Problems in point charge electrodynamics

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Abstract

This thesis consists of two parts. In part I we consider a discrepancy in the derivation of the electromagnetic self force for a point charge. The self force is given by the Abraham-von Laue vector, which consists of the radiation reaction term proportional to the 4-acceleration, and the Schott term proportional to the 4-jerk. In the point charge framework the self force can be defined as an integral of the Liénard-Wiechert stress 3-forms over a suitably defined worldtube. In order to define such a worldtube it is necessary to identify a map which associates a unique point along the worldline of the source with every field point off the worldline. One choice of map is the Dirac time, which gives rise to a spacelike displacement vector field and a Dirac tube with spacelike caps. Another choice is the retarded time, which gives rise to a null displacement vector field and a Bhabha tube with null caps. In previous calculations which use the Dirac time the integration yields the complete self force, however in previous calculations which use the retarded time the integration produces only the radiation reaction term and the Schott term is absent. We show in this thesis that the Schott term may be obtained using a null displacement vector providing certain conditions are realized.

Part II comprises an investigation into a problem in accelerator physics. In a high energy accelerator the cross-section of the beampipe is not continuous and there exist geometric discontinuities such as collimators and cavities. When a relativistic bunch of particles passes such a discontinuity the field generated by a leading charge can interact with the wall and consequently affect the motion of

trailing charges. The fields acting on the trailing charges are known as (geometric) wakefields. We model a bunch of particles as a one dimensional continuum of point charges and by calculating the accumulated Liénard-Wiechert fields we address the possibility of reducing wakefields at a collimator interface by altering the path of the beam prior to collimation. This approach is facilitated by the highly relativistic regime in which lepton accelerators operate, where the Coulomb field given from the Liénard-Wiechert potential is highly collimated in the direction of motion. It will be seen that the potential reduction depends upon the ratio of the bunch length to the width of the collimator aperture as well as the relativistic factor and path of the beam. Given that the aperture of the collimator is generally on the order of millimetres we will see that for very short bunches, on the order of hundredths of a picosecond, a significant reduction is achieved.

Author's declaration

I declare that the original ideas contained in this thesis are the result of my own work conducted in collaboration with my supervisor Dr Jonathan Gratus. An article based on the ideas in Part I has been published in Journal of Mathematical Physics (JMP) [1]. A letter describing the key results of Part II has been submitted to European Physical Letters (EPL)[2].

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Guide to Notation

All fields will be regarded as sections of tensor bundles over appropriate domains of Minkowski space $\mathcal{M}$. Sections of the tangent bundle over $\mathcal{M}$ will be denoted $\Gamma \mathrm{T}\mathcal{M}$ while sections of the bundle of exterior p-forms will be denoted $\Gamma\Lambda^p\mathcal{M}$. Given a single worldline C in free space sections over the whole of spacetime excluding the worldline will be written $\Gamma\mathrm{T}(\mathcal{M}\backslash C)$ and $\Gamma\Lambda^p(\mathcal{M}\backslash C)$. We use the SI unit convention. Appendix A provides a brief summary of the dimensions of various mathematical objects.

Chapter 1

Introduction

In this chapter we introduce the defining characteristics of Minkowski space, namely the metric and the affine structure, and the fundamental equations of Maxwell-Lorentz electrodynamics. We use the term *Maxwell-Lorentz electrodynamics* to denote the microscopic vacuum Maxwell equations, first derived by Lorentz from the macroscopic Maxwell equations (see [3, 4]) and often called the *Maxwell-Lorentz equations*, together with the Lorentz force equation. We introduce the general form of the electromagnetic stress 3-forms and show that they give rise to a set of conservation laws. A brief introduction to the necessary mathematics can be found in Appendix B.

1.1 Minkowski space

Definition 1.1.1. Minkowski space is the pseudo-Euclidean space defined by the pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is the four dimensional real vector space $\mathbb{R}^4$ and g is the Minkowski metric. With respect to a global Lorentzian coordinate basis (y^0, y^1, y^2, y^3) on $\mathcal{M}$ the Minkowski metric is defined by

$$g = -dy^0 \otimes dy^0 + dy^1 \otimes dy^1 + dy^2 \otimes dy^2 + dy^3 \otimes dy^3. \tag{1.1}$$

Lemma 1.1.2. *Given a new set of coordinates* (z^0, z^1, z^2, z^3) *on* $\mathcal{M}$ *we write* g *in* *terms of the new basis using the transformations (B.19),*

$$g = g_{ab} dy^a \otimes dy^b = g_{ab} \frac{\partial y^a}{\partial z^c} \frac{\partial y^b}{\partial z^d} dz^c \wedge dz^d = g^{(z)}_{cd} dz^c \wedge dz^d \tag{1.2}$$

where

$$g^{(z)}_{cd} = g_{ab} \frac{\partial y^a}{\partial z^c} \frac{\partial y^b}{\partial z^d}. \tag{1.3}$$

Lemma 1.1.3. *The coordinate basis (y^0, y^1, y^2, y^3) on $\mathcal{M}$ naturally gives rise to the basis (dy^0, dy^1, dy^2, dy^3) of $1-$forms on $\mathrm{T}^*\mathcal{M}$. This forms a g-orthonormal basis and from definition B.2.5 it follows*

$$\star 1 = dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \tag{1.4}$$

is the volume form on $\mathcal{M}$. In terms of a different coordinate basis (z^0, z^1, z^2, z^3) it is given by

$$\star 1 = \sqrt{|\det(g^z)|} dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3 \tag{1.5}$$

where g^z is the matrix of metric components $g^{(z)}_{ab}$ defined by (1.3).

Definition 1.1.4. Let V be a vector field over set of points M. If the map

$$M \times M \longrightarrow V, \qquad (x, y) \longrightarrow x - y, \tag{1.6}$$

exists and satisfies

$$\begin{aligned} &1. \quad \text{For all} \quad x \in M, \quad \text{for all} \quad v \in V \\ &\quad\;\; \text{there exists} \quad y \in M \quad \text{such that} \quad y - x = v, \\ &2. \quad \text{For all} \quad x, y, z \in M, \quad (x - y) + (z - x) = z - y, \end{aligned} \tag{1.7}$$

then M is an affine space.

For any integer n the space $\mathbb{R}^n$ is affine. It follows that Minkowski space is an affine space. In some calculations it will be necessary to endow Minkowski space with an origin, thus transforming it into a vector space, however the results of such calculations will not depend on the vector space structure but only the affine structure.

1.2 Maxwell-Lorentz Equations

The equations which describe the interaction between matter and the electromagnetic field were first formulated by Maxwell in 1865[5]. Maxwell's equations form a continuum theory of electrodynamics due to their origins in macroscopic experiment. In this thesis we are interested in the interaction of point charges and their fields, therefore we need equations which are valid on the microscopic scale.

Definition 1.2.1. The Maxwell-Lorentz equations, or the microscopic vacuum Maxwell equations are given by

$$d\mathcal{F} = 0 \tag{1.8}$$

$$\epsilon_0 d \star \mathcal{F} = \mathcal{J} \tag{1.9}$$

where $\mathcal{F} \in \Gamma\Lambda^2\mathcal{M}$ is the electromagnetic 2-form, $\mathcal{J} \in \Gamma\Lambda^3\mathcal{M}$ is the current 3-form and ϵ_0 is the permittivity of free space. If we introduce the 1-form potential

$$\mathcal{A} \in \Gamma\Lambda^1\mathcal{M} \qquad \text{such that} \qquad \mathcal{F} = d\mathcal{A}, \tag{1.10}$$

then (1.8) and (1.9) reduce to the single equation

$$\epsilon_0 d \star d\mathcal{A} = \mathcal{J}, \tag{1.11}$$

where (1.8) is satisfied automatically because the double action of the exterior derivative is zero.

Definition 1.2.2. In terms of distributional forms (see C.1.2)

$$\mathcal{A}^D \in \Gamma_D \Lambda^1 \mathcal{M}, \qquad \mathcal{F}^D = d\mathcal{A}^D, \tag{1.12}$$

Maxwell's second equation is given by

$$\epsilon_0 d \star d\mathcal{A}^D[\varphi] = \mathcal{J}^D[\varphi] = \int_{\mathcal{M}} \varphi \wedge \mathcal{J}. \tag{1.13}$$

where $\varphi \in \Gamma_0 \Lambda^1 \mathcal{M}$ is any test 1-form (see C.1.1).

$3 + 1$ decomposition

Definition 1.2.3. Given any velocity vector field $U \in \Gamma \mathrm{T} \mathcal{M}$ satisfying $g(U, U) = -1$, the electromagnetic 2-form $\mathcal{F}$ may be written

$$\mathcal{F} = \widetilde{\mathcal{E}} \wedge \widetilde{U} + c\mathbf{B}, \tag{1.14}$$

where $\widetilde{\mathcal{E}} \in \Gamma \Lambda^1 \mathcal{M}$ and $\mathbf{B} \in \Gamma \Lambda^2 \mathcal{M}$ are the electric 1-form and magnetic 2-form associated with U and $\mathcal{F}$, and satisfy

$$i_U \widetilde{\mathcal{E}} = i_U \mathbf{B} = 0. \tag{1.15}$$

Here $\widetilde{\ }$ is the metric dual operator defined by (B.29) and c is the speed of light in a vacuum.

Lemma 1.2.4. *According to observers whose worldlines coincide with integral curves of U, the electric field $\mathcal{E} \in \Gamma \mathrm{T} \mathcal{M}$ is given by*

$$\mathcal{E} = \widetilde{i_U \mathcal{F}}, \tag{1.16}$$

Proof of 1.2.4. Follows trivially from (1.14) ☐

Definition 1.2.5. We may use the vector field U to write the Minkowski metric g in terms of a metric $\underline{g}$ on the instantaneous 3-spaces

$$g = -\widetilde{U} \otimes \widetilde{U} + \underline{g}. \tag{1.17}$$

Let $\#$ be the Hodge map associated with the instantaneous 3-space such that for $\alpha \in \Gamma\Lambda^p\mathcal{M}$

$$\# : \Gamma\Lambda^p\mathcal{M} \to \Gamma\Lambda^{3-p}\mathcal{M}, \qquad \alpha \mapsto \#\alpha = (-1)^{p+1} i_U \star \alpha \tag{1.18}$$

The Minkowski Hodge dual is then given by

$$\star\alpha = (-1)^p \widetilde{U} \wedge \#\alpha. \tag{1.19}$$

Lemma 1.2.6. *The Hodge dual of $\mathcal{F}$ is given by*

$$\star\mathcal{F} = \#\widetilde{\mathcal{E}} - c\#\mathbf{B} \wedge \widetilde{U} \tag{1.20}$$

Proof of 1.2.6.

$$\star\mathcal{F} = \star(\widetilde{\mathcal{E}} \wedge \widetilde{U}) + c \star \mathbf{B}, \tag{1.21}$$

It follows from (1.18) that if $\alpha \in \Lambda^1\mathcal{M}$ then $\#\alpha = i_U \star \alpha = \star(\alpha \wedge \widetilde{U})$. Thus $\star(\widetilde{\mathcal{E}} \wedge \widetilde{U}) = \#\widetilde{\mathcal{E}}$. Similarly it follows from (1.19) that if $\beta \in \Lambda^2\mathcal{M}$ then $\star\beta = \widetilde{U} \wedge \#\beta$, thus $\star\mathbf{B} = \widetilde{U} \wedge \#\mathbf{B}$. □

Lemma 1.2.7. *Let $\widetilde{\mathcal{B}} = -\#\mathbf{B}$ where $\mathcal{B} \in \Gamma\mathrm{T}\mathcal{M}$ is the magnetic field, then according to observers whose worldlines coincide with integral curves of U it is given by*

$$\mathcal{B} = \frac{1}{c}\widetilde{i_U \star \mathcal{F}}, \qquad . \tag{1.22}$$

Proof of 1.2.7. Consider (1.20). Since $i_U\#\alpha = i_U\#\beta = 0$ it follows that $i_U \star \mathcal{F} = -c\#\mathbf{B}$. □

Lemma 1.2.8. *In terms of $\mathcal{E}$ and $\mathcal{B}$ the 2-forms $\mathcal{F}$ and $\star\mathcal{F}$ are given by*

$$\mathcal{F} = \widetilde{\mathcal{E}} \wedge \widetilde{U} - c\#\widetilde{\mathcal{B}} = \widetilde{\mathcal{E}} \wedge \widetilde{U} - c \star (\widetilde{\mathcal{B}} \wedge \widetilde{U}), \tag{1.23}$$

$$\star\mathcal{F} = \#\widetilde{\mathcal{E}} + c\widetilde{\mathcal{B}} \wedge \widetilde{U} = \star(\widetilde{\mathcal{E}} \wedge \widetilde{U}) + c\widetilde{\mathcal{B}} \wedge \widetilde{U}. \tag{1.24}$$

Proof of 1.2.8. Since $\widetilde{\mathcal{B}} = -\#\mathbf{B}$ and $\#\#\mathbf{B} = \mathbf{B}$ it follows that $\#\widetilde{\mathcal{B}} = -\mathbf{B}$. Substituting this into (1.14) yields (1.23). Similarly substituting the first relation into (1.20) yields (1.24). □

The Lorentz Force

Definition 1.2.9. Let $C : I \subset \mathbb{R} \to \mathcal{M}$ be the proper time parameterized inextendible worldline of a point particle with observed rest mass m and charge q. For $\tau \in I$

$$\dot{C} = C_*(d/d\tau), \qquad \ddot{C} = \nabla_{\dot{C}}\dot{C}, \qquad \text{and} \qquad \dddot{C} = \nabla_{\dot{C}}\nabla_{\dot{C}}\dot{C} \tag{1.25}$$

are the velocity, acceleration and jerk of the particle respectively. Here the push-forward map $_*$ is defined by (B.79)-(B.83) and ∇ is the Levi-Civita connection (see B.2). In this introductory chapter and in Part II we assign the dimension of time to proper time τ such that

$$g(\dot{C}, \dot{C}) = -c^2, \tag{1.26}$$

However the reader should note that Part I we will find it convenient to assign the dimension of length to proper time so that

$$g(\dot{C}, \dot{C}) = -1, \tag{1.27}$$

For further details about dimensions see appendix A.

Lemma 1.2.10.

$$g(\dot{C}, \ddot{C}) = 0, \tag{1.28}$$

$$\text{and} \qquad g(\dot{C}, \dddot{C}) = -g(\ddot{C}, \ddot{C}). \tag{1.29}$$

Proof of 1.2.10. Equation (1.28) follows by differentiating (1.26) with respect to τ. Similarly, equation (1.29) follows by differentiating (1.28). □

Definition 1.2.11. The force on a point particle with worldline $C(\tau)$ due to an external field $\mathrm{F}_{\mathrm{ext}} \in \Gamma\Lambda^2\mathcal{M}$ is given by the Lorentz force f_{L}, where

$$f_{\mathrm{L}} \in \Gamma\mathrm{T}\mathcal{M}, \qquad f_{\mathrm{L}} = \frac{q}{c}\widetilde{i_{\dot{C}}\mathrm{F}_{\mathrm{ext}}}. \tag{1.30}$$

In 1916 Lorentz writes[6]

> Like our former equations [Maxwell's equations], it is got by generalizing the results of electromagnetic experiments

1.3 Conservation Laws

Definition 1.3.1. A vector field V is a Killing field if it satisfies

$$\mathcal{L}_V g = 0. \tag{1.31}$$

In terms of coordinate basis $\{y^i\}$ the metric may be written $g = g_{ab}(y^i)dy^a \otimes dy^b$, thus for vector field $V = \frac{\partial}{\partial y^a}$ the left hand side of (1.31) yields

$$\mathcal{L}_{\partial_{\mathrm{K}}} g = \frac{\partial g_{ab}}{\partial y^{\mathrm{K}}} dy^a \otimes dy^b, \tag{1.32}$$

where $\partial_{\mathrm{K}} = \frac{\partial}{\partial y^{\mathrm{K}}}$. In Minkowski space $g_{00} = -1$ and $g_{ab} = \delta^a_b$ for $a = 1, 2, 3$. Thus for the four translational vectors $\frac{\partial}{\partial y^0}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}$ (1.31) is trivially satisfied. In fact there are 10 killing vector fields on Minkowski space.

Let V be a Killing vector, then another property of Killing fields we shall use is

$$\mathcal{L}_V \star = \star \mathcal{L}_V, \tag{1.33}$$

Definition 1.3.2. The electromagnetic stress 3-forms $\mathcal{S}_{\mathrm{K}} \in \Gamma\Lambda^3\mathcal{M}$ are given by

$$\mathcal{S}_{\mathrm{K}} = \frac{\epsilon_0}{2c}\big(i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge \star\mathcal{F} - i_{\partial_{\mathrm{K}}} \star \mathcal{F} \wedge \mathcal{F}\big) \tag{1.34}$$

where $\partial_{\mathrm{K}} = \frac{\partial}{\partial y^{\mathrm{K}}}$ are the four translational Killing vectors. These 3-forms can be obtained from the Lagrangian density for the electromagnetic field using Noether's theorem, see [7] for a detailed exposition. The stress forms are related to the symmetric stress-energy-momentum tensor $\mathcal{T} \in \Gamma\bigotimes^{[\mathbb{V},\mathbb{V}]}\mathcal{M}$ by

$$\mathcal{T}^{a\mathrm{K}} = i_{\frac{\partial}{\partial y^a}} \star \mathcal{S}_{\mathrm{K}}, \qquad \mathcal{S}_{\mathrm{K}} = \star\Big(\big(\mathcal{T}(dy^{\mathrm{K}}, -)\big)^{\sim}\Big) \tag{1.35}$$

where $\mathcal{T} = \mathcal{T}^{ab}\dfrac{\partial}{\partial y^a} \otimes \dfrac{\partial}{\partial y^b}$.

Lemma 1.3.3. *The stress forms satisfy*

$$d\mathcal{S}_{\mathrm{K}} = -\frac{1}{c} i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge \mathcal{J}, \tag{1.36}$$

and thus for any source free region $N \subset \mathcal{M}$

$$d\mathcal{S}_{\mathrm{K}} = 0. \tag{1.37}$$

Proof of 1.3.3.

$$
\begin{aligned}
d\mathcal{S}_{\mathrm{K}} =& \frac{\epsilon_0}{2c} d\big(i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge \star\mathcal{F} - i_{\partial_{\mathrm{K}}} \star \mathcal{F} \wedge \mathcal{F}\big) \\
=& \frac{\epsilon_0}{2c}\big(d i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge \star\mathcal{F} - i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge d \star \mathcal{F} - d i_{\partial_{\mathrm{K}}} \star \mathcal{F} \wedge \mathcal{F} + i_{\partial_{\mathrm{K}}} \star \mathcal{F} \wedge d\mathcal{F}\big) \\
=& \frac{\epsilon_0}{2c}\big(d i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge \star\mathcal{F} - d i_{\partial_{\mathrm{K}}} \star \mathcal{F} \wedge \mathcal{F}\big) - \frac{\epsilon_0}{2c} i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge d \star \mathcal{F}.
\end{aligned}
\tag{1.38}
$$

From (1.8) and (B.73) it follows that

$$
\mathcal{L}_{\partial_{\mathrm{K}}}\mathcal{F} = d i_{\partial_{\mathrm{K}}}\mathcal{F}. \tag{1.39}
$$

Using (1.39), (B.2.11) and (1.33) respectively yields

$$
d i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge \star\mathcal{F} = \mathcal{F} \wedge \star d i_{\partial_{\mathrm{K}}}\mathcal{F} = \mathcal{F} \wedge \star\mathcal{L}_{\partial_{\mathrm{K}}}\mathcal{F} = \mathcal{F} \wedge \mathcal{L}_{\partial_{\mathrm{K}}} \star \mathcal{F} = \mathcal{F} \wedge d i_{\partial_{\mathrm{K}}} \star \mathcal{F} + \mathcal{F} \wedge i_{\partial_{\mathrm{K}}} d \star \mathcal{F} \tag{1.40}
$$

Substituting (1.40) into (1.38) yields

$$
\begin{aligned}
d\mathcal{S}_{\mathrm{K}} =& \frac{\epsilon_0}{2c}\big(\mathcal{F} \wedge d i_{\partial_{\mathrm{K}}} \star \mathcal{F} + \mathcal{F} \wedge i_{\partial_{\mathrm{K}}} d \star \mathcal{F} - d i_{\partial_{\mathrm{K}}} \star \mathcal{F} \wedge \mathcal{F}\big) - \frac{\epsilon_0}{2c} i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge d \star \mathcal{F} \\
=& \frac{\epsilon_0}{2c}\big(\mathcal{F} \wedge i_{\partial_{\mathrm{K}}} d \star \mathcal{F} - i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge d \star \mathcal{F}\big)
\end{aligned}
\tag{1.41}
$$

Since $\mathcal{F}$ is a 2-form and $d \star \mathcal{F}$ is a 3-form it follows that

$$
i_{\partial_{\mathrm{K}}}(\mathcal{F} \wedge d \star \mathcal{F}) = i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge d \star \mathcal{F} + \mathcal{F} \wedge i_{\partial_{\mathrm{K}}} d \star \mathcal{F} = 0. \tag{1.42}
$$

Thus substituting $\mathcal{F} \wedge i_{\partial_{\mathrm{K}}} d \star \mathcal{F} = -i_{\partial_{\mathrm{K}}}\mathcal{F} \wedge d \star \mathcal{F}$ and (1.9) into (1.41) yields result.
□

Lemma 1.3.4. *For any source free region $N \subset \mathcal{M}$*

$$
\int_{\partial\mathcal{N}} \mathcal{S}_{\mathrm{K}} = \int_{\mathcal{N}} d\mathcal{S}_{\mathrm{K}} = 0. \tag{1.43}
$$

Proof of 1.3.4. Follows trivially from (1.37) and Stokes' theorem (B.103). □

Lemma 1.3.5. *If U is a timelike Killing vector then applying the 3+1 decomposition yields*

$$\mathcal{S}_U = \epsilon_0 \widetilde{\mathcal{E}} \wedge \widetilde{\mathcal{B}} \wedge \widetilde{U} + \frac{\epsilon_0}{2c}(\widetilde{\mathcal{E}} \wedge \#\widetilde{\mathcal{E}} + c^2 \widetilde{\mathcal{B}} \wedge \#\widetilde{\mathcal{B}}), \tag{1.44}$$

where $\epsilon_0 \widetilde{\mathcal{E}} \wedge \widetilde{\mathcal{B}}$ is the Poynting 2-form, and $\frac{\epsilon_0}{2c}(\widetilde{\mathcal{E}} \wedge \#\widetilde{\mathcal{E}} + c^2 \widetilde{\mathcal{B}} \wedge \#\widetilde{\mathcal{B}})$ the energy density 3-form.

Proof of 1.3.5.

Using definition 1.3.2

$$\mathcal{S}_U = \frac{\epsilon_0}{2c}\big(i_{\partial_U}\mathcal{F} \wedge \star\mathcal{F} - i_{\partial_U} \star \mathcal{F} \wedge \mathcal{F}\big)$$

Substituting (1.24) and (1.23) and using the relations (1.16) and (1.22) yields

$$\begin{aligned}\mathcal{S}_U =& \frac{\epsilon_0}{2c}\big(\widetilde{\mathcal{E}} \wedge (\#\widetilde{\mathcal{E}} + c\widetilde{\mathcal{B}} \wedge \widetilde{U}) - c\widetilde{\mathcal{B}} \wedge (\widetilde{\mathcal{E}} \wedge \widetilde{U} - c\#\widetilde{\mathcal{B}})\big),\\ =& \frac{\epsilon_0}{2c}\Big(\widetilde{\mathcal{E}} \wedge \#\widetilde{\mathcal{E}} + c^2\widetilde{\mathcal{B}} \wedge \#\widetilde{\mathcal{B}}\Big) + \epsilon_0 \widetilde{\mathcal{E}} \wedge \widetilde{\mathcal{B}} \wedge \widetilde{U}.\end{aligned}$$

□

1.4 The source $\mathcal{J}$ for a point charge

We now consider the particular form of the current 3-form $\mathcal{J} \in \Lambda^3\mathcal{M}$ for a point charge. We use notation $\mathrm{J} = \mathcal{J}_{\text{point charge}}$ in order to emphasize that J is a particular choice for $\mathcal{J}$. The source is located only on the worldline of the particle therefore we expect the source distribution $\mathrm{J}^D \in \Gamma_D\Lambda^3\mathcal{M}$ to have the form of a Dirac delta distribution.

Definition 1.4.1. Given the four 0-form distributions $j^a \in \Gamma_D\Lambda^0\mathcal{M}$, where for $x \in \mathcal{M}$

$$j^a(x) = q\int_\tau \dot{C}^a(\tau)\delta^{(4)}(x - C(\tau))d\tau, \tag{1.45}$$

we define the distributional current vector field by

$$j = j^a(x)\frac{\partial}{\partial y^a}, \tag{1.46}$$

The distributions $j^a(x)$ are non-zero only when $x = C(\tau)$. The 3-form $\mathrm{J} \in \Lambda^3\mathcal{M}$ is given by

$$\mathrm{J} = \star\widetilde{j}. \tag{1.47}$$

Lemma 1.4.2. *The current* 3*-form distribution* $\mathrm{J}^D \in \Gamma_D\Lambda^3\mathcal{M}$ *is given by*

$$\mathrm{J}^D[\varphi] = q\int_I C^*\varphi \tag{1.48}$$

for any test 1*-form* $\varphi \in \Gamma_0\Lambda^1\mathcal{M}$.

Proof of 1.4.2.

From (1.47) the distribution J^D is given by

$$\begin{aligned}
\mathrm{J}^D[\varphi] &= \int_{\mathcal{M}} \star\widetilde{j} \wedge \varphi, \\
&= \int_{\mathcal{M}} i_j \star 1 \wedge \varphi, \\
&= \int_{\mathcal{M}} j^a i_{\frac{\partial}{\partial x^a}}\varphi \star 1, \\
&= \int_{\mathcal{M}} j^a \varphi_a \star 1.
\end{aligned}$$

Substitution of (1.45) yields

$$\begin{aligned}
\mathrm{J}^D[\varphi] &= q\int_{\mathcal{M}}\int_{\tau} \dot{C}^a(\tau)\delta^{(4)}(x - C(\tau))d\tau\varphi_a \star 1 \\
&= q\int_{\tau} \dot{C}^a(\tau)\varphi_a(C(\tau))d\tau \\
&= q\int \varphi_a(C(\tau))\frac{dC^a}{d\tau}d\tau \\
&= q\int \varphi_a(C(\tau))dC^a \\
&= q\int \varphi_a(C(\tau))C^*(dy^a) \\
&= q\int_I C^*\varphi
\end{aligned}$$

where $C^*(y^a) = y^a \circ C = C^a$. □

1.5 Worldline geometry

Given the proper time parameterized inextendible worldline

$$C : I \subset \mathbb{R} \to \mathcal{M}, \quad \tau \mapsto C(\tau), \tag{1.49}$$

we required a way to locally map each point $x \in (\mathcal{M}\backslash C)$ to a unique point $C(\tau'(x))$ along the worldline. Consider the region $N = \widetilde{N}\backslash C$ where $\widetilde{N} \subset \mathcal{M}$ is a local neighborhood of the worldline. The affine structure of $\mathcal{M}$ permits the construction of a unique displacement vector $Z|_x$ defined as the difference between the two points (see figure 1.1),

$$Z|_x = x - C(\tau'(x)). \tag{1.50}$$

Note that the definition of Z only requires the affine structure of $\mathcal{M}$. It does not require $\mathcal{M}$ to be converted into a vector space by assigning an origin.

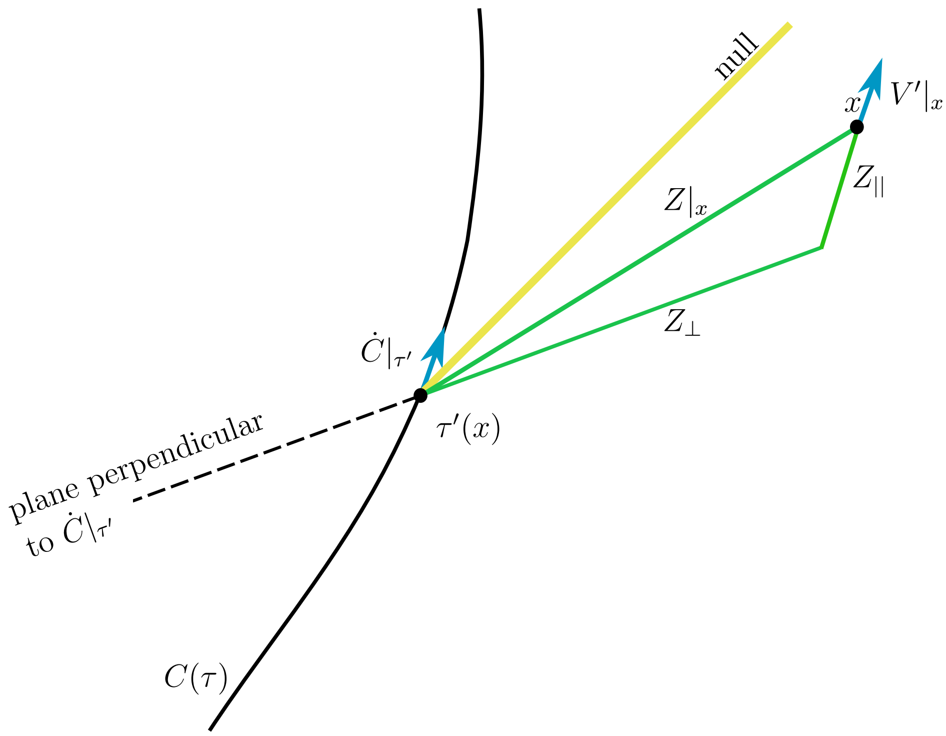


Figure 1.1: Displacement vector $Z|_x$

We may construct a local vector field $Z \in \Gamma \mathrm{T} N$ such that

$$Z = Z^a \frac{\partial}{\partial y^a}, \qquad \text{where} \qquad Z^a = x^a - C^a(\tau'(x)). \tag{1.51}$$

for all $x \in N$. Here $y^a(x) = x^a$.

Since Z is defined for every $x \in N$ the only requirement needed to define Z completely is to fix $\tau'(x)$. We are free to choose $\tau'(x)$ in any way we like however particular choices are beneficial for certain problems. In one choice the vector Z lies in the plane perpendicular to $\dot{C}(\tau'(x))$ (see figure 1.2). In this case we use the notation $\tau' = \tau_D$, where τ_D is the *Dirac time*. The Dirac time associates each point $x \in N$ with the time $\tau_D(x)$ given by the solution to

$$g\big(x - C(\tau_D(x)), \dot{C}(\tau_D(x))\big) = 0. \tag{1.52}$$

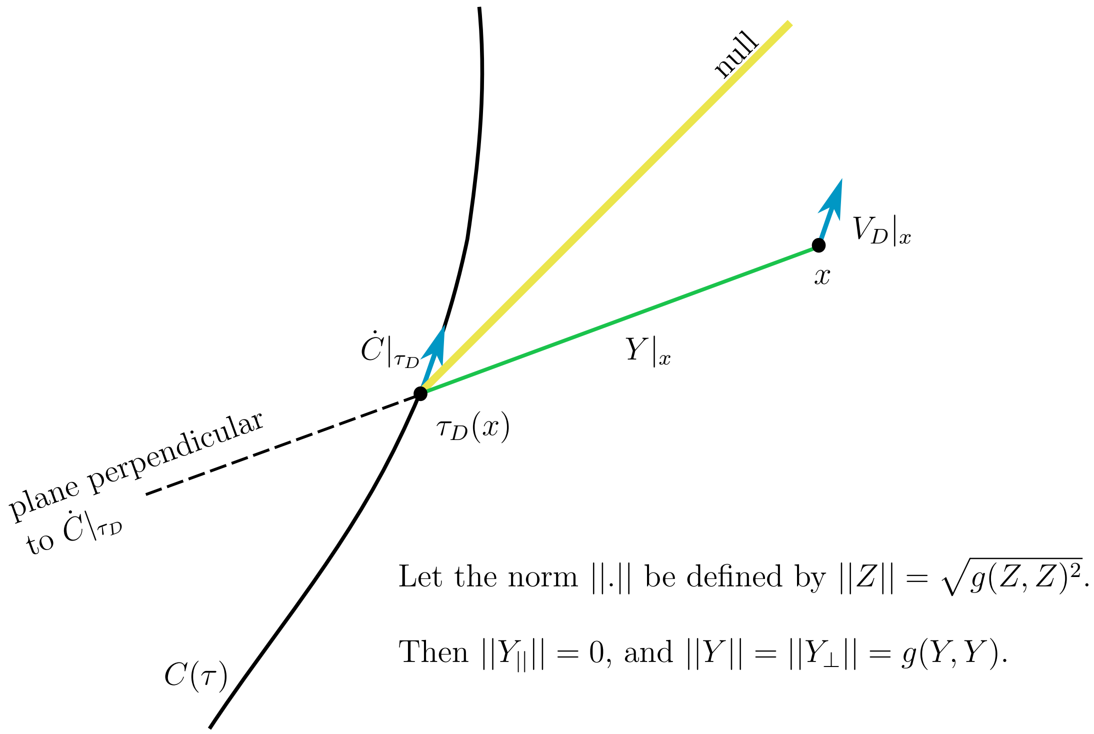


Figure 1.2: Displacement vector $Y|_x$.

In this case $Z|_x = Z_\perp = x - C(\tau_D(x))$. We use the special notation

$$Y = x - C(\tau_D(x)). \tag{1.53}$$

The map $\tau_D : \mathcal{M} \to C$ is not unique for every $x \in \mathcal{M}$, for example in figure 1.3 we see that a single point can be mapped to multiple points along the worldline. However for a sufficiently small neighborhood $N \subset (\mathcal{M} \backslash C)$ uniqueness can be ensured. In appendix D we explore this geometry further.

In another choice the vector Z lies on the null cone (see figure 1.4). In this case we use the notation $\tau' = \tau_r$, where τ_r is the *retarded time*. The retarded time associates a point $x \in (\mathcal{M} \backslash C)$ with the time $\tau_r(x)$ given by the solution to

$$g\big(x - C(\tau_r(x)), x - C(\tau_r(x))\big) = 0, \qquad x^0 > C^0(\tau_r(x)) \tag{1.54}$$

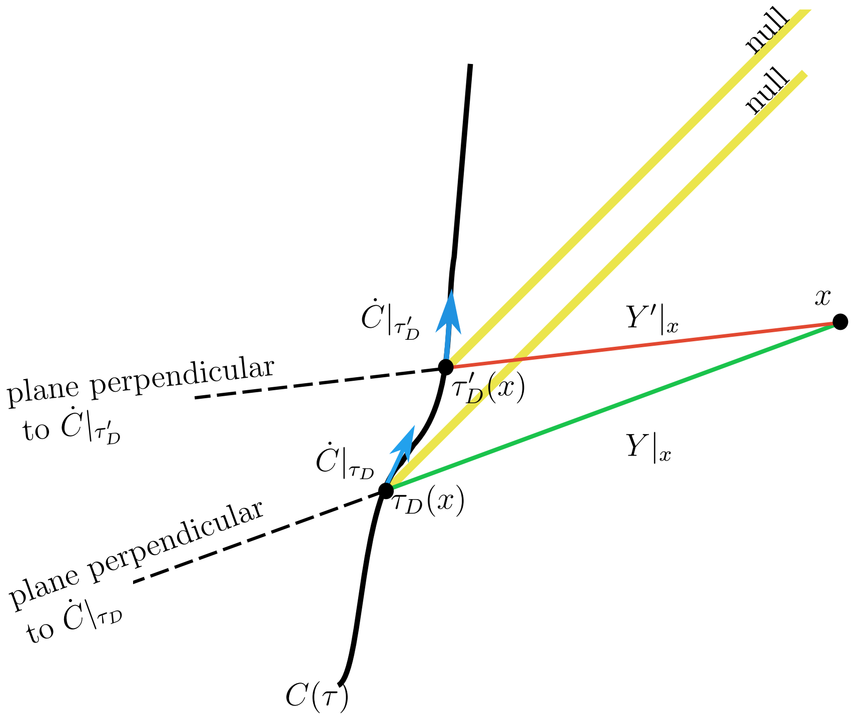


Figure 1.3: Globally the map τ_D is non-unique.

