Exploring the Edges of Electromagnetism using Extreme Fields

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Abstract

This thesis considers aspects of nonlinear electromagnetism and the effects of spin under the influence of extreme fields. Born-Infeld-like theories are studied in the context of possible slow light experiments. Maximum amplitude plasma waves are considered as a possible testing ground for nonlinear electrodynamics with regards to electron energy gain. Finally the effects of the coupling between the electromagnetic field and the spin of a relativistic classical particle are considered via a new derivation of the relativistic Stern-Gerlach and Thomas-Bargmann-Michel-Telegdi equations. These equations are then paired with the Nakano-Tulczyjew condition and, as the Stern-Gerlach-type terms in the equations of motion are most prominent in a field with a high field gradient, the impact of spin is investigated in the context of a maximum amplitude plasma wave.

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Last, but by no means least, my heartfelt gratitude to my family for their endless support, love and patience in accepting my love for a subject I could never truly explain. I would never have gotten this far without you.

Declaration

I declare that this thesis is my own work carried out in collaboration with my PhD supervisor Dr. David A. Burton. No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification at this or any other institute of learning.

The constant magnetic field portion of the work presented in Chapter 3 was published in EPL in 2012 (see Ref. [1]).

“There is nothing like looking, if you want to find something. You certainly usually find something, if you look, but it is not always quite the something you were after.”

J. R. R. Tolkien, *The Hobbit*

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Chapter 1

Introduction

A brief note on units and conventions: this thesis uses units where $\epsilon_0 = \mu_0 = c = 1$ *unless otherwise stated (see Section 5.3.4), and the flat spacetime metric* η *has signature* $\{-, +, +, +\}$*. The Einstein summation convention is used throughout: Latin indices are summed over 0 to 3, i.e.* $X^a Y_a = X^0 Y_0 + X^1 Y_1 + X^2 Y_2 + X^3 Y_3$*. Notation* $\{x, y, z\}$ *is used to denote a set containing elements* x, y, z*. Quantities with indices will have these indices in italics, whereas quantities with labels will be in normal text; for instance the electron current* j_e*.*

The classical theory of electrodynamics is one of the most celebrated physical theories in terms of its usefulness in physics and engineering. It is known, however, that classical Maxwell theory is not without its problems (see Ref. [2] for discussion); since the Coulomb law is a key part of the theory, the electric field of charged particles is $\sim 1/\vec{r}$, which diverges as one approaches the particle itself. This leads to singular self-energy of classical charged particles. Quantum electrodynamics (QED) also has similar issues, and hence the theory requires methods such as renormalisation and regularisation to avoid these singularities.

Maxwell theory is called a *linear* theory of electromagnetism, since its Lagrangian density depends on the electromagnetic Lorentz invariants

$$X = \vec{E}^2 - \vec{B}^2, \tag{1.1}$$

$$\text{and} \quad Y = 2\vec{E} \cdot \vec{B}, \tag{1.2}$$

in a linear fashion (in fact $L_{\rm M} = X/2$). It is equivalent to say that the constitutive relations are linear in X and Y since

$$\vec{D} = \frac{\partial L}{\partial \vec{E}}, \qquad \vec{H} = -\frac{\partial L}{\partial \vec{B}}, \tag{1.3}$$

(and clearly with $L_{\rm M}$, the linear relations $\vec{D} = \vec{E}$ and $\vec{H} = \vec{B}$ are retrieved).

Nonlinear electrodynamics originated in the early 20th century with the aim of classically improving upon some of the failings of classical electrodynamics by introducing a nonlinear dependence of the Lagrangian on X and Y. The most famous of these theories developed in the last century is Born-Infeld theory [3]. There are also nonlinear theories of electrodynamics arising from quantum mechanical approaches, such as Euler-Heisenberg theory [4], which arises from one loop calculations of the quantum vacuum. With experiments such as the Extreme Light Infrastructure (ELI) [5] and the European High Power laser Energy Research facility (HiPER) [6] approaching completion, with anticipated laser field strength approaching 10^{25} Wcm^{-2} [7], for the first time it may be possible to test for any nonlinearity of electrodynamics outside of the predictions of QED through effects such as photon-photon scattering [8]. For a review of QED and nonlinear electrodynamics see Ref. [9].

In the early 20th century, before the development of renormalised QED, Max Born and Leopold Infeld attempted to fix the problem of the infinite self-energy of the electron by extending Maxwell electrodynamics into nonlinearity. As previously stated, Maxwell theory can be written as the Lagrangian density $L_{\rm M} = X/2$. Born and Infeld decided to keep the theory manifestly Lorentz invariant and hence wrote their theory in terms of the electromagnetic invariants X and Y, arriving at the Lagrangian density

$$L_{\rm BI} = b^2 \left(1 - \sqrt{-\det\left(\eta + \frac{F}{b}\right)}\right) \tag{1.4}$$

$$= b^2 \left(1 - \sqrt{1 - \frac{X}{b^2} - \frac{Y^2}{4b^4}}\right). \tag{1.5}$$

Here η is the flat spacetime metric, F is the electromagnetic 2-form and the constant b acts as a dimensional scale constant, determining the energy scale at which the non-linearities become significant (and serving to give a maximum to the electric field). The main difference between this new theory of electromagnetism and Maxwell theory was that electron now had a finite self energy due to the limit on the maximum possible electric field $E^{\mathrm{BI}}_{\mathrm{max}} \sim b$. Significant attention was not given to Born-Infeld theory at the time however due to advances of QED, which shifted the focus of the theoretical physics community away from classical modifications of Maxwell theory and into the field quantisation of Maxwell-Dirac theory.

Quantum electrodynamics is most often a perturbative theory[1] of the quantum vacuum, using operator theory or functional path integrals to calculate probability amplitudes. The specifics of QED are beyond the scope of this thesis, although the successes of QED in predicting phenomena like the Lamb shift of electron orbitals and the anomalous magnetic moment of the electron[2] are an indicator that any nonlinearity of the underlying classical electromagnetic Lagrangian must be very small in such regimes. Thus the parameter b in the Born-Infeld Lagrangian (1.5) must be such that the non-linearities only become significant at much higher electric fields.

While quantum electrodynamics attempted to fully quantise Maxwell-Dirac theory, Hans Euler and Werner Heisenberg used a semi-classical approach. By incorporating the quantisation via operator theory and assuming that the electromagnetic fields were classical (for more detail, see (the translation of) the original paper by Euler and Heisenberg [4]), resulting in an effective (one loop) Lagrangian:

$$\begin{aligned} L^{(1)}_{\mathrm{EH}} = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-im_e^2 s}\Bigg(&\frac{2}{3}(q_e s)^2 X - 1 \\ &+(q_e s)^2\frac{|Y|}{2}\cot\left[q_e s\sqrt{X+\sqrt{X^2+\frac{Y^2}{4}}}\right]\coth\left[q_e s\sqrt{-X+\sqrt{X^2+\frac{Y^2}{4}}}\right]\Bigg). \end{aligned} \tag{1.6}$$

[1]Since the vast majority of non-perturbative problems in QED appear to be impossible to solve analytically.

[2]First found by Schwinger in 1948 [10] and as of 1996 known analytically up to third order in the fine structure constant α_{fs} [11].

The Euler-Heisenberg Lagrangian describes the phenomenon of vacuum polarisation, which occurs when the electric field is almost large enough to separate the virtual electron-positron pairs of the quantum vacuum. Euler-Heisenberg theory has gained a considerable following and is still studied widely today (for more details, see Ref. [12]).

To give some comparison of how this theory compares with Born-Infeld theory, it is interesting to note that the weak-field approximations of the two Lagrangians are

$$L_{\text{EH}}^{\text{Weak}} = \frac{2\alpha_{\text{fs}}^2 \hbar^3}{45 m_e^4}[X^2 + \frac{7}{4}Y^2], \tag{1.7}$$

$$L_{\text{BI}}^{\text{Weak}} = \frac{X}{2} + \frac{1}{8b^2}[X^2 + Y^2]. \tag{1.8}$$

Firstly it is clear that while Born-Infeld includes the Maxwell Lagrangian $X/2$, the Euler-Heisenberg Lagrangian does not. This is because the Euler-Heisenberg Lagrangian is an *additional* contribution to the Maxwell Lagrangian while Born-Infeld theory is a replacement, which *becomes* Maxwell theory in the limit $b \to \infty$. Secondly, the nonlinear contributions of (1.7) and (1.8) are not the same, so the theories are distinct (seen via the different Y^2 coefficients).

Indeed, it has been suggested [13] that a quantum Born-Infeld theory should display the effects of Euler-Heisenberg theory, and hence the overall electromagnetic Lagrangian should be $L_{\text{eff}} \approx L_{\text{BI}} + L_{\text{EH}}$. This has implications for tests of nonlinear electrodynamics; Ref. [13] has shown that background magnetic field can be used to test vacuum birefringence, the absence of which only the Born-Infeld Lagrangian (among regular nonlinear Lagrangians) is known to demonstrate. The lack of birefringence in Born-Infeld theory is one of the properties uncovered by Boillat [14] and Plebanski [15], whose study of the theory's wave propagation properties contributed to a resurgence of interest in Born-Infeld theory in the 1970s. Their discovery that Born-Infeld theory alone among the class of (non-singular) Lagrangians $L(X, Y)$ ensured the absence of birefringence meant that Born-Infeld theory demonstrates exceptional causal properties (single light cones) and absence of shock waves (see Ref. [16] for more details).

Further interest developed as work in string/M theory showed that the low energy dynamics of strings and branes share similarities with Born-Infeld theory

[17], leading to more recent work (see for instance Refs. [18, 19]). This Born-Infeld Lagrangian motivates the maximum electric field via replacing the Born-Infeld constant b with κ via $b = \frac{1}{\kappa}$ in (1.5), where $\frac{1}{\kappa} \sim$ the string tension (of unknown value). Hence the scales on which the nonlinearities become significant are unknown. It is hoped, however, that performing experiments such as the slow light experiment [1, 20, 21] (using strong magnetic fields in an optical cavity), or by using the high field strengths of experiments such as at ELI [5], it will be possible to determine the constant κ and confirm that such classical phenomena play a role alongside corrections expected from quantum effects.

This thesis investigates the uses of extreme fields to test the edges of known physics, starting with Chapter 3[1], which extends the slow light experiment [20, 21] to explore the properties of Born-Infeld theory relative to a family of similar nonlinear theories of electromagnetism. This chapter investigates the propagation of plane waves through regions of constant magnetic fields in order to argue that the experiment should be modified to have magnetic fields with nonzero components parallel to the wave's own magnetic field. Similarly experiments involving plane waves propagating though regions of constant electric field are recommended to include a component of electric field parallel to the wave's own electric field. The results are then considered in the context of the desirability of a nonlinear theory to retain properties of Maxwell theory such as electric-magnetic duality invariance [22].

Chapter 4 then moves to study the energy gained by a charged particle in an electric field in the context of distinguishing nonlinear electromagnetic theories. The context chosen for this investigation is that of electron energy gain in maximum amplitude plasma waves[2]; an extension of the work done in Ref. [23]. By appealing to the stress balance law rather than the field equations method used in Ref. [23], the electron energy gain in a maximum amplitude plasma wave is shown to be dependent on only the mass of the particle and the speed of the plasma wave. Though this *appears* to be independent of theory, Chapter 3 indicates

[1] Chapter 3 is a more detailed account of the work presented in the publication in EPL, Ref. [1].

[2] Note that the maximum field strength here is due to field-matter interaction, not due to Born-Infeld etc. (see Chapter 4.)

that this speed will depend on the background field and the theory of electromagnetism. Since repeating the calculation for Chapter 3 would require delving into advanced numerics, this is left for future study and the thesis progresses to investigate other areas open to analytical approaches.

The final part of this thesis focuses on the effects of spin on a classical charged particle in an electromagnetic field. Both spin and radiation reaction (the force on an accelerating charged particle due to its own emitted radiation) are considered to be small effects [24]. Radiation reaction forces are being considered at present as the cumulative radiative contributions of accelerated electron bunches are expected to play a role in future accelerators. The effects of Stern-Gerlach forces on charged particles in high field situations such as those of maximum amplitude plasma waves have not, however, received much attention.

Chapter 5 shows a new derivation (via de Rham currents and balance laws) of the covariant Stern-Gerlach and Thomas-Bargmann-Michel-Telegdi (TBMT) equations [25, 26] of motion for a relativistic spinning charged particle, and then proceeds to investigate the motion of charged particles in the electromagnetic field produced by the maximum amplitude plasma wave discussed in Chapter 4. By perturbing around a known exact solution trajectory, the perturbative solutions are found to be linearly unstable. Since the particular solution in question is orthogonal to the motion of the plasma electrons and is unstable, the electrons following such trajectories could cause undesirable properties in (for instance) the bunching properties of electrons in laser wakefield accelerators. These trajectories exist only when spin is taken into account and since the electrons are non-accelerating, the radiation reaction forces are negligible; hence the Stern-Gerlach forces are shown to be important.

Chapter 2

Introduction to Differential Geometry

This chapter introduces the mathematical notation and machinery used throughout this thesis. This chapter is not intended to be a rigorous introduction to differential geometry and exterior calculus; the intention is simply to establish the conventions required to follow the calculations in the proceeding chapters. For a more expansive introduction to the relevant topics, see Refs. [27–31].

2.1 Introduction

Many of the calculations in this thesis are presented in the coordinate-free language of differential forms. This chapter will introduce the basic framework and concepts required to follow these calculations.

Spacetime is modelled as a smooth Lorentzian manifold; that is an n-dimensional pseudo-Riemannian manifold $\mathcal{M}$ on which a metric of signature $(1,(n-1))$ is defined. The signature of the metric can be thought of as the relative numbers of negative and positive signs in the metric terms. This requirement on the metric is imposed in order to distinguish the temporal coordinate from the spatial coordinates.

Coordinates x^a are sets of maps taking points in $\mathcal{M}$ to real numbers. In general no one coordinate system can cover the entire manifold, though in the case of Minkowski space this is not so, as is described later.

Vector fields (4-vector fields) are introduced as $V = V^a \partial_a$. Here the Einstein summation convention is used, summing a from 0 to 3; ∂_a is a basis vector, pointing in the direction of increasing x^a. In general, a set of basis vectors of a space is called a frame and is written $\{X_a\}$.

Vectors can be evaluated on scalar functions f and h, and obey the Leibniz rule:

$$V(f) = V^a \frac{\partial f}{\partial x^a}, \tag{2.1}$$

$$V(fh) = fV(h) + hV(f). \tag{2.2}$$

The metric g is a rank 2 tensor that takes two vectors and gives a real number as an analogue of the standard vector dot product:

$$g(U, V) = g_{ab} U^a V^b. \tag{2.3}$$

Orthonormal frames satisfy $g(X_a, X_b) = \eta_{ab}$, where

$$\eta_{ab} = \begin{cases} -1 & \text{for } a = b = 0, \\ 1 & \text{for } a = b = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The metric is nondegenerate, f-linear and symmetric, that is for vectors U, V and functions f, h

$$g(fU, hV) = fhg(U, V), \tag{2.5}$$

$$g(U, V) = 0 \quad \text{for all } U \text{ then } V = 0. \tag{2.6}$$

The metric allows the classification of three kinds of vector field:

- Timelike $g(V, V) < 0$,
- Spacelike $g(V, V) > 0$,
- Null (lightlike) $g(V, V) = 0$.

If two vectors U and V satisfy $g(U,V)=0$, they are said to be orthogonal.

The simplest example of a spacetime is Minkowski spacetime; this has one set of coordinates covering the entire manifold, which are the standard coordinates $\{t,x,y,z\}$ with the straightforward metric $g_{ab}=\eta_{ab}$. On Minkowski spacetime the natural frame is $\{\partial_t,\partial_x,\partial_y,\partial_z\}$. Special relativity is the study of physics in Minkowski spacetime; there are no gravitational effects in this theory.

1-form fields take vectors and give real numbers; they are elements of the dual space, which is a vector space. The object $g(V,-)$ is an example of a 1-form, called the metric dual of V and is written $\widetilde{V}$. Every 1-form can be written as the dual of a vector and vice versa, since the metric is nondegenerate (2.6). The square of the dual operation is the identity map and allows the definition of the dual metric $\widetilde{g}$, which acts on two 1-forms α, β via

$$\widetilde{g}(\alpha,\beta)=g\left(\widetilde{\alpha},\widetilde{\beta}\right). \tag{2.7}$$

In inertial Cartesian coordinates $\{x^a\}$ on Minkowski spacetime, $g_{ab}=\eta_{ab}$ and $\widetilde{g}^{ab}=\eta^{ab}$ where $[\eta^{ab}]=[\eta_{ab}]$. Vectors and 1-forms can act upon each other via

$$\alpha(V)=g\left(\widetilde{\alpha},V\right)=\widetilde{g}\left(\alpha,\widetilde{V}\right)=\widetilde{V}\left(\widetilde{\alpha}\right). \tag{2.8}$$

In general for every frame $\{X_a\}$ there is a naturally dual coframe e^a, a basis for 1-forms, where $e^a(X_b)=\delta^a_b$, where δ^a_b is the Kronecker delta. The coframe naturally dual to an orthonormal frame is called an orthonormal coframe. With an orthonormal frame-coframe pair, $\widetilde{X_a}=e_a$ where $e_a=\eta_{ab}e^b$ and so $\widetilde{X_0}=-e^0$, $\widetilde{X}_1=e^1$, $\widetilde{X}_2=e^2$, $\widetilde{X}_3=e^3$. On Minkowski spacetime, the natural orthonormal frame is $\{\partial_t,\partial_x,\partial_y,\partial_z\}$ with naturally dual orthonormal coframe $\{dt,dx,dy,dz\}$.

There are also higher degree forms: the wedge product (also called the exterior product) $\wedge$ combines two 1-forms to make a 2-form; using notation where $\alpha^{(1)}$ indicates a 1-form etc.

$$\alpha^{(1)}\wedge\beta^{(1)}=\omega^{(2)}. \tag{2.9}$$

For scalar functions f, the wedge product satisfies

$$\alpha\wedge(\beta_1+\beta_2)=\alpha\wedge\beta_1+\alpha\wedge\beta_2, \tag{2.10}$$

$$\alpha\wedge f\beta=f\alpha\wedge\beta=f(\alpha\wedge\beta), \tag{2.11}$$

$$\alpha^{(1)}\wedge\beta^{(1)}=-\beta^{(1)}\wedge\alpha^{(1)}, \tag{2.12}$$

and in particular, $\alpha^{(1)} \wedge \alpha^{(1)} = 0$. Functions (scalar fields) are also called 0-forms, though 0-forms can also have indices (for instance the components of a vector).

The basis of 2-forms is hence $\{dx^a \wedge dx^b$, for $1 \leq a \leq n$ and $a < b \leq n\}$. For instance on Minkowski spacetime, the basis of 2-forms can be written $\{dt \wedge dx$, $dt \wedge dy$, $dt \wedge dz$, $dx \wedge dy$, $dx \wedge dz$, $dy \wedge dz\}$. In order to avoid double counting, 2-forms $\alpha^{(2)}$ are written using the summation convention as

$$\alpha^{(2)} = \frac{1}{2}\alpha_{ab} dx^a \wedge dx^b = \frac{1}{2}\alpha_{ab} dx^{ab}, \tag{2.13}$$

where the last notation is used in this thesis when brevity is called for.

Higher degree forms can also be constructed using the wedge product. A p-form is made by wedging together p 1-forms; the degree of a p-form is p, and (2.12) is extended to higher degree forms via

$$\alpha^{(p)} \wedge \beta^{(q)} = (-1)^{pq} \beta^{(q)} \wedge \alpha^{(p)}, \tag{2.14}$$

where the superscript label on $\alpha^{(p)}$ simply indicates that α is a p-form, used when the degree of the form is important. A general p-form is written in terms of the appropriate basis

$$\alpha^{(p)} = \frac{1}{p!}\alpha_{\underbrace{a\ldots b}_{p \text{ indices}}} \overbrace{dx^a \wedge \ldots dx^b}^{p \text{ 1-forms}} = \frac{1}{p!}\alpha_{a\ldots b} dx^{a\ldots b}. \tag{2.15}$$

Note that due to the properties of the wedge product, an n-dimensional manifold can only support forms of degree n or less. Forms of degree n on an n-dimensional manifold are called top forms. Attempting to wedge a non-zero form to a top form returns zero. For instance Minkowski spacetime can support 0-forms to 4-forms, but not p-forms with $p > 4$.

2.2 The Exterior Derivative, Internal Contraction and Hodge Map

The exterior derivative d increases the degree of a form by one. On 0-forms f, d acts via

$$df = \frac{\partial f}{\partial x^a} dx^a. \tag{2.16}$$

The exterior derivative acts in general on a wedge product of a p-form $\alpha^{(p)}$ and any form β via

$$d(\alpha^{(p)} \wedge \beta) = (d\alpha^{(p)}) \wedge \beta + (-1)^p \alpha^{(p)} \wedge (d\beta). \tag{2.17}$$

In particular, $d^2 = 0$. Any form which satisfies $d\alpha = 0$ is called a closed form, and applying d to a top form $\alpha^{(n)}$ results in $d\alpha^{(n)} = 0$.

The internal contraction operator i_V reduces the degree of a form by 1 via contraction on vector V. As should be expected, applying the internal contraction to a 0-form gives 0. Applying i_V to a 1-form α gives the contraction;

$$i_V \alpha = V^a \alpha_a, \tag{2.18}$$

and the internal contraction operator commutes with the wedge product via

$$i_V \left(\alpha^{(p)} \wedge \beta\right) = i_V \alpha^{(p)} \wedge \beta + (-1)^p \alpha^{(p)} \wedge i_V \beta, \tag{2.19}$$

where β is of arbitrary degree. Hence i_V can be applied to any form. The internal contraction obeys

$$i_{fV} \alpha = f i_V \alpha, \tag{2.20}$$

$$i_U i_V \alpha = -i_V i_U \alpha, \tag{2.21}$$

and the wedge product and internal contraction satisfy the identity

$$e^a \wedge i_{X_a} \alpha^{(p)} = p \alpha^{(p)}. \tag{2.22}$$

The Hodge operator $\star$ maps p-forms to $(n-p)$-forms on n-dimensional manifolds; it is distributive and obeys

$$\star(f\alpha) = f \star \alpha, \tag{2.23}$$

for 0-forms f. Applying the Hodge map twice to a p-form on an n-dimensional manifold gives

$$\star \star \alpha^{(p)} = (-1)^{p(n-p)} \frac{det(g_{ab})}{|det(g_{ab})|} \alpha^{(p)}, \tag{2.24}$$

and in particular on Minkowski spacetime,

$$\star\star\alpha = \begin{cases} \alpha & \text{for } \deg(\alpha) \text{ odd}, \\ -\alpha & \text{for } \deg(\alpha) \text{ even}, \end{cases} \tag{2.25}$$

so that $\star$ is almost self-inverse.

The object $\star 1$ is a special top form on a manifold as it defines the orientation of the manifold. The object $\star 1$ is called the volume form, and for orthonormal coframe $\{e^a\}$, it can be written

$$\star 1 = e^1 \wedge \dots e^n, \tag{2.26}$$

which on Minkowski spacetime with the natural orthonormal frame is simply

$$\star 1 = dt \wedge dx \wedge dy \wedge dz. \tag{2.27}$$

The Hodge map is defined on p-forms inductively via

$$p = 0: \quad \star f = f \star 1 \tag{2.28}$$

$$p = 1: \quad \star\alpha = \star(1 \wedge \alpha) = i_{\widetilde{\alpha}} \star 1. \tag{2.29}$$

$$\dots$$

Using the Hodge map, the dot product may be generalised to forms of *equal degree* via

$$\alpha \cdot \beta = \star^{-1}(\alpha \wedge \star\beta), \tag{2.30}$$

though it is sometimes helpful to use the component notation

$$\alpha^{(p)} \cdot \beta^{(p)} = \frac{1}{p!}\alpha^{n_1 \dots n_p}\beta_{n_1 \dots n_p}. \tag{2.31}$$

Note that for 1-forms, this can be rewritten as the more convenient metric product

$$\alpha \cdot \beta = \star^{-1}(\alpha \wedge \star\beta) = i_{\widetilde{\alpha}}\beta = \widetilde{g}(\alpha, \beta) = \alpha^a \beta_a. \tag{2.32}$$

Two helpful identities involving the Hodge map $\star$ are

$$\star(i_V \alpha^{(p)}) = (-1)^{p+1}\widetilde{V} \wedge \star\alpha^{(p)}, \tag{2.33}$$

$$\alpha^{(p)} \wedge \star\beta^{(p)} = \beta^{(p)} \wedge \star\alpha^{(p)}, \tag{2.34}$$

where the latter is known as the star-pivot.

2.3 Differentiation on Manifolds: Lie Derivatives and Connections

The Lie derivative $\mathcal{L}_V$ with respect to vector V acts on vector U via

$$(\mathcal{L}_V U)\, f = V\,(Uf) - U\,(Vf)\,, \tag{2.35}$$

for some 0-form f. The Lie derivative can be applied to differential forms α via the Cartan identity

$$\mathcal{L}_V \alpha = d i_V \alpha + i_V d\alpha, \tag{2.36}$$

in particular on 0-forms f

$$\mathcal{L}_V f = i_V df = Vf. \tag{2.37}$$

A clear consequence of (2.36) is that the exterior derivative d commutes with the Lie derivative: $d\mathcal{L}_V = \mathcal{L}_V d$.

The Lie derivative obeys

$$\mathcal{L}_V(U+W) = \mathcal{L}_V U + \mathcal{L}_V W, \tag{2.38}$$

$$\mathcal{L}_V(fU) = (\mathcal{L}_V f)U + f\mathcal{L}_V U, \tag{2.39}$$

and commutes with contractions via

$$\mathcal{L}_V i_U \alpha = i_U \mathcal{L}_V \alpha + i_{\mathcal{L}_V U}\alpha, \tag{2.40}$$

for any p-form α. In particular, the Lie derivative acts on wedge and tensor products by a Leibniz rule:

$$\mathcal{L}_V(T\otimes S) = (\mathcal{L}_V T)\otimes S + T\otimes(\mathcal{L}_V S), \tag{2.41}$$

$$\mathcal{L}_V(\alpha\wedge\beta) = (\mathcal{L}_V \alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_V \beta). \tag{2.42}$$

For example the Lie derivative can be applied to a metric product:

$$\mathcal{L}_V(g(U,W)) = (\mathcal{L}_V g)(U,W) + g(\mathcal{L}_V U, W) + g(U, \mathcal{L}_V W). \tag{2.43}$$

Equation (2.43) also allows the definition of a *Killing vector* K:

$$\mathcal{L}_K g = 0. \tag{2.44}$$

Killing vectors preserve the metric and indicate symmetries; each Killing vector corresponds to a symmetry of the spacetime. In Minkowski spacetime for instance, the Killing vectors $\{\partial_t, \partial_x, \partial_y, \partial_z\}$ (translations) correspond to energy and momentum conservation respectively, while $\{x\partial_y - y\partial_x, y\partial_z - z\partial_y, x\partial_z - z\partial_x\}$ (rotations) correspond to angular momentum conservation and $\{x\partial_t + t\partial_x, y\partial_t + t\partial_y, z\partial_t + t\partial_z\}$ (boosts) correspond to another conserved quantity[1]. Killing vectors also have the property that $\mathcal{L}_K \star = \star \mathcal{L}_K$.

Connections ∇_V allow differentiation along prescribed vector field V. There are different kinds of connection and each one encodes information as to how the vector being acted on is transported along the vector V; whether it is rotated or not for instance.

Connections are distributive in both arguments as well as obeying

$$\nabla_U(fV) = \nabla_U(f)V + f\nabla_U(V), \tag{2.45}$$

$$\nabla_{fU}(V) = f\nabla_U(V), \tag{2.46}$$

for 0-forms f and vectors U and V where

$$\nabla_U(f) = Uf, \tag{2.47}$$

and hence $\nabla_U f = \mathcal{L}_U f$.

The Levi-Civita connection is a particular kind of connection; for a prescribed metric the Levi-Civita connection is uniquely defined as the only connection satisfying metric compatibility and is torsion-free. In other words, ∇ satisfies

$$\nabla_U g(V, W) = g(\nabla_U V, W) + g(V, \nabla_U W), \tag{2.48}$$

$$\nabla_U V - \nabla_V U = UV - VU = \mathcal{L}_U V, \tag{2.49}$$

[1] Since in relativistic mechanics two different observers may not agree on which is a Lorentz boost and which is a rotation, the conserved quantity for both of these together is sometimes considered to be conservation of 4-angular momentum, just as the translational Killing vectors give conservation of energy and 3-momentum, hence 4-momentum. The quantity conserved in a given frame by boosts is sometimes called “centre of energy” or “centre of momentum”.

for all vectors U, V, W. Metric compatible connections preserve information about lengths and angles under transport, while torsion induces additional rotation. Metric compatible connections such as the Levi-Civita connection also commute with the Hodge map, that is $\star\nabla_V = \nabla_V\star$, since the Hodge map $\star$ depends only on the metric.

On Minkowski spacetime, with the rectilinear inertial frame $\{\partial_a\}$ and coframe $\{dx^a\}$, the connection is defined on vectors $U = U^a\partial_a$ and 1-forms $\alpha = \alpha_a dx^a$ by

$$\nabla_V U = V(U^a)\partial_a, \tag{2.50}$$

$$\nabla_V \alpha = V(\alpha_a)dx^a, \tag{2.51}$$

and in particular $\nabla_V dx^a = 0$. The connection commutes with contractions via

$$\nabla_V i_U \alpha = i_U \nabla_V \alpha + i_{\nabla_V U}\alpha, \tag{2.52}$$

and the connection can be applied to wedge products via

$$\nabla_V(\alpha \wedge \beta) = (\nabla_V \alpha) \wedge \beta + \alpha \wedge (\nabla_V \beta). \tag{2.53}$$

A parallel vector U satisfies $\nabla_V U = 0$ for all V, and hence $i_U\nabla_V = \nabla_V i_U$.

The Levi-Civita connection allows Killing's equation to be written

$$g(U, \nabla_V K) + g(V, \nabla_U K) = 0, \tag{2.54}$$

for all vectors U and V; this is equivalent to (2.44), seen via (2.49) and (2.48).

2.4 The Tangent to a Curve

Curves C in spacetime are used to denote (for instance) trajectories of particles. At each point on the curve $C(\tau)$ there is a tangent vector $\dot{C}(\tau)$ defined, where τ is the curve parameter. In coordinate system $\{x^a\}$ this can be written

$$\dot{C}(\tau) = \frac{dC^a}{d\tau}\frac{\partial}{\partial x^a}\bigg|_{C(\tau)}. \tag{2.55}$$

As with the vector fields above, the metric can be used to classify curves. All massive particle trajectories are timelike and have $g(\dot{C}, \dot{C}) < 0$ for all τ. Particles travelling at the speed of light have trajectories satisfying $g(\dot{C},\dot{C}) = 0$. Curves satisfying $g(\dot{C},\dot{C}) > 0$ would represent particles travelling faster than the speed of light and hence are non-physical trajectories.

A curve C is an *integral curve* of a vector field V if at each point p on C, $\dot{C}\Big|_p = V|_p$, as Figure 2.1 illustrates.

In order to model physical trajectories C on differential manifolds, the normalisation condition

$$g(\dot{C},\dot{C}) = -1, \tag{2.56}$$

is imposed in order to maintain the length of the time-like 4-vector $\dot{C}$. For such a normalisation, the curve parameter τ is called the proper time.

Note that since the magnitude of $\dot{C}$ is constant, the 4-acceleration $\ddot{C}$, which on flat spacetime with global Lorentzian coordinates is given by

$$\ddot{C} = \frac{d}{d\tau}\dot{C} = \frac{d^2C^a}{d\tau^2}\frac{\partial}{\partial x^a}\Big|_{C(\tau)}, \tag{2.57}$$

must be orthogonal to the velocity $\dot{C}$ since:

$$g\left(\frac{d}{d\tau}\dot{C},\dot{C}\right) = -g\left(\dot{C},\frac{d}{d\tau}\dot{C}\right) = 0, \tag{2.58}$$

and since the metric is symmetric. Note that for 0-forms f, the connection $\nabla_{\dot{C}}$ acts as a simple derivative along the curve via

$$\nabla_{\dot{C}} f = \dot{C}f = \frac{d}{d\tau}f. \tag{2.59}$$

Metric compatible connections like the Levi-Civita connection also satisfy the identity

$$\nabla_V \widetilde{V} = i_V d\widetilde{V}, \tag{2.60}$$

for normalised V, i.e. $g(V,V)$ is constant.

Given any vector V, the parallel and orthogonal projection operators $\Pi^{\|}_V$ and $\Pi^{\perp}_V$ can be defined on 1-form $\alpha^{(1)}$ as follows;

$$\Pi^{\|}_V \alpha^{(1)} = -\alpha^{(1)}(V)\widetilde{V}, \tag{2.61}$$

$$\Pi^{\perp}_V \alpha^{(1)} = \alpha^{(1)} - \Pi^{\|}_V \alpha^{(1)}. \tag{2.62}$$

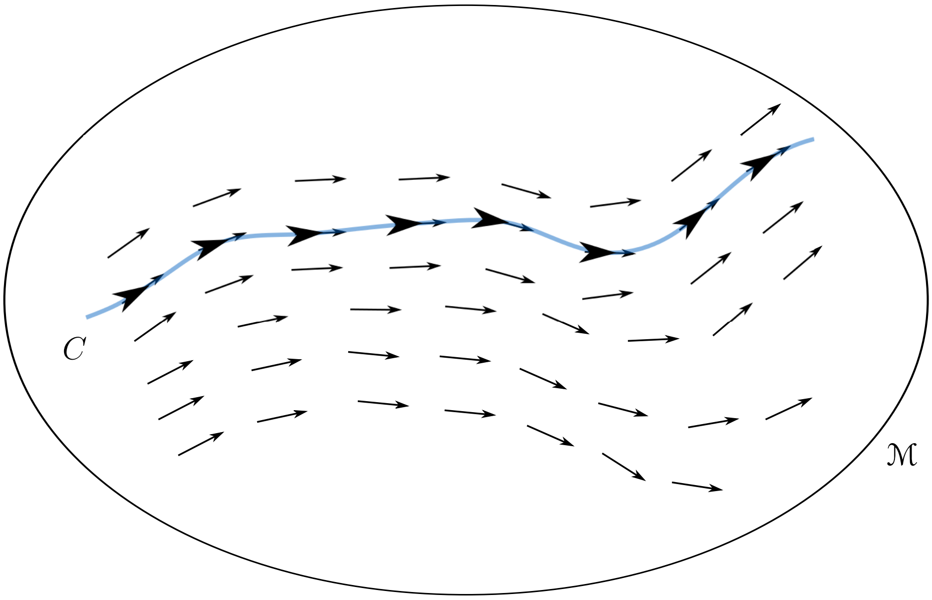

Figure 2.1: Illustration of C (blue curve), an integral curve of a field represented by a field of black arrows on manifold $\mathcal{M}$.

Clearly the parallel projection operator $\Pi_V^{\parallel}$ projects out the components of α perpendicular to $\widetilde{V}$, whereas the orthogonal projection operator $\Pi_V^{\perp}$ projects out the parallel part. In component notation these are given by

$$\left(\Pi_V^{\parallel}\right)^a{}_b = -V^a V_b, \tag{2.63}$$

$$\left(\Pi_V^{\perp}\right)^a{}_b = \eta^a{}_b - \left(\Pi_V^{\parallel}\right)^a{}_b. \tag{2.64}$$

2.5 Integration

On an n-dimensional manifold $\mathcal{M}$ with coordinate system x^a, an n-form (top form) $\alpha = f dx^1 \wedge \ldots \wedge dx^n$ can be integrated via

$$\int_{\mathcal{M}} \alpha = \int_{\mathcal{M}} f dx^1 \wedge \ldots \wedge dx^n \tag{2.65}$$

$$= \int \ldots \int f(x^1 \ldots x^n) dx^1 \ldots dx^n. \tag{2.66}$$

From here it is clear why $\star 1$ is known as the volume form; it is also important to specify the volume form since, for instance with global Lorentzian coordinates, the distinction between $dt \wedge dx \wedge dy \wedge dz$ and $dx \wedge dt \wedge dy \wedge dz$ is an overall sign with regards to orientation of the volume. One of the most powerful results in exterior differential calculus is the generalised Stokes' theorem on $(n-1)$-forms α:

$$\int_{\mathcal{M}} d\alpha = \int_{\partial\mathcal{M}} \alpha, \tag{2.67}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$. The generalised Stokes' theorem (2.67) contains both the usual Gauss' divergence theorem and the usual Stokes' theorem.

An example of integrating a form is as follows. Consider integrating a 1-form α over a curve $C(\tau)$ on the manifold; in this case the integral can be written

$$\int_C \alpha = \int_0^1 (i_{\dot{C}}\alpha)\, d\tau, \tag{2.68}$$

where the endpoints of the curve are $x^a = C^a(\tau = 0)$ and $x^a = C^a(\tau = 1)$.

2.6 De Rham Currents

De Rham currents are a class of linear functional; they act on functions and return numbers. De Rham currents act on test forms, which are differential forms that are both smooth (infinitely differentiable) and have compact support on the manifold in question[1]. Test functions are denoted $\hat{f}$ and test p-forms are denoted $\hat{\varphi}^{(p)}$. There are two kinds of de Rham current: regular distributions and submanifold distributions.

Regular distributions are associated with differential forms as follows: given a smooth p-form α and a test form $\hat{\varphi}^{(q)}$, the distribution α_D associated with α is

$$\alpha_D[\hat{\varphi}^{(q)}] = \begin{cases} \int_{\mathcal{M}} \alpha \wedge \hat{\varphi}^{(q)} & \text{if } q = n - p, \\ 0 & \text{if } q \neq n - p, \end{cases} \tag{2.69}$$

where n is the dimension of the manifold $\mathcal{M}$. Here α_D has degree $n - p$.

[1]Functions with compact support are zero outside some finite region; hence at the boundaries of the manifold test functions must be zero.

Submanifold distributions are also useful in physics; in particular on a space-time manifold, the 3-current C_D acts on the test 1-form $\hat{\varphi}^{(1)}$ via

$$C_D[\hat{\varphi}^{(1)}] = \int_C \hat{\varphi}^{(1)}. \tag{2.70}$$

All de Rham currents T_D satisfy the following identities

$$dT_D[\hat{\varphi}] = -(-1)^p T_D[d\hat{\varphi}], \tag{2.71}$$

$$(T_D \wedge \alpha)[\hat{\varphi}] = T_D[\alpha \wedge \hat{\varphi}], \tag{2.72}$$

leading to the properties

$$i_V T_D[\hat{\varphi}] = -(-1)^p T_D[i_V \hat{\varphi}], \tag{2.73}$$

$$(\star T_D)[\hat{\varphi}] = (-1)^{p(n-p)} T_D[\star\hat{\varphi}], \tag{2.74}$$

where p is the degree of T_D and n is the dimension of the manifold. For submanifold distributions, there is one more property seen from (2.71):

$$C_D[d\hat{\varphi}] = \partial C_D[\hat{\varphi}], \tag{2.75}$$

where C represents a curve over the the manifold.

2.7 Physics on Differential Manifolds

Physics uses the language of differential forms in order to represent quantities such as the electromagnetic 2-form (also called the Faraday 2-form) F. Given an observer $\dot{C}$, this 2-form can be written

$$F = E \wedge \widetilde{\dot{C}} + \star(B \wedge \widetilde{\dot{C}}), \tag{2.76}$$

where E is the 1-form $E = E_x dx + E_y dy + E_z dz$, where the components E_x, E_x, E_z are the components of the electric field 3-vector in the frame of the observer with worldline C (likewise for magnetic field B). E and B, the 1-forms representing the electric and magnetic fields measured by the observer, are defined uniquely in terms of F via

$$E = i_{\dot{C}} F, \qquad B = -i_{\dot{C}} \star F. \tag{2.77}$$

For instance in Minkowski spacetime in the lab frame i.e. $\dot{C} = \partial_t$, F is written

$$F = E_x dt \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz - B_x dy \wedge dz - B_y dz \wedge dx - B_z dx \wedge dy. \tag{2.78}$$

In order to manifestly satisfy the Maxwell equation $dF = 0$, the Faraday 2-form can be written as an exact form $F = dA$, where A is the electromagnetic potential 1-form, related to the traditional electric and magnetic potentials Φ and $\vec{A}$ via $\widetilde{A} = \left(\Phi, \vec{A}\right)$.

The electromagnetic invariants X and Y can also be written concisely in terms of F, via

$$X = \star\left(F \wedge \star F\right), \tag{2.79}$$

$$Y = \star\left(F \wedge F\right), \tag{2.80}$$

which upon computation gives the standard results (1.1) and (1.2).

The stress-energy-momentum tensor (also simply called the stress tensor) T is a rank 2 tensor containing information about energy density, momentum density and stress of any event as measured by any observer in spacetime.

The stress-energy-momentum forms $\mathcal{T}_a$ (also simply called the stress forms) are related to the stress tensor via

$$\mathcal{T}_a = \star(T(-, X_a)). \tag{2.81}$$

Chapter 3

Properties of Born-Infeld-like Theories in Strong Fields

3.1 Introduction

As mentioned in Chapter 1, Born-Infeld theory has been the focus of some interest in recent years [13, 20, 21, 23, 32, 33] due to its uncommonly good physical properties [14, 15] and the fact that string theory predicts Born-Infeld electromagnetism as an effective Lagrangian for low energy branes [17]. String theory also, however, motivates a larger family of possible Lagrangians and this thesis conjectures that "Born-Infeld-like" Lagrangians of the form

$$L = \mathcal{F}(X + \lambda Y^2), \tag{3.1}$$

may be relevant, where $\mathcal{F}$ is a smooth function, X and Y are the electromagnetic invariants (1.1) and (1.2) and λ is a parameter of the theory. Hence it would be desirable to know whether Born-Infeld theory can be set apart from the other members of its family, (3.1), by some physical experiment.

