{rad} \mathscr{T}_2 = \mathscr{R}_1 \cap \mathscr{C}_2 = \left\{\begin{pmatrix}0 & \alpha_{1,2}\\ 0 & 0\end{pmatrix} : \alpha_{1,2} \in \mathbb{K}\right\} \arrow[d, dash] & \\
 & \{0\}, &
\end{tikzcd}$$

where $\operatorname{rad} \mathscr{T}_2$ denotes the Jacobson radical of $\mathscr{T}_2$, and the lines denote proper inclusions with the larger ideal at the top. $\square$

Proof of Proposition 1.6. Endow $\mathscr{T}_2$ with an algebra norm. (Since $\mathscr{T}_2$ is finite-dimensional, all norms on it are equivalent, so it does not matter which one we choose.) Then $\mathscr{T}_2$ is a standard example of a non-amenable Banach algebra, for instance because the map

$$\begin{pmatrix}\alpha_{1,1} & \alpha_{1,2}\\ 0 & \alpha_{2,2}\end{pmatrix} \mapsto \begin{pmatrix}0 & \alpha_{1,2}\\ 0 & 0\end{pmatrix}, \quad \mathscr{T}_2 \to \operatorname{rad} \mathscr{T}_2,$$

is a bounded derivation which is not inner (and its codomain $\operatorname{rad} \mathscr{T}_2$ is a dual Banach $\mathscr{T}_2$-bimodule because it is a finite-dimensional two-sided ideal of $\mathscr{T}_2$). Moreover, the map

$$A \mapsto \psi(A) + \mathscr{K}(Z), \quad \mathscr{T}_2 \to \mathscr{B}(Z)/\mathscr{K}(Z), \tag{3.3}$$

where ψ is given by (3.2), is an algebra isomorphism, which is automatically bounded because its domain is finite-dimensional, so that $\mathscr{T}_2$ is isomorphic to a quotient of $\mathscr{B}(Z)$, and hence the conclusion follows from [7, Proposition 2.8.64(ii)]. $\square$

Remark 3.1. The proof of Proposition 1.6 shows that the algebra homomorphism φ given by (3.1) is bounded because it is the composition of the quotient homomorphism of $\mathscr{B}(Z)$ onto $\mathscr{B}(Z)/\mathscr{K}(Z)$ with the inverse of the isomorphism (3.3).

4. Approximate identities: the proof of Theorem 1.8

Recall that, for $n \in \mathbb{N}$, $P_{(0,n]}|X_{\mathrm{AH}}$ is the n^{th} canonical projection associated with the shrinking FDD $(M_k)_{k\in\mathbb{N}} = (\operatorname{span}\{d_\gamma : \gamma \in \Delta_k^{\mathrm{AH}}\})_{k\in\mathbb{N}}$ for X_{AH}. Clearly the subspace Y is $P_{(0,n]}$-invariant, and the restriction $P_{(0,n]}|Y$ is the n^{th} canonical projection associated with the shrinking FDD $(\operatorname{span}\{d_\gamma : \gamma \in \Delta'_k\})_{k=2}^\infty$ for Y. Consequently Lemma 2.1 implies the following result, which establishes all the positive statements concerning the existence of bounded approximate identities in Theorem 1.8.

Proposition 4.1. (i) *The sequence*

$$\left(\begin{pmatrix}I_{X_{\mathrm{AH}}} & 0\\ 0 & P_{(0,n]}|Y\end{pmatrix}\right)_{n\in\mathbb{N}}$$

is a bounded left approximate identity in $\mathscr{M}_1$.

(ii) *The sequence*
$$\left(\begin{pmatrix} P_{(0,\,n]}|X_{\mathrm{AH}} & 0 \\ 0 & I_Y \end{pmatrix}\right)_{n\in\mathbb{N}}$$
is a bounded right approximate identity in $\mathscr{M}_2$.

(iii) *The sequence*
$$\left(\begin{pmatrix} P_{(0,\,n]}|X_{\mathrm{AH}} & 0 \\ 0 & P_{(0,\,n]}|Y \end{pmatrix}\right)_{n\in\mathbb{N}}$$
is a bounded two-sided approximate identity in $\mathscr{K}(Z)$.

The non-existence statements in Theorem 1.8 will all be easy consequences of the following lemma.

Lemma 4.2. *The inclusion map* $J\colon Y \to X_{\mathrm{AH}}$ *has distance* 1 *to the ideal of compact operators, in the sense that*
$$\inf\bigl\{\|J-K\| : K \in \mathscr{K}(Y, X_{\mathrm{AH}})\bigr\} = 1.$$

Proof. The right-hand side dominates the left-hand side because $\|J\| = 1$.