In this case $Z|_x = x - C(\tau_r(x))$. We use the special notation

$$X = x - C(\tau_r(x)). \tag{1.55}$$

There is another possible choice in which Z is a vector in the advanced null cone at x (see figure 1.4). In this case we use the notation $\tau' = \tau_a$, where τ_a is the *advanced time*. The advanced time associates a point $x \in (\mathcal{M} \backslash C)$ with the time $\tau_a(x)$ given by the solution to

$$g\big(x - C(\tau_a(x)), x - C(\tau_a(x))\big) = 0, \qquad x^0 < C^0(\tau_r(x)) \tag{1.56}$$

In this case $Z|_x = x - C(\tau_a(x))$. We use the special notation

$$W = x - C(\tau_a(x)). \tag{1.57}$$

The maps $\tau_r(x)$ and $\tau_a(x)$ are not necessarily defined for all $X \in \mathcal{M}$. Figure 1.6 shows the path of a curve undergoing constant acceleration. The backwards light

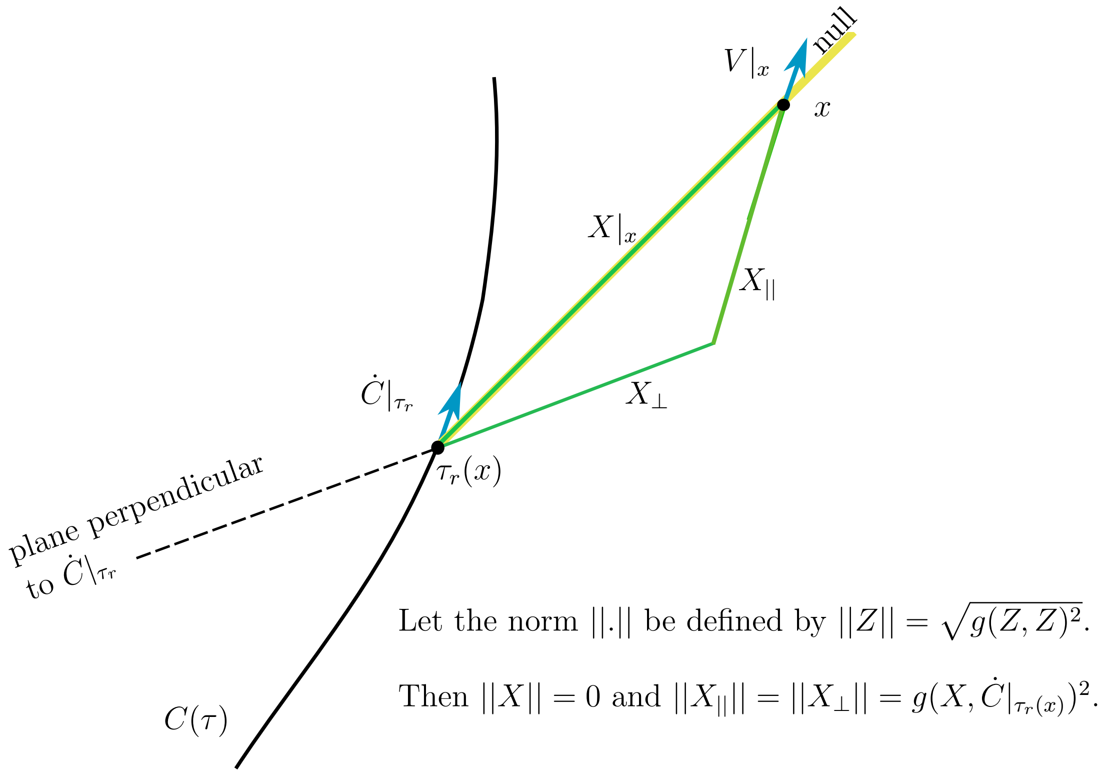


Figure 1.4: Displacement vector $X|_x$.

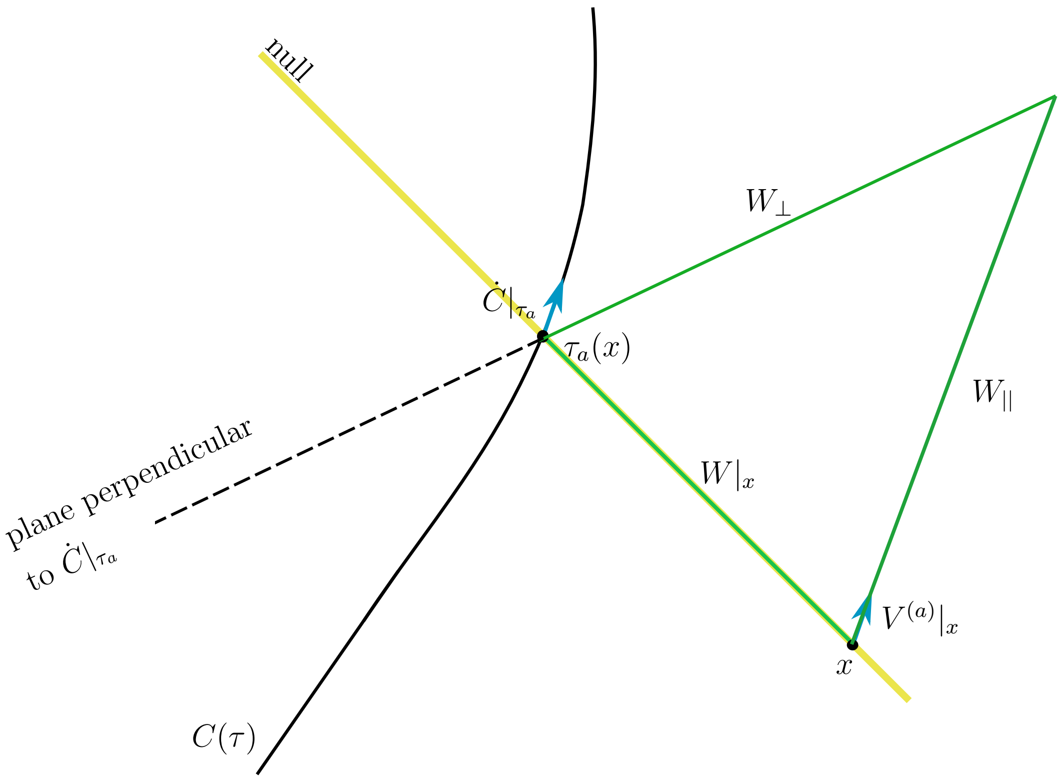


Figure 1.5: Displacement vector $W|_x$

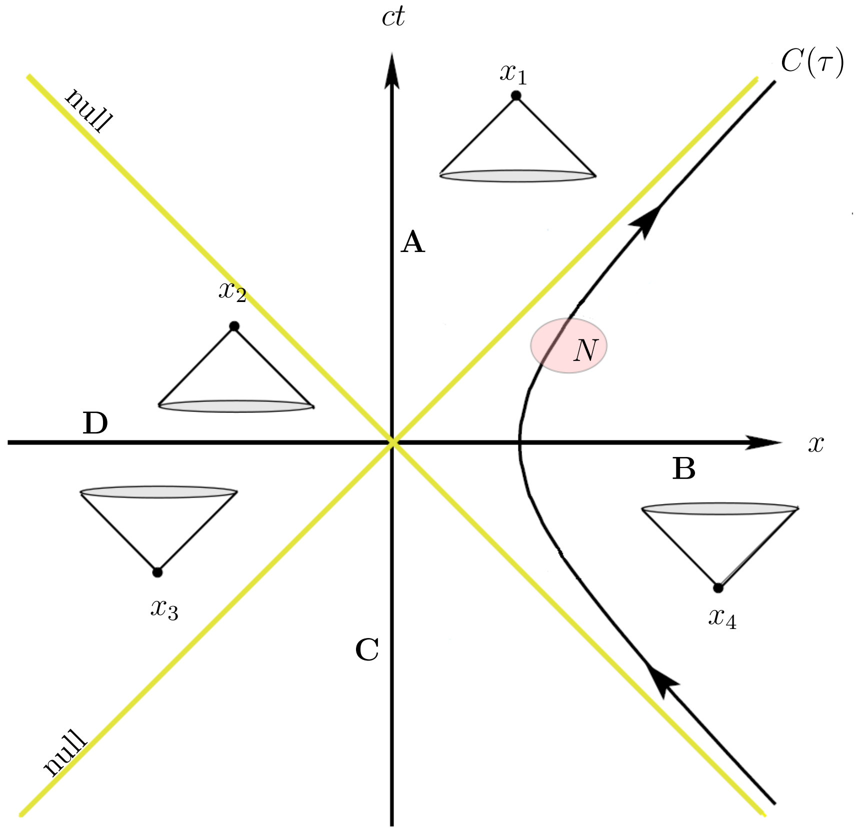


Figure 1.6: Globally the retarded and advanced times are not necessarily defined for all $x \in (\mathcal{M} \backslash C)$.

cone from an arbitrary point in quadrant **A** or **B** intersects the worldline once in quadrant **B**, hence the retarded map $\tau_r(x)$ is well defined in **A** and **B**. However the backwards light cone from any point in quadrants **C** or **D** will never intersect the worldline, therefore the map $\tau_r(x)$ is not defined for $x \in \mathbf{C}$ or $x \in \mathbf{D}$. We can ensure the existence and uniqueness of the maps τ_r and τ_a by working exclusively in a sufficiently small (and appropriately chosen) neighbourhood $N \subset (\mathcal{M} \backslash C)$.

Null geometry

In this section we explore further the consequences of choosing $\tau' = \tau_r$. This map is particularly suited to electromagnetic phenomena which propagate on the light cone. We call the resulting geometry *null geometry*. We begin by consolidating equations (1.49), (1.51), (1.54) and (1.55).

Definition 1.5.1. Given the one-parameter curve $C(\tau)$ which traces the path of a point charge in spacetime, then for every field point $x \in (\mathcal{M}\backslash C)$ there is at most one point $\tau_r(x)$ at which the worldline crosses the retarded light-cone with apex at x.

$$C : \mathbb{R} \to \mathcal{M}, \quad \tau \mapsto C(\tau) \tag{1.58}$$

$$\tau_r : \mathcal{M} \to \mathbb{R}, \quad x \mapsto \tau_r(x) \tag{1.59}$$

Definition 1.5.2. The null vector $X \in \Gamma\mathrm{T}(\mathcal{M}\backslash C)$ is given by the difference between the field point x and the worldline point $C(\tau_r(x))$

$$X|_x = x - C\big(\tau_r(x)\big), \tag{1.60}$$

where

$$g(X, X) = g\big(x - C(\tau_r(x)), x - C(\tau_r(x))\big) = 0. \tag{1.61}$$

Definition 1.5.3. The vector fields $V, A, \dot{A} \in \Gamma\mathrm{T}(\mathcal{M}\backslash C)$ are defined as

$$V|_x = \dot{C}^j(\tau_r(x))\frac{\partial}{\partial y^j}, \quad A|_x = \ddot{C}^j(\tau_r(x))\frac{\partial}{\partial y^j} \quad \text{and} \quad \dot{A}|_x = \dddot{C}^j(\tau_r(x))\frac{\partial}{\partial y^j}, \tag{1.62}$$

hence from lemma 1.2.10 it follows

$$g(V, V) = -c^2, \quad g(A, V) = 0, \quad \text{and} \quad g(\dot{A}, V) = -g(A, A). \tag{1.63}$$

Definition 1.5.4. We define the normalized null vector field by

$$n = \frac{X}{R}, \qquad \text{where} \qquad R = -g(X, V) \tag{1.64}$$

The normalized vector satisfies

$$g(n,n)=0 \qquad \text{and} \qquad g(n,V)=-1. \tag{1.65}$$

Lemma 1.5.5. *The exterior derivative of the retarded proper time τ_r is given by*

$$d\tau_r = \frac{\widetilde{X}}{g(X,V)}. \tag{1.66}$$

Proof of 1.5.5.

Definition 1.5.2 requires only the affine structure of $\mathcal{M}$. For the following proof we demand the stronger requirement that the points $x \in \mathcal{M}$ and $C(\tau_r(x)) \in C(\tau) \subset \mathcal{M}$ are attributed with a vector structure on $\mathcal{M}$, such that

$$\mathbf{x} \in \Gamma\mathrm{T}\mathcal{M}, \qquad \mathbf{x}|_x = x^a\frac{\partial}{\partial y^a} \qquad \text{and} \qquad \mathbf{C} \in \Gamma\mathrm{T}\mathcal{M}, \qquad \mathbf{C}|_x = C^a(\tau_r(x))\frac{\partial}{\partial y^a}, \tag{1.67}$$

however the result (1.66) requires only the affine structure.

We begin with the light cone condition

$$\begin{aligned} 0 &= g(X,X), \\ &= g(\mathbf{x}-\mathbf{C},\mathbf{x}-\mathbf{C}), \\ &= g(\mathbf{x}-\mathbf{C},\mathbf{x}) - g(\mathbf{x}-\mathbf{C},\mathbf{C}), \\ &= g(\mathbf{x},\mathbf{x}) - 2g(\mathbf{C},\mathbf{x}) + g(\mathbf{C},\mathbf{C}). \end{aligned} \tag{1.68}$$

Therefore

$$\begin{aligned} 0 &= d\Big[g(\mathbf{x},\mathbf{x}) - 2g(\mathbf{C},\mathbf{x}) + g(\mathbf{C},\mathbf{C})\Big], \\ &= dg(\mathbf{x},\mathbf{x}) - 2dg(\mathbf{C},\mathbf{x}) + dg(\mathbf{C},\mathbf{C}). \end{aligned} \tag{1.69}$$

Now the first term in (1.69) yields

$$\begin{aligned} dg(\mathbf{x},\mathbf{x}) &= d(g_{ab}x^a x^b), \\ &= g_{ab}(dx^a)x^b + g_{ab}x^a(dx^b), \end{aligned}$$

Note that $x^a = y^a(x)$ thus $dx^a = dy^a$ and therefore

$$\begin{aligned} dg(\mathbf{x},\mathbf{x}) &= x_a dy^a + x_a dy^a, \\ &= 2x_a dy^a, \\ &= 2\widetilde{\mathbf{x}}. \end{aligned} \tag{1.70}$$

Similarly the second term yields

$$\begin{aligned} dg(\mathbf{C},\mathbf{x}) &= d(g_{ab}x^a C^b(\tau_r)), \\ &= g_{ab}(dx^a)C^b(\tau_r) + g_{ab}x^a d(C^b(\tau_r)), \\ &= C_a(\tau_r)dy^a + x_a d(C^a(\tau_r)), \end{aligned} \tag{1.71}$$

where $d(C^a(\tau_r)) = \dfrac{d}{d\tau_r}(C^a(\tau_r))d\tau_r = V^a d\tau_r$, therefore

$$\begin{aligned} dg(\mathbf{C},\mathbf{x}) &= \widetilde{\mathbf{C}} + x_a V^a d\tau_r, \\ &= \widetilde{\mathbf{C}} + g(\mathbf{x},V)d\tau_r. \end{aligned} \tag{1.72}$$

The third term gives

$$\begin{aligned} dg(\mathbf{C},\mathbf{C}) &= d(g_{ab}C^a(\tau_r)C^b(\tau_r)), \\ &= (dC^a(\tau_r))g_{ab}C^b(\tau_r) + (dC^a(\tau_r))g_{ab}C^b(\tau_r), \\ &= 2C_a(\tau_r)V^a d\tau_r, \\ &= 2g(\mathbf{C},V)d\tau_r. \end{aligned} \tag{1.73}$$

Substituting (1.70), (1.72) and (1.73) into (1.69) yields

$$0 = 2\widetilde{\mathbf{x}} - 2\Big(\widetilde{\mathbf{C}} + g(\mathbf{x}, V)d\tau_r\Big) + 2g(\mathbf{C}, V)d\tau_r,$$

therefore

$$\begin{aligned}
2(\widetilde{\mathbf{x}} - \widetilde{\mathbf{C}}) &= 2\Big(g(\mathbf{x}, V) - g(\mathbf{C}, V)\Big)d\tau_r, \\
\widetilde{\mathbf{x}} - \widetilde{\mathbf{C}} &= g(\mathbf{x} - \mathbf{C}, V)d\tau_r,
\end{aligned}$$

and upon rearrangement yields

$$d\tau_r = \frac{\widetilde{\mathbf{x}} - \widetilde{\mathbf{C}}}{g(X, V)} = \frac{\widetilde{X}}{g(X, V)}. \tag{1.74}$$

□

Lemma 1.5.6.

$$d\widetilde{V} = d\tau_r \wedge \widetilde{A} \tag{1.75}$$

Proof of 1.5.6.

$$\begin{aligned}
d\widetilde{V} &= \frac{dV_a}{d\tau_r} d\tau_r \wedge dy^a \\
&= A_a d\tau_r \wedge dy^a \\
&= d\tau_r \wedge \widetilde{A}
\end{aligned}$$

□

Corollary 1.5.7.

$$d\widetilde{V} = \frac{\widetilde{X} \wedge \widetilde{A}}{g(X, V)} \tag{1.76}$$

Proof of 1.5.7. Follows directly from (1.74) and (1.75). □

Lemma 1.5.8.

$$d(\star\widetilde{V}) = \frac{g(X,A)}{g(X,V)} \star 1 \tag{1.77}$$

Proof of 1.5.8.

$$\begin{aligned}
d(\star\widetilde{V}) &= dV^a \wedge i_{\frac{\partial}{\partial y^a}} \star 1 \\
&= A^a d\tau_r \wedge i_{\frac{\partial}{\partial y^a}} \star 1 \\
&= d\tau_r \wedge \star\tilde{A} \\
&= \frac{\widetilde{X} \wedge \star\tilde{A}}{g(X,V)} \\
&= \frac{g(X,A)}{g(X,V)} \star 1
\end{aligned}$$

□

Lemma 1.5.9.

$$d \star (\widetilde{X} \wedge \widetilde{V}) = \frac{\widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{A})}{g(X,V)} - 3 \star \widetilde{V} \tag{1.78}$$

Proof of 1.5.9.

$$\begin{aligned}
d\star(\widetilde{X}\wedge\widetilde{V}) &= d\star(X_a dy^a \wedge V_b dy^b)\\
&= d(V_b X_a \star(dy^a\wedge dy^b))\\
&= d(V_b X_a)\wedge\star(dy^a\wedge dy^b) + V_b X_a d\star(dy^a\wedge dy^b)\\
&= dV_b\wedge X_a\star(dy^a\wedge dy^b) + V_b dX_a\wedge\star(dy^a\wedge dy^b)\\
&= dV_b\wedge\star(\widetilde{X}\wedge dy^b) + dX_a\wedge\star(dy^a\wedge\widetilde{V})\\
&= \frac{dV_b}{d\tau_r}d\tau_r\wedge\star(\widetilde{X}\wedge dy^b) - d(g^{ab}X_b)\wedge i_{\partial_{y^a}}\star\widetilde{V}\\
&= A_b d\tau_r\wedge\star(\widetilde{X}\wedge dy^b) - dX^a\wedge i_{\partial_{y^a}}\star\widetilde{V}\\
&= d\tau_r\wedge\star(\widetilde{X}\wedge\widetilde{A}) - dy^a\wedge i_{\partial_{y^a}}\star\widetilde{V} + dC^a\wedge i_{\partial_{y^a}}\star\widetilde{V}\\
&= d\tau_r\wedge\star(\widetilde{X}\wedge\widetilde{A}) - dy^a\wedge i_{\partial_{y^a}}\star\widetilde{V} + d\tau_r\wedge\star(\widetilde{V}\wedge\widetilde{V})\\
&= \frac{\widetilde{X}\wedge\star(\widetilde{X}\wedge\widetilde{A})}{g(X,V)} - dy^a\wedge i_{\partial_{y^a}}\star\widetilde{V}
\end{aligned}$$

Need also to show that

$$dy^a\wedge i_{\partial_{y^a}}\star\widetilde{V} = 3\star\widetilde{V} \tag{1.79}$$

Let $\widetilde{V} = V_a dy^a$, then

$$\star\widetilde{V} = V_a g^{ab} i_{\partial_{y^b}}\star 1 = V^b i_{\partial_{y^b}}\star 1$$

Substituting $\star 1 = dy^0\wedge dy^1\wedge dy^2\wedge dy^3$ and contracting yields

$$\star\widetilde{V} = V_0 dy^0\wedge dy^2\wedge dy^3 - V_1 dy^0\wedge dy^2\wedge dy^3 + V_2 dy^0\wedge dy^1\wedge dy^3 - V_3 dy^0\wedge dy^1\wedge dy^2 \tag{1.80}$$

Thus

$$
\begin{aligned}
dy^a \wedge i_{\partial_{y^a}} \star \widetilde{V} = &- V_1 dy^0 \wedge dy^2 \wedge dy^3 + V_2 dy^0 \wedge dy^1 \wedge dy^3 - V_3 dy^0 \wedge dy^1 \wedge dy^2 \\
&- V_0 dy^1 \wedge dy^2 \wedge dy^3 - V_2 dy^1 \wedge dy^0 \wedge dy^3 + V_3 dy^1 \wedge dy^0 \wedge dy^2 \\
&+ V_0 dy^2 \wedge dy^1 \wedge dy^3 + V_1 dy^2 \wedge dy^0 \wedge dy^3 - V_3 dy^2 \wedge dy^0 \wedge dy^1 \\
&- V_0 dy^3 \wedge dy^1 \wedge dy^2 - V_1 dy^3 \wedge dy^0 \wedge dy^2 + V_2 dy^3 \wedge dy^0 \wedge dy^1
\end{aligned}
\tag{1.81}
$$

Collecting terms in (1.81) and comparing with (1.80) yields (1.79). □

Lemma 1.5.10.

$$
d \star (\widetilde{X} \wedge \widetilde{A}) = \frac{\widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{\dot{A}})}{g(X,V)} + \frac{\widetilde{X} \wedge \star(\widetilde{A} \wedge \widetilde{V})}{g(X,V)} - 3 \star \widetilde{A} \tag{1.82}
$$

Proof of 1.5.10.

$$
\begin{aligned}
d \star (\widetilde{X} \wedge \widetilde{A}) &= d(X_a A_b \star (dy^a \wedge dy^b)) \\
&= d(X_a A_b) \wedge \star(dy^a \wedge dy^b) + X_a A_b d \star (dy^a \wedge dy^b) \\
&= (dX_a) \wedge A_b \star (dy^a \wedge dy^b) + X_a dA_b \wedge \star(dy^a \wedge dy^b) \\
&= dX_a \wedge \star(dy^a \wedge \widetilde{A}) + dA_b \wedge \star(\widetilde{X} \wedge dy^b) \\
&= -d(g^{ab} X_a) \wedge i_{\partial_{y^b}} \star \widetilde{A} + \dot{A}_b d\tau_r \wedge \star(\widetilde{X} \wedge dy^b) \\
&= -dX^a \wedge i_{\partial_{y^a}} \star \widetilde{A} + \frac{\widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{\dot{A}})}{g(X,V)} \\
&= -dy^a \wedge i_{\partial_{y^a}} \star \widetilde{A} + dC^a \wedge i_{\partial_{y^a}} \star \widetilde{A} + \frac{\widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{\dot{A}})}{g(X,V)} \\
&= -3 \star \widetilde{A} + V^a d\tau_r \wedge i_{\partial_{y^a}} \star \widetilde{A} + \frac{\widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{\dot{A}})}{g(X,V)} \\
&= -3 \star \widetilde{A} + \frac{\widetilde{X} \wedge \star(\widetilde{A} \wedge \widetilde{V})}{g(X,V)} + \frac{\widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{\dot{A}})}{g(X,V)}
\end{aligned}
$$

□

Lemma 1.5.11.

$$dg(A,A) = 2g(A,\dot{A})d\tau_r \tag{1.83}$$

$$dg(X,V) = \widetilde{V} + \Big(\frac{g(X,A)+c^2}{g(X,V)}\Big)\widetilde{X} \tag{1.84}$$

$$dg(X,A) = \widetilde{A} + \Big(\frac{g(X,\dot{A})}{g(X,V)}\Big)\widetilde{X} \tag{1.85}$$

Proof of 1.5.11.

Proof of (1.83)

$$\begin{aligned} dg(A,A) =& d(g_{ab}\ddot{C}^a(\tau_r)\ddot{C}^b(\tau_r)) \\ &= 2g_{ab}d\ddot{C}^a\ddot{C}^b \\ &= 2g_{ab}\dddot{C}^a\ddot{C}_b d\tau_r \\ &= 2g(A,\dot{A})d\tau_r \end{aligned}$$

Proof of (1.84)

$$\begin{aligned} dg(X,V) &= dg(x - C(\tau_r), V) \\ &= d\Big[g(x,V) - g(C(\tau_r),V)\Big] \\ &= dg(x,V) - dg(C(\tau_r),V) \\ &= d(g_{ab}x^aV^b) - d(g_{ab}C^a(\tau_r)V^b) \\ &= g_{ab}V^b dy^a + g_{ab}x^a dV^b - g_{ab}V^b dC^a(\tau_r) - g_{ab}C^a(\tau_r)dV^b \\ &= \widetilde{V} + g(x,A)d\tau_r - g(V,V)d\tau_r - g(C(\tau_r),A)d\tau_r \\ &= \widetilde{V} + \Big[g(x - C(\tau_r),A) - g(V,V)\Big]d\tau_r \\ &= \widetilde{V} + \Big[g(X,A) - g(V,V)\Big]d\tau_r \end{aligned}$$

Substituting (1.26) yields result.

Proof of (1.85)

$$\begin{aligned} dg(X,A) &= dg(x - C(\tau_r), A) \\ &= d\Big[g(x,A) - g(C(\tau_r),A)\Big] \\ &= dg(x,A) - dg(C(\tau_r),A) \\ &= d(g_{ab}x^aA^b) - d(g_{ab}C^a(\tau_r)A^b) \\ &= g_{ab}A^bdy^a + g_{ab}x^adA^b - g_{ab}A^bdC^a(\tau_r) - g_{ab}C^a(\tau_r)dA^b \\ &= \widetilde{A} + g(x,\dot{A})d\tau_r - g(A,V)d\tau_r - g(C(\tau_r),\dot{A})d\tau_r \\ &= \widetilde{A} + \Big[g(x - C(\tau_r),\dot{A}) - g(A,V)\Big]d\tau_r \\ &= \widetilde{A} + \Big[g(X,\dot{A}) - g(A,V)\Big]d\tau_r \end{aligned}$$

Substituting (1.28) yields result. □

1.6 Newman-Unti coordinates (τ, R, θ, ϕ)

We introduce a system of coordinates adapted to the null worldline geometry. The coordinates were first introduced in a general form for arbitrary manifolds by Temple in 1938 [8], where they are referred to as *optical coordinates.* In 1963 Newman and Unti [9] claim to introduce a new coordinate system "intrinsically attached to an arbitrary timelike worldline", however the coordinate system they investigate is none other than the specialization of Temple's coordinates to Minkowski space. Since in this thesis we work explicitly with Minkowski space we have chosen to refer to the coordinates as Newman-Unti (N-U) coordinates in the spirit of Galt'sov and Spirin [10], however the general class of coordinates should be attributed to Temple. Similar coordinates were used by Trautman and Robinson [11] in their work on gravitational waves, and in the 1980's Ellis [12] and others use similar coordinates in problems in relativistic cosmology where they are called *Observational coordinates.* Other variations on the name include *retarded coordinates, null* *geodesic coordinates and lightcone coordinates.*

We recall from (1.60) that

$$X = x - C(\tau) = -\frac{R}{\alpha}\Big(\frac{\partial}{\partial y^0} + \sin(\theta)\cos(\phi)\frac{\partial}{\partial y^1} + \sin(\theta)\sin(\phi)\frac{\partial}{\partial y^2} + \cos\theta\frac{\partial}{\partial y^3}\Big). \tag{1.86}$$

Definition 1.6.1. Given the global Lorentzian frame (y^0, y^1, y^2, y^3) on $\mathcal{M}$, the Newman-Unti coordinates (τ, R, θ, ϕ) are defined by the coordinate transformation,

$$\begin{aligned} y^0 &= C^0(\tau) - \frac{R}{\alpha}, \\ y^1 &= C^1(\tau) - \frac{R}{\alpha}\sin(\theta)\cos(\phi), \\ y^2 &= C^2(\tau) - \frac{R}{\alpha}\sin(\theta)\sin(\phi), \\ \text{and} \qquad y^3 &= C^3(\tau) - \frac{R}{\alpha}\cos(\theta), \end{aligned} \tag{1.87}$$

where $\alpha \in \Gamma\Lambda^0\mathcal{M}$ is defined by

$$\begin{aligned} \alpha(\tau, \theta, \phi) &= -\frac{g(X, \dot{C}(\tau))}{g(X, \partial_{y^0})} \\ &= -\dot{C}^0(\tau) + \dot{C}^1(\tau)\sin(\theta)\cos(\phi) + \dot{C}^2(\tau)\sin(\theta)\sin(\phi) + \dot{C}^3(\tau)\cos(\theta). \end{aligned} \tag{1.88}$$

From (1.87) and (1.88) it follows

$$R = -g(X, \dot{C}(\tau)) \qquad \text{and} \qquad \tau = \tau_r(x(\tau, R, \theta, \phi)). \tag{1.89}$$

The spherical coordinates θ and ϕ are given naturally from the global Lorentzian frame (y^0, y^1, y^2, y^3).

Lemma 1.6.2. *In Newman-Unti coordinates the vector fields X and V are given by*

$$X = R\frac{\partial}{\partial R} \tag{1.90}$$

and

$$V = \frac{\partial}{\partial \tau} + X\frac{\dot{\alpha}}{\alpha} \tag{1.91}$$

Proof of 1.6.2.

Proof of (1.90) Differentiating the coordinate transformation (1.87) with respect to R yields

$$\begin{aligned}
\frac{\partial}{\partial R} &= \frac{\partial y^0}{\partial R}\frac{\partial}{\partial y^0} + \frac{\partial y^1}{\partial R}\frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial R}\frac{\partial}{\partial y^2} + \frac{\partial y^3}{\partial R}\frac{\partial}{\partial y^3} \\
&= -\frac{1}{\alpha}\Big(\frac{\partial}{\partial y^0} + \sin(\theta)\cos(\phi)\frac{\partial}{\partial y^1} + \sin(\theta)\sin(\phi)\frac{\partial}{\partial y^2} + \cos\theta\frac{\partial}{\partial y^3}\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
X &= x - C(\tau) \\
&= -\frac{R}{\alpha}\Big(\frac{\partial}{\partial y^0} + \sin(\theta)\cos(\phi)\frac{\partial}{\partial y^1} + \sin(\theta)\sin(\phi)\frac{\partial}{\partial y^2} + \cos\theta\frac{\partial}{\partial y^3}\Big) \\
&= R\frac{\partial}{\partial R}
\end{aligned}$$

Proof of (1.91)

$$\begin{aligned}
\frac{\partial}{\partial \tau} =& \frac{\partial y^0}{\partial \tau}\frac{\partial}{\partial y^0} + \frac{\partial y^1}{\partial \tau}\frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial \tau}\frac{\partial}{\partial y^2} + \frac{\partial y^3}{\partial \tau y}\frac{\partial}{\partial y^3} \\
=& \Big(\dot{C}^0(\tau) - R\frac{\partial}{\partial \tau}(\frac{1}{\alpha})\Big)\frac{\partial}{\partial y^0} + \Big(\dot{C}^1(\tau) - R\sin(\theta)\cos(\phi)\frac{\partial}{\partial \tau}(\frac{1}{\alpha})\Big)\frac{\partial}{\partial y^1} \\
&+ \Big(\dot{C}^2(\tau) - R\sin(\theta)\sin(\phi)\frac{\partial}{\partial \tau}(\frac{1}{\alpha})\Big)\frac{\partial}{\partial y^2} + \Big(\dot{C}^3(\tau) - R\cos(\theta)\frac{\partial}{\partial \tau}(\frac{1}{\alpha})\Big)\frac{\partial}{\partial y^3} \\
=& \dot{C}^a(\tau)\frac{\partial}{\partial y^a} + \frac{\partial X}{\partial \tau} \\
=& V - X\frac{\dot{\alpha}}{\alpha} \\
\therefore V =& \frac{\partial}{\partial \tau} + X\frac{\dot{\alpha}}{\alpha}
\end{aligned}$$

□

Lemma 1.6.3. *In Newman-Unti coordinate the Minkowski metric $g \in \bigotimes^{[\mathbb{F},\mathbb{F}]}\mathbf{M}$ is given by*

$$\begin{aligned} g =& (-c^2 + 2R\frac{\dot{\alpha}}{\alpha})d\tau \otimes d\tau - (d\tau \otimes dR + dR \otimes d\tau) \\ &+ \frac{R^2}{\alpha^2} d\theta \otimes d\theta + \frac{R^2}{\alpha^2}\sin(\theta)^2 d\phi \otimes d\phi, \end{aligned} \tag{1.92}$$

and inverse metric $g^{-1} \in \bigotimes^{[\mathbb{V},\mathbb{V}]}\mathbf{M}$ is given by

$$\begin{aligned} g^{-1} =& -\frac{-c^2\alpha + 2R\dot{\alpha}}{\alpha}\frac{\partial}{\partial R} \otimes \frac{\partial}{\partial R} + \frac{\alpha^2}{R^2}\frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta} \\ &+ \frac{\alpha^2}{R^2\sin(\theta)^2}\frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial \phi} - (\frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial R} + \frac{\partial}{\partial R} \otimes \frac{\partial}{\partial \tau}). \end{aligned} \tag{1.93}$$

Let $z^0 = \tau, \quad z^1 = R, \quad z^2 = \theta, \quad z^3 = \phi$, then the matrices $G = G_{ab} = g(\partial_{z^a}, \partial_{z^b})$ and $G^{-1} = G^{-1}_{ab} = g^{-1}(dz^a, dz^b)$ are given by

$$G = \begin{pmatrix} \frac{-c^2\alpha + 2R\dot{\alpha}}{\alpha} & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{R^2}{\alpha^2} & 0 \\ 0 & 0 & 0 & \frac{R^2\sin(\theta)^2}{\alpha^2} \end{pmatrix}, \tag{1.94}$$

and

$$G^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -\frac{-c^2\alpha + 2R\dot{\alpha}}{\alpha} & 0 & 0 \\ 0 & 0 & \frac{\alpha^2}{R^2} & 0 \\ 0 & 0 & 0 & \frac{\alpha^2}{R^2\sin(\theta)^2} \end{pmatrix} \tag{1.95}$$

Proof of 1.6.3. Differentiation of the coordinate transformation (1.87) gives

$$\begin{aligned}
dy^0 &= \Big(\dot{C}^0(\tau) + R\frac{\dot{\alpha}}{\alpha^2}\Big)d\tau - \frac{1}{\alpha}dR + R\frac{\alpha_\theta}{\alpha^2}d\theta + R\frac{\alpha_\phi}{\alpha^2}d\phi, \\
dy^1 &= \Big(\dot{C}^1(\tau) + R\sin(\theta)\cos(\phi)\frac{\dot{\alpha}}{\alpha^2}\Big)d\tau - \frac{\sin(\theta)\cos(\phi)}{\alpha}dR \\
&\quad + \frac{R}{\alpha^2}\big(\alpha_\theta \sin(\theta)\cos(\phi) - \alpha\cos(\theta)\cos(\phi)\big)d\theta + \frac{R}{\alpha^2}\big(\alpha_\phi \sin(\theta)\cos(\phi) - \alpha\cos(\theta)\cos(\phi)\big)d\phi, \\
dy^2 &= \Big(\dot{C}^2(\tau) + R\sin(\theta)\sin(\phi)\frac{\dot{\alpha}}{\alpha^2}\Big)d\tau - \frac{\sin(\theta)\sin(\phi)}{\alpha}dR \\
&\quad + \frac{R}{\alpha^2}\big(\alpha_\theta \sin(\theta)\sin(\phi) - \alpha\cos(\theta)\sin(\phi)\big)d\theta + \frac{R}{\alpha^2}\big(\alpha_\phi \sin(\theta)\sin(\phi) - \alpha\sin(\theta)\cos(\phi)\big)d\phi, \\
dy^3 &= \Big(\dot{C}^3(\tau) + R\cos(\theta)\frac{\dot{\alpha}}{\alpha^2}\Big)d\tau - \frac{\cos(\theta)}{\alpha}dR + \frac{R}{\alpha^2}\big(\alpha_\theta\cos(\theta). + \alpha\sin(\theta)\big)d\theta + \frac{R}{\alpha^2}\cos(\theta)\alpha_\phi d\phi
\end{aligned} \tag{1.96}$$

where

$$\dot{\alpha} = \frac{\partial\alpha}{\partial\tau}, \qquad \alpha_\theta = \frac{\partial\alpha}{\partial\theta}, \qquad \alpha_\phi = \frac{\partial\alpha}{\partial\phi} \tag{1.97}$$

Substitution of (1.96) into (1.1) yields (1.92). The dual metric (1.93) follows from (1.95). □

Lemma 1.6.4. *The 1-forms $\widetilde{X}, \widetilde{V} \in \Gamma\Lambda^1(\mathcal{M}\backslash C)$ are given by*

$$\widetilde{X} = -Rd\tau \tag{1.98}$$

$$\widetilde{V} = \frac{R\dot{\alpha} - c^2\alpha}{\alpha}d\tau - dR \tag{1.99}$$

Proof of 1.6.4.

$$\begin{aligned}
\widetilde{X} &= Rg(\frac{\partial}{\partial R}, -) \\
&= -Rd\tau \\
\widetilde{V} &= g(\frac{\partial}{\partial\tau} + X\frac{\dot{\alpha}}{\alpha}, -) \\
&= \frac{R\dot{\alpha} - c^2\alpha}{\alpha}d\tau - dR
\end{aligned}$$

□

Lemma 1.6.5.

$$\widetilde{A} =(R\frac{\dot{\alpha}^2}{\alpha^2})d\tau - \frac{\dot{\alpha}}{\alpha}dR + R\frac{\dot{\alpha}\alpha_\theta - \alpha\dot{\alpha}_\theta}{\alpha^2}d\theta + R\frac{\dot{\alpha}\alpha_\phi - \alpha\dot{\alpha}_\phi}{\alpha^2}d\phi \tag{1.100}$$

$$\widetilde{\dot{A}} =(g(\dot{A}, V) + R\frac{\dot{\alpha}\ddot{\alpha}}{\alpha^2})d\tau - \frac{\ddot{\alpha}}{\alpha}dR + R\frac{\ddot{\alpha}\alpha_\theta - \alpha\ddot{\alpha}_\theta}{\alpha^2}d\theta + R\frac{\ddot{\alpha}\alpha_\phi - \alpha\ddot{\alpha}_\phi}{\alpha^2}d\phi \tag{1.101}$$

Proof of 1.6.5.

$$\begin{aligned}
\widetilde{A} &= \frac{dV_a}{d\tau}dy^a \\
&= -\ddot{C}^0(\tau)dy^0 + \ddot{C}^1(\tau)dy^1 + \ddot{C}^2(\tau)dy^2 + \ddot{C}^3(\tau)dy^3 \\
\widetilde{\dot{A}} &= \frac{dA_a}{d\tau}dy^a \\
&= -\dddot{C}^0(\tau)dy^0 + \dddot{C}^1(\tau)dy^1 + \dddot{C}^2(\tau)dy^2 + \dddot{C}^3(\tau)dy^3
\end{aligned}$$

result follows on substitution of (1.96). □

Lemma 1.6.6.

$$\widetilde{d\tau} = -\frac{\partial}{\partial R} \qquad \text{and} \qquad \widetilde{dR} = -\frac{\partial}{\partial \tau} + \left(c^2 - 2R\frac{\dot{\alpha}}{\alpha}\right)\frac{\partial}{\partial R} \tag{1.102}$$

Proof of 1.6.6. Follows from (B.29) and (1.93). □

Lemma 1.6.7.

$$A =\Big(R\frac{\dot{\alpha}}{\alpha} - c^2\Big)\frac{\dot{\alpha}}{\alpha}\frac{\partial}{\partial R} + \frac{\dot{\alpha}}{\alpha}\frac{\partial}{\partial \tau} + \frac{\dot{\alpha}\alpha_\theta - \alpha\dot{\alpha}_\theta}{R}\frac{\partial}{\partial \theta} + \frac{\dot{\alpha}\alpha_\phi - \alpha\dot{\alpha}_\phi}{R\sin^2(\theta)}\frac{\partial}{\partial \phi} \tag{1.103}$$

Proof of 1.6.7. follows from (B.30), (1.93) and (1.101). □

Lemma 1.6.8.

$$g(X, A) = -R\frac{\dot{\alpha}}{\alpha} \tag{1.104}$$

Proof of 1.6.8. Follows by substitution of (1.90) and (1.103) into (1.92). □

Lemma 1.6.9.

$$\star 1 = \frac{R^2 \sin(\theta)}{\alpha^2} d\tau \wedge dR \wedge d\theta \wedge d\phi \tag{1.105}$$

Proof of 1.6.9. Follows from (1.5) and (1.87). □

In appendix E we present a different coordinate system which we have called adapted N-U coordinates. They will be used in Part II of this thesis.