It is already well known that when passing through a region of background electromagnetic field, the speed of an electromagnetic wave changes in nonlinear electrodynamics [14, 15, 20, 21]. In particular, Ref. [20] (and [21]) showed that in a constant magnetic field on flat spacetime, the phase speed of a plane Born-Infeld wave is given by

$$v = \sqrt{\frac{1 - \kappa^2 B_{\mathrm{L}}^2}{1 - \kappa^2 B^2}}, \tag{3.2}$$

where κ is the Born-Infeld constant, B_{L} is the longitudinal component of the background magnetic field and B^2 is the modulus squared of the background field. Similarly for a background electric field, they showed that the phase speed of an EM wave is

$$v = \sqrt{1 - \kappa^2 \left(E^2 - E_{\mathrm{L}}^2\right)}. \tag{3.3}$$

Since the Born-Infeld constant κ ($\sim$ (string tension)$^{-1}$) is a parameter of unknown size, a measurement of a slow Born-Infeld wave could not only validate Born-Infeld theory as the successor to Maxwell theory, but also help to pin down an elusive unknown of string theory.

In this chapter the exact Born-Infeld solutions and slow light experiment of Refs. [20, 21] are extended to investigate the properties of Born-Infeld theory relative to its family (3.1) of similar theories. After studying various field configurations, it is concluded that a more general configuration of electromagnetic field than the slow light experiment proposed by Ref. [21] would provide a more effective theory discriminant.

The work presented in Section 3.3 has been published in EPL - see [1].

3.2 Born-Infeld Waves as Exact Solutions to the Field Equations

Firstly it is demonstrative to show that the Born-Infeld plane wave introduced in [20, 21] is indeed a solution to the field equations. The vacuum Born-Infeld equations are

$$dF = 0, \tag{3.4}$$

$$d \star G_{\mathrm{BI}} = 0, \tag{3.5}$$

where F is the electromagnetic 2-form and G_{BI} is the excitation 2-form given by

$$G_{\mathrm{BI}} = \frac{1}{\sqrt{1 - \kappa^2 X - \frac{\kappa^4}{4} Y^2}} \left(F - \frac{\kappa^2 Y}{2} \star F \right), \tag{3.6}$$

since the Born-Infeld Lagrangian can be written

$$L_{\mathrm{BI}} = \frac{1}{\kappa^2} \left(1 - \sqrt{1 - \kappa^2 X - \frac{\kappa^4}{4} Y^2} \right). \tag{3.7}$$

As described in Chapter 2, the electromagnetic invariants X and Y can be written in terms of the Faraday 2-form F:

$$X = \star(F \wedge \star F), \qquad \text{(2.79 revisited)}$$

$$Y = \star(F \wedge F). \qquad \text{(2.80 revisited)}$$

The exact Born-Infeld wave solution given in [21], an electromagnetic plane wave propagating at constant speed v through a constant magnetic field B in flat spacetime, is given by

$$F = \mathcal{E}(z - vt)\,(dz - vdt) \wedge dx - Bdy \wedge dz. \tag{3.8}$$

Now to show that this F solves the field equations (3.4) and (3.5); the former is satisfied automatically by lieu of the dependence of $\mathcal{E}$ on $z - vt$ alone. The latter equation is not so trivial.

The electromagnetic invariants for the Born-Infeld wave (3.8) are given by

$$X = \mathcal{E}^2(v^2 - 1) - B^2, \tag{3.9}$$

$$Y = -2B\mathcal{E}v. \tag{3.10}$$

As $\star \star F = -F$ on Minkowski spacetime, $\star G_{\mathrm{BI}}$ can be written

$$\star G_{\mathrm{BI}} = \frac{1}{\sqrt{1 - \kappa^2 X - \frac{\kappa^4}{4} Y^2}} \left(\star F + \frac{\kappa^2 Y}{2} F \right). \tag{3.11}$$

Using the abbreviated notation $dx^{ab} = dx^a \wedge dx^b$, where $dx^0 = dt$, $dx^1 = dx$, $dx^2 = dy$ and $dx^3 = dz$, X, Y and F are substituted into $\star G_{\mathrm{BI}}$, resulting in

$$\star G_{\mathrm{BI}} = \frac{B\left[v^2\kappa^2\mathcal{E}^2 - 1\right] dx^{01} + \mathcal{E} dx^{02} + \kappa^2 B v \mathcal{E}^2 dx^{13} + \mathcal{E} v \left[1 + \kappa^2 B^2\right] dx^{23}}{\sqrt{\left[1 + \kappa^2 B^2\right] + \kappa^2 \mathcal{E}^2 \left(1 - v^2[1 + \kappa^2 B^2]\right)}}. \tag{3.12}$$

Since a solution of the Born-Infeld equation $d\star G_{\mathrm{BI}} = 0$ is sought, in order to proceed a velocity v is chosen such that square root divisor is simply a constant. Choosing to set the coefficient of $\mathcal{E}^2$ in the denominator equal to zero results in the choice of phase velocity given in [21], i.e. $v = \frac{1}{\sqrt{1+\kappa^2 B^2}}$, leading to

$$\frac{1}{\sqrt{[1+\kappa^2 B^2] + \kappa^2 \mathcal{E}^2 \left(1 - v^2[1+\kappa^2 B^2]\right)}} = v. \tag{3.13}$$

Thus $\star G_{\mathrm{BI}}$ becomes

$$\star G_{\mathrm{BI}} = v\left(B\left[v^2\kappa^2\mathcal{E}^2 - 1\right]dx^{01} + \mathcal{E}dx^{02} + \kappa^2 Bv\mathcal{E}^2 dx^{13} + \mathcal{E}v^{-1}dx^{23}\right). \tag{3.14}$$

Applying the exterior derivative d and noting that the only non-constant parameter is $\mathcal{E} = \mathcal{E}(z - vt)$,

$$d\star G_{\mathrm{BI}} = d\mathcal{E} \wedge \left[dx^{23} + vdx^{02} + \left(2\kappa^2 Bv^2\mathcal{E}\right)\left(dx^{13} + vdx^{01}\right)\right]. \tag{3.15}$$

Now to consider $d\mathcal{E} = d\mathcal{E}(z - vt)$;

$$\begin{aligned} d\mathcal{E}(z - vt) &= \partial_t\mathcal{E}(z - vt)dt + \partial_z\mathcal{E}(z - vt)dz \\ &= \mathcal{E}'\left(dx^3 - vdx^0\right), \end{aligned} \tag{3.16}$$

where $\mathcal{E}' = \frac{d\mathcal{E}(\xi)}{d\xi}$ and $\xi = z - vt$. Substituting this into (3.15), it is clear (due to the fact that $dx^i \wedge dx^i = 0$) that (3.5) is satisfied. Hence the solution $F = \mathcal{E}(z-vt)\,(dz - vdt)\wedge dx - Bdy\wedge dz$ corresponding to a plane Born-Infeld wave in a constant magnetic field B is indeed a solution to the field equations given that the wave travels with constant velocity $v = (1+\kappa^2 B^2)^{-\frac{1}{2}}$, confirming the prior work of Refs. [20, 21].

3.3 Solving the Field Equations with Constant Background Magnetic Fields

Having shown that the Born-Infeld plane wave is a solution to the Born-Infeld field equations, a reasonable question to ask is the following: is Born-Infeld the only theory whose nonlinear field equations possess exact plane wave solutions?

The general electromagnetic field equations are

$$dF = 0, \tag{3.17}$$

$$d \star G = 0, \tag{3.18}$$

where G is given by

$$G = 2\left(\frac{\partial L}{\partial X}F - \frac{\partial L}{\partial Y} \star F\right), \tag{3.19}$$

and

$$L = L(X, Y) \tag{3.20}$$

is the electromagnetic Lagrangian of the theory in question. The electromagnetic Lagrangian is assumed to be dependent only on the electromagnetic field invariants, X and Y, defined by (2.79) and (2.80).

3.3.1 Background Magnetic Field Parallel to the Wave's Electric Field

Firstly, consider the electromagnetic plane wave propagating at constant speed v through a constant magnetic field B, oriented so as to be parallel to the wave's electric field (in this instance in the x-direction). Then F, X and Y remain as in the previous section, i.e.

$$F = \mathcal{E}(z - vt)\,(dz - vdt) \wedge dx - Bdy \wedge dz, \tag{3.21}$$

$$X = \mathcal{E}^2\left(v^2 - 1\right) - B^2, \tag{3.22}$$

$$Y = -2B\mathcal{E}v. \tag{3.23}$$

Since using (3.21), (3.22) and (3.23) clearly restricts solutions to a subspace of X and Y, it is prudent to use different notation to denote the restricted and unrestricted Lagrangians. Hence the notation L for free Lagrangians and

$$\hat{L} = L|_{\substack{X=\mathcal{E}^2(v^2-1)-B^2 \\ Y=-2B\mathcal{E}v}}, \tag{3.24}$$

and so on for the restrictions of L and its derivatives. The aim of this section is to use the field equations in order to arrive at a partial differential equation (P.D.E.)

for the Lagrangian L exclusively in terms of the electromagnetic invariants X and Y and to solve the resulting P.D.E. for a theory of nonlinear electromagnetism.

Substituting F, $\star F$ and $d\star F$ into $\frac{1}{2}d\star G$ using abbreviated notation as before;

$$\frac{1}{2}d\star G = \widehat{\frac{\partial L}{\partial X}}\frac{\mathcal{E}'}{\gamma^2}dx^{023} + \left(-Bd\widehat{\frac{\partial L}{\partial X}} - v\mathcal{E}d\widehat{\frac{\partial L}{\partial Y}}\right)\wedge dx^{01} + \mathcal{E}d\widehat{\frac{\partial L}{\partial X}}\wedge dx^{02}$$
$$+\left(-\mathcal{E}d\widehat{\frac{\partial L}{\partial Y}}\right)\wedge dx^{13} + \left(v\mathcal{E}d\widehat{\frac{\partial L}{\partial X}} - Bd\widehat{\frac{\partial L}{\partial Y}}\right)\wedge dx^{23}. \tag{3.25}$$

Noting that $\frac{\partial L}{\partial X}$ is a 0-form, then applying the exterior derivative d results in

$$d\frac{\partial L}{\partial X} = \partial_a\frac{\partial L}{\partial X}dx^a. \tag{3.26}$$

As $L = L(X, Y)$, use of the chain rule results in

$$d\frac{\partial L}{\partial X} = \left(\frac{\partial^2 L}{\partial X^2}\frac{\partial X}{\partial x^a} + \frac{\partial^2 L}{\partial X\partial Y}\frac{\partial Y}{\partial x^a}\right)dx^a. \tag{3.27}$$

Thus in order to write the derivatives in the field equation (3.25) in terms of X and Y alone, the derivatives $\partial_a X$ and $\partial_a Y$ are needed. As both $\widehat{X}$ and $\widehat{Y}$ depend only on the variable quantity $\mathcal{E} = \mathcal{E}(z - vt)$, it can immediately be seen that $\widehat{\partial_x X} = \widehat{\partial_x Y} = \widehat{\partial_y X} = \widehat{\partial_y Y} = 0$. Noting that $\gamma = \frac{1}{\sqrt{1-v^2}}$ is the Lorentz factor of the wave, the remaining derivatives are

$$\widehat{\partial_a X}dx^a = -\frac{2\mathcal{E}}{\gamma^2}\partial_a\mathcal{E}dx^a, \tag{3.28}$$
$$\widehat{\partial_a Y}dx^a = -2vB\partial_a\mathcal{E}dx^a, \tag{3.29}$$

leaving the $\mathcal{E}$ derivative

$$\begin{aligned}\partial_a\mathcal{E}dx^a &= \partial_t\mathcal{E}dt + \partial_z\mathcal{E}dz \\ &= -v\mathcal{E}'dt + \mathcal{E}'dz = \mathcal{E}'(dz - vdt).\end{aligned} \tag{3.30}$$

Combining this with (3.28) and (3.29) and inserting into (3.27) (and similarly for the Y derivatives),

$$d\widehat{\frac{\partial L}{\partial X}} = -2\mathcal{E}'\left(\frac{\mathcal{E}}{\gamma^2}\widehat{\frac{\partial^2 L}{\partial X^2}} + vB\widehat{\frac{\partial^2 L}{\partial X\partial Y}}\right)(dz - vdt), \tag{3.31}$$
$$d\widehat{\frac{\partial L}{\partial Y}} = -2\mathcal{E}'\left(\frac{\mathcal{E}}{\gamma^2}\widehat{\frac{\partial^2 L}{\partial X\partial Y}} + vB\widehat{\frac{\partial^2 L}{\partial Y^2}}\right)(dz - vdt). \tag{3.32}$$

Hence inserting (3.31) and (3.32) into (3.25) simplifies $\frac{1}{2}d\star G$ to

$$\frac{1}{2}d\star G = 2B\mathcal{E}'\left(\frac{\mathcal{E}}{\gamma^2}\widehat{\frac{\partial^2 L}{\partial X^2}} + vB\widehat{\frac{\partial^2 L}{\partial X\partial Y}}\right)dx^{013} + \frac{\mathcal{E}'}{\gamma^2}\left(\widehat{\frac{\partial L}{\partial X}} - \frac{2\mathcal{E}}{\gamma^2}\widehat{\frac{\partial^2 L}{\partial X^2}} - 2v^2B^2\gamma^2\widehat{\frac{\partial^2 L}{\partial Y^2}} - 4vB\mathcal{E}\widehat{\frac{\partial^2 L}{\partial X\partial Y}}\right)dx^{023}. \quad (3.33)$$

Hence the field equations are satisfied if two conditions are satisfied: assuming $B \neq 0$, $\mathcal{E}' \neq 0$ and $v \neq c$, the first condition (from the dx^{013} component of (3.33)) is

$$\frac{\mathcal{E}}{\gamma^2}\widehat{\frac{\partial^2 L}{\partial X^2}} + vB\widehat{\frac{\partial^2 L}{\partial X\partial Y}} = 0, \quad (3.34)$$

and the second (from the dx^{023} component of (3.33)) is

$$\widehat{\frac{\partial L}{\partial X}} - \frac{2\mathcal{E}}{\gamma^2}\widehat{\frac{\partial^2 L}{\partial X^2}} - 2v^2B^2\gamma^2\widehat{\frac{\partial^2 L}{\partial Y^2}} - 4vB\mathcal{E}\widehat{\frac{\partial^2 L}{\partial X\partial Y}} = 0. \quad (3.35)$$

The field equations are still not yet written in terms of X and Y alone, hence the substitution $\mathcal{E} = -\frac{Y}{2Bv}$ is used to reduce these equations functions of (and derivatives with respect to) X and Y. Making the assumption that this result can be extended outside the solution subspace given by (3.24), the hats are removed. Hence (3.34) and (3.35) become two conditions that the Lagrangian must meet in order to satisfy the nonlinear field equations. These condition are:

$$Y\frac{\partial^2 L}{\partial X^2} - 2v^2B^2\gamma^2\frac{\partial^2 L}{\partial X\partial Y} = 0, \quad (3.36)$$

$$\frac{\partial L}{\partial X} + \frac{Y}{vB\gamma^2}\frac{\partial^2 L}{\partial X^2} - 2v^2B^2\gamma^2\frac{\partial^2 L}{\partial Y^2} + 2Y\frac{\partial^2 L}{\partial X\partial Y} = 0. \quad (3.37)$$

Integrating (3.36) with respect to X gives

$$Y\frac{\partial L}{\partial X} - 2v^2B^2\gamma^2\frac{\partial L}{\partial Y} = \mathcal{F}_1(Y), \quad (3.38)$$

where $\mathcal{F}_1(Y)$ is some (unknown) function of Y. Assuming that this is an integrable function, with $\frac{d\mathcal{F}_2(Y)}{dY} = \mathcal{F}_1(Y)$, this equation can be written

$$\partial_Y\left(L - \mathcal{F}_2(Y)\right) - \frac{1}{2B^2v^2\gamma^2}Y\partial_X\left(L - \mathcal{F}_2(Y)\right) = 0, \quad (3.39)$$

and, by inspection, this is solved by

$$L - \mathcal{F}_2(Y) = \mathcal{F}_3\left(X + \frac{Y^2}{4B^2v^2\gamma^2}\right). \tag{3.40}$$

Inputting this solution into (3.37) restricts $\mathcal{F}_2$ further to a linear function of Y, hence plane waves (3.21) solve the field equations of Lagrangians of the form

$$L = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}_3\left(X + \frac{Y^2}{4B^2v^2\gamma^2}\right), \tag{3.41}$$

where $\frac{1}{4B^2v^2\gamma^2}$ is a constant, as are the (integration) constants $\mathcal{C}_1$ and $\mathcal{C}_2$.

Notice that this class of solutions contains the Born-Infeld field system (choosing $v = \frac{1}{\sqrt{1+\kappa^2 B^2}}$ as before), and the Maxwellian system in the limit that the constant $\frac{1}{4B^2v^2\gamma^2}$ becomes zero; i.e. $\kappa \to 0$ or $v \to 1$ ($\gamma \to \infty$) as expected. Both of these cases also require $\mathcal{C}_1$ and $\mathcal{C}_2$ to be zero. As $Y \star 1 = -F \wedge F$ is a closed form, the $\mathcal{C}_2 Y$ term in fact does not contribute to the Maxwell action regardless of the value of $\mathcal{C}_2$.

Hence, given the assumption that the solutions of equations (3.34) and (3.35) on the (X, Y) subspace described by (3.22) and (3.23) are also valid outside this (X, Y) subset, it follows that all Lagrangians of the form

$$L = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}_3\left(X + \lambda Y^2\right), \tag{3.42}$$

where λ is a constant of the theory, satisfy the field equations for the wave (3.21), with phase speed $v = \frac{1}{\sqrt{1+4\lambda B^2}}$. In other words all Born-Infeld-like theories, that is theories with Lagrangians of the form (3.1), support the plane wave solution (3.21).

It is important to note that this is not a complete set of all possible theories to which (3.21) is an exact solution; there could be terms in the family (3.42) which are zero when X and Y are given by (3.22) and (3.23) but are non-zero outside this solution subspace.

3.3.2 Background Magnetic Field Orthogonal to the Wave's Magnetic Field with Transverse Electric Component

Now it is natural to ask if the result of the previous section remains true if the magnetic field is extended to include a component parallel to the wave vector. In order to include this component of the magnetic field, it is necessary to include an extra longitudinal electric field component:

$$\begin{aligned} F = {} & \mathcal{E}(z - vt)\,(dz - vdt) \wedge dx - B_x dy \wedge dz - B_z dx \wedge dy \\ & + \Omega_1 \mathcal{E}(z - vt) dt \wedge dz, \end{aligned} \tag{3.43}$$

where Ω_1 is a real constant. This extra term is present to allow for some interaction of the wave with the background field as in Ref. [20]. As before, the first field equation $dF = 0$ is automatically satisfied, leaving only the second field equation to satisfy. By the definition of the excitation 2-form (3.19), the field equation (3.18) becomes (on substitution of the 2-form (3.43))

$$\begin{aligned} \frac{1}{2} \star G = {} & -\left(B_x \partial_X L + \mathcal{E} v \partial_Y L\right) dt \wedge dx + \partial_X L \mathcal{E} dt \wedge dy \\ & - \left(B_z \partial_X L - \Omega_1 \mathcal{E} \partial_Y L\right) dt \wedge dz - \left(\Omega_1 \mathcal{E} \partial_X L + B_z \partial_Y L\right) dx \wedge dy \\ & - \partial_Y L \mathcal{E} dx \wedge dz + \left(\mathcal{E} v \partial_X L - B_x \partial_Y L\right) dy \wedge dz. \end{aligned} \tag{3.44}$$

Proceeding as before, the electromagnetic invariants in this case are

$$X = \left(\Omega_1^2 - \gamma^{-2}\right) \mathcal{E}^2 - \left(B_x^2 + B_z^2\right), \tag{3.45}$$

$$Y = 2\mathcal{E}\left(\Omega_1 B_z - v B_x\right), \tag{3.46}$$

and as in the previous section, the restricted and unrestricted Lagrangians are denoted by $\widehat{L}$ and L respectively (but using (3.45) and (3.46) as the subspace restriction). Substituting (3.45) and (3.46) into (3.44) and applying $\star d$ results the 1-form $\star d \star G$:

$$\begin{aligned}
\star d \star G = \mathcal{E}' \Bigg[& 2\Omega_1 \widehat{\frac{\partial L}{\partial X}} + 4\Omega_1 \mathcal{E}^2 (\Omega_1^2 - \gamma^{-2}) \widehat{\frac{\partial^2 L}{\partial X^2}} + 4B_z(\Omega_1 B_z - vB_x) \widehat{\frac{\partial^2 L}{\partial Y^2}} \\
& +4\mathcal{E}(B_z(2\Omega_1^2 - \gamma^{-2}) - vB_x\Omega_1) \widehat{\frac{\partial^2 L}{\partial X \partial Y}} \Bigg] dt \\
+ \mathcal{E}' \Bigg[& -2\gamma^{-2} \widehat{\frac{\partial L}{\partial X}} - 4\mathcal{E}^2 \gamma^{-2} (\Omega_1^2 - \gamma^{-2}) \widehat{\frac{\partial^2 L}{\partial X^2}} - 4vB_x(\Omega_1 B_z - vB_x) \widehat{\frac{\partial^2 L}{\partial Y^2}} \\
& +8\mathcal{E} \left(\frac{B_z}{2} \Omega_1 \gamma^{-2} + vB_x(\gamma^{-2} + \frac{1}{2}\Omega_1^2) \right) \widehat{\frac{\partial^2 L}{\partial X \partial Y}} \Bigg] dx \\
+ \mathcal{E}' \Bigg[& -4B_x \mathcal{E} \left(\Omega_1^2 - \gamma^{-2} \right) \widehat{\frac{\partial^2 L}{\partial X^2}} - 4B_x \left(\Omega_1 B_z - vB_x \right) \widehat{\frac{\partial^2 L}{\partial X \partial Y}} \Bigg] dy \\
- \frac{\mathcal{E}'}{v} \Bigg[& 2\Omega_1 \widehat{\frac{\partial L}{\partial X}} + 4\Omega_1 \mathcal{E}^2 (\Omega_1^2 - \gamma^{-2}) \widehat{\frac{\partial^2 L}{\partial X^2}} + 4B_z(\Omega_1 B_z - vB_x) \widehat{\frac{\partial^2 L}{\partial Y^2}} \\
& +4\mathcal{E}(B_z(2\Omega_1^2 - \gamma^{-2}) - vB_x\Omega_1) \widehat{\frac{\partial^2 L}{\partial X \partial Y}} \Bigg] dz. \qquad (3.47)
\end{aligned}$$

In order to satisfy the field equation (3.18) (equivalently $\star d \star G = 0$), each of the components of (3.47) must independently be equal to zero. Since the dt and dz coefficients are multiples of each other, this results in three independent equations (again assuming $\mathcal{E}' \neq 0$):

$$\begin{aligned}
\Omega_1 \widehat{\frac{\partial L}{\partial X}} + 2\Omega_1 \mathcal{E}^2 (\Omega_1^2 - \gamma^{-2}) \widehat{\frac{\partial^2 L}{\partial X^2}} + 2B_z(\Omega_1 B_z - vB_x) \widehat{\frac{\partial^2 L}{\partial Y^2}} & \\
+2\mathcal{E}(B_z(2\Omega_1^2 - \gamma^{-2}) - vB_x\Omega_1) \widehat{\frac{\partial^2 L}{\partial X \partial Y}} & = 0, \qquad (3.48) \\
\gamma^{-2} \widehat{\frac{\partial L}{\partial X}} + 2\mathcal{E}^2 \gamma^{-2} (\Omega_1^2 - \gamma^{-2}) \widehat{\frac{\partial^2 L}{\partial X^2}} + 2vB_x(\Omega_1 B_z - vB_x) \widehat{\frac{\partial^2 L}{\partial Y^2}} & \\
-4\mathcal{E} \left(\frac{B_z}{2} \Omega_1 \gamma^{-2} + vB_x(\gamma^{-2} + \frac{1}{2}\Omega_1^2) \right) \widehat{\frac{\partial^2 L}{\partial X \partial Y}} & = 0, \qquad (3.49) \\
B_x \mathcal{E} \left(\Omega_1^2 - \gamma^{-2} \right) \widehat{\frac{\partial^2 L}{\partial X^2}} + B_x \left(\Omega_1 B_z - vB_x \right) \widehat{\frac{\partial^2 L}{\partial X \partial Y}} & = 0. \qquad (3.50)
\end{aligned}$$

Since these equations still contain explicit $\mathcal{E}$ dependence, the substitution

$$\mathcal{E} = \frac{Y}{2\left(\Omega_1 B_z - vB_x\right)} \qquad (3.51)$$

is made[1] with the assumption as before to extend outside the solution subspace given by (3.45) and (3.46), resulting in the (final) P.D.E.s to solve:

$$\Omega_1 \frac{\partial L}{\partial X} + Y^2 \frac{\Omega_1(\Omega_1^2 - \gamma^{-2})}{2\left(\Omega_1 B_z - vB_x\right)^2} \frac{\partial^2 L}{\partial X^2} + 2B_z(\Omega_1 B_z - vB_x)\frac{\partial^2 L}{\partial Y^2}$$
$$+Y\frac{(B_z(2\Omega_1^2 - \gamma^{-2}) - vB_x\Omega_1)}{(\Omega_1 B_z - vB_x)} \frac{\partial^2 L}{\partial X \partial Y} = 0, \quad (3.52)$$

$$\gamma^{-2}\frac{\partial L}{\partial X} + Y^2 \frac{(\Omega_1^2 - \gamma^{-2})}{2\gamma^2\left(\Omega_1 B_z - vB_x\right)^2} \frac{\partial^2 L}{\partial X^2} + 2vB_x(\Omega_1 B_z - vB_x)\frac{\partial^2 L}{\partial Y^2}$$
$$-Y\frac{(B_z\Omega_1\gamma^{-2} + vB_x(2\gamma^{-2} + \Omega_1^2))}{(\Omega_1 B_z - vB_x)} \frac{\partial^2 L}{\partial X \partial Y} = 0, \quad (3.53)$$

$$B_x Y \frac{(\Omega_1^2 - \gamma^{-2})}{2\left(\Omega_1 B_z - vB_x\right)} \frac{\partial^2 L}{\partial X^2} + B_x\left(\Omega_1 B_z - vB_x\right) \frac{\partial^2 L}{\partial X \partial Y} = 0. \quad (3.54)$$

The simplest approach to solving this system of equations is via (3.54): assuming $B_x \neq 0$ and integrating once with respect to X gives

$$Y \frac{(\Omega_1^2 - \gamma^{-2})}{2\left(\Omega_1 B_z - vB_x\right)} \frac{\partial L}{\partial X} + \left(\Omega_1 B_z - vB_x\right) \frac{\partial L}{\partial Y} = \mathcal{F}_1(Y), \quad (3.55)$$

for some function $\mathcal{F}_1$. This is clearly an analogue of (3.38), and the equation is solved to give

$$L = \mathcal{F}_2(Y) + \mathcal{F}_3\left(X + \left(\frac{\gamma^{-2} - \Omega_1^2}{4(\Omega_1 B_z - vB_x)^2}\right) Y^2\right). \quad (3.56)$$

Inserting this into the other two components of $\star d \star G$ (3.52) and (3.53) restricts the family of Lagrangians to

$$L = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}_3\left(X + \lambda Y^2\right), \quad (3.57)$$

$$\text{with} \qquad \Omega_1 = \frac{B_z}{v\gamma^2 B_x}, \quad (3.58)$$

where

$$v^2 = \frac{1 + 4\lambda B_z^2}{1 + 4\lambda(B_x^2 + B_z^2)}. \quad (3.59)$$

[1]It is also possible to replace the $\mathcal{E}^2$ terms with an X-like term, resulting in different P.D.E.s. The above substitution is made for simplicity.

Notice that if $\lambda = \frac{\kappa^2}{4} = \frac{1}{4b^2}$, where $\kappa = 1/b$ is the Born-Infeld constant, then condition (3.59) agrees with equation (17) & (18) of Ref. [20].

By using the polar decomposition $B_x = B\sin\theta$ and $B_z = B\cos\theta$ for the angle θ between the wavevector and the magnetic field, (3.58) and (3.59) become

$$\Omega_1 = \frac{\cot\theta}{v\gamma^2} \quad \text{and} \quad v^2 = 1 - \frac{4\lambda B^2 \sin^2\theta}{1+4\lambda B^2}, \tag{3.60}$$

so it is clear that the background field has maximum effect on the wave when $\theta = \frac{\pi}{2}$, i.e. $B_z = \Omega_1 = 0$, and $v^2 = \frac{1}{1+4\lambda B_x^2}$ as before.

3.3.3 Background Magnetic Field in an Arbitrary Direction

Performing a similar calculation with all three components of the magnetic field active proves to be much more difficult. Indeed no simple F can be written such that a family of Lagrangians can be derived from the field equations. However Ref. [20] shows that a wave of the form

$$\begin{aligned} F = {}& \mathcal{E}(z-vt)\,(dz - vdt)\wedge dx - B_x dy\wedge dz - B_y dz\wedge dx - B_z dx\wedge dy \\ & + \Omega_2 \mathcal{E}(z-vt) dt\wedge dz, \end{aligned} \tag{3.61}$$

where Ω_2 is a coupling constant, satisfies the *Born-Infeld* field equations (the general field equations (3.17) and (3.18) with $L = L_{BI}$) with the constants

$$\Omega_2 = \frac{B_x B_z}{v\gamma^2\left(B_x^2 + B_y^2\right)}, \tag{3.62}$$

$$\text{and} \quad v^2 = \frac{1-4\lambda B_z^2}{1+4\lambda\left(B_x^2+B_y^2+B_z^2\right)}. \tag{3.63}$$

In fact using these constants and a Lagrangian of the form found previously, i.e. $L = \mathcal{F}(X + \lambda Y^2)$ with $\lambda = \frac{\kappa}{4}$,

$$L = \mathcal{F}\left(X + \frac{\kappa^2}{4} Y^2\right), \tag{3.64}$$

the dt and dz components of $\star d\star G$ are zero immediately and the dx and dy components are actually multiples of each other. The resulting condition on (3.64) is then

$$-\gamma^{-2}\mathcal{F}'\left(X+\frac{\kappa^2}{4}Y^2\right)+2\frac{\left(B_x^2+B_y^2-\gamma^{-2}B_y\mathcal{E}\right)^2}{B_x^2+By^2}\mathcal{F}''\left(X+\frac{\kappa^2}{4}Y^2\right)=0. \quad (3.65)$$

Now parametrising the argument of the function as $\Lambda = X+\frac{\kappa^2}{4}Y^2$, then inserting X and Y explicitly, $\mathcal{E}$ can be written in terms of Λ;

$$\mathcal{E}=\frac{B_x^2+B_y^2\pm\sqrt{\left(B_x^2+B_y^2\right)\left(v^2(B_x^2+B_y^2)-\gamma^{-2}(B_z^2+\Lambda)\right)}}{\gamma^{-2}B_y}. \quad (3.66)$$

Substituting either solution into the differential equation gives

$$-\mathcal{F}'(\Lambda)+2\left[v^2\gamma^2\left(B_x^2+B_y^2\right)-B_z^2-\Lambda\right]\mathcal{F}''(\Lambda)=0, \quad (3.67)$$

$$\text{i.e.}\qquad \mathcal{F}'(\Lambda)-\frac{2}{\kappa^2}\left(1-\kappa^2\Lambda\right)\mathcal{F}''(\Lambda)=0. \quad (3.68)$$

The solution to this equation is simply

$$L=\mathcal{F}(\Lambda)=C_1+\frac{C_2}{\kappa}\sqrt{1-\kappa^2\Lambda}, \quad (3.69)$$

and choosing $C_1=\frac{1}{\kappa^2}$ and $C_2=-\frac{1}{\kappa}$, the Born-Infeld Lagrangian emerges[1]. Hence the Born-Infeld Lagrangian does indeed satisfy the field equations for a wave of the form (3.61), and is the only Lagrangian of the form (3.64) whose field equations are solved by the wave (3.61) with the constants (3.62) and (3.63).

3.4 Solving the Field Equations with Constant Background Electric Fields

3.4.1 Background Electric Field Parallel to the Wave's Magnetic Field

Having considered a nonlinear wave in a constant magnetic field, it is natural to ask if the presence of a constant background *electric* field instead of a magnetic

[1]For a more detailed method, see Section 3.4.3 for an in-depth analogous calculation.

field[1] makes a difference to which theories have exact plane wave solutions that satisfy the field equations (3.17) and (3.18). Hence using the same wave profile as in the previous sections but using a constant electric field E_y, the appropriate F is

$$F = \mathcal{E}\left(dz - vdt\right) \wedge dx + E_y dt \wedge dy, \tag{3.70}$$

with invariants

$$X = E_y^2 - \gamma^{-2}\mathcal{E}^2, \tag{3.71}$$

$$Y = -2E_y\mathcal{E}. \tag{3.72}$$

As in the previous section, the four components of $\star d \star G$ are written in terms of X and Y by substituting the $\mathcal{E}$ terms for Y via (in this instance) $\mathcal{E} = -\frac{Y}{2E_y}$. The notation $\widehat{L}$ is again used to show the restriction to the subspace where X and Y are given by (3.71) and (3.72). Since the dt and dz components of $\star d \star G$ are zero, all that remains is

$$\begin{aligned}\star d \star G = \frac{1}{\gamma^2}\left[\frac{1}{E_y^2\gamma^2}Y^2\widehat{\frac{\partial^2 L}{\partial X^2}} - 4Y\widehat{\frac{\partial^2 L}{\partial Y \partial X}} + 4\gamma^2 E_y^2\widehat{\frac{\partial^2 L}{\partial Y^2}} - 2\widehat{\frac{\partial L}{\partial X}}\right] dx \\ + 2v\left[-\gamma^{-2}Y\widehat{\frac{\partial^2 L}{\partial X^2}} + 2E_y^2\left(\widehat{\frac{\partial^2 L}{\partial Y \partial X}}\right)\right] dy.\end{aligned} \tag{3.73}$$

Field equation (3.17) is satisfied automatically once again, and field equation (3.18) implies that the two coefficients in (3.73) must also be zero. Integration of the dy component, with the same previous assumption of extension beyond the subspace given by (3.71) and (3.72), gives

$$L = \mathcal{F}_1(Y) + \mathcal{F}_2\left(X + \frac{1}{4\gamma^2 E_y^2}Y^2\right), \tag{3.74}$$

for smooth functions $\mathcal{F}_1$ and $\mathcal{F}_2$. Insertion into the equation corresponding to the dx component restricts $\mathcal{F}_1$ gives the resultant Lagrangian as

$$L = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}_2\left(X + \lambda Y^2\right), \tag{3.75}$$

[1]Note that it is not possible to use a duality transform to adapt the B-field case into the constant E-field case. The duality transform entangles the varying $\mathcal{E}$ into all of the components, so that constant components are impossible.

where $\lambda = \frac{1}{4\gamma^2 E_y^2}$, in analogue to (3.42), or equivalently

$$v^2 = 1 - 4\lambda E_y^2. \tag{3.76}$$

Hence the larger the electric field parallel to the electromagnetic wave's magnetic component, the lower the phase speed of the wave.

3.4.2 Background Electric Field Orthogonal to the Wave's Electric Field with Transverse Electric Component

Extending the background E field to the $y-z$ plane, F is of the form

$$F = \mathcal{E}\,(dz - vdt) \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz + \Omega_3 \mathcal{E} dt \wedge dz, \tag{3.77}$$

with constant Ω_3 and invariants

$$X = E_y^2 + E_z^2 + 2E_z\Omega_3\mathcal{E} + \left(\Omega_3^2 - \gamma^{-2}\right)\mathcal{E}^2, \tag{3.78}$$

$$Y = -2E_y\mathcal{E}. \tag{3.79}$$

Upon substituting $\mathcal{E} = -\frac{Y}{2E_y}$ and as before extending beyond the subspace where X and Y are given by (3.78) and (3.79), the field equations can be studied. Consulting $d\star G = 0$, one of the four components is zero and the three remaining linearly independent equations, corresponding to the dx, dy and dz components of $\star d \star G$, are (assuming $v \neq 0, \mathcal{E}' \neq 0$)

$$-2\frac{\partial L}{\partial X} + \frac{1}{\gamma^2 E_y^2}\left(2E_yE_z\Omega_3 - \left({\Omega_3}^2 - \gamma^{-2}\right)Y\right)Y\frac{\partial^2 L}{\partial X^2} + 4{E_y}^2\frac{\partial^2 L}{\partial Y^2}$$
$$-2\left(2E_yE_z\Omega_3 - \left(\Omega_3^2 - 2\gamma^{-2}\right)Y\right)\frac{\partial^2 L}{\partial Y \partial X} = 0, \tag{3.80}$$

$$-2v\left(2E_yE_z\Omega_3 - \left({\Omega_3}^2 + \gamma^{-2}\right)Y\right)\frac{\partial^2 L}{\partial X^2} + 4vE_y^2\frac{\partial^2 L}{\partial Y \partial X} = 0, \tag{3.81}$$

$$-2vE_y^2\Omega_3\frac{\partial L}{\partial X} - \frac{v}{E_y^2}\left(2E_yE_z + \Omega_3 Y\right)\left(2E_yE_z\Omega_3 - \left({\Omega_3}^2 - \gamma^{-2}\right)Y\right)\frac{\partial^2 L}{\partial X^2}$$
$$+2v\left(2E_yE_z + \Omega_3 Y\right)\frac{\partial^2 L}{\partial Y \partial X} = 0. \tag{3.82}$$

Integration of (3.81), the dy component of $\star d \star G$, results in

$$L = \mathcal{F}_1(Y) + \mathcal{F}_2\left(X + \frac{E_z}{E_y}\Omega_3 Y - \frac{\Omega_3^2 - \gamma^{-2}}{4E_y^2}Y^2\right), \tag{3.83}$$

for smooth functions $\mathcal{F}_1$ and $\mathcal{F}_2$. As before, insertion into the equation corresponding to the dx component restricts $\mathcal{F}_1$, leading to the condition $\Omega_3 = 0$ and a linear $\mathcal{F}_1$, i.e.

$$L = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}_2\left(X + \lambda Y^2\right), \tag{3.84}$$

where $\lambda = -\frac{\Omega_3^2 - \gamma^{-2}}{4E_y^2}$ or equivalently

$$v^2 = 1 - 4\lambda E_y^2, \tag{3.85}$$

identical to (3.76). Interestingly, note the lack of E_z dependence here; no matter the strength of the electric field parallel to the direction of propagation of the wave, the speed of the wave is unchanged.

3.4.3 Background Electric Field in an Arbitrary Direction

Extending the background electric field to an arbitrary direction relative to the wave, i.e. using the electromagnetic 2-form

$$\begin{aligned} F = {} & \mathcal{E}\left(dz - vdt\right) \wedge dx + E_x dt \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz \\ & + \Omega_4 \mathcal{E}\left(dt \wedge dz\right), \end{aligned} \tag{3.86}$$

as before, again results in difficulties. There is no simple dy component of $\star d \star G$ as in the previous cases, and hence there is no obvious way to proceed analytically. Using Maple software it is possible to show that the three linearly independent components of $\star d \star G$ give the three solutions

$$L_1 = \mathcal{F}_1\left(X - v\frac{E_x}{E_y}Y + \frac{1}{4\gamma^2 E_y^2}Y^2\right), \tag{3.87}$$

$$L_2 = \mathcal{F}_2\left(X + v\frac{E_y}{E_x}Y\right), \tag{3.88}$$

$$L_3 = \mathcal{F}_3 + \mathcal{F}_4\left(X + \frac{1}{E_y}\left(E_z\Omega_4 - vE_x\right)Y - \frac{1}{4E_y^2}\left(\Omega_4^2 - \gamma^{-2}\right)Y^2\right). \tag{3.89}$$

Each of these solutions however only satisfies one of the three necessary conditions (one of the components of $\star d \star G = 0$) and hence it is necessary to plug each of these into the remaining two equations in order to acquire solutions satisfying all three equations.

In order for L_1 to satisfy the field equations, $\Omega_4 = E_x = 0$, i.e. L_1 reverts to the E_y alone case, (see (3.75)). For L_2 to satisfy the field equations, E_x and E_y must be equal to zero, i.e. $L_2 = 0$. L_3 satisfies the field equations if $\mathcal{F}_3$ is linear and either

a) $E_y = 0$ so that $L_3 = \mathcal{F}_5(Y)$ or

b) $E_x = \Omega_4 = 0$ so that

$$L_3 = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}_4 \left(X + \frac{1}{4\gamma^2 E_y^2} Y^2 \right). \tag{3.90}$$

Hence there is no clear way to derive a family of theories supporting a wave passing through a region of arbitrarily aligned electric field. However as per Ref. [20], it is again possible to show that wave (3.86) is a solution to the Born-Infeld equations as in the magnetic field case. Hence, given the constants (from Ref. [20])

$$\Omega_4 = \frac{E_x E_z}{v\gamma^2(E_x^2 + E_y^2)} \tag{3.91}$$

$$\text{and} \quad v^2 = 1 - \kappa^2(E_x^2 + E_y^2), \tag{3.92}$$

it is possible to show that the Born-Infeld Lagrangian satisfies the field equations via the following method.