On the other hand, given $\varepsilon > 0$ and $K \in \mathscr{K}(Y, X_{\mathrm{AH}})$, we can find $n \in \mathbb{N}$ such that $\|K - P_{(0,\,n]}K\| \leqslant \varepsilon/2$ by Lemma 2.1(i). Riesz's lemma (see, *e.g.*, [6, Lemma 1.1.1]) implies that there exists a unit vector $y \in Y$ such that $\|y - P_{(0,\,n]}x\|_\infty \geqslant 1 - \varepsilon/2$ for each $x \in X_{\mathrm{AH}}$, and hence
$$\|J-K\| \geqslant \|(J-K)y\|_\infty \geqslant \|y - P_{(0,\,n]}Ky\|_\infty - \|Ky - P_{(0,\,n]}Ky\|_\infty \geqslant 1-\varepsilon,$$
from which the conclusion follows. ☐

Proof of Theorem 1.8(i). For each
$$T = \begin{pmatrix} \alpha_{1,1}I_{X_{\mathrm{AH}}} + K_{1,1} & \alpha_{1,2}J + K_{1,2} \\ K_{2,1} & K_{2,2} \end{pmatrix} \in \mathscr{M}_1,$$
where $\alpha_{1,1}, \alpha_{1,2} \in \mathbb{K}$ and $K_{1,1}, \ldots, K_{2,2}$ are compact, we have
$$\left\|\begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} T\right\| = \left\|\begin{pmatrix} -JK_{2,1} & J - JK_{2,2} \\ 0 & 0 \end{pmatrix}\right\| \geqslant \|J - JK_{2,2}\| \geqslant 1$$
by Lemma 4.2. Hence $\mathscr{M}_1$ has no right approximate identity.

The other statements are proved similarly. ☐

5. Maximal left ideals of $\mathscr{B}(Z)$: the proof of Theorem 1.9

The key ingredient in our proof of Theorem 1.9, besides the properties of X_{AH} and Y stated in Theorems 1.1 and 1.2, is the following extension theorem of Grothendieck (see [12, pp. 559–560], or [19, Theorem 1]), which applies to compact operators into X_{AH} or Y because they are both isomorphic preduals of ℓ_1.

Theorem 5.1 (Grothendieck)**.** *Let* E *be a subspace of a Banach space* F*, and let* G *be a Banach space whose dual space is isomorphic to* $L_1(\mu)$ *for some measure* μ*. Then every compact operator from* E *into* G *has an extension to a compact operator from* F *into* G*.*

We shall also require the following elementary observation regarding the maximal two-sided ideals

$$\mathscr{R}_1 = \left\{ \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ 0 & 0 \end{pmatrix} : \alpha_{1,1},\, \alpha_{1,2} \in \mathbb{K} \right\} \qquad \text{and} \qquad \mathscr{C}_2 = \left\{ \begin{pmatrix} 0 & \alpha_{1,2} \\ 0 & \alpha_{2,2} \end{pmatrix} : \alpha_{1,2},\, \alpha_{2,2} \in \mathbb{K} \right\}$$

of $\mathscr{T}_2$ that were introduced in the proof of Theorem 1.4. This observation is probably well known, but we include a short proof for completeness.

Lemma 5.2. *The ideals $\mathscr{R}_1$ and $\mathscr{C}_2$ are the only maximal left ideals of $\mathscr{T}_2$.*

Proof. Both $\mathscr{R}_1$ and $\mathscr{C}_2$ have codimension one in $\mathscr{T}_2$, so that they are maximal as left ideals.

Let $\mathscr{L}$ be any maximal left ideal of $\mathscr{T}_2$. As noted in the proof of Theorem 1.4, the Jacobson radical of $\mathscr{T}_2$ is given by

$$\operatorname{rad} \mathscr{T}_2 = \left\{ \begin{pmatrix} 0 & \alpha_{1,2} \\ 0 & 0 \end{pmatrix} : \alpha_{1,2} \in \mathbb{K} \right\}.$$

This ideal is not maximal as a left ideal because it is properly contained in $\mathscr{R}_1$ and $\mathscr{C}_2$. Hence the definition of the Jacobson radical as the intersection of all the maximal left ideals of $\mathscr{T}_2$ implies that $\mathscr{L}$ contains $\operatorname{rad} \mathscr{T}_2$ properly, and consequently we can find

$$\begin{pmatrix} \alpha_{1,1} & 0 \\ 0 & \alpha_{2,2} \end{pmatrix} \in \mathscr{L}$$

with either $\alpha_{1,1} \neq 0$ or $\alpha_{2,2} \neq 0$. In the first case we conclude that

$$\begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \beta/\alpha_{1,1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{1,1} & 0 \\ 0 & \alpha_{2,2} \end{pmatrix} \in \mathscr{L} \qquad (\beta \in \mathbb{K}),$$

so that $\mathscr{R}_1 \subseteq \mathscr{L}$, and hence $\mathscr{R}_1 = \mathscr{L}$ by the maximality of $\mathscr{R}_1$. A similar argument shows that $\mathscr{L} = \mathscr{C}_2$ in the second case. $\square$

Proof of Theorem 1.9. As in the proof of Theorem 1.4, we see that $\mathscr{L} \mapsto \varphi^{-1}[\mathscr{L}]$ defines an order isomorphism of the lattice of left ideals of $\mathscr{T}_2$ onto the lattice of closed left ideals of $\mathscr{B}(Z)$ that contain $\mathscr{K}(Z)$. By [8, Corollary 4.1], every non-fixed, maximal left ideal of $\mathscr{B}(Z)$ contains $\mathscr{E}(Z)$ and hence $\mathscr{K}(Z)$, and therefore Lemma 5.2 shows that $\mathscr{M}_1 = \varphi^{-1}[\mathscr{R}_1]$ and $\mathscr{M}_2 = \varphi^{-1}[\mathscr{C}_2]$ are the only non-fixed, maximal left ideals of $\mathscr{B}(Z)$.