1.7 The Liénard-Wiechert field

The Liénard-Wiechert potential is the solution to the Maxwell-Lorentz equations when the source $\mathcal{J} \in \Gamma\Lambda^3\mathcal{M}$ is given by J, the current 3-form for a point charge moving arbitrarily in free space (1.47). In this section the Liénard-Wiechert potential and associated fields are given in term of the null geometry formalism developed in the proceeding section. We use the notation A for the Liénard-Wiechert 1-form potential as a special case for $\mathcal{A} \in \Gamma\Lambda^1\mathcal{M}$. It is the solution to (1.10) given the source J.

Definition 1.7.1. The **Liénard-Wiechert Potential** of the point charge at point $x \in (\mathcal{M}\backslash C)$ is given by

$$\mathrm{A}|_x \in \Gamma\Lambda^1(\mathcal{M}\backslash C), \qquad \mathrm{A}|_x = \frac{q}{4\pi\epsilon_0}\frac{\widetilde{V}}{g(V,X)}. \tag{1.106}$$

We associate with A the 1-form distribution A^D defined by its action on test 3-form $\varphi \in \Gamma_0\Lambda^3\mathcal{M}$ by

$$\mathrm{A}^D \in \Gamma_D\Lambda^1\mathcal{M}, \qquad \mathrm{A}^D[\varphi] = \int_{\mathcal{M}} \varphi \wedge \mathrm{A}. \tag{1.107}$$

Lemma 1.7.2. *The electromagnetic* 2*-form attained by substituting* $\mathcal{A} = \mathrm{A}$ *in*

(1.10) will be called the Liénard-Wiechert 2-form and is given by

$$\mathrm{F} \in \Gamma\Lambda^2(\mathcal{M}\backslash C), \qquad \mathrm{F}|_x = \frac{q}{4\pi\epsilon_0}\frac{g(X,V)\widetilde{X}\wedge\widetilde{A} - g(X,A)\widetilde{X}\wedge\widetilde{V} - c^2\widetilde{X}\wedge\widetilde{V}}{g(X,V)^3}. \tag{1.108}$$

Proof of 1.7.2.

$$\begin{aligned}\mathrm{F} = d\mathrm{A} &= \frac{q}{4\pi\epsilon_0}d\left(\frac{\widetilde{V}}{g(V,X)}\right)\\ &= \frac{q}{4\pi\epsilon_0}d\left(\frac{1}{g(X,V)}\right)\widetilde{V} + \frac{q}{4\pi\epsilon_0}\frac{1}{g(X,V)}d(\widetilde{V})\\ &= \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{g(X,V)^2}dg(X,V) + \frac{1}{g(X,V)}d(\widetilde{V})\right]\end{aligned} \tag{1.109}$$

Substituting (1.84) and (1.76) into (1.109) yields

$$\mathrm{F} = \frac{q}{4\pi\epsilon_0}\frac{g(X,V)\widetilde{X}\wedge\widetilde{A} - g(X,A)\widetilde{X}\wedge\widetilde{V} - c^2\widetilde{X}\wedge\widetilde{V}}{g(X,V)^3} \tag{1.110}$$

□

Definition 1.7.3. We associate with F the regular 2-form distribution F^D defined by its action on test 2-form $\varphi \in \Gamma_0\Lambda^2\mathcal{M}$ by

$$\mathrm{F}^D \in \Gamma_D\Lambda^2\mathcal{M} \qquad \mathrm{F}^D[\varphi] = \int_{\mathcal{M}} \varphi\wedge\mathrm{F}. \tag{1.111}$$

Readers familiar with the 3-vector notation for the Electric and Magnetic Liénard-Wiechert fields can look at lemma 6.1.7 to see how these relate to F.

Definition 1.7.4. We split the Liénard-Wiechert 2-form F into two terms where

$$\mathrm{F}_{\mathrm{R}} = \frac{q}{4\pi\epsilon_0}\frac{g(X,V)\widetilde{X}\wedge\widetilde{A} - g(X,A)\widetilde{X}\wedge\widetilde{V}}{g(X,V)^3} \tag{1.112}$$

will be referred to as the radiation term, and

$$\mathrm{F_C} = \frac{q}{4\pi\epsilon_0}\frac{-c^2\widetilde{X}\wedge\widetilde{V}}{g(X,V)^3} \tag{1.113}$$

will be referred to as the Coulomb term.

Lemma 1.7.5. *The Liénard-Wiechert potential (1.106) satisfies the Lorentz gauge condition*

$$d\star \mathrm{A} = 0. \tag{1.114}$$

Proof of 1.7.5.

Let

$$\kappa = \frac{q}{4\pi\epsilon_0}, \tag{1.115}$$

then

$$\begin{aligned}\frac{1}{\kappa}d\star\mathrm{A} &= d\Big(\frac{\star\widetilde{V}}{g(X,V)}\Big)\\ &= -\frac{1}{g(V,X)^2}dg(V,X)\wedge\star\widetilde{V} + \frac{1}{g(V,X)}d(\star\widetilde{V})\end{aligned}$$

Substituting (1.84) and (1.77) yields

$$\frac{1}{\kappa}d\star\mathrm{A} = -\frac{1}{g(V,X)^2}\widetilde{V}\wedge\star\widetilde{V} - \Big(\frac{g(X,A)+c^2}{g(X,V)^3}\Big)\widetilde{X}\wedge\star\widetilde{V} + \frac{1}{g(V,X)}\big(\frac{g(X,A)}{g(X,V)}\star 1\big)$$

and using lemma B.2.7 gives

$$\frac{1}{\kappa}d\star\mathrm{A} = \frac{c^2}{g(V,X)^2}\star 1 - \Big(\frac{g(X,A)+c^2}{g(X,V)^2}\Big)\star 1 + \frac{g(X,A)}{g(X,V)^2}\star 1 = 0 \tag{1.116}$$

□

Lemma 1.7.6. *Off the worldline the Liénard-Wiechert potential satisfies*

$$d\star d\mathrm{A} = d\star \mathrm{F} = 0.$$

Thus given arbitrary region $N \subset (\mathcal{M}\backslash C)$ it follows that

$$\int_N \varphi \wedge d\star \mathrm{F} = 0 \tag{1.117}$$

for any test 1-form $\varphi \in \Gamma_0\Lambda^1\mathcal{M}$.

Proof of 1.7.6.

From (1.108)

$$\frac{1}{\kappa}\star \mathrm{F} = \frac{g(X,V)\star(\widetilde{X}\wedge\widetilde{A}) - (g(X,A)+c^2)\star(\widetilde{X}\wedge\widetilde{V})}{g(X,V)^3}$$

Thus

$$\begin{aligned}\frac{1}{\kappa}d\star \mathrm{F} =& d\Big(\frac{\star(\widetilde{X}\wedge\widetilde{A})}{g(X,V)^2}\Big) - d\Big(\frac{(g(X,A)+c^2)\star(\widetilde{X}\wedge\widetilde{V})}{g(X,V)^3}\Big)\\ =&\frac{d\star(\widetilde{X}\wedge\widetilde{A})}{g(X,V)^2} + d\Big(\frac{1}{g(X,V)^2}\Big)\wedge\star(\widetilde{X}\wedge\widetilde{A}) - \frac{(g(X,A)+c^2)}{g(X,V)^3}d\star(\widetilde{X}\wedge\widetilde{V})\\ &- d\Big(\frac{g(X,A)+c^2}{g(X,V)^3}\Big)\star(\widetilde{X}\wedge\widetilde{V})\end{aligned} \tag{1.118}$$

Using the chain rule for differentiation yields

$$\begin{aligned} d\Big(\frac{1}{gX,V^2}\Big) &= -\frac{2dg(X,V)}{g(X,V)^2}\\ \text{and}\quad d\Big(\frac{g(X,A)+c^2}{g(X,V)^3}\Big) &= \frac{dg(X,A)}{g(X,V)^3} - \frac{3(g(X,A)+c^2)}{g(X,V)^4}dg(X,V)\end{aligned} \tag{1.119}$$

Substituting (1.78), (1.82), (1.84), (1.85) and (1.119) into (1.118) and using lemma B.2.9 yields result. □

Lemma 1.7.7. *In Newman-Unti coordinates the Liénard-Wiechert potential* $\mathrm{A} \in$

$\Gamma\Lambda^1(\mathcal{M}\backslash C)$ and the electromagnetic 2-form $\mathrm{F} \in \Gamma\Lambda^2(\mathcal{M}\backslash C)$ are given by

$$\mathrm{A} = -\frac{q}{4\pi\epsilon_0}\Big(\Big(\frac{c^2}{R} - \frac{\dot\alpha}{\alpha}\Big)d\tau + \frac{1}{R}dR\Big), \tag{1.120}$$

$$\mathrm{F_R} = \frac{q}{4\pi\epsilon_0}\Big(\frac{\alpha\dot\alpha_\theta - \dot\alpha\alpha_\theta}{\alpha^2}d\tau\wedge d\theta + \frac{\alpha\dot\alpha_\phi - \dot\alpha\alpha_\phi}{\alpha^2}d\tau\wedge d\phi\Big), \tag{1.121}$$

and

$$\mathrm{F_C} = \frac{q}{4\pi\epsilon_0}\frac{c^2}{R^2}d\tau\wedge dR. \tag{1.122}$$

It follows from theorems C.2.1 and C.2.2 that the distributional 1-form $\mathrm{A}_D \in \Gamma_D\Lambda^1\mathcal{M}$ and distributional 2-form $\mathrm{F}_D \in \Gamma_D\Lambda^2\mathcal{M}$ are well defined.

Proof of 1.7.7.

Equations (1.120) and (1.122) follow by substitution of (1.89) and (1.99) into (1.106) and (1.113). Equation (1.121) follows by substitution of (1.89), (1.99), (1.101) and (1.6.8) into (1.112). □

Lemma 1.7.8.

$$\star\,\mathrm{F_R} = \frac{q}{4\pi\epsilon_0}\Big(\frac{\alpha_\phi\dot\alpha - \alpha\dot\alpha_\phi}{\alpha^2\sin(\theta)}dR\wedge d\theta - \frac{\sin(\theta)(\alpha_\theta\dot\alpha - \alpha\dot\alpha_\theta)}{\alpha^2}dR\wedge d\phi\Big),$$

and

$$\star\,\mathrm{F_C} = \frac{q}{4\pi\epsilon_0}\frac{c^2\sin(\theta)}{\alpha^2}d\theta\wedge d\phi. \tag{1.123}$$

Proof of 1.7.8. Follows from definition B.2.6, lemma 1.6.6 and the equations (1.105), (1.121) and (1.122). □

Lemma 1.7.9. *The distributional Liénard-Wiechert field $\mathrm{F}^D \in \Gamma_D\Lambda^2\mathcal{M}$ satisfies*

$$\epsilon_0 d\star\mathrm{F}^D[\varphi] = \mathrm{J}^D[\varphi], \tag{1.124}$$

for any test 1-form $\varphi \in \Gamma_0\Lambda^1\mathcal{M}$.

Proof of 1.7.9. First we consider the form of φ close to the worldline. A general test 1-form $\varphi \in \Gamma_0\Lambda^1\mathcal{M}$ is given in Minkowski coordinates by

$$\varphi = \varphi_i(y^0, y^1, y^2, y^3)dy^i \tag{1.125}$$

Now making transformation (1.6.1) to Newman-Unti coordinates, such that

$$\varphi = \hat{\varphi}_\tau(\tau, R, \theta, \phi)d\tau + \hat{\varphi}_R(\tau, R, \theta, \phi)dR + \hat{\varphi}_\theta(\tau, R, \theta, \phi)d\theta + \hat{\varphi}_\phi(\tau, R, \theta, \phi)d\phi \tag{1.126}$$

yields

$$\begin{aligned}
\hat{\varphi}_\tau =& Y^0_\tau(\tau) + Y^1_\tau(\tau, \theta, \phi)R, \\
\hat{\varphi}_R =& Y^0_R(\tau, \theta, \phi), \\
\hat{\varphi}_\theta =& Y^1_\theta(\tau, \theta, \phi)R, \\
\hat{\varphi}_\phi =& Y^1_\phi(\tau, \theta, \phi)R,
\end{aligned} \tag{1.127}$$

where Y^0_τ is a bounded function of τ and the rest of the Y^l_i's are bounded functions of τ, θ and ϕ. Thus to zero order in R

$$\varphi = Y^0_\tau d\tau + Y^0_R dR + \mathcal{O}(R), \tag{1.128}$$

and

$$d\varphi = -\frac{\partial Y^0_\tau}{\partial \theta} d\tau \wedge d\theta - \frac{\partial Y^0_\tau}{\partial \phi} d\tau \wedge d\phi + \frac{\partial Y^0_R}{\partial \tau} d\tau \wedge dR - \frac{\partial Y^0_R}{\partial \theta} dR \wedge d\theta - \frac{\partial Y^0_R}{\partial \phi} dR \wedge d\phi + \mathcal{O}(R). \tag{1.129}$$

Now definition C.1.4 yields

$$\begin{aligned}
d \star \mathrm{F}^D[\varphi] &= \star \mathrm{F}^D[d\varphi] \\
&= \int_{\mathcal{M}} d\phi \wedge \star \mathrm{F}
\end{aligned} \tag{1.130}$$

We split the integral over $\mathcal{M}$ into a region away from the worldline and a region containing the worldline. Let the four dimensional region $\mathbb{B} \subset \mathcal{M}$ be defined in

N-U coordinates by

$$\mathbb{B} = \{ \quad \tau, R, \theta, \phi \quad | \quad \tau \in I, \quad 0 \leq R \leq k, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi \quad \} \tag{1.131}$$

where I is the domain of C. The boundary $\partial(\mathcal{M}\backslash C) = \partial\mathbb{B}$ is given by

$$\partial\mathbb{B} = \{ \quad \tau, R, \theta, \phi \quad | \quad \tau \in I, \quad R = k, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi \quad \}. \tag{1.132}$$

We calculate $d\star\mathcal{F}^D[\phi]$ with the assumption that $k \to 0$ so that the approximation (1.128) remains valid

$$\begin{aligned} d \star \mathrm{F}^D[\varphi] &= \star\mathrm{F}^D[d\varphi] \\ &= \int_{\mathcal{M}} d\phi \wedge \star\mathrm{F} \\ &= \int_{\mathcal{M}\backslash\mathbb{B}} d\varphi \wedge \star\mathrm{F} + \int_{\mathbb{B}} d\varphi \wedge \star\mathrm{F} \\ &= \int_{\mathcal{M}\backslash\mathbb{B}} d(\varphi \wedge \star\mathrm{F}) + \int_{\mathcal{M}\backslash\mathbb{B}} \varphi \wedge d \star \mathrm{F} + \int_{\mathbb{B}} d\varphi \wedge \star\mathrm{F} \end{aligned} \tag{1.133}$$

The second term in (1.133) vanishes due to lemma (1.7.6). Consider the third term.

$$\int_{\mathbb{B}} d\varphi \wedge \star\mathrm{F} = \int_{\mathbb{B}} d\varphi \wedge \star\mathrm{F_C} + \int_{\mathbb{B}} d\varphi \wedge \star\mathrm{F_R} \tag{1.134}$$

Using (1.123) and (1.129) yields

$$\begin{aligned} \int_{\mathbb{B}} d\varphi \wedge \star\mathrm{F} =& \frac{q}{4\pi\epsilon_0}\Big(\int_{\mathcal{M}} \frac{\partial Y^0_R}{\partial\tau} \frac{c^2 \sin(\theta)}{\alpha^2} d\tau \wedge dR \wedge d\theta \wedge d\phi \\ &+ \int_{\mathcal{M}} \frac{\partial Y^0_\tau}{\partial\phi} \frac{\alpha_\phi \dot{\alpha} - \alpha\dot{\alpha}_\phi}{\alpha^2 \sin(\theta)} d\tau \wedge dR \wedge d\theta \wedge d\phi \\ &+ \int_{\mathcal{M}} \frac{\partial Y^0_\tau}{\partial\theta} \frac{\sin(\theta)(\alpha\dot{\alpha}_\theta - \alpha_\theta\dot{\alpha})}{\alpha^2} d\tau \wedge dR \wedge d\theta \wedge d\phi \end{aligned} \tag{1.135}$$

All three terms vanish under integration with respect to R when $k \to 0$, therefore the third term in (1.133) vanishes. Finally we consider the first term. We note that R is constant on the boundary and therefore $dR = 0$. By Stokes' Theorem

$$\begin{aligned}\int_{\mathcal{M}\backslash\mathbb{B}} d(\varphi \wedge \star\mathrm{F}) &= \int_{\partial\mathbb{B}} \varphi \wedge \star\mathrm{F} \\ &= \int_{\partial\mathbb{B}} \varphi \wedge \star\mathrm{F}_{\mathrm{C}} + \int_{\partial\mathbb{B}} \varphi \wedge \star\mathrm{F}_{\mathrm{R}}\end{aligned} \tag{1.136}$$

The second term vanishes because $dR = 0$. We are left with the first term,

$$\int_{\partial\mathbb{B}} \varphi \wedge \star\mathrm{F}_{\mathrm{C}} = \frac{q}{4\pi\epsilon_0} \int_{\partial\mathbb{B}} Y^0_\tau(\tau) \frac{c^2 \sin(\theta)}{\alpha^2} d\tau \wedge d\theta \wedge d\phi \tag{1.137}$$

$$= \frac{q}{4\pi\epsilon_0} \int_{\tau=-\infty}^{\infty} Y^0_\tau(\tau) \Big(\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{c^2 \sin(\theta)}{\alpha^2} d\theta d\phi \Big) d\tau \tag{1.138}$$

Let $I = \displaystyle\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{c^2 \sin(\theta)}{\alpha^2} d\theta d\phi$, then substituting (1.88) we obtain

$$I = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{c^2 \sin(\theta)}{(-\dot{C}^0 + \dot{C}^1 \sin(\theta)\cos(\phi) + \dot{C}^2 \sin(\theta)\sin(\phi) + \dot{C}^3 \cos(\theta))^2} d\theta d\phi \tag{1.139}$$

Let $z = e^{i\phi}$ such that

$$\sin(\phi) = \frac{1}{2i}(z - z^{-1}), \qquad \cos(\phi) = \frac{1}{2}(z + z^{-1}), \qquad d\phi = \frac{dz}{iz} \tag{1.140}$$

Substitution yields

$$I = c^2 \int_{\mu(0,1)} \frac{4i \sin(\theta) z}{((-i\dot{C}^2 + \dot{C}^1)\sin(\theta) z^2 + (2\dot{C}^3 \cos(\theta) - 2\dot{C}^0) z + (i\dot{C}^2 + \dot{C}^1)\sin(\theta))^2} dz \wedge d\theta \tag{1.141}$$

where $\mu(0,1)$ represents circle of radius 1 centred at the origin. The quadratic:

$$(-i\dot{C}^2 + \dot{C}^1)\sin(\theta) z^2 + (2\dot{C}^3 \cos(\theta) - 2\dot{C}^0) z + (i\dot{C}^2 + \dot{C}^1)\sin(\theta) = 0 \tag{1.142}$$

has roots at

$$\frac{1}{\sin(\theta)(-i\dot{C}^2+\dot{C}^1)}\Bigg(-\dot{C}^3\cos(\theta)+\dot{C}^0 \pm\sqrt{\dot{C}^0-2\dot{C}^{0^2}\dot{C}^3\cos(\theta)+\dot{C}^{3^2}\cos^2(\theta)-\dot{C}^{1^2}-\dot{C}^{2^2}+\cos^2(\theta)\dot{C}^{1^2}+\cos(\theta)^2\dot{C}^{2^2}}\Bigg) \tag{1.143}$$

Denoting the roots by $\alpha(+),\beta(-)$ yields,

$$I=\int_{\mu(0,1)}\frac{4iz\sin(\theta)}{(z-\alpha)^2(z-\beta)^2}dzd\theta \tag{1.144}$$

$|\alpha|>1$ therefore it lies outside the contour. The residue of I at $z=\beta$ is given by

$$res=\frac{4i\sin(\theta)(\alpha+\beta)}{(-\beta+\alpha)^3}, \tag{1.145}$$

Therefore by the residue theorem (see for example [13]),

$$I=2\pi c^2\int_0^\pi\frac{\sin(\theta)(\dot{C}^3\cos(\theta)-\dot{C}^0)}{((\dot{C}^3\cos(\theta)-\dot{C}^0)^2+(\dot{C}^{2^2}+\dot{C}^{1^2})\sin^2(\theta))^{\frac{3}{2}}}d\theta \tag{1.146}$$

and integration using a computer gives

$$I=\frac{4\pi c^2}{-\dot{C}^{0^2}+\dot{C}^{1^2}+\dot{C}^{2^2}+\dot{C}^{3^2}}=4\pi\frac{c^2}{c^2} \tag{1.147}$$

hence

$$\int_{\partial\mathbb{B}}\varphi\wedge\star\mathrm{F_C}=\frac{q}{\epsilon_0}\int_{\tau=-\infty}^{\infty}\mathrm{Y}_\tau^0(\tau)d\tau=\frac{q}{\epsilon_0}\int_{\tau=-\infty}^{\infty}\hat{\varphi}_\tau(\tau)d\tau=\frac{q}{\epsilon_0}\int_I C^*\phi, \tag{1.148}$$

and comparison with lemma 1.4.2 yields

$$\epsilon_0 d*\mathrm{F}^D[\phi]=q\int_I C^*\phi=\mathrm{J}^D[\phi] \tag{1.149}$$

□

PART I

The self force and the Schott term discrepancy

Chapter 2

Introduction

In chapter 1 we assign the dimension of time to proper time τ so that (1.26) is satisfied. We will return to this convention in Part II where we derive the electric and magnetic fields in the standard 3-vector notation. In Part I it is convenient to assign the dimension of length to τ so that (1.27) is satisfied. For details see appendix A.

2.1 The self force, mass renormalization and the equation of motion

It is a consequence of Maxwell-Lorentz electrodynamics that any source of an electromagnetic field will be subject to interaction with that field. The resulting force on the source is known as the *self force*. For a charged particle undergoing inertial motion the self force is zero, however for an accelerating charge the force is non-zero and tends to act as a damping term [14]. It is well known that an accelerating charge loses energy due to the emission of radiation, where the instantaneous loss of momentum due to radiation, $\dot{\mathrm{P}}_{\mathrm{RAD}} \in \Gamma\mathrm{T}\mathcal{M}$, is given by the Larmor-Abraham formula [15]

$$\dot{\mathrm{P}}_{\mathrm{RAD}} = \frac{q^2}{6\pi\epsilon_0} g(\ddot{C}, \ddot{C})\dot{C}. \tag{2.1}$$

The negative of this force is the *radiation reaction* force, which must be a contribution to the self force. This has led many authors to use the term *radiation reaction* synonymously with self force, however in the fully relativistic case there is an extra term in the self force in addition to the negative of (2.1). This additional term is known as the *Schott term*, and has lead to some controversy. The fully relativistic self force is given by the Abraham-von Laue vector [16]

$$f_{\text{self}} = \frac{q^2}{6\pi\epsilon_0}(\dddot{C} - g(\ddot{C},\ddot{C})\dot{C}), \tag{2.2}$$

where the Schott term is third order with respect to the worldline.

The zeroth component of the radiation reaction force is Larmor's equation for the rate of radiation. The spatial part is proportional to the negative of the Newtonian velocity and may be interpreted as the radiation reaction force of the particle. The physical nature of the Schott term has been a topic for debate. Its presence leads to two interesting results: i) the self force can vanish even when the radiation rate is non-zero, for example in the case of uniform circular motion, and ii) the self force can be non-zero even when there is momentarily no radiation being emitted. Thus the identification of the whole of (2.2) as a radiation reaction force would be misleading. The Schott term is a total derivative, so it does not correspond to an irreversible loss of momentum by the particle, but plays an important role in the momentum balance between the radiation and the particle [4].

With the self force given by (2.2) the resulting equation of motion for a charged particle undergoing arbitrary motion is given by the Abraham-Lorentz-Dirac (ALD) equation

$$m\nabla_{\dot{C}}\dot{C} = f_{\text{L}} + f_{\text{self}} + f_{\text{ext}}, \tag{2.3}$$

where m is the observed rest mass of the particle, $f_{\text{L}} \in \Gamma\mathrm{T}\mathcal{M}$ is the Lorentz force due to the external field, $f_{\text{ext}} \in \Gamma\mathrm{T}\mathcal{M}$ is the force due to non-electromagnetic effects [1], and $f_{\text{self}} \in \Gamma \mathrm{T}\mathcal{M}$ is given by (2.2). All three forces on the right of (2.3) are vector fields with support on finite closed regions of the worldline[2] thus Rohrlich's dynamic asymptotic condition [4],

$$\lim_{|\tau|\to\infty} \nabla_{\dot{C}}\dot{C} = 0, \tag{2.4}$$

is satisfied. The third order nature of the Schott term has instilled doubts about the validity of the ALD equation, since it leads to particular classes of solution which are foreign to classical physics. These solutions include *preacceleration*, where a particle may begin to accelerate before a force has been applied, and *runaway solutions*, where a particle may continue to accelerate exponentially even for a static force (see [4, 20, 21]). Further doubts about the validity of the ALD equation are raised by the fact that there remains to this day no derivation of (2.3) which is completely free from ambiguity. The most widely known difficulty is that of mass renormalization.

The origin of mass renormalization can be found in the early attempts to calculate the self force based on extended models for the electron. At the dawn of the twentieth century the limitations imposed by quantum physics were unknown and it was widely believed the dynamics of an electron could be established by supposing a classical model for the particle. The model was based on the idea of a macroscopic charged object reduced to the microscopic scale. There is an inherent problem with this approach because macroscopic charged objects are stable only because of the intermolecular forces binding them together. As an elementary particle the electron is necessarily devoid of these forces, thus within such a model the particle would have a tendency to blow itself apart due to the mutual repulsion of its volume elements. The solution of this difficulty, proposed by Poincaré, was to postulate the existence of an additional cohesive force which would exactly cancel

[1] In general these are not known but could include effects due to gravity or collision with neutral particles. It is common to assume $f_{\text{ext}} = 0$.

[2] When looking at the solution of this equation it is often useful to consider the external force (EM or non-EM) to be a pulse [17, 18, 19], however this is a mathematical idealization.

the repulsion. This cohesive force would enable the electron to remain stable, however it would by definition have no effect on the motion of the particle and its physical nature would remain unknown.

If we accept the Poincaré's hypothesis and assume an extended model for the electron, then the self force may be calculated using the Lorentz force law. It is possible to calculate the Lorentz force acting on a particular volume element due to the rest of the charge distribution. The self force is then given by net force on the particle due to the respective Lorentz forces on each of the volume elements. In order to calculate this force it is necessary to postulate an additional condition on the model, that of rigidity. The most common notion of rigidity is that of Born rigidity, where the particle is rigid in its rest frame. In the early 1900s Lorentz [22] and Schott[23], amongst others, were able to calculate the resulting force for a number of different charge distributions. Non-relativistically, for a Born rigid , spherically symmetric charge distribution instantaneously at rest, the calculation yields[4]

$$\underline{f}_{\text{self}} \approx -\frac{2}{3c^2} q\kappa U \ddot{\underline{x}} + \frac{2}{3c^3} q\kappa \dddot{\underline{x}} - \frac{2}{3c^2} q\kappa \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{d^n \ddot{\underline{x}}}{c^n dt^n} \mathcal{O}(r^{n-1}), \tag{2.5}$$

where $\underline{f}_{\text{self}}$ is a 3-vector, $\dot{\underline{x}}$ is the 3-acceleration of the charge and the dot denotes differentiation with respect to time. The constant κ is defined by (1.115) and r denotes the radius of the distribution. The constant U is given by,

$$U = \int\int \frac{n(\underline{x})n(\underline{x}')}{r} d^3x d^3x' \tag{2.6}$$

where $n(\underline{x})/q$ is the normalized charge distribution. In the limit $r \to 0$, i.e. the point charge limit, the terms in the summation vanish. The resulting equation of motion for $r \to 0$ is given by

$$m_0 \ddot{\underline{x}} \approx -\frac{2}{3c^2} q\kappa U \ddot{\underline{x}} + \frac{2}{3c^3} q\kappa \dddot{\underline{x}} + \underline{f}_{\text{L}} + \underline{f}_{\text{ext}}, \tag{2.7}$$

where m_0 is the *bare mass* and $\underline{f_{\mathrm{L}}}$ and $\underline{f_{\mathrm{ext}}}$ are 3-vectors. We notice the first term on the right hand side is proportional to the acceleration of the electron. This led to the identification of the coefficient $m_e = \frac{2}{3c^2}q\kappa U$ as an electromagnetic contribution to the observed rest mass of the particle. This enables the term to be shifted to the left hand side of (2.7), resulting in the equation of motion

$$m\underline{\ddot{x}} = \frac{2}{3c^3}q\kappa\underline{\dddot{x}} + \underline{f_{\mathrm{L}}} + \underline{f_{\mathrm{ext}}}, \tag{2.8}$$

where $m = m_0 + m_e$ is the observed rest mass given by the sum of the electromagnetic mass and bare mass. This is known as the Lorentz-Abraham equation, and is the non-relativistic limit of (2.3). The process of shifting the term $m_e\underline{\dot{x}}$ to the left hand side is known as *mass renormalization.* In the point charge approach mass renormalization is still required, however the electromagnetic mass of the point particle is found to be infinite. This means the bare mass must be assumed to be negatively infinite in order to leave a finite observed mass and a meaningful equation of motion. This process of adding two infinite quantities to give a finite mass is undesirable and brings into question the validity of the resulting equation of motion. [3].

With the advance of physics since the early twentieth century it is now clear that any notion of rigidity is incompatible with special relativity. It is also known that electrons and other charged elementary particles exhibit wave particle duality and other quantum behavior. This has lead to almost complete abandonment of the macroscopic model in favor of other models which do not cling to the idea of miniature classical distributions of charge. The simplest such model is that of a point charge. However if we adopt the point charge model from the outset it is not obvious how to define the self force because the Liénard-Wiechert field is singular at the position of the particle. In 1938 Dirac proposed a method by which the self

[3]There have been attempts to eradicate mass normalization, for example see [24], where different mathematical techniques are used to cancel the singular terms, however there remains no physical justification. The inability of classical physics to consistently treat field divergences has lead to further needs for renormalization in quantum field theory.

force arises as an integral of the stress-energy-momentum tensor associated with the Liénard-Wiechert field. In 1973 Rohrlich writes [25]

> Whatever one may think today of Dirac's reasons in developing a classical theory of a point electron, it is by many contemporary views (and I completely concur), the correct thing to do: if one does not wish to exceed the applicability limits of classical (i.e., non-quantum) physics one cannot explore the electron down to distances so short that its structure (whatever it might be) would become apparent. Thus for the classical physicist the electron is a point charge within his limits of observation.

2.2 The point charge approach and the Schott term discrepancy

Within the point model framework the components of the instantaneous change in electromagnetic 4-momentum $\dot{\mathrm{P}}_{\mathrm{EM}} \in \Gamma\mathrm{T}\mathcal{M}$ arise as integrals of the Liénard-Wiechert stress 3-forms over a suitable three dimensional domain of spacetime. This instantaneous change in 4-momentum is identified as the negative of the self force but with an additional singular term which can be discarded by mass renormalization. In Dirac's calculation the domain is the side Σ_T^{D} of a narrow tube, of spatial radius $R_{\mathrm{D}0}$, enclosing a section of the worldline C. See FIG. 2.1. The displacement vector Y defining the Dirac tube is spacelike, therefore the Liénard-Wiechert potential is not naturally given in terms of the Dirac time $\tau_D(x)$. However in appendix D we show that for small $R_D = g(Y,Y)$, and hence small $\tau_D - \tau_r$ due to (D.20), it can be expressed as the series

$$\begin{aligned}\frac{1}{\kappa}\mathrm{A}|_x = &-\frac{V_D}{R_D} + \big(A_D + \frac{1}{2}g(n_D, A_D)V_D\big)\\ &+ \Big(V_D\big(\frac{1}{8}g(A_D,A_D) - \frac{1}{8}g(n_D,A_D)^2 - \frac{1}{3}g(n_D,\dot{A}_D)\big) - \frac{1}{2}\dot{A}_D - \frac{1}{2}g(n_D,A_D)A_D\Big)R_D\\ &+ \mathcal{O}(R_D^2),\end{aligned} \tag{D.29}$$

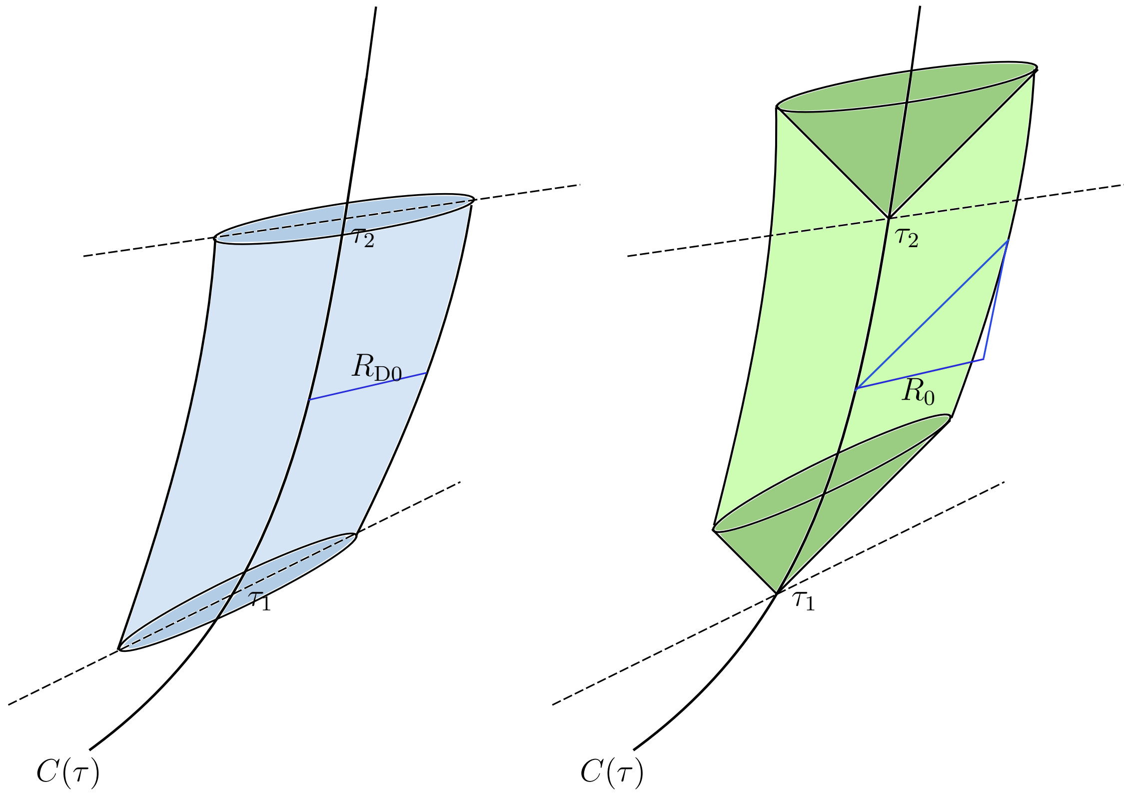


Figure 2.1: The Dirac tube (left) and Bhabha tube (right)

where the vector fields $V_D, A_D, \dot{A}_D \in \Gamma\mathrm{T}(\mathcal{M}\backslash C)$ are defined as

$$V_D|_x = \dot{C}^j(\tau_D(x))\frac{\partial}{\partial y^j}, \quad A_D|_x = \ddot{C}^j(\tau_D(x))\frac{\partial}{\partial y^j} \quad \text{and} \quad \dot{A}_D|_x = \dddot{C}^j(\tau_D(x))\frac{\partial}{\partial y^j}, \tag{D.4}$$

When using a Dirac tube the integration of the stress 3-forms gives for the instantaneous EM 4-momentum [17, 26, 10, 24]

$$-\dot{\mathrm{P}}^{\mathrm{D}}_{\mathrm{EM}} = q\kappa\Big(\frac{2}{3}\big(\dddot{C} - g(\ddot{C},\ddot{C})\dot{C}\big) - \lim_{R_{\mathrm{D0}}\to 0}\frac{1}{2R_{\mathrm{D0}}}\ddot{C}\Big), \tag{2.9}$$

This is the Abraham-von Laue vector with the additional singular term which depends on the shrinking of the Dirac tube onto the worldline.