Firstly, given the electromagnetic 2-form F (3.86) with invariants

$$X = \left(\Omega_4^2 - \gamma^{-2}\right) \mathcal{E}^2 + 2\left(\Omega_4 E_z - v E_x\right) \mathcal{E} + E_x^2 + E_y^2 + E_z^2, \tag{3.93}$$

$$Y = -2 E_y \mathcal{E}, \tag{3.94}$$

the usual $\star d \star G$ equations are derived. Then Ω_4 from (3.91) is inserted, and the choice $L = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}\left(X + \frac{\kappa^2}{4} Y^2\right)$ is made, where

$$\kappa^2 = \frac{1}{\gamma^2 \left(E_x^2 + E_y^2\right)} \tag{3.95}$$

from (3.92). Here it becomes clear that the four equations (from the four components of $\star d \star G$) are all linearly dependent hence the problem collapses to solving

$$-2\frac{1}{\gamma^2}\mathcal{F}'\left(X+\frac{\kappa^2}{4}Y^2\right)+a\mathcal{F}''\left(X+\frac{\kappa^2}{4}Y^2\right)=0, \tag{3.96}$$

where

$$a=\frac{\left[v^2\left(E_x^2+E_y^2+E_z^2\right)-E_z^2\right]\left[2vE_y\left(E_x^2+E_y^2\right)-E_x\gamma^{-2}Y\right]^2}{v^2E_y^2\left(E_x^2+E_y^2\right)^2}. \tag{3.97}$$

The argument of $\mathcal{F}$ is then parametrised via $X+\frac{\kappa^2}{4}Y^2=\Lambda$, which via insertion of X (3.93) and Y (3.94) enables $\mathcal{E}$ to be written in terms of Λ:

$$\mathcal{E}=-\frac{v\gamma^2\left(E_x^2+E_y^2\right)}{E_x}\pm v\gamma^2\frac{\sqrt{\left(E_x^2+E_y^2-\gamma^{-2}\Lambda\right)\left(v^2\left(E_x^2+E_y^2+E_z^2\right)-E_z^2\right)}}{E_x\left(v^2\left(E_x^2+E_y^2+E_z^2\right)-E_z^2\right)}. \tag{3.98}$$

Inserting Y in terms of $\mathcal{E}$ and hence in terms of Λ into (3.96) (and (3.97)) simplifies the condition to

$$-\frac{1}{\gamma^2}\mathcal{F}'(\Lambda)+2\left(E_x^2+E_y^2-\gamma^{-2}\Lambda\right)\mathcal{F}''(\Lambda)=0, \tag{3.99}$$

and recalling that $E_x^2+E_y^2=\kappa^{-2}\gamma^{-2}$ via (3.95), the condition becomes

$$-\mathcal{F}'(\Lambda)+2\left(\kappa^{-2}-\Lambda\right)\mathcal{F}''(\Lambda)=0. \tag{3.100}$$

This is simple to solve, and yields

$$\mathcal{F}=\mathcal{C}_3+\mathcal{C}_4\sqrt{\Lambda-\frac{1}{\kappa^2}}=\mathcal{C}_3+\mathcal{C}_4\sqrt{X+\frac{\kappa^2}{4}Y^2-\frac{1}{\kappa^2}}, \tag{3.101}$$

which is simply the Born-Infeld Lagrangian using $\mathcal{C}_3=\frac{1}{\kappa^2}$ and $\mathcal{C}_4=-i\frac{1}{\kappa^2}$.

Hence while the direct approach does not yield any Lagrangians whose field equations are satisfied by the plane wave in an arbitrary configuration of a background electric field, it is possible to show that this wave *does* satisfy the Born-Infeld field equations.

3.5 Duality Transform Invariance

The previous sections have used a wave propagating through a region of constant background electromagnetic field in order to test the properties of Born-Infeld-like theories with Lagrangians of the form

$$L = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}\left(X + \lambda Y^2\right). \tag{3.42 revisited}$$

While it is argued that the presence of certain components of field (in the above configuration, B_y or E_x) would aid in discriminating between these theories, it is also possible that invoking other laws or invariances could be of assistance. This section shows that the family of Lagrangians (3.42) can be reduced even down to a single member of the family.

When extending Maxwell theory into nonlinearity, it is necessary to consider which properties of linear Maxwell theory should be preserved. For instance by writing Lagrangians in terms of the Lorentz invariants X and Y, overall Lorentz-transform invariance can be preserved. Now consider electric-magnetic duality transformations, i.e.

$$\mathbf{E} \to \cos(\vartheta)\mathbf{E} - \sin(\vartheta)\mathbf{B}, \tag{3.102}$$

$$\mathbf{B} \to \cos(\vartheta)\mathbf{B} + \sin(\vartheta)\mathbf{E}, \tag{3.103}$$

for some real constant ϑ and the usual electric and magnetic field 3-vectors $\mathbf{E}$ and $\mathbf{B}$, under which vacuum Maxwell theory is invariant. This duality invariance can be thought of as a consequence of special relativity (applying Lorentz transformations to electric fields results in magnetic fields etc.) and again also has interest from string theory, since electric-magnetic duality is a 4 dimensional reduction of S-duality, which switches the strong and weak string couplings (see page 374 of Ref. [34]). This property may be maintained by elevating electric-magnetic duality invariance to a fundamental property of the electromagnetic field. Following the work of Ref. [22], the covariant generalisation of the electric-magnetic duality transforms can be written

$$F_\vartheta = \cos(\vartheta)F + \sin(\vartheta) \star G, \tag{3.104}$$

$$G_\vartheta = \cos(\vartheta)G + \sin(\vartheta) \star F. \tag{3.105}$$

Hence

$$\begin{aligned}\star\left(F_\vartheta \wedge F_\vartheta\right) &= \star\left[(\cos(\vartheta)F + \sin(\vartheta)\star G) \wedge (\cos(\vartheta)F + \sin(\vartheta)\star G)\right] \\ &= \star\left[\cos^2(\vartheta)F \wedge F - \sin^2(\vartheta)G \wedge G + 2\sin(\vartheta)\cos(\vartheta)F \wedge \star G\right]\end{aligned} \tag{3.106}$$

and similarly

$$\begin{aligned}\star\left(G_\vartheta \wedge G_\vartheta\right) &= \star\left[(\cos(\vartheta)G + \sin(\vartheta)\star F) \wedge (\cos(\vartheta)G + \sin(\vartheta)\star F)\right] \\ &= \star\left[-\sin^2(\vartheta)F \wedge F + \cos^2(\vartheta)G \wedge G + 2\sin(\vartheta)\cos(\vartheta)F \wedge \star G\right].\end{aligned} \tag{3.107}$$

Subtracting (3.107) from (3.106):

$$\star\left(F_\vartheta \wedge F_\vartheta\right) - \star\left(G_\vartheta \wedge G_\vartheta\right) = \star\left(F \wedge F\right) - \star\left(G \wedge G\right). \tag{3.108}$$

Hence the quantity $\star\left(F_\vartheta \wedge F_\vartheta\right) - \star\left(G_\vartheta \wedge G_\vartheta\right)$ is independent of ϑ, and can be written

$$\star\left(F_\vartheta \wedge F_\vartheta\right) - \star\left(G_\vartheta \wedge G_\vartheta\right) = \mathcal{C}, \tag{3.109}$$

where $\mathcal{C}$ is independent of ϑ. Since this is true for any choice of ϑ and Maxwell electrodynamics has the relationship $F = G$, (3.109) is satisfied for $\mathcal{C} = 0$ case. To preserve this, $\mathcal{C} = 0$ is assumed from this point. The condition

$$\star\left(F \wedge F\right) = \star\left(G \wedge G\right) \tag{3.110}$$

is known as the Gaillard-Zumino condition (first considered in [22], but first used with $\mathcal{C} = 0$ in [35], though more straightforward to see in [36]). Using the definition of G (3.19), this condition becomes the P.D.E.

$$Y = 4Y\left[\left(\frac{\partial L}{\partial X}\right)^2 - \left(\frac{\partial L}{\partial Y}\right)^2\right] - 8X\frac{\partial L}{\partial X}\frac{\partial L}{\partial Y}. \tag{3.111}$$

Now consider which members of the family (3.42) satisfy the proposed condition of electric-magnetic duality invariance; the assumption that $\mathcal{C}_2 = 0$ results in

$$Y\left(1 - 4\left(1 - 4\lambda\left(X + \lambda Y^2\right)\right)\left(\mathcal{F}'\left(X + \lambda Y^2\right)\right)^2\right) = 0, \tag{3.112}$$

which is solved algebraically to give

$$\mathcal{F}'\left(X+\lambda Y^2\right) = \pm\frac{1}{\sqrt{1-4\lambda\left(X+\lambda Y^2\right)}}, \tag{3.113}$$

$$\text{i.e.}\quad \mathcal{F}\left(X+\lambda Y^2\right) = \mp\frac{1}{4\lambda}\sqrt{1-4\lambda\left(X+\lambda Y^2\right)} + \mathcal{C}_3. \tag{3.114}$$

Choosing the *negative* sign here ensures that, in the weak field case, Maxwell theory is retrieved from L. Choosing $\lambda = \frac{1}{4}\kappa^2$ and the integration constant $\mathcal{C}_3 = \frac{1}{\kappa^2}$, results in the Born-Infeld Lagrangian

$$L = \frac{1}{\kappa^2}\left(1-\sqrt{1-\kappa^2 X - \frac{\kappa^4 Y^2}{4}}\right). \tag{3.115}$$

Hence the only member of the family $\mathcal{F}\left(X+\lambda Y^2\right)$ that satisfies electric-magnetic duality invariance (with the duality constant $\mathcal{C}=0$ as per the Gaillard-Zumino condition) is the Born-Infeld Lagrangian.

3.6 Summary

This chapter has shown that a plane electromagnetic wave travelling through a region of constant magnetic field (F of the form (3.61)) is an exact solution of the field equations of the family of theories with Lagrangians given by

$$L = \mathcal{C}_1 + \mathcal{C}_2 Y + \mathcal{F}\left(X+\lambda Y^2\right), \tag{3.42 revisited}$$

so long as B_y (the component of the background field parallel to the wave's magnetic field) is zero. The same can be said for a plane EM wave travelling through a region of constant electric field (F of the form (3.86)) so long as E_x (the component of the background field parallel to the wave's electric field) is zero. The speed of the wave does not depend on the theory in question, and hence there is no way to distinguish between theories of the form (3.42) using a slow-light experiment such as those considered in Ref. [21] or [20] without imposing electric-magnetic duality invariance. Insisting on electric-magnetic duality invariance restricts this family to just Born-Infeld theory.

Inclusion of a non-zero B_y (or E_x) component to the background field means that (3.61) (or (3.86)) is no longer a solution to the field equations generated by the family of theories (3.42). The only theory found whose equations these waves solve was Born-Infeld theory. Hence if one aims to distinguish Born-Infeld from the family (3.42), it is desirable to ensure that the background field includes a magnetic component parallel to the wave's own magnetic field or an electric component parallel to the wave's own electric field.

Sections 3.3 and 3.5 have been published in EPL in 2012 (see Ref. [1]). Section 3.4 is also original work, but has yet to be published.

Chapter 4

Electron Energy Gain in a Maximum Amplitude Plasma Wave

4.1 Introduction

Since nonlinear electromagnetic theories such as Born-Infeld and Euler-Heisenberg are not equivalent to Maxwell theory at high energy scales, it is important to consider scenarios with the potential to distinguish between nonlinear theories. Such a potential experiment is considering the energy gained by an electron in a maximum amplitude plasma wave.

Sufficiently short, high-intensity laser pulses can form longitudinal waves within the electrons of a plasma. Such oscillations in the plasma electrons travel with speed comparable to the group speed of the laser pulse. Not all plasma electrons form this wave, however; some of the free plasma electrons are caught up in the wave and accelerated by its high fields. When large numbers of electrons are accelerated the wave breaks due to damping. This wave breaking is fundamentally nonlinear and hence is an ideal place to study extensions to Maxwell theory. This chapter hence focuses on a plasma near to wave-breaking. Since upcoming lasers [5, 6] are hoped to be powerful enough to investigate quantum phenomena, it could also be possible to investigate whether the effects of nonlinear classical theories need to be accounted for first.

Some preliminary work on the subject of electrons in maximum amplitude plasma waves has already been done [23, 37] in the context of Born-Infeld theory, though only an estimate for the electron energy gain was found. This chapter aims to study this energy gain not only for Born-Infeld theory but for a general non-linear theory with Lagrangian $L(X, Y)$, where X and Y are the electromagnetic invariants, by appealing to the stress balance law (see Appendix A for motivation). Additionally, the presence of a background magnetic field is considered.

4.2 Maximum Amplitude Plasma Waves

4.2.1 Preliminaries

In order to find the energy gained by an electron in a half-wavelength of a maximum amplitude plasma wave, it is necessary to first set up several tools which will be needed later. This chapter is inspired by Ref. [23], though uses a different approach for the main calculation and the final result is more general.

Since this chapter is investigating the properties of a plasma wave propagating along the z-direction with velocity v, it is helpful to use the orthonormal coframe $\{\gamma d\zeta, dx, dy, \gamma d\xi\}$ where $\xi = z - vt$ is the wave's phase, $\zeta = -t + vz$ and $\gamma = \frac{1}{\sqrt{1-v^2}}$ is the Lorentz factor of the plasma wave. This orthonormal coframe is adapted to the wave frame[1] just as the coframe $\{dt, dx, dy, dz\}$ is adapted to the lab frame.

This chapter assumes a cold plasma, and since the scales involved are such that the electron motion is much greater than that of the ions, the plasma ions are considered to be a stationary background. As such the plasma ion worldlines are the trajectories of $V_{\text{ion}} = \frac{\partial}{\partial t}$ and the plasma ion density n_{ion} is a constant.

[1] In the wave frame there is no time evolution of the wave.

The plasma electrons constituting the wave[1] have worldlines given by the trajectories of the vector field $V_{\rm e}$ with number density $n_{\rm e}$. Using the quasi-static approximation, $V_{\rm e}$ is supposed to have the form

$$\widetilde{V}_{\rm e} = \nu(\xi)d\zeta + \chi(\xi)d\xi, \tag{4.1}$$

for some smooth functions ν and χ. Insistence that $V_{\rm e}$ be normalised according to $g(V_{\rm e}, V_{\rm e}) = -1$ results in

$$\begin{aligned}\widetilde{V}_{\rm e} &= \nu d\zeta - \sqrt{\nu^2-\gamma^2}d\xi & (4.2)\\ &= \left(-\nu + v\sqrt{\nu^2-\gamma^2}\right)dt + \left(v\nu - \sqrt{\nu^2-\gamma^2}\right)dz. & (4.3)\end{aligned}$$

Note that ν must be positive in order for $V_{\rm e}$ to be future-pointing. Hence the electrons move slower than the plasma wave except at the wave-breaking limit when the electrons catch the wave (when $\nu = \gamma$).

4.2.2 Introducing the Plasma Wave

Consider a wave of plasma electrons with a background of plasma ions in a region of constant magnetic field $\mathbf{B} = (B_x, B_y, B_z)$. Hence the Faraday 2-form of the plasma wave is

$$F = E(\xi)dt\wedge dz - B_x dy\wedge dz - B_y dz\wedge dx - B_z dx\wedge dy, \tag{4.4}$$

and the electromagnetic invariants are

$$\begin{aligned}X &= E^2 - (B_x^2 + B_y^2 + B_z^2), & (4.5)\\ Y &= 2EB_z. & (4.6)\end{aligned}$$

The appropriate[2] field equations and the Lorentz-force equation are

$$\begin{aligned}dF &= 0, & (4.7)\\ d\star G &= -q_{\rm e}n_{\rm e}\star\widetilde{V}_{\rm e} - q_{\rm ion}n_{\rm ion}\star\widetilde{V}_{\rm ion}, & (4.8)\\ \nabla_{V_{\rm e}}\widetilde{V}_{\rm e} &= \frac{q_{\rm e}}{m_{\rm e}} i_{V_{\rm e}}F, & (4.9)\end{aligned}$$

[1]The free plasma electrons constituting the accelerated electrons which cause the wave-breaking will not be modelled in this chapter.

[2]These are well established, but Appendix B shows the variation of a sample Lagrangian to justify them.

where G is the excitation 2-form

$$G = 2\left(\frac{\partial L}{\partial X}F - \frac{\partial L}{\partial Y}\star F\right). \tag{3.19 revisited}$$

Since $g(V_\mathrm{e}, V_\mathrm{e}) = -1$, it is possible to rewrite $\nabla_{V_\mathrm{e}}\widetilde{V}_\mathrm{e} = i_{V_\mathrm{e}}d\widetilde{V}_\mathrm{e}$ and thus (4.9) becomes

$$i_{V_\mathrm{e}}d\widetilde{V}_\mathrm{e} = \frac{q_\mathrm{e}}{m_\mathrm{e}}i_{V_\mathrm{e}}F. \tag{4.10}$$

Inserting (4.4) and (4.3) into (4.10) results in the four conditions

$$\left(-\nu + v\sqrt{\nu^2-\gamma^2}\right)\left(\frac{q_\mathrm{e}}{m_\mathrm{e}}E - \frac{1}{\gamma^2}\nu'\right) = 0, \tag{4.11}$$

$$\frac{q_\mathrm{e}}{m_\mathrm{e}}\left(v\nu - \sqrt{\nu^2-\gamma^2}\right)B_y = 0, \tag{4.12}$$

$$\frac{q_\mathrm{e}}{m_\mathrm{e}}\left(v\nu - \sqrt{\nu^2-\gamma^2}\right)B_x = 0, \tag{4.13}$$

$$\left(v\nu - \sqrt{\nu^2-\gamma^2}\right)\left(\frac{q_\mathrm{e}}{m_\mathrm{e}}E - \frac{1}{\gamma^2}\nu'\right) = 0, \tag{4.14}$$

which for a non-constant ν result in $B_x = B_y = 0$ and

$$E = \frac{m_\mathrm{e}\nu'}{q_\mathrm{e}\gamma^2}. \tag{4.15}$$

It is also possible to relate the plasma electron number density n_e to the background ion density n_ion via (4.8). As F (and hence G) depends on ξ alone, $d\star G$ must be of the form $d\xi \wedge \ldots$, thus wedging $d\xi$ to the source part of (4.8) results in the condition:

$$d\xi \wedge \left(q_\mathrm{e}n_\mathrm{e}\star\widetilde{V}_\mathrm{e} + q_\mathrm{ion}n_\mathrm{ion}\star\widetilde{V}_\mathrm{ion}\right) = 0. \tag{4.16}$$

Breaking $d\xi$ into $dz - vdt$ and inserting (4.3) and $V_\mathrm{ion} = \frac{\partial}{\partial t}$ into (4.16) results in

$$q_\mathrm{e}n_\mathrm{e}(v\nu - \sqrt{\nu^2-\gamma^2})dz\wedge\star dz - v\left(q_\mathrm{e}n_\mathrm{e}(-\nu + v\sqrt{\nu^2-\gamma^2}) - q_\mathrm{ion}n_\mathrm{ion}\right)dt\wedge\star dt = 0, \tag{4.17}$$

and since $dz\wedge\star dz = \star 1 = -dt\wedge\star dt$,

$$q_\mathrm{e}n_\mathrm{e}(-\sqrt{\nu^2-\gamma^2} + v\nu) + v\left(q_\mathrm{e}n_\mathrm{e}(-\nu + v\sqrt{\nu^2-\gamma^2}) - q_\mathrm{ion}n_\mathrm{ion}\right) = 0, \tag{4.18}$$

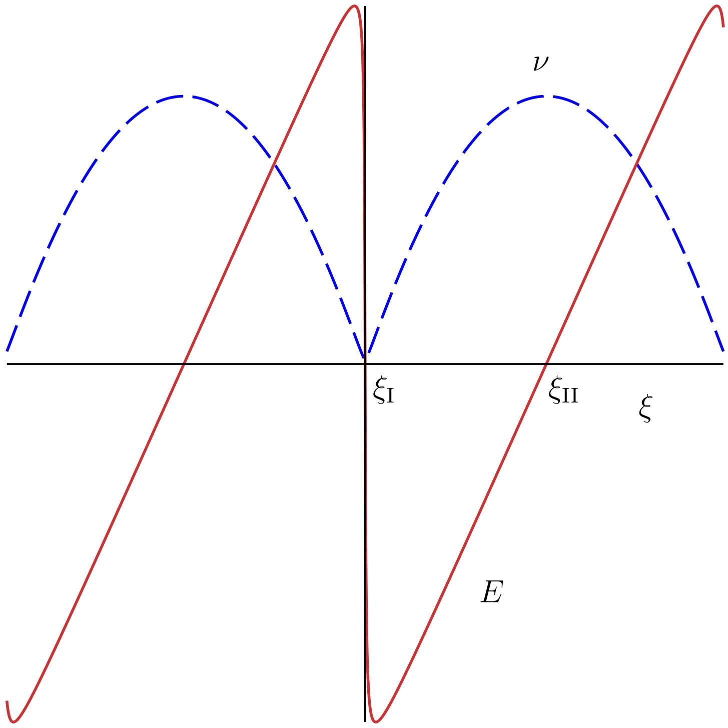

Figure 4.1: Electric field (red) and ν (blue) plotted along ξ (not to scale). Since the electron has a negative charge, (4.15) gives $E \sim -\nu'$. The points ξ_{I} and ξ_{II} are also shown (adapted from Ref. [37]).

and solving for n_{e} gives

$$n_{\mathrm{e}} = -\frac{v\gamma^2 q_{\mathrm{ion}} n_{\mathrm{ion}}}{q_{\mathrm{e}}\sqrt{\nu^2 - \gamma^2}}. \tag{4.19}$$

The overall sign on this term will be positive since the charge of the electron $q_e = -e$, where e is the elementary charge[1].

[1] Equation (4.19) can also be written in terms of the degree of ionisation, $Z = -\frac{q_{\mathrm{ion}}}{q_{\mathrm{e}}}$ thus: $n_{\mathrm{e}} = \frac{v\gamma^2 Z n_{\mathrm{ion}}}{\sqrt{\nu^2 - \gamma^2}}$.

4.2.3 Solving the Stress Balance Equation

To summarise the previous section, the information acquired so far is as follows:

$$F = Edt \wedge dz - B_z dx \wedge dy, \tag{4.20}$$

$$E = \frac{m_{\mathrm{e}}}{q_{\mathrm{e}}\gamma^2}\nu', \tag{4.15 revisited}$$

$$\widetilde{V}_{\mathrm{e}} = \left(-\nu + v\sqrt{\nu^2 - \gamma^2}\right) dt + \left(v\nu - \sqrt{\nu^2 - \gamma^2}\right) dz, \tag{4.3 revisited}$$

$$n_{\mathrm{e}} = -\frac{v\gamma^2 q_{\mathrm{ion}} n_{\mathrm{ion}}}{q_{\mathrm{e}}\sqrt{\nu^2 - \gamma^2}}. \tag{4.19 revisited}$$

One more tool is necessary to proceed; the stress-energy-momentum 3-forms for a cold plasma[1] in a nonlinear electromagnetic theory are given by

$$\mathcal{T}_a = i_{X_a} F \wedge \star G + i_{X_a} \star L + m_{\mathrm{e}} n_{\mathrm{e}} i_{X_a} \widetilde{V}_{\mathrm{e}} \star \widetilde{V}_{\mathrm{e}}, \tag{4.21}$$

which, on Killing frame X_a given by $\{\partial_t, \partial_x, \partial_y, \partial_z\}$, obey the balance law [2]

$$d\mathcal{T}_a = q_{\mathrm{ion}} n_{\mathrm{ion}} i_{V_{\mathrm{ion}}} i_{X_a} F \star 1. \tag{4.22}$$

Inserting F (4.20) and $\widetilde{V}_{\mathrm{e}}$ (4.3) into the four stress form components τ_{X_a} gives

$$\mathcal{T}_0 = \mathcal{F}_1(\xi) dx \wedge dy \wedge dz + \mathcal{F}_2(\xi) dt \wedge dx \wedge dy, \tag{4.23}$$

$$\mathcal{T}_1 = \mathcal{F}_3(\xi) dt \wedge dy \wedge dz, \tag{4.24}$$

$$\mathcal{T}_2 = \mathcal{F}_4(\xi) dt \wedge dx \wedge dz, \tag{4.25}$$

$$\mathcal{T}_3 = \mathcal{F}_5(\xi) dt \wedge dx \wedge dy + \mathcal{F}_6(\xi) dx \wedge dy \wedge dz, \tag{4.26}$$

[1] See Appendix B for motivation of this via variation of a sample action.

[2] For justification, see Appendix B.5.

where

$$\mathcal{F}_1(\xi) = -2E\left(E\frac{\partial L}{\partial X} + B_z\frac{\partial L}{\partial Y}\right) + L - m_\mathrm{e}n_\mathrm{e}\left(-\nu + v\sqrt{\nu^2-\gamma^2}\right)^2, \tag{4.27}$$

$$\mathcal{F}_2(\xi) = -m_\mathrm{e}n_\mathrm{e}\left(-\nu + v\sqrt{\nu^2-\gamma^2}\right)\left(v\nu - \sqrt{\nu^2-\gamma^2}\right), \tag{4.28}$$

$$\mathcal{F}_3(\xi) = 2B_z\left(E\frac{\partial L}{\partial Y} - B_z\frac{\partial L}{\partial X}\right) - L, \tag{4.29}$$

$$\mathcal{F}_4(\xi) = -2B_z\left(E\frac{\partial L}{\partial Y} - B_z\frac{\partial L}{\partial X}\right) - L, \tag{4.30}$$

$$\mathcal{F}_5(\xi) = 2E\left(E\frac{\partial L}{\partial X} + B_z\frac{\partial L}{\partial Y}\right) - L - m_\mathrm{e}n_\mathrm{e}\left(v\nu - \sqrt{\nu^2-\gamma^2}\right)^2, \tag{4.31}$$

$$\mathcal{F}_6(\xi) = -m_\mathrm{e}n_\mathrm{e}\left(v\nu - \sqrt{\nu^2-\gamma^2}\right)\left(-\nu + v\sqrt{\nu^2-\gamma^2}\right). \tag{4.32}$$

Since the only variable quantity in any $\mathcal{F}_n$ is $\xi = z - vt$, $d\mathcal{F}_n = \frac{\partial \mathcal{F}_n}{\partial \xi}dz - v\frac{\partial \mathcal{F}_n}{\partial \xi}dt$, and thus

$$\begin{aligned} d\mathfrak{T}_0 &= d\mathcal{F}_1(\xi)\wedge dx\wedge dy\wedge dz + d\mathcal{F}_2(\xi)\wedge dt\wedge dx\wedge dy \\ &= -\frac{\partial}{\partial \xi}\left(v\mathcal{F}_1(\xi) + \mathcal{F}_2(\xi)\right)\star 1, \end{aligned} \tag{4.33}$$

$$d\mathfrak{T}_1 = d\mathcal{F}_3(\xi)\wedge dt\wedge dy\wedge dz = 0, \tag{4.34}$$

$$d\mathfrak{T}_2 = d\mathcal{F}_4(\xi)\wedge dt\wedge dx\wedge dz = 0, \tag{4.35}$$

$$\begin{aligned} d\mathfrak{T}_3 &= d\mathcal{F}_5(\xi)\wedge dt\wedge dx\wedge dy + d\mathcal{F}_6(\xi)\wedge dx\wedge dy\wedge dz \\ &= -\frac{\partial}{\partial \xi}\left(\mathcal{F}_5(\xi) + v\mathcal{F}_6(\xi)\right)\star 1. \end{aligned} \tag{4.36}$$

The four components of the RHS of (4.22) with $V_\mathrm{ion} = \frac{\partial}{\partial t}$ are

$$q_\mathrm{ion}n_\mathrm{ion}i_{V_\mathrm{ion}}i_{X_0}F\star 1 = 0, \tag{4.37}$$

$$q_\mathrm{ion}n_\mathrm{ion}i_{V_\mathrm{ion}}i_{X_1}F\star 1 = 0, \tag{4.38}$$

$$q_\mathrm{ion}n_\mathrm{ion}i_{V_\mathrm{ion}}i_{X_2}F\star 1 = 0, \tag{4.39}$$

$$q_\mathrm{ion}n_\mathrm{ion}i_{V_\mathrm{ion}}i_{X_3}F\star 1 = -q_\mathrm{ion}n_\mathrm{ion}E\star 1. \tag{4.40}$$

Hence the x and y components of (4.22) are immediately satisfied leaving the t and z components:

$$d\mathfrak{T}_0 = -\frac{\partial}{\partial \xi}\left(v\mathcal{F}_1(\xi) + \mathcal{F}_2(\xi)\right)\star 1 = 0, \tag{4.41}$$

$$d\mathfrak{T}_3 = -\frac{\partial}{\partial \xi}\left(\mathcal{F}_5(\xi) + v\mathcal{F}_6(\xi)\right)\star 1 = -q_\mathrm{ion}n_\mathrm{ion}E\star 1. \tag{4.42}$$

Since $E = \frac{m_{\rm e}}{q_{\rm e}\gamma^2}\nu' = \frac{m_{\rm e}}{q_{\rm e}\gamma^2}\frac{d\nu}{d\xi}$, these equations can be written

$$d\mathfrak{T}_0 = -\frac{\partial}{\partial\xi}\left(v\mathcal{F}_1(\xi) + \mathcal{F}_2(\xi)\right) \star 1 = 0, \tag{4.43}$$

$$d\mathfrak{T}_3 = -\frac{\partial}{\partial\xi}\left(\mathcal{F}_5(\xi) + v\mathcal{F}_6(\xi) - m_{\rm e}n_{\rm ion}\frac{q_{\rm ion}}{q_{\rm e}}\frac{\nu}{\gamma^2}\right) \star 1 = 0. \tag{4.44}$$

Inserting the functions $\mathcal{F}_n$, E (4.15) and $n_{\rm e}$ (4.19), it becomes clear that (4.43) and (4.44) are multiples of one another; thus the stress balance law (4.22) reduces to the single partial differential equation

$$\frac{\partial}{\partial\xi}\left[2\left(\frac{m_{\rm e}\nu'}{q_{\rm e}\gamma^2}\right)^2\frac{\partial L}{\partial X} + 2\frac{m_{\rm e}\nu'}{q_{\rm e}\gamma^2}B_z\frac{\partial L}{\partial Y} - L - m_{\rm e}n_{\rm ion}\frac{q_{\rm ion}}{q_{\rm e}}\left(\nu - v\sqrt{\nu^2-\gamma^2}\right)\right] = 0. \tag{4.45}$$

A brief aside: using X and Y given by (4.5) and (4.6), the factors in (4.45) in front of the derivatives of the Lagrangian can be rewritten as;

$$\frac{d}{d\xi}\left[2\left(X + B_z^2\right)\frac{\partial L}{\partial X} + Y\frac{\partial L}{\partial Y} - L - m_{\rm e}n_{\rm ion}\frac{q_{\rm ion}}{q_{\rm e}}\left(\nu - \sqrt{\nu^2-\gamma^2}\right)\right]\Bigg|_{\mathsf{S}} = 0, \tag{4.46}$$

where S is the subspace defined by $X = E^2 - B_z^2$, $Y = 2EB_z$. This equation is consistent with the prior result in Ref. [37] though now with extra terms due to the background magnetic field, though Ref. [37] used the field equations instead of working from the stress balance law (4.22).

Returning to (4.45), integration gives

$$\left[2\left(\frac{m_{\rm e}\nu'}{q_{\rm e}\gamma^2}\right)^2\frac{\partial L}{\partial X} + 2\frac{m_{\rm e}\nu'}{q_{\rm e}\gamma^2}B_z\frac{\partial L}{\partial Y} - L - m_{\rm e}n_{\rm ion}\frac{q_{\rm ion}}{q_{\rm e}}\left(\nu - v\sqrt{\nu^2-\gamma^2}\right)\right] = \mathcal{C}, \tag{4.47}$$

for some integration constant $\mathcal{C}$ (to be found). As in [37], the square root in (4.45) places a lower bound on ν. For a maximum amplitude plasma wave, ν attains its lowest possible value (see Figure 4.1), which the square root term shows to be $\nu_{\rm I} = \gamma$. With the assumption that ν attains its lowest possible value some $\xi_{\rm I}$, i.e. $\nu(\xi_{\rm I}) = \nu_{\rm I} = \gamma$ and $\nu'_{\rm I} = \frac{d\nu}{d\xi}\Big|_{\xi=\xi_{\rm I}} = 0$, (4.47) can be evaluated to find $\mathcal{C}$ during a maximum amplitude oscillation. The first two terms vanish on turning points of ν, leaving

$$-m_{\rm e}n_{\rm ion}\frac{q_{\rm ion}}{q_{\rm e}}\gamma - L_0 = \mathcal{C}, \tag{4.48}$$

where $L_0 = L|_{\nu=\gamma,\nu'=0}$ (or $= L|_{X=-B^2,Y=0}$ on S). Hence (4.47) becomes

$$2\left(\frac{m_e\nu'}{q_e\gamma^2}\right)^2\frac{\partial L}{\partial X} + 2\frac{m_e\nu'}{q_e\gamma^2}B_z\frac{\partial L}{\partial Y} - (L - L_0) - m_e n_{ion}\frac{q_{ion}}{q_e}\left(\nu - \gamma - v\sqrt{\nu^2-\gamma^2}\right) = 0. \quad (4.49)$$

It is now possible to use (4.49) to find the turning points of ν, ν_I and ν_{II}, in the maximum amplitude oscillation. Since these are turning points of ν, they correspond to the zeroes of E and hence half a wavelength of the plasma wave. In order to find these turning points, it is necessary to substitute $\nu' = 0$ into (4.49), resulting in

$$-m_e n_{ion}\frac{q_{ion}}{q_e}\left(\nu - \gamma - v\sqrt{\nu^2-\gamma^2}\right) = 0,$$
$$\text{i.e.} \qquad \nu_\pm = \gamma^3(1 \pm v^2). \quad (4.50)$$

Hence the lower value is $\nu_I = \gamma$ and the upper value is $\nu_{II} = \gamma^3(1+v^2)$, and

$$\nu_{II} - \nu_I = 2v^2\gamma^3. \quad (4.51)$$

4.3 Relativistic Energy Gain

Now it is possible to calculate the energy gained by the test electron in a maximum amplitude plasma wave. The relativistic energy of a particle of mass m_e and charge q_e with trajectory $\dot{C}$ in the inertial frame of observer $\mathcal{U}$ is defined by

$$W_{\mathcal{U}} = -m_e g(\mathcal{U}, \dot{C}). \quad (4.52)$$

In order to find the energy gained by a charged particle in an electromagnetic field F, consider the following.

As seen in previously, the Lorentz force equation satisfied by the plasma electrons is written

$$\nabla_{V_e}\widetilde{V}_e = \frac{q_e}{m_e} i_{V_e} F, \quad \text{(4.9 revisited)}$$

where V_e describes the motion of the electrons, F the electromagnetic field of the plasma wave and ∇ is the Levi-Civita connection. A test electron inserted into the system[1] then must obey the equation of motion

$$\nabla_{\dot{C}}\widetilde{\dot{C}} = \frac{q_\mathrm{e}}{m_\mathrm{e}} i_{\dot{C}} F, \tag{4.53}$$

where $C(\tau)$ is the curve representing the trajectory of the test electron, satisfying the normalisation condition $g(\dot{C}, \dot{C}) = -1$. Contracting (4.53) on a parallel[2] unit timelike future pointing vector $\mathcal{U}$ and noting that the interior contraction is antisymmetric gives:

$$\nabla_{\dot{C}}\left(g(\mathcal{U}, \dot{C})\right) = -\frac{q_\mathrm{e}}{m_\mathrm{e}} i_{\dot{C}} i_{\mathcal{U}} F. \tag{4.54}$$

Since τ is defined along $\dot{C}$, this can be rewritten

$$\frac{d}{d\tau} g(\mathcal{U}, \dot{C}) = \frac{q_\mathrm{e}}{m_\mathrm{e}} i_{\mathcal{U}} i_{\dot{C}} F = -\frac{q_\mathrm{e}}{m_\mathrm{e}} i_{\dot{C}} i_{\mathcal{U}} F. \tag{4.55}$$

By integrating over the interval $[\tau_\mathrm{I}, \tau_\mathrm{II}]$ and noting that for a 1-form α,

$$\int_C \alpha = \int_{\tau_\mathrm{I}}^{\tau_\mathrm{II}} i_{\dot{C}} \alpha \, d\tau, \tag{4.56}$$

(4.55) can be written

$$\int_{\tau_\mathrm{I}}^{\tau_\mathrm{II}} \frac{d}{d\tau} g(\mathcal{U}, \dot{C}) \, d\tau = -\frac{q_\mathrm{e}}{m_\mathrm{e}} \int_C i_{\mathcal{U}} F. \tag{4.57}$$

From (4.52), this is simply

$$\Delta W_{\mathcal{U}} = q \int_C i_{\mathcal{U}} F. \tag{4.58}$$

4.3.1 Energy Gain in a Maximum Amplitude Plasma Wave

The equation for the change in energy experienced by a charged particle in an electromagnetic field described by F requires an inertial observer to act as a frame of reference. It is most convenient to choose the wave frame for $\mathcal{U}$; as

[1] The test electron's effect on the overall system is assumed negligible.

[2] A parallel vector $\mathcal{U}$ satisfies $\nabla\mathcal{U} = 0$, and hence $i_{\mathcal{U}}\nabla_V = \nabla_V i_{\mathcal{U}}$.

mentioned in Section 4.2.1, the orthonormal coframe $\{\gamma d\zeta, dx, dy, \gamma d\xi\}$ is adapted to an observer moving in the z-direction with speed v and is ideal for the wave frame. Hence choosing $\mathcal{U} = \gamma \widetilde{d\zeta}$ and rewriting F in the wave frame: $F = E\gamma^2 d\xi \wedge d\zeta - B_z dx \wedge dy$, (4.58) becomes

$$\Delta W = q_e \gamma \int_C i_{\widetilde{d\zeta}} \left(E\gamma^2 d\xi \wedge d\zeta - B_z dx \wedge dy \right) = q_e \gamma \int_C E d\xi. \tag{4.59}$$

Since the form of E is already known from (4.15), the integral can be written using the chain rule to give

$$\Delta W = q_e \gamma \int_{\nu_I}^{\nu_{II}} \frac{m_e}{q_e \gamma^2} d\nu = \frac{m_e}{\gamma}(\nu_{II} - \nu_I), \tag{4.60}$$

for some ν_I and ν_{II}.

For a maximum amplitude plasma wave over a half-wavelength, the value of $\nu_{II} - \nu_I$ is known (see (4.51)) and hence

$$\Delta W = 2 m_e \gamma^2 v^2. \tag{4.61}$$

This value for the energy gained by the test electron was found in Ref. [37] as an estimate of the energy gained for Born-Infeld theory in the background-field-free case, but *in the lab frame* not in the wave frame as is the case here. It can now be asserted that (4.61) is exact not only for Born-Infeld theory, but for any electromagnetic theory with Lagrangian $L(X, Y)$. Also, the presence of a background magnetic field B_z does not change the amount of energy gained by the electron even though the background magnetic field affects the plasma wave.

4.4 Summary

This chapter has shown that the energy gained by a test electron in a maximum amplitude plasma wave bathed in a constant longitudinal background field is dependent only on the group speed of the plasma wave and the mass of the electron.

This result was found by appealing to the stress balance law (4.22) and then choosing the electric field E and the plasma electron density n_e such that the field equation (4.8) and the Lorentz-force equation (4.9) were satisfied. Finally, by finding the value of ν at the two zeroes of the electric field, ν_{I} and ν_{II}, the energy in the wave frame was calculated via (4.58).

Additional components of the background magnetic field perpendicular to the propagation of the wave were found not to solve the equation of motion (4.9) and hence inclusion of these components will require modification of the ansatz of the plasma electron fluid (4.2).

Since the group speed of the driving laser pulse (and the subsequent speed of the plasma wave) will depend on the background fields and the electromagnetic theory in question (see Chapter 3), the energy gain of an electron in a maximum amplitude is nonlinear theory dependent. Since, however, finding the dependence of the group velocity of the laser pulse will almost certainly require extensive numerical study, precisely how the energy gain of an electron in a maximum amplitude plasma wave depends on the electromagnetic theory is left for future study.

Chapter 5

Relativistic Spinning Particles

5.1 Introduction

Alongside the effects of classical nonlinear electrodynamical theories, such as Born-Infeld theory, there are other effects that must be taken into account when considering strong fields such as ELI [5] and HiPER [6], or for instance when accelerating over short distances, such as laser wakefield acceleration [38].

Two such effects are the Stern-Gerlach-type forces and radiation reaction. Radiation reaction, the interaction of the radiation emitted when a charged particle undergoes acceleration and the particle itself, has been taken into consideration in various particle-in-cell (PIC) codes by studying the Landau-Lifshitz equation[1] and is known [39] to become important when optical laser intensities exceed 5×10^{22} W cm^{-2}. However the impact of the quantum mechanical spin of particles in high field environments, such as maximum amplitude plasma waves, is generally neglected despite the estimation that the Stern-Gerlach forces can be of (and indeed above) the order of the radiation reaction terms (see Section 2 of Ref. [24]).

The concept of attempting to model a quantum mechanical electron as an analogue of a covariant classical spinning particle is not a new one. There have

[1]The inconsistencies of radiation reaction theory with regards to Lorentz-Abraham-Dirac versus Landau-Lifshitz are not within the scope of this thesis.

been various approaches from the work of Frenkel [40] and Thomas [41] in the 1920s through the work of Nakano [42], Tulczyjew [43], Dixon [44–46], Corben [47, 48], Suttorp and de Groot [49, 50] and Ellis [51] in the 1950-70s. The approaches used to derive these equations are varied, but this chapter aims to contribute a new method via an approach using de Rham currents and distributional methods.

After deriving the equations of motion, this Chapter also studies the particular situation of the motion of a classical electron in a maximum amplitude plasma wave. Using the equations of motion with the electric field used in Chapter 4, solutions where the impact of spin is greater than the radiation reaction force are explored. Such a case is pinpointed and found to have adverse consequences for the size of electron bunches in proposed laser-plasma wakefield accelerators [38].