(i). For each $T = (T_{j,k})_{j,k=1}^2 \in \mathscr{M}_1$, the operator $T_{2,2}$ is compact, so that it has a compact extension $\widetilde{T}_{2,2}\colon X_{\mathrm{AH}} \to Y$ by Theorems 1.2(iii) and 5.1. Moreover, we may express $T_{1,2}$ in the form $T_{1,2} = \alpha_{1,2}J + K_{1,2}$, where $\alpha_{1,2} \in \mathbb{K}$ and $K_{1,2}\colon Y \to X_{\mathrm{AH}}$ is compact, and then another application of Theorem 5.1 gives a compact operator $\widetilde{K}_{1,2}\colon X_{\mathrm{AH}} \to X_{\mathrm{AH}}$ that extends $K_{1,2}$. Hence we have

$$T = \begin{pmatrix} T_{1,1} & 0 \\ T_{2,1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & T_{1,2} \\ 0 & T_{2,2} \end{pmatrix} = T \begin{pmatrix} I_{X_{\mathrm{AH}}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_{1,2} I_{X_{\mathrm{AH}}} + \widetilde{K}_{1,2} & 0 \\ \widetilde{T}_{2,2} & 0 \end{pmatrix} \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix},$$

which shows that $\mathscr{M}_1$ is generated as a left ideal by the pair of operators given by (1.5).

On the other hand, to see that $\mathscr{M}_1$ is not generated as a left ideal by a single bounded operator, assume the contrary, and let $R = (R_{j,k})_{j,k=1}^2$ be a generator of $\mathscr{M}_1$. Take

$\alpha_{1,1}, \alpha_{1,2} \in \mathbb{K}$ and compact operators $K_{1,1}$ and $K_{1,2}$ such that $R_{1,1} = \alpha_{1,1} I_{X_{\mathrm{AH}}} + K_{1,1}$ and $R_{1,2} = \alpha_{1,2} J + K_{1,2}$. Since the operators given by (1.5) both belong to $\mathscr{M}_1$, we can find bounded operators $S = (S_{j,k})_{j,k=1}^2$ and $T = (T_{j,k})_{j,k=1}^2$ on Z such that

$$\begin{pmatrix} I_{X_{\mathrm{AH}}} & 0 \\ 0 & 0 \end{pmatrix} = SR \qquad \text{and} \qquad \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} = TR. \tag{5.1}$$

Write $S_{1,1} = \beta I_{X_{\mathrm{AH}}} + U_{1,1}$ and $T_{1,1} = \gamma I_{X_{\mathrm{AH}}} + V_{1,1}$, where $\beta, \gamma \in \mathbb{K}$ and $U_{1,1}$ and $V_{1,1}$ are compact. The first part of (5.1) implies that $\beta\alpha_{1,1} = 1$ and $\beta\alpha_{1,2} = 0$, so that necessarily $\alpha_{1,2} = 0$, while the second part shows that $\gamma\alpha_{1,1} = 0$ and $\gamma\alpha_{1,2} = 1$. This is clearly impossible, and hence $\mathscr{M}_1$ cannot be generated as a left ideal by a single operator.

(ii). Assume towards a contradiction that $\mathscr{M}_2$ is the left ideal of $\mathscr{B}(Z)$ generated by the operators $R_1, \ldots, R_n$ for some $n \in \mathbb{N}$. The definition (1.3) of $\mathscr{M}_2$ implies that, for each $j \in \{1, \ldots, n\}$, we can find $\beta_j, \gamma_j \in \mathbb{K}$ and $K_j \in \mathscr{K}(Z)$ such that $R_j = S_j + K_j$, where

$$S_j = \begin{pmatrix} 0 & \beta_j J \\ 0 & \gamma_j I_Y \end{pmatrix}. \tag{5.2}$$

By [8, Corollary 4.7], the operator

$$\Psi\colon\ z \mapsto (R_1 z, \ldots, R_n z), \quad Z \to Z^n,$$

is bounded below, and it is thus an upper semi-Fredholm operator; that is, Ψ has finite-dimensional kernel and closed image. Since the set of upper semi-Fredholm operators is closed under compact perturbations (see, *e.g.*, [6, Corollary 1.3.7]), the operator

$$S\colon\ z \mapsto \Psi z - (K_1 z, \ldots, K_n z) = (S_1 z, \ldots, S_n z), \quad Z \to Z^n,$$

is also an upper semi-Fredholm operator, so that its kernel is finite-dimensional. This, however, contradicts the fact that $S(x, 0) = (0, \ldots, 0)$ for each $x \in X_{\mathrm{AH}}$ by (5.2). ☐

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