An alternative approach, first used by Bhabha[27] in 1939 , is to integrate the Liénard-Wiechert stress forms over the side Σ_{T} of the Bhabha tube with spatial radius R_0. The principal advantage of this approach is that the displacement vector X is lightlike and as a result the Liénard-Wiechert potential can written explicitly,

$$\frac{1}{\kappa}\mathrm{A}|_x = \frac{\widetilde{V}}{g(V,X)}. \tag{1.106}$$

It follows that the corresponding stress 3-forms can also be written explicitly. However previous articles which use a Bhabha tube to evaluate the instantaneous EM 4-momentum give the following expression [24, 27, 21, 28, 20]

$$-\dot{\mathrm{P}}_{\mathrm{EM}} = -q\kappa\Big(\frac{2}{3}g(\ddot{C},\ddot{C})\dot{C} + \lim_{R_0\to 0}\frac{1}{2R_0}\ddot{C}\Big) = -\dot{\mathrm{P}}^{\mathrm{D}}_{\mathrm{EM}} - \frac{2}{3}q\kappa\dddot{C}. \tag{2.10}$$

This is the radiation reaction force with the additional singular term which depends on the shrinking of the Bhabha tube onto the worldline. The Schott term $2q\kappa\frac{\dddot{C}}{3}$ is missing from the approaches employing the Bhabha tube. In 2006 Gal'tsov and Spirin [10] draw attention to this discrepancy. They claim the Schott term should arise directly from the electromagnetic stress-energy-momentum tensor and provide a derivation using Dirac geometry in order to show this. However they propose the missing term in (2.10) is a consequence of the null geometry used to define the Bhabha tube. We show in this thesis that the term may be obtained using null geometry providing certain conditions are realized.

2.3 Regaining the Schott term

Addition to non-EM momentum

The standard approach which has been used in articles [24, 27, 21, 28], is to simply add the term to the non-EM momentum of the particle. This method will give the correct form for the ALD equation, however it is not physically justified since the self force is by nature an electromagnetic effect.

We suppose a balance of momentum

$$\dot{\mathrm{P}}_{\mathrm{PART}} + \dot{\mathrm{P}}_{\mathrm{EM}} = f_{\mathrm{ext}} \tag{2.11}$$

where total momentum has been separated into electromagnetic contribution P_{EM} and non-electromagnetic contribution P_{PART}, and $\dot{P} = \nabla_{\dot{C}} P$. All the external forces acting on the particle are denoted by f_{ext}. A suitable choice for the non-electromagnetic momentum P_{PART} has to be made. Most external forces f_{ext}, including the Lorentz force, are orthogonal to $\dot{C}$:

$$g(f_{\mathrm{ext}}, \dot{C}) = 0. \tag{2.12}$$

For such an external force, if (2.9) is obtained then a natural choice for P_{PART} is

$$P_{\mathrm{PART}} = m_0 \dot{C}. \tag{2.13}$$

This is the correct term for the 4-momentum of a particle if its spin has been neglected. Combining (2.9), (2.11) and (2.13) gives

$$\begin{aligned} m_0 \ddot{C} &= f_{\mathrm{ext}} - \dot{\mathrm{P}}^{\mathrm{D}}_{\mathrm{EM}} \\ &= f_{\mathrm{ext}} - q\kappa\Big(\tfrac{2}{3} g(\ddot{C}, \ddot{C})\dot{C} + \dddot{C} - \lim_{R_{\mathrm{D}0}\to 0} \frac{1}{2R_{\mathrm{D}0}} \ddot{C}\Big). \end{aligned} \tag{2.14}$$

Thus assuming the observed rest mass m to be given by

$$m = m_0 + \lim_{R_{\mathrm{D}0}\to 0} \frac{q\kappa}{2R_{\mathrm{D}0}} \ddot{C} \tag{2.15}$$

we satisfy the orthogonality condition (1.28). By contrast, if (2.10) is obtained one cannot set

$$m_0 \ddot{C} = f_{\mathrm{ext}} - \dot{\mathrm{P}}_{\mathrm{EM}} \qquad \text{and} \qquad m = m_0 + \lim_{R_0\to 0} \frac{q\kappa}{2R_0} \ddot{C} \tag{2.16}$$

and satisfy (1.28). This has lead some authors [24, 27, 21, 28] to add an ad hoc term to the non-electromagnetic contribution to the force.

$$\dot{P}^{\mathrm{B}}_{\mathrm{PART}} = m_0 \ddot{C} + \tfrac{2}{3} q \kappa \dddot{C}. \tag{2.17}$$

This ad hoc term will ensure the orthogonality condition is satisfied and hence compensate for the missing Schott term.

Regaining the term by careful analysis of limits

We will show that the calculation of the self force using null geometry requires three limits to be taken (see figure 2.2), the shrinking of the Bhabha tube Σ_{T} onto the worldline C i.e. $R_0 \to 0$, and the bringing together of the lightlike caps Σ_1 and Σ_2 onto the lightlike cone with vertex $C(\tau_0)$ i.e. $\tau_1 \to \tau_0$ $\tau_2 \to \tau_0$, where τ_0 is the proper time at which we wish to evaluate the self force (see FIG.2.1). We therefore have the freedom to choose the order of these limits. We choose to let the three limits take place simultaneously, subject to the constraint that

$$\lambda = \lim_{\substack{R_0 \to 0 \\ \tau_1 \to \tau_0 \\ \tau_2 \to \tau_0}} \left(\frac{\tau_1 + \tau_2 - 2\tau_0}{4R_0} \right) \tag{2.18}$$

where $\lambda \in \mathbb{R}$ is finite. This gives the self force as

$$f_{\mathrm{self}} = -q\kappa \left(\tfrac{2}{3} g(\ddot{C}, \ddot{C}) \dot{C} + \lambda \dddot{C} + \lim_{R_0 \to 0} \frac{1}{2R_0} \ddot{C} \right) \tag{2.19}$$

which is in agreement with $f^{\mathrm{D}}_{\mathrm{self}}$ if $\lambda = -\frac{2}{3}$, hence the Schott term arises by direct integration of the stress forms using null geometry.

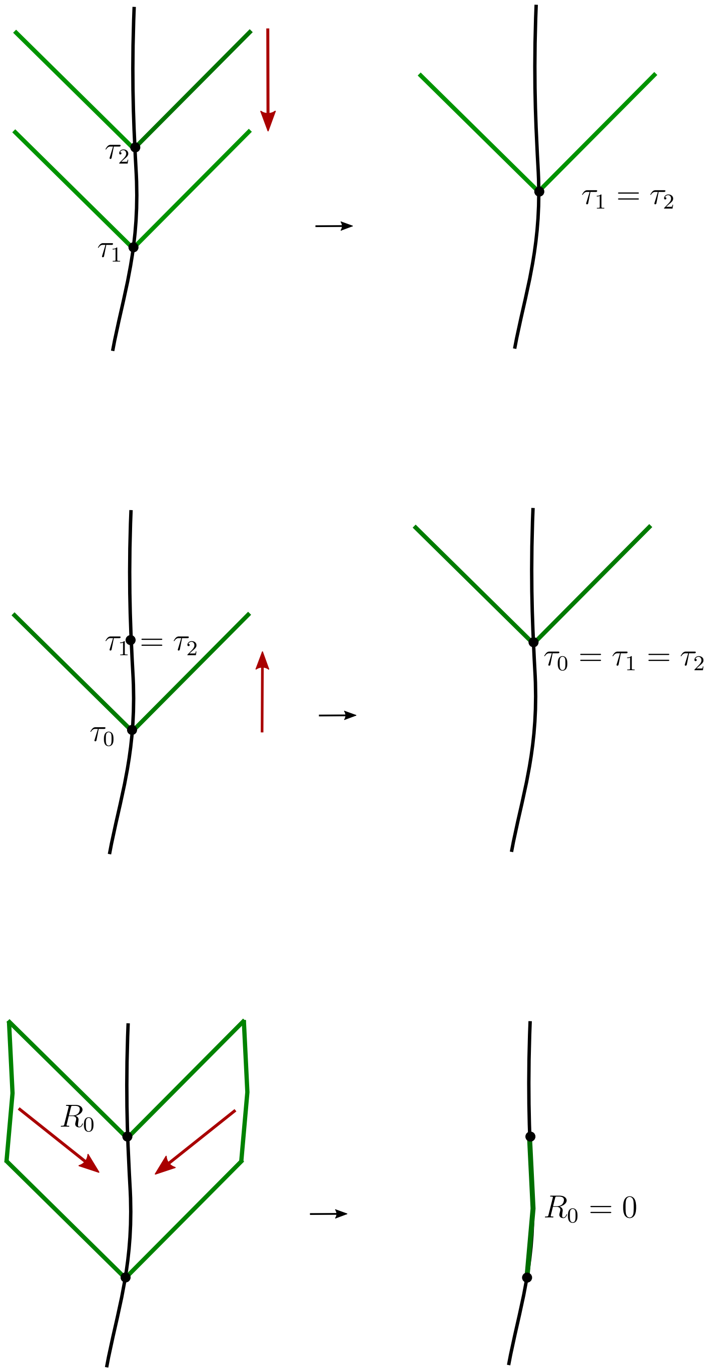


Figure 2.2: Three independent limits are required. The converging of the caps $\tau_1 \to \tau_2$, the movement of the apex of the squashed tube to the point where the self force will be evaluated $\tau_2 \to \tau_0$, and the shrinking of the radius $R_0 \to 0$.

Chapter 3

Defining the self force for a point charge

In this chapter we formally define the Dirac and Bhabha tubes. We also present the definition of the self force based on conservation of four momentum within a Bhabha tube. For this definition we take the limit as the tube approaches an arbitrary point on the worldline.

3.1 The Dirac and Bhabha tubes

Definition 3.1.1. Consider the region $N = \widetilde{N} \backslash C$ where $\widetilde{N} \subset \mathcal{M}$ is a local neighborhood of the worldline. Suppose the two continuous maps

$$\tau' : N \to \mathbb{R}, \quad x \mapsto \tau'(x) \tag{3.1}$$

$$R' : N \to \mathbb{R}^+, \quad x \mapsto R'(x), \tag{3.2}$$

are well defined for all $x \in N$. Here $\mathbb{R}^+$ denotes the positive real numbers and we shall call $R'(x)$ the displacement of x from $C(\tau'(x))$. Furthermore for $\lambda \in \mathbb{R}^+$ let

$$\begin{aligned} &\tau'\Big(\lambda\Big(x - C\big(\tau'(x)\big)\Big) + C\big(\tau'(x)\big)\Big) = \tau'(x), \\ \text{and} \quad &R'\Big(\lambda\Big(x - C\big(\tau'(x)\big)\Big) + C\big(\tau'(x)\big)\Big) = \lambda R'(x). \end{aligned} \tag{3.3}$$

These relations ensure that for an arbitrary point τ_0 on the worldline with $C(\tau_0) \in \widetilde{N}$

$$\lim_{x \to C(\tau_0)} \tau'(x) = \tau_0 \qquad \text{and} \qquad \lim_{x \to C(\tau_0)} R'(x) = 0 \tag{3.4}$$

Since N is open there exist values $\tau_{\min}, \tau_{\max}, R_{\max} \in N$ such that the 4-region

$$\mathbb{S} = \left\{ \quad x \quad \middle| \quad \tau_{\min} < \tau'(x) < \tau_{\max}, \quad 0 < R'(x) < R'_0 \quad \right\}, \tag{3.5}$$

where $R'_0 < R_{\max}$, is well defined .

Definition 3.1.2. The 3-boundary of this region $\mathbb{T} = \partial\mathbb{S}$ is a known as a **world-tube** and is defined by $\mathbb{T} = \Sigma'_1 \cup \Sigma'_2 \cup \Sigma'_{\mathrm{T}}$ where for $i = 1, 2$

$$\Sigma'_i = \left\{ \quad x \quad \middle| \tau'(x) = \tau_i, \quad 0 < R'(x) \leq R'_0 \quad \right\}, \tag{3.6}$$

where R'_0 and τ_i are constants and $\tau_i = \tau'(x)$ for all $x \in \Sigma'_i$.

$$\Sigma'_{\mathrm{T}} = \left\{ \quad x \quad \middle| \quad \tau_2 \leq \tau' \leq \tau_1, \quad R' = R'_0 \quad \right\}. \tag{3.7}$$

We call Σ'_i the caps of the worldtube $\mathbb{T}$ and they are surfaces of constant τ' whose boundaries are topological 2-sheres. We call Σ'_{T} the side of $\mathbb{T}$ and it is a timelike surface of constant $R' = R'_0$ topologically equivalent to a cylinder. We call τ' the *worldline map* associated with $\mathbb{T}'$ and R' the *displacement map.*

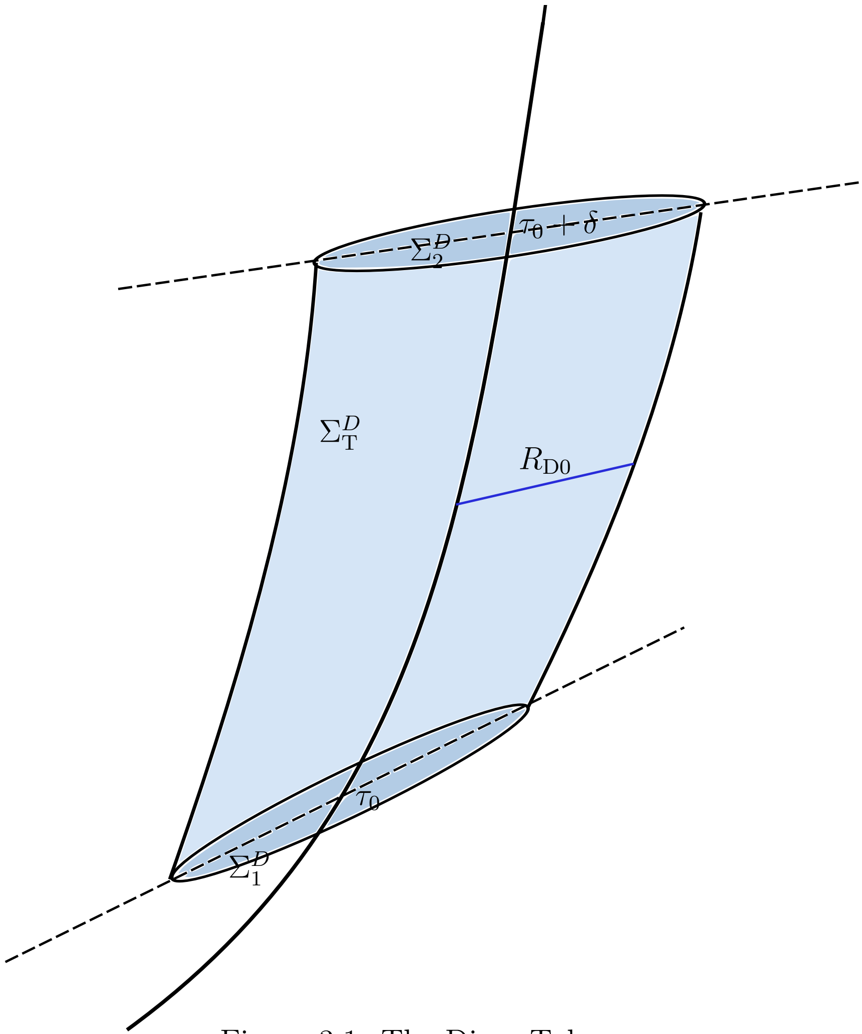


Figure 3.1: The Dirac Tube

Lemma 3.1.3. *When using Dirac geometry* $\tau' = \tau_D$ *and* $R' = R_D = \sqrt{g(Y,Y)}$*. The surfaces* $\Sigma_i^{\rm D}$ *are subregions of the planes of simultaneity according to an observer comoving at* $C(\tau_i)$*. The* **Dirac tube** $\mathbb{T}_D$ *is defined by* $\Sigma_1^{\rm D} \cup \Sigma_2^{\rm D} \cup \Sigma_{\rm T}^{\rm D}$ *where for* $i = 1, 2$

$$\Sigma_i^{\rm D} = \Big\{ C(\tau_i) + Y \quad \Big| \quad g(Y, \dot{C}) = 0, \quad g(Y,Y) < (R_{\rm D0})^2 \Big\},$$
$$\Sigma_{\rm T}^{\rm D} = \Big\{ C(\tau) + Y \quad \Big| \quad g(Y, \dot{C}) = 0 \quad g(Y,Y) = (R_{\rm D0})^2, \quad \tau_1 \le \tau \le \tau_2 \Big\}.$$

The parameter $R_{\rm D0} > 0$ *is a measure of the cross-sectional radius of the Dirac tube, see figure 3.1.*

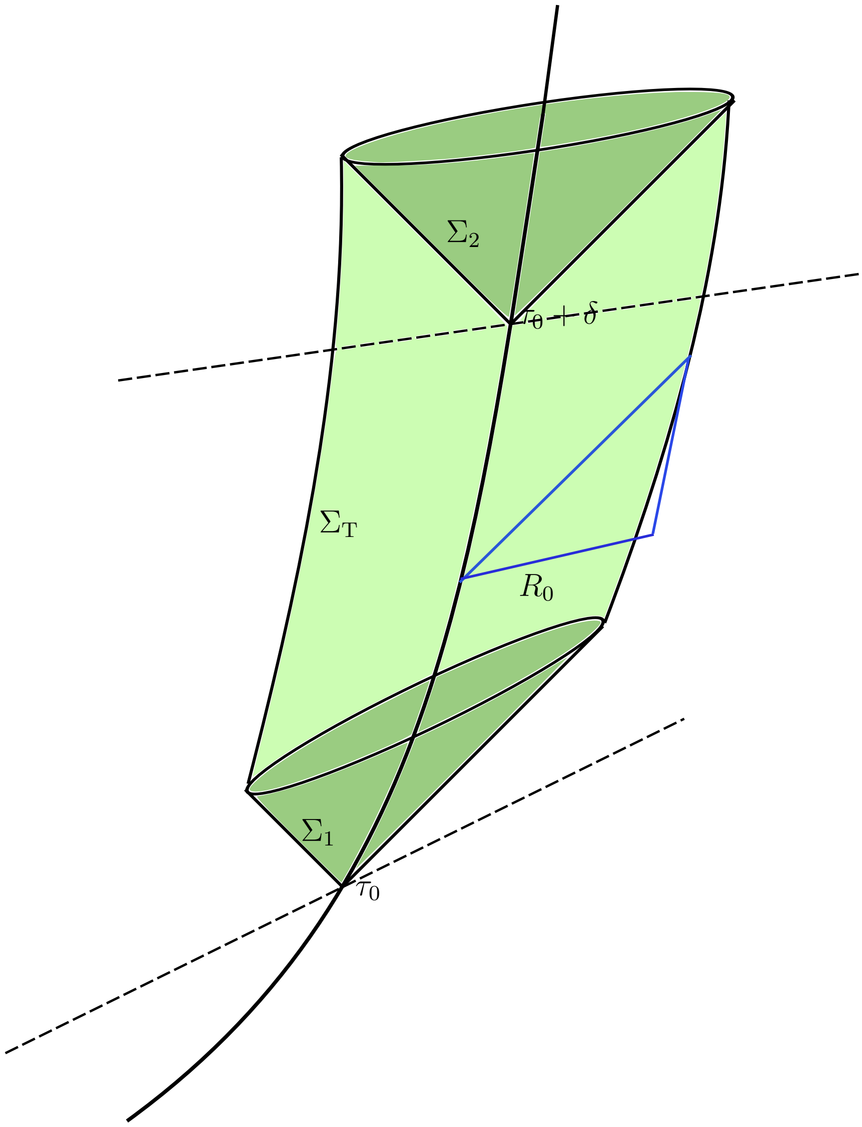


Figure 3.2: The Bhabha Tube

Lemma 3.1.4. *When using null geometry $\tau' = \tau_r$ and $R' = R = -g(X, V)$. The surfaces Σ_i are subregions of the forward null cones at $C(\tau_i)$. The* **Bhabha tube** *$\mathbb{T}_B$ is given by $\Sigma_1 \cup \Sigma_2 \cup \Sigma_{\mathrm{T}}$ where for $i = 1, 2$*

$$\Sigma_i = \left\{ C(\tau_i) + X \;\middle|\; g(X, X) = 0, \quad -g(X, \dot{C}) < R_0 \right\},$$
$$\Sigma_{\mathrm{T}} = \left\{ C(\tau) + X \;\middle|\; g(X, X) = 0, \quad -g(X, \dot{C}) = R_0, \quad \tau_1 \leq \tau \leq \tau_2 \right\}. \quad (3.8)$$

The parameter $R_0 > 0$ is a measure of the cross-sectional radius of the Bhabha tube, see figure 3.2.

3.2 Conservation of 4-momentum

Lemma 3.2.1. *Consider figure 3.3. The two Bhabha tubes* $\mathbb{T}^{\text{in}}$ *and* $\mathbb{T}^{\text{out}}$ *given by*

$$\begin{aligned} \mathbb{T}^{\text{in}} =& \Sigma_1^{\text{in}} \cup \Sigma_2^{\text{in}} \cup \Sigma_{\text{T}}^{\text{in}}, \\ \text{and} \qquad \mathbb{T}^{\text{out}} =& \Sigma_1^{\text{out}} \cup \Sigma_2^{\text{out}} \cup \Sigma_{\text{T}}^{\text{out}}, \end{aligned} \tag{3.9}$$

have different radii R^{in} *and* R^{out}*. The surfaces* Σ_1^{diff} *and* Σ_2^{diff} *are the differences between the caps of the two tubes,*

$$\begin{aligned} \Sigma_1^{\text{diff}} =& \Sigma_1^{\text{out}} \backslash \Sigma_1^{\text{in}} \\ \text{and} \qquad \Sigma_2^{\text{diff}} =& \Sigma_2^{\text{out}} \backslash \Sigma_2^{\text{in}}. \end{aligned} \tag{3.10}$$

Let the 4-region $\mathbb{S}$ *enclosed by the two tubes be finite and source free, with boundary*

$$\partial \mathbb{S} = \Sigma_1^{\text{diff}} - \Sigma_2^{\text{diff}} + \Sigma_{\text{T}}^{\text{out}} - \Sigma_{\text{T}}^{\text{in}}. \tag{3.11}$$

then the following relation is true

$$\int_{\Sigma_1^{\text{diff}}} \text{S}_{\text{K}} - \int_{\Sigma_2^{\text{diff}}} \text{S}_{\text{K}} = \int_{\Sigma_{\text{T}}^{\text{in}}} \text{S}_{\text{K}} - \int_{\Sigma_{\text{T}}^{\text{out}}} \text{S}_{\text{K}} \tag{3.12}$$

Proof of 3.2.1. Using Stokes Theorem (B.103) and (1.37) it follows that

$$\begin{aligned} 0 =& \int_N d\text{S}_{\text{K}} = \int_{\partial N} \text{S}_{\text{K}} \\ =& \int_{\Sigma_1^{\text{diff}}} \text{S}_{\text{K}} - \int_{\Sigma_2^{\text{diff}}} \text{S}_{\text{K}} + \int_{\Sigma_{\text{T}}^{\text{out}}} \text{S}_{\text{K}} - \int_{\Sigma_{\text{T}}^{\text{in}}} \text{S}_{\text{K}}, \end{aligned} \tag{3.13}$$

□

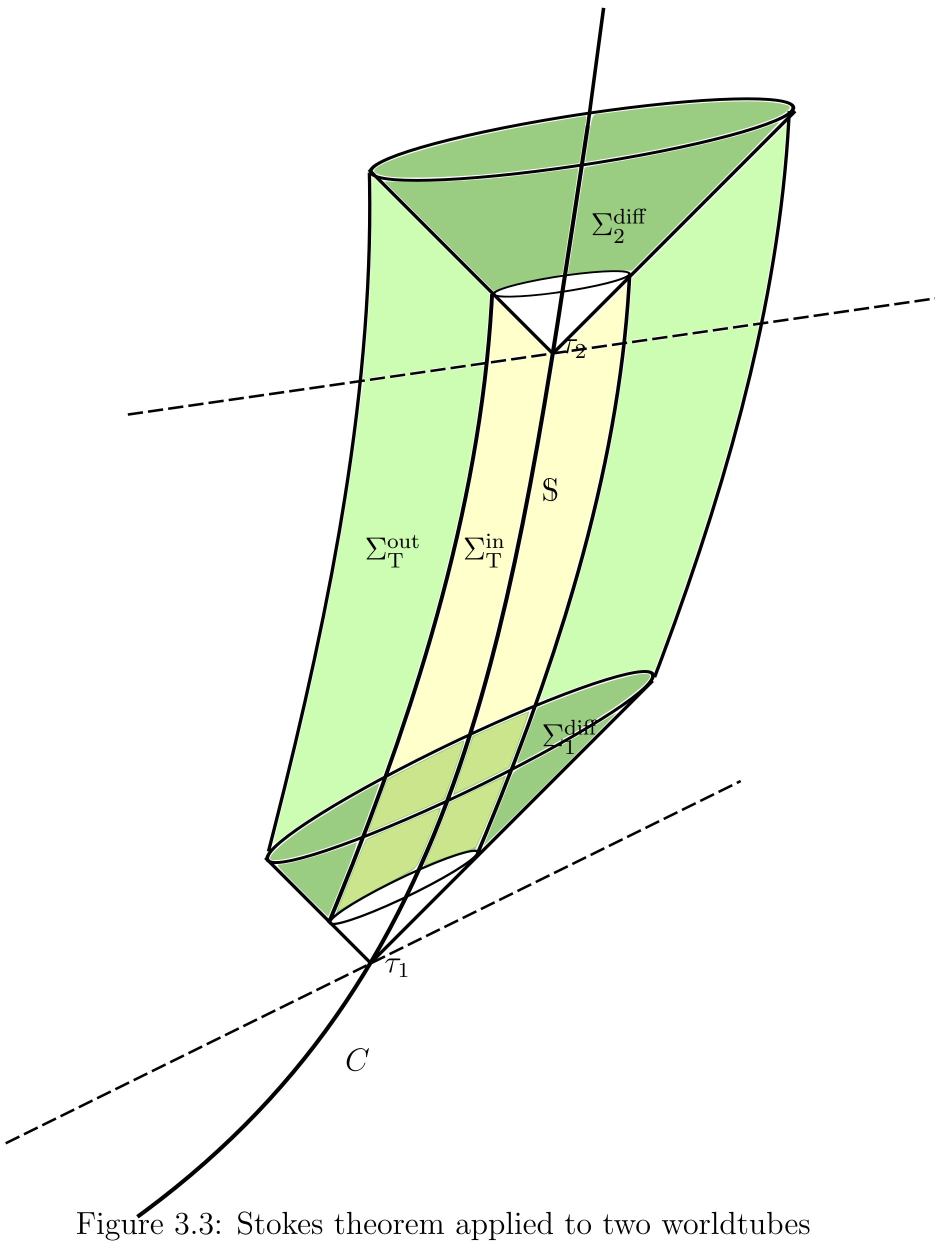


Figure 3.3: Stokes theorem applied to two worldtubes

3.3 The instantaneous force at an arbitrary point on the worldline

Definition 3.3.1. The k-component of 4-momentum flux $\mathrm{P}_{\mathrm{K}}^{(\Sigma)} \in \Gamma\Lambda^0\mathbf{M}$ through an arbitrary source-free 3-surface $\Sigma \subset \mathcal{M}$ is defined by

$$\mathrm{P}_{\mathrm{K}}^{(\Sigma)} = \int_{\Sigma} \mathrm{S}_{\mathrm{K}}, \tag{3.14}$$

We cannot use definition 3.3.1 to give the flux of momentum through the caps Σ_i of the Bhabha tube because they are not source free; there is a singularity at the point intersected by the worldline. Instead in the following we employ Stokes theorem in order to heuristically justify defining the difference in momentum between the two caps as an integral over the side Σ_{T}.

Definition 3.3.2. Consider the Bhabha tube $\mathbb{T} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_{\mathrm{T}}$ with $R = R_0$ where R_0 is constant. Let $\tau_1 = \tau_r(\Sigma_1)$ and $\tau_2 = \tau_r(\Sigma_2)$ with $\tau_2 > \tau_1$. The instantaneous change in 4-momentum at arbitrary proper time τ_0 is defined by

$$\dot{\mathrm{P}}_{\mathrm{K}}(\tau_0) = \lim_{\substack{R_0 \to 0 \\ \tau_1 \to \tau_0 \\ \tau_2 \to \tau_0}} \left(\frac{1}{\tau_2 - \tau_1} \int_{\Sigma_{\mathrm{T}}} \mathrm{S}_{\mathrm{K}} \right). \tag{3.15}$$

This definition is justified heuristically as follows. Inspired by definition 3.3.1 we wish to write

$$\Delta \mathrm{P}_{\mathrm{K}} = \mathrm{P}_{\mathrm{K}}^{(\Sigma_2)} - \mathrm{P}_{\mathrm{K}}^{(\Sigma_1)} = \int_{\Sigma_{\mathrm{T}}} \mathrm{S}_{\mathrm{K}} \tag{3.16}$$

However the integrals $\mathrm{P}_{\mathrm{K}}^{(\Sigma_1)}$ and $\mathrm{P}_{\mathrm{K}}^{(\Sigma_2)}$ are both infinite [1]. Ignoring this, we assert

$$\dot{\mathrm{P}}_{\mathrm{K}}(\tau_0) = \lim_{\substack{R_0 \to 0 \\ \tau_1 \to \tau_0 \\ \tau_2 \to \tau_0}} \left(\frac{1}{\tau_2 - \tau_1} \left(\mathrm{P}_{\mathrm{K}}^{(\Sigma_2)} - \mathrm{P}_{\mathrm{K}}^{(\Sigma_1)} \right) \right) \tag{3.17}$$

Inserting (3.16) into (3.17) yields (3.15).

[1] There are methods by which these infinities can be avoided, for example Rowe [29] uses distribution theory in order to eradicate singularity, and Norton [24] uses a method where the zero limit is not used.

Definition 3.3.3. The vector $\dot{\mathrm{P}}_{\mathrm{EM}}(\tau_0) \in T_{C(\tau_0)}\mathcal{M}$ is defined by

$$\dot{\mathrm{P}}_{\mathrm{EM}}(\tau_0) = \dot{\mathrm{P}}_{\mathrm{K}}(\tau_0) g^{\mathrm{K}l} \frac{\partial}{\partial y^l} \tag{3.18}$$

where $g^{\mathrm{K}l} = g^{-1}(dy^{\mathrm{K}}, dy^l)$ and g^{-1} is the inverse metric on $\mathcal{M}$. Since τ_0 is arbitrary there is an induced vector field $\dot{\mathrm{P}}_{\mathrm{EM}}$ on the curve C.

Lemma 3.3.4. *Let Liénard-Wiechert stress 3-forms* $\mathrm{S}_{\mathrm{K}} \in \Gamma\Lambda^3\mathcal{M}$ *be given by*

$$\mathrm{S}_{\mathrm{K}} = \mathrm{S}^{\mathrm{R}}_{\mathrm{K}} + \mathrm{S}^{\mathrm{C}}_{\mathrm{K}} + \mathrm{S}^{\mathrm{R\,C}}_{\mathrm{K}} \tag{3.19}$$

where

$$\begin{aligned}
\mathrm{S}^{\mathrm{R}}_{\mathrm{K}} =& \frac{\epsilon_0}{2}\big(i_{\partial_{\mathrm{K}}}\mathrm{F}_{\mathrm{R}} \wedge \star\mathrm{F}_{\mathrm{R}} - i_{\partial_{\mathrm{K}}} \star \mathrm{F}_{\mathrm{R}} \wedge \mathrm{F}_{\mathrm{R}}\big)\\
\mathrm{S}^{\mathrm{C}}_{\mathrm{K}} =& \frac{\epsilon_0}{2}\big(i_{\partial_{\mathrm{K}}}\mathrm{F}_{\mathrm{C}} \wedge \star\mathrm{F}_{\mathrm{C}} - i_{\partial_{\mathrm{K}}} \star \mathrm{F}_{\mathrm{C}} \wedge \mathrm{F}_{\mathrm{C}}\big)\\
\mathrm{S}^{\mathrm{RC}}_{\mathrm{K}} =& \frac{\epsilon_0}{2}\big(i_{\partial_{\mathrm{K}}}\mathrm{F}_{\mathrm{C}} \wedge \star\mathrm{F}_{\mathrm{R}} - i_{\partial_{\mathrm{K}}} \star \mathrm{F}_{\mathrm{C}} \wedge \mathrm{F}_{\mathrm{R}} + i_{\partial_{\mathrm{K}}}\mathrm{F}_{\mathrm{R}} \wedge \star\mathrm{F}_{\mathrm{C}} - i_{\partial_{\mathrm{K}}} \star \mathrm{F}_{\mathrm{R}} \wedge \mathrm{F}_{\mathrm{C}}\big)
\end{aligned} \tag{3.20}$$

Then

$$\begin{aligned}
\mathrm{S}^{\mathrm{R}}_{\mathrm{K}} =& q^2\frac{g(A,A) - g(n,A)^2}{16\pi^2\epsilon_0 R^2} n_{\mathrm{K}} \star \widetilde{n},\\
\mathrm{S}^{\mathrm{C}}_{\mathrm{K}} =& \frac{q^2}{16\pi^2\epsilon_0 R^4}\big(n_{\mathrm{K}} \star \widetilde{V} + V_{\mathrm{K}} \star \widetilde{n} - n_{\mathrm{K}} \star \widetilde{n} + g_{\mathrm{K}a} \star dx^a\big),\\
\mathrm{S}^{\mathrm{RC}}_{\mathrm{K}} =& -\frac{q^2}{8\pi^2\epsilon_0 R^3}\big(A_{\mathrm{K}} \star \widetilde{n} + n_{\mathrm{K}} \star \widetilde{A} + g(A,n)(V_{\mathrm{K}} \star \widetilde{n} + n_{\mathrm{K}} \star \widetilde{V} - 2n_{\mathrm{K}} \star \widetilde{n})\big).
\end{aligned} \tag{3.21}$$

Note that the factor of c in (1.34) is absent from (3.20). This is due to our decision to assign the dimension of length to proper time.

Proof of 3.3.4.

We will show only for for $\mathrm{S}^{\mathrm{R}}_{\mathrm{K}}$, the other results follow similarly. From (1.112) we have

$$
\begin{aligned}
\frac{1}{\kappa} i_{\partial_{\mathrm{K}}} \mathrm{F}_{\mathrm{R}} =& \frac{1}{g(X,V)^2} i_{\partial_{\mathrm{K}}} (\widetilde{X} \wedge \widetilde{A}) - \frac{g(X,A)}{g(X,V)^3} i_{\partial_{\mathrm{K}}} (\widetilde{X} \wedge \widetilde{V}) \\
=& \frac{1}{g(X,V)^2} (X_{\mathrm{K}} \widetilde{A} - A_{\mathrm{K}} \widetilde{X}) - \frac{g(X,A)}{g(X,V)^3} (X_{\mathrm{K}} \widetilde{V} - V_{\mathrm{K}} \widetilde{X})
\end{aligned}
$$

where κ is defined by (1.115). Thus

$$
\begin{aligned}
\frac{1}{\kappa^2} i_{\partial_{\mathrm{K}}} \mathrm{F}_{\mathrm{R}} \wedge \star \mathrm{F}_{\mathrm{R}} =& \frac{1}{g(X,V)^4} \Big(X_{\mathrm{K}} \widetilde{A} \wedge \star(\widetilde{X} \wedge \widetilde{A}) - A_{\mathrm{K}} \widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{A}) \Big) \\
& - \frac{g(X,A)}{g(X,V)^5} \Big(X_{\mathrm{K}} \widetilde{A} \wedge \star(\widetilde{X} \wedge \widetilde{V}) - A_{\mathrm{K}} \widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{V}) \Big) \\
& - \frac{g(X,A)}{g(X,V)^5} \Big(X_{\mathrm{K}} \widetilde{V} \wedge \star(\widetilde{X} \wedge \widetilde{A}) - V_{\mathrm{K}} \widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{A}) \Big) \\
& + \frac{g(X,A)^2}{g(X,V)^6} \Big(X_{\mathrm{K}} \widetilde{V} \wedge \star(\widetilde{X} \wedge \widetilde{V}) - V_{\mathrm{K}} \widetilde{X} \wedge \star(\widetilde{X} \wedge \widetilde{V}) \Big)
\end{aligned} \tag{3.22}
$$

Now using lemma B.2.9 yields

$$
\begin{aligned}
\frac{1}{\kappa^2} i_{\partial_{\mathrm{K}}} \mathrm{F}_{\mathrm{R}} \wedge \star \mathrm{F}_{\mathrm{R}} =& \frac{1}{g(X,V)^4} \Big(X_{\mathrm{K}} g(A,A) \star \widetilde{X} - X_{\mathrm{K}} g(A,X) \star \widetilde{A} - A_{\mathrm{K}} g(X,A) \star \widetilde{X} \Big) \\
& - \frac{g(X,A)}{g(X,V)^5} \Big(- X_{\mathrm{K}} g(A,X) \star \widetilde{V} - A_{\mathrm{K}} g(V,X) \star \widetilde{X} \Big) \\
& - \frac{g(X,A)}{g(X,V)^5} \Big(- X_{\mathrm{K}} g(V,X) \star \widetilde{A} - V_{\mathrm{K}} g(A,X) \star \widetilde{X} \Big) \\
& + \frac{g(X,A)^2}{g(X,V)^6} \Big(- X_{\mathrm{K}} \star \widetilde{X} - X_{\mathrm{K}} g(V,X) \star \widetilde{V} - V_{\mathrm{K}} g(V,X) \star \widetilde{X} \Big)
\end{aligned}
$$

Expanding and cancelling like terms yields

$$
\frac{1}{\kappa^2} i_{\partial_{\mathrm{K}}} \mathrm{F}_{\mathrm{R}} \wedge \star \mathrm{F}_{\mathrm{R}} = \left(g(A,A) - \frac{g(X,A)^2}{g(X,V)^2} \right) \frac{X_{\mathrm{K}} \star \widetilde{X}}{g(X,V)^4} \tag{3.23}
$$

Now we need to calculate the second term in $\mathrm{S}_{\mathrm{K}}^{\mathrm{R}}$ in (3.20). From (1.112) we have

$$\begin{aligned}\frac{1}{\kappa^2} i_{\partial_{\mathrm{K}}} \star \mathrm{F}_{\mathrm{R}} \wedge \mathrm{F}_{\mathrm{R}} =& \frac{1}{g(X,V)^4}\Big(i_{\partial_{\mathrm{K}}} \star (\widetilde{X} \wedge \widetilde{A}) \wedge \widetilde{X} \wedge \widetilde{A}\Big) \\ &- \frac{g(X,A)}{g(X,V)^5}\Big(i_{\partial_{\mathrm{K}}} \star (\widetilde{X} \wedge \widetilde{A}) \wedge \widetilde{X} \wedge \widetilde{V}\Big) \\ &- \frac{g(X,A)}{g(X,V)^5}\Big(i_{\partial_{\mathrm{K}}} \star (\widetilde{X} \wedge \widetilde{V}) \wedge \widetilde{X} \wedge \widetilde{A}\Big) \\ &+ \frac{g(X,A)^2}{g(X,V)^6}\Big(i_{\partial_{\mathrm{K}}} \star (\widetilde{X} \wedge \widetilde{V}) \wedge \widetilde{X} \wedge \widetilde{V}\Big)\end{aligned} \tag{3.24}$$

Now using lemma B.2.10 yields

$$\begin{aligned}i_{\partial_{\mathrm{K}}} \star (\widetilde{X} \wedge \widetilde{A}) \wedge \widetilde{X} \wedge \widetilde{A} =& \big(A_{\mathrm{K}} g(X,A) - X_{\mathrm{K}} g(A,A)\big) \star \widetilde{X} + X_{\mathrm{K}} g(X,A) \star \widetilde{A} - g(X,A)^2 g_{\mathrm{K}a} \star dy^a \\ i_{\partial_{\mathrm{K}}} \star (\widetilde{X} \wedge \widetilde{A}) \wedge \widetilde{X} \wedge \widetilde{V} =& g(A,X) V_{\mathrm{K}} \star \widetilde{X} + X_{\mathrm{K}} g(X,V) \star \widetilde{A} - g(X,A) g(X,V) g_{\mathrm{K}a} \star dy^a \\ i_{\partial_{\mathrm{K}}} \star (\widetilde{X} \wedge \widetilde{V}) \wedge \widetilde{X} \wedge \widetilde{A} =& g(A,X) X_{\mathrm{K}} \star \widetilde{V} - X_{\mathrm{K}} g(X,A) \star \widetilde{X} - g(X,A) g(X,V) g_{\mathrm{K}a} \star dy^a \\ i_{\partial_{\mathrm{K}}} \star (\widetilde{X} \wedge \widetilde{V}) \wedge \widetilde{X} \wedge \widetilde{V} =& \big(V_{\mathrm{K}} g(X,V) + X_{\mathrm{K}}\big) \star \widetilde{X} + X_{\mathrm{K}} g(X,V) \star \widetilde{V} - g(X,V)^2 g_{\mathrm{K}a} \star dy^a\end{aligned} \tag{3.25}$$

Substituting (3.25) into (3.24) and expanding yields

$$\frac{1}{\kappa^2} i_{\partial_{\mathrm{K}}} \star \mathrm{F}_{\mathrm{R}} \wedge \mathrm{F}_{\mathrm{R}} = \left(\frac{g(X,A)^2}{g(X,V)^2} - g(A,A)\right) \frac{X_{\mathrm{K}} \star \widetilde{X}}{g(X,V)^4} \tag{3.26}$$

Thus

$$\begin{aligned}\mathrm{S}_{\mathrm{K}}^{\mathrm{R}} =& \kappa^2 \frac{\epsilon_0}{2}\big(i_{\partial_{\mathrm{K}}} \mathrm{F}_{\mathrm{R}} \wedge \star \mathrm{F}_{\mathrm{R}} - i_{\partial_{\mathrm{K}}} \star \mathrm{F}_{\mathrm{R}} \wedge \mathrm{F}_{\mathrm{R}}\big) \\ =& \frac{q^2}{16\pi^2 \epsilon_0}\left(g(A,A) - \frac{g(X,A)^2}{g(X,V)^2}\right) \frac{X_{\mathrm{K}} \star \widetilde{X}}{g(X,V)^4}\end{aligned} \tag{3.27}$$

□

Chapter 4

The resulting expression

In this chapter we give the result of carrying out the integration in definition (3.3.2). We use a computer to carry out the calculation and make use of the Newman-Unti coordinate system. The input code for the MAPLE mathematical software can be found in appendix F. Here we state the result.