Since this chapter includes objects which have different aspects, the notation will be made clear as follows: 3-vectors will be denoted with an arrow $\vec{V}$, 4-vectors V (with the appropriate metric dual 1-form as $\widetilde{V}$) and distributions will be written with a subscript V_D.

5.2 Deriving Equations of Motion for a Classical Spinning Particle Using Distributions

This section contains a new derivation of the equations of motion for a relativistic spinning charged particle via an approach using de Rham currents. In order to begin, it is necessary to find the distributional analogues of physical quantities (analogous to moving from a continuum to particle approach).

5.2.1 Writing Polarisation and Magnetisation as de Rham Currents

Consider a system of a charged continuum with polarisation $\vec{\mathfrak{p}}$ and magnetisation $\vec{\mathfrak{m}}$ 3-vectors given by

$$\vec{\mathfrak{p}}(\vec{r},t) = n(\vec{r},t)\vec{\mu}_{\mathrm{e}}(\vec{r},t), \tag{5.1}$$

$$\vec{\mathfrak{m}}(\vec{r},t) = n(\vec{r},t)\vec{\mu}_{\mathrm{m}}(\vec{r},t), \tag{5.2}$$

where n is the particle number density and $\vec{\mu}_{\mathrm{e}}$ and $\vec{\mu}_{\mathrm{m}}$ are the electric and magnetic dipole moments respectively. Using the fact that the excitation 2-form G can be written $G = F + \Pi$, the polarisation 2-form Π is introduced:

$$\Pi = -\widetilde{V} \wedge \widetilde{\mathfrak{p}} + \#\widetilde{\mathfrak{m}}. \tag{5.3}$$

Here $\widetilde{\mathfrak{p}}$ and $\widetilde{\mathfrak{m}}$ are the 1-form metric duals of their vector equivalents and the $\#$ notation is shorthand for

$$\#\alpha = \star(\widetilde{V} \wedge \alpha), \tag{5.4}$$

where V is the 4-vector describing the motion of the fluid.

In order to move from the continuum model to a single-particle model, de Rham currents[1] are introduced. Firstly, in order to establish the notation in a simple setting[2], it is assumed that the fluid is at rest and hence described by $V = \partial_t$. Then the distributional current associated with the worldline of a particle is introduced via;

$$\int_{\mathcal{M}} \hat{f} n \star 1 \rightarrow \int_C \hat{f} dt, \tag{5.5}$$

analogous to using the particle density as a Dirac delta function to only pick out the integral over the particle worldline C rather than integrating over the full manifold $\mathcal{M}$. Here C is the curve representing the worldline of the particle with constant x, y, z (due to the temporary choice of V) and $\hat{f}$ is a test form. Since the aim of this method is to induce the equation of motion of a particle from a fluid description, C is assumed to be an integral curve of V. As in previous chapters, spacetime is assumed to be flat, i.e. the metric is η_{ab}.

In order to find the appropriate distributions for the particle versions of the magnetisation and polarisation, consider the following. Given a Killing 3-vector $\vec{K} \in \{\vec{i}, \vec{j}, \vec{k}\}$, where $\vec{i} \cdot \vec{i} = 1$, $\vec{j} \cdot \vec{j} = 1$, $\vec{k} \cdot \vec{k} = 1$ and $\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$, it is

[1] See Section 2.6 for the essentials or [30] for detail.

[2] It is simple to extend this to any vector V, though it distracts from the point of the method.

natural to associate

$$\left(\vec{\mathfrak{p}}\cdot\vec{K}\right)_D[\hat{f}\star 1]=\int_C \vec{\mu}_{\mathrm{e}}\cdot\vec{K}\hat{f}dt, \tag{5.6}$$

$$\left(\vec{\mathfrak{m}}\cdot\vec{K}\right)_D[\hat{f}\star 1]=\int_C \vec{\mu}_{\mathrm{m}}\cdot\vec{K}\hat{f}dt, \tag{5.7}$$

where $\cdot$ represents the usual scalar product on 3-vectors and $\left(\vec{\mathfrak{p}}\cdot\vec{K}\right)_D$ represents the distribution associated with the scalar $\vec{\mathfrak{p}}\cdot\vec{K}$. Expanding on the first of these equations:

$$\begin{aligned}\left(\vec{\mathfrak{p}}\cdot\vec{K}\right)_D\star 1[\hat{f}] &= \int_C \vec{\mu}_{\mathrm{e}}\cdot\vec{K}\hat{f}dt\\ &= C_D[\vec{\mu}_{\mathrm{e}}\cdot\vec{K}\hat{f}dt]\\ &= -\left(\vec{\mu}_{\mathrm{e}}\cdot\vec{K}\right)(dt\wedge C_D)[\hat{f}].\end{aligned} \tag{5.8}$$

Stripping off the test function and noting that in this case $\dot{C}=\partial_t$ and $\star\star 1=-1$:

$$\left(\vec{\mathfrak{p}}\cdot\vec{K}\right)_D=-\left(\vec{\mu}_{\mathrm{e}}\cdot\vec{K}\right)\star\left(\widetilde{\dot{C}}\wedge C_D\right). \tag{5.9}$$

Introducing the 4-vector $\mu_{\mathrm{e}}=\mu_{\mathrm{e}x}\partial_x+\mu_{\mathrm{e}y}\partial_y+\mu_{\mathrm{e}z}\partial_z$, where $\mu_{\mathrm{e}x}$ is the x-component of the vector $\vec{\mu}_{\mathrm{e}}$ etc., (5.9) for $A=\{1,2,3\}$ can be written

$$\begin{aligned}\mathfrak{p}_D &= \left(\vec{\mathfrak{p}}\cdot\vec{K}_A\right)_D dx^A\\ &= \left(i_{\dot{C}}\star C_D\right)\widetilde{\mu_{\mathrm{e}}},\end{aligned} \tag{5.10}$$

and similarly

$$\mathfrak{m}_D=\left(i_{\dot{C}}\star C_D\right)\widetilde{\mu_{\mathrm{m}}}. \tag{5.11}$$

Hence the polarisation distribution (analogous to (5.3)) can be written;

$$\Pi_D=-\widetilde{V}\wedge\mathfrak{p}_D+\#\mathfrak{m}_D \tag{5.12}$$

$$=-\widetilde{\dot{C}}\wedge i_{\dot{C}}\star C_D\wedge\widetilde{\mu_{\mathrm{e}}}+\star(\widetilde{\dot{C}}\wedge i_{\dot{C}}\star C_D\wedge\widetilde{\mu_{\mathrm{m}}}), \tag{5.13}$$

since C is an integral curve of V. Since it is possible to simplify the above expression using $\widetilde{\dot{C}}\wedge i_{\dot{C}}\star C_D=-\star C_D$, the polarisation distribution can be written in the succinct form

$$\Pi_D=\star C_D\wedge\widetilde{\mu_{\mathrm{e}}}-\star(\star C_D\wedge\widetilde{\mu_{\mathrm{m}}}) \tag{5.14}$$

$$=\star C_D\wedge\widetilde{\mu_{\mathrm{e}}}-i_{\mu_{\mathrm{m}}}C_D. \tag{5.15}$$

5.2.2 Writing Free and Bound Currents as de Rham Currents

The field equations for a continuum are given by

$$dF = 0, \tag{5.16}$$
$$d \star F = d \star G - d \star \Pi \tag{5.17}$$
$$= j^{\text{free}} + j^{\text{bound}}, \tag{5.18}$$

where the current j^{bound} encapsulates the information regarding the currents inside the particles. The currents can hence be written

$$j^{\text{free}} = d \star G = -qn \star \widetilde{V}_{\text{m}}, \tag{5.19}$$
$$j^{\text{bound}} = -d \star \Pi, \tag{5.20}$$

for matter described by vector field V_{m} with number density n and charge q. Currents (5.19) and (5.20) may be used as a basis for constructing the particle distributions[1] j_D^{free} and j_D^{bound} using the general form of (5.5), i.e.

$$\int_{\mathcal{M}} \hat{f} n \star 1 \to -\int_C \hat{f}\widetilde{\dot{C}} = \int_{\tau_{\min}}^{\tau_{\max}} \hat{f} d\tau, \tag{5.21}$$
$$\text{since} \quad \int_C \alpha^{(1)} = \int_{\tau_{\min}}^{\tau_{\max}} i_{\dot{C}}\alpha^{(1)} d\tau, \tag{5.22}$$

where τ is the curve parameter, the proper time of the particle, running from τ_{min} to τ_{max} to define the whole curve $C(\tau)$. Acting on a test 1-form $\hat{\varphi}^{(1)}$ using j_D^{free} gives

$$j_D^{\text{free}}[\hat{\varphi}^{(1)}] = \int_{\mathcal{M}} -qn \star \widetilde{V}_{\text{m}} \wedge \hat{\varphi}^{(1)}$$
$$= \int_{\mathcal{M}} qn\hat{\varphi}^{(1)}(V_{\text{m}}) \star 1,$$

[1]The free current distribution corresponds to the usual motion of the particle, whereas the bound current distribution indicates some kind of internal structure to the particle to incorporate the moments μ_{e} and μ_{m}.

since $\hat{\varphi}^{(1)} \star \widetilde{V}_{\mathrm{m}} = i_{V_{\mathrm{m}}} \hat{\varphi}^{(1)} \star 1$. Using (5.21), and noting that C is an integral curve of V_{m}, this becomes

$$j_D^{\mathrm{free}}[\hat{\varphi}^{(1)}] = -\int_C q i_{\dot{C}} \hat{\varphi}^{(1)} \widetilde{\dot{C}} = \int_C q\hat{\varphi}^{(1)}, \tag{5.23}$$

since the integral projects out the $\dot{C}$-orthogonal components of the integrand, and hence

$$j_D^{\mathrm{free}} = qC_D. \tag{5.24}$$

As for the bound current j^{bound}, it is helpful to first introduce a polarisation 2-form

$$\Sigma = -\widetilde{\dot{C}} \wedge \widetilde{\mu_{\mathrm{e}}} + \star(\widetilde{\dot{C}} \wedge \widetilde{\mu_{\mathrm{m}}}), \tag{5.25}$$

analogous to the relationship between E and B and the Faraday 2-form. From this, note that $\widetilde{\mu_{\mathrm{e}}} = i_{\dot{C}}\Sigma$ and $\widetilde{\mu_{\mathrm{m}}} = i_{\dot{C}} \star \Sigma$ and hence j_D^{bound} can be written

$$\begin{aligned} j_D^{\mathrm{bound}} &= -d \star \Pi_D = -d \star (\star C_D \wedge \widetilde{\mu_{\mathrm{e}}} - \star(\star C_D \wedge \widetilde{\mu_{\mathrm{m}}})) \\ &= -d \star (\star C_D \wedge i_{\dot{C}}\Sigma) - d\,(\star C_D \wedge i_{\dot{C}} \star \Sigma)\,. \end{aligned} \tag{5.26}$$

5.2.3 The Stress Balance Law

There are several balance laws motivated via considering the invariances of a general class of actions[1] that can nevertheless be independently considered for systems without explicit actions. One of these balance laws is the stress balance equation

$$d\mathfrak{T}^a = i_{X^a} F \wedge j^{\mathrm{free}} + i_{X^a} F \wedge j^{\mathrm{bound}}, \tag{5.27}$$

where $\mathfrak{T}^a$ are the stress-energy-momentum 3-forms of the classical spinning charged particle and $\{X^a\}$ represents a Killing frame. In order to adapt this law for use on a single particle, consider the distributional analogue of (5.27):

$$d\mathfrak{T}^a_D = i_{X^a} F \wedge j_D^{\mathrm{free}} + i_{X^a} F \wedge j_D^{\mathrm{bound}}, \tag{5.28}$$

[1]See Appendix A for details.

where $\mathcal{T}^a_D$ (to be found) are the stress distributions associated with stress 3-forms $\mathcal{T}^a$, and the current distributions are defined as in the previous section. In order to use this balance law, consider consider source terms individually. Acting on a test function $\hat{f}$, the free current component of (5.28) can be written

$$i_{X^a}F \wedge j_D^{\text{free}}[\hat{f}] = \int_{\tau_{\min}}^{\tau_{\max}} -q i_{\dot{C}} i_{X^a} F \hat{f} d\tau, \tag{5.29}$$

where $j_D^{\text{free}} = qC_D$ has been used. The bound current term of (5.28) is not so trivial however and requires more analysis. Acting on a test form $\hat{f}$,

$$i_{X^a}F \wedge j_D^{\text{bound}}[\hat{f}] = \left(d \star (\star C_D \wedge i_{\dot{C}}\Sigma) \wedge i_{X^a}F + d\,(\star C_D \wedge i_{\dot{C}} \star \Sigma) \wedge i_{X^a}F\right)[\hat{f}]. \tag{5.30}$$

Consider the first term of (5.30); using the properties of de Rham currents (see Section 2.7), it is instructive to rewrite this in detail:

$$\begin{aligned} d \star (\star C_D \wedge i_{\dot{C}}\Sigma) \wedge i_{X^a}F[\hat{f}] &= d \star (\star C_D \wedge i_{\dot{C}}\Sigma)\,[\hat{f} i_{X^a}F] \\ &= - \star (\star C_D \wedge i_{\dot{C}}\Sigma) \left[d\left(\hat{f} i_{X^a}F\right)\right] \\ &= -\,(\star C_D \wedge i_{\dot{C}}\Sigma) \left[\star d\left(\hat{f} i_{X^a}F\right)\right] \\ &= - \star C_D \left[i_{\dot{C}}\Sigma \wedge \star d\left(\hat{f} i_{X^a}F\right)\right] \\ &= C_D \left[\star\left(i_{\dot{C}}\Sigma \wedge \star d\left(\hat{f} i_{X^a}F\right)\right)\right]. \end{aligned} \tag{5.31}$$

Similarly, the second term can be rewritten;

$$\begin{aligned} d\,(\star C_D \wedge i_{\dot{C}} \star \Sigma) \wedge i_{X^a}F[\hat{f}] &= d\,(\star C_D \wedge i_{\dot{C}} \star \Sigma)\,[\hat{f} i_{X^a}F] \\ &= -\,(\star C_D \wedge i_{\dot{C}} \star \Sigma) \left[d\left(\hat{f} i_{X^a}F\right)\right] \\ &= - \star C_D \left[i_{\dot{C}} \star \Sigma \wedge d\left(\hat{f} i_{X^a}F\right)\right] \\ &= C_D \left[\star\left(i_{\dot{C}} \star \Sigma \wedge d\left(\hat{f} i_{X^a}F\right)\right)\right]. \end{aligned} \tag{5.32}$$

Using (5.31) and (5.32), (5.30) becomes

$$\begin{aligned} i_{X^a}F \wedge j_D^{\text{bound}}[\hat{f}] &= \int_C \left\{\star\left(i_{\dot{C}}\Sigma \wedge \star d\left(\hat{f} i_{X^a}F\right)\right) + \star\left(i_{\dot{C}} \star \Sigma \wedge d\left(\hat{f} i_{X^a}F\right)\right)\right\} \\ &= \int_{\tau_{\min}}^{\tau_{\max}} \left\{- \star\left(\widetilde{\dot{C}} \wedge i_{\dot{C}}\Sigma \wedge \star d(\hat{f} i_{X^a}F)\right)\right. \\ &\qquad \left. - \star\left(\widetilde{\dot{C}} \wedge i_{\dot{C}} \star \Sigma \wedge d(\hat{f} i_{X^a}F)\right)\right\} d\tau. \end{aligned} \tag{5.33}$$

Since $\widetilde{\dot{C}} \wedge i_{\dot{C}}\Sigma = \widetilde{\dot{C}} \wedge \widetilde{\mu_{\rm e}}$ and $\widetilde{\dot{C}} \wedge i_{\dot{C}} \star \Sigma = \star^{-1} \star (\widetilde{\dot{C}} \wedge \widetilde{\mu_{\rm m}}) = - \star \#_C \widetilde{\mu_{\rm m}}$, it is possible to simplify (5.33) via star-pivoting the first term

$$\begin{aligned} i_{X^a}F \wedge j_D^{\rm bound}[\hat{f}] = &\int_{\tau_{\min}}^{\tau_{\max}} \Big\{ -\star \left(d(\hat{f} i_{X^a}F) \wedge \star \left(\widetilde{\dot{C}} \wedge i_{\dot{C}}\Sigma \right) \right) \\ &\qquad - \star \left(d(\hat{f} i_{X^a}F) \wedge \star \left(\widetilde{\dot{C}} \wedge i_{\dot{C}} \star \Sigma \right) \right) \Big\} d\tau \\ = &\int_{\tau_{\min}}^{\tau_{\max}} \star \left(d(\hat{f} i_{X^a}F) \wedge \star\Sigma \right) d\tau. \end{aligned} \tag{5.34}$$

Expanding out the exterior derivative using the Leibniz rule gives

$$i_{X^a}F \wedge j_D^{\rm bound}[\hat{f}] = \int_{\tau_{\min}}^{\tau_{\max}} \left\{ \star \left(d\hat{f} \wedge i_{X^a}F \wedge \star\Sigma \right) + \star \left(\mathcal{L}_{X^a}F \wedge \star\Sigma \right) \hat{f} \right\} d\tau, \tag{5.35}$$

since $dF = 0$.

Consider now the $d\hat{f}$ term of (5.35); this can be split into its $\dot{C}$-parallel and -orthogonal parts via

$$d\hat{f} = \Pi^{\parallel}_{\dot{C}} d\hat{f} + \Pi^{\perp}_{\dot{C}} d\hat{f}, \tag{5.36}$$

to give

$$\begin{aligned} \int_{\tau_{\min}}^{\tau_{\max}} \star \left(d\hat{f} \wedge i_{X^a}F \wedge \star\Sigma \right) d\tau = &\int_{\tau_{\min}}^{\tau_{\max}} \Big\{ \star \left(\Pi^{\parallel}_{\dot{C}} d\hat{f} \wedge i_{X^a}F \wedge \star\Sigma \right) \\ &\qquad + \star \left(\Pi^{\perp}_{\dot{C}} d\hat{f} \wedge i_{X^a}F \wedge \star\Sigma \right) \Big\} d\tau, \end{aligned} \tag{5.37}$$

where the parallel and orthogonal parts of $d\hat{f}$ are given by

$$\Pi^{\parallel}_{\dot{C}} d\hat{f} = -i_{\dot{C}} d\hat{f} \widetilde{\dot{C}}, \tag{5.38}$$

$$\Pi^{\perp}_{\dot{C}} d\hat{f} = d\hat{f} + i_{\dot{C}} d\hat{f} \widetilde{\dot{C}}. \tag{5.39}$$

The $\dot{C}$-parallel term corresponds to the components along the worldline, whereas the $\dot{C}$-orthogonal terms correspond to components off the worldline of the particle. Note that since $i_{\dot{C}} d\hat{f} = \frac{d\hat{f}}{d\tau}$, for curve parameter τ, the parallel part of (5.37) can be written

$$-\int_{\tau_{\min}}^{\tau_{\max}} \star \left(\widetilde{\dot{C}} \wedge i_{X^a}F \wedge \star\Sigma \right) i_{\dot{C}} d\hat{f} d\tau = \int_{\tau_{\min}}^{\tau_{\max}} i_{\dot{C}} d \star \left(\widetilde{\dot{C}} \wedge i_{X^a}F \wedge \star\Sigma \right) \hat{f} d\tau, \tag{5.40}$$

using integration by parts and the fact that $\hat{f}$ has compact support. Then since $i_{\dot{C}}dh = \nabla_{\dot{C}}h$ for 0-form h and

$$\star(\widetilde{\dot{C}} \wedge i_{X^a}F \wedge \star\Sigma) = i_{\widetilde{i_{X^a}F}} i_{\dot{C}}\Sigma = (i_{X^a}F) \cdot i_{\dot{C}}\Sigma, \tag{5.41}$$

where $\cdot$ represents the generalised scalar product on forms[1], (5.35) can be written

$$\begin{aligned} i_{X^a}F \wedge j_D^{\text{bound}}[\hat{f}] &= \int_{\tau_{\min}}^{\tau_{\max}} \Big\{\star\left(\Sigma \wedge \star\mathcal{L}_{X^a}F\right)\hat{f} + i_{\dot{C}}d\star\left(\widetilde{\dot{C}} \wedge i_{X^a}F \wedge \star\Sigma\right)\hat{f} \\ &\qquad + \star\left(\Pi_{\dot{C}}^{\perp}d\hat{f} \wedge i_{X^a}F \wedge \star\Sigma\right)\Big\}\, d\tau \\ &= \int_{\tau_{\min}}^{\tau_{\max}} \Big\{-\Sigma \cdot \left(\mathcal{L}_{X^a}F\right)\hat{f} - \nabla_{\dot{C}}((i_{X^a}F) \cdot i_{\dot{C}}\Sigma)\hat{f} \\ &\qquad + \star\left(\Pi_{\dot{C}}^{\perp}d\hat{f} \wedge i_{X^a}F \wedge \star\Sigma\right)\Big\}\, d\tau. \end{aligned} \tag{5.42}$$

This can then be added to the free current term (5.29) to give

$$\begin{aligned} d\mathcal{T}_D^a[\hat{f}] &= i_{X^a}F \wedge j_D^{\text{free}}[\hat{f}] + i_{X^a}F \wedge j_D^{\text{bound}}[\hat{f}] \\ &= \int_{\tau_{\min}}^{\tau_{\max}} \Big\{-q i_{\dot{C}} i_{X^a}F\hat{f} - \Sigma \cdot \left(\mathcal{L}_{X^a}F\right)\hat{f} - \nabla_{\dot{C}}((i_{X^a}F) \cdot i_{\dot{C}}\Sigma)\hat{f} \\ &\qquad + \star\left(\Pi_{\dot{C}}^{\perp}d\hat{f} \wedge i_{X^a}F \wedge \star\Sigma\right)\Big\}\, d\tau. \end{aligned} \tag{5.43}$$

5.2.4 Choosing the Stress-Energy-Momentum Distributions

The stress-energy-momentum distributions are chosen[2] to be of the form

$$\mathcal{T}_D^a = -g(\pi, X^a)C_D + \varsigma^a \wedge \star C_D, \tag{5.44}$$

where π is a candidate momentum 4-vector[3] and ς^a are 2-form coefficients to the off-worldline components attached to $\star C_D$. These coefficients are included in

[1] Where $\alpha \cdot \beta = \star^{-1}\left(\alpha \wedge \star\beta\right)$ for forms α and β of equal degree.

[2] Note that this is simply a choice for the stress forms; it is possible that another choice for the stress forms may lead to different equations of motion than those shown in the subsequent sections.

[3] The physical meaning of this candidate momentum vector is expanded on in Section 5.2.7.

order to absorb the $\Pi^{\perp}_{\dot{C}} d\hat{f}$ term of the bound current (see (5.43)). Applying the exterior derivative to $\varsigma^a \wedge \star C_D$ acting on a test function $\hat{f}$ gives

$$\begin{aligned} d\left(\varsigma^a \wedge \star C_D\right)[\hat{f}] = \varsigma^a \wedge \star C_D[d\hat{f}] &= -\int_{\tau_{\min}}^{\tau_{\max}} \star\left(\varsigma^a \wedge d\hat{f} \wedge \widetilde{\dot{C}}\right) d\tau \\ &= -\int_{\tau_{\min}}^{\tau_{\max}} \star\left(\Pi^{\perp}_{\dot{C}} d\hat{f} \wedge \Pi^{\perp}_{\dot{C}} \varsigma^a \wedge \widetilde{\dot{C}}\right) d\tau. \end{aligned} \tag{5.45}$$

Hence for this to absorb the $d\hat{f}$ term from (5.43), ς^a must satisfy

$$-\Pi^{\perp}_{\dot{C}} d\hat{f} \wedge \Pi^{\perp}_{\dot{C}} \varsigma^a \wedge \widetilde{\dot{C}} = \Pi^{\perp}_{\dot{C}} d\hat{f} \wedge i_{X^a} F \wedge \star\Sigma, \tag{5.46}$$

which after contracting along $\dot{C}$ and rearranging becomes

$$\Pi^{\perp}_{\dot{C}} d\hat{f} \wedge \left[\Pi^{\perp}_{\dot{C}} \varsigma^a - i_{\dot{C}}\left(i_{X^a} F \wedge \star\Sigma\right)\right] = 0. \tag{5.47}$$

The LHS of (5.47) is a 3-form in the $\dot{C}$-orthogonal projection of the space of 3-forms. Since $\Pi^{\perp}_{\dot{C}} d\hat{f}$ can be chosen to be any of the three members of a frame of $\dot{C}$-orthogonal forms, the condition on ς^a becomes

$$\Pi^{\perp}_{\dot{C}} \varsigma^a = i_{\dot{C}}\left(i_{X^a} F \wedge \star\Sigma\right). \tag{5.48}$$

Hence one example of a 2-form ς^a satisfying this criterion is when $i_{\dot{C}} \varsigma^a = 0$ so that $\Pi^{\perp}_{\dot{C}} \varsigma^a = \varsigma^a$, giving

$$\varsigma^a = i_{\dot{C}}\left(i_{X^a} F \wedge \star\Sigma\right). \tag{5.49}$$

Hence let the stress distributions be

$$\mathcal{T}^a_D = -g(\pi, X^a) C_D + i_{\dot{C}}\left(i_{X^a} F \wedge \star\Sigma\right) \wedge \star C_D. \tag{5.50}$$

5.2.5 Deriving the Equation of Motion for the Candidate Momentum $\boldsymbol{\pi}$

Returning now to the stress balance equation (5.43), the choices for the stress-energy-momentum distribution $\mathcal{T}^a_D$ (5.50) are inserted, hence removing the $\Pi^{\perp}_{\dot{C}} d\hat{f}$ term from the integral through the careful choice of ς^a (5.49):

$$-d(g(\pi, X^a) C_D)[\hat{f}] = \int_{\tau_{\min}}^{\tau_{\max}} \left\{-q i_{\dot{C}} i_{X^a} F - \Sigma \cdot (\mathcal{L}_{X^a} F) - \nabla_{\dot{C}}((i_{X^a} F) \cdot i_{\dot{C}} \Sigma)\right\} \hat{f} d\tau. \tag{5.51}$$

Noting that $dC_D = 0$, the LHS of this equation can be written:

$$\begin{aligned} -d(g(\pi, X^a)C_D)[\hat{f}] &= -di_{X^a}\widetilde{\pi} \wedge C_D[\hat{f}] \\ &= \int_C di_{X^a}\widetilde{\pi}\hat{f} \\ &= \int_{\tau_{\min}}^{\tau_{\max}} \nabla_{\dot{C}} i_{X^a}\widetilde{\pi}\hat{f}d\tau, \end{aligned} \tag{5.52}$$

since $i_{\dot{C}}dh = \nabla_{\dot{C}}h$ for 0-form h. The stress balance law (5.51) is now

$$\int_{\tau_{\min}}^{\tau_{\max}} \nabla_{\dot{C}} i_{X^a}\widetilde{\pi}\hat{f}d\tau = \int_{\tau_{\min}}^{\tau_{\max}} \left\{-qi_{\dot{C}}i_{X^a}F - \Sigma \cdot (\mathcal{L}_{X^a}F) - \nabla_{\dot{C}}((i_{X^a}F) \cdot i_{\dot{C}}\Sigma)\right\} \hat{f}d\tau, \tag{5.53}$$

and hence by stripping off the integral, the test function, and gathering up the derivatives, the stress balance equation becomes

$$\nabla_{\dot{C}} \left(i_{X^a}\widetilde{\pi} + (i_{X^a}F) \cdot i_{\dot{C}}\Sigma\right) = -qi_{\dot{C}}i_{X^a}F - \Sigma \cdot (\mathcal{L}_{X^a}F). \tag{5.54}$$

5.2.6 Deriving the Equation of Motion for the Spin 2-Form Components S^{ab}

To study the evolution of the spin of the particle, another balance law is invoked, the spin balance law[1]

$$d\sigma^{ab} = \frac{1}{2}\left(dx^a \wedge \mathcal{T}^b - dx^b \wedge \mathcal{T}^a\right), \tag{5.55}$$

for a Killing frame X^a given by $\{\partial_t, \partial_x, \partial_y, \partial_z\}$ and where σ^{ab} are the spin 3-forms. As in the case of the stress balance law, the distributional analogue of this equation is invoked, i.e.

$$d\sigma_D^{ab} = \frac{1}{2}\left(dx^a \wedge \mathcal{T}_D^b - dx^b \wedge \mathcal{T}_D^a\right). \tag{5.56}$$

Since the stress distributions $\mathcal{T}_D^a$ have been specified in the previous section, the RHS can be analysed, whereas the spin distributions σ_D^{ab} are to be specified.

[1] See Appendix A for details.

Substituting the stress distributions (5.50) into one of the terms of (5.56) acting on a test function $\hat{f}$ gives

$$dx^a \wedge \mathcal{T}^b_D[\hat{f}] = -\left(\pi^b dx^a \wedge C_D\right)[\hat{f}] - \star C_D[dx^a \wedge i_{\dot{C}}\left(i_{X^b}F \wedge \star\Sigma\right)\hat{f}], \tag{5.57}$$

which can be written in integral form as

$$dx^a \wedge \mathcal{T}^b_D[\hat{f}] = \int_{\tau_{\min}}^{\tau_{\max}} \left[\left(\pi^b \dot{C}^a\right) + i_{\dot{C}} \star \left(dx^a \wedge i_{\dot{C}}\left(i_{X^b}F \wedge \star\Sigma\right)\right)\right] \hat{f} d\tau. \tag{5.58}$$

The second term in this expression can be simplified by using the properties of the Hodge map and internal contraction to give

$$i_{\dot{C}} \star \left(dx^a \wedge i_{\dot{C}}\left(i_{X^b}F \wedge \star\Sigma\right)\right) = -(i_{\dot{C}} i_{X^b}F)(i_{\dot{C}} i_{X^a}\Sigma) + i_{\dot{C}} i_{X^a} \star \left[\star(\Sigma \wedge \tilde{\dot{C}}) \wedge i_{X^b}F\right], \tag{5.59}$$

and with more manipulation, it is possible to write

$$\begin{aligned} i_{\dot{C}} i_{X^a} \star \left[\star(\Sigma \wedge \tilde{\dot{C}}) \wedge i_{X^b}F\right] = &-(i_{X^a} i_{\dot{C}}\Sigma)(i_{\dot{C}} i_{X^b}F) + \dot{C}^a (i_{X^b}F) \cdot i_{\dot{C}}\Sigma \\ &+ (i_{X^b}F) \cdot i_{X^a}\Sigma. \end{aligned} \tag{5.60}$$

Substituting this back into (5.59) gives

$$\begin{aligned} i_{\dot{C}} \star \left(dx^a \wedge i_{\dot{C}}\left(i_{X^b}F \wedge \star\Sigma\right)\right) &= \dot{C}^a (i_{X^b}F) \cdot i_{\dot{C}}\Sigma + (i_{X^b}F) \cdot i_{X^a}\Sigma \\ &= (i_{X^b}F) \cdot \left[\dot{C}^a \dot{C}^c i_{X_c}\Sigma + i_{X^a}\Sigma\right] \\ &= (i_{X^b}F) \cdot \left[\dot{C}^a \dot{C}^c + \eta^{ac}\right] i_{X_c}\Sigma. \end{aligned} \tag{5.61}$$

The object in the square brackets is the projection operator $\left(\Pi^{\perp}_{\dot{C}}\right)^{ac}$, and hence

$$i_{\dot{C}} \star \left(dx^a \wedge i_{\dot{C}}\left(i_{X^b}F \wedge \star\Sigma\right)\right) = (i_{X^b}F) \cdot \left[\left(\Pi^{\perp}_{\dot{C}}\right)^{ac} i_{X_c}\Sigma\right]. \tag{5.62}$$

Substituting (5.62) into (5.58) gives

$$dx^a \wedge \mathcal{T}^b_D[\hat{f}] = \int_{\tau_{\min}}^{\tau_{\max}} \left[\left(\pi^b \dot{C}^a\right) + (i_{X^b}F) \cdot \left[\left(\Pi^{\perp}_{\dot{C}}\right)^{ac} i_{X_c}\Sigma\right]\right] \hat{f} d\tau, \tag{5.63}$$

hence the the balance law (5.56) can be written

$$\begin{aligned} d\sigma^{ab}_D = \frac{1}{2}\int_{\tau_{\min}}^{\tau_{\max}} &\Big(\pi^b \dot{C}^a + (i_{X^b}F) \cdot \left[\left(\Pi^{\perp}_{\dot{C}}\right)^{ac} i_{X_c}\Sigma\right] \\ &-\pi^a \dot{C}^b - (i_{X^a}F) \cdot \left[\left(\Pi^{\perp}_{\dot{C}}\right)^{bc} i_{X_c}\Sigma\right]\Big) d\tau \hat{f}, \end{aligned} \tag{5.64}$$

or equivalently

$$d\sigma_D^{ab}[\hat{f}] = \int_{\tau_{\min}}^{\tau_{\max}} \frac{1}{2}\Big[\dot{C}^a\left(\pi^b + (i_{X^b}F)\cdot i_{\dot{C}}\Sigma\right) + (i_{X^b}F)\cdot i_{X^a}\Sigma \\ - \dot{C}^b\left(\pi^a + (i_{X^a}F)\cdot i_{\dot{C}}\Sigma\right) - (i_{X^a}F)\cdot i_{X^b}\Sigma\Big]\hat{f}\,d\tau. \quad (5.65)$$

Analogous to (5.21) (and indeed (5.24)), suppose that σ_D^{ab} are of the form $\sigma_D^{ab} = \frac{1}{2}S^{ab}C_D$ where the 0-forms S^{ab} represent the components of the spin 2-form. Then $d\sigma_D^{ab}$ acting on a test form can be written

$$d\sigma_D^{ab}[\hat{f}] = \frac{1}{2}S^{ab}C_D[d\hat{f}] = \int_C \frac{1}{2}S^{ab}d\hat{f}, \quad (5.66)$$

which can be rewritten

$$d\sigma_D^{ab}[\hat{f}] = -\int_{\tau_{\min}}^{\tau_{\max}} \nabla_{\dot{C}}\frac{1}{2}S^{ab}\hat{f}\,d\tau, \quad (5.67)$$

using integration by parts and since the boundary terms vanish because $\hat{f}$ has compact support. Hence the spin balance law (5.64) becomes

$$-\int_{\tau_{\min}}^{\tau_{\max}} \nabla_{\dot{C}}\frac{1}{2}S^{ab}\hat{f}\,d\tau = \int_{\tau_{\min}}^{\tau_{\max}} \frac{1}{2}\Big[\dot{C}^a\left(\pi^b + (i_{X^b}F)\cdot i_{\dot{C}}\Sigma\right) + (i_{X^b}F)\cdot i_{X^a}\Sigma \\ - \dot{C}^b\left(\pi^a + (i_{X^a}F)\cdot i_{\dot{C}}\Sigma\right) - (i_{X^a}F)\cdot i_{X^b}\Sigma\Big]\hat{f}\,d\tau. \quad (5.68)$$

Stripping off the integrals and test forms, the resulting equation for S^{ab} is

$$\nabla_{\dot{C}}S^{ab} = -\dot{C}^a\left(\pi^b + (i_{X^b}F)\cdot i_{\dot{C}}\Sigma\right) - (i_{X^b}F)\cdot i_{X^a}\Sigma \\ + \dot{C}^b\left(\pi^a + (i_{X^a}F)\cdot i_{\dot{C}}\Sigma\right) + (i_{X^a}F)\cdot i_{X^b}\Sigma. \quad (5.69)$$

5.2.7 Relating Momenta π and P

The two equations of motion for a classical spinning charged particle are given by (5.54) and (5.69), i.e.

$$\nabla_{\dot{C}}\left(i_{X^a}\widetilde{\pi} + (i_{X^a}F)\cdot i_{\dot{C}}\Sigma\right) = -q i_{\dot{C}} i_{X^a}F - \Sigma\cdot\left(\mathcal{L}_{X^a}F\right), \quad \text{(5.54 revisited)}$$

$$\nabla_{\dot{C}}S^{ab} = -\dot{C}^a\left(\pi^b + (i_{X^b}F)\cdot(i_{\dot{C}}\Sigma)\right) - (i_{X^b}F)\cdot(i_{X^a}\Sigma) \\ + \dot{C}^b\left(\pi^a + (i_{X^a}F)\cdot(i_{\dot{C}}\Sigma)\right) + (i_{X^a}F)\cdot(i_{X^b}\Sigma). \quad \text{(5.69 revisited)}$$

The component forms of (5.54) and (5.69) given by

$$\nabla_{\dot{C}}\left(\pi^a - F^{ab}\Sigma_{bc}\dot{C}^c\right) = -qF^{ab}\dot{C}_b - \frac{1}{2}\Sigma^{bc}\partial^a F_{bc}, \tag{5.70}$$

$$\nabla_{\dot{C}}S^{ab} = -\dot{C}^a\left(\pi^b - F^{bc}\Sigma_{cd}\dot{C}^d\right) + \dot{C}^b\left(\pi^a - F^{ac}\Sigma_{cd}\dot{C}^d\right) + F^{bc}\Sigma_c{}^a - F^{ac}\Sigma_c{}^b. \tag{5.71}$$

These equations agree with those derived by Suttorp and de Groot, i.e. equations (38) and (39) in Ref. [49] with a simple matching of symbols[1], given that the momenta satisfy the condition

$$\pi^a = P^a + F^{ab}\Sigma_{bc}\left(\frac{P^c}{\dot{C}_d P^d} + \dot{C}^c\right), \tag{5.72}$$

where P^a are the components of the momentum used by Suttorp and de Groot.

As to what is behind this relationship between π and P, note that the definition of the momentum P used by Suttorp and de Groot [49] (and similarly used in the Nakano-Tulczyjew condition (see Section 5.2.8)) is

$$P^a(\lambda) = -\int_{\Sigma_\lambda} T^{ab}N_b \star \widetilde{N}, \tag{5.73}$$

where T^{ab} is the stress-energy-momentum tensor and Σ_λ is a family of spacelike hyperplanes for different values of λ along C (see Figure 5.1) with unit normal

[1]The equations given in Suttorp and de Groot [49] are

$$dP^\alpha/ds = (e/c)F^{\alpha\beta}(X)U_\beta + \frac{1}{2}\{\partial^\alpha F^{\beta\gamma}\}M_{\beta\gamma} + (d/ds)\{F^{\alpha\beta}(X)M_{\beta\gamma}P^\gamma/U_\epsilon P^\epsilon\},$$

$$dS^{\alpha\beta}/ds = F^{\alpha\gamma}(X)M_\gamma{}^\beta - F^{\beta\gamma}(X)M_\gamma{}^\alpha + P^\alpha U^\beta - P^\beta U^\alpha -$$
$$- F^{\alpha\gamma}(X)M_{\gamma\epsilon}P^\epsilon U^\beta/U_\zeta P^\zeta + F^{\beta\gamma}(X)M_{\gamma\epsilon}P^\epsilon U^\alpha/U_\zeta P^\zeta.$$

The above equations are written as presented in Ref. [49] and hence the notation in this footnote is independent of the notation in the rest of this thesis. The relations between these quantities and those in the rest of the thesis are as follows: particle velocity U^α corresponds to $\dot{C}^a$, dipole components $M_{\alpha\beta}$ correspond to Σ_{ab}, spin components $S^{\alpha\beta}$ correspond to S^{ab}. The momentum P^α is related to the candidate momentum π^a as is shown in (5.72). Also important to note is that the electromagnetic field tensor used in Ref. [49] is of the *opposite* sign to that used in this thesis, i.e. $F^{\alpha\beta}(X)$ corresponds to $-F^{ab}$.

$N = \frac{P}{|P|}$. Since the stress-energy-momentum tensor T^{ab} is related to the stress-energy-momentum forms $\mathcal{T}^a$ via $\mathcal{T}^a = \star(T(X^a, -))$, note that for test 0-form $\hat{f}$,

$$\int_{\mathcal{M}} \mathcal{T}^a \wedge \widetilde{N}\hat{f} = \int_{\mathcal{M}} \star(T(X^a, -)) \wedge \widetilde{N}\hat{f}, \tag{5.74}$$

which via a star-pivot and a manipulation of the interior operator gives

$$\int_{\mathcal{M}} \mathcal{T}^a \wedge \widetilde{N}\hat{f} = -\int_{\mathcal{M}} (T(X^a, N)) \star 1\hat{f}. \tag{5.75}$$

The vector N is normalised as $g(N, N) = -1$, so the volume form can be written $\star 1 = -\widetilde{N} \wedge \star\widetilde{N}$ and hence

$$\int_{\mathcal{M}} \mathcal{T}^a \wedge \widetilde{N}\hat{f} = \int_{\mathcal{M}} (T(X^a, N))\, \widetilde{N} \wedge \star\widetilde{N}\hat{f}. \tag{5.76}$$

Since $\widetilde{N} = -\frac{d\lambda}{|d\lambda|}$, the integral can be split into a piece along the worldline C and another over the hyperplane Σ_λ via

$$\begin{aligned} \int_{\mathcal{M}} \mathcal{T}^a \wedge \widetilde{N}\hat{f} &= -\int_C \frac{d\lambda}{|d\lambda|} \int_{\Sigma_\lambda} T^{ab} N_b \star \widetilde{N}\hat{f} \\ &= -\int_C P^a \frac{d\lambda}{|d\lambda|}\hat{f}, \end{aligned} \tag{5.77}$$

and stripping off the test forms yields the relation

$$\mathcal{T}^a_D \wedge \widetilde{N} = -P^a C_D \wedge \widetilde{N} \tag{5.78}$$

with $N^a = \frac{P^a}{|P|}$.