4.1 Arbitrary co-moving frame

Setting $\tau = \tau_0 + \delta$ we expand $\mathcal{S}_{\mathrm{K}}$ in powers of δ. We adapt the global Lorentz frame such that

$$\begin{aligned} & y^j(C(\tau_0)) = 0 \quad \text{for} \quad j = 0,1,2,3 \\ \text{and} \qquad & \dot{C}(\tau_0) = \frac{\partial}{\partial y^0}, \quad \ddot{C}(\tau_0) = a\frac{\partial}{\partial y^3}, \quad \dddot{C}(\tau_0) = b^j\frac{\partial}{\partial y^j} \end{aligned} \tag{4.1}$$

where $a, b^j \in \mathbb{R}$ are constants given by

$$a = \sqrt{g(\ddot{C}(\tau_0), \ddot{C}(\tau_0))}, \qquad b^j = dy^j(\dddot{C}(\tau_0)) \tag{4.2}$$

and from (1.28)

$$b^0 = a^2$$

Thus expanding $\dot{C}$ and $\ddot{C}$ we have

$$
\begin{aligned}
\dot{C}(\delta+\tau_0) &= \Big(1+\frac{b^0}{2}\delta^2\Big)\frac{\partial}{\partial y^0}+\frac{b^1}{2}\delta^2\frac{\partial}{\partial y^1}+\frac{b^2}{2}\delta^2\frac{\partial}{\partial y^2}+\Big(a\delta+\frac{b^3}{2}\delta^2\Big)\frac{\partial}{\partial y^3}+\mathcal{O}(\delta^3),\\
\ddot{C}(\delta+\tau_0) &= b^0\delta\frac{\partial}{\partial y^0}+b^1\delta\frac{\partial}{\partial y^1}+b^2\delta\frac{\partial}{\partial y^2}+\Big(a+b^3\delta\Big)\frac{\partial}{\partial y^3}+\mathcal{O}(\delta^2)
\end{aligned}
\tag{4.3}
$$

From (1.62) and (1.89) we have

$$
V|_{(\delta+\tau_0,R,\theta,\phi)}=\dot{C}(\delta+\tau_0) \qquad \text{and} \qquad A|_{(\delta+\tau_0,R,\theta,\phi)}=\ddot{C}(\delta+\tau_0) \tag{4.4}
$$

It is useful to express V and A in mixed coordinates, with the basis vectors in terms of the global Lorentz coordinates, but the coefficients expressed in terms of the Newman-Unti coordinates.

The result

Here we outline the steps taken in the MAPLE code. We begin with the expression (1.120) for the Liénard-Wiechert potential A in Newman-Unti coordinates. Taking the exterior derivative we obtain the field 2−form F and its Hodge dual $\star$F. We obtain expressions for the four translational Killing vectors $\frac{\partial}{\partial y^k}$ in Newman-Unti coordinates and using (1.34) we obtain expressions for the four electromagnetic stress 3−forms $\mathrm{S_K}$. Substituting the expansions (4.3) into these expressions we obtain the integrands, and finally using (4.14) we integrate over Σ_{T}. The result is

$$
\begin{aligned}
\frac{1}{q\kappa}\int_{\Sigma_{\mathrm{T}}} S_0 &= -\frac{1}{4}b^0\frac{\delta_2^2-\delta_1^2}{R_0}-\frac{2}{3}a^2(\delta_2-\delta_1)-\frac{2}{3}ab^3(\delta_2^2-\delta_1^2)+\mathcal{O}(\delta_1^3)+\mathcal{O}(\delta_2^3),\\
\frac{1}{q\kappa}\int_{\Sigma_{\mathrm{T}}} S_1 &= +\frac{1}{4}b^1\frac{\delta_2^2-\delta_1^2}{R_0}+\mathcal{O}(\delta_1^3)+\mathcal{O}(\delta_2^3),\\
\frac{1}{q\kappa}\int_{\Sigma_{\mathrm{T}}} S_2 &= +\frac{1}{4}b^2\frac{\delta_2^2-\delta_1^2}{R_0}+\mathcal{O}(\delta_1^3)+\mathcal{O}(\delta_2^3),\\
\frac{1}{q\kappa}\int_{\Sigma_{\mathrm{T}}} S_3 &= +\frac{1}{4}b^3\frac{\delta_2^2-\delta_1^2}{R_0}+\frac{1}{2}a\frac{\delta_2-\delta_1}{R_0}+\frac{1}{3}a^3(\delta_2^2-\delta_1^2)+\mathcal{O}(\delta_1^3)+\mathcal{O}(\delta_2^3)
\end{aligned}
\tag{4.5}
$$

where

$$\delta_1 = \tau_1 - \tau_0, \qquad \delta_2 = \tau_2 - \tau_0$$

and κ is given by (1.115).

Combining (4.5) into a single expression and using (3.15) and (3.18) we obtain the following expression for $\dot{\mathrm{P}}(\tau_0) \in \mathrm{T}_{C(\tau_0)}\mathcal{M}$

$$\frac{1}{q\kappa}\dot{\mathrm{P}}(\tau_0) = \tfrac{2}{3}a^2\tfrac{\partial}{\partial y^0} + \lim_{R_0\to 0}\frac{1}{2R_0}a\tfrac{\partial}{\partial y^3} + \lim_{\substack{R_0\to 0\\ \tau_1\to\tau_0\\ \tau_2\to\tau_0}}\Big(\frac{\tau_1+\tau_2-2\tau_0}{4R_0}\Big)b^j\tfrac{\partial}{\partial y^j} + \mathcal{O}\big(\delta_1^2\big) + \mathcal{O}\big(\delta_2^2\big). \tag{4.6}$$

Hence from the definition of λ (2.18) and (4.1),

$$\frac{1}{q\kappa}\dot{\mathrm{P}}(\tau_0) = +\tfrac{2}{3}g\big(\ddot{C}(\tau_0),\ddot{C}(\tau_0)\big)\dot{C}(\tau_0) + \lim_{R_0\to 0}\frac{1}{2R_0}\ddot{C}(\tau_0) + \lambda\dddot{C}(\tau_0) + \mathcal{O}\big(\delta_1^2\big) + \mathcal{O}\big(\delta_2^2\big). \tag{4.7}$$

The first term in (4.7) is the standard radiation reaction term and the second term is the singular term to be renormalized. The third term is proportional to $\dddot{C}(\tau_0)$ and therefore may be recognised as the Schott term providing the coefficient is well defined in the limit.

If λ is chosen to be finite it follows immediately that all higher order terms in the series vanish. This is because R_0^{-1} is the most divergent power of R_0 appearing in the series. Mathematically we are free to choose λ to diverge, in which case higher order terms could be made finite. However this would require extra renormalization in order to accommodate the λ terms and the resulting equation of motion would not resemble the Lorentz-Abraham-Dirac equation.

Choosing λ to be finite yields for $\dot{\mathrm{P}}_{\mathrm{EM}}(\tau) \in \mathrm{T}_{C(\tau)}\mathcal{M}$

$$\frac{1}{q\kappa}\dot{\mathrm{P}}_{\mathrm{EM}} = \tfrac{2}{3}g(\ddot{C},\ddot{C})\dot{C} + \lambda\dddot{C} + \lim_{R_0\to 0}\frac{1}{2R_0}\ddot{C}. \tag{4.8}$$

The value of λ may be fixed by satisfying the orthogonality condition (1.28),

$$0 = \frac{1}{q\kappa} g(\dot{\mathrm{P}}_{\mathrm{EM}}, \dot{C}) = -\tfrac{2}{3} g(\ddot{C}, \ddot{C}) + \lambda g(\dddot{C}, \dot{C}) = -(\tfrac{2}{3} + \lambda) g(\ddot{C}, \ddot{C}). \tag{4.9}$$

Therefore $\lambda = -\frac{2}{3}$ and the final covariant expression for f_{self} is given by

$$f_{\mathrm{self}} = -\dot{\mathrm{P}}_{\mathrm{EM}} = \tfrac{2}{3}\kappa\big(\dddot{C} - g(\ddot{C}, \ddot{C})\big)\dot{C} - \lim_{R_0 \to 0} \frac{\kappa}{2R_0} \ddot{C}, \tag{4.10}$$

which is identical to (2.9).

4.2 Conclusion

We have shown that the complete self force may be obtained directly from the Liénard-Wiechert stress 3-forms when using the null geometry with the Bhabha tube as the domain of integration. This eliminates the need to introduce the extra ad hoc term in (2.17). It also proves the reason for the missing term in previous calculations is the procedure followed in taking the limits, and not the nature of the coordinates used as proposed by Gal'tsov and Spirin [10].

We have seen that a requirement for the term to appear is that the ratio of limits λ, which describes the way in which the Bhabha tube is collapsed onto the worldline, is made finite. This is a natural choice because it demands δ_1, δ_2 and R_0 to be the same order of magnitude. The specific value $\lambda = -\frac{2}{3}$ is fixed by the orthogonality condition (1.28), however the physical justification for imposing this particular geometry on the Bhabha tube is currently unknown.

Consider figure 2.2. It is easily seen that definition (3.3.2) is equivalent to

$$\dot{\mathrm{P}}_{\mathrm{K}}(\tau_0) = \lim_{\substack{R_0 \to 0 \\ \tau_1 \to \tau_2 \\ \tau_2 \to \tau_0}} \left(\frac{1}{\tau_1 - \tau_2} \int_{\Sigma_{\mathrm{T}}} \mathrm{S}_{\mathrm{K}} \right). \tag{4.11}$$

It turns out that the limit $\tau_1 \to \tau_2$ may be taken before the other two limits. This results in the following lemma.

Lemma 4.2.1.

$$\dot{\mathrm{P}}_{\mathrm{K}}(\tau_0) = \lim_{\substack{R_0\to 0\\ \tau_1\to\tau_0}} \int_{S^2(\tau_1)} i_{\frac{\partial}{\partial\tau}} \mathrm{S}_{\mathrm{K}} \tag{4.12}$$

where $S^2(\tau_1)$ is the 2-sphere given in Newman-Unti coordinates by

$$S^2 = \Big\{(\tau, R, \theta, \phi)\Big|\tau = \tau_1, \quad R = R_0, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi\Big\} \tag{4.13}$$

Proof of 4.2.1. In Newman-Unti coordinates the side Σ_{T} of the Bhabha tube is given by

$$\Sigma_{\mathrm{T}} = \Big\{(\tau, R, \theta, \phi)\Big|\tau_1 \le \tau \le \tau_2, \quad R = R_0, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi\Big\}. \tag{4.14}$$

It follows from (4.11)that

$$\begin{aligned}\dot{\mathrm{P}}_{\mathrm{K}}(\tau_0) &= \lim_{\substack{R_0\to 0\\ \tau_2\to\tau_1\\ \tau_1\to\tau_0}} \left(\frac{1}{\tau_1-\tau_2}\int_{\tau=\tau_1}^{\tau_2}\int_{S^2(\tau)} \mathrm{S}_{\mathrm{K}}(\tau, R_0, \theta, \phi)\right)\\ &= \lim_{\substack{R_0\to 0\\ \tau_1\to\tau_0}}\left(\lim_{\tau_2\to\tau_1}\left(\frac{1}{\tau_1-\tau_2}\int_{\tau=\tau_1}^{\tau_2}\int_{S^2(\tau)} \mathrm{S}_{\mathrm{K}}(\tau, R_0, \theta, \phi)\right)\right)\end{aligned} \tag{4.15}$$

Applying theorem (B.4.4) yields result. □

Lemma 4.2.1 shows the important step in the calculation is taking the limits $R_0 \to 0$ and $\tau_1 = \tau_2 \to \tau_0$ simultaneously. If we use (4.12) instead of (3.3.2) for our definition of the self force then we obtain the following in place of (4.6),

$$\frac{1}{q\kappa}\dot{\mathrm{P}}(\tau_0) = \tfrac{2}{3}a^2\tfrac{\partial}{\partial y^0} + \lim_{R_0\to 0}\frac{1}{2R_0}a\tfrac{\partial}{\partial y^3} + \lim_{\substack{R_0\to 0\\ \tau_1\to\tau_0}}\left(\frac{\tau_1-\tau_0}{2R_0}\right)b^j\tfrac{\partial}{\partial y^j} + \mathcal{O}\big(\delta_1^2\big) + \mathcal{O}\big(\delta_2^2\big). \tag{4.16}$$

and the orthogonality condition (1.28) yields the key result

$$\lim_{\substack{R_0\to 0\\ \tau_1\to\tau_0}}\left(\frac{\tau_1-\tau_0}{2R_0}\right) = \lim_{\substack{R_0\to 0\\ \delta_1\to 0}}\left(\frac{\delta_1}{2R_0}\right) = -\frac{2}{3}. \tag{4.17}$$

In Dirac's calculation $\tau_1 = \tau_0$ and so the Schott term arises naturally without having to take this limit. However when using Dirac geometry the Liénard-Wiechert potential and stress forms have to be expanded in a Taylor series about the retarded time τ_r (see appendix D.2). In this process the relationship (D.20) is used which gives a relationship between $R_{\mathrm{D}0}$ and $\delta_r = \tau_D - \tau_r$ which is similar to (4.17).

PART II

A new approach to the reduction of wakefields

Chapter 5

Introduction

5.1 Collimation and Wakefields in a particle accelerator

It is common for accelerators to have bunches of order 10^8 particles or more. For example, ALICE, at Daresbury Laboratory, uses bunches with bunch charges of 20pC to 80pC, which represents approximately 1.25×10^8 to 5×10^8 electrons. As the bunch traverses the accelerator some of these particles will be perturbed from the ideal orbit or trajectory. This may be due to collective instabilities elsewhere in the beam or deflection due to residual gas that could not be removed from the vacuum chamber. In addition particles from the wall of the beam pipe can be accelerated in a process known as *self injection*. All of these stray particles will form a low density region of charge around the beam which is called the *beam halo*.

The presence of a large beam halo is generally undesirable. In colliders the halo particles reduce the accuracy of measurements at the interaction region, and in medical accelerators they can cause severe consequences as a result of highly energetic particles missing the desired target. In order to remove the halo from a beam specially designed apparatus called collimating systems are used.

In general collimating systems incorporate regions where the cross-sectional area of the beam pipe is reduced. Collimators are specific sections of the beam pipe which undergo a narrowing in one or both of the transverse dimensions. There are many possible configurations depending on the design requirements of individual projects. In high energy accelerators the presence of collimators can also have an adverse effect on the beam due to so called *wakefields*. Electromagnetic fields due to highly relativistic particle beams can interact with the walls of the collimator and induce *image charges* on the wall. The fields resulting from these image charges are known as wakefields and can effect the motion of trailing charges, often inducing instabilities and emittance growth. Generally fields caused by large scale geometric discontinuities, for example in cavities and collimators, are known as *geometric wakefields* and fields caused by resistivity in the wall are known as *resistive wakefields*.

5.2 Present approaches to the reduction of wakefields

There is much interest in methods for reducing the geometric wakefields produced by a charged bunch of particles passing through a collimator. The customary approach is to reduce the taper angle of the collimator. Early work on the calculation of wakefields from smoothly tapered structures was pioneered by Yokoya [30], Warnock [31] and Stupakov [32, 33]. More recent investigations by Stupakov, Bane and Zagorodnov [34, 35, 36] and Podobedov and Krinsky [37, 38] have also looked at the effect of altering the transverse cross section of the collimator. A detailed analysis of the numerical and analytic calculation of collimator wakefields, including an informative introduction to the topic, may be found in [39]. All of the present methods for minimizing wakefields rely on altering the geometry of the collimator. In this thesis we propose a new method where the trajectory of the beam is altered.

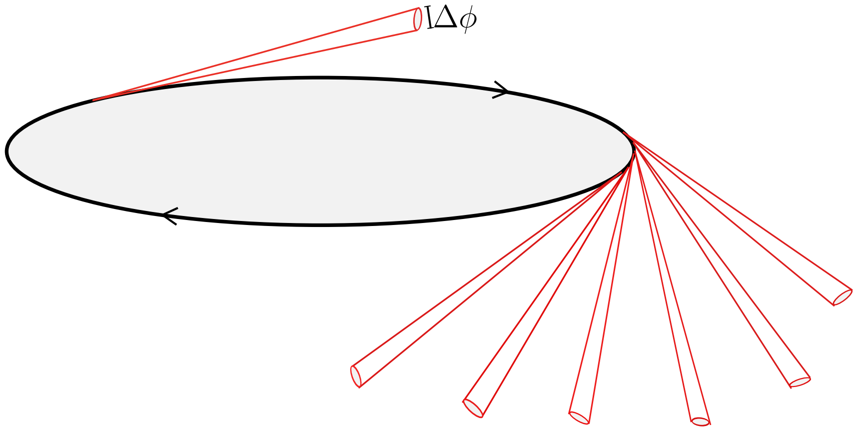


Figure 5.1: Synchrotron radiation

5.3 The relativistic Liénard-Wiechert field

A relativistic particle undergoing nonlinear acceleration will generate a radiation field primarily in the instantaneous direction of motion (see figure 5.1). This is known as **synchrotron radiation**. For high γ-factors the bulk of the field lies inside an angle $\Delta\phi \sim 1/\gamma$ where $\Delta\phi$ is the angle from the direction of motion. By contrast, a relativistic particle with constant velocity will generate no radiation field. This is easily seen from the form of $\mathrm{F_R}$ in equation (1.112) since when the acceleration is zero this term vanishes. It is well known in accelerator physics that the Coulomb field $\mathrm{F_C}$ generated by a relativistic particle moving with constant velocity is flattened towards the plane orthogonal to the direction of motion, and is often called a **pancake field** (see figure 5.2).

Consider figure 5.3. The magnitude of the Coulombic $\mathrm{F_C}$ and radiative $\mathrm{F_R}$ Liénard-Wiechert fields are plotted as height above the sphere for high γ . In both cases the field is distributed in a narrow spike protruding from the sphere in a small angle from the direction of motion. In both cases the bulk of the field lies inside an angle $\Delta\phi \sim 1/\gamma$ from the direction of motion.

At first glance the plot of the Coulomb field seems to contradict figure 5.2 which shows the field flattened in the transverse plane. It is reasonable to ask how these two radically different behaviours can be consistent. Consider a particle

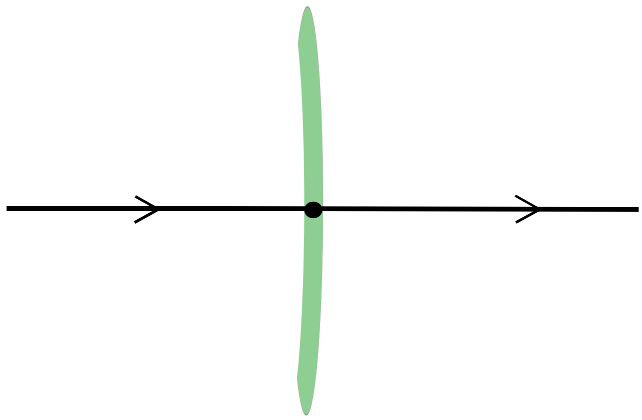


Figure 5.2: The pancake field

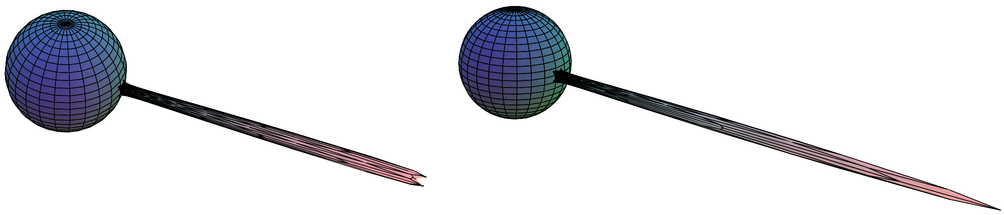


Figure 5.3: The magnitude of the Coulomb and radiative fields for a high γ, given as height above the sphere. The bulk of the fields is in the direction of motion.

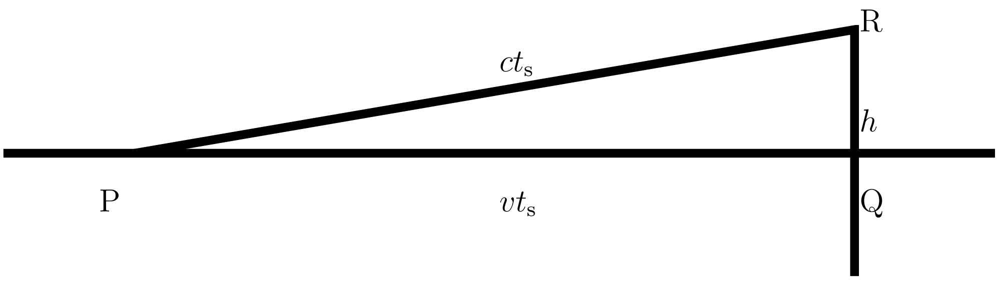


Figure 5.4: Showing the communication between a particle and its pancake

moving at velocity v along the horizontal line PQ in figure 5.4. Let R be a point in the pancake a distance h from the particle, when the particle is at Q. The last point at which the particle could communicate with the point R is at P, a length vt_s from Q. Here t_s is the time it takes for light to travel from P to R and also the time for the particle to travel from P to Q. Then $\|PR\| = ct_\mathrm{s}$ and $\|PQ\| = vt_\mathrm{s}$. Thus $(ct_\mathrm{s})^2 = h^2 + (vt_\mathrm{s})^2$. Hence $h^2 = c^2 {t_\mathrm{s}}^2(1 - v^2/c^2) = c^2 {t_\mathrm{s}}^2/\gamma^2$ so $t_\mathrm{s} = \gamma h/c$ and $\|PQ\| = \gamma hv/c$. Thus a particle needs to have travelled in a straight line for a length $\|PQ\| = \gamma hv/c$ in order for a pancake of radius h to develop. Looking at the fields which originate at P and arrive at R, they are at an angle approximately $\|RQ\| \, / \, \|PR\| = 1/\gamma$. This is consistent with figure 5.3.

5.4 Proposal

We investigate the possibility of reducing wakefields in accelerators by placing structures which give rise to geometric wakefields, such as collimators and cavities, directly after a bending dipole. We model a beam of charged particles as a one dimensional continuum of point charges undergoing the same motion in space, but at a different time. In our analysis we envisage a collimator as the source of wakefields and we calculate the field strength at the entrance of the collimator due to the collective Liénard-Wiechert field of the particles in the beam. We do not consider any boundary conditions imposed by the beam pipe or the collimator itself. We propose the new method of reduction of wakefields should be used parasitically on existing bends so that there is no additional beam disruption due to coherent synchrotron radiation wakefields (CSR wakes) or loss or energy due to synchrotron radiation (SR). In particular an accelerator which requires the following:

- Short bunches (much shorter bunch length that the aperture of the collimator).
- The bending of the bunches, via the use of dipoles.

- Collimation.

can achieve the collimation for free, i.e. with no additional loss of energy or disruption to the bunches from geometric or CSR wakes, by placing the collimator just after the bend.

We have seen that in order for a pancake of radius h to develop the particle needs to have travelled in a straight line for a distance $\gamma hv/c$. For highly relativistic motion this is approximately γh. Our proposed method of reduction of wakefields relies on this result. The idea is to bend the beam slightly before it enters the collimator (see figure 5.5). Most of the Coulomb field generated by the particle before the bend will continue in a straight line. By sufficiently enlarging the beam pipe in this direction the wakefield due to this part of the field can be neglected. If the distance, Z, of the straight line segment from the terminus of the bend to the centre of the collimator is sufficiently small, then the resulting pancake field will be too small to reach the sides of the structure. Of course bending the beam will generate additional radiation fields, however by judicious choice of geometry of the beam these can be minimized.

Let h denote half the aperture of the collimator and let L represent the spatial length of the bunch. The following two scenarios will be considered:

- Long smooth bunches where $L > h$ and any variation in the density of the bunch is over length scales longer than h,

- Bunches where variation in density is over short length scales less than about $0.2h$. This includes the case of very short bunches where $L \ll 0.2h$.

These two scenarios are both applicable to present day machines, where the bunch length depends upon the specific objectives and engineering considerations of individual projects.

In chapter 6 we show that the coherent electric (magnetic) fields due to a bunch modeled as a 1D continuum of point particles are given by the convolution of the

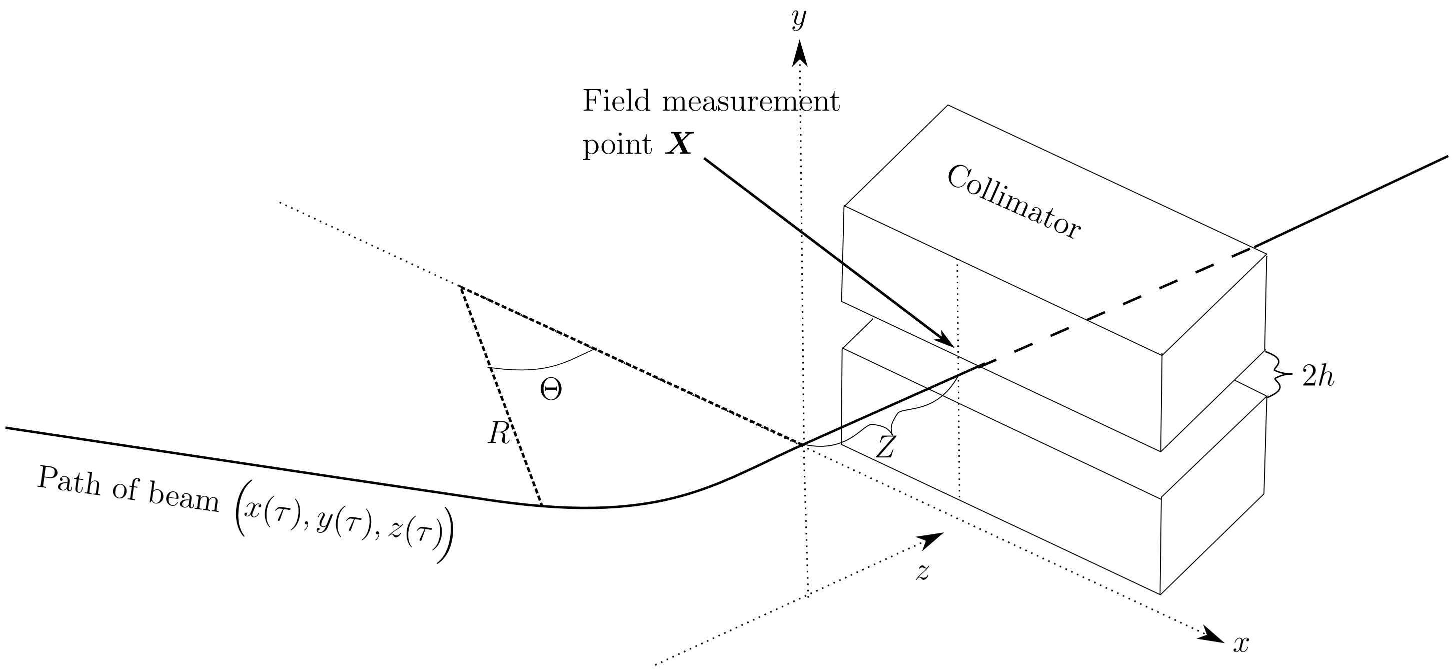


Figure 5.5: Setup for beam trajectory and collimator

electric (magnetic) field due to a single particle with the charge profile. In chapter 7 we carry out a numerical investigation using the mathematical software MAPLE. Assuming a Gaussian charge profile we minimize the electric field generated by the bunch by calculating the field due to different beam trajectories. Having optimized the trajectory we calculate the electric field at a specific point, representing a point on the collimator wall, for a selection of different bunch lengths which are attainable at present day facilities. Calculating the secondary electromagnetic fields generated by the collimator is a boundary value problem, hence calculating the full wakefield kick due to the collimator and a bent beam would require detailed knowledge of the geometry and material properties of the beam pipe. This will not be undertaken in this thesis. However we will show that the field incident on the boundary may be reduced by a factor of 7, and since the wakefields are, to a large extent, proportional to the fields at the boundary, the field in the beam pipe will automatically be reduced by approximately the same factor. We will find that for short bunches, or bunches with large amounts of micro-bunching, it is possible to make a significant reduction in wakefields. This is applicable to present day free

Table 5.1: Bunch lengths for some modern colliders and FELs

Collider	Year of Commissioning	Bunch length [ps]
SLC, SLAC	1989	3
ILC	$\geq$ 2015	1
CLIC	$\geq$ 2025	0.15
Free Electron Laser		Min. bunch length [ps]
FLASH, DESY	2005	0.05
LCLS, SLAC	2009	0.008
XFEL, DESY	2014	0.08

electron lasers, which employ bunch compressors to produce very short bunches, for example in LCLS $L/c \approx 0.008$ps. Assuming a collimator of half aperture $h = 0.5$mm then in this case $L = 0.0048h$. It turns out that electromagnetic fields due to long smooth bunches may not be reduced significantly. In many present day colliders the bunches are designed to be long and smooth, however in the future short bunch colliders may be desirable (see Table 5.1).

Chapter 6

The field of a 1D continuum of point charges

In this chapter we consider the field generated by a 1D continuum of point charges on an arbitrary trajectory. The key result is that the electric field for the continuum is given by the convolution of the electric field for a single point charge with the charge profile. This result will be used in the next chapter where we adopt the 1D continuum as our model for a bunch of particles in an accelerator.

6.1 The Liénard-Wiechert field in 3-vector notation

Definition 6.1.1. Given a choice of time coordinate t such that $\frac{\partial}{\partial t}$ is Killing we can write $\mathcal{M} = \mathbb{R} \times \underline{\mathcal{M}}$, where $\underline{\mathcal{M}}$ is Euclidean three space. We denote the points $x \in (\mathcal{M} \backslash C)$ and $C(\tau) \in \mathcal{M}$ by

$$x = (cT, \boldsymbol{X}), \qquad \text{and} \qquad C(\tau) = (ct, \boldsymbol{x}) \tag{6.1}$$

where $T, t \in \mathbb{R}$ and $\boldsymbol{X}, \boldsymbol{x} \in \underline{\mathcal{M}}$. The null displacement vector X is given by

$$X = (cT - ct, \boldsymbol{X} - \boldsymbol{x}), \qquad \text{where} \qquad t = \gamma\tau_r(cT, \boldsymbol{X}) \tag{6.2}$$

where the difference $\boldsymbol{X} - \boldsymbol{x}$ is a 3-vector at point $\boldsymbol{X} \in \underline{\mathcal{M}}$. It follows from the definition of τ_r that $T > t$.

Definition 6.1.2. The spatial displacement between the field point $\boldsymbol{X}$ and the emission point $\boldsymbol{x}$ will be denoted by

$$\boldsymbol{r} = ||\boldsymbol{X} - \boldsymbol{x}||, \tag{6.3}$$

where $||.||$ is the Euclidean norm. We define the unit 3-vector $\boldsymbol{n} \in \mathrm{T}_{\boldsymbol{X}}\underline{\mathcal{M}}$ by

$$\boldsymbol{n} = \frac{\boldsymbol{X} - \boldsymbol{x}}{||\boldsymbol{X} - \boldsymbol{x}||} = \frac{\boldsymbol{X} - \boldsymbol{x}}{\boldsymbol{r}}, \qquad \boldsymbol{n} \cdot \boldsymbol{n} = 1 \tag{6.4}$$

where the dot denotes the standard scalar product.

Lemma 6.1.3. *It follows from the null condition that*

$$\boldsymbol{r} = cT - ct \tag{6.5}$$

Proof of 6.1.3.

$$\begin{aligned} 0 = g(X, X) =& g\big((cT - ct, \boldsymbol{X} - \boldsymbol{x}), (cT - ct, \boldsymbol{X} - \boldsymbol{x})\big) \\ =& -(cT - ct)^2 + ||\boldsymbol{X} - \boldsymbol{x}||^2 \end{aligned}$$

Thus

$$||cT - ct|| = ||\boldsymbol{X} - \boldsymbol{x}|| = \boldsymbol{r} \tag{6.6}$$

The result (6.5) follows from noticing $T > t$. □

It follows trivially from definitions 6.1.1 and 6.1.2 and lemma 6.1.3 that the null 4-vector is given by

$$X = \boldsymbol{r}(1, \boldsymbol{n}) \tag{6.7}$$

Definition 6.1.4. The normalized Newtonian velocity $\boldsymbol{\beta}$ and acceleration $\boldsymbol{a}$ are defined by

$$\boldsymbol{\beta} = \frac{1}{c}\frac{d\boldsymbol{x}}{dt} = \frac{1}{c\gamma}\frac{d\boldsymbol{x}}{d\tau}, \qquad \text{and} \qquad \boldsymbol{a} = \frac{d\boldsymbol{\beta}}{dt} = \frac{1}{\gamma}\frac{d\boldsymbol{\beta}}{d\tau} \tag{6.8}$$

Lemma 6.1.5. *Thus the* 4*-vectors* $\dot{C}, \ddot{C} \in \mathrm{T}_{C(\tau)}\mathcal{M}$ *are given by*

$$\begin{aligned} &\dot{C} = c\gamma(1, \boldsymbol{\beta}) \\ \text{and} \qquad &\ddot{C} = c\gamma^4(\boldsymbol{a} \cdot \boldsymbol{\beta})(1, \boldsymbol{\beta}) + c\gamma^2(0, \boldsymbol{a}) \end{aligned} \tag{6.9}$$

Proof of 6.1.5. First note that

$$\frac{d\gamma}{dt} = \frac{d}{dt}(1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta})^{-\frac{1}{2}} = -\frac{1}{2}(1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta})^{-\frac{3}{2}}\frac{d}{dt}(-\boldsymbol{\beta} \cdot \boldsymbol{\beta}),$$

thus since $\frac{d}{dt}(-\boldsymbol{\beta} \cdot \boldsymbol{\beta}) = -2\boldsymbol{a} \cdot \boldsymbol{\beta}$ it follows that

$$\frac{d\gamma}{dt} = \gamma^3 \boldsymbol{a} \cdot \boldsymbol{\beta} \tag{6.10}$$

From (6.1.1) we have $C = (ct, \boldsymbol{x})$. Thus

$$\dot{C} = \frac{dC}{d\tau} = \gamma\frac{dC}{dt} = \gamma(c, \frac{d\boldsymbol{x}}{dt})$$

Also

$$\ddot{C} = \frac{d\dot{C}}{d\tau} = \gamma\frac{d\dot{C}}{dt} = \gamma c\frac{d\gamma}{dt}(1, \boldsymbol{\beta}) + \gamma^2 c\frac{d}{dt}(1, \boldsymbol{\beta}).$$

Substituting (6.10) and (6.1.4) yields the result 6.1.5. $\square$

Lemma 6.1.6. *The following relations are true*

$$g(X,\dot{C}) = \boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta}-1)$$
$$\text{and} \qquad g(X,\ddot{C}) = \boldsymbol{r}c\gamma^4(\boldsymbol{\beta}\,\boldsymbol{.}\,\boldsymbol{n}-1)(\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{\beta}) + \boldsymbol{r}c\gamma^2(\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{n}) \tag{6.11}$$

Proof of 6.1.6.

$$g(X,\dot{C}) = g\big(\boldsymbol{r}(1,\boldsymbol{n}), c\gamma(1,\boldsymbol{\beta})\big) = \boldsymbol{r}c\gamma(\boldsymbol{\beta}\,\boldsymbol{.}\,\boldsymbol{n}-1)$$

Also

$$\begin{aligned} g(X,\ddot{C}) =& g\big(\boldsymbol{r}(1,\boldsymbol{n}), \gamma c\frac{d\gamma}{dt}(1,\boldsymbol{\beta})\big) + g\big(\boldsymbol{r}(1,\boldsymbol{n}), \gamma^2 c(0,\boldsymbol{a})\big) \\ =& \boldsymbol{r}c\gamma^4\big((\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{\beta})\boldsymbol{\beta}\,\boldsymbol{.}\,\boldsymbol{n} - (\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{\beta})\big) + \boldsymbol{r}c\gamma^2(\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{n}). \end{aligned}$$

$\square$

Lemma 6.1.7. *In chapter 1 equations (1.16) and (1.22) we define the electric and magnetic 1-forms $\widetilde{\mathcal{E}}, \widetilde{\mathcal{B}} \in \Gamma T\mathcal{M}$ for a timelike observer curve U. Given a coordinate chart (y^0, y^1, y^2, y^3) let $U = \partial_{y^0}$ and let $\mathrm{E} = \mathrm{E_C} + \mathrm{E_R}$ where*

$$\widetilde{\mathrm{E}}_{\mathrm{C}} = i_{\partial_{y^0}}\mathrm{F_C} \qquad \text{and} \qquad \widetilde{\mathrm{E}}_{\mathrm{R}} = i_{\partial_{y^0}}\mathrm{F_R}. \tag{6.12}$$

If $\widetilde{\mathrm{E}}_{\mathrm{C}} = \mathrm{E}_{\mathrm{C}a}dy^a$ and $\widetilde{\mathrm{E}}_{\mathrm{R}} = \mathrm{E}_{\mathrm{R}a}dy^a$ for $a = 1,2,3$ then

$$\mathrm{E}_{\mathrm{C}a} = \frac{q}{4\pi\epsilon_0}\frac{(\boldsymbol{n}-\boldsymbol{\beta})_a}{\boldsymbol{r}^2\gamma^2(1-\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta})^3} \qquad \text{and} \qquad \mathrm{E}_{\mathrm{R}a} = \frac{q}{4\pi\epsilon_0}\frac{(\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a})_a}{\boldsymbol{r}c\gamma(1-\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta})^3}. \tag{6.13}$$

Proof of 6.1.7. Let $\mathrm{F_C} = \mathrm{F}_{\mathrm{C}ab}dz^a \wedge dz^b$, then from (1.113) and (6.11) it follows that

$$\frac{1}{\kappa}\mathrm{F}_{\mathrm{C}ab} = \frac{-c^2(X_a\dot{C}_b - X_b\dot{C}_a)}{(rc\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta}-1))^3}. \tag{6.14}$$

where $\kappa = \frac{q}{4\pi\epsilon_0}$. Thus

$$\frac{1}{\kappa}\mathrm{E}_{\mathrm{C}a} = \frac{1}{\kappa}\mathrm{F}_{\mathrm{C}a0} = \frac{-c^2(X_a\dot{C}_0 - X_0\dot{C}_a)}{(rc\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^3}. \tag{6.15}$$

It follows from (6.7) and (6.9) that

$$X_0 = -\boldsymbol{r}, \qquad X_a = \boldsymbol{r}\boldsymbol{n}_a, \qquad \dot{C}_0 = -c\gamma \qquad \text{and} \qquad \dot{C}_a = c\gamma\boldsymbol{\beta}_a,$$

thus

$$\frac{1}{\kappa}\mathrm{E}_{\mathrm{C}a} = \frac{c^3\gamma\boldsymbol{r}\boldsymbol{n}_a - \boldsymbol{r}c^3\gamma\boldsymbol{\beta}_a}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^3} = \frac{(\boldsymbol{n} - \boldsymbol{\beta})_a}{\boldsymbol{r}^2\gamma^2(1 - \boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta})^3} \tag{6.16}$$

Similarly let $\mathrm{F}_{\mathrm{R}} = \mathrm{F}_{\mathrm{R}ab}dz^a \wedge dz^b$. It follows from (1.112) and (6.11) that

$$\begin{aligned}\frac{1}{\kappa}\mathrm{F}_{\mathrm{R}ab} =& \frac{\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1)(X_a\ddot{C}_b - X_b\ddot{C}_a)}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^3} - \frac{\boldsymbol{r}c\gamma\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{a}(X_a\dot{C}_b - X_b\dot{C}_a)}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^3} \\ =& \frac{(X_a\ddot{C}_b - X_b\ddot{C}_a)}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^2} - \frac{\boldsymbol{r}c\gamma\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{a}(X_a\dot{C}_b - X_b\dot{C}_a)}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^3}\end{aligned}$$

Thus

$$\frac{1}{\kappa}\mathrm{E}_{\mathrm{R}a} = \frac{1}{\kappa}\mathrm{F}_{\mathrm{R}a0} = \frac{(X_a\ddot{C}_0 - X_0\ddot{C}_a)}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^2} - \frac{\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{a})(X_a\dot{C}_0 - X_0\dot{C}_a)}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^3}\boldsymbol{.} \tag{6.17}$$

It follows from (6.9) that

$$\ddot{C}_0 = -c\gamma^4\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{\beta}\text{and} \qquad \ddot{C}_a = c\gamma^2\boldsymbol{a}_a + c\gamma^4(\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{\beta})\boldsymbol{\beta}_a, \tag{6.18}$$

thus the first term in (6.17) yields

$$\begin{aligned}\text{first term} =& \frac{-\boldsymbol{r}c\gamma^4(\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{\beta})\boldsymbol{n}_a + \boldsymbol{r}\big(c\gamma^2\boldsymbol{a}_a + c\gamma^4(\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{\beta})\boldsymbol{\beta}_a\big)}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1))^2} \\ =& \frac{\boldsymbol{a}_a}{\boldsymbol{r}c(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1)^2} + \frac{\gamma^2(\boldsymbol{a}\,\boldsymbol{.}\,\boldsymbol{\beta})(\boldsymbol{\beta}_a - \boldsymbol{n}_a)}{\boldsymbol{r}c(\boldsymbol{n}\,\boldsymbol{.}\,\boldsymbol{\beta} - 1)^2}\end{aligned} \tag{6.19}$$

and the second term yields

$$\text{second term} = -\frac{\big((-\boldsymbol{r}c\gamma\boldsymbol{n}_a + \boldsymbol{r}c\gamma\boldsymbol{\beta}_a)(\boldsymbol{r}c\gamma^4(\boldsymbol{a}\centerdot\boldsymbol{\beta})(\boldsymbol{\beta}\centerdot\boldsymbol{n}-1) + \boldsymbol{r}c\gamma^2(\boldsymbol{a}\centerdot\boldsymbol{n}))\big)}{(\boldsymbol{r}c\gamma(\boldsymbol{n}\centerdot\boldsymbol{\beta}-1))^3}$$
$$= \frac{\gamma^2(\boldsymbol{n}_a-\boldsymbol{\beta}_a)(\boldsymbol{a}\centerdot\boldsymbol{\beta})}{\boldsymbol{r}c(\boldsymbol{n}\centerdot\boldsymbol{\beta}-1)^2} + \frac{(\boldsymbol{n}_a-\beta_a)(\boldsymbol{a}\centerdot\boldsymbol{n})}{\boldsymbol{r}c(\boldsymbol{n}\centerdot\boldsymbol{\beta}-1)^3}$$

Thus adding the two term yields

$$\frac{1}{\kappa}\mathrm{E}_{\mathrm{R}a} = \frac{(\boldsymbol{n}\centerdot\boldsymbol{\beta}-1)\boldsymbol{a}_a}{\boldsymbol{r}c(\boldsymbol{n}\centerdot\boldsymbol{\beta}-1)^3} + \frac{(\boldsymbol{n}_a-\beta_a)(\boldsymbol{a}\centerdot\boldsymbol{n})}{\boldsymbol{r}c(\boldsymbol{n}\centerdot\boldsymbol{\beta}-1)^3}$$

The result follows from the rule for triple vector products. □

Definition 6.1.8. Similarly let $\widetilde{\mathrm{B}} = \widetilde{\mathrm{B}}_{\mathrm{C}} + \widetilde{\mathrm{B}}_{\mathrm{R}}$ where

$$\mathcal{B}_{\mathrm{C}} = \frac{1}{c}\widetilde{i_{\partial_{z^0}} \star \mathrm{F}_{\mathrm{C}}} \qquad \text{and} \qquad \mathcal{B}_{\mathrm{R}} = \frac{1}{c}\widetilde{i_{\partial_{z^0}} \star \mathrm{F}_{\mathrm{R}}}, \tag{6.20}$$

Then if $\widetilde{\mathrm{B}}_{\mathrm{C}} = \mathrm{B}_{\mathrm{C}a}dy^a$ and $\widetilde{\mathrm{B}}_{\mathrm{R}} = \mathrm{B}_{\mathrm{R}a}dy^a$ it can be show that

$$\mathcal{B}_{\mathrm{C}a} = \frac{1}{c}(\boldsymbol{n}\times\boldsymbol{E}_{\mathrm{C}})_a \qquad \text{and} \qquad \mathcal{B}_{\mathrm{R}a} = \frac{1}{c}(\boldsymbol{n}\times\boldsymbol{E}_{\mathrm{R}})_a \tag{6.21}$$

6.2 The model of a beam

We model our bunch of particles as a one dimensional bunch where each particle undergoes the same motion in space but at a different time. This bunch is moving at a constant speed with relativistic factor γ. Let ν label the points in the bunch, which will be called body points. The profile of the bunch is given by $\rho(\nu)$.[1]

Definition 6.2.1. Let $\boldsymbol{x}_\nu(\tau)$ represent the position of body point ν at proper time τ, and for each body point ν let

$$t_\nu(\tau) = (\tau+\nu)/\gamma. \tag{6.22}$$

[1] Note that ν has the dimension of time.

Definition 6.2.2. The retarded time for the body point ν corresponding to the fields measured at $\boldsymbol{X}$ at laboratory time T is denoted by $\hat{\tau}(\boldsymbol{X}, T, \nu)$. Similarly the arrival time at $\boldsymbol{X}$ of the field generated by body point ν at proper time τ is denoted by $\hat{T}(\nu, \tau, \boldsymbol{X})$.

For the $\nu = 0$ particle we define

$$\hat{\tau}_0(\boldsymbol{X}, T) = \hat{\tau}(\boldsymbol{X}, T, 0) \quad \text{and} \quad \hat{T}_0(\tau, \boldsymbol{X}) = \hat{T}(0, \tau, \boldsymbol{X}). \tag{6.23}$$

Lemma 6.2.3. *It follows that*

$$\hat{\tau}_0(\boldsymbol{X}, \hat{T}(\nu, \tau, \boldsymbol{X}) - \nu/\gamma) = \tau. \tag{6.24}$$

and

$$\hat{\tau}(\boldsymbol{X}, T, \nu) = \hat{\tau}_0(\boldsymbol{X}, T - \nu/\gamma). \tag{6.25}$$

Proof of 6.2.3. The retarded time condition is given by

$$cT - ct_\nu\big(\hat{\tau}(\boldsymbol{X}, T, \nu)\big) = \big\|\boldsymbol{X} - \boldsymbol{x}_\nu\big(\hat{\tau}(\boldsymbol{X}, T, \nu)\big)\big\|, \tag{6.26}$$

and hence

$$cT - c\hat{\tau}(\boldsymbol{X}, T, \nu)/\gamma - c\nu/\gamma = \big\|\boldsymbol{X} - \boldsymbol{x}_\nu\big(\hat{\tau}(\boldsymbol{X}, T, \nu)\big)\big\|. \tag{6.27}$$

Thus

$$\begin{aligned} c\hat{T}(\nu, \tau, \boldsymbol{X}) &= ct_\nu(\tau) + \|\boldsymbol{X} - \boldsymbol{x}_\nu(\tau)\| \\ &= c(\tau + \nu)/\gamma + \|\boldsymbol{X} - \boldsymbol{x}_\nu(\tau)\|. \end{aligned} \tag{6.28}$$

From (6.27) and (6.28)

$$\begin{aligned} cT &= c\big(\hat{\tau}(\boldsymbol{X},T,\nu)+\nu\big)/\gamma + \big\|\boldsymbol{X} - \boldsymbol{x}_\nu\big(\hat{\tau}(\boldsymbol{X},T,\nu)\big)\big\| \\ &= c\hat{T}(\nu,\hat{\tau}(\boldsymbol{X},T,\nu),\boldsymbol{X}). \end{aligned} \tag{6.29}$$

Since $\hat{T}$ is increasing and the range of $\hat{\tau}$ is from $-\infty$ to $+\infty$ it follows that $\hat{T}$ and $\hat{\tau}$ are inverse to each other, yielding (6.29) and

$$\hat{\tau}(\boldsymbol{X},\hat{T}(\nu,\tau,\boldsymbol{X}),\nu) = \tau. \tag{6.30}$$

Now $\hat{T}(\nu,\tau,\boldsymbol{X})$ and $\hat{\tau}(\boldsymbol{X},T,\nu)$ may be written in terms of $\hat{T}_0(\tau,\boldsymbol{X})$ and $\hat{\tau}_0(\boldsymbol{X},T)$. From (6.28)

$$\hat{T}(\nu,\tau,\boldsymbol{X}) = \hat{T}_0(\tau,\boldsymbol{X}) + \nu/\gamma. \tag{6.31}$$

From (6.29), (6.30) and (6.23),

$$\hat{T}_0(\hat{\tau}_0(\boldsymbol{X},T),\boldsymbol{X}) = T \tag{6.32}$$

and

$$\hat{\tau}_0(\boldsymbol{X},\hat{T}_0(\tau,\boldsymbol{X})) = \tau. \tag{6.33}$$

Substituting (6.31) into (6.33) leads to

$$\hat{\tau}_0(\boldsymbol{X},\hat{T}(\nu,\tau,\boldsymbol{X}) - \nu/\gamma) = \tau. \tag{6.34}$$

Substituting $\tau = \hat{\tau}(\boldsymbol{X},T,\nu)$ and using (6.29) yields

$$\hat{\tau}(\boldsymbol{X},T,\nu) = \hat{\tau}_0(\boldsymbol{X},T-\nu/\gamma). \tag{6.35}$$

□

Statistics for independent identical distributions

Definition 6.2.4. We assume the ν for each particle has the identical distributions $\rho(\nu)$, where

$$\int \rho(\nu) d\nu = 1 \tag{6.36}$$

That is the probability that particle k has displacement ν_k is $\rho(\nu_k)d\nu$.

Definition 6.2.5. Given a function of $H(\nu_1, \ldots, \nu_N)$ of all the random variables we define the expectation of H as

$$\langle H \rangle = \int d\nu_1 \rho(\nu_1) \cdots \int d\nu_N \rho(\nu_N) H(\nu_1, \ldots, \nu_N) \tag{6.37}$$

Lemma 6.2.6. *For a function which is simply the sum of functions* $\sum_{k=1}^{N} h(\nu_k)$ *we have*

$$\left\langle \sum_{k=1}^{N} h(\nu_k) \right\rangle = N \langle h \rangle_{1\mathrm{P}} \tag{6.38}$$

where

$$\langle h \rangle_{1\mathrm{P}} = \int \rho(\nu) h(\nu)\, d\nu \tag{6.39}$$

is the one particle expectation.

Proof of 6.2.6.

$$\begin{aligned} \left\langle \sum_{k=1}^{N} h(\nu_k) \right\rangle &= \int d\nu_1 \rho(\nu_1) \cdots \int d\nu_N \rho(\nu_N) \sum_{k=1}^{N} h(\nu_k) \\ &= \sum_{k=1}^{N} \int d\nu_1 \rho(\nu_1) \cdots \int d\nu_N \rho(\nu_N) h(\nu_k) \\ &= \sum_{k=1}^{N} \int d\nu_k \rho(\nu_k) h(\nu_k) = N \int d\nu\, \rho(\nu) h(\nu) \end{aligned}$$

□

Lemma 6.2.7. *The expectation sum of product of the two functions*

$$H(\nu_1,\ldots,\nu_N)=\Big(\sum_{k=1}^{N}h(\nu_k)\Big)\Big(\sum_{m=1}^{N}g(\nu_m)\Big)$$

is given by

$$\langle H\rangle = N\langle hg\rangle_{1\mathrm{P}}+(N^2-N)\langle h\rangle_{1\mathrm{P}}\langle g\rangle_{1\mathrm{P}} \tag{6.40}$$

This is important since it corresponds to components of the energy, momentum and stress of the electromagnetic field. In particular, the energy of the electromagnetic field is determined in section 6.4.

Proof of 6.2.7.

$$\begin{aligned}
\langle H\rangle =&\Big\langle\Big(\sum_{k=1}^{N}h(\nu_k)\Big)\Big(\sum_{m=1}^{N}g(\nu_m)\Big)\Big\rangle=\sum_{k=1}^{N}\sum_{m=1}^{N}\langle h(\nu_k)g(\nu_m)\rangle\\
=&\sum_{k=1}^{N}\sum_{m=1}^{N}\int d\nu_1\rho(\nu_1)\cdots\int d\nu_N\rho(\nu_N)h(\nu_k)g(\nu_m)\\
=&\sum_{k=1}^{N}\sum_{m=k}\int d\nu_1\rho(\nu_1)\cdots\int d\nu_N\rho(\nu_N)h(\nu_k)g(\nu_m)\\
&+\sum_{k=1}^{N}\sum_{m\neq k}\int d\nu_1\rho(\nu_1)\cdots\int d\nu_N\rho(\nu_N)h(\nu_k)g(\nu_m)\\
=&\sum_{k=1}^{N}\int d\nu_k\rho(\nu_k)h(\nu_k)g(\nu_k)+\sum_{k=1}^{N}\sum_{m\neq k}\int d\nu_k\rho(\nu_k)\int d\nu_m\rho(\nu_m)h(\nu_k)g(\nu_m)\\
=&N\int d\nu\rho(\nu)h(\nu)g(\nu)+\sum_{k=1}^{N}\sum_{m\neq k}\Big(\int d\nu_k\rho(\nu_k)h(\nu_k)\Big)\Big(\int d\nu_m\rho(\nu_m)g(\nu_m)\Big)\\
=&N\langle hg\rangle_{1\mathrm{P}}+\sum_{k=1}^{N}\sum_{m\neq k}\langle h\rangle_{1\mathrm{P}}\langle g\rangle_{1\mathrm{P}}=N\langle hg\rangle_{1\mathrm{P}}+(N^2-N)\langle h\rangle_{1\mathrm{P}}\langle g\rangle_{1\mathrm{P}}
\end{aligned}$$

Note the structure of the expectation of H, in particular the appearance of N and N^2-N. □

Lemma 6.2.8. *The one particle expectation of a shifted function is given by*

$$\big\langle g(T-\gamma^{-1}\nu)\big\rangle_{1\mathrm{P}} = \int \rho_{\mathrm{Lab}}(T-T')g(T')dT'. \tag{6.41}$$

Proof of 6.2.8.

$$\begin{aligned}\big\langle g(T-\gamma^{-1}\nu)\big\rangle_{1\mathrm{P}} &= \int \rho(\nu)g(T-\gamma^{-1}\nu)\,d\nu \\ &= \int \gamma\rho\big(\gamma(T-T')\big)g(T')dT' \\ &= \int \rho_{\mathrm{Lab}}(T-T')g(T')dT'\end{aligned}$$

where $T' = T-\gamma^{-1}\nu$, and

$$\rho_{\mathrm{Lab}}(T) = \gamma\rho(\gamma T) \tag{6.42}$$

is the charge density as measured in the laboratory frame. □

6.3 Expectation of electric and magnetic fields

Definition 6.3.1. For a particle of charge q undergoing arbitrary motion $\boldsymbol{x}(\tau)$, where τ is the particle's proper time, the Liénard-Wiechert fields at point $\boldsymbol{X}$ and time T are given [40] by

$$\boldsymbol{E}(\boldsymbol{X},T) = \mathrm{E}\Big(\boldsymbol{X}-\boldsymbol{x}(\tau_R),\boldsymbol{\beta}(\tau_R),\boldsymbol{a}(\tau_R)\Big) \tag{6.43}$$

and

$$\boldsymbol{B}(\boldsymbol{X},T) = \mathrm{B}\Big(\boldsymbol{X}-\boldsymbol{x}(\tau_R),\boldsymbol{\beta}(\tau_R),\boldsymbol{a}(\tau_R)\Big). \tag{6.44}$$

where E and B are defined in lemma 6.1.7 and definition 6.1.8 respectively.

For the body point ν the Liénard-Wiechert electric and magnetic fields at point

$\boldsymbol{X}$ and time T are given by substituting $\tau_R = \hat{\tau}(\boldsymbol{X}, T, \nu)$ into (6.43),

$$\boldsymbol{E}(\boldsymbol{X}, T, \nu) = \mathrm{E}\Big(\boldsymbol{X} - \boldsymbol{x}\big(\hat{\tau}(\boldsymbol{X}, T, \nu)\big), \boldsymbol{\beta}\big(\hat{\tau}(\boldsymbol{X}, T, \nu)\big), \boldsymbol{a}\big(\hat{\tau}(\boldsymbol{X}, T, \nu)\big)\Big)$$

and likewise for $\boldsymbol{B}(\boldsymbol{X}, T, \nu)$.

Let $\boldsymbol{E_0}(\boldsymbol{X}, T)$ be the electric field at point $\boldsymbol{X}$ and time T due to the body point $\nu = 0$ given by

$$\boldsymbol{E_0}(\boldsymbol{X}, T) = \mathrm{E}\Big(\boldsymbol{X} - \boldsymbol{x}\big(\hat{\tau}_0(\boldsymbol{X}, T)\big), \boldsymbol{\beta}\big(\hat{\tau}_0(\boldsymbol{X}, T)\big), \boldsymbol{a}\big(\hat{\tau}_0(\boldsymbol{X}, T)\big)\Big)$$

Using (6.25) it follows

$$\boldsymbol{E}(\boldsymbol{X}, T, \nu) = \boldsymbol{E_0}(\boldsymbol{X}, T - \nu/\gamma). \tag{6.45}$$

and

$$\boldsymbol{B}(\boldsymbol{X}, T, \nu) = \boldsymbol{B_0}(\boldsymbol{X}, T - \nu/\gamma). \tag{6.46}$$

Lemma 6.3.2. *The total electric and magnetic fields at time T at the point $\boldsymbol{X}$ are given by*

$$\boldsymbol{E}_{\mathrm{Tot}}(\boldsymbol{X}, T, \nu_1, \dots, \nu_N) = \sum_{k=1}^{N} \boldsymbol{E}(\boldsymbol{X}, T, \nu_k) \tag{6.47}$$

and

$$\boldsymbol{B}_{\mathrm{Tot}}(\boldsymbol{X}, T, \nu_1, \dots, \nu_N) = \sum_{k=1}^{N} \boldsymbol{B}(\boldsymbol{X}, T, \nu_k). \tag{6.48}$$

It follows that

$$\langle \boldsymbol{E}_{\mathrm{Tot}}(\boldsymbol{X}, T, \nu_1, \dots, \nu_N) \rangle = N \int \rho(\nu) \boldsymbol{E_0}(\boldsymbol{X}, T - \nu/\gamma)\, d\nu \tag{6.49}$$

and

$$\langle \boldsymbol{B}_{\text{Tot}}(\boldsymbol{X}, T, \nu_1, \ldots, \nu_N) \rangle = N \int \rho(\nu) \boldsymbol{B_0}(\boldsymbol{X}, T - \nu/\gamma)\, d\nu. \tag{6.50}$$

Proof of 6.3.2.

The result follows from lemma 6.2.6 and equations (6.45) and (6.46). □

Let total electric field at the point $\boldsymbol{X}$ at time T be given by

$$\boldsymbol{E}_{\text{Tot}}(\boldsymbol{X}, T) = \frac{1}{N} \langle \boldsymbol{E}_{\text{Tot}}(\boldsymbol{X}, T, \nu_1, \ldots, \nu_N) \rangle \tag{6.51}$$

We notice that (6.45) and (6.46) are functions where the dependence on ν is simply shifted $g(T - \gamma^{-1}\nu)$. Thus by lemma 6.2.8 it follows

$$\begin{aligned}
\boldsymbol{E}_{\text{Tot}}(\boldsymbol{X}, T) &= \int \rho(\nu) \boldsymbol{E}(\boldsymbol{X}, T, \nu) d\nu \\
&= \int \rho(\nu) \boldsymbol{E_0}(\boldsymbol{X}, T - \nu/\gamma) d\nu \\
&= \int \gamma \rho\big(\gamma(T - T')\big) \boldsymbol{E_0}(\boldsymbol{X}, T') dT' \\
&= \int \rho_{\text{Lab}}(T - T') \boldsymbol{E_0}(\boldsymbol{X}, T') dT',
\end{aligned}$$

where $T' = T - \nu/\gamma$, and $q\rho_{\text{Lab}}(T) = q\gamma\rho(\gamma T)$ is the charge density as measured in the laboratory frame. Thus the key result is that the total electric field is given by the convolution

$$\boldsymbol{E}_{\text{Tot}}(\boldsymbol{X}, T) = \int \rho_{\text{Lab}}(T - T') \boldsymbol{E_0}(\boldsymbol{X}, T') dT'. \tag{6.52}$$

The above can be repeated for the total magnetic field $\boldsymbol{B}_{\text{Tot}}(\boldsymbol{X}, T)$. Clearly $\boldsymbol{E_0}(\boldsymbol{X}, T')$ will depend on the energy of the beam γ and the path of the beam $\boldsymbol{x}(\tau)$.

6.4 Expectation of field energy and coherence

Definition 6.4.1. The energy of the electromagnetic field at time T at the point $\boldsymbol{X}$ for the N particles is defined as the expectation

$$\phi(\boldsymbol{X},T) = \left\langle \|\boldsymbol{E}_{\mathrm{Tot}}(\boldsymbol{X},T,\nu_1,\dots,\nu_N)\|^2 + \|\boldsymbol{B}_{\mathrm{Tot}}(\boldsymbol{X},T,\nu_1,\dots,\nu_N)\|^2 \right\rangle \tag{6.53}$$

Lemma 6.4.2.

$$\phi(\boldsymbol{X},T) = N\phi_{\mathrm{inc}}(\boldsymbol{X},T) + (N^2-N)\phi_{\mathrm{coh}}(\boldsymbol{X},T) \tag{6.54}$$

where the incoherent field is given by

$$\phi_{\mathrm{inc}}(\boldsymbol{X},T) = \left\langle \|\boldsymbol{E}(\boldsymbol{X},T,\nu)\|^2 + \|\boldsymbol{B}(\boldsymbol{X},T,\nu)\|^2 \right\rangle_{\mathrm{1P}} \tag{6.55}$$

and the coherent field is given by

$$\phi_{\mathrm{coh}}(\boldsymbol{X},T) = \|\boldsymbol{E}_{\mathrm{cts}}(\boldsymbol{X},T)\|^2 + \|\boldsymbol{B}_{\mathrm{cts}}(\boldsymbol{X},T)\|^2 \tag{6.56}$$

where the one particle continuous electromagnetic fields are given by

$$\boldsymbol{E}_{\mathrm{cts}}(\boldsymbol{X},T) = \langle \boldsymbol{E}(\boldsymbol{X},T,\nu)\rangle_{\mathrm{1P}} \qquad \textit{and} \qquad \boldsymbol{B}_{\mathrm{cts}}(\boldsymbol{X},T) = \langle \boldsymbol{B}(\boldsymbol{X},T,\nu)\rangle_{\mathrm{1P}} \tag{6.57}$$

I.e. $\boldsymbol{E}_{\mathrm{cts}}(\boldsymbol{X},T)$ *and* $\boldsymbol{B}_{\mathrm{cts}}(\boldsymbol{X},T)$ *correspond to the electric and magnetic fields due to a continuous distributions of charge with distribution given by* $\rho(\nu)$.

Proof of 6.4.2. Expanding (6.53) we see that this is simply a sum of products

$$\begin{aligned}\phi(\boldsymbol{X},T) =& \sum_i^3 \langle E_{i,\mathrm{Tot}}(\boldsymbol{X},T,\nu_1,\dots,\nu_N)\ E_{i,\mathrm{Tot}}(\boldsymbol{X},T,\nu_1,\dots,\nu_N)\rangle \\ &+ \sum_i^3 \langle B_{i,\mathrm{Tot}}(\boldsymbol{X},T,\nu_1,\dots,\nu_N)\ B_{i,\mathrm{Tot}}(\boldsymbol{X},T,\nu_1,\dots,\nu_N)\rangle\end{aligned}$$

where $E_{i,\mathrm{Tot}}$ is the i'th component of $\boldsymbol{E}_{\mathrm{Tot}}$. From (6.40) we have

$$\begin{aligned}\phi(\boldsymbol{X},T) = N\sum_i^3 \left\langle E_i(\boldsymbol{X},T,\nu)^2\right\rangle_{1\mathrm{P}} + (N^2-N)\sum_i^3 \langle E_i(\boldsymbol{X},T,\nu)\rangle_{1\mathrm{P}}{}^2 \\ + N\sum_i^3 \left\langle B_i(\boldsymbol{X},T,\nu)^2\right\rangle_{1\mathrm{P}} + (N^2-N)\sum_i^3 \langle B_i(\boldsymbol{X},T,\nu)\rangle_{1\mathrm{P}}{}^2\end{aligned}$$

where $E_i(\boldsymbol{X},T,\nu)$ is the i'th component of $\boldsymbol{E}(\boldsymbol{X},T,\nu)$. □

We've already seen from (6.45) and (6.46) that the electric and magnetic fields are simply shifted functions so we can use (6.41) to give the coherent and incoherent fields in terms of convolutions

$$\phi_{\mathrm{inc}}(\boldsymbol{X},T) = \int \rho_{\mathrm{Lab}}(T-T')\Big(\left\|\boldsymbol{E}_{\mathbf{0}}(\boldsymbol{X},T')\right\|^2 + \left\|\boldsymbol{B}_{\mathbf{0}}(\boldsymbol{X},T')\right\|^2\Big)dT' \tag{6.58}$$

and

$$\begin{aligned}\boldsymbol{E}_{\mathrm{cts}}(\boldsymbol{X},T) &= \int \rho_{\mathrm{Lab}}(T-T')\boldsymbol{E}_{\mathbf{0}}(\boldsymbol{X},T')dT' \\ \text{and}\quad \boldsymbol{B}_{\mathrm{cts}}(\boldsymbol{X},T) &= \int \rho_{\mathrm{Lab}}(T-T')\boldsymbol{B}_{\mathbf{0}}(\boldsymbol{X},T')dT'\end{aligned} \tag{6.59}$$

Chapter 7

Numerical results

In this chapter we present the results of a numerical investigation carried out with the mathematical software MAPLE. The relevant code can be found in appendix G. In the following we give a brief outline of the calculations involved and state the main results.

7.1 The field at a fixed point $\boldsymbol{X}$ for a single particle

Consider figure 5.5. Half the aperture of the collimator is given by distance h. We have seen (figure 5.4) that for high γ, a pancake of radius h can develop only if the particle has been travelling in a straight line over a displacement of at least $\gamma hv/c \approx \gamma h$. Therefore for our proposal to be effective the distance Z in figure 5.5 should be less than γh. Preliminary results show that the optimum value for Z is $Z \lesssim 10h$, with the field varying little with lower values, thus in the following analysis we fix the field measurement point $\boldsymbol{X} = (0, h, 10h)$ and consider the magnitude of the electric field at $\boldsymbol{X}$ due a single particle approaching and passing through the collimator. In all calculations we use $q = -1.60217 \times 10^{-19}$C.

We consider the path constructed from a straight line followed by an arc of a circle of radius R followed by another straight line. Observe that this path is unrealistic since it would require large magnets to remove the *magnetic leakage*. Magnetic leakage is defined as the passage of magnetic flux outside the path along which it can do useful work. In general in a bending dipole the magnetic leakage causes the path of the charge to be slightly smoothed out at the ends of the dipole, so that in a real bending magnet the path of the charge would not be precisely the arc of a circle. We assume that the smoothing of the path corresponding to real dipoles would not significantly change the nature of the result.

Let Θ denote the angle of arc. The coordinate system is chosen so that the direction of the second straight line is along the z axis and the arc is in the $x-z$ plane, finishing at the origin. We refer to this trajectory as the *pre-bent* trajectory in contradistinction with that of a particle approaching from $(x, y, z) = (0, 0 - \infty)$ on a straight line towards the origin, which we refer to as the *straight* trajectory.

The pre-bent trajectory is given by $\boldsymbol{x}(\tau) = (x(\tau), y(\tau), z(\tau))$ where

$$x(\tau) = \begin{cases} R(\cos\Theta - 1) + (\Theta R + \gamma v\tau)\sin\Theta \quad \text{for} \quad -\infty < \tau < -R\Theta/\gamma v \\ R\Big(\cos(\gamma v\tau/R) - 1\Big) \quad \text{for} \quad -R\Theta/\gamma v < \tau < 0 \\ 0 \quad \text{for} \quad 0 < \tau < \infty, \end{cases}$$

$$y(\tau) = \begin{cases} 0 \quad \text{for} \quad -\infty < \tau < \infty, \end{cases} \tag{7.1}$$

$$\text{and} \quad z(\tau) = \begin{cases} -R\sin\Theta + (\Theta R + \gamma v\tau)\cos\Theta \quad \text{for} \quad -\infty < \tau < -R\Theta/\gamma v \\ R\sin(\gamma v\tau/R) \quad \text{for} \quad -R\Theta/\gamma v < \tau < 0 \\ \gamma v\tau \quad \text{for} \quad 0 < \tau < \infty. \end{cases}$$

The straight trajectory is given by

$$(x(\tau), y(\tau), z(\tau)) = (0, 0, \gamma v\tau) \quad \text{for} \quad -\infty < \tau < \infty. \tag{7.2}$$

Calculating the field at $\boldsymbol{X}$ due to a specific path

We use a coordinate system $\{\tau, \mathsf{r}, \hat{\theta}, \hat{\phi}\}$ adapted from the Newman-Unti coordinates $\{\tau, R, \theta, \phi\}$. The coordinate transformation is given by (E.2). Comparison with (1.87) yields $\mathsf{r} = \frac{R}{\alpha}$ where $R = -g(X, V)$ is the Newman-Unti radial parameter. We require the electric and magnetic fields due to a particle on a given trajectory. For fixed field point $(\boldsymbol{X}, T) = (X_0, Y_0, Z_0, T_0)$ there exist parameters $\hat{\mathsf{r}}$, $\hat{\theta}$, and $\hat{\phi}$ which satisfy

$$
\begin{aligned}
cT_0 &= C^0(\tau) + \hat{\mathsf{r}} \\
X_0 &= C^1(\tau) + \hat{\mathsf{r}} \sin(\hat{\theta}) \cos(\hat{\phi}) \\
Y_0 &= C^2(\tau) + \hat{\mathsf{r}} \sin(\hat{\theta}) \sin(\hat{\phi}) \\
Z_0 &= C^3(\tau) + \hat{\mathsf{r}} \cos(\hat{\theta}).
\end{aligned} \tag{7.3}
$$

Rearranging yields the relations

$$
\begin{aligned}
\hat{\mathsf{r}} =& \sqrt{(X_0 - C^1)^2 + (Y_0 - C^2)^2 + (Z_0 - C^3)^2} \\
cT_0(\tau) =& \hat{\mathsf{r}} + C^0 \\
\cos(\hat{\theta}) =& \frac{Z_0 - C^3}{\hat{\mathsf{r}}} \\
\sin(\hat{\theta}) =& \frac{\sqrt{(X_0 - C^1)^2 + (Y_0 - C^2)^2}}{\hat{\mathsf{r}}} \\
\cos(\hat{\phi}) =& \frac{X_0 - C^1}{\sqrt{(X_0 - C^1)^2 + (Y_0 - C^2)^2}} \\
\sin(\hat{\phi}) =& \frac{Y_0 - C^2}{\sqrt{(X_0 - C^1)^2 + (Y_0 - C^2)^2}}
\end{aligned} \tag{7.4}
$$

These relations can be substituted into the expressions (E.24) for the radiative $\mathrm{E_R}(\tau, \mathsf{r}, \theta, \phi)$ and Coulombic $\mathrm{E_C}(\tau, \mathsf{r}, \theta, \phi)$ electric fields (or magnetic fields). This gives the electric field (magnetic field) as a function of the components C^0, C^1, C^2, C^3 and the coordinates T_0, X_0, Y_0, Z_0.

When considering the electric field due to a particle on a specific trajectory we need only substitute the correct components for C. For example in order to calculate the electric field for the pre-bent path we consider the three sections of the path independently. For each of the three intervals in (7.1) we input the trajectory by defining components

$$\begin{aligned} C^0 &= \gamma\tau \\ C^1(\tau) &= x(\tau) \\ C^2(\tau) &= y(\tau) \\ C^3(\tau &= z(\tau) \end{aligned} \tag{7.5}$$

where the corresponding values for $x(\tau), y(\tau)$ and $z(\tau)$ are defined in (7.1). See appendix G lines 131-147.