Hence given the stress tensor $\mathcal{T}^a_D$, the momentum 4-vector components $\mathcal{P}^a$ crossing the leaves of a local spacetime foliation with unit future-pointing timelike normal 4-vector $\mathfrak{n}$ are given by

$$\mathcal{T}^a_D \wedge \widetilde{\mathfrak{n}} = -\mathcal{P}^a C_D \wedge \widetilde{\mathfrak{n}}, \tag{5.79}$$

the relationship between P and π begins to clear. Suttorp and de Groot [49] (as well as Nakano [42] and Tulczyjew [43]) chose the foliation such that the normal N restricted to C pointed in the direction of the particle's 4-momentum. In this

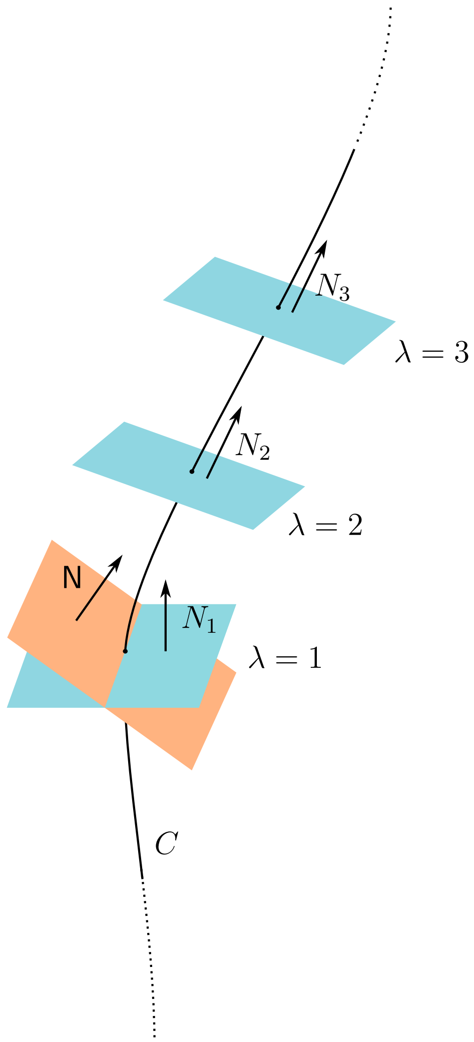

Figure 5.1: Illustrating multiple possible foliations with normals N, N. N here is parallel to $\dot{C}$, similar to the foliation which gave π^a rather than P^a. Several such slices along the worldline are shown, each with a corresponding constant λ.

chapter, the foliation was chosen such that the normal points in the direction of the worldline C. Indeed, examination of (5.79) using $\mathfrak{n} = P$ and $\mathcal{T}^a_D$ given by (5.50) yields

$$\mathcal{T}^a_D \wedge \widetilde{P} = -P^a C_D \wedge \widetilde{P} \tag{5.80}$$

$$(-g(\pi, X^a)C_D + i_{\dot{C}}\left(i_{X^a}F \wedge \star\Sigma\right) \wedge \star C_D) \wedge \widetilde{P} = -P^a C_D \wedge \widetilde{P}, \tag{5.81}$$

which after expanding and star-pivoting the second term on the LHS becomes

$$-g(\pi, X^a)C_D \wedge \widetilde{P} - C_D \wedge \star\left(i_{\dot{C}}\left(i_{X^a}F \wedge \star\Sigma\right) \wedge \widetilde{P}\right) = -P^a C_D \wedge \widetilde{P}. \tag{5.82}$$

Since $C_D \wedge \alpha = -C_D \wedge \left(\widetilde{\alpha} \cdot \dot{C}\right) \widetilde{\dot{C}}$ for 1-forms α, the distributions can be stripped from (5.82) leaving

$$\pi^a \left(P \cdot \dot{C}\right) + i_{\dot{C}} \star \left(i_{\dot{C}}\left(i_{X^a}F \wedge \star\Sigma\right) \wedge \widetilde{P}\right) = P^a \left(P \cdot \dot{C}\right). \tag{5.83}$$

The second term of (5.83) can be simplified as follows:

$$i_{\dot{C}} \star \left(i_{\dot{C}}\left(i_{X^a}F \wedge \star\Sigma\right) \wedge \widetilde{P}\right) = i_{\dot{C}} \star i_{\dot{C}}\left(i_{X^a}F \wedge \star\Sigma \wedge \Pi^{\perp}_{\dot{C}}\widetilde{P}\right) \tag{5.84}$$

$$= i_{\dot{C}} \star i_{\dot{C}}\left(\Sigma \wedge \star\left(i_{X^a}F \wedge \Pi^{\perp}_{\dot{C}}\widetilde{P}\right)\right) \tag{5.85}$$

$$= i_{\dot{C}} \star i_{\dot{C}}\left(-i_{\widetilde{i_{X^a}F}}\, i_{\Pi^{\perp}_{\dot{C}}P}\Sigma \star 1\right), \tag{5.86}$$

using a star-pivot and properties of the internal contraction. Since $i_{\widetilde{i_{X^a}F}}\, i_{\Pi^{\perp}_{\dot{C}}P}\Sigma$ is a 0-form and $\star\star\widetilde{\dot{C}} = \widetilde{\dot{C}}$, (5.86) is simply

$$i_{\widetilde{i_{X^a}F}}\, i_{\Pi^{\perp}_{\dot{C}}P}\Sigma = F^{ab}\left(\Pi^{\perp}_{\dot{C}}P\right)^c \Sigma_{cb}, \tag{5.87}$$

which substituting back into (5.83) gives

$$\pi^a \left(P \cdot \dot{C}\right) + F^{ab}\left(\Pi^{\perp}_{\dot{C}}P\right)^c \Sigma_{cb} = P^a \left(P \cdot \dot{C}\right). \tag{5.88}$$

Noting that

$$\left(\Pi^{\perp}_{\dot{C}}P\right)^c = P^c + \left(P \cdot \dot{C}\right)\dot{C}^c, \tag{5.89}$$

equation (5.88) can be rewritten to become

$$\pi^a = P^a - F^{ab}\Sigma_{cb}\left(\frac{P^c}{\left(P \cdot \dot{C}\right)} + \dot{C}^c\right), \tag{5.90}$$

which is nothing more than relationship (5.72) after an adjustment of indices.

Having confirmed that these equations match those found in the literature, a choice remains as to which momentum to use. Since the momentum P is preferred by the form of the Nakano-Tulczyjew condition, henceforth the equations of motion will be stated in terms of P, rather than π.

5.2.8 Spin Conditions and Relating the Magnetic Dipole Moment to Quantum Mechanical Spin

The equations of motion in terms of momentum P^a and spin S^{ab} are given in component form by

$$\frac{d}{d\tau}\left(P^a + \frac{F^{ab}\Sigma_{bc}P^c}{\dot{C}^d P_d}\right) = -qF^{ab}\dot{C}_b - \frac{1}{2}\Sigma^{bc}\partial^a F_{bc}, \tag{5.91}$$

$$\frac{d}{d\tau}S^{ab} = -\dot{C}^a\left(P^b + \frac{F^{bc}\Sigma_{cd}P^d}{\dot{C}^e P_e}\right) + \dot{C}^b\left(P^a + \frac{F^{ac}\Sigma_{cd}P^d}{\dot{C}^e P_e}\right) + F^{bc}\Sigma_c{}^a - F^{ac}\Sigma_c{}^b. \tag{5.92}$$

These equations are not, however, a complete system. A relationship between P^a and S^{ab} is required, as well as a relationship between P^a and $\dot{C}^a$, and additional information about Σ^{ab}. The relationship between P^a and S^{ab} has been considered before and two main choices have come to light; firstly the Frenkel condition [40] $i_{\dot{C}}S = 0 = \dot{C}^a S_{ab}$ and then the Nakano-Tulczyjew [42, 43] condition $i_P S = 0 = P^a S_{ab}$. The Frenkel condition, whilst being simple and intuitive, has been found to have issues when modelled fully; for instance particles have helical solutions in field-free systems (sometimes called Zittebewegung) [47, 48] and hence the Nakano-Tulczyjew condition is preferred (see for instance Dixon [44–46] and Suttorp and de Groot [49]).

A particle with quantum mechanical spin has a magnetic dipole moment related to the spin by the gyromagnetic ratio $\frac{\mathfrak{g}q}{2M_0}$. Hence it is supposed that the electric dipole moment μ_e of the particle is zero and hence $\Sigma^{ab} = \frac{\mathfrak{g}q}{2M_0}S^{ab}$ where $\mathfrak{g}$

is the so-called g-factor of the particle. The system of equations is now

$$\frac{d}{d\tau}\left(P^a + \frac{\mathfrak{g}q}{2M_0}\frac{F^{ab}S_{bc}P^c}{\dot{C}^d P_d}\right) = -qF^{ab}\dot{C}_b - \frac{\mathfrak{g}q}{4M_0}S^{bc}\partial^a F_{bc}, \tag{5.93}$$

$$\frac{d}{d\tau}S^{ab} = -\dot{C}^a\left(P^b + \frac{\mathfrak{g}q}{2M_0}\frac{F^{bc}S_{cd}P^d}{\dot{C}^e P_e}\right) + \dot{C}^b\left(P^a + \frac{\mathfrak{g}q}{2M_0}\frac{F^{ac}S_{cd}P^d}{\dot{C}^e P_e}\right) + \frac{\mathfrak{g}q}{2M_0}F^{bc}S_c{}^a - \frac{\mathfrak{g}q}{2M_0}F^{ac}S_c{}^b, \tag{5.94}$$

$$P^a S_{ab} = 0. \tag{5.95}$$

Using the Nakano-Tulczyjew condition (5.95), the system simplifies somewhat:

$$\frac{d}{d\tau}P^a = -qF^{ab}\dot{C}_b - \frac{\mathfrak{g}q}{4M_0}S^{bc}\partial^a F_{bc}, \tag{5.96}$$

$$\frac{d}{d\tau}S^{ab} = -\dot{C}^a P^b + \dot{C}^b P^a + \frac{\mathfrak{g}q}{2M_0}F^{bc}S_c{}^a - \frac{\mathfrak{g}q}{2M_0}F^{ac}S_c{}^b, \tag{5.97}$$

$$P^a S_{ab} = 0. \tag{5.95 revisited}$$

The Nakano-Tulczyjew condition (5.95) can also be differentiated in order to find the relationship between P and $\dot{C}$ (as per the method in [49]):

$$\frac{d}{d\tau}P^a S_{ab} + P^a\frac{d}{d\tau}S_{ab} = 0, \tag{5.98}$$

and inserting (5.96) and (5.97) gives

$$-qF^{ac}\dot{C}_c S_{ab} - \frac{\mathfrak{g}q}{4M_0}\left(\partial^a F_{cd}\right)S^{cd}S_{ab} - P^a\dot{C}_a P_b + \dot{C}_b P^a P_a - \frac{\mathfrak{g}q}{2M_0}P^a F_a{}^c S_{cb} = 0. \tag{5.99}$$

Dividing through by $P^e\dot{C}_e$ results in the condition that P must satisfy:

$$P_b = -q\frac{F^{ac}\dot{C}_c S_{ab}}{P^e\dot{C}_e} - \frac{\mathfrak{g}q}{4M_0}\frac{\left(\partial^a F_{cd}\right)S^{cd}S_{ab}}{P^e\dot{C}_e} + \frac{\dot{C}_b P^a P_a}{P^e\dot{C}_e} - \frac{\mathfrak{g}q}{2M_0}\frac{P^a F_a{}^c S_{cb}}{P^e\dot{C}_e}. \tag{5.100}$$

The $P^a P_a$ term can be replaced as follows: by multiplying (5.99) by $\dot{C}^b$ and noting that $\dot{C}^a\dot{C}_a = -1$, $P^a P_a$ is found to be of the form

$$P^a P_a = -qF^{ac}\dot{C}_c S_{ab}\dot{C}^b - \frac{\mathfrak{g}q}{4M_0}\left(\partial^a F_{cd}\right)S^{cd}S_{ab}\dot{C}^b - \left(P^a\dot{C}_a\right)^2 - \frac{\mathfrak{g}q}{2M_0}P^a F_a{}^c S_{cb}\dot{C}^b. \tag{5.101}$$

Inserting this into the momentum condition (5.100) and rearranging gives:

$$P_b = -P^a\dot{C}_a\dot{C}_b - \frac{\left(S_{ab} + S_{ad}\dot{C}^d\dot{C}_b\right)}{P^e\dot{C}_e}\left(qF^{ac}\dot{C}_c + \frac{\mathfrak{g}q}{4M_0}\partial^a F_{cd}S^{cd} + \frac{\mathfrak{g}q}{2M_0}P^cF_c{}^a\right). \tag{5.102}$$

Hence the full system of equations of a particle with spin is

$$\frac{d}{d\tau}P^a = -qF^{ab}\dot{C}_b - \frac{\mathfrak{g}q}{4M_0}S^{bc}\partial^a F_{bc}, \tag{5.96 revisited}$$

$$\frac{d}{d\tau}S^{ab} = -\dot{C}^aP^b + \dot{C}^bP^a + \frac{\mathfrak{g}q}{2M_0}F^{bc}S_c{}^a - \frac{\mathfrak{g}q}{2M_0}F^{ac}S_c{}^b, \tag{5.97 revisited}$$

$$P^aS_{ab} = 0, \tag{5.95 revisited}$$

$$P^a = -P^b\dot{C}_b\dot{C}^a - \frac{\left(S_b{}^a + S_{bd}\dot{C}^d\dot{C}^a\right)}{P^e\dot{C}_e}\left(qF^{bc}\dot{C}_c + \frac{\mathfrak{g}q}{4M_0}\partial^b F_{cd}S^{cd} + \frac{\mathfrak{g}q}{2M_0}P^cF_c{}^b\right). \tag{5.102 revisited}$$

Note that the first term of (5.96) on the RHS corresponds to the usual Lorentz force, while the second term corresponds to a Stern-Gerlach field-spin interaction, particularly evident in electromagnetic fields with high field gradients. The equation (5.97) is a generalisation of the Thomas-Bargmann-Michel-Telegdi (TBMT) equation (see Ref. [25] or p561-565 of Ref. [26]).

5.2.9 Linearising the Equations of Motion for an Electron

Consider now the motion of a classical electron with charge $q = q_{\mathrm{e}}$, rest mass $M_0 = m_{\mathrm{e}}$ and $\mathfrak{g} = 2$. The equations of motion are hence

$$\frac{d}{d\tau}P^a = -q_{\mathrm{e}}F^{ab}\dot{C}_b - \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}\partial^a F_{bc}, \tag{5.103}$$

$$\frac{d}{d\tau}S^{ab} = -\dot{C}^aP^b + \dot{C}^bP^a + \frac{q_{\mathrm{e}}}{m_{\mathrm{e}}}F^{bc}S_c{}^a - \frac{q_{\mathrm{e}}}{m_{\mathrm{e}}}F^{ac}S_c{}^b, \tag{5.104}$$

$$P^aS_{ab} = 0, \tag{5.105}$$

$$P^a = -P^b\dot{C}_b\dot{C}^a - \frac{\left(S_b{}^a + S_{bd}\dot{C}^d\dot{C}^a\right)}{P^e\dot{C}_e}\left(q_{\mathrm{e}}F^{bc}\dot{C}_c + \frac{q_{\mathrm{e}}}{m_{\mathrm{e}}}\partial^b F_{cd}S^{cd} + \frac{q_{\mathrm{e}}}{m_{\mathrm{e}}}P^cF_c{}^b\right). \tag{5.106}$$

These equations of motion do not explicitly give P^a in terms of $\dot{C}^a$, since (5.106) is merely a condition that P^a must satisfy. In order to negate this problem, the equations of motion are linearised. There are two options for linearisation; linearise the equations in F^{ab} similar to the approach used in Ref. [49] or linearise in S^{ab}, since spin is inherently such a small quantity. Since the F^{ab}-linearised equations of motion are already considered in Ref. [49], this chapter is concerned with linearising in S^{ab}.

Firstly consider that in the spin-free case, the equations of motion are

$$\frac{d}{d\tau}P^a = -qF^{ab}\dot{C}_b, \tag{5.107}$$

$$0 = -\dot{C}^a P^b + \dot{C}^b P^a, \tag{5.108}$$

$$P^a = -P^b\dot{C}_b\dot{C}^a = m_e\dot{C}^a. \tag{5.109}$$

So with the ansatz $P^a = m_e\dot{C}^a + \mathsf{P}^a(S) + \mathcal{O}(S^2)$, where $\mathsf{P}^a(S)$ contains terms of first order in S^{ab}, consider the P^a condition (5.106). This yields

$$\mathsf{P}^a = \mathsf{P}^b\dot{C}_b\dot{C}^a + \mathcal{O}(S^2) \tag{5.110}$$

and hence to first order in S^{ab}, P^a must be parallel to $\dot{C}^a$. Hence the ansatz becomes

$$P^a = \left(m_e + \mathsf{M}(S)\right)\dot{C}^a, \tag{5.111}$$

where $\mathsf{M}(S) = -\mathsf{P}^a\dot{C}_a$ is first order in S^{ab}. M can be found explicitly since

$$m_e + \mathsf{M}(S) = -P^a\dot{C}_a \tag{5.112}$$

and noting that the rest mass m_e is constant, i.e. $\frac{dm_e}{d\tau} = 0$,

$$\begin{aligned}\frac{d\mathsf{M}(S)}{d\tau} &= -\frac{dP^a}{d\tau}\dot{C}_a - P^a\frac{d\dot{C}_a}{d\tau} \\ &= q_e\dot{C}_aF^{ab}\dot{C}_b + \frac{q_e}{2m_e}S^{bc}\dot{C}_a\partial^aF_{bc} - \left(m_e + \mathsf{M}(S)\right)\dot{C}^a\frac{d\dot{C}_a}{d\tau}.\end{aligned} \tag{5.113}$$

The first term here is zero since F^{ab} is antisymmetric and the last term is zero since the 4-velocity $\dot{C}^a$ and 4-acceleration $\ddot{C}^a$ of the particle are orthogonal. For the remaining term, noting that $\dot{C}_a\partial^a = \frac{d}{d\tau}$,

$$S^{bc}\dot{C}_a\partial^aF_{bc} = S^{bc}\frac{d}{d\tau}F_{bc} = \frac{d}{d\tau}\left(S^{bc}F_{bc}\right) - F_{bc}\frac{d}{d\tau}S^{bc}, \tag{5.114}$$

and substituting (5.104) with (5.111) gives

$$S^{bc}\dot{C}_a\partial^a F_{bc} = \frac{d}{d\tau}\left(S^{bc}F_{bc}\right) - 2\frac{q_{\mathrm{e}}}{m_{\mathrm{e}}}F_{bc}F^{cd}S_d{}^b. \tag{5.115}$$

Since $F_{bc}F^{cd}$ is symmetric, $F_{bc}F^{cd}S_d{}^b = 0$; hence

$$S^{bc}\dot{C}_a\partial^a F_{bc} = \frac{d}{d\tau}\left(S^{bc}F_{bc}\right), \tag{5.116}$$

so mass $\mathsf{M}(S)$ can be written

$$\mathsf{M}(S) = \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}F_{bc} + \mathcal{O}(S^2). \tag{5.117}$$

The ansatz for P^a to first order in S^{ab} is thus

$$P^a = \left(m_{\mathrm{e}} + \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}F_{bc}\right)\dot{C}^a. \tag{5.118}$$

Inserting (5.118) into (5.104) gives

$$\frac{d}{d\tau}S^{ab} = \frac{q_{\mathrm{e}}}{m_{\mathrm{e}}}\left(F^{bc}S_c{}^a - F^{ac}S_c{}^b\right) + \mathcal{O}(S^2). \tag{5.119}$$

In order to find the analogous equation for P^a, note that applying $\frac{d}{d\tau}$ to (5.118) gives

$$\begin{aligned}\frac{d}{d\tau}P^a &= \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}\frac{d}{d\tau}\left(S^{bc}F_{bc}\right)\dot{C}^a + \left(m_{\mathrm{e}} + \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}F_{bc}\right)\frac{d}{d\tau}\dot{C}^a \\ &= \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}\dot{C}^a\dot{C}_d\partial^d F_{bc} + \left(m_{\mathrm{e}} + \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}F_{bc}\right)\frac{d}{d\tau}\dot{C}^a,\end{aligned} \tag{5.120}$$

using (5.118). Substituting (5.120) into (5.103) and rearranging gives

$$\frac{d}{d\tau}\dot{C}^a = \frac{1}{m_{\mathrm{e}} + \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}F_{bc}}\left[-q_{\mathrm{e}}F^{ab}\dot{C}_b - \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}\partial^a F_{bc} - \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}}S^{bc}\dot{C}^a\dot{C}_d\partial^d F_{bc}\right]. \tag{5.121}$$

Noting that $S^{bc}\partial^a F_{bc} + S^{bc}\dot{C}^a\dot{C}_d\partial^d F_{bc} = S^{bc}\left(\Pi^{\perp}_{\dot{C}}\right)^{ad}\partial_d F_{bc}$, linearising (5.121) results in

$$\frac{d}{d\tau}\dot{C}^a = -\left(1 - \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}^2}S^{bc}F_{bc}\right)\frac{q_{\mathrm{e}}}{m_{\mathrm{e}}}F^{ab}\dot{C}_b - \frac{q_{\mathrm{e}}}{2m_{\mathrm{e}}^2}S^{bc}\left(\Pi^{\perp}_{\dot{C}}\right)^{ad}\partial_d F_{bc} + \mathcal{O}(S^2). \tag{5.122}$$

To first order in S^{ab}, the system of equations governing the motion of a classical electron with spin are hence

$$\frac{d}{d\tau}\dot{C}^a = -\left(1 - \frac{q_\text{e}}{2m_\text{e}^2}S^{bc}F_{bc}\right)\frac{q_\text{e}}{m_\text{e}}F^{ab}\dot{C}_b - \frac{q_\text{e}}{2m_\text{e}^2}S^{bc}\left(\Pi_{\dot{C}}^{\perp}\right)^{ad}\partial_d F_{bc}, \tag{5.123}$$

$$\frac{d}{d\tau}S^{ab} = \frac{q_\text{e}}{m_\text{e}}\left(F^{bc}S_c{}^a - F^{ac}S_c{}^b\right), \tag{5.124}$$

with momentum and spin satisfying the conditions

$$P^a = \left(m_\text{e} + \frac{q_\text{e}}{2m_\text{e}}S^{bc}F_{bc}\right)\dot{C}^a, \tag{5.118 revisited}$$

$$P^a S_{ab} = 0. \tag{5.95 revisited}$$

Interestingly, since to first order in S^{ab} the momentum is parallel to the velocity, satisfying the Nakano-Tulczyjew condition also satisfies the Frenkel condition $\dot{C}^a S_{ab} = 0$.

Hence in order for the Stern-Gerlach terms of (5.123) to be noticeable, a physical situation with high field gradient is required.

5.3 Motion of an Electron with Spin in a Plasma Wave

Since the electric field generated by a plasma wave near wave breaking has very large gradients, the behaviour of the equations of motion due to the spin terms should be noticeable in this context. This section hence applies the equations of motion (5.123) and (5.124) to a system of a plasma wave moving in the z-direction as in Section 4.2. Note that since the equation for the plasma wave from Chapter 4 is used, the spin effects of the particles that make up the wave itself are neglected.

Hence the system contains plasma wave electrons with 4-velocity 1-form

$$\widetilde{V}_\text{e} = \nu(\xi)d\zeta - \sqrt{\nu(\xi)^2 - \gamma^2}d\xi, \tag{4.2 revisited}$$

where γ is the Lorentz factor of the wave, ν is a dimensionless quantity akin to the electric potential, and $\xi = z - vt$, $\zeta = -t + vz$ are coordinates for the plasma wave frame moving in the z direction with speed v. The electric field of the plasma wave is $E_z = \frac{m_\mathrm{e}}{q_\mathrm{e}\gamma^2}\nu'(\xi)$ where ν satisfies the equation (4.49). Despite the fact that Chapter 4 considers a nonlinear electrodynamical theory, this chapter uses only the linear Maxwell Lagrangian $L = L_\mathrm{M} = X/2$. By neglecting the higher order terms that arise from nonlinear electrodynamics, it is possible to focus solely on the effects of the Stern-Gerlach-like terms in the equations of motion. Thus (4.49) for Maxwell electrodynamics is simply

$$\frac{m_\mathrm{e}^2}{2q_\mathrm{e}^2\gamma^4}\nu'^2 - m_\mathrm{e} Z n_\mathrm{ion}\left(v\sqrt{\nu^2-\gamma^2} - \nu + \gamma\right) = 0, \tag{5.125}$$

for a maximum amplitude wave, where $Z = -\frac{q_\mathrm{ion}}{q_\mathrm{e}} = \frac{q_\mathrm{ion}}{e}$ is the degree of ionisation of the plasma. For a system where the only nonzero electromagnetic field component is

$$E_z = \frac{m_\mathrm{e}}{q_\mathrm{e}\gamma^2}\nu', \tag{4.15 revisited}$$

i.e.

$$F_{03} = -F_{30} = E_z \quad \text{and} \quad F_{ab} = 0 \quad \text{otherwise}, \tag{5.126}$$

the components $\{C^0, C^1, C^2, C^3\}$ i.e. $\{t, x, y, z\}$ of the equation of motion (5.123) become

$$\ddot{C}^0 = -\left(1 - \frac{q_\mathrm{e}}{m_\mathrm{e}^2}S^{03}F_{03}\right)\frac{q_\mathrm{e}}{m_\mathrm{e}}F^{03}\dot{C}_3 - \frac{q_\mathrm{e}}{m_\mathrm{e}^2}\left(v + \dot{C}^0\left(\dot{C}^3 - v\dot{C}^0\right)\right)F'_{03}S^{03}, \tag{5.127}$$

$$\ddot{C}^1 = -\frac{q_\mathrm{e}}{m_\mathrm{e}^2}\dot{C}^1\left(\dot{C}^3 - v\dot{C}^0\right)F'_{03}S^{03}, \tag{5.128}$$

$$\ddot{C}^2 = -\frac{q_\mathrm{e}}{m_\mathrm{e}^2}\dot{C}^2\left(\dot{C}^3 - v\dot{C}^0\right)F'_{03}S^{03}, \tag{5.129}$$

$$\ddot{C}^3 = \left(1 - \frac{q_\mathrm{e}}{m_\mathrm{e}^2}S^{03}F_{03}\right)\frac{q_\mathrm{e}}{m_\mathrm{e}}F^{03}\dot{C}_0 - \frac{q_\mathrm{e}}{m_\mathrm{e}^2}\left(1 + \dot{C}^3\left(\dot{C}^3 - v\dot{C}^0\right)\right)F'_{03}S^{03}, \tag{5.130}$$

since $\frac{d}{d\tau}\dot{C} = \ddot{C}$ and

$$\begin{aligned}\dot{C}^d\partial_d F_{03} &= \dot{C}^0\partial_0 F_{03} + \dot{C}^3\partial_3 F_{03}\\ &= \left(\dot{C}^3 - v\dot{C}^0\right)F'_{03}.\end{aligned} \tag{5.131}$$

Interestingly the S_{03} component is the only spin component that appears in these equations and hence the only component of (5.124) it is necessary to consider is the S_{03} one. For completeness, however, all components of the spin equation of motion are

$$\dot{S}^{01} = -\frac{q_e}{m_e}F^{03}{S_3}^1, \qquad \dot{S}^{13} = -\frac{q_e}{m_e}F^{03}{S_0}^1, \tag{5.132}$$
$$\dot{S}^{02} = -\frac{q_e}{m_e}F^{03}{S_3}^2, \qquad \dot{S}^{23} = -\frac{q_e}{m_e}F^{03}{S_0}^2, \tag{5.133}$$
$$\dot{S}^{03} = 0, \qquad \dot{S}^{12} = 0, \tag{5.134}$$

the solutions to which are found to be

$$S^{01} = -\mathcal{C}_3 \cosh \mathcal{X} - \mathcal{C}_4 \sinh \mathcal{X}, \qquad S^{13} = \mathcal{C}_4 \cosh \mathcal{X} + \mathcal{C}_3 \sinh \mathcal{X}, \tag{5.135}$$
$$S^{02} = -\mathcal{C}_5 \cosh \mathcal{X} - \mathcal{C}_6 \sinh \mathcal{X}, \qquad S^{23} = \mathcal{C}_6 \cosh \mathcal{X} + \mathcal{C}_5 \sinh \mathcal{X}, \tag{5.136}$$
$$S^{03} = \mathcal{C}_1, \qquad S^{12} = \mathcal{C}_2, \tag{5.137}$$

where $\mathcal{X} = -\frac{q_e}{m_e}\int F^{03}d\tau = \frac{1}{\gamma^2}\int \frac{d\mu}{d\xi}d\tau$ and each $\mathcal{C}_n$ is simply an integration constant. Since, however, the equations of motion depend only on S^{03}, the main fact of importance is that S^{03} is constant. The other components are irrelevant so long as the constants $\mathcal{C}_n$ are chosen to satisfy the Frenkel condition $\dot{C}_a S^{ab} = 0$. Henceforth it is assumed that S^{03} and S^{12} are arbitrary and that the constants $\mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$ have been chosen to satisfy these conditions. Choosing S^{03} to be a non-zero constant, the spin equations can hence be neglected, condensing the system of equations to (5.127), (5.128), (5.129) and (5.130). Converting these into the wave frame presents some simplification. The wave-frame coordinates are $\{\gamma\zeta, x, y, \gamma\xi\}$ for a plasma wave travelling at speed v in the z direction, where $\zeta = -t + vz$ and $\xi = z - vt$. Hence writing $C^\xi = C^3 - vC^0$ and $C^\zeta = -C^0 + vC^3$ the equations of motion (5.127), (5.128), (5.129) and (5.130) become

$$\ddot{C}^0 = -\left(1 - \frac{q_e}{m_e^2}S^{03}F_{03}\right)\frac{q_e}{m_e}F^{03}\dot{C}^3 - \frac{q_e}{m_e^2}\left(v + \dot{C}^0\dot{C}^\xi\right)F'_{03}S^{03}, \tag{5.138}$$
$$\ddot{C}^1 = -\frac{q_e}{m_e^2}\dot{C}^1\dot{C}^\xi F'_{03}S^{03}, \tag{5.139}$$
$$\ddot{C}^2 = -\frac{q_e}{m_e^2}\dot{C}^2\dot{C}^\xi F'_{03}S^{03}, \tag{5.140}$$
$$\ddot{C}^3 = -\left(1 - \frac{q_e}{m_e^2}S^{03}F_{03}\right)\frac{q_e}{m_e}F^{03}\dot{C}^0 - \frac{q_e}{m_e^2}\left(1 + \dot{C}^3\dot{C}^\xi\right)F'_{03}S^{03}, \tag{5.141}$$

and since $\ddot{C}^\xi = \ddot{C}^3 - v\ddot{C}^0$ and $\ddot{C}^\zeta = -\ddot{C}^0 + v\ddot{C}^3$, these equations can be written

$$\ddot{C}^\zeta = \left(1 - \frac{q_\mathrm{e} S^{03}}{m_\mathrm{e}^2} F_{03}\right) \frac{q_\mathrm{e}}{m_\mathrm{e}} F^{03} \dot{C}^\xi - \frac{q_\mathrm{e} S^{03}}{m_\mathrm{e}^2} \dot{C}^\zeta \dot{C}^\xi F'_{03}, \tag{5.142}$$

$$\ddot{C}^1 = -\frac{q_\mathrm{e} S^{03}}{m_\mathrm{e}^2} \dot{C}^1 \dot{C}^\xi F'_{03}, \tag{5.143}$$

$$\ddot{C}^2 = -\frac{q_\mathrm{e} S^{03}}{m_\mathrm{e}^2} \dot{C}^2 \dot{C}^\xi F'_{03}, \tag{5.144}$$

$$\ddot{C}^\xi = \left(1 - \frac{q_\mathrm{e} S^{03}}{m_\mathrm{e}^2} F_{03}\right) \frac{q_\mathrm{e}}{m_\mathrm{e}} F^{03} \dot{C}^\zeta - \frac{q_\mathrm{e} S^{03}}{m_\mathrm{e}^2} \left(\gamma^{-2} + (\dot{C}^\xi)^2\right) F'_{03}. \tag{5.145}$$

Substituting in (4.15), the equations of motion for a classical electron with spin in the electric field of a maximum amplitude plasma wave are

$$\ddot{C}^\zeta = -\left(1 - \frac{S^{03}}{m_\mathrm{e}\gamma^2}\nu'\right) \frac{\nu'}{\gamma^2} \dot{C}^\xi - \frac{S^{03}}{m_\mathrm{e}\gamma^2} \dot{C}^\zeta \dot{C}^\xi \nu'', \tag{5.146}$$

$$\ddot{C}^1 = -\frac{S^{03}}{m_\mathrm{e}\gamma^2} \dot{C}^1 \dot{C}^\xi \nu'', \tag{5.147}$$

$$\ddot{C}^2 = -\frac{S^{03}}{m_\mathrm{e}\gamma^2} \dot{C}^2 \dot{C}^\xi \nu'', \tag{5.148}$$

$$\ddot{C}^\xi = -\left(1 - \frac{S^{03}}{m_\mathrm{e}\gamma^2}\nu'\right) \frac{\nu'}{\gamma^2} \dot{C}^\zeta - \frac{S^{03}}{m_\mathrm{e}\gamma^2} \left(\gamma^{-2} + (\dot{C}^\xi)^2\right) \nu'', \tag{5.149}$$

where ν satisfies (5.125), and to distinguish the derivatives, dots represent proper time derivatives and dashes represent derivatives with respect to ξ.

5.3.1 A Particular Solution to the Equations of Motion for a Plasma

In order to simplify notation, note that on the worldline C of the particle, $C^1 = x$, $C^2 = y$ etc. Since the equations of motion of a particle must be restricted only to its worldline, the equations of motion (5.146)-(5.149) are written

$$\ddot{\zeta} = -\left(1 - \bar{S}\nu'\right) \frac{\nu'}{\gamma^2} \dot{\xi} - \bar{S}\dot{\zeta}\dot{\xi}\nu'', \tag{5.150}$$

$$\ddot{x} = -\bar{S}\dot{x}\dot{\xi}\nu'', \tag{5.151}$$

$$\ddot{y} = -\bar{S}\dot{y}\dot{\xi}\nu'', \tag{5.152}$$

$$\ddot{\xi} = -\left(1 - \bar{S}\nu'\right) \frac{\nu'}{\gamma^2} \dot{\zeta} - \bar{S}\left(\gamma^{-2} + (\dot{\xi})^2\right) \nu'', \tag{5.153}$$

where $\bar{S} = \frac{S^{03}}{m_e\gamma^2}$.

Since a general solution to these equations is not apparent, a particular solution is sought in order to investigate the impact of spin on the path of a particle. Clearly a particular solution can be found for constant $\xi = \xi_C$. These solutions correspond to particles moving at the same speed as the plasma wave in the z-direction so the particle experiences a constant electric field $E_C = \frac{m_e}{q_e\gamma^2}\frac{d\nu}{d\xi}\Big|_{\xi=\xi_C}$. This section is hence concerned with the stability of this particular solution.

Firstly, the particular solution for constant ξ (equal to ξ_C), is given by solving the equations (5.150)-(5.153) with $\dot{\xi} = \ddot{\xi} = 0$:

$$\ddot{\zeta} = 0, \tag{5.154}$$

$$\ddot{x} = 0, \tag{5.155}$$

$$\ddot{y} = 0, \tag{5.156}$$

$$0 = -\left(1 - \bar{S}\nu'_C\right)\frac{\nu'_C}{\gamma^2}\dot{\zeta} - \bar{S}\gamma^{-2}\nu''_C, \tag{5.157}$$

where $\nu'_C = \frac{d\nu}{d\xi}\Big|_{\xi=\xi_C}$ and so on. Integrating these equations gives

$$\zeta(\tau) = -\frac{\bar{S}}{\left(1 - \bar{S}\nu'_C\right)}\frac{\nu''_C}{\nu'_C}\tau + \zeta_0, \tag{5.158}$$

$$x(\tau) = \dot{x}_0\tau + x_0, \tag{5.159}$$

$$y(\tau) = \dot{y}_0\tau + y_0, \tag{5.160}$$

$$\xi = \xi_C, \tag{5.161}$$

where ζ_0, x_0, y_0, $\dot{x}_0$ and $\dot{y}_0$ are also constants. It is possible to substitute one of $\dot{x}_0$, $\dot{y}_0$ for the other since the particular solution is normalised, i.e.

$$-\left(\frac{\bar{S}}{\left(1 - \bar{S}\nu'_C\right)}\frac{\nu''_C}{\nu'_C}\right)^2 + \dot{x}_0^2 + \dot{y}_0^2 = -1. \tag{5.162}$$

Note that it is impossible to satisfy the normalisation condition with $\bar{S} = 0$; i.e. this normalisable trajectory does not exist for a spinless particle. Similarly the derivatives ν'_C and ν''_C must be nonzero. In order to simplify notation it is assumed that $\dot{x}_0$ and $\dot{y}_0$ have been chosen to satisfy (5.162). Since the equations

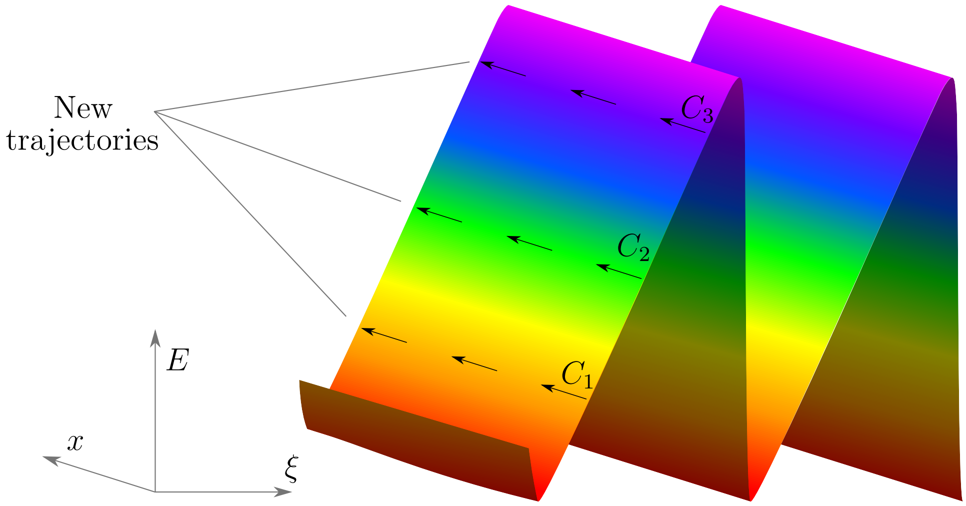

Figure 5.2: Illustration of several trajectories C_1, C_2, C_3 given by different choices of ξ_{C}. Whilst the plasma electrons travel along ξ, these solutions travel transverse to the wave's velocity, "surfing" along the wave.

of motion depend on the velocity and not position of the particles (other than ξ_{C}), the particular solution considered henceforth is

$$\zeta_{\mathrm{sol}}(\tau) = -\frac{\bar{S}}{(1 - \bar{S}\nu'_{\mathrm{C}})}\frac{\nu''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\tau, \tag{5.163}$$

$$x_{\mathrm{sol}}(\tau) = \dot{x}_0\tau, \tag{5.164}$$

$$y_{\mathrm{sol}}(\tau) = \dot{y}_0\tau, \tag{5.165}$$

$$\xi_{\mathrm{sol}} = \xi_{\mathrm{C}}. \tag{5.166}$$

This solution corresponds to a particle travelling in the z-direction with speed v but with the transverse motion given by x_0 and y_0; in essence such a particle would "surf" along the wave. Three sample trajectories are shown in Figure 5.2. Note that due to the normalisation condition (5.162), the transverse trajectories cannot exist for all ξ_C: the minima and maxima of the wave correspond to zeroes of ν'_C for instance. These choices of ν'_C lead to no solutions of the normalisation condition and are hence invalid.