In the Maple code we have written a procedure which will take a selection of variable input parameters and output any field as a function of τ. See (G.1). The variable input parameters are the the components C^0, C^1, C^2, C^3 and a list of numerical values for the parameters X_0, Y_0, Z_0 and R, Θ, γ as well as an initial value for τ.

Lab time

The ranges of τ for the three different trajectories are obtained by substituting the numerical input values for R, Θ and γ into the intervals in (7.1). Thus for a given set of input variables we are able to plot any desired field component for a particular section of the path against τ for the range of τ appropriate to that section. In order to plot the field component against τ for the whole path we simply display the three plots corresponding to the three sections of the trajectory on the same graph.

The lab time $T_0(\tau)$ is a different function of τ for each of the three sections. This follows from (7.3). For a particular section we may obtain $T_0(\tau)$ by substituting our variable input parameters into (7.3) and thus we may plot any desired field against T_0 for that section of path by plotting the field and the time T_0 as parametric equations in τ. To plot the field over the whole range of T_0 we simply display all three plots on the same graph as before.

Optimizing the values of $\boldsymbol{R}$ and $\boldsymbol{\Theta}$

We have control over the variable parameters R and Θ. In order to establish the optimum set of parameters to minimize the field at $\boldsymbol{X}$ we calculate the peak energy of the electric field $||\boldsymbol{E_0}(\boldsymbol{X}, T)||$ for a range of T, performing a parameter sweep for a selection of values for R and Θ. We set $\gamma = 1000$, which represents an energy level easily obtained in modern accelerators. The procedure used in MAPLE is given in G.2 and the results are displayed in figure 7.1. We have displayed three different views of the same graph. The numerical values for the electric field are absent because we are interested only in the relative values for the different trajectories. The values for Θ and R used in these plots are a selection of the values tested, however they are sufficient to show the trend.

Recall that R determines the curvature of the bend and Θ determines the length of the bend. Looking at the first graph we see that in the far corner of the graph, where Θ and R are at a minimum, the field is at a maximum. As we approach the near region of the graph the magnitude decreases very rapidly with increasing Θ. We interpret this as follows. For a short bend the radial distance from the point $\boldsymbol{X}$ to the continuation of the straight section will be small, and thus the pancake which developed on the straight section will be strongly encountered at point $\boldsymbol{X}$. For a longer bend this contribution will be greatly reduced due to both the increased radial distance of $\boldsymbol{X}$ from the continuation of the straight section and the increased longitudinal distance of $\boldsymbol{X}$ from the terminus of the straight section. The latter distance is important because once the straight section ends and the bend begins the pancake is no longer travelling with the particle and the field strength within the pancake is decreasing. In consequence we interpret the ridge in the graph where the steep section ends as the cut off where the point X no longer encounters a significant field due to the pancake.

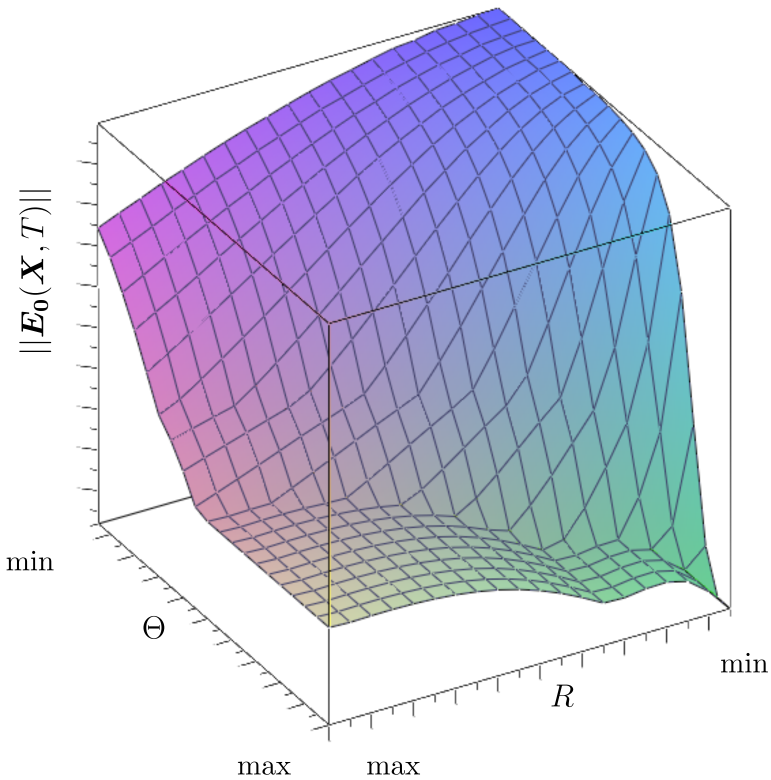
	R	Θ
min	500	1/95
	1000	1/90
	1500	1/85
	2000	1/80
	2500	1/75
	3000	1/70
	3500	1/65
	4000	1/60
	4500	1/55
	5000	1/50
	5500	1/45
	6000	1/40
	6500	1/35
	7000	1/30
	7500	1/25
	8000	1/20
	8500	1/15
	9000	1/10
	9500	1/5
max	10000	1

Table 7.1: Input values for R and Θ.

In reality we cannot adopt the smallest R and largest Θ because they are impractical in the design considerations of real machines. We chose to restrict the trajectory to the values $\Theta = 0.13$rad and $R = 0.5$m because the bend is sufficient to reduce the field at X while also maintaining a minimal length and intensity in order to suppress radiation and CSR wakes. In addition a bend of this size would be practical from an engineering perspective.

The field due to a particle on the optimised pre-bent trajectory compared with that of a particle on the straight trajectory

Consider the two cases given in figure 7.2 in which $\gamma = 1000$, $x = 0$, $y = h$ and $z = 10h$. In the straight line case the peak field is $\approx 75\text{Vm}^{-1}$ and the majority of the field arrives within an interval of 0.015ps. In fact it is easy to show that



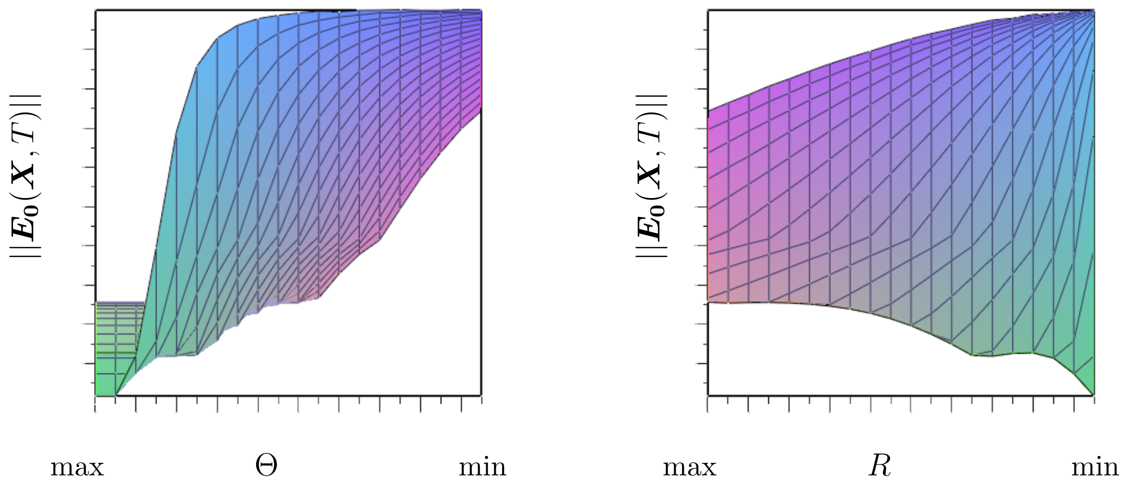


Figure 7.1: Field strength for different values of R and Θ. We see clearly that the minimum field energy occurs when the R is at its minimum value and Θ is at its maximum value.

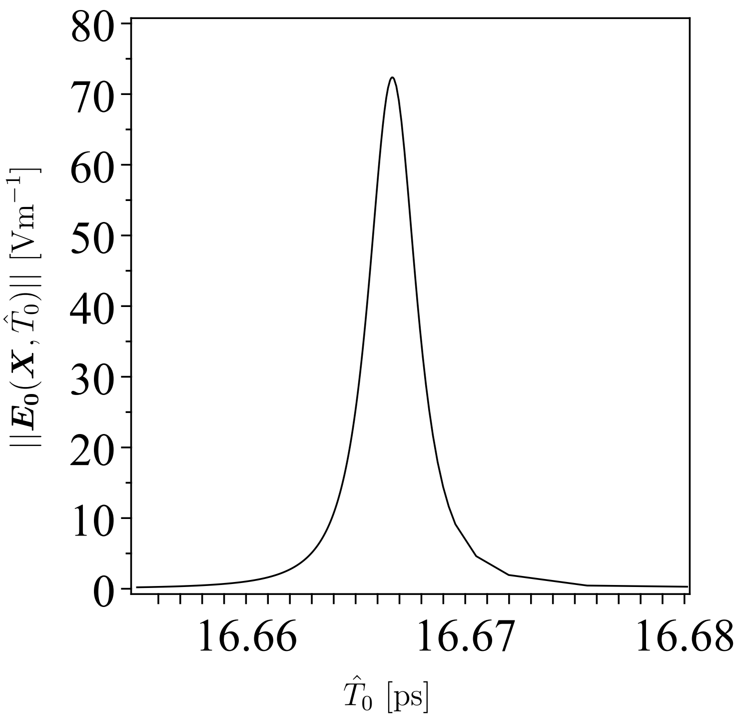
for a straight line path the peak field increases with γ and the width decreases with γ leading to the classic pancake. By contrast for the pre-bent case the peak field is significantly reduced to only $\approx 7.7\mathrm{Vm}^{-1}$, however the interval over which the field arrives is now 0.35ps for the right hand peak, and 0.1ps for the left hand peak. The reason for these two peaks is that the left hand peak is the coulomb field due to the first straight line segment, whereas the second peak is due to the radiation from the circular part of the beam path. The discontinuity is a result of the discontinuity in acceleration for this trajectory. Repeating the calculation with higher γ-factors does not significantly change the height or shape of the second peak.

Figure 7.3 shows the cartesian components of the electric field for the particle on the pre-bent trajectory. We see that the field is largely in the x and y directions, with the peak field in the y direction. By contrast for a particle on the straight trajectory the y component dominates with the z component negligible and the x component zero. This means that for a straight trajectory the field is primarily directed in the transverse direction as expected. The nonzero x component in the pre-bent case is due to the incident angle of the initial straight section as well as the radiation caused by the bend.

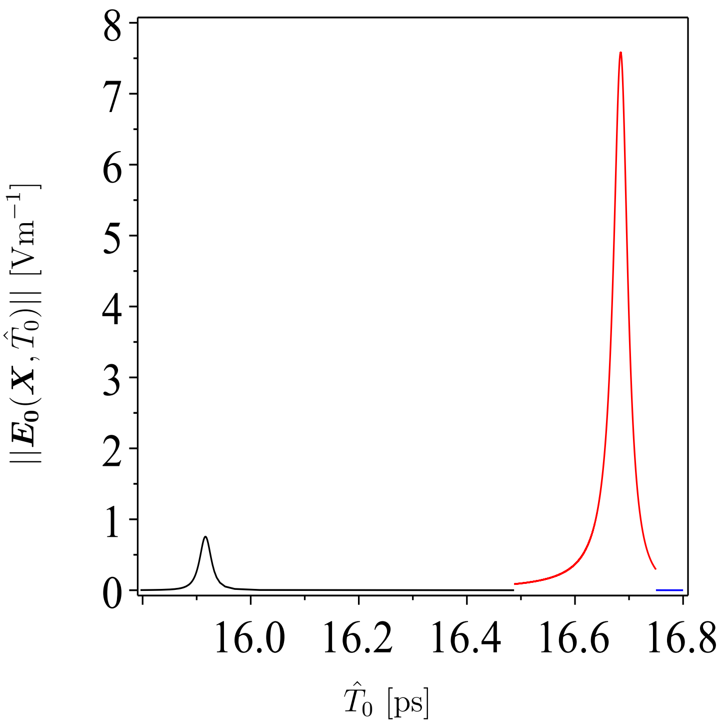
7.2 The coherent field at $\boldsymbol{X}$ due to a bunch

Consider a bunch modelled as a one dimensional continuum of point particles with a low density halo. The fields generated in the halo will not be considered. The one dimensional continuum is a good model for beams with transverse dimension significantly smaller than the bunch length. The assumption is made that the majority of the bunch charge is contained in the one-dimensional core and only the halo is removed by collimation. Within this model, each particle in the core undergoes the same motion in space but at a different time and is moving at a constant speed with relativistic factor γ.

Using (6.52) we calculate the field due to a Gaussian particle distribution ρ_{Lab}



Straight path



Pre-bent path

Figure 7.2: The electric field strength $||\boldsymbol{E_0}(\boldsymbol{X}, T)||$ at $\boldsymbol{X} = (0, h, 10h)$, with $h =$ 0.5mm, due to a body point following a straight path along the z-axis and a body point following the *pre-bent* path given in (7.1) with $\Theta = 0.13$rad and $R = 0.5$m.

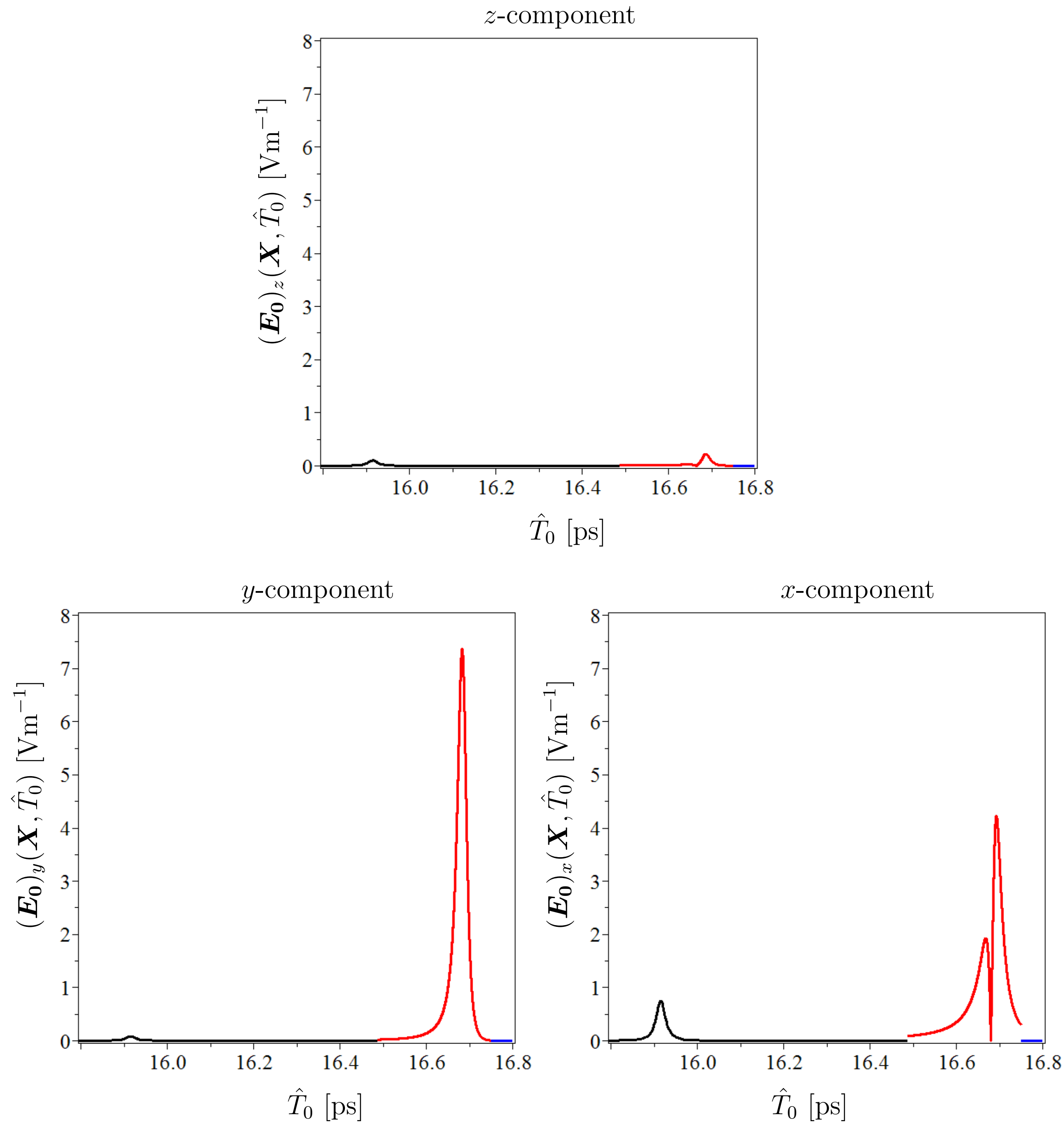


Figure 7.3: Electric field components $(\boldsymbol{E_0})_x, (\boldsymbol{E_0})_y$ and $(\boldsymbol{E_0})_z$ at point $\boldsymbol{X} = (0, h, 10h)$, with $h = 0.5$mm, due to a body point following a *pre-bent* path with $\Theta = 0.13$rad and $R = 0.5$m.

Table 7.2: Peak field strength for different sized bunches with h=0.5mm.

Bunch Length		Peak $\|\|\boldsymbol{E}_{\mathrm{Tot}}(\boldsymbol{X},T)\|\|$ [Vm^{-1}]	
L[h]	(L/c)[ps]	straight	pre-bent
1.8×10^{1}	3.00×10^{0}	1.97×10^{-1}	1.97×10^{-1}
6.00×10^{-1}	1.00×10^{0}	5.91×10^{-1}	5.89×10^{-1}
9.00×10^{-2}	1.50×10^{-1}	3.93×10^{0}	3.48×10^{0}
4.80×10^{-2}	8.00×10^{-2}	7.33×10^{0}	5.27×10^{0}
3.00×10^{-2}	5.00×10^{-2}	1.16×10^{1}	6.36×10^{0}
4.80×10^{-3}	8.00×10^{-3}	5.12×10^{1}	7.53×10^{0}

for the two cases in figure 7.2. The code for the convolution can be found in G.4. We input the time $T_0 = t$ at which the convolution should be centered, the bunch length (FWHM of the Gaussian) as upper and lower values of T_0, and the number of points N over which the samples should be taken. The procedure can be summed up in the following steps which are followed in a do loop for $j = 1..N$.

- Define $\rho_{\mathrm{Lab}} := (t, a, b) -> 1/(a\sqrt(2\pi)) \exp\left((-(t-b)^2)/(2a^2)\right)$

- Solve $T_0(\tau_j) = t_j$ for τ_j, where $t_j = t - a - (b-a)/N)(j + 1/2)$ and a and b are the upper and lower bounds on the range of T_0 respectively.

- Substitute $\tau = \tau_j$ into the electric field $\mathrm{E}(T_0(\tau))$ to give the field strength at time $T_0 = t_j$

- Multiply $\mathrm{E}(t_j)$ by $\rho_{\mathrm{Lab}}(t_j)$, where ρ_{Lab} is a specific Gaussian defined by inputting FWHM.

- Sum the result over j, sum $= \sum_{j=1}^{N} \mathrm{E}(t_j)\rho_{\mathrm{Lab}}(t_j)$

- Normalize by dividing sum by $\sum_{j=1}^{N} \rho_{\mathrm{Lab}}(t_j)$

Table 7.2 displays the results for a selection of bunch lengths which are attainable in some present day machines.

7.2.1 Long bunches

If the bunch is long and smooth, i.e. longer than the collimator aperture, so that there is no significant change in ρ_{Lab} over the width of $\boldsymbol{E_0}(\boldsymbol{X}, T')$, then $\boldsymbol{E_0}(\boldsymbol{X}, T')$ may be crudely regarded as a δ-function and $\boldsymbol{E}_{\text{Tot}}(\boldsymbol{X}, T)$ is given by

$$\boldsymbol{E}_{\text{Tot}}(\boldsymbol{X}, T) \approx \rho_{\text{Lab}}(T) \int \boldsymbol{E_0}(\boldsymbol{X}, T') dT'. \tag{7.6}$$

Integration of $\boldsymbol{E_0}(\boldsymbol{X}, T')$ for the straight and pre-bent trajectories reveals that

$$||\boldsymbol{E}_{\text{Tot}}(\boldsymbol{X}, T)|| \approx \frac{q}{2\pi\epsilon_0 c} \frac{\rho_{\text{Lab}}(T)}{||\boldsymbol{X}||} \tag{7.7}$$

This value of $\boldsymbol{E}_{\text{Tot}}$ is independent of R and Θ for all paths where R is large compared to L. To see why this is the case consider our one dimensional beam of particles as a continuous flow of charge, similar to a line charge in a wire but without the background ions. The fields due to this flow may be calculated using the Biot-Savart law. Since $h \ll R$ the field is dominated by the nearby current and hence no variation of R, Θ or Z will alter the fields. We find that calculations using the Biot-Savart Law agree very closely with equation (7.7), thus providing verification of our code.

7.2.2 Short bunches

If the beam has bunches of length $L \lesssim 0.05h$ then it follows from (6.52) and figure 7.2 that a considerable reduction in fields is possible. If ρ_{Lab} has full width at half maximum $L/c = 0.008$ps with corresponding bunch length $L = 0.0048h$, then the peak value for the total electric field in the straight line case is given by $\approx 51.2\text{Vm}^{-1}$. By contrast, in the pre-bent case the peak value for the total electric field is $\approx 7.5\text{Vm}^{-1}$, giving an approximate factor of 7 reduction in field. This is approaching the maximal factor of 10 improvement one can achieve with $\gamma = 1000$, which occurs when the bunch length is small enough that the convolution gives the peak values for the fields in figure 7.2. With higher energies and shorter bunch lengths the radiation peak remains unchanged, whereas the electric field for the straight path grows linearly with γ. Thus even greater improvements can be made.

7.3 Conclusion

In our analysis we have chosen a specific point $\boldsymbol{X} = (0, 0.5\text{mm}, 5\text{mm})$ and minimized the field at this point. We have shown that the magnitude of the electric field due to a single particle can be considerably reduced by altering the path of the beam, however the duration for which the field is non-zero is increased. We have used this result to show that the coherent field for a short bunch can be reduced significantly by bending the beam, with reductions of up to 85% feasible for some present day FELs and future colliders. No reduction in the coherent field can be made for long smooth bunches. We assume the coherent field will dominate the incoherent field because of the N Vs N^2 behaviour given in (6.54), however the incoherent fields are always present.

If the field point $\boldsymbol{X}$ is instead displaced in the positive x direction, then a significant increase in field strength is observed. This increase results from both the Coulomb field from the straight section of the path before the arc and the radiation from the circular part of the path.

Consider figure 7.4. The beam has been pre-bent from the left before passing through the origin of the graph, hence while on the bend the direction of motion is in the positive x direction. The magnitude of the field is shown as a contour plot. We see the magnitude increasing as we pass from the origin along the x-axis and then decreasing again after a very dense region. This pattern is what we expect to find from the synchrotron radiation emitted from the bend. The darkest parts of the graph are where the majority of the synchrotron radiation passed through the x-y plane. There are four discrete spots; two very dark spots at approximately $x = 1.4$mm and two slightly less intense spots to the left of these at $x = 1.0$mm. The two dark spots are approximately ten times the magnitude of the other two.

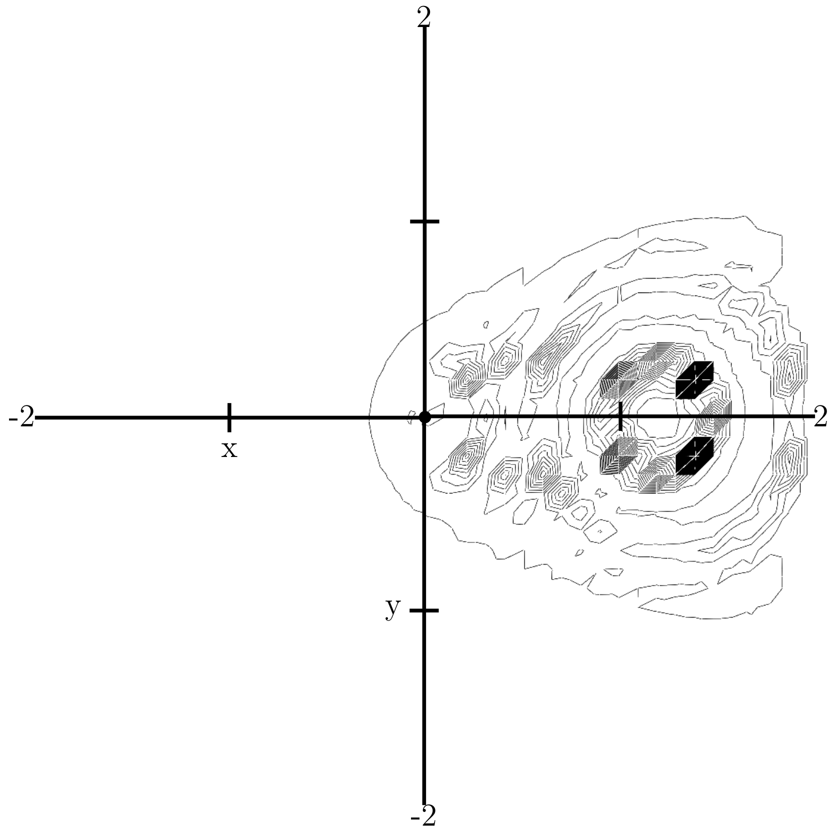


Figure 7.4: Contour plot for the maximum fields $||\boldsymbol{E_0}(\boldsymbol{X}, T)||$ in the (x, y) plane transverse to beam at $z = Z$. The plot represents a 4mm×4mm region around the beam. The beam has been bent from the left before passing though the origin of the graph. Twenty timesteps were taken and the lack of smoothness is due to numerical errors. Most of the field is between 0Vm^{-1} and 100Vm^{-1} in magnitude however the two black spots represent regions where the field is approximately 1000 times greater.

All the remaining field shown in the plot is negligible in comparison to these four peaks. It will be necessary to alter the shape of the collimator to avoid the high field regions interacting with the material in the collimator. This need not affect the efficacy of the collimator to remove the halo, for example see figure 7.5.

One criticism of our work is the fact that we have used the Liénard-Wiechert field, which is strictly accurate only for a particle in free space. In the accelerator community the following formula is often employed to describe the field of a bunch traversing a circular beam pipe

$$E_r = \frac{2\lambda}{r}, \tag{7.8}$$

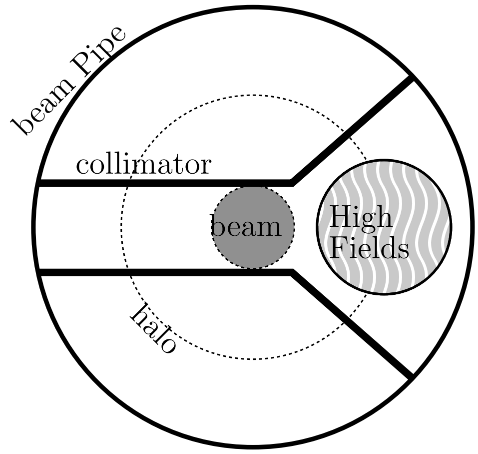


Figure 7.5: Modified collimator in the plane transverse to the path of the beam.

where λ is the longitudinal charge distribution and r is the radial distance from the axis. This formula assumes $\sigma_z \geq r/\gamma$, where σ_z is the rms bunch length. For typical machines this is the normal regime, for example with $r = 1cm$, $\gamma = 1000$ then $\sigma_z \geq 10\mu m$. In our investigation we have shown that the most substantial reductions in wakefield are expected for very small bunch lengths, therefore even without the difficulty introduced by the bend this formula would be inappropriate. The correct form for the field in a curved beam pipe is a boundary value problem depending on the intricate geometry of the beampipe, see for example [41].

In this investigation we have adopted the *rigid beam approximation* so that the charge profile remains constant throughout the whole trajectory. This approximation is generally adopted in calculating the field generated by a relativistic beam traveling in a straight line, however for a pre-bent trajectory there will be CSR wakes and energy loss due to radiation which in general will disrupt the charge profile. In order to determine whether there will be an advantage in bending the beam before collimation it will be necessary to calculate the net effect of bending plus reduced collimation wakes compared with collimation wakes on a straight beam trajectory. It is probable that the adverse effects of implementing an additional bend will be too severe for there to be any advantage in this new approach. However all accelerators, even linacs, already have to bend the beam using dipoles in certain places. Therefore it seems natural to place a collimator directly after a bending magnet in order to prevent additional adverse effects. The optimum design of the beam path, beam tube and collimator shape, for particular machines will require a combination of analytic, numerical and experimental research. Clearly long tapers will reduce the advantage gained by bending the beam since it will give time for the pancake to form. However it may be advantageous to use a short taper.

Appendices

Appendix A

Dimensional Analysis

The SI base units are given by

$$
\begin{aligned}
L &= \text{metre}, \quad m \\
T &= \text{second}, \quad s \\
M &= \text{kilogram}, \quad kg \\
A &= \text{Ampere}, \quad A
\end{aligned} \tag{A.1}
$$

It is more convenient to use the dimension of charge Q instead of A, with derived unit the Coulomb $C = sA$. We use square brackets to denote the dimensions of the enclosed object. The constants ϵ_0, μ_0 where $c^{-2} = \epsilon_0\mu_0$ have dimensions

$$
\begin{aligned}
[\epsilon_0] =& \frac{Q^2T^2}{ML^3} \\
[\mu_0] =& \frac{ML}{Q^2}
\end{aligned} \tag{A.2}
$$

The electric current 3-form has the dimension of charge $[\underset{(3)}{\mathcal{J}}] = Q$. Here the (3) denotes the degree of the differential form. Dimensions of the electric and magnetic fields are given by

$$[\underset{(0)}{\mathcal{E}^i}] = \frac{ML}{QT^2}, \qquad [\mathcal{E}] = \frac{M}{QT^2} \qquad \text{and} \qquad [\underset{(1)}{\widetilde{\mathcal{E}}}] = \frac{ML^2}{QT^2}$$
$$[\underset{(0)}{\mathcal{B}^i}] = \frac{M}{QT}, \qquad [\mathcal{B}] = \frac{M}{LQT^2} \qquad \text{and} \qquad [\underset{(1)}{\widetilde{\mathcal{B}}}] = \frac{ML}{QT} \tag{A.3}$$

and dimensions of the electromagnetic 1-from potential $\mathcal{A}$ and 2-forms $\mathcal{F}$ and $\star\mathcal{F}$ are given by

$$[\underset{(1)}{\mathcal{A}}] = [\underset{(2)}{\mathcal{F}}] = [\underset{(2)}{\star\mathcal{F}}] = \frac{ML^3}{QT^2} \tag{A.4}$$

Also

$$[\underset{(3)}{\mathcal{S}_{\mathrm{K}}}] = [\underset{(0)}{\mathrm{P}_{\mathrm{K}}}] = \frac{ML}{T} \qquad \text{and} \qquad [\underset{(0)}{\dot{\mathrm{P}}_{\mathrm{K}}}] = [\text{force}] = \frac{ML}{T^2}. \tag{A.5}$$

Base quantities have dimensions

$$[x^a] = [dx^a] = L, \qquad \text{and} \qquad \left[\frac{\partial}{\partial x^a}\right] = \frac{1}{L} \tag{A.6}$$

such that $[g] = L^2$ and $[g^{-1}] = \frac{1}{L^2}$.

A.1 Dimensions in chapter 1 and Part II

We choose proper time τ to have the dimension of time T such that

$$[C^a(\tau)] = L, \qquad [\dot{C}^a(\tau)] = \frac{L}{T}, \qquad [\ddot{C}^a(\tau)] = \frac{L}{T^2} \tag{A.7}$$

It follows

$$
\begin{aligned}
[X] =& [x^a - C^a(\tau)]\Big[\frac{\partial}{\partial x^a}\Big] = 1, \qquad [\widetilde{X}] = [g(-,X)] = L^2,\\
[V] =& [\dot{C}^a(\tau)]\Big[\frac{\partial}{\partial x^a}\Big] = \frac{1}{T}, \qquad [\widetilde{V}] = [g(-,V)] = \frac{L^2}{T},\\
[A] =& [\ddot{C}^a(\tau)]\Big[\frac{\partial}{\partial x^a}\Big] = \frac{1}{T^2}, \qquad [\widetilde{A}] = [g(-,A)] = \frac{L^2}{T^2}
\end{aligned}
\tag{A.8}
$$

and

$$
[g(V,V)] = \frac{L^2}{T^2}, \qquad [g(X,V)] = \frac{L^2}{T}, \qquad [g(X,A)] = \frac{L^2}{T^2}
\tag{A.9}
$$

A.2 Dimensions in Part I

We choose proper time τ to have the dimension of time L such that

$$
[C^a(\tau)] = L, \qquad [\dot{C}^a(\tau)] = 1, \qquad [\ddot{C}^a(\tau)] = \frac{1}{L}
\tag{A.10}
$$

It follows

$$
\begin{aligned}
[X] =& [x^a - C^a(\tau)]\Big[\frac{\partial}{\partial x^a}\Big] = 1, \qquad [\widetilde{X}] = [g(-,X)] = L^2,\\
[V] =& [\dot{C}^a(\tau)]\Big[\frac{\partial}{\partial x^a}\Big] = \frac{1}{L}, \qquad [\widetilde{V}] = [g(-,V)] = L,\\
[A] =& [\ddot{C}^a(\tau)]\Big[\frac{\partial}{\partial x^a}\Big] = \frac{1}{L^2}, \qquad [\widetilde{A}] = [g(-,A)] = 1
\end{aligned}
\tag{A.11}
$$

and

$$
[g(V,V)] = 1, \qquad [g(X,V)] = L, \qquad [g(X,A)] = 1
\tag{A.12}
$$

Appendix B

Differential Geometry

In this appendix we present a brief introduction to the geometric constructs and notations encountered in the thesis. This introduction is by no means complete, and for a deeper understanding of the subject we refer the reader to the vast collection of introductory books on the subject, of which [42, 43] are good examples.

B.1 Tensor Fields and differential forms

Vector fields

Definition B.1.1. Let $\mathbf{M}$ be an arbitrary differential manifold of dimension m, and x an arbitrary point in $\mathbf{M}$. The **space of smooth real valued c^∞ functions** over $\mathbf{M}$ is denoted $\mathcal{F}(\mathbf{M})$,

$$f \in \mathcal{F}(\mathbf{M}) \quad \text{implies} \quad f : \mathbf{M} \to \mathbb{R}, \qquad x \mapsto f(x) \tag{B.1}$$

Definition B.1.2. A (contravariant) **vector at a point** $V|_x$ is a map from $\mathcal{F}(\mathbf{M})$ to $\mathbb{R}$,

$$V|_x : \mathcal{F}(\mathbf{M}) \to \mathbb{R}, \qquad f \mapsto V|_x(f), \tag{B.2}$$

which satisfies

$$
\begin{aligned}
V|_x(f+g) &= V|_x(f) + V|_x(g), \\
V|_x(\lambda f) &= \lambda V|_x(f), \\
\text{and} \qquad V|_x(fg) &= V|_x(f)g + fV|_x(g),
\end{aligned} \tag{B.3}
$$

where $f, g \in \mathcal{F}(\mathbf{M})$ and λ is an arbitrary scalar.

Definition B.1.3. For every $x \in \mathbf{M}$ the **tangent space** to $\mathbf{M}$ at point x, written $\mathrm{T}_x\mathbf{M}$, is the m dimensional vector space whose elements are the vectors at x, i.e. $V|_x \in \mathrm{T}_x\mathbf{M}$. The **tangent bundle** is the $2m$ dimensional manifold given by the set theoretic union of the tangent spaces $\mathrm{T}_x\mathbf{M}$ for all $x \in \mathbf{M}$,

$$
\mathrm{T}\mathbf{M} = \bigcup_{x \in \mathbf{M}} \mathrm{T}_x\mathbf{M}. \tag{B.4}
$$

We call $\mathbf{M}$ the base space.

Definition B.1.4. Given the projection map

$$
\pi : \mathrm{T}\mathbf{M} \to \mathbf{M}, \qquad V|_x \mapsto x, \tag{B.5}
$$

a **section of the tangent bundle** is a continuous map

$$
\begin{aligned}
&V : \mathbf{M} \to \mathrm{T}\mathbf{M}, \qquad x \mapsto V|_x \\
\text{such that} \qquad &\pi(V|_x) = x \qquad \text{for all} \qquad x \in \mathbf{M}.
\end{aligned} \tag{B.6}
$$

The map V identifies a vector at a point for each point in the base space[1], therefore when acting on a smooth function it is the map

$$
V : \mathcal{F}(\mathbf{M}) \to \mathbb{R}, \qquad f \mapsto V(f) \tag{B.7}
$$

[1] We have used the notation $V|_x$ to denote a vector at a point as well as a vector field evaluated at a point

with the properties (B.3). A smooth section of the tangent bundle is a **vector field** . The space of vector fields over $\mathbf{M}$ is written $\Gamma\mathrm{T}\mathbf{M}$.

Definition B.1.5. Given a local coordinate basis y^a on $\mathbf{M}$ there exists an induced local basis $\frac{\partial}{\partial y^a}$ on $\mathrm{T}\mathbf{M}$. In terms of this basis a vector field $V \in \mathrm{T}\mathbf{M}$ is given by

$$V = V^a \frac{\partial}{\partial y^a} \tag{B.8}$$

where the Einstein summation convention is used for $a = 1..m$ and V^a are smooth functions on $\mathbf{M}$. In terms of a different local coordinate basis z^a

$$V = V^a \frac{\partial}{\partial y^a} = V^a \frac{\partial z^b}{\partial y^a} \frac{\partial}{\partial z^b}. \tag{B.9}$$

where V^a are the components of V in the y^a coordinate basis and $V^a \partial z^b / \partial y^a$ are the components of V in the z^b coordinate basis.