5.3.2 Perturbing around the Particular Solution

In order to perturb around these solutions, consider

$$\zeta(\tau) = \zeta_{\rm sol}(\tau) + \varepsilon\Delta\zeta(\tau) = -\frac{\bar{S}}{\left(1-\bar{S}\nu'_{\rm C}\right)}\frac{\nu''_{\rm C}}{\nu'_{\rm C}}\tau + \varepsilon\Delta\zeta(\tau), \tag{5.167}$$

$$x(\tau) = x_{\rm sol}(\tau) + \varepsilon\Delta x(\tau) = \dot{x}_0\tau + \varepsilon\Delta x(\tau), \tag{5.168}$$

$$y(\tau) = y_{\rm sol}(\tau) + \varepsilon\Delta y(\tau) = \dot{y}_0\tau + \varepsilon\Delta y(\tau), \tag{5.169}$$

$$\xi(\tau) = \xi_{\rm sol} + \varepsilon\Delta\xi(\tau) = \xi_{\rm C} + \varepsilon\Delta\xi(\tau), \tag{5.170}$$

where ε is a small constant and the Δ terms correspond to perturbations. Substituting (5.170) into ν and its derivatives and taking Taylor series gives:

$$\nu(\xi_{\rm C} + \varepsilon\Delta\xi) = \nu_C + \varepsilon\nu'_{\rm C}\Delta\xi + \mathcal{O}(\varepsilon^2), \tag{5.171}$$

$$\nu'(\xi_{\rm C} + \varepsilon\Delta\xi) = \nu'_{\rm C} + \varepsilon\nu''_{\rm C}\Delta\xi + \mathcal{O}(\varepsilon^2), \tag{5.172}$$

$$\nu''(\xi_{\rm C} + \varepsilon\Delta\xi) = \nu''_{\rm C} + \varepsilon\nu'''_{\rm C}\Delta\xi + \mathcal{O}(\varepsilon^2). \tag{5.173}$$

Hence inserting the perturbed solutions (5.167)-(5.170) along with ν and its derivatives (5.171)-(5.173) into the equations of motion (5.150)-(5.153) can be simplified to

$$\varepsilon\Delta\ddot{\zeta} = \left[-\left(1-\bar{S}\nu'_{\rm C}\right)\frac{\nu'_{\rm C}}{\gamma^2} + \frac{\bar{S}^2}{1-\bar{S}\nu'_{\rm C}}\frac{(\nu''_{\rm C})^2}{\nu'_{\rm C}}\right]\varepsilon\Delta\dot{\xi} + \mathcal{O}(\varepsilon^2), \tag{5.174}$$

$$\varepsilon\Delta\ddot{x} = \left[-\bar{S}\dot{x}_0\nu''_{\rm C}\right]\varepsilon\Delta\dot{\xi} + \mathcal{O}(\varepsilon^2), \tag{5.175}$$

$$\varepsilon\Delta\ddot{y} = \left[-\bar{S}\dot{y}_0\nu''_{\rm C}\right]\varepsilon\Delta\dot{\xi} + \mathcal{O}(\varepsilon^2), \tag{5.176}$$

$$\varepsilon\Delta\ddot{\xi} = \left[-(1-\bar{S}\nu'_{\rm C})\frac{\nu'_{\rm C}}{\gamma^2}\right]\varepsilon\Delta\dot{\zeta} + \left[\left(\frac{1-2\bar{S}\nu'_{\rm C}}{1-\bar{S}\nu'_{\rm C}}\right)\bar{S}\frac{(\nu''_{\rm C})^2}{\gamma^2\nu'_{\rm C}} - \bar{S}\frac{\nu'''_{\rm C}}{\gamma^2}\right]\varepsilon\Delta\xi + \mathcal{O}(\varepsilon^2). \tag{5.177}$$

Equation (5.174) can be integrated to give

$$\Delta\dot{\zeta} = \left[-\left(1-\bar{S}\nu'_{\rm C}\right)\frac{\nu'_{\rm C}}{\gamma^2} + \frac{\bar{S}^2}{1-\bar{S}\nu'_{\rm C}}\frac{(\nu''_{\rm C})^2}{\nu'_{\rm C}}\right]\Delta\xi + \mathcal{O}(\varepsilon), \tag{5.178}$$

and hence (5.177) can be written

$$\begin{aligned}\varepsilon\Delta\ddot{\xi} = &\left[-(1-\bar{S}\nu'_{\rm C})\frac{\nu'_{\rm C}}{\gamma^2}\right]\left[-\left(1-\bar{S}\nu'_{\rm C}\right)\frac{\nu'_{\rm C}}{\gamma^2} + \frac{\bar{S}^2}{1-\bar{S}\nu'_{\rm C}}\frac{(\nu''_{\rm C})^2}{\nu'_{\rm C}}\right]\varepsilon\Delta\xi \\ &+ \left[\left(\frac{1-2\bar{S}\nu'_{\rm C}}{1-\bar{S}\nu'_{\rm C}}\right)\bar{S}\frac{(\nu''_{\rm C})^2}{\gamma^2\nu'_{\rm C}} - \bar{S}\frac{\nu'''_{\rm C}}{\gamma^2}\right]\varepsilon\Delta\xi + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{5.179}$$

Collecting like terms, this can be written

$$\varepsilon\ddot{\Delta\xi} = \left[(1-\bar{S}\nu'_{\mathrm{C}})^2\frac{(\nu'_{\mathrm{C}})^2}{\gamma^4} + \left(\frac{1-\bar{S}\nu'_{\mathrm{C}}}{\nu'_{\mathrm{C}}} - \frac{\bar{S}}{1-\bar{S}\nu'_{\mathrm{C}}}\right)\frac{\bar{S}(\nu''_{\mathrm{C}})^2}{\gamma^2} - \bar{S}\frac{\nu'''_{\mathrm{C}}}{\gamma^2}\right]\varepsilon\Delta\xi + \mathcal{O}(\varepsilon^2), \tag{5.180}$$

hence the equations of motion to first order in ε are

$$\ddot{\Delta\zeta} = \left[-\left(1-\bar{S}\nu'_{\mathrm{C}}\right)\frac{\nu'_{\mathrm{C}}}{\gamma^2} + \frac{\bar{S}^2}{1-\bar{S}\nu'_{\mathrm{C}}}\frac{(\nu''_{\mathrm{C}})^2}{\nu'_{\mathrm{C}}}\right]\dot{\Delta\xi}, \tag{5.181}$$

$$\ddot{\Delta x} = \left[-\bar{S}\dot{x}_0\nu''_{\mathrm{C}}\right]\dot{\Delta\xi}, \tag{5.182}$$

$$\ddot{\Delta y} = \left[-\bar{S}\dot{y}_0\nu''_{\mathrm{C}}\right]\dot{\Delta\xi}, \tag{5.183}$$

$$\ddot{\Delta\xi} = \left[(1-\bar{S}\nu'_{\mathrm{C}})^2\frac{(\nu'_{\mathrm{C}})^2}{\gamma^4} + \left(\frac{1-\bar{S}\nu'_{\mathrm{C}}}{\nu'_{\mathrm{C}}} - \frac{\bar{S}}{1-\bar{S}\nu'_{\mathrm{C}}}\right)\frac{\bar{S}(\nu''_{\mathrm{C}})^2}{\gamma^2} - \bar{S}\frac{\nu'''_{\mathrm{C}}}{\gamma^2}\right]\Delta\xi. \tag{5.184}$$

Clearly these equations can be written in the form

$$\ddot{\Delta\zeta} = \mathcal{A}_1\dot{\Delta\xi}, \tag{5.185}$$

$$\ddot{\Delta x} = \mathcal{A}_2\dot{\Delta\xi}, \tag{5.186}$$

$$\ddot{\Delta y} = \mathcal{A}_3\dot{\Delta\xi}, \tag{5.187}$$

$$\ddot{\Delta\xi} = \mathcal{A}_4\Delta\xi, \tag{5.188}$$

where the constants $\mathcal{A}_n$ are given by

$$\mathcal{A}_1 = -\left(1-\bar{S}\nu'_{\mathrm{C}}\right)\frac{\nu'_{\mathrm{C}}}{\gamma^2} + \frac{\bar{S}^2}{1-\bar{S}\nu'_{\mathrm{C}}}\frac{(\nu''_{\mathrm{C}})^2}{\nu'_{\mathrm{C}}}, \tag{5.189}$$

$$\mathcal{A}_2 = -\bar{S}\dot{x}_0\nu''_{\mathrm{C}}, \tag{5.190}$$

$$\mathcal{A}_3 = -\bar{S}\dot{y}_0\nu''_{\mathrm{C}}, \tag{5.191}$$

$$\mathcal{A}_4 = (1-\bar{S}\nu'_{\mathrm{C}})^2\frac{(\nu'_{\mathrm{C}})^2}{\gamma^4} + \left(\frac{1-\bar{S}\nu'_{\mathrm{C}}}{\nu'_{\mathrm{C}}} - \frac{\bar{S}}{1-\bar{S}\nu'_{\mathrm{C}}}\right)\frac{\bar{S}(\nu''_{\mathrm{C}})^2}{\gamma^2} - \bar{S}\frac{\nu'''_{\mathrm{C}}}{\gamma^2}. \tag{5.192}$$

Since these $\mathcal{A}_n$ are constant for a given choice of ξ_{C}, the equations of motion prove simple to integrate. Clearly $\dot{\Delta\chi} = \mathcal{A}_1\Delta\xi + \mathcal{C}_1$, $\dot{\Delta x} = \mathcal{A}_2\Delta\xi + \mathcal{C}_2$ and $\dot{\Delta y} = \mathcal{A}_3\Delta\xi + \mathcal{C}_3$ for some integration constants $\mathcal{C}_n$ and

$$\Delta\xi = \mathcal{C}_4 e^{\sqrt{\mathcal{A}_4}\tau} + \mathcal{C}_5 e^{-\sqrt{\mathcal{A}_4}\tau}. \tag{5.193}$$

The full set of perturbed solutions are hence

$$\Delta\zeta = \mathcal{A}_1 \frac{\mathcal{C}_4}{\sqrt{\mathcal{A}_4}} e^{\sqrt{\mathcal{A}_4}\tau} + \mathcal{A}_1 \frac{\mathcal{C}_5}{\sqrt{\mathcal{A}_4}} e^{-\sqrt{\mathcal{A}_4}\tau}, \tag{5.194}$$

$$\Delta x = \mathcal{A}_2 \frac{\mathcal{C}_4}{\sqrt{\mathcal{A}_4}} e^{\sqrt{\mathcal{A}_4}\tau} + \mathcal{A}_2 \frac{\mathcal{C}_5}{\sqrt{\mathcal{A}_4}} e^{-\sqrt{\mathcal{A}_4}\tau}, \tag{5.195}$$

$$\Delta y = \mathcal{A}_3 \frac{\mathcal{C}_4}{\sqrt{\mathcal{A}_4}} e^{\sqrt{\mathcal{A}_4}\tau} + \mathcal{A}_3 \frac{\mathcal{C}_5}{\sqrt{\mathcal{A}_4}} e^{-\sqrt{\mathcal{A}_4}\tau}, \tag{5.196}$$

$$\Delta\xi = \mathcal{C}_4 e^{\sqrt{\mathcal{A}_4}\tau} + \mathcal{C}_5 e^{-\sqrt{\mathcal{A}_4}\tau}, \tag{5.197}$$

where the $\mathcal{A}_n$ are defined (5.189)-(5.192) and the $\mathcal{C}_n$ are integration constants. In general the perturbations also contain a linear component in τ, though these are omitted here for brevity.

5.3.3 Stability of the Perturbed Solutions

The stability of the system thus depends on the sign of $\mathcal{A}_4$, that is (5.192). If $\mathcal{A}_4$ is positive, the perturbations will diverge exponentially (unless the integration constant $\mathcal{C}_4$ is zero); if $\mathcal{A}_4$ is negative, the perturbations will oscillate. Hence $\mathcal{A}_4$ must be studied more closely;

$$\mathcal{A}_4 = (1 - \bar{S}\nu'_{\mathrm{C}})^2 \frac{(\nu'_{\mathrm{C}})^2}{\gamma^4} + \left(\frac{1 - \bar{S}\nu'_{\mathrm{C}}}{\nu'_{\mathrm{C}}} - \frac{\bar{S}}{1 - \bar{S}\nu'_{\mathrm{C}}} \right) \frac{\bar{S}(\nu''_{\mathrm{C}})^2}{\gamma^2} - \bar{S}\frac{\nu'''_{\mathrm{C}}}{\gamma^2}, \tag{5.192 revisited}$$

however attempting to find the overall sign of this quantity is not a simple task. In order to simplify matters, (5.192) is written

$$\mathcal{A}_4 = \left(\frac{\nu'_{\mathrm{C}}}{\gamma^2}\right)^2 + \left(\left(\frac{\nu''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\right)^2 - 2\left(\frac{\nu'_{\mathrm{C}}}{\gamma}\right)^2 - \frac{\nu'''_{\mathrm{C}}}{\nu'_{\mathrm{C}}} \right) \frac{\bar{S}\nu'_{\mathrm{C}}}{\gamma^2} + \mathcal{O}(\bar{S}^2), \tag{5.198}$$

and then it is assumed that that $\mathcal{O}(\bar{S}^2)$ can be neglected[1] since $\bar{S}$ is assumed to be small. As the square root of (5.192) is given by

$$\sqrt{\mathcal{A}_4} = \frac{|\nu'_{\mathrm{C}}|}{\gamma^2} + \frac{1}{2}\frac{\nu'_{\mathrm{C}}}{|\nu'_{\mathrm{C}}|} \left(\left(\frac{\nu''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\right)^2 - 2\left(\frac{\nu'_{\mathrm{C}}}{\gamma}\right)^2 - \frac{\nu'''_{\mathrm{C}}}{\nu'_{\mathrm{C}}} \right) \bar{S} + \mathcal{O}(\bar{S}^2), \tag{5.199}$$

[1]The validity of this assumption is tested in the next section.

the first exponential of (5.197) is

$$e^{\sqrt{\mathcal{A}_4}}\tau = \exp\left[\left(\frac{|\nu'_{\mathrm{C}}|}{\gamma^2} + \frac{1}{2}\frac{\nu'_{\mathrm{C}}}{|\nu'_{\mathrm{C}}|}\left(\left(\frac{\nu''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\right)^2 - 2\left(\frac{\nu'_{\mathrm{C}}}{\gamma}\right)^2 - \frac{\nu'''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\right)\bar{S} + \mathcal{O}(\bar{S}^2)\right)\tau\right]$$
$$= e^{\frac{|\nu'_{\mathrm{C}}|}{\gamma^2}\tau} \times \exp\left[\frac{1}{2}\frac{\nu'_{\mathrm{C}}}{|\nu'_{\mathrm{C}}|}\left(\left(\frac{\nu''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\right)^2 - 2\left(\frac{\nu'_{\mathrm{C}}}{\gamma}\right)^2 - \frac{\nu'''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\right)\bar{S}\tau\right] \times e^{\mathcal{O}(\bar{S}^2)}. \quad (5.200)$$

Since $\bar{S}$ is considered to be a small parameter, $\exp\left[\mathcal{O}(\bar{S}^2)\right] \approx 1$ and

$$e^{\sqrt{\mathcal{A}_4}}\tau \approx \left(1 + N_{\mathrm{C}}\bar{S}\tau\right) e^{\frac{|\nu'_{\mathrm{C}}|}{\gamma^2}\tau}, \quad (5.201)$$

to first order in $\bar{S}$, where $N_{\mathrm{C}} = \frac{1}{2}\frac{\nu'_{\mathrm{C}}}{|\nu'_{\mathrm{C}}|}\left(\left(\frac{\nu''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\right)^2 - 2\left(\frac{\nu'_{\mathrm{C}}}{\gamma}\right)^2 - \frac{\nu'''_{\mathrm{C}}}{\nu'_{\mathrm{C}}}\right)$. Hence the ξ perturbation (5.197) to first order in $\bar{S}$ is

$$\Delta\xi = \mathcal{C}_4\left(1 + N_{\mathrm{C}}\bar{S}\tau\right) e^{\frac{|q_e E_{\mathrm{C}}|}{m_e}\tau} + \mathcal{C}_5\left(1 - N_{\mathrm{C}}\bar{S}\tau\right) e^{-\frac{|q_e E_{\mathrm{C}}|}{m_e}\tau}. \quad (5.202)$$

Hence the perturbation $\Delta\xi$ is unstable (to first order in S_{03}) as the first exponential will diverge as τ increases, unless $\mathcal{C}_4 = 0$. Since the other three perturbations are closely linked to $\Delta\xi$, the complete perturbation is also divergent (unless $\mathcal{C}_4$=0).

5.3.4 Consistency Check

To confirm that the assumption made in the previous section is valid, this section confirms the relative sizes of the $\mathcal{O}(\bar{S}^2)$ terms of $\mathcal{A}_4$ relative to the zeroth and first order terms.

Hence consider the $\mathcal{O}(\bar{S}^2)$ part of $\mathcal{A}_4$, given by subtracting the zeroth and first order terms (5.198) from the full expression (5.192) and dropping the C subscripts for notational simplicity:

$$\mathcal{A}_4(S^2) = \frac{\bar{S}^2}{\gamma^2(1 - \bar{S}\nu')}\left[(1 - \bar{S}\nu')\frac{(\nu')^4}{\gamma^2} - (2 - \bar{S}\nu')(\nu'')^2\right]. \quad (5.203)$$

Since the assumption was made that the second order and above terms were much smaller than the first and zeroth order, dividing (5.203) by (5.198) gives the size of the second order terms relative to the first and zeroth order. The relative size

R of the second order and higher terms compared to the first and zeroth order terms of $\mathcal{A}_4$ can hence be written

$$R = \frac{\bar{S}^2\nu'}{(1-\bar{S}\nu')}\left[\frac{(\nu')^4 - 2\gamma^2(\nu'')^2 + \bar{S}\nu'\left(\gamma^2(\nu'')^2 - (\nu')^4\right)}{(\nu')^3 + \bar{S}\left(\gamma^2(\nu'')^2 - 2(\nu')^4 - \gamma^2\nu'''\nu'\right)}\right]. \tag{5.204}$$

Analytical investigation of (5.204) is not a simple task, however, and in order to progress numerical methods must be used. In order to find appropriate values for the quantities involved, the ν' terms can be replaced with electric field E terms, and similarly ν'' and ν''' can be replaced via the plasma wave equation (5.125) and its derivatives. Consulting equation (5.125) constraining ν,

$$\frac{m_{\mathrm{e}}^2}{2q_{\mathrm{e}}^2\gamma^4}\nu'^2 - m_{\mathrm{e}}Zn_{\mathrm{ion}}\left(v\sqrt{\nu^2-\gamma^2} - \nu + \gamma\right) = 0, \qquad \text{(5.125 revisited)}$$

differentiation yields the relationships

$$\nu'' = \frac{q_{\mathrm{e}}^2\gamma^4 Zn_{\mathrm{ion}}}{m_{\mathrm{e}}}\left(v\frac{\nu}{\sqrt{\nu^2-\gamma^2}} - 1\right), \tag{5.205}$$

$$\nu''' = -\frac{q_{\mathrm{e}}^2\gamma^6 vZn_{\mathrm{ion}}}{m_{\mathrm{e}}}\frac{\nu'}{(\nu^2-\gamma^2)^{3/2}}. \tag{5.206}$$

Solving (5.125) algebraically for ν in terms of ν' gives

$$\nu = -\gamma^2\left(k_1(\nu')^2 - \gamma\right) \pm \gamma^2 v\sqrt{\left(k_1(\nu')^2-\gamma\right)^2 - 1}, \tag{5.207}$$

where $k_1 = \frac{m_{\mathrm{e}}}{2q_{\mathrm{e}}^2\gamma^4 Zn_{\mathrm{ion}}}$. Hence the system of equations depends on the quantities $S^{03} = -S_{03} = -S_{tz}$, m_{e}, v, q_{e}, Z, n_{ion}, E. However, scaling ξ via the substitution $\xi = \hat{\xi}\bar{S}$ so that $\frac{d}{d\xi} = \frac{1}{\bar{S}}\frac{d}{d\hat{\xi}}$, and so on for the higher derivatives, allows (5.204) to be written

$$R = \frac{\hat{\nu'}}{(1-\hat{\nu'})}\left[\frac{(\hat{\nu'})^4 - 2\gamma^2(\hat{\nu''})^2 + \hat{\nu'}\left(\gamma^2(\hat{\nu''})^2 - (\hat{\nu'})^4\right)}{(\hat{\nu'})^3 + \gamma^2(\hat{\nu''})^2 - 2(\hat{\nu'})^4 - \gamma^2\hat{\nu'''}\hat{\nu'}}\right], \tag{5.208}$$

where $\hat{\nu'} = \frac{d}{d\hat{\xi}}\nu(\xi)$. Thus ν and its higher derivatives become

$$\nu = -\gamma^2\left(k_2(\hat{\nu'})^2 - \gamma\right) \pm \gamma^2 v\sqrt{\left(k_2(\hat{\nu'})^2-\gamma\right)^2 - 1} = \nu_\pm, \tag{5.209}$$

$$\hat{\nu''} = \frac{1}{2k_2}\left(v\frac{\nu}{\sqrt{\nu^2-\gamma^2}} - 1\right), \tag{5.210}$$

$$\hat{\nu'''} = -\frac{1}{2k_2}v\gamma^2\frac{\hat{\nu'}}{(\nu^2-\gamma^2)^{3/2}}, \tag{5.211}$$

where $k_2 = \frac{k_1}{S^2}$. Hence R now depends on the three parameters

$$v \to \frac{v}{c}, \tag{5.212}$$
$$\hat{\nu'} = -\frac{q_\mathrm{e}}{m_\mathrm{e}^2} E S_{tz} \to -\frac{q_\mathrm{e}}{m_\mathrm{e}^2 c^3} E S_{tz}, \tag{5.213}$$
$$k_2 = \frac{m_\mathrm{e}^3}{2 q_\mathrm{e}^2 Z n_\mathrm{ion} S_{tz}^2} \to \frac{m_\mathrm{e}^3 c^4 \epsilon_0}{2 q_\mathrm{e}^2 Z n_\mathrm{ion} S_{tz}^2}, \tag{5.214}$$

where the final expressions are given in SI units via restoration of the speed of light c and the permittivity of free space ϵ_0. Introducing the Schwinger limit $E_\mathrm{S} = \frac{m_\mathrm{e}^2 c^3}{q_\mathrm{e} \hbar}$ and the maximum electric field[1] for a cold plasma, $E_\mathrm{max} = c\sqrt{\frac{2(\gamma-1) m_\mathrm{e} Z n_\mathrm{ion}}{\epsilon_0}}$, the free parameters can then be written

$$v \to \frac{v}{c}, \tag{5.212 revisited}$$
$$\hat{\nu'} \to -\frac{E}{E_\mathrm{S}} \frac{S_{tz}}{\hbar}, \tag{5.215}$$
$$k_2 \to \left(\frac{E_\mathrm{S}}{E_\mathrm{max}}\right)^2 \left(\frac{\hbar}{S_{tz}}\right)^2 (\gamma - 1). \tag{5.216}$$

Writing $k_3 = \frac{\gamma-1}{k_2}$, ν and its derivatives are given by

$$\nu = -\gamma^2 \left(\frac{\gamma-1}{k_3} (\hat{\nu'})^2 - \gamma\right) \pm \gamma^2 v \sqrt{\left(\frac{\gamma-1}{k_3} (\hat{\nu'})^2 - \gamma\right)^2 - 1}, \tag{5.217}$$
$$\hat{\nu''} = \frac{1}{2} \frac{k_3}{\gamma - 1} \left(v \frac{\nu}{\sqrt{\nu^2 - \gamma^2}} - 1\right), \tag{5.218}$$
$$\hat{\nu'''} = -\frac{1}{2} \frac{k_3}{\gamma - 1} v \gamma^2 \frac{\hat{\nu'}}{(\nu^2 - \gamma^2)^{3/2}}, \tag{5.219}$$

[1] The idea of a wavebreaking limit is well known (see Ref. [52]). The wave-breaking limit may be obtained from (5.125) by integrating from ξ_I, the minimum of ν and hence a zero of E, to ξ_II, the maximum of E and turning point of ν'. Since $\nu_\mathrm{I} = \gamma$ (from (5.125)) and $\nu_\mathrm{II} = \gamma^2$ (from the derivative of (5.125)), the result follows.

which along with (5.208) form a system dependent on the parameters

$$v \to \frac{v}{c}, \tag{5.212 revisited}$$

$$\hat{\nu}' \to -\frac{E}{E_{\mathrm{S}}}\frac{S_{tz}}{\hbar}, \tag{5.220}$$

$$k_3 \to \left(\frac{E_{\mathrm{max}}}{E_S}\right)^2 \left(\frac{S_{tz}}{\hbar}\right)^2. \tag{5.221}$$

Then since $E << E_{\mathrm{S}}$, $E_{\mathrm{max}} < E_{\mathrm{S}}$ and $S_{tz} \sim \hbar$, $-1 < \hat{\nu}' < 1$ and $0 < k_3 < 1$. It should be noted that the square root in (5.217) is real and non-zero since

$$\left(\frac{\gamma - 1}{k_3}(\hat{\nu}')^2 - \gamma\right)^2 = \left((\gamma - 1)\frac{E^2}{E_{\mathrm{M}}^2} - \gamma\right)^2, \tag{5.222}$$

leaving the condition

$$\nu > \gamma, \tag{5.223}$$

in order to keep the square root in (5.218) and (5.219) real and non-zero[1].

Figures 5.3 and 5.4 show that there are indeed regions for which the second order terms are smaller than the first and zeroth order; in particular the regions in Figure 5.3(b,d,e) and Figure 5.4(b,d,e) coloured from blue to green are ideal[2]. While the assumption that the second and higher order terms is clearly not valid for *all* values of v, k_3 and $\hat{\nu}'$, it is not difficult to find parameters such that the expansion, and hence the conclusions of Section 5.3.3, are valid.

[1]The case $\nu < -\gamma$ is neglected since $\nu > 0$ in order to keep the 4-velocity of the plasma electrons future-pointing.

[2]It is important to note that some of the regions where $|R|$ in these plots becomes very large (red) may be artefacts of numerical error. However, the presence and relative abundance of $|R| < 1$ is all that is required for this consistency check.

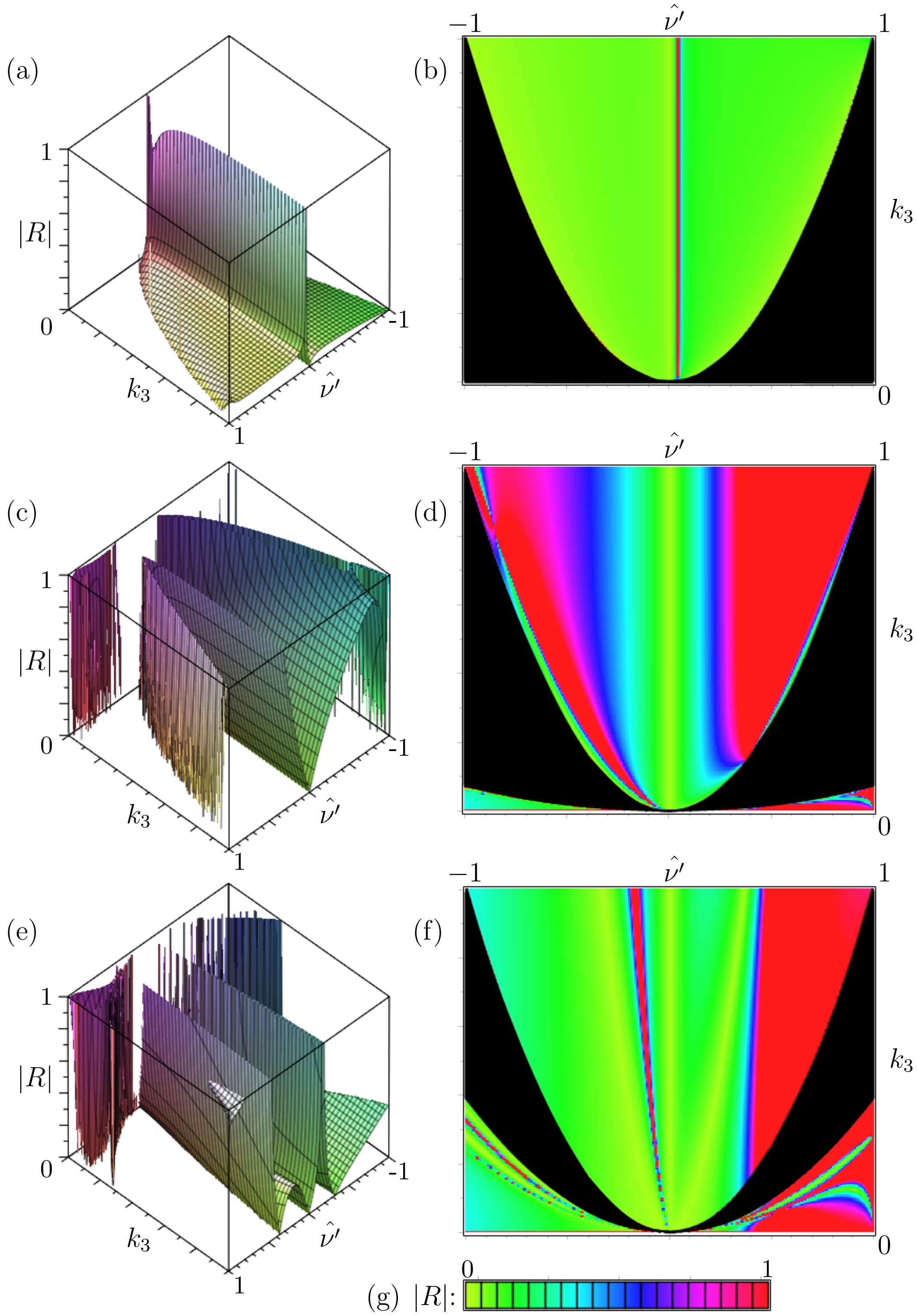

Figure 5.3: (a,c,e): The size of R with $\nu = \nu_+$ across a range of the free parameters $\hat{\nu}' = -\frac{E}{E_S}\frac{S_{tz}}{\hbar}$ and $k_3 = \left(\frac{E_{\max}}{E_S}\right)^2\left(\frac{S_{tz}}{\hbar}\right)^2$ over a range of speeds v: $0.1c$ (a), $0.5c$ (c) and $0.9c$ (e). (b,d,f): Heat charts showing detail of (a,c,e). The black region where $|R|$ becomes imaginary is excluded by $E < E_{\max}$. (g): Key for (b,d,f).

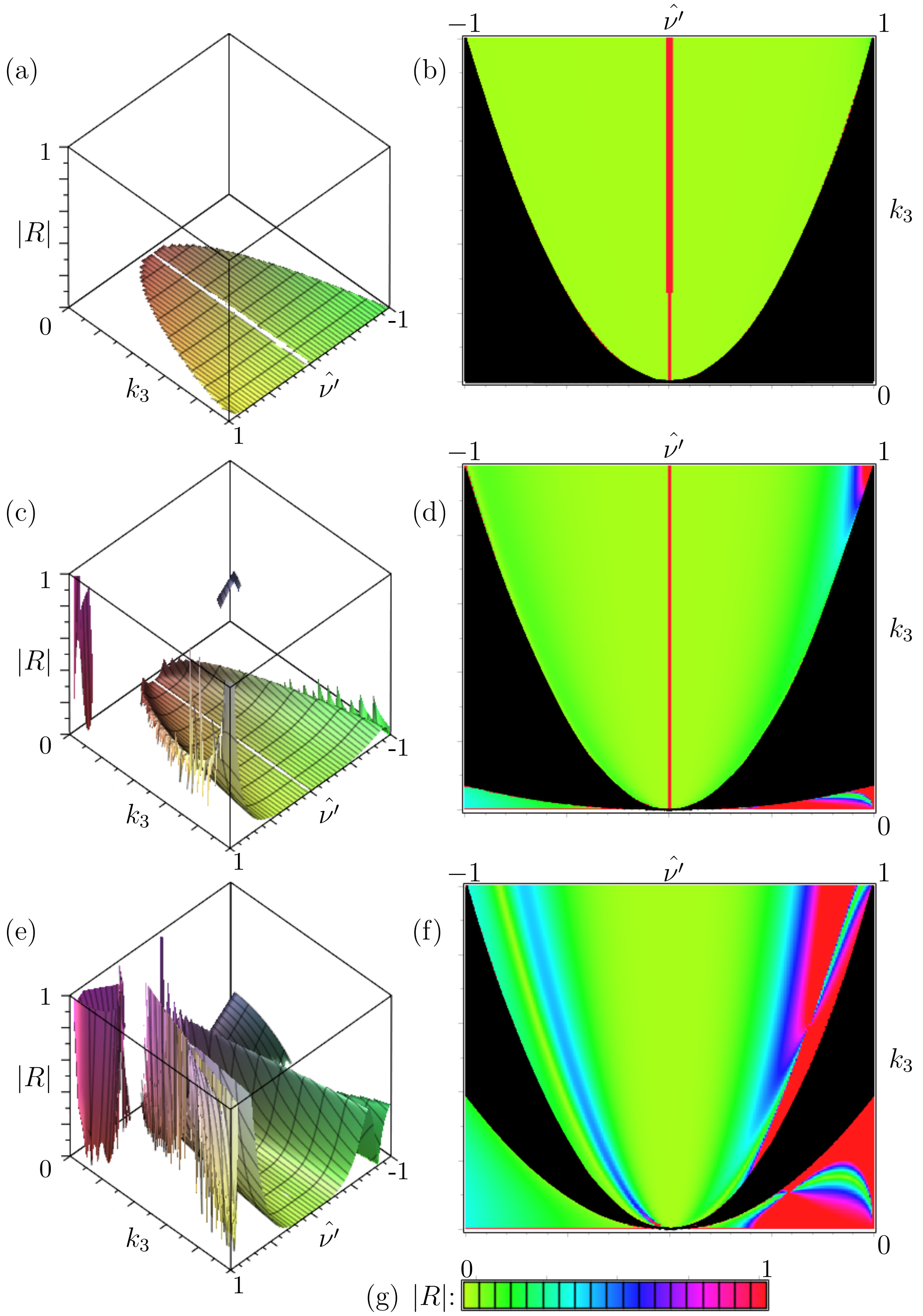

Figure 5.4: (a,c,e): The size of R with $\nu = \nu_-$ across a range of the free parameters $\hat{\nu}' = -\frac{E}{E_S}\frac{S_{tz}}{\hbar}$ and $k_3 = \left(\frac{E_{\max}}{E_S}\right)^2\left(\frac{S_{tz}}{\hbar}\right)^2$ over a range of speeds v: $0.1c$ (a), $0.5c$ (c) and $0.9c$ (e). (b,d,f): Heat charts showing detail of (a,c,e). The black region where $|R|$ becomes imaginary is excluded by $E < E_{\max}$. (g): Key for (b,d,f).

5.4 Summary

This chapter has shown an alternative derivation of the covariant Stern-Gerlach and TBMT equations governing the motion of a classical particle with spin under the influence of electromagnetic fields. By using de Rham currents a pair of covariant equations of motion were derived using the stress and spin balance laws. By comparison with known equations in Ref. [49], the equations of motion found were seen to be equivalent to that found in the literature up to choice of 4-momentum. The pair of equations were then converted to use the momentum P used in the wider literature and in particular used in the Nakano-Tulczyjew condition (5.95), which was added to complete the system. By linearising the equations of motion in the spin components S^{ab}, equations for $\frac{d}{d\tau}\dot{C}^a$ and $\frac{d}{d\tau}S^{ab}$ were obtained.

Using the maximum amplitude plasma wave from Chapter 4, a solution of the equations of motion was sought such that the spin of a particle was significant. Such a solution was found for a trajectory moving transverse to the motion of the plasma electrons; the spinning particle moved with constant speed and hence it is suggested that radiation reaction will not play a significant role. This particular solution was investigated and found to be linearly unstable; a fact which could (for instance) affect the quality of electron bunches for proposed laser-plasma wakefield accelerators (see [38] for an outline of laser wakefield acceleration), since electrons can move in and out of these trajectories once they have reached speed equal to the phase speed of the wave. Since these trajectories are linearly unstable it is likely that these solutions will only contribute on large timescales, causing electron bunches to spread out in the transverse directions.

Chapter 6

Conclusion

This thesis has explored several aspects of electrodynamics in extreme situations; both at high electromagnetic field strength and at high electromagnetic field gradient. A new derivation of the relativistic equations of motion for a spinning charged particle in a background electromagnetic field (the relativistic Stern-Gerlach and TBMT equations) has also been presented.

Through studying the properties of plane waves in constant background fields, Chapter 4 sought to discriminate between the members of the family of Born-Infeld-like theories, that is nonlinear electrodynamical theories whose Lagrangians are of the form

$$L = \mathcal{F}(X + \lambda Y^2). \qquad \text{(3.1 revisited)}$$

It was shown that plane electromagnetic waves in constant background magnetic fields, solutions that satisfy the nonlinear field equations of Born-Infeld theory, also satisfy the nonlinear field equations of all *Born-Infeld-like theories* (3.1), unless the background magnetic field had a nonzero component parallel to the electromagnetic wave's own magnetic field. Similarly with a plane wave in a constant background electric field, the Born-Infeld-like family's field equations are satisfied unless the background field includes a nonzero component parallel to the wave's own electric field, in which case only the Born-Infeld field equations are known to be solved. It is therefore recommended that these components of field be active in any slow light experiment seeking to distinguish members of the family (3.1) from one another. It is also noted that the only member of the family of nonlinear theories (3.1) that satisfies electric-magnetic duality invariance, i.e. the Gaillard-Zumino condition, is the Born-Infeld Lagrangian. Hence Born-Infeld theory is the only duality invariant nonlinear electromagnetic theory whose field equations are solved by plane electromagnetic waves in background fields of arbitrary direction.

Chapter 4 studied the phenomenon of maximum amplitude plasma waves in order to find the energy gained by an electron in half a wavelength of such a wave. By doing so, it was hoped that the result would prove to be a theory discriminant, with different energy gain in different theories. While the result in the wave frame was found to be strikingly simple, i.e. energy gain $\mathcal{W}$ is

$$\Delta\mathcal{W} = 2m_{\mathrm{e}}v^2\gamma^2, \qquad \text{(4.61 revisited)}$$

where γ is the Lorentz factor of the plasma wave with speed v, the relationship between the plasma wave speed v and the nonlinear theories (and background fields) is not known in the context of a plasma. Indeed such an investigation is likely to involve significant numerical machinery and is hence left for future study.

Chapter 5 returned to areas of promising analytic study by investigating the equations of motion for a charged classical particle with spin in a background electromagnetic field. Firstly, however, the equations of motion were derived using a new approach involving de Rham currents and the balance laws for the stress-energy-momentum $\mathcal{T}^a$ and spin σ^{ab} 3-forms:

$$d\mathcal{T}^a = i_{X^a}F \wedge j^{\mathrm{free}} + i_{X^a}F \wedge j^{\mathrm{bound}}, \qquad \text{(5.28 revisited)}$$

$$d\sigma^{ab} = \frac{1}{2}\left(dx^a \wedge \mathcal{T}^b - dx^b \wedge \mathcal{T}^a\right). \qquad \text{(5.55 revisited)}$$

Upon comparison with existing equations in Suttorp and de Groot [49], the newly derived equations of motion were found to be consistent up to choice of momentum.

Using the gyromagnetic ratio to relate the quantum mechanical spin to the classical dipole moment, the equations of motion for a classical electron with spin were acquired. For ease of use the equations were linearised in the spin 2-form components S^{ab} and the Nakano-Tulczyjew condition was used to complete the system.

In order to demonstrate a situation in which the Stern-Gerlach-like terms in the equations of motion would play a significant role (more so than for instance the effects of radiation reaction terms) the maximum amplitude plasma wave studied in Chapter 4 was considered once again. A particular solution of the equations of motion were found where the test electron propagated in a direction transverse to the motion of the plasma wave; a solution which exists only for non-zero spin. Since this particular solution has constant speed, the impact of radiation reaction should be negligible and hence the effects of the spin could prove important in situations where only longitudinal electron motion is desired.

Appendix A

Noether Identities

A.1 Noether Identities from an Action

Noether identities can be regarded as balance laws obtained from local invariances of an action. This appendix shows how U(1), SO(1,3) and local diffeomorphism invariance for an extended particle leads to balance laws used in chapters 4 and 5.

For any infinitesimal transformation δ_u, the action can be written [53, 54]

$$\delta_u \mathcal{S}[e, \omega, A, \Phi] = \int_{\mathcal{M}} \left[\mathcal{T}_a \wedge \delta_u e^a + \sigma_a{}^b \wedge \delta_u \omega^a{}_b + j_{\mathrm{e}} \wedge \delta_u A + \mathsf{E} \wedge \delta_u \Phi \right], \qquad \text{(A.1)}$$

where $\mathcal{T}_a$ are the stress 3-forms, e is the coframe, $\sigma_a{}^b$ are the spin 3-forms, $\omega^a{}_b$ are the connection 1-forms corresponding to the metric compatible connection ∇, j_{e} is the electric current, A is the electromagnetic potential 1-form and E is the Euler-Lagrange equation of Φ, the matter field of the extended particle. Since the connection ∇ is metric compatible, it can be shown that the connection 1-forms are antisymmetric.

A.1.1 U(1) Invariance

Consider the U(1) gauge invariance of the electromagnetic field $A \to A + df$. Introduce a 1-parameter family $A_\varepsilon = A + \varepsilon \delta_{U(1)} A$ where $\delta_{U(1)} A = df$ is some small variation with compact support on $\mathcal{M}$. The action variation $\delta_{\mathrm{U}(1)}\mathcal{S}$ is defined as

$$\delta_{\mathrm{U}(1)}\mathcal{S} \equiv \frac{d}{d\varepsilon}\mathcal{S}\left[e,\ \omega,\ A + \varepsilon\delta_{U(1)}A,\ \Phi\right]\Big|_{\varepsilon=0}, \tag{A.2}$$

which can be expanded using (A.1) to give

$$\begin{aligned} \delta_{\mathrm{U}(1)}\mathcal{S} &= \int_{\mathcal{M}} j_{\mathrm{e}} \wedge \delta_{\mathrm{U}(1)}A \\ &= \int_{\mathcal{M}} j_{\mathrm{e}} \wedge df \\ &= \int_{\mathcal{M}} f dj_{\mathrm{e}}, \end{aligned} \tag{A.3}$$

where the last step uses integration by parts and the fact that f has compact support on $\mathcal{M}$. Requiring $\delta_{\mathrm{U}(1)}\mathcal{S} = 0$ then gives (since (A.3) holds for any f with compact support) the conservation of electric current: $dj_{\mathrm{e}} = 0$.

A.1.2 SO(1,3) Invariance

A.1.2.1 Variations $\delta_{\mathrm{SO}(1,3)}e^a$ and $\delta_{\mathrm{SO}(1,3)}\omega^a{}_b$

Consider the SO(1,3) Lorentz group invariance of the spacetime metric. Lorentz transforms $\Lambda^a{}_b$ are given by

$$\Lambda^a{}_b = \delta^a{}_b + \varepsilon W^a{}_b, \tag{A.4}$$

where $W^a_{\ b}$ transform the frame/coframe such that the metric product $g(X_a, X_b)$ is unchanged. Consider the infinitesimal SO(1,3) transformation $e^a \to e^a + \varepsilon W^a_{\ b} e^b$ and introduce a 1-parameter family $e^a_\varepsilon = e^a + \varepsilon\delta_{\mathrm{SO}(1,3)}e^a$ where $\delta_{\mathrm{SO}(1,3)}e^a = W^a_{\ b}e^b$ is some variation with compact support on $\mathcal{M}$. Since the metric is invariant under frame transformations, $W^a_{\ b}$ must satisfy $\delta_{\mathrm{SO}(1,3)}g = 0$, i.e.

$$\begin{aligned} \delta_{\mathrm{SO}(1,3)}g &= \delta_{\mathrm{SO}(1,3)}\eta_{ab}e^a \otimes e^b + \eta_{ab}\delta_{\mathrm{SO}(1,3)}e^a \otimes e^b + \eta_{ab}e^a \otimes \delta_{\mathrm{SO}(1,3)}e^b \\ &= \eta_{ab}W^a{}_c e^c \otimes e^b + \eta_{ab}e^a \otimes W^b{}_c e^c \\ &= \left(W_{ba} + W_{ab}\right) e^a \otimes e^b, \end{aligned} \tag{A.5}$$

and thus in order to satisfy $\delta_{\mathrm{SO}(1,3)}g = 0$, $W^a_{\ b}$ must be antisymmetric, that is $W_{ba} = -W_{ab}$.

The transformation $\delta_{\mathrm{SO}(1,3)}X_a$ can be found by insisting that the condition $e^a(X_b) = \delta^a{}_b$, where $\delta^a{}_b$ is the Kronecker delta, is invariant under the infinitesimal SO(1,3) transformation. Thus

$$\frac{d}{d\varepsilon}(e^a_\varepsilon(X_{\varepsilon b}))\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\delta^a{}_b\bigg|_{\varepsilon=0}, \tag{A.6}$$

where $e^a_\varepsilon = e^a + \varepsilon\delta_{\mathrm{SO}(1,3)}e^a$, $X_{\varepsilon b} = X_b + \varepsilon\delta_{\mathrm{SO}(1,3)}X_b$ and $\delta_{\mathrm{SO}(1,3)}e^a = W^a_{\ b}e^b$. Expanding (A.6) results in the condition

$$e^a(\delta_{\mathrm{SO}(1,3)}X_b) + \delta_{\mathrm{SO}(1,3)}e^a(X_b) = 0, \tag{A.7}$$

which noting that $\delta_{\mathrm{SO}(1,3)}e^a = W^a_{\ b}e^b$ results in

$$\delta_{\mathrm{SO}(1,3)}X_a = -W^b{}_a X_b = W_a{}^b X_b, \tag{A.8}$$

since W_{ab} is antisymmetric.

The connection 1-forms are defined via the connection ∇ acting on frame X_a

$$\nabla_{X_a}X_b = \omega^c{}_b(X_a)X_c. \tag{A.9}$$

Hence by applying the SO(1,3) variation to both sides of (A.9),

$$\begin{aligned}\nabla_{\delta_{\mathrm{SO}(1,3)}X_a}X_b + \nabla_{X_a}\delta_{\mathrm{SO}(1,3)}X_b = \delta_{\mathrm{SO}(1,3)}\omega^c{}_b(X_a)X_c + \omega^c{}_b(\delta_{\mathrm{SO}(1,3)}X_a)X_c \\ + \omega^c{}_b(X_a)\delta_{\mathrm{SO}(1,3)}X_c,\end{aligned} \tag{A.10}$$

and since $\nabla_{fX_a} = f\nabla_{X_a}$, using (A.9) a common term is found and removed:

$$\nabla_{X_a}\delta_{\mathrm{SO}(1,3)}X_b = \delta_{\mathrm{SO}(1,3)}\omega^c{}_b(X_a)X_c + \omega^c{}_b(X_a)\delta_{\mathrm{SO}(1,3)}X_c. \tag{A.11}$$

Using (A.8), this is

$$\nabla_{X_a}(W_b{}^cX_c) = \delta_{\mathrm{SO}(1,3)}\omega^c{}_b(X_a)X_c + \omega^c{}_b(X_a)W_c{}^dX_d, \tag{A.12}$$

and applying the Leibniz rule to the LHS:

$$\begin{aligned}\nabla_{X_a}(W_b{}^cX_c) &= (\nabla_{X_a}W_b{}^c)\,X_c + W_b{}^c\,(\nabla_{X_a}X_c) \\ &= X_aW_b{}^cX_c + W_b{}^c\omega^d{}_c(X_a)X_d.\end{aligned} \tag{A.13}$$

Hence (A.12) is rearranged to give

$$\delta_{\mathrm{SO}(1,3)}\omega^c{}_b(X_a)X_c = X_a W_b{}^c X_c + W_b{}^c \omega^d{}_c(X_a)X_d - \omega^c{}_b(X_a)W_c{}^d X_d. \tag{A.14}$$

Inspecting the components of (A.14) gives

$$\delta_{\mathrm{SO}(1,3)}\omega^h{}_b(X_a) = X_a W_b{}^h + W_b{}^c \omega^h{}_c(X_a) - \omega^c{}_b(X_a)W_c{}^h. \tag{A.15}$$

Since $Vf = df(V)$, wedging with e^a gives

$$\delta_{\mathrm{SO}(1,3)}\omega^h{}_b = dW_b{}^h + W_b{}^c \omega^h{}_c - \omega^c{}_b W_c{}^h. \tag{A.16}$$

Introducing the covariant exterior derivative D, which acts on $\alpha^{a\ldots b}{}_{c\ldots d}$ via

$$\begin{aligned} D\alpha^{a\ldots b}{}_{c\ldots d} = d\alpha^{a\ldots b}{}_{c\ldots d} + \omega^a{}_e \alpha^{e\ldots b}{}_{c\ldots d} \ldots + \omega^b{}_e \alpha^{a\ldots e}{}_{c\ldots d} \\ - \omega^e{}_c \alpha^{a\ldots b}{}_{e\ldots d} \ldots - \omega^e{}_d \alpha^{a\ldots b}{}_{c\ldots e}, \end{aligned} \tag{A.17}$$

it is clear that

$$\delta_{\mathrm{SO}(1,3)}\omega^a{}_b = DW_b{}^a = -DW^a_{\ b}. \tag{A.18}$$

A.1.2.2 The Spin Noether Identity

The action variation $\delta_{\mathrm{SO}(1,3)}\mathcal{S}$ is defined as

$$\delta_{\mathrm{SO}(1,3)}\mathcal{S} \equiv \frac{d}{d\varepsilon}\mathcal{S}\left[e + \varepsilon\delta_{\mathrm{SO}(1,3)}e,\ \omega + \varepsilon\delta_{\mathrm{SO}(1,3)}\omega,\ A,\ \Phi\right]\Big|_{\varepsilon=0}, \tag{A.19}$$

which can be expanded using (A.1) to give

$$\delta_{\mathrm{SO}(1,3)}\mathcal{S} = \int_{\mathcal{M}} \left[\mathcal{T}_a \wedge \delta_{\mathrm{SO}(1,3)}e^a + \sigma_a{}^b \wedge \delta_{\mathrm{SO}(1,3)}\omega^a{}_b\right]. \tag{A.20}$$

Then (A.20) can be written in terms of $W^a_{\ b}$:

$$\delta_{\mathrm{SO}(1,3)}\mathcal{S} = \int_{\mathcal{M}} \left[\mathcal{T}_a \wedge W^a_{\ b}e^b - \sigma_a{}^b \wedge DW^a_{\ b}\right], \tag{A.21}$$

since $W_b{}^a = -W^a_{\ b}$ because W_{ab} is antisymmetric. Then, since

$$\int_{\mathcal{M}} \sigma_a{}^b \wedge DW^a_{\ b} = \int_{\mathcal{M}} D(\sigma_a{}^b W^a_{\ b}) + \int_{\mathcal{M}} W^a_{\ b} D\sigma_a{}^b, \tag{A.22}$$

where

$$\int_{\mathcal{M}} D(\sigma_a{}^b W^a_{\ b}) = \int_{\mathcal{M}} d(\sigma_a{}^b W^a_{\ b}) = 0, \tag{A.23}$$

and the final step follows because $W^a_{\ b}$ has compact support on $\mathcal{M}$. So

$$\int_{\mathcal{M}} \sigma_a{}^b \wedge DW^a_{\ b} = \int_{\mathcal{M}} W^a_{\ b} D\sigma_a{}^b, \tag{A.24}$$

and hence

$$\delta_{\mathrm{SO}(1,3)}\mathcal{S} = \int_{\mathcal{M}} \left[\mathcal{T}_a \wedge e^b - D\sigma_a{}^b\right] W^a_{\ b}. \tag{A.25}$$

Since $W^a_{\ b}$ is antisymmetric, the symmetric part of $\left[\mathcal{T}_a \wedge e^b - D\sigma_a{}^b\right]$ is projected out, leaving the antisymmetric part:

$$\begin{aligned}\delta_{\mathrm{SO}(1,3)}\mathcal{S} &= \frac{1}{2}\int_{\mathcal{M}} \left[\mathcal{T}_a \wedge e^b - D\sigma_a{}^b - \mathcal{T}^b \wedge e_a + D\sigma^b{}_a\right] W^a_{\ b} \\ &= \frac{1}{2}\int_{\mathcal{M}} \left[\mathcal{T}_a \wedge e^b - 2D\sigma_a{}^b - \mathcal{T}^b \wedge e_a\right] W^a_{\ b},\end{aligned} \tag{A.26}$$

where the last step uses the fact that the spin 3-forms satisfy $\sigma_{ab} = -\sigma_{ba}$. Insisting that $\delta_{\mathrm{SO}(1,3)}\mathcal{S} = 0$ then gives the identity

$$D\sigma_a{}^b = \frac{1}{2}\left(\mathcal{T}_a \wedge e^b - \mathcal{T}^b \wedge e_a\right), \tag{A.27}$$

A.1.3 Local Diffeomorphism Invariance

Diffeomorphisms are isomorphisms on smooth manifolds. Lie derivatives correspond to infinitesimal diffeomorphisms; hence considering an action invariant under local diffeomorphisms is equivalent to considering an action invariant under the Lie derivative $\mathcal{L}_{\mathcal{W}}$, where the components of $\mathcal{W}$ have compact support on $\mathcal{M}$.

Hence using (A.1),

$$\delta_{\mathrm{Diff}(\mathcal{M})}\mathcal{S} = \int_{\mathcal{M}} \left[\mathcal{T}_a \wedge \mathcal{L}_{\mathcal{W}} e^a + \sigma_a{}^b \wedge \mathcal{L}_{\mathcal{W}} \omega^a_{\ b} + j_{\mathrm{e}} \wedge \mathcal{L}_{\mathcal{W}} A + \mathsf{E} \wedge \mathcal{L}_{\mathcal{W}} \Phi\right]. \tag{A.28}$$

Using Cartan's identity,

$$\begin{aligned}\mathcal{L}_{\mathcal{W}} A &= d i_{\mathcal{W}} A + i_{\mathcal{W}} dA \\ &= d i_{\mathcal{W}} A + i_{\mathcal{W}} F \\ &= \delta_{\mathrm{U}(1)} A + i_{\mathcal{W}} F,\end{aligned} \tag{A.29}$$

since $di_{\mathcal{W}}A = d(i_{\mathcal{W}}A)$, analogous to df. Introducing the structure equations [27] defining the torsion 2-forms T^a and curvature 2-forms $R^a{}_b$,

$$de^a = T^a - \omega^a{}_b \wedge e^b \tag{A.30}$$

$$d\omega^a{}_b = R^a{}_b - \omega^a{}_c \wedge \omega^c{}_b, \tag{A.31}$$

consider the expression

$$\begin{aligned}\mathcal{L}_{\mathcal{W}} e^a &= di_{\mathcal{W}} e^a + i_{\mathcal{W}} de^a \\ &= di_{\mathcal{W}} e^a + i_{\mathcal{W}} \left(T^a - \omega^a{}_b \wedge e^b\right) \\ &= di_{\mathcal{W}} e^a + i_{\mathcal{W}} T^a - i_{\mathcal{W}} \omega^a{}_b e^b + \omega^a{}_b i_{\mathcal{W}} e^b \\ &= Di_{\mathcal{W}} e^a + i_{\mathcal{W}} T^a - i_{\mathcal{W}} \omega^a{}_b e^b.\end{aligned} \tag{A.32}$$

Since $i_{\mathcal{W}}\omega^a{}_b$ is an element of the algebra $\mathfrak{so}(1,3)$ (as the connection 1-forms ω_{ab} are antisymmetric) and since the components of X have compact support on $\mathcal{M}$, (A.32) can be written

$$\mathcal{L}_{\mathcal{W}} e^a = Di_{\mathcal{W}} e^a + i_{\mathcal{W}} T^a - \delta_{\mathrm{SO}(1,3)} e^a, \tag{A.33}$$

where $\delta_{\mathrm{SO}(1,3)} e^a = i_{\mathcal{W}}\omega^a{}_b e^b$. Finally the connection 1-form term:

$$\begin{aligned}\mathcal{L}_{\mathcal{W}} \omega^a{}_b &= di_{\mathcal{W}} \omega^a{}_b + i_{\mathcal{W}} d\omega^a{}_b \\ &= di_{\mathcal{W}} \omega^a{}_b + i_{\mathcal{W}} \left(R^a{}_b - \omega^a{}_c \wedge \omega^c{}_b\right) \\ &= Di_{\mathcal{W}} \omega^a{}_b + i_{\mathcal{W}} R^a{}_b \\ &= i_{\mathcal{W}} R^a{}_b - \delta_{\mathrm{SO}(1,3)} \omega^a{}_b,\end{aligned} \tag{A.34}$$

where $\delta_{\mathrm{SO}(1,3)}\omega^a{}_b = -Di_{\mathcal{W}}\omega^a{}_b$ (see (A.18)).

Inserting (A.29), (A.33) and (A.34) into the variation of the action under local diffeomorphisms (A.28) gives

$$\begin{aligned}\delta_{\mathrm{Diff}(\mathcal{M})}\mathcal{S} = \int_{\mathcal{M}} \big[&\mathcal{T}_a \wedge Di_{\mathcal{W}} e^a + \mathcal{T}_a \wedge i_{\mathcal{W}} T^a + \sigma_a{}^b \wedge i_{\mathcal{W}} R^a{}_b + j_{\mathrm{e}} \wedge i_{\mathcal{W}} F + \mathsf{E} \wedge \mathcal{L}_{\mathcal{W}} \Phi \\ &-\mathcal{T}_a \wedge \delta_{\mathrm{SO}(1,3)} e^a - \sigma_a{}^b \wedge \delta_{\mathrm{SO}(1,3)} \omega^a{}_b + j_{\mathrm{e}} \wedge \delta_{\mathrm{U}(1)} A\big] .\end{aligned} \tag{A.35}$$

The gauge invariances in the previous sections remove the last three terms:

$$\delta_{\mathrm{Diff}(\mathcal{M})}\mathcal{S} = \int_{\mathcal{M}} \left[\mathcal{T}_a \wedge Di_{\mathcal{W}} e^a + \mathcal{T}_a \wedge i_{\mathcal{W}} T^a + \sigma_a{}^b \wedge i_{\mathcal{W}} R^a{}_b + j_{\mathrm{e}} \wedge i_{\mathcal{W}} F + \mathsf{E} \wedge \mathcal{L}_{\mathcal{W}} \Phi\right] . \tag{A.36}$$

Since vector components $\mathcal{W}^a = i_{\mathcal{W}} e^a$ have compact support on $\mathcal{M}$, the covariant exterior derivative D can be shifted to the stress tensor via integration by parts and Stokes' theorem:

$$\delta_{\text{Diff}(\mathcal{M})}\mathcal{S} = \int_{\mathcal{M}} \left[D\mathcal{T}_a \mathcal{W}^a + \mathcal{T}_a \wedge i_{\mathcal{W}} T^a + \sigma_a{}^b \wedge i_{\mathcal{W}} R^a{}_b + j_{\text{e}} \wedge i_{\mathcal{W}} F + \mathsf{E} \wedge \mathcal{L}_{\mathcal{W}} \Phi \right], \tag{A.37}$$

and stripping the components from $\mathcal{W}$ gives

$$\delta_{\text{Diff}(\mathcal{M})}\mathcal{S} = \int_{\mathcal{M}} \left[D\mathcal{T}_a + \mathcal{T}_b \wedge i_{X_a} T^b + \sigma_b{}^c \wedge i_{X_a} R^b{}_c + j_{\text{e}} \wedge i_{X_a} F + \mathsf{E} \wedge \mathcal{L}_{X_a} \Phi \right] \mathcal{W}^a. \tag{A.38}$$

Hence requiring $\delta_{\text{Diff}(\mathcal{M})}\mathcal{S} = 0$ gives the balance law

$$D\mathcal{T}_a = -\mathcal{T}_b \wedge i_{\mathcal{W}_a} T^b - \sigma_b{}^c \wedge i_{\mathcal{W}_a} R^b{}_c - j_{\text{e}} \wedge i_{\mathcal{W}_a} F - \mathsf{E} \wedge \mathcal{L}_{\mathcal{W}_a} \Phi. \tag{A.39}$$

A.1.4 In Lorentz Coordinates on Minkowski Spacetime

Since this thesis deals with flat spacetime, the three balance laws (A.27) and (A.39) are significantly simplified. Choosing the Levi-Civita connection gives $T^a = 0$ since the Levi-Civita connection is torsion-free. Minkowski spacetime has no curvature, so $R^a{}_b = 0$, and choosing the orthonormal coframe $\{e^a = dx^a\}$ where $\{x^a\}$ are inertial cartesian coordinates, the connection 1-forms $\omega^a{}_b$ are also zero. Finally, since E correspond to the Euler-Lagrange equation for Φ, $\mathsf{E} = 0$. The balance laws hence take the simple form

$$d\sigma_a{}^b = \frac{1}{2}\left(\mathcal{T}_a \wedge e^b - \mathcal{T}^b \wedge e_a \right), \tag{A.40}$$

$$d\mathcal{T}_a = -j_{\text{e}} \wedge i_{X_a} F. \tag{A.41}$$

Note that the balance law used in Chapter 4 uses a fluid model instead of the extended particle model above, and also has non-background electromagnetic fields to consider. The stress tensor in this section does not include electromagnetic stress terms (see Section B.4.4 and B.5 for details).

Appendix B

Action variation for Nonlinear Electrodynamics

A sample action $\mathcal{S}$ for a cold plasma is given by

$$\mathcal{S}[A, e, \phi] = \int_{\mathcal{M}} \left(-L(X, Y) \star 1 + m_{\mathrm{e}} \sqrt{j_{\mathrm{e}} \cdot j_{\mathrm{e}}} \star 1 + q_{\mathrm{e}} A \wedge j_{\mathrm{e}} + q_{\mathrm{ion}} A \wedge j_{\mathrm{ion}} \right), \quad \text{(B.1)}$$

where the first term encapsulates the electromagnetic field theory, the second the mass energies of the electrons, the third coupling of the electrons and the electromagnetic field and the final term couples the electromagnetic field to a background of ions. The arguments of the action are A, the electromagnetic potential 1-form, e the coframe and ϕ, a map between the spacetime manifold $\mathcal{M}$ and a body manifold $\mathcal{B}$. The map ϕ is an example of a matter field Φ from the previous appendix. The electromagnetic part of the Lagrangian is written in terms of the invariants

$$X = \star(F \wedge \star F), \quad \text{(B.2)}$$

$$Y = \star(F \wedge F), \quad \text{(B.3)}$$

where $F = dA$ is the electromagnetic 2-form defined on the 4-dimensional spacetime manifold $\mathcal{M}$. By using variational principles, several quantities of physical importance can be found, including the field equations, the stress tensor and the Lorentz equation. Note that in this appendix lower case Latin indices indicate an

Einstein sum over 0 to 3 associated with coordinates on $\mathcal{M}$, whereas upper case Latin indices are summed 1 to 3 associated with coordinates on $\mathcal{B}$.

B.1 Preliminaries

B.1.1 Pullbacks

Consider two manifolds $\mathcal{M}$ and $\mathcal{N}$ (of different dimension) with a mapping ψ between them such that ψ maps a point p in $\mathcal{M}$ to a point q in $\mathcal{N}$:

$$\psi : \mathcal{M} \to \mathcal{N}, \tag{B.4}$$

$$p \to q = \psi(p). \tag{B.5}$$

Each manifold also has associated with it a set of coordinates $\hat{x}^\mu$ and $\hat{y}^a$ respectively, which are injective maps taking points in their manifold to a set of real numbers. For instance on m-dimensional manifold $\mathcal{M}$ there are m coordinate maps $\hat{x}^\mu$:

$$\hat{x}^\mu : \mathcal{M} \to \mathbb{R}^m \tag{B.6}$$

$$p \to \{x^\mu_p\} = \{\hat{x}^\mu(p)\}. \tag{B.7}$$

There are also a set of a maps $\hat{\psi}^a$ which relate the two sets of coordinates, defined by

$$\psi : \mathbb{R}^m \to \mathbb{R}^n \tag{B.8}$$

$$\{x^\mu\} \to \{y^a\} = \{\hat{\psi}^a(\{x^\mu\})\}. \tag{B.9}$$

Given this structure, the *pullback* of $f \in \mathcal{N}$ is defined as

$$\psi^* f = f \circ \psi \tag{B.10}$$

$$\text{i.e. } p \in \mathcal{M} \to (\psi^* f)(p) = f(\psi(p)), \tag{B.11}$$

so-called as the map $\psi^* f$ now maps from $\mathcal{M}$ to $\mathbb{R}$ instead of from $\mathcal{N}$ to $\mathbb{R}$, so the pullback map pulls objects from $\mathcal{N}$ back onto $\mathcal{M}$. The pullback can be generalised to forms by using the properties

$$\psi^* df = d(\psi^* f), \tag{B.12}$$

$$\psi^*(hdf) = \psi^* h\ \psi^* df, \tag{B.13}$$

$$\psi^*(\alpha + \beta) = \psi^*\alpha + \psi^*\beta, \tag{B.14}$$

$$\psi^*(\alpha \wedge \beta) = \psi^*\alpha \wedge \psi^*\beta, \tag{B.15}$$

$$\psi^*(S \otimes T) = \psi^* S \otimes \psi^* T. \tag{B.16}$$

B.1.2 Defining j_e and ϕ

The electron current j_e and the map ϕ of (B.1) are defined as follows. Given that $\mathcal{M}$ is a 4-dimensional flat spacetime manifold over which the normalised vector field V_e (i.e. $g(V_e, V_e) = -1$) representing the worldlines of electrons is defined, let $\mathcal{B}$ be a 3-dimensional manifold such that each integral curve of V_e in $\mathcal{M}$ is mapped to a point in $\mathcal{B}$ (see Figure B.1). For more on body manifolds (also called "material" manifolds), see Ref. [55], though the concept was first introduced by Maugin [56].

Now consider the map ϕ defined between the two manifolds $\mathcal{M}$ and $\mathcal{B}$ to be a *submersion*, i.e.

$$\phi : \mathcal{M} \to \mathcal{B} \tag{B.17}$$

$$p \to \phi(p) \tag{B.18}$$

$$x^a \to y^A = \phi^A(p), \tag{B.19}$$

where x^a and y^A are the coordinates on $\mathcal{M}$ and $\mathcal{B}$ respectively, and $d\phi^A$ is non-vanishing by definition[1]. From the relationship between $\mathcal{B}$ and $\mathcal{M}$, it is clear that the map ϕ contains information about the vector field V_e.

Now define a 3-form Θ on $\mathcal{B}$:

$$\Theta = \frac{1}{3!}\Theta_{ABC}(y) dy^A \wedge dy^B \wedge dy^C. \tag{B.20}$$

[1]Since ϕ is a submersion.

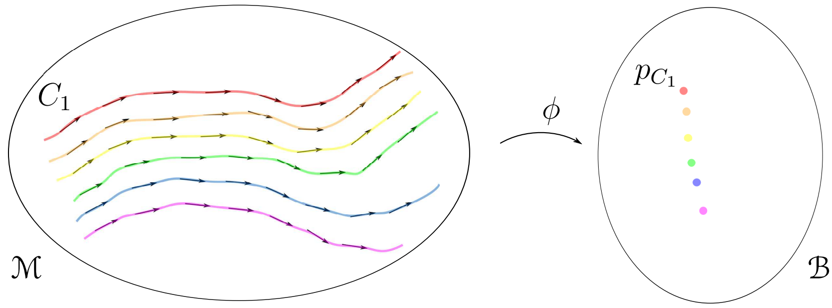

Figure B.1: Illustration of the relationship between the integral curves of vector field V on manifold $\mathcal{M}$ to points in the body manifold $\mathcal{B}$. Each integral curve of V (represented by a different colour) is mapped to a point in $\mathcal{B}$. For instance integral curve C_1 is mapped to point p_{C_1} in $\mathcal{B}$. The pullback map ϕ^* encodes the vector field V in the way the points and curves are related.

Since Θ is a top form on $\mathcal{B}$, it follows that $d\Theta = 0$. Introduce a 3-form j_{e} on $\mathcal{M}$ derived by pulling back Θ:

$$j_{\text{e}} = \phi^*\Theta \tag{B.21}$$

$$= \frac{1}{3!}\left(\Theta_{ABC} \circ \phi\right) d\phi^A \wedge d\phi^B \wedge d\phi^C. \tag{B.22}$$

Noting that $\phi^* dy^A = d\phi^A$, which implies that $V_{\text{e}}\phi^A = 0$ as $V_{\text{e}}\phi^A = d\phi^A(V_{\text{e}})$ and ϕ^A is constant along the integral curves of V_{e},

$$i_{V_{\text{e}}}\left(\phi^*\Theta\right) = i_{V_{\text{e}}}\left(\frac{1}{3!}\left(\Theta_{ABC} \circ \phi\right) d\phi^A \wedge d\phi^B \wedge d\phi^C\right) = 0, \tag{B.23}$$

i.e. $i_{V_{\text{e}}} j_{\text{e}} = 0$. However since j_{e} is a 3-form, i.e. the Hodge dual of a 1-form, say α,

$$i_{V_{\text{e}}} \star \alpha = \star\left(\alpha \wedge \widetilde{V}_{\text{e}}\right) = 0, \tag{B.24}$$

and applying the inverse Hodge map to both sides, it is clear that $\alpha \wedge \widetilde{V}_{\text{e}} = 0$, i.e. $\alpha = n_{\text{e}}\widetilde{V}_{\text{e}}$ for some 0-form n_{e}. The 3-form j_{e} can hence be written

$$j_{\text{e}} = n_{\text{e}} \star \widetilde{V}_{\text{e}}. \tag{B.25}$$

Interpreting n_{e} as the proper number density of the electron fluid, $n_{\text{e}} \geq 0$ is required and is satisfied by choosing Θ and ϕ appropriately.

Since the dot product between two 2-forms is defined as

$$\alpha\cdot\beta = \star^{-1}(\alpha \wedge \star\beta), \tag{B.26}$$

using (B.25),

$$j_{\mathrm{e}}\cdot j_{\mathrm{e}} = n_{\mathrm{e}}^2 \star^{-1} \left(\star\widetilde{V}_{\mathrm{e}} \wedge \widetilde{V}_{\mathrm{e}}\right). \tag{B.27}$$

Consider now the expression

$$i_{V_{\mathrm{e}}} \left(\star 1 \wedge \widetilde{V}_{\mathrm{e}}\right) = (i_{V_{\mathrm{e}}} \star 1) \wedge \widetilde{V}_{\mathrm{e}} + \star 1 \wedge i_{V_{\mathrm{e}}}\widetilde{V}_{\mathrm{e}}. \tag{B.28}$$

The LHS of this equation is identically zero (since $\star 1$ is a top form), and on the RHS, $i_{V_{\mathrm{e}}}\widetilde{V}_{\mathrm{e}} = g(V_{\mathrm{e}}, V_{\mathrm{e}}) = -1$. Hence

$$(i_{V_{\mathrm{e}}} \star 1) \wedge \widetilde{V}_{\mathrm{e}} = \star 1, \tag{B.29}$$

and therefore $n_{\mathrm{e}} = \sqrt{j_{\mathrm{e}} \cdot j_{\mathrm{e}}}$.

B.2 The Field Equations

In order to find the field equations, the action (B.1) is varied with respect to the electromagnetic potential 1-form A. Introduce a 1-parameter family of 1-forms A_ε, i.e. for every value of ε in a range (for example $\varepsilon \in (-1, 1)$) there exists a 1-form associated with this value. A_ε is chosen to be

$$A_\varepsilon = A + \varepsilon\delta A, \tag{B.30}$$

where δA is the variation of A. Variational methods aim to find the stationary 'points' (in fact 1-forms) of the action $\mathcal{S}$:

$$\frac{d}{d\varepsilon}\mathcal{S}[A_\varepsilon, e, \phi]\bigg|_{\varepsilon=0} = 0. \tag{B.31}$$

For brevity, the quantity in (B.31) is named $\delta_A\mathcal{S}$, where $\mathcal{S}$ is the action B.1. As $X(A_\varepsilon) = X_\varepsilon$ and $Y(A_\varepsilon) = Y_\varepsilon$, where X and Y are given by (B.2) and (B.3) with $F = dA$, the first step is to apply the chain rule on the electromagnetic term. The purely matter term contains no A dependence and hence the variation yields no result and the variation of the coupling of the potential A with the electron and ion currents is trivial;

$$\delta_A \mathcal{S} = \int_{\mathcal{M}} \left(- \left[\frac{\partial L}{\partial X} \delta_A X + \frac{\partial L}{\partial Y} \delta_A Y \right] \star 1 + \delta A \wedge (q_{\mathrm{e}} j_{\mathrm{e}} + q_{\mathrm{ion}} j_{\mathrm{ion}}) \right). \tag{B.32}$$

Simplifying $\delta_A X$:

$$\delta_A X \star 1 = \delta_A (X \star 1) = \delta_A \star X = \delta_A (\star \star (F \wedge \star F)), \tag{B.33}$$

and noting

$$\star \star \alpha = \frac{\det g_{ab}}{|\det g_{ab}|} (-1)^{(n-q)q} \alpha, \tag{B.34}$$

where α is a q-form on an n-dimensional manifold $\mathcal{M}$ ($n = 4$ in this thesis),

$$\delta_A X \star 1 = \delta_A (\star \star (F \wedge \star F)) = -\delta_A (F \wedge \star F), \tag{B.35}$$

and similarly

$$\delta_A Y \star 1 = -\delta_A (F \wedge F). \tag{B.36}$$

Hence (B.32) can be rewritten

$$\delta_A \mathcal{S} = \int_{\mathcal{M}} \left(\left[\frac{\partial L}{\partial X} \delta_A (F \wedge \star F) + \frac{\partial L}{\partial Y} \delta_A (F \wedge F) \right] + \delta A \wedge (q_{\mathrm{e}} j_{\mathrm{e}} + q_{\mathrm{ion}} j_{\mathrm{ion}}) \right). \tag{B.37}$$

Expanding the wedge products:

$$\begin{aligned} \delta_A (F \wedge \star F) &= \frac{d}{d\varepsilon} (d(A_\varepsilon) \wedge \star d(A_\varepsilon)) \Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} (dA \wedge \star dA + dA \wedge \star \varepsilon d\delta A + \varepsilon d\delta A \wedge \star dA + \varepsilon d\delta A \wedge \star \varepsilon d\delta A) \Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} (dA \wedge \star dA + \varepsilon (dA \wedge \star d\delta A + d\delta A \wedge \star dA) + \varepsilon^2 d\delta A \wedge \star d\delta A) \Big|_{\varepsilon=0} \\ &= (dA \wedge \star d\delta A + d\delta A \wedge \star dA + 2\varepsilon d\delta A \wedge \star d\delta A)|_{\varepsilon=0} \\ &= dA \wedge \star d\delta A + d\delta A \wedge \star dA \\ &= 2 d\delta A \wedge \star dA, \end{aligned} \tag{B.38}$$

where the final step comes about using the star-pivot, that is for forms α and β of equal degree, $\alpha \wedge \star \beta = \beta \wedge \star \alpha$. Similarly

$$\delta_A (F \wedge F) = 2 d\delta A \wedge dA, \tag{B.39}$$

so that (B.37) is now

$$\delta_A \mathcal{S} = \int_{\mathcal{M}} \left(2 \left[\frac{\partial L}{\partial X} d\delta A \wedge \star dA + \frac{\partial L}{\partial Y} d\delta A \wedge dA \right] + \delta A \wedge (q_{\mathrm{e}} j_{\mathrm{e}} + q_{\mathrm{ion}} j_{\mathrm{ion}}) \right). \tag{B.40}$$

The key method in variational calculations is to separate the δA term from the other terms. The two terms in the square brackets in the above expression can be written as

$$\begin{aligned} \frac{\partial L}{\partial X} d\delta A \wedge \star dA = d\left(\frac{\partial L}{\partial X}\delta A \wedge \star dA\right) - d\left(\frac{\partial L}{\partial X}\right) \wedge \delta A \wedge \star dA \\ + \frac{\partial L}{\partial X}\delta A \wedge d \star dA, \end{aligned} \tag{B.41}$$

and

$$\frac{\partial L}{\partial Y} d\delta A \wedge dA = d\left(\frac{\partial L}{\partial Y}\delta A \wedge dA\right) - d\left(\frac{\partial L}{\partial Y}\right) \wedge \delta A \wedge dA + \frac{\partial L}{\partial Y}\delta A \wedge d^2 A. \tag{B.42}$$

Hence choosing δA with compact support so that it vanishes on the boundary of the manifold, two terms can be removed from the action integral (by using Stokes' theorem on forms), leaving

$$\begin{aligned} \delta_A \mathcal{S} = \int_{\mathcal{M}} \Bigg(2 \Bigg[-d\left(\frac{\partial L}{\partial X}\right) \wedge \delta A \wedge \star dA + \frac{\partial L}{\partial X}\delta A \wedge d \star dA - d\left(\frac{\partial L}{\partial Y}\right) \wedge \delta A \wedge dA \\ + \frac{\partial L}{\partial Y}\delta A \wedge d^2 A \Bigg] + L_{qj} \Bigg), \end{aligned} \tag{B.43}$$

where for brevity the charge-current term $\delta A \wedge (q_{\mathrm{e}} j_{\mathrm{e}} + q_{\mathrm{ion}} j_{\mathrm{ion}})$ is relabelled L_{qj}. Permuting the wedge products using $\alpha^{(p)} \wedge \beta^{(q)} = (-1)^{pq} \beta^{(q)} \wedge \alpha^{(p)}$,

$$\begin{aligned} \delta_A \mathcal{S} = \int_{\mathcal{M}} \Bigg(-2 \Bigg[d\left(\frac{\partial L}{\partial X}\right) \wedge \star dA + \frac{\partial L}{\partial X} d \star dA + d\left(\frac{\partial L}{\partial Y}\right) \wedge dA + \frac{\partial L}{\partial Y} d^2 A \Bigg] \wedge \delta A \\ + L_{qj} \Bigg). \end{aligned} \tag{B.44}$$

Then, as

$$\begin{aligned} L_{q_{\mathrm{e}} j_{\mathrm{e}}} &= \delta A \wedge (q_{\mathrm{e}} j_{\mathrm{e}} + q_{\mathrm{ion}} j_{\mathrm{ion}}) \\ &= -(q_{\mathrm{e}} j_{\mathrm{e}} + q_{\mathrm{ion}} j_{\mathrm{ion}}) \wedge \delta A, \end{aligned} \tag{B.45}$$

and as the variation δA can be *any* smooth variation of A with compact support,

$$2\left(d\left(\frac{\partial L}{\partial X}\right)\wedge\star F+\frac{\partial L}{\partial X}d\star F+d\left(\frac{\partial L}{\partial Y}\right)\wedge F+\frac{\partial L}{\partial Y}dF\right)+(q_\mathrm{e}j_\mathrm{e}+q_\mathrm{ion}j_\mathrm{ion})=0. \tag{B.46}$$

This can be written

$$d\star 2\left(\frac{\partial L}{\partial X}F-\frac{\partial L}{\partial Y}\star F\right)+(q_\mathrm{e}j_\mathrm{e}+q_\mathrm{ion}j_\mathrm{ion})=0. \tag{B.47}$$

Hence (as $F=dA$) the nonlinear generalisation of the Maxwell equations are given by;

$$dF=0,\qquad d\star G=-q_\mathrm{e}j_\mathrm{e}-q_\mathrm{ion}j_\mathrm{ion}, \tag{B.48}$$

where

$$G=2\left(\frac{\partial L}{\partial X}F-\frac{\partial L}{\partial Y}\star F\right). \tag{B.49}$$

B.3 The Lorentz Force Equation

To find the Lorentz force equation, the action (B.1) is varied with respect to the map ϕ between $\mathcal{M}$ and the body manifold $\mathcal{B}$ (effectively varying the structure of V_e whilst maintaining the normalisation condition $g(V_\mathrm{e},V_\mathrm{e})=-1$). As only the current j_e is not invariant under variation of ϕ,

$$\delta_\phi\mathcal{S}[A,\phi,e]=\int_\mathcal{M}\left[\delta_\phi(m_\mathrm{e}\sqrt{j_\mathrm{e}\cdot j_\mathrm{e}}\star 1)+\delta_\phi(q_\mathrm{e}A\wedge j_\mathrm{e})\right], \tag{B.50}$$

and using the chain rule,

$$\delta_\phi\mathcal{S}[A,\phi,e]=\int_\mathcal{M}\left[m_\mathrm{e}\frac{(\delta_\phi j_\mathrm{e})\cdot j_\mathrm{e}}{\sqrt{j_\mathrm{e}\cdot j_\mathrm{e}}}\star 1+q_\mathrm{e}A\wedge\delta_\phi j_\mathrm{e}\right]. \tag{B.51}$$

Since the dot product is defined on forms of equal degree by $\alpha\cdot\beta\star 1=\alpha\wedge\star\beta$,

$$\begin{aligned}\delta_\phi\mathcal{S}[A,\phi,e]&=\int_\mathcal{M}\left[m_\mathrm{e}\frac{\delta_\phi j_\mathrm{e}\wedge\star j_\mathrm{e}}{\sqrt{j_\mathrm{e}\cdot j_\mathrm{e}}}+q_\mathrm{e}A\wedge\delta_\phi j_\mathrm{e}\right]\\&=\int_\mathcal{M}\left[q_\mathrm{e}A-m_\mathrm{e}\frac{\star j_\mathrm{e}}{\sqrt{j_\mathrm{e}\cdot j_\mathrm{e}}}\right]\wedge\delta_\phi j_\mathrm{e}.\end{aligned} \tag{B.52}$$

Note that the current is defined in (B.22) as the pullback of a top form Θ on $\mathcal{B}$, where coordinates in $\mathcal{B}$ are denoted with upper case Latin letters running from 1 to 3. Focussing [1] on $\delta_\phi j_\mathrm{e}$:

$$\begin{aligned}\delta_\phi j_\mathrm{e} &= \delta_\phi\left(\frac{1}{3!}\left(\Theta_{ABC}\circ\phi\right)d\phi^A\wedge d\phi^B\wedge d\phi^C\right)\\ &= \frac{1}{3!}\left(\frac{\partial\Theta_{ABC}}{\partial y^E}\circ\phi\right)\delta\phi^E d\phi^A\wedge d\phi^B\wedge d\phi^C + \frac{1}{2!}\left(\Theta_{ABC}\circ\phi\right)d\delta\phi^A\wedge d\phi^B\wedge d\phi^C.\end{aligned} \tag{B.53}$$

This is equivalent to the statement

$$\delta_\phi j_\mathrm{e} = i_\mathsf{W} dj_\mathrm{e} + di_\mathsf{W} j_\mathrm{e}, \tag{B.54}$$

where $\widetilde{V}_\mathrm{e}(\mathsf{W}) = 0$. To see this, consider

$$i_\mathsf{W} dj_\mathrm{e} + di_\mathsf{W} j_\mathrm{e} = \delta\phi^A i_{\mathsf{W}_A} dj_\mathrm{e} + d\delta_\phi^A i_{\mathsf{W}_A} j_\mathrm{e}, \tag{B.55}$$

where the frame $\{V_\mathrm{e}, \mathsf{W}_A\}$ is naturally dual to coframe $\{-\widetilde{V}_\mathrm{e}, d\phi^A\}$, i.e.

$$d\phi^A(\mathsf{W}_B) = \delta^A{}_B, \tag{B.56}$$

$$\widetilde{V}_\mathrm{e}(\mathsf{W}_A) = 0. \tag{B.57}$$

Since the current j_e is the pullback of a top form from the body manifold $\mathcal{B}$, $dj_\mathrm{e} = 0$. It is instructive, however, to deconstruct this term in order to show the relation (B.54) from (B.53). The first term of (B.55) is

$$\begin{aligned}\delta\phi^E i_{\mathsf{W}_E} dj_\mathrm{e} &= \delta\phi^E i_{\mathsf{W}_E} d\left[\frac{1}{3!}\left(\Theta_{ABC}\circ\phi\right)d\phi^A\wedge d\phi^B\wedge d\phi^C\right]\\ &= \delta\phi^E i_{\mathsf{W}_E}\left[\frac{1}{3!}d\left(\Theta_{ABC}\circ\phi\right)\wedge d\phi^A\wedge d\phi^B\wedge d\phi^C\right].\end{aligned} \tag{B.58}$$

Applying d to $(\Theta_{ABC}\circ\phi)$ gives

$$d\left(\Theta_{ABC}\circ\phi\right) = \frac{\partial\Theta_{ABC}\circ\phi}{\partial\phi^E}d\phi^E, \tag{B.59}$$

and hence

$$\delta\phi^F i_{\mathsf{W}_F} dj_\mathrm{e} = \delta\phi^F\frac{1}{3!}\frac{\partial\Theta_{ABC}\circ\phi}{\partial\phi^E} i_{\mathsf{W}_F}\left[d\phi^E\wedge d\phi^A\wedge d\phi^B\wedge d\phi^C\right], \tag{B.60}$$

[1] N.b. Used in this calculation is the fact that $\Theta_{ABC} = -\Theta_{BAC}$ is totally antisymmetric.

where

$$i_{\mathsf{W}_F}(d\phi)^{EABC} = \delta_F^E (d\phi)^{ABC} - \delta_F^A (d\phi)^{EBC} + \delta_F^B (d\phi)^{EAC} - \delta_F^C (d\phi)^{EAB}, \quad \text{(B.61)}$$

and $(d\phi)^{ABC}$ is shorthand for $d\phi^A \wedge d\phi^B \wedge d\phi^C$. It follows

$$\delta\phi^F i_{\mathsf{W}_F} dj_{\mathrm{e}} = \delta\phi^F \frac{1}{3!}\frac{\partial \Theta_{ABC}\circ\phi}{\partial\phi^E}\left[\delta_F^E (d\phi)^{ABC} - \delta_F^A (d\phi)^{EBC} \right. \\ \left. + \delta_F^B (d\phi)^{EAC} - \delta_F^C (d\phi)^{EAB}\right]. \quad \text{(B.62)}$$

By the fact that Θ is totally antisymmetric, (B.62) can be rewritten

$$\delta\phi^F i_{\mathsf{W}_F} dj_{\mathrm{e}} = \frac{1}{3!}\frac{\partial \Theta_{ABC}\circ\phi}{\partial\phi^E}\delta\phi^E (d\phi)^{ABC} - \frac{1}{2!}\frac{\partial \Theta_{ABC}\circ\phi}{\partial\phi^E}\delta\phi^A (d\phi)^{EBC}. \quad \text{(B.63)}$$

Now the second term of (B.55):

$$d\delta_\phi^F i_{\mathsf{W}_F} j_{\mathrm{e}} = d\left(\delta\phi^F \frac{1}{2!}\left(\Theta_{ABF}\circ\phi\right)(d\phi)^{AB}\right) \\ = d\delta\phi^F \wedge \frac{1}{2!}\left(\Theta_{ABF}\circ\phi\right)(d\phi)^{AB} + \delta\phi^F \frac{1}{2!}\frac{\partial\Theta_{ABF}\circ\phi}{\partial\phi^E}(d\phi^{EAB}). \quad \text{(B.64)}$$

Hence by relabelling and permuting indices,

$$i_{\mathsf{W}} dj_{\mathrm{e}} + d i_{\mathsf{W}} j_{\mathrm{e}} = \frac{1}{3!}\left(\frac{\partial\Theta_{ABC}}{\partial y^F}\circ\phi\right)\delta\phi^F (d\phi)^{ABC} + \frac{1}{2!}\left(\Theta_{ABC}\circ\phi\right) d\delta\phi^A \wedge (d\phi)^{BC}, \quad \text{(B.65)}$$

and therefore

$$\delta_\phi j_{\mathrm{e}} = i_{\mathsf{W}} dj_{\mathrm{e}} + d i_{\mathsf{W}} j_{\mathrm{e}}. \quad \text{(B.66)}$$

Since $j_{\mathrm{e}} = \phi^*\Theta$, clearly $dj_{\mathrm{e}} = d\phi^*\Theta = \phi^* d\Theta = 0$ as Θ is a top form on $\mathcal{B}$. Hence from (B.66), clearly $\delta_\phi j_{\mathrm{e}} = d i_{\mathsf{W}} j_{\mathrm{e}}$. Using this fact, the variation of the action, (B.52), becomes

$$\delta_\phi \mathcal{S}[A,\phi,e] = \int_{\mathcal{M}} \left[q_{\mathrm{e}} A - m_{\mathrm{e}} \frac{\star j_{\mathrm{e}}}{\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}}\right] \wedge d i_{\mathsf{W}} j_{\mathrm{e}}. \quad \text{(B.67)}$$

To proceed, consider the integral

$$\int_{\mathcal{M}} d\left(\left[q_{\mathrm{e}} A - m_{\mathrm{e}} \frac{\star j_{\mathrm{e}}}{\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}}\right] \wedge i_{\mathsf{W}} j_{\mathrm{e}}\right) = \int_{\mathcal{M}} d\left[q_{\mathrm{e}} A - m_{\mathrm{e}} \frac{\star j_{\mathrm{e}}}{\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}}\right] \wedge i_{\mathsf{W}} j_{\mathrm{e}} \\ - \int_{\mathcal{M}} \left[q_{\mathrm{e}} A - m_{\mathrm{e}} \frac{\star j_{\mathrm{e}}}{\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}}\right] \wedge d i_{\mathsf{W}} j_{\mathrm{e}}. \quad \text{(B.68)}$$

By Stokes' theorem, however, the LHS of this equation is zero as the components W^A are chosen to have compact support on $\mathcal{M}$. Thus (B.67) becomes

$$\delta_\phi \mathcal{S} = \int_{\mathcal{M}} \left(m_e i_{\mathsf{W}} j_e \wedge \left(d(q_e A) - d\frac{\star j_e}{\sqrt{j_e \cdot j_e}} \right) \right). \tag{B.69}$$

Since $j_e = n_e \star \widetilde{V}_e$ and $\sqrt{j_e \cdot j_e} = n_e$,

$$\begin{aligned} \delta_\phi \mathcal{S} &= \int_{\mathcal{M}} \left(i_{\mathsf{W}} j_e \wedge \left(d(q_e A) - m_e d\widetilde{V}_e \right) \right) \\ &= \int_{\mathcal{M}} \left(\delta\phi^A i_{\mathsf{W}_A} j_e \wedge \left(d(q_e A) - m_e d\widetilde{V}_e \right) \right). \end{aligned} \tag{B.70}$$

Now, requiring (as per usual in variational calculus) that $\delta_\phi \mathcal{S} = 0$ for suitable variations of $\delta\phi^A$ results in the condition

$$i_{\mathsf{W}_A} j_e \wedge d\left(q_e A - m_e \widetilde{V}_e \right) = 0. \tag{B.71}$$

Substituting in $j_e = n_e \star \widetilde{V}_e$ gives

$$\begin{aligned} & i_{\mathsf{W}_A} n_e \star \widetilde{V}_e \wedge d\left(q_e A - m_e \widetilde{V}_e \right) = 0 \\ \text{so} \quad & i_{\mathsf{W}_A} i_{V_e} \star 1 \wedge d\left(q_e A - m_e \widetilde{V}_e \right) = 0. \end{aligned} \tag{B.72}$$

Now consider

$$\begin{aligned} i_{\mathsf{W}_A} i_{V_e} \left(\star 1 \wedge d\left(q_e A - m_e \widetilde{V}_e \right) \right) = {} & i_{\mathsf{W}_A} i_{V_e} \star 1 \wedge d\left(q_e A - m_e \widetilde{V}_e \right) \\ & + \star 1 \wedge i_{\mathsf{W}_A} i_{V_e} d\left(q_e A - m_e \widetilde{V}_e \right), \end{aligned} \tag{B.73}$$

and as the LHS ($\star 1$ is a top form) (B.72) can be rewritten

$$\star 1 \wedge i_{\mathsf{W}_A} i_{V_e} d\left(q_e A - m_e \widetilde{V}_e \right) = 0, \tag{B.74}$$

and star-pivoting gives

$$i_{\mathsf{W}_A} i_{V_e} d\left(q_e A - m_e \widetilde{V}_e \right) = 0. \tag{B.75}$$

Now consider the 1-form $\alpha = i_{V_e} d\left(q_e A - m_e \widetilde{V}_e \right)$. Two things become apparent; firstly $i_{V_e}\alpha = 0$ as $i_{V_e} i_{V_e} = 0$ and secondly from (B.75), $i_{\mathsf{W}_A}\alpha = 0$. However since $\{V_e, \mathsf{W}_1, \mathsf{W}_2, \mathsf{W}_3\}$ forms a frame[1] on $\mathcal{M}$ then α must be zero and hence

$$i_{V_e} d\left(q_e A - m_e \widetilde{V}_e \right) = 0, \tag{B.76}$$

[1]Since W_A span the V-orthogonal subspace of the tangent space of $\mathcal{M}$ by (B.56), (B.57) .

and since $dA = F$,

$$i_{V_e} d\widetilde{V}_e = \frac{q_e}{m_e} i_{V_e} F. \tag{B.77}$$

Using the identity $d = e^a \wedge \nabla_{X_a}$, it can be shown that $i_{V_e} d\widetilde{V}_e = \nabla_{V_e}\widetilde{V}_e$ for Levi-Civita ∇:

$$\begin{aligned} i_{V_e} d\widetilde{V}_e &= i_{V_e}\left(e^a \wedge \nabla_{X_a}\widetilde{V}_e\right) \\ &= \left(i_{V_e} e^a\right) \nabla_{X_a}\widetilde{V}_e - e^a \left(i_{V_e} \nabla_{X_a}\widetilde{V}_e\right) \\ &= (V_e)^a \nabla_{X_a}\widetilde{V}_e - e^a \left(\nabla_{X_a}\widetilde{V}_e\right)(V_e). \end{aligned} \tag{B.78}$$

Now note two things; firstly as $(V_e)^a$ is a 0-form it can moved into the first argument of the connection via $f\nabla_{V_e} = \nabla_{fV_e}$. Secondly since $\widetilde{A}(B) = g(A, B)$ and ∇ is metric compatible, rewrite $\nabla_{X_a}\widetilde{V}_e = \widetilde{\nabla_{X_a} V_e}$. Hence

$$i_{V_e} d\widetilde{V}_e = \nabla_{(V_e)^a X_a}\widetilde{V}_e - e^a g\left(\nabla_{X_a} V_e, V_e\right). \tag{B.79}$$

The metric compatibility of ∇ gives

$$\nabla_A \left(g(B, C)\right) = g\left(\nabla_A B, C\right) + g\left(B, \nabla_A C\right), \tag{B.80}$$

and hence

$$\nabla_{X_a} \left(g(V_e, V_e)\right) = 2g\left(\nabla_{X_a} V_e, V_e\right). \tag{B.81}$$

Using this, (B.79) can be rewritten as

$$i_{V_e} d\widetilde{V}_e = \nabla_{V_e}\widetilde{V}_e - \frac{1}{2} e^a \nabla_{X_a}\left(g(V_e, V_e)\right), \tag{B.82}$$

and since $g(V_e, V_e) = -1$, the second term is zero, leaving $i_{V_e} d\widetilde{V}_e = \nabla_{V_e}\widetilde{V}_e$. Thus (B.77) becomes

$$\nabla_{V_e}\widetilde{V}_e = \widetilde{\nabla_{V_e} V_e} = \frac{q_e}{m_e} i_{V_e} F, \tag{B.83}$$

the covariant Lorentz force equation for an electron fluid.

B.4 The Stress 3-Forms

In order to find the stress-energy momentum 3-forms, note that as per Appendix A, varying the Action with respect to the orthonormal coframe e results in

$$\begin{aligned}\delta_e \mathcal{S} &= \frac{d}{d\varepsilon}\mathcal{S}[A, e_\varepsilon, \phi]\bigg|_{\varepsilon=0} \\ &= \int_{\mathcal{M}} \mathcal{T}_a \wedge \delta e^a, \end{aligned}\tag{B.84}$$

since action (B.1) does not depend on the connection 1-forms $\omega^a{}_b$. The stress form in Appendix A does not include electromagnetic stress components since the Lagrangian used in that appendix only has a background electromagnetic field and the kinetic term $L(X,Y)\star 1$ for A is not included. This appendix will derive the expression for the stress form (including electromagnetic components) for action (B.1).

Since only terms involving the Hodge map $\star$ in the action (B.1) are not invariant under orthonormal coframe variation,

$$\delta_e \mathcal{S}[A,e,\phi] = \int_{\mathcal{M}} \delta_e(-L\star 1 + m_{\mathrm{e}}\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}\star 1). \tag{B.85}$$

Expanding this using the fact that the variation operator has properties of a derivative gives

$$\begin{aligned}\delta_e \mathcal{S}[A,e,\phi] &= \int_{\mathcal{M}} \left[-\delta_e(L)\star 1 - L\delta_e(\star 1) + m_{\mathrm{e}}\delta_e\left(\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}\right)\star 1 + m_{\mathrm{e}}\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}\delta_e(\star 1)\right] \\ &= \int_{\mathcal{M}} \Bigg[-\left(\frac{\partial L}{\partial X}\delta_e(X) + \frac{\partial L}{\partial Y}\delta_e(Y)\right)\star 1 - L\delta_e(\star 1) \\ &\qquad + m_{\mathrm{e}}\delta_e\left(\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}\right)\star 1 + m_{\mathrm{e}}\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}\delta_e(\star 1)\Bigg]. \end{aligned}\tag{B.86}$$

Now the variations of X, Y and $\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}}$ with respect to the coframe can be considered separately.

B.4.1 The Variation $\delta_e X$

Consider the orthonormal coframe variation of X:

$$\delta_e X = \delta_e\left(\star(F\wedge\star F)\right). \tag{B.87}$$

Rewriting F in terms of the coframe $F = \frac{1}{2}F_{ab}e^{ab}$;

$$\delta_e X = \delta_e \left(\star \left(\frac{1}{2}F_{ab}e^{ab} \wedge \star \frac{1}{2}F_{cd}e^{cd} \right) \right)$$
$$= \frac{1}{4}\delta_e \left(F_{ab}F_{cd} \star (e^{ab} \wedge \star e^{cd}) \right), \tag{B.88}$$

and using the differential nature of the variation operator,

$$\delta_e X = \frac{1}{4}F_{cd}\star(e^{ab}\wedge\star e^{cd})\delta_e(F_{ab})+\frac{1}{4}F_{ab}\star(e^{ab}\wedge\star e^{cd})\delta_e(F_{cd})+\frac{1}{4}F_{cd}F_{ab}\delta_e(\star(e^{ab}\wedge\star e^{cd})). \tag{B.89}$$

Consider now the second term in (B.89). Relabelling indices ($a \leftrightarrow c$ and $b \leftrightarrow d$):

$$\frac{1}{4}F_{cd} \star (e^{cd} \wedge \star e^{ab})\delta_e(F_{ab}), \tag{B.90}$$

which is identical to the first term in (B.89), seen by using the star-pivot;

$$\frac{1}{4}F_{cd} \star (e^{cd} \wedge \star e^{ab})\delta_e(F_{ab}) = \frac{1}{4}F_{cd} \star (e^{ab} \wedge \star e^{cd})\delta_e(F_{ab}), \tag{B.91}$$

hence (B.89) simplifies to

$$\delta_e X = \frac{1}{2}F_{cd} \star (e^{ab} \wedge \star e^{cd})\delta_e(F_{ab}) + \frac{1}{4}F_{cd}F_{ab}\delta_e(\star(e^{ab} \wedge \star e^{cd})). \tag{B.92}$$

In order to proceed, the Levi-Civita alternating symbol ϵ^{abcd} is introduced, where

$$\epsilon_{abcd} = \begin{cases} 1 & \text{if } abcd \text{ is an even permutation of 0123,} \\ -1 & \text{if } abcd \text{ is an odd permutation of 0123,} \\ 0 & \text{if } abcd \text{ is not a permutation of 0123 .} \end{cases} \tag{B.93}$$

The Levi-Civita alternating symbol is necessary to write the Hodge map in terms of the orthonormal coframe; the volume element $\star 1$ is written

$$\star 1 = \frac{1}{4!}\epsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d = \frac{1}{4!}\epsilon_{abcd}e^{abcd}, \tag{B.94}$$

for instance. Contracting $\star 1$ on vector X_f,

$$i_{X_f} \star 1 = \frac{1}{3!}\epsilon_{fbcd}e^{bcd}. \tag{B.95}$$

Since

$$i_{X_f} \star 1 = \star \widetilde{X_f}, \tag{B.96}$$

and

$$\widetilde{X_f} = \eta_{ab} e^a e^b(X_f) = \eta_{af} e^a. \tag{B.97}$$

Hence $\star e^a$ is simply

$$\star e^a = \frac{1}{3!}\eta^{af}\epsilon_{fbcd}e^{bcd} = \frac{1}{3!}\epsilon^a{}_{bcd}e^{bcd}. \tag{B.98}$$

Similarly, $\star e^{ab}$ can also be written in terms of the Levi-Civita alternating symbol:

$$\star e^{ab} = \frac{1}{2!}\epsilon^{ab}{}_{cd}e^{cd}. \tag{B.99}$$

Considering the final term in (B.92),

$$\begin{aligned} \delta_e(\star(e^{ab} \wedge \star e^{cd})) &= \delta_e\left(\star\left(e^{ab} \wedge \frac{1}{2}\epsilon^{cd}{}_{gh}e^{gh}\right)\right) \\ &= \delta_e\left(\frac{1}{2}\epsilon^{cd}{}_{gh} \star (e^{abgh})\right) \\ &= \delta_e\left(\frac{1}{2}\epsilon^{cd}{}_{gh}\epsilon^{abgh}\right), \end{aligned} \tag{B.100}$$

which is zero, as the alternating symbols ϵ are invariant with respect to changes of orthonormal coframe. Thus

$$\delta_e X = \frac{1}{2}F_{cd} \star (e^{ab} \wedge \star e^{cd})\delta_e(F_{ab}). \tag{B.101}$$

As $\delta_e(F_{ab})$ is just a scalar, it can be taken inside the Hodge map:

$$\delta_e X = \frac{1}{2}F_{cd} \star (\delta_e(F_{ab})e^{ab} \wedge \star e^{cd}). \tag{B.102}$$

In order to proceed further, consider

$$\delta_e(F) = \frac{1}{2}\delta_e(F_{ab}e^{ab}) = \frac{1}{2}\delta_e(F_{ab})e^{ab} + \frac{1}{2}F_{ab}\delta_e(e^{ab}). \tag{B.103}$$

By expanding the final term of this equation, as well as permuting indices,

$$\delta_e(F) = \frac{1}{2}\delta_e(F_{ab})e^{ab} + F_{ab}\delta e^a \wedge e^b. \tag{B.104}$$

Since F is independent of coframe, $\delta_e(F)$ is zero, hence

$$\delta_e(F_{ab})e^{ab} = -2F_{ab}\delta e^a \wedge e^b. \tag{B.105}$$

Substituting (B.105) into (B.102) gives

$$\delta_e X = -F_{ab}F_{cd}\star(\delta e^a \wedge e^b \wedge \star e^{cd}). \tag{B.106}$$

The placement of $\delta_e X$ in the action equation (B.86) allows for further simplification;

$$\delta_e(X)\star 1 = -F_{ab}F_{cd}\star(\delta e^a \wedge e^b \wedge \star e^{cd})\star 1. \tag{B.107}$$

Because $\star(\delta e^a \wedge e^b \wedge \star e^{cd})$ is a 0-form, it can be buried in $\star 1$. This results in a double Hodge map, which for a 4-form on a 4-dimensional manifold is just given by the expression

$$\star\star\,\alpha = -\alpha, \tag{B.108}$$

hence

$$\begin{aligned}\delta_e(X)\star 1 &= F_{ab}F_{cd}\delta e^a \wedge e^b \wedge \star e^{cd}\\ &= 2F_{ab}\delta e^a \wedge e^b \wedge \star F.\end{aligned} \tag{B.109}$$

B.4.2 The Variation $\delta_e Y$

Consider the orthonormal coframe variation of Y:

$$\delta_e Y = \delta_e\left(\star(F\wedge F)\right). \tag{B.110}$$

As in the previous section, rewrite F in terms of the orthonormal coframe;

$$\begin{aligned}\delta_e Y &= \delta_e\left(\star\left(\frac{1}{2}F_{ab}e^{ab}\wedge\frac{1}{2}F_{cd}e^{cd}\right)\right)\\ &= \delta_e\left(F_{ab}F_{cd}\star(e^{abcd})\right),\end{aligned} \tag{B.111}$$

and using the differential nature of the variation operator,

$$\delta_e Y = \frac{1}{4}F_{cd}\star(e^{abcd})\delta_e(F_{ab}) + \frac{1}{4}F_{ab}\star(e^{abcd})\delta_e(F_{cd}) + \frac{1}{4}F_{cd}F_{ab}\delta_e(\star(e^{abcd})). \tag{B.112}$$

Similar to the working in the previous section, the second term in (B.112) is equal to the first term and the third term is identically zero due to the invariance of the Levi-Civita symbols ϵ under orthonormal coframe variation. Thus

$$\delta_e Y = \frac{1}{2}F_{cd}\star(e^{abcd})\delta_e(F_{ab}) \tag{B.113}$$

$$= \frac{1}{2}F_{cd}\star(\delta_e(F_{ab})e^{abcd}). \tag{B.114}$$

From the previous section, substitute in (B.105), that is

$$\delta_e(F_{ab})e^{ab} = -2F_{ab}\delta e^a \wedge e^b, \tag{B.115}$$

hence (B.114) becomes

$$\begin{aligned} \delta_e Y &= -F_{ab}F_{cd} \star (\delta e^a \wedge e^{bcd}) && \text{(B.116)} \\ &= -2F_{ab} \star (\delta e^a \wedge e^b \wedge F). && \text{(B.117)} \end{aligned}$$

The placement of $\delta_e Y$ in the action equation (B.86) allows for further simplification:

$$\delta_e(Y) \star 1 = -F_{ab} \star (\delta e^a \wedge e^b \wedge F) \star 1. \tag{B.118}$$

Again, since $\star(\delta e^a \wedge e^b \wedge F)$ is a 0-form, it can be buried in the $\star 1$:

$$\delta_e(Y) \star 1 = 2F_{ab}\delta e^a \wedge e^b \wedge F. \tag{B.119}$$

B.4.3 The Variation $\delta_e\left(\sqrt{j_\mathrm{e}\cdot j_\mathrm{e}}\right)$

Consider the orthonormal coframe variation of $n_\mathrm{e} = \sqrt{j_\mathrm{e}\cdot j_\mathrm{e}}$:

$$\begin{aligned} \delta_e n_\mathrm{e} = \delta_e\sqrt{j_\mathrm{e}\cdot j_\mathrm{e}} &= \frac{1}{2}\frac{1}{\sqrt{j_\mathrm{e}\cdot j_\mathrm{e}}}\delta_e(j_\mathrm{e}\cdot j_\mathrm{e}) \\ &= \frac{1}{2n_\mathrm{e}}\delta_e\left(\star^{-1}(j_\mathrm{e} \wedge \star j_\mathrm{e})\right). && \text{(B.120)} \end{aligned}$$

As $j_\mathrm{e} \wedge \star j_\mathrm{e}$ is a 4-form,

$$\delta_e n_\mathrm{e} = \frac{1}{2n_\mathrm{e}}\delta_e\left(- \star (j_\mathrm{e} \wedge \star j_\mathrm{e})\right). \tag{B.121}$$

Rewriting j_e in terms of the orthonormal coframe $j_\mathrm{e} = \frac{1}{3!}(j_\mathrm{e})_{abc}e^{abc}$:

$$\begin{aligned} \delta_e n_\mathrm{e} &= \frac{1}{2n_\mathrm{e}}\frac{1}{36}\delta_e\left(- \star \left((j_\mathrm{e})_{abc}e^{abc} \wedge \star(j_\mathrm{e})_{dfg}e^{dfg}\right)\right) \\ &= -\frac{1}{2n_\mathrm{e}}\frac{1}{36}\delta_e\left((j_\mathrm{e})_{abc}(j_\mathrm{e})_{dfg} \star \left(e^{abc} \wedge \star e^{dfg}\right)\right), && \text{(B.122)} \end{aligned}$$

and using the differential nature of the variation operator,

$$\begin{aligned} \delta_e n_\mathrm{e} = -\frac{1}{2n_\mathrm{e}}\frac{1}{36}\Big[&\delta_e\left((j_\mathrm{e})_{abc}\right)(j_\mathrm{e})_{dfg} \star \left(e^{abc} \wedge \star e^{dfg}\right) + (j_\mathrm{e})_{abc}\delta_e\left((j_\mathrm{e})_{dfg}\right) \star \left(e^{abc} \wedge \star e^{dfg}\right) \\ &+(j_\mathrm{e})_{abc}(j_\mathrm{e})_{dfg}\delta_e\left(\star(e^{abc} \wedge \star e^{dfg})\right)\Big]. && \text{(B.123)} \end{aligned}$$

Relabelling indices and star-pivoting the second term gives the first term. The third term gives no contribution, similar to in the $\delta_e X$ and $\delta_e Y$ cases. Hence

$$\delta_e n_{\mathrm{e}} = -\frac{1}{2n_{\mathrm{e}}}\frac{1}{18}\delta_e\left(j_{abc}\right) j_{def} \star \left(e^{abc} \wedge \star e^{def}\right), \tag{B.124}$$

and bringing the variation part into the Hodge dual gives

$$\delta_e n_{\mathrm{e}} = -\frac{1}{2n_{\mathrm{e}}}\frac{1}{18}(j_{\mathrm{e}})_{dfg} \star \left(\delta_e\left((j_{\mathrm{e}})_{abc}\right) e^{abc} \wedge \star e^{dfg}\right). \tag{B.125}$$

Now consider $\delta_e(j_{\mathrm{e}})$:

$$\begin{aligned}\delta_e(j_{\mathrm{e}}) &= \frac{1}{3!}\delta_e\left((j_{\mathrm{e}})_{abc}\right) e^{abc} + \frac{1}{3!}(j_{\mathrm{e}})_{abc}\delta_e\left(e^{abc}\right) = 0\\ \text{i.e.}\quad \delta_e\left((j_{\mathrm{e}})_{abc}\right) e^{abc} &= -(j_{\mathrm{e}})_{abc}\delta_e\left(e^{abc}\right).\end{aligned} \tag{B.126}$$

Applying this to (B.125) results in

$$\begin{aligned}\delta_e n_{\mathrm{e}} &= \frac{1}{2n_{\mathrm{e}}}\frac{1}{18}(j_{\mathrm{e}})_{dfg} \star \left((j_{\mathrm{e}})_{abc}\delta_e\left(e^{abc}\right) \wedge \star e^{dfg}\right)\\ &= \frac{1}{2n_{\mathrm{e}}}\frac{1}{3!}(j_{\mathrm{e}})_{dfg} \star \left((j_{\mathrm{e}})_{abc}\delta e^{a} \wedge e^{bc} \wedge \star e^{dfg}\right)\\ &= \frac{1}{2n_{\mathrm{e}}} \star \left((j_{\mathrm{e}})_{abc}\delta e^{a} \wedge e^{bc} \wedge \star j_{\mathrm{e}}\right).\end{aligned} \tag{B.127}$$

The placement of $\delta_e n_{\mathrm{e}}$ in the action equation (B.86) allows for further simplification;

$$\delta_e n_{\mathrm{e}} \star 1 = \frac{1}{2n_{\mathrm{e}}} \star \left((j_{\mathrm{e}})_{abc}\delta e^{a} \wedge e^{bc} \wedge \star j_{\mathrm{e}}\right) \star 1. \tag{B.128}$$

Because $\star\left((j_{\mathrm{e}})_{abc}\delta e^{a} \wedge e^{bc} \wedge \star j_{\mathrm{e}}\right)$ is a 0-form, it can be buried in $\star 1$. This results in a double Hodge map, which for a 4-form on a 4 dimensional manifold is just given by the expression

$$\star\star\alpha = -\alpha, \tag{B.129}$$

hence

$$\delta_e n_{\mathrm{e}} \star 1 = \delta_e\sqrt{j_{\mathrm{e}}\cdot j_{\mathrm{e}}} \star 1 = -\frac{1}{2n_{\mathrm{e}}}(j_{\mathrm{e}})_{abc}\delta e^{a} \wedge e^{bc} \wedge \star j_{\mathrm{e}}. \tag{B.130}$$

B.4.4 Finding the Stress Tensor

Returning to (B.86) and inserting (B.109), (B.119) and (B.130),

$$\delta_e \mathcal{S} = \int_{\mathcal{M}} \left[-\frac{\partial L}{\partial X} 2F_{ab} \delta e^a \wedge e^b \wedge \star F - \frac{\partial L}{\partial Y} 2F_{ab} \delta e^a \wedge e^b \wedge F - L\delta_e(\star 1) \right.$$
$$\left. - \frac{m_\mathrm{e}}{2n_\mathrm{e}} (j_\mathrm{e})_{abc} \delta e^a \wedge e^{bc} \wedge \star j_\mathrm{e} + m_\mathrm{e} n_\mathrm{e} \delta_e(\star 1) \right]. \tag{B.131}$$

The terms involving the variation of $\star 1$ are substituted via

$$\delta_e(\star 1) = \delta e^a \wedge \star e_a, \tag{B.132}$$

so that

$$\begin{aligned} \delta_e \mathcal{S} &= \int_{\mathcal{M}} \delta e^a \wedge \left[-\frac{\partial L}{\partial X} 2F_{ab} e^b \wedge \star F - \frac{\partial L}{\partial Y} 2F_{ab} e^b \wedge F - L \star e_a \right. \\ &\qquad \left. - \frac{m_\mathrm{e}}{2n_\mathrm{e}} (j_\mathrm{e})_{abc} e^{bc} \wedge \star j_\mathrm{e} + m_\mathrm{e} n_\mathrm{e} \star e_a \right] \\ &= \int_{\mathcal{M}} \delta e^a \wedge \left[-2\frac{\partial L}{\partial X} i_{X_a} F \wedge \star F - 2\frac{\partial L}{\partial Y} i_{X_a} F \wedge F - L \star e_a \right. \\ &\qquad \left. + m_\mathrm{e} \left(n_\mathrm{e} \star e_a - \frac{1}{n_\mathrm{e}} i_{X_a} j_\mathrm{e} \wedge \star j_\mathrm{e} \right) \right]. \end{aligned} \tag{B.133}$$

Rewriting the matter piece of the action in terms of the vector field V_e using $j_\mathrm{e} = n_\mathrm{e} \star \widetilde{V}_\mathrm{e}$,

$$\begin{aligned} m_\mathrm{e} \left(n_\mathrm{e} \star e_a - \frac{1}{n_\mathrm{e}} i_{X_a} j_\mathrm{e} \wedge \star j_\mathrm{e} \right) &= m_\mathrm{e} \left(n_\mathrm{e} i_{X_a} \star 1 - n_\mathrm{e} i_{X_a} \star \widetilde{V}_\mathrm{e} \wedge \star\star \widetilde{V}_\mathrm{e} \right) \\ &= m_\mathrm{e} n_\mathrm{e} \left(i_{X_a} \star 1 - i_{X_a} \star \widetilde{V}_\mathrm{e} \wedge \widetilde{V}_\mathrm{e} \right). \end{aligned} \tag{B.134}$$

The first term of (B.134) can be rewritten using $n_\mathrm{e}^2 = j_\mathrm{e} \cdot j_\mathrm{e}$. From this, note

$$\begin{aligned} \star 1 &= \frac{1}{n_\mathrm{e}^2} \star (j_\mathrm{e} \cdot j_\mathrm{e}) = \frac{1}{n_\mathrm{e}^2} j_\mathrm{e} \wedge \star j_\mathrm{e} \\ &= \star \widetilde{V}_\mathrm{e} \wedge \widetilde{V}_\mathrm{e}, \end{aligned} \tag{B.135}$$

and thus

$$\begin{aligned} m_\mathrm{e} \left(n_\mathrm{e} \star e_a - \frac{1}{n_\mathrm{e}} i_{X_a} j_\mathrm{e} \wedge \star j_\mathrm{e} \right) &= m_\mathrm{e} n_\mathrm{e} \left(i_{X_a} \left(\star \widetilde{V}_\mathrm{e} \wedge \widetilde{V}_\mathrm{e} \right) - i_{X_a} \star \widetilde{V}_\mathrm{e} \wedge \widetilde{V}_\mathrm{e} \right) \\ &= m_\mathrm{e} n_\mathrm{e} \left(- \star \widetilde{V}_\mathrm{e} \left(i_{X_a} \widetilde{V}_\mathrm{e} \right) \right) \\ &= -m_\mathrm{e} n_\mathrm{e} i_{X_a} \widetilde{V}_\mathrm{e} \wedge \star \widetilde{V}_\mathrm{e}. \end{aligned} \tag{B.136}$$

Hence (B.133) can be written

$$\delta_e \mathcal{S} = \int_{\mathcal{M}} \delta e^a \wedge \left[-2\frac{\partial L}{\partial X} i_{X_a} F \wedge \star F - 2\frac{\partial L}{\partial Y} i_{X_a} F \wedge F - L \star e_a - m_e n_e i_{X_a} \widetilde{V}_e \star \widetilde{V}_e \right]$$
$$= \int_{\mathcal{M}} \left[2\frac{\partial L}{\partial X} i_{X_a} F \wedge \star F + 2\frac{\partial L}{\partial Y} i_{X_a} F \wedge F + L \star e_a + m_e n_e i_{X_a} \widetilde{V}_e \star \widetilde{V}_e \right] \wedge \delta e^a, \tag{B.137}$$

and by (B.84), the stress 3-forms $\mathcal{T}_a$ can be written in terms of the coframe variation as

$$\delta_e \mathcal{S} = \int_{\mathcal{M}} \mathcal{T}_a \wedge \delta e^a, \tag{B.138}$$

and comparing with (B.137), the stress forms are clealry

$$\mathcal{T}_a = 2\frac{\partial L}{\partial X} i_{X_a} F \wedge \star F + 2\frac{\partial L}{\partial Y} i_{X_a} F \wedge F + L \star e_a + m_e n_e i_{X_a} \widetilde{V}_e \star \widetilde{V}_e. \tag{B.139}$$

Since the excitation 2-form G is

$$G = 2\frac{\partial L}{\partial X} F - 2\frac{\partial L}{\partial Y} \star F, \tag{B.140}$$

the stress 3-forms can be written more simply:

$$\mathcal{T}_a = i_{X_a} F \wedge \star G + i_{X_a} \star L + m_e n_e i_{X_a} \widetilde{V}_e \star \widetilde{V}_e, \tag{B.141}$$

where (B.141) the electromagnetic components and the purely matter contribution, $\mathcal{T}_a^{\text{Matter}} = m_e n_e g(V_e, X_a) \star \widetilde{V}_e$.

B.5 Stress Balance Equation of Chapter 4

This section obtains the stress balance equation used in Chapter 4 satisfied by the nonlinear stress forms (see Section B.4.4) for a plasma with background ions.

The stress tensor associated with Killing vector K for a nonlinear theory with Lagrangian of the form $L(X, Y)$ is given by

$$\mathcal{T}_K = i_K F \wedge \star G + i_K \star L + m_e n_e i_K V_e \star \widetilde{V}_e$$
$$= i_K F \wedge \star G + i_K \star L + \frac{m_e}{q_e}(i_K V_e) j_e, \tag{B.142}$$

where

$$G = 2\left(\frac{\partial L}{\partial X}F - \frac{\partial L}{\partial Y}\star F\right), \tag{B.143}$$

with the field equations

$$d\star G = -q_{\mathrm{e}}j_{\mathrm{e}} - q_{\mathrm{ion}}j_{\mathrm{ion}}. \tag{B.144}$$

Taking the exterior derivative of the stress forms (B.142) gives

$$d\mathfrak{T}_K = d(i_K F)\wedge\star G - i_K F\wedge d\star G + d\left(i_K\star L\right) + d\left(\frac{m_{\mathrm{e}}}{q_{\mathrm{e}}}(i_K V_{\mathrm{e}})j_{\mathrm{e}}\right). \tag{B.145}$$

Since Cartan's identity gives $d i_K\alpha = \mathcal{L}_K\alpha - i_K d\alpha$ and as both $dF = 0$ and $d\star L = 0$, (B.145) can be written

$$\begin{aligned} d\mathfrak{T}_K &= \mathcal{L}_K F\wedge\star G - i_K F\wedge d\star G + \mathcal{L}_K\star L + d\left(\frac{m_{\mathrm{e}}}{q_{\mathrm{e}}}(i_K V_{\mathrm{e}})j_{\mathrm{e}}\right) \\ &= \mathcal{L}_K F\wedge\star G - i_K F\wedge d\star G + \star\mathcal{L}_K L + d\left(\frac{m_{\mathrm{e}}}{q_{\mathrm{e}}}(i_K V_{\mathrm{e}})j_{\mathrm{e}}\right), \end{aligned} \tag{B.146}$$

where the final step uses the fact that for Killing K, $\mathcal{L}_K\star = \star\mathcal{L}_K$. The expression (B.146) is to be analysed term by term. Firstly consider the Lie derivative on the Lagrangian

$$\star\mathcal{L}_K L = \star\left(\partial_X L\mathcal{L}_K X + \partial_Y L\mathcal{L}_K Y\right), \tag{B.147}$$

which after substituting X and Y in terms of F and recalling that $\star\mathcal{L}_K = \mathcal{L}_K\star$ gives

$$\begin{aligned} \star\mathcal{L}_K L &= \star\left(\partial_X L\mathcal{L}_K\star(F\wedge\star F) + \partial_Y L\mathcal{L}_K\star(F\wedge F)\right) \\ &= \star\left(\partial_X L\star\mathcal{L}_K(F\wedge\star F) + \partial_Y L\star\mathcal{L}_K(F\wedge F)\right) \\ &= -\partial_X L\mathcal{L}_K(F\wedge\star F) - \partial_Y L\mathcal{L}_K(F\wedge F). \end{aligned} \tag{B.148}$$

Expanding the Lie derivative in the first term in (B.148),

$$\mathcal{L}_K(F\wedge\star F) = (\mathcal{L}_K F)\wedge\star F + F\wedge(\mathcal{L}_K\star F) = (\mathcal{L}_K F)\wedge\star F + F\wedge(\star\mathcal{L}_K F), \tag{B.149}$$

and upon a star-pivot these terms are clearly identical (similarly for the other term in (B.148)), hence

$$\begin{aligned}\mathcal{L}_K \star L = \star i_K dL &= -2\partial_X L \mathcal{L}_K(F) \wedge \star F - \partial_Y L \mathcal{L}_K(F) \wedge F \\ &= -\mathcal{L}_K F \wedge (2\partial_X L \star F + 2\partial_Y L F) \\ &= -\mathcal{L}_K F \wedge \star G. \end{aligned} \tag{B.150}$$

This term cancels with the first term in (B.146). Hence

$$d\mathfrak{T}_K = -i_K F \wedge d \star G + d\left(\frac{m_e}{q_e}(i_K \widetilde{V}_e) j_e\right). \tag{B.151}$$

Recalling that the electron current is closed (that is $dj_e = 0$), the final term of (B.146) can be written

$$\begin{aligned} d\left(\frac{m_e}{q_e}(i_K \widetilde{V}_e) j_e\right) &= \frac{m_e}{q_e} d\left(i_K \widetilde{V}_e\right) \wedge j_e \\ &= m_e n_e d\left(i_K \widetilde{V}_e\right) \wedge i_{V_e} \star 1. \end{aligned} \tag{B.152}$$

Now exploiting properties of the interior derivative,

$$i_{V_e}\left(d\left(i_K \widetilde{V}_e\right)\right) \wedge \star 1 - d\left(i_K \widetilde{V}_e\right) \wedge i_{V_e} \star 1 = i_{V_e}\left(m_e n_e d\left(i_K \widetilde{V}_e\right) \wedge \star 1\right) = 0, \tag{B.153}$$

to rewrite (B.152) further:

$$\begin{aligned} d\left(\frac{m_e}{q_e}(i_K \widetilde{V}_e) j_e\right) &= m_e n_e i_{V_e}\left(d\left(i_K \widetilde{V}_e\right)\right) \wedge \star 1 \\ &= m_e n_e \nabla_{V_e}\left(i_K \widetilde{V}_e\right) \wedge \star 1. \end{aligned} \tag{B.154}$$

Since Killing's equation is

$$g(X, \nabla_Y K) + g(Y, \nabla_X K) = 0, \tag{B.155}$$

for all vectors X and Y, clearly $g(V_e, \nabla_{V_e} K) = 0$. Then, by (2.52), $\nabla_{V_e} i_K \widetilde{V}_e = i_K \widetilde{\nabla_{V_e} V_e}$ for Killing vectors K. Thus (B.154) is

$$d\left(\frac{m_e}{q_e}(i_K \widetilde{V}_e) j_e\right) = m_e n_e i_K \widetilde{\nabla_{V_e} V_e} \star 1. \tag{B.156}$$

Substituting in the covariant Lorentz force equation (B.83) into (B.156) gives

$$d\left(\frac{m_{\rm e}}{q_{\rm e}}(i_K\widetilde{V}_{\rm e})j_{\rm e}\right) = m_{\rm e}n_{\rm e}i_K(\frac{q_{\rm e}}{m_{\rm e}}i_{V_{\rm e}}F)\star 1 = -q_{\rm e}n_{\rm e}\left(i_{V_{\rm e}}i_K F\right)\star 1, \tag{B.157}$$

and substituting this back into (B.151) results in

$$d\mathfrak{T}_K = -i_K F\wedge d\star G - q_{\rm e}n_{\rm e}\left(i_{V_{\rm e}}i_K F\right)\star 1. \tag{B.158}$$

Since $d\star G = -q_{\rm e}n_{\rm e}\star\widetilde{V}_{\rm e} - q_{\rm ion}n_{\rm ion}\star\widetilde{V_{\rm ion}}$, the third term of (B.151) can be written;

$$\begin{aligned} i_K F\wedge d\star G &= -i_K F\wedge\left(q_{\rm e}n_{\rm e}\star\widetilde{V}_{\rm e} + q_{\rm ion}n_{\rm ion}\star\widetilde{V_{\rm ion}}\right) \\ &= -q_{\rm e}n_{\rm e}i_K F\wedge\star\widetilde{V}_{\rm e} - q_{\rm ion}n_{\rm ion}i_K F\wedge\widetilde{\star V_{\rm ion}} \\ &= -q_{\rm e}n_{\rm e}(i_{V_{\rm e}}i_K F)\star 1 - q_{\rm ion}n_{\rm ion}(i_{V_{\rm ion}}i_K F)\star 1. \end{aligned} \tag{B.159}$$

The electron piece here cancels with a term from (B.158), leaving

$$d\mathfrak{T}_K = q_{ion}n_{\rm ion}i_{V_{\rm ion}}i_K F\star 1, \tag{B.160}$$

as required.

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