Differential 1-forms

Definition B.1.6. The space dual to the tangent space $\mathrm{T}_x\mathbf{M}$ is called the cotangent space at x and is denoted by $\mathrm{T}^*_x\mathbf{M}$. Elements $\zeta|_x \in \mathrm{T}^*_x\mathbf{M}$ are covariant vectors or covectors at x and satisfy

$$\zeta|_x : \mathrm{T}_x\mathbf{M} \to \mathbb{R}, \qquad V \mapsto \zeta|_x(V), \tag{B.10}$$

with the properties

$$\begin{aligned} \zeta|_x(V|_x + W|_x) &= \zeta|_x(V|_x) + \zeta|_x(W|_x), \\ \zeta|_x(fV|_x) &= f\zeta|_x(V|_x). \end{aligned} \tag{B.11}$$

Definition B.1.7. The **cotangent bundle** is the $2m$ dimensional manifold $\mathrm{T}^*\mathbf{M}$

defined as the set theoretic union of the cotangent spaces $\mathrm{T}^*_x\mathbf{M}$ for all $x \in \mathbf{M}$,

$$\mathrm{T}^*\mathbf{M} = \bigcup_{x\in\mathbf{M}} \mathrm{T}^*_x\mathbf{M}. \tag{B.12}$$

Definition B.1.8. Given the projection map

$$\pi_c : \mathrm{T}^*\mathbf{M} \to \mathbf{M}, \qquad \zeta|_x \mapsto x, \tag{B.13}$$

a **section of the cotangent bundle** is a continuous map

$$\begin{aligned} \zeta : \mathbf{M} \to \mathrm{T}^*\mathbf{M}, \qquad & x \mapsto \zeta|_x \\ \text{such that} \qquad \pi_c(\zeta|_x) = x \qquad & \text{for all} \qquad x \in \mathbf{M}. \end{aligned} \tag{B.14}$$

The map ζ identifies a covector at a point for each point in the base space. When acting on a vector field it is the map

$$\zeta : \Gamma\mathrm{T}\mathbf{M} \to \mathcal{F}(\mathbf{M}), \qquad V \mapsto \zeta(V) \tag{B.15}$$

with the properties

$$\begin{aligned} \zeta(V+W) &= \zeta(V) + \zeta(W), \\ \zeta(fV) &= f\zeta(V). \end{aligned} \tag{B.16}$$

A smooth section of the tangent bundle is called a **covector field** or **(differential) 1-form** . The space of 1-forms is written $\Gamma\mathrm{T}^*\mathbf{M}$.

Lemma B.1.9. *The duality of* $\mathrm{T}_x\mathbf{M}$ *and* $\mathrm{T}^*_x\mathbf{M}$ *demands*

$$\zeta|_x(V|_x) = V|_x(\zeta|_x) \tag{B.17}$$

where $V|_x(\zeta|_x)$ *satisfies the reversal of (B.11) with respect to vectors and covectors.*

Definition B.1.10. Given a local coordinate basis y^a on $\mathbf{M}$ there exists an induced local basis dy^a on $\mathrm{T}^*\mathbf{M}$. In terms of this basis a differential 1-form $\zeta \in \mathrm{T}^*\mathbf{M}$ is given by

$$\zeta = \zeta_a dy^a \tag{B.18}$$

where ζ_a are smooth functions on $\mathbf{M}$. In terms of a different local coordinate basis z^a

$$\zeta = \zeta_a dy^a = \zeta_a \frac{\partial y^a}{\partial z^b} dz^b. \tag{B.19}$$

where ζ^a are the components of ζ in the y^a coordinate basis and $\zeta_a \frac{\partial y^a}{\partial z^b}$ are the components of ζ in the z^a coordinate basis.

Tensor fields

Definition B.1.11. The degree of an arbitrary tensor will be represented as an ordered list s of 0 or more entries. Each entry is either the symbol $\mathbb{F}$ (for 1-form) or $\mathbb{V}$ (for vector) e.g. $s = [\mathbb{V}, \mathbb{F}, \mathbb{F}, \mathbb{V}]$. The space of tensors of degree s over $\mathbf{M}$ is denoted $\bigotimes^s \mathbf{M}$ with sections $\Gamma \bigotimes^s \mathbf{M}$. Let the tangent space and the cotangent space be denoted

$$\mathrm{T}\mathbf{M} = \bigotimes^{[\mathbb{V}]}\mathbf{M} \qquad \text{and} \qquad \mathrm{T}^*\mathbf{M} = \bigotimes^{[\mathbb{F}]}\mathbf{M} \tag{B.20}$$

then arbitrary degree tensors are constructed using the tensor product,

$$\otimes : \bigotimes^s\mathbf{M} \times \bigotimes^t\mathbf{M} \to \bigotimes^{[s,t]}\mathbf{M}, \qquad (\mathbf{T}, \mathbf{S}) \mapsto \mathbf{T} \otimes \mathbf{S} \tag{B.21}$$

where s and t are ordered lists and $[s,t]$ is simply the concatenation of the two lists. For example given vector field $V \in \Gamma\mathrm{T}\mathbf{M}$ and 1-forms $\alpha, \beta \in \Gamma\mathrm{T}^*\mathbf{M}$ we may define a degree $[\mathbb{V}, \mathbb{F}, \mathbb{F}]$ tensor field by

$$V \otimes \alpha \otimes \beta \in \Gamma \bigotimes^{[\mathbb{V},\mathbb{F},\mathbb{F}]} \mathbf{M}. \tag{B.22}$$

The tensor product satisfies

$$\begin{aligned} \mathbf{T} \otimes (\mathbf{S} \otimes \mathbf{R}) =& (\mathbf{T} \otimes \mathbf{S}) \otimes \mathbf{R} \\ (\mathbf{T}_1 + \mathbf{T}_2) \otimes \mathbf{S} =& \mathbf{T}_1 \otimes \mathbf{S} + \mathbf{T}_2 \otimes \mathbf{S} \end{aligned} \tag{B.23}$$

and

$$f(\mathbf{T} \otimes \mathbf{S}) = (f\mathbf{T}) \otimes \mathbf{S} = \mathbf{T} \otimes (f\mathbf{S}) \tag{B.24}$$

for $\mathbf{T}, \mathbf{T}_1, \mathbf{T}_2 \in \bigotimes^s \mathbf{M}$, $\mathbf{S} \in \bigotimes^t \mathbf{M}$, $\mathbf{R} \in \bigotimes^u \mathbf{M}$ and $f \in \mathcal{F}(\mathbf{M})$. The dual space of $\bigotimes^s \mathbf{M}$ is denoted $\bigotimes^{\bar{s}} \mathbf{M}$, where $\bar{s}$ is the list obtained by interchanging the symbols $\mathbb{F}$ and $\mathbb{V}$ in s. The total contraction of elements in $\bigotimes^{\bar{s}} \mathbf{M}$ with elements in $\bigotimes^s \mathbf{M}$ is written

$$\bigotimes^{\bar{s}} \mathbf{M} \times \bigotimes^s \mathbf{M} \to \mathcal{F}(\mathbf{M}), \qquad (\mathbf{T}, \mathbf{R}) \mapsto \mathbf{T} : \mathbf{R} \tag{B.25}$$

where $\mathbf{T} \in \bigotimes^{\bar{s}} \mathbf{M}$ and $\mathbf{R} \in \bigotimes^s \mathbf{M}$. It is defined inductively via

$$V : \alpha = \alpha : V = \alpha(V) \qquad \text{where} \qquad \alpha \in \bigotimes^{[\mathbb{F}]} \mathbf{M} \qquad \text{and} \qquad V \in \bigotimes^{[\mathbb{V}]}, \tag{B.26}$$

and extended to arbitrary tensors by

$$(\mathbf{S} \otimes \mathbf{T}) : (\mathbf{R} \otimes \mathbf{U}) = (\mathbf{S} : \mathbf{R})(\mathbf{T} : \mathbf{U}) \tag{B.27}$$

The metric

Definition B.1.12. Of special importance is the symmetric, non-degenerate degree $[\mathbb{F},\mathbb{F}]$ tensor field $g \in \bigotimes^{[\mathbb{F},\mathbb{F}]} \mathbf{M}$ with the properties

$$\begin{aligned} & g(V,W) = g(W,V), \\ \text{and} \quad & g(V,W) = 0 \quad \text{for all} \quad V \neq 0 \Rightarrow W = 0. \end{aligned} \tag{B.28}$$

This tensor is called the metric. It provides an isomorphism between the covariant and contravariant vector fields. The metric dual $\widetilde{V}$ of vector field V is the differential 1-form given by

$$\widetilde{V} = g(V, -), \tag{B.29}$$

where $(-)$ denotes an empty argument. There exists a symmetric, non-degenerate degree $[\mathbb{V},\mathbb{V}]$ tensor field $g^{-1} \in \bigotimes^{[\mathbb{V},\mathbb{V}]} \mathbf{M}$ which satisfies

$$g^{-1}(\widetilde{V},\widetilde{W}) = g(V,W) \quad \text{and} \quad V = g^{-1}(\widetilde{V}, -). \tag{B.30}$$

Given the local coordinate basis y^a on $\mathbf{M}$ the metric g is given by

$$g = g_{ab} dy^a \otimes dy^b \tag{B.31}$$

where the functions g_{ab} are determined by

$$g_{ab} = g\Big(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\Big) \tag{B.32}$$

Differential p-forms

Definition B.1.13. An important subspace of $\bigotimes \mathbf{M}$ is the space of totally anti-symmetric degree $[\mathbb{F}_1, ..., \mathbb{F}_p]$ tensors denoted $\Lambda^p\mathbf{M}$. The space of smooth sections of $\Lambda^p\mathbf{M}$ is denoted $\Gamma\Lambda^p\mathbf{M}$ and elements $\Psi \in \Gamma\Lambda^p\mathbf{M}$ are called (differential) p-

forms. For example for $\nu, \omega \in \Gamma \bigotimes^{[\mathbb{F}]} \mathbf{M}$ the totally antisymmetric part of the degree $[\mathbb{F}, \mathbb{F}]$ tensor field $\nu \otimes \omega$ is the difference

$$\frac{1}{2}(\nu \otimes \omega - \omega \otimes \nu) = \Psi \tag{B.33}$$

since reversing the positions of ν and ω yields

$$\frac{1}{2}(\omega \otimes \nu - \nu \otimes \omega) = -\Psi, \tag{B.34}$$

threfore $\Psi \in \Gamma\Lambda^2\mathbf{M}$ is a differential 2-form. The space of differential 0-forms $\Gamma\Lambda^0\mathbf{M}$ is defined as the space of smooth functions over $\mathbf{M}$, i.e. $\mathcal{F}(\mathbf{M}) = \Gamma\Lambda^0\mathbf{M}$ and the space of differential 1-forms $\Gamma\mathrm{T}^*\mathbf{M} = \Gamma \bigotimes^{[\mathbb{F}]} \mathbf{M}$ is now also written as as $\Gamma\Lambda^1\mathbf{M}$.

Higher degree forms are obtained using the **exterior or wedge product**. The wedge product of $\alpha \in \Gamma\Lambda^p\mathbf{M}$ and $\beta \in \Gamma\Lambda^q\mathbf{M}$ is the map

$$\wedge : \Gamma\Lambda^p\mathbf{M} \times \Gamma\Lambda^q\mathbf{M} \quad \rightarrow \quad \Gamma\Lambda^{p+q}\mathbf{M}, \qquad \alpha, \beta \quad \mapsto \quad \alpha \wedge \beta, \tag{B.35}$$

where the $(p+q)$-form $\alpha \wedge \beta$ is the totally antisymmetric part of the tensor field $\alpha \otimes \beta$. The wedge product satisfies

$$\begin{aligned} \alpha \wedge (\beta \wedge \gamma) =& (\alpha \wedge \beta) \wedge \gamma, \\ (\alpha_1 + \alpha_2) \wedge \beta =& \alpha_1 \wedge \beta + \alpha_2 \wedge \beta, \end{aligned} \tag{B.36}$$

and

$$f(\alpha \wedge \beta) = (f\alpha) \wedge \beta = \alpha \wedge (f\beta), \tag{B.37}$$

and

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \tag{B.38}$$

for $\alpha, \alpha_1, \alpha_2 \in \Lambda^p\mathbf{M}$, $\beta \in \Lambda^q\mathbf{M}$, $\gamma \in \Lambda^r\mathbf{M}$ and $f \in \mathcal{F}(\mathbf{M})$. It follows any arbitrary p-form can be reduced to a linear superposition of wedge products of p differential 1-forms. Given local coordinate basis y^a on $\mathbf{M}$,

$$\alpha \in \Gamma\Lambda^p\mathbf{M}, \qquad \alpha = \alpha_{a_1 a_2 .. a_p} dy^{a_1} \wedge dy^{a_2} \wedge .. \wedge dy^{a_p}. \tag{B.39}$$

B.2 Differential operators

Definition B.2.1. For the following definitions it is useful to define the map

$$\begin{aligned} \eta : \Lambda^p\mathbf{M} \to \Lambda^p\mathbf{M}, &\qquad \alpha \mapsto \eta(\alpha), \\ \text{where} \quad \eta(\alpha) = (-1)^p\alpha &\quad \text{for all} \quad \alpha \in \Lambda^p\mathbf{M}. \end{aligned} \tag{B.40}$$

Exterior derivative

Definition B.2.2. The **exterior derivative** of a differential form is the map

$$d : \Gamma\Lambda^p\mathbf{M} \to \Gamma\Lambda^{p+1}\mathbf{M}, \qquad \alpha \mapsto d\alpha, \tag{B.41}$$

where

$$\begin{aligned} df(V) &= V(f), \\ d(\alpha \wedge \beta) &= d\alpha \wedge \beta + \eta(\alpha) \wedge d\beta, \\ d(d\alpha) &= 0 \end{aligned} \tag{B.42}$$

for all $f \in \Gamma\Lambda^0\mathbf{M}$, $V \in \Gamma\mathrm{T}\mathbf{M}$, $\alpha \in \Gamma\Lambda^p\mathbf{M}$ and $\beta \in \Gamma\Lambda^q\mathbf{M}$

Lemma B.2.3. *For 1-form $\nu \in \Gamma\Lambda^1\mathbf{M}$ and vector fields $V, W \in \Gamma\mathrm{T}\mathbf{M}$ the following relation holds*

$$d\nu(V, W) = V(\nu(W)) - W(\nu(V)) - \nu([V, W]), \tag{B.43}$$

where $[,]$ *denotes the* Lie Bracket.

Proof of B.2.3. It is sufficient to prove for the $1-form$ $\nu = fdg$, where $f, g \in \mathcal{F}(\mathbf{M})$. In this case $d\nu = df \wedge dg$ and thus

$$\begin{aligned} d\nu(V,W) =& df \wedge dg(V,W), \\ =& \frac{1}{2}\big(df(V)dg(W) - dg(V)df(W)\big), \\ =& \frac{1}{2}\big(V(f)W(g) - V(g)W(f)\big). \end{aligned} \tag{B.44}$$

Here $d\nu(V,W)$ is the action of the $2-$form $d\nu$ on the ordered pair of vector fields (V,W). Also

$$\begin{aligned} V(\nu(W)) - W(\nu(V)) - \nu([V,W]) & \\ =& V(fdg(W)) - W(fdg(V)) - fdg([V,W]) \\ =& V(f(W(g))) - W(fV(g)) - f[V,W](g) \\ =& V(f)W(g) + fV(W(g)) - W(f)V(g) \\ & - fW(V(g)) - f[V,W](g) \\ =& V(f)W(g) - W(f)V(g) \\ =& 2d\nu(V,W) \end{aligned} \tag{B.45}$$

□

Interior contraction

Definition B.2.4. The **interior contraction** of a p-form with respect to a vector field is defined by

$$i : \Gamma\mathrm{T}\mathbf{M} \times \Gamma\Lambda^p\mathbf{M} \to \Gamma\Lambda^{p-1}\mathbf{M}, \qquad V, \alpha \mapsto i_V\alpha \tag{B.46}$$

where

$$
\begin{aligned}
i_V \nu &= \nu(V) \\
i_V(\alpha \wedge \beta) &= i_V \alpha \wedge \beta + \eta(\alpha) \wedge i_V \beta \\
i_V i_V \alpha &= 0
\end{aligned} \tag{B.47}
$$

for all $V \in \Gamma\mathrm{T}\mathbf{M}$, $\nu \in \Gamma\Lambda^1\mathbf{M}$, $\alpha \in \Gamma\Lambda^p\mathbf{M}$ and $\beta \in \Gamma\Lambda^q\mathbf{M}$.

Hodge Dual

Definition B.2.5. Introduce a g-orthonormal frame e^a such that

$$
g = g_{ab} dy^a \otimes dy^b = \eta_{ab} e^a \otimes e^b \tag{B.48}
$$

where $\eta_{ab} = \pm 1$ for $a = b$ and $\eta_{ab} = 0$ for $a \neq b$. Then

$$
\star 1 \in \Lambda^m \mathbf{M}, \qquad \star 1 = e^0 \wedge e^1 \wedge .. \wedge e^{m-1}, \tag{B.49}
$$

is the **volume form** on $\mathbf{M}$.

Definition B.2.6. The **Hodge dual** is the map

$$
\star : \Lambda^p \mathbf{M} \to \Lambda^{m-p} \mathbf{M}, \qquad \alpha \mapsto \star \alpha \tag{B.50}
$$

where

$$
\begin{aligned}
\star(1) &= \star 1 \\
\star \nu &= i_{\widetilde{\nu}} \star 1, \\
\star(\alpha \wedge \nu) &= i_{\widetilde{\nu}} \star \alpha, \\
\star(f\alpha) &= f \star \alpha.
\end{aligned} \tag{B.51}
$$

for all $\nu \in \Gamma\Lambda^1\mathbf{M}$, $\alpha \in \Gamma\Lambda^p\mathbf{M}$ and $\beta \in \Gamma\Lambda^q\mathbf{M}$. Properties (B.39) and (B.2.6) define the action of the Hodge dual on any arbitrary p-form.

Lemma B.2.7. *Given two vector fields $V, W \in \Gamma\mathbf{TM}$ the following relation is true*

$$\widetilde{V} \wedge \star\widetilde{W} = g(V, W) \star 1 \tag{B.52}$$

Proof of B.2.7.

$$\begin{aligned} \widetilde{V} \wedge \star\widetilde{W} =& \widetilde{V} \wedge i_W \star 1 \\ =& V_a W^b dz^a \wedge i_{\frac{\partial}{\partial z^b}} \star 1 \end{aligned} \tag{B.53}$$

Using the alternating Leibniz rule (B.47) yields

$$\begin{aligned} 0 = i_{\partial z^b}(dz^a \wedge \star 1) = i_{\partial z^b} dz^a \wedge \star 1 - dz^a \wedge i_{\partial z^b} \star 1 \\ \text{and thus} \qquad dz^a \wedge i_{\partial z^b} \star 1 = \delta^a_b \star 1. \end{aligned} \tag{B.54}$$

Substituting (B.54) into (B.53) yields

$$\begin{aligned} \widetilde{V} \wedge \star\widetilde{W} =& V_b W^b \star 1 \\ =& g(V, W) \star 1 \end{aligned} \tag{B.55}$$

□

Lemma B.2.8. *Given $\nu \in \Gamma\Lambda^1\mathbf{M}$ and $\alpha \in \Gamma\Lambda^p\mathbf{M}$ the following is true*

$$\nu \wedge \star\alpha = - \star(i_{\widetilde{\nu}} \alpha^\eta) \tag{B.56}$$

where $\alpha^\eta = \eta(\alpha)$.

Proof of B.2.8. By lemma B.2.7 it is clearly true for $\deg(\alpha) = 1$. Assume true for

$\deg(\alpha) = p$, then for $\omega \in \Gamma\Lambda^1\mathbf{M}$

$$\begin{aligned}\star i_{\widetilde{\nu}}(\alpha \wedge \omega) &= \star\,(i_{\widetilde{\nu}}\alpha \wedge \omega) + \star(\alpha^\eta \wedge i_{\widetilde{\nu}}\omega)\\ &= \star\,(i_{\widetilde{\nu}}\alpha \wedge \omega) + g(\widetilde{\omega}, \widetilde{\nu}) \star \alpha^\eta\\ &= \star\,(i_{\widetilde{\nu}}\alpha \wedge \omega) + i_{\widetilde{\omega}}(\nu \wedge \star\alpha^\eta) + \nu \wedge i_{\widetilde{\omega}} \star \alpha^\eta\end{aligned} \tag{B.57}$$

Evaluating the first two terms yields

$$\begin{aligned}\star(i_{\widetilde{\nu}}\alpha \wedge \omega) + i_{\widetilde{\omega}}(\nu \wedge \star\alpha^\eta) &= \star\,(\omega \wedge i_{\widetilde{\nu}}\alpha^\eta) - i_{\widetilde{\omega}}(\star(i_{\widetilde{\nu}}\alpha))\\ &= \star\,(\omega \wedge i_{\widetilde{\nu}}\alpha^\eta) - \star(i_{\widetilde{\nu}}\alpha \wedge \omega)\\ &= \star\,(\omega \wedge i_{\widetilde{\nu}}\alpha^\eta) - \star(\omega \wedge i_{\widetilde{\nu}}\alpha^\eta)\\ &= 0\end{aligned} \tag{B.58}$$

Thus

$$-\star i_{\widetilde{\nu}}(\alpha \wedge \omega) = -\,\nu \wedge i_{\widetilde{\omega}} \star \alpha^\eta = \nu \wedge \star(\alpha \wedge \omega)^\eta \tag{B.59}$$

□

Lemma B.2.9. *Given 1-forms $\nu, \omega, \alpha \in \Gamma\Lambda^1\mathbf{M}$ the following is true*

$$\nu \wedge \star(\omega \wedge \alpha) = g(\widetilde{\nu}, \widetilde{\alpha}) \star \omega - g(\widetilde{\nu}, \widetilde{\omega}) \star \alpha \tag{B.60}$$

Proof of B.2.9. By lemma B.2.8 we have

$$\begin{aligned}
\nu \wedge \star(\omega \wedge \alpha) &= - \star(i_{\widetilde{\nu}}(\omega \wedge \alpha) \\
&= - \star(i_{\widetilde{\nu}}\omega \wedge \alpha - \omega \wedge i_{\widetilde{\nu}}\alpha) \\
&= \star\left(g(\widetilde{\nu}, \widetilde{\alpha})\omega\right) - \star(g(\widetilde{\nu}, \widetilde{\omega})\alpha) \\
&= g(\widetilde{\nu}, \widetilde{\alpha}) \star \omega - g(\widetilde{\nu}, \widetilde{\omega}) \star \alpha
\end{aligned}$$

□

Lemma B.2.10. *Given one forms $\alpha, \beta, \gamma, \nu, \omega \in \Lambda^1\mathbf{M}$ the following is true*

$$\begin{aligned}
i_{\widetilde{\omega}} \star (\alpha \wedge \beta) \wedge \gamma \wedge \nu =& \big(\beta(\widetilde{\gamma})\omega(\widetilde{\nu}) - \omega(\widetilde{\gamma})\beta(\widetilde{\nu})\big) \star \alpha \\
&+ \big(\omega(\widetilde{\gamma})\alpha(\widetilde{\nu}) - \alpha(\widetilde{\gamma})\omega(\widetilde{\nu})\big) \star \beta \\
&+ \big(\alpha(\widetilde{\gamma})\beta(\widetilde{\nu}) - \beta(\widetilde{\gamma})\alpha(\widetilde{\nu})\big) \star \omega
\end{aligned} \tag{B.61}$$

Proof of B.2.10.

$$\begin{aligned}
i_{\widetilde{\omega}} \star (\alpha \wedge \beta) \wedge \gamma \wedge \nu &= \star (\alpha \wedge \beta \wedge \omega) \wedge \gamma \wedge \nu \\
&= - \gamma \wedge \star(\alpha \wedge \beta \wedge \omega) \wedge \nu
\end{aligned} \tag{B.62}$$

Using (B.2.8) yields

$$\begin{aligned}
\gamma \wedge \star(\alpha \wedge \beta \wedge \omega) &= - \star\big(i_{\widetilde{\gamma}}(-\alpha \wedge \beta \wedge \omega)\big) \\
&= \star \left(i_{\widetilde{\gamma}}(\alpha \wedge \beta \wedge \omega)\right) \\
&= \star \left(\alpha(\widetilde{\gamma})\beta \wedge \omega - \beta(\widetilde{\gamma})\alpha \wedge \omega + \omega(\widetilde{\gamma})\alpha \wedge \beta\right)
\end{aligned} \tag{B.63}$$

Thus substituting (B.63) into (B.62) yields

$$i_{\widetilde{\omega}} \star (\alpha \wedge \beta) \wedge \gamma \wedge \nu = -\nu \wedge \star\big(\alpha(\widetilde{\gamma})\beta \wedge \omega - \beta(\widetilde{\gamma})\alpha \wedge \omega + \omega(\widetilde{\gamma})\alpha \wedge \beta\big) \tag{B.64}$$

Now using (B.2.8) again yields

$$\begin{aligned}
i_{\widetilde{\omega}} \star (\alpha \wedge \beta) \wedge \gamma \wedge \nu =& \star \Big(i_{\widetilde{\omega}}\big(\alpha(\widetilde{\gamma})\beta \wedge \omega\big) - i_{\widetilde{\omega}}\big(\beta(\widetilde{\gamma})\alpha \wedge \omega\big) + i_{\widetilde{\omega}}\big(\omega(\widetilde{\gamma})\alpha \wedge \beta\big)\Big) \\
=&\big(\beta(\widetilde{\gamma})\omega(\widetilde{\nu}) - \omega(\widetilde{\gamma})\beta(\widetilde{\nu})\big) \star \alpha \\
&+ \big(\omega(\widetilde{\gamma})\alpha(\widetilde{\nu}) - \alpha(\widetilde{\gamma})\omega(\widetilde{\nu})\big) \star \beta \\
&+ \big(\alpha(\widetilde{\gamma})\beta(\widetilde{\nu}) - \beta(\widetilde{\gamma})\alpha(\widetilde{\nu})\big) \star \omega
\end{aligned}$$

□

Lemma B.2.11. *For two forms $\alpha, \beta \in \Gamma\Lambda^p\mathbf{M}$ of the same degree the following is true*

$$\alpha \wedge \star\beta = \beta \wedge \star\alpha. \tag{B.65}$$

Proof of B.2.11. By (B.2.7) it is clearly true for $\deg(\alpha) = \deg(\beta) = 1$. Assume true for $\deg(\alpha) = \deg(\beta) = p$, then for for $\alpha \in \Gamma\Lambda^{p+1}\mathbf{M}$, $\beta \in \Gamma\Lambda^p\mathbf{M}$ and $\nu \in \Gamma\Lambda^1\mathbf{M}$ we have

$$\begin{aligned}
\alpha \wedge \star(\beta \wedge \nu) =& \alpha \wedge i_{\widetilde{\nu}} \star \beta \\
=& i_{\widetilde{\nu}}(\alpha^\eta \wedge \star\beta) - i_{\widetilde{\nu}}\alpha^\eta \wedge \star\beta \\
=& -\beta \wedge \star i_{\widetilde{\nu}}\alpha^\eta \\
=& \beta \wedge \nu \wedge \star\alpha
\end{aligned} \tag{B.66}$$

□

Lie Derivative

Definition B.2.12. The Lie derivative is the the map

$$\mathcal{L} : \Gamma\mathrm{T}\mathbf{M} \times \Gamma\bigotimes^{[s]}\mathbf{M} \to \Gamma\bigotimes^{[s]}\mathbf{M}, \qquad V, \mathbf{T} \mapsto \mathcal{L}_V\mathbf{T}, \tag{B.67}$$

which is additive linear in both arguments

$$\mathcal{L}_V(\mathbf{T}+\mathbf{S}) = \mathcal{L}_V\mathbf{T} + \mathcal{L}_V\mathbf{S},$$
$$\mathcal{L}_{V+W}\mathbf{T} = \mathcal{L}_V\mathbf{T} + \mathcal{L}_W\mathbf{T}, \tag{B.68}$$

has the properties

$$\mathcal{L}_V f = V(f),$$
$$\mathcal{L}_V W = [V,W], \tag{B.69}$$

and obeys the Leibniz rule for tensor products, wedge products and contractions

$$\mathcal{L}_V(\mathbf{T}\otimes\mathbf{S}) = \mathcal{L}_V\mathbf{T}\otimes\mathbf{S} + \mathbf{T}\otimes\mathcal{L}_V\mathbf{S}, \tag{B.70}$$
$$\mathcal{L}_V(\alpha\wedge\beta) = \mathcal{L}_V\alpha\wedge\beta + \alpha\wedge\mathcal{L}_V\beta, \tag{B.71}$$
$$\mathcal{L}_V(\alpha(W)) = \mathcal{L}_V\alpha(W) + \alpha(\mathcal{L}_V W). \tag{B.72}$$

Lemma B.2.13. *Cartan's formula*

$$\mathcal{L}_V = di_V + i_V d \tag{B.73}$$

Proof of B.2.13. Trivial for 0-forms. First prove for 1-form $\nu \in \Lambda^1\mathbf{M}$. From (B.72) and (B.43)

$$\begin{aligned}
\mathcal{L}_V\nu(W) =& \mathcal{L}_V(\nu(W)) - \nu(\mathcal{L}_V W)\\
=& V(\nu(W)) - \nu([V,W])\\
=& 2d\nu(V,W) + W(\nu(V))\\
=& i_V d\nu(W) + d(\nu(V))(W)\\
=& i_V d\nu(W) + di_V\nu(W)\\
=& (i_V d\nu + di_V\nu)(W)
\end{aligned} \tag{B.74}$$

Now assume true for p-form $\alpha \in \Gamma\Lambda^p\mathbf{M}$, show true for $(p+1)$-form $\alpha \wedge \nu$.

$$\begin{aligned}
(i_V d + d i_V)(\alpha \wedge \nu) =& i_V(d\alpha \wedge \nu + (-1)^p \alpha \wedge d\nu) + d(i_V\alpha \wedge \nu + (-1)^p \alpha \wedge i_V \nu) \\
=& i_V(d\alpha \wedge \nu) + (-1)^p i_V(\alpha \wedge d\nu) + d(i_V\alpha \wedge \nu) + (-1)^p d(\alpha \wedge i_V\nu) \\
=& i_V d\alpha \wedge \nu + ((-1)^{p+1} + (-1)^p) d\alpha \wedge i_V\nu + d i_V \alpha \wedge \nu \\
&+ ((-1)^{p-1} + (-1)^p) i_V\alpha \wedge d\nu + (-1)^{2p}(\alpha \wedge i_V d\nu + \alpha \wedge d i_V \nu) \\
=& (i_V d + d i_V)\alpha \wedge \nu + \alpha \wedge (i_V d + d i_V)\nu \\
=& \mathcal{L}_V \alpha \wedge \nu + \alpha \wedge \mathcal{L}_V \nu \\
=& \mathcal{L}_V(\alpha \wedge \nu)
\end{aligned}$$

Thus by induction true for all p-forms. □

Lemma B.2.14. *The Lie derivative commutes with the exterior derivative*

$$\mathcal{L}_V d = d\mathcal{L}_V \tag{B.75}$$

Proof of B.2.14. follows trivially from B.73 □

Levi-Civita Connection

An affine connection is a map

$$\nabla : \Gamma\mathrm{T}\mathbf{M} \times \Gamma\bigotimes^{[s]}\mathbf{M} \quad \rightarrow \quad \Gamma\bigotimes^{[s]}\mathbf{M}, \qquad (V, \mathbf{T}) \mapsto \nabla_V \mathbf{T} \tag{B.76}$$

which is additive linear in both arguments,

$$\begin{aligned}
\nabla_V(\mathbf{T} + \mathbf{S}) &= \nabla_V \mathbf{T} + \nabla_V \mathbf{S}, \\
\nabla_{V+W}\mathbf{T} &= \nabla_V \mathbf{T} + \nabla_W \mathbf{T},
\end{aligned} \tag{B.77}$$

and satisfies

$$\begin{aligned}
&\nabla_V f = V(f),\\
&\nabla_{fV}\mathbf{T} = f\nabla_V\mathbf{T},\\
&\nabla_V(\mathbf{T}\otimes\mathbf{S}) = \nabla_V\mathbf{T}\otimes\mathbf{S} + \mathbf{T}\otimes\nabla_V\mathbf{S},\\
&\nabla_V(\alpha\wedge\beta) = \nabla_V\alpha\wedge\beta + \alpha\wedge\nabla_V\beta.
\end{aligned} \tag{B.78}$$

The Levi-Civita connection on $\mathbf{M}$ is the unique torsion free metric compatible affine connection.

B.3 Pushforwards, pullbacks and curves

Pushforward map

Definition B.3.1. Given differentiable manifolds $\mathbf{M}$ and $\mathbf{N}$ and the smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{N}, \quad x \longmapsto \phi(x)$, the **pushforward** of a vector at a point $V|_x \in \mathrm{T}_x\mathbf{M}$ with respect to ϕ is the map

$$\phi_* : \mathrm{T}_x\mathbf{M} \longrightarrow \mathrm{T}_{\phi(x)}\mathbf{N}, \qquad V|_x \longmapsto \phi_* V|_x, \tag{B.79}$$

where

$$\begin{aligned}
\phi_* V|_x(f) &= V|_x(f\circ\phi),\\
\phi_*(V|_x + W|_x) &= \phi_* V|_x + \phi_* W|_x,\\
\text{and}\quad \phi_*(V|_x W|_x) &= \phi_* V|_x \phi_* W|_x
\end{aligned} \tag{B.80}$$

for $V|_x, W|_x \in \mathrm{T}_x\mathbf{M}$ and $f \in \Gamma\Lambda^0\mathbf{N}$. The pushforward of a vector at a point is naturally extended using (B.6) to obtain the pushforward of a vector field

$$\phi_* : \Gamma\mathrm{T}\mathbf{M} \longrightarrow \Gamma\mathrm{T}\mathbf{N}, \qquad V \longmapsto \phi_* V. \tag{B.81}$$

Let $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{O}$ be differential manifolds and let $\phi_* : \mathrm{T}_x\mathbf{M} \longrightarrow \mathrm{T}_{\phi(x)}\mathbf{N}$ and $\psi_* : \mathrm{T}_x\mathbf{N} \longrightarrow \mathrm{T}_{\phi(x)}\mathbf{O}$, then

Lemma B.3.2. *The composition of the pushforwards is the pushforward of the composition*

$$\psi_* \circ \phi_* = (\psi \circ \phi)_* \tag{B.82}$$

Proof of B.3.2.

$$\begin{aligned}(\psi \circ \phi)_* V(f) &= V(f \circ \psi \circ \phi)\\ &= \phi_* V(f \circ \psi)\\ &= \psi_* \phi_* V(f)\end{aligned}$$

□

Lemma B.3.3. *Let $(x^1, x^2, ..., x^m)$ be a coordinate basis of $\mathbb{R}^M$ and $(y^1, y^2, ..., y^n)$ a coordinate basis of $\mathbb{R}^N$. Given a diffeomorphism ϕ where*

$$\begin{aligned}&\phi : \mathbb{R}^M \longrightarrow \mathbb{R}^N,\\ &x = x^a(x) = (x^1(x), .., x^m(x)) \quad \longmapsto \quad \phi(x) = y^b(\phi(x^1(x), .., x^m(x))),\end{aligned}$$

then

$$\phi_* \frac{\partial}{\partial x^a}\Big|_x = \frac{\partial \phi^b}{\partial x^a}\frac{\partial}{\partial y^b}, \qquad \textit{where} \qquad \phi^b = y^b \circ \phi. \tag{B.83}$$

Proof of B.3.3.

$$\begin{aligned}\phi_* \frac{\partial}{\partial x^a}\Big|_x (f) &= \frac{\partial}{\partial x^a}\Big|_x (f \circ \phi)\\ &= \frac{\partial}{\partial x^a}\Big|_x (f \circ y^b \circ \phi)\\ &= \frac{\partial}{\partial x^a} f(y^b(\phi(x)))\end{aligned}$$

where $y^b(\phi(x)) = \phi^b$ by definition. Evaluating using the chain rule yields

$$\begin{aligned}\frac{\partial}{\partial x^a}(f(\phi^b)) &= \frac{\partial \phi^b}{\partial x^a}\frac{\partial f}{\partial y^b} \\ &= \frac{\partial \phi^b}{\partial x^a}\frac{\partial}{\partial y^b}(f)\end{aligned}$$

□

Pullback map

Definition B.3.4. For the smooth maps $\phi : \mathbf{M} \to \mathbf{N}$ and $f : \mathbf{N} \to \mathbb{R}$ the **pullback** of f with respect to ϕ is given by the composition:

$$\phi^* f = \psi \circ f \tag{B.84}$$

Lemma B.3.5. *For the smooth maps $\phi : \mathbf{M} \to \mathbf{N}$, $\psi : \mathbf{N} \to \mathbf{O}$, and $f : \mathbf{N} \to \mathbb{R}$ the* **pullback** *of the composition is the composition of the pullbacks*

$$(\phi \circ \psi)^* = \phi^* \circ \psi^* \tag{B.85}$$

Proof of B.3.5.

$$\begin{aligned}\psi^*(\phi^* f) &= (f \circ \phi) \circ \psi \\ &= f \circ \phi \circ \psi \\ &= (\phi \circ \psi)^* f\end{aligned}$$

□

Definition B.3.6. The pullback of a differential p-form with respect to the smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{N}, \quad x \longmapsto \phi(x)$ is given by:

$$\phi^* : \Gamma\Lambda^p\mathbf{N} \longrightarrow \Gamma\Lambda^p\mathbf{M}, \qquad \alpha \longmapsto \phi^*\alpha \tag{B.86}$$

where

$$\begin{aligned} \phi^*(\alpha+\beta) &= \phi^*\alpha + \phi^*\beta \\ \phi^*(\alpha\wedge\beta) &= \phi^*\alpha \wedge \phi^*\beta \end{aligned} \tag{B.87}$$

for all $\alpha \in \Gamma\Lambda^p\mathbf{N}$ and $\beta \in \Gamma\Lambda^q\mathbf{N}$.

Definition B.3.7. The pullback of a 1-form acting on a vector field is the 1-form acting on the pushforward of the vector field

$$\phi^* df(V) = df(\phi_* V), \tag{B.88}$$

for all $f \in \Gamma\Lambda^0\mathbf{N}$ and $V \in \Gamma\mathrm{T}\mathbf{M}$.

Lemma B.3.8. *the pullback commutes with the exterior derivative*

$$d\phi^*\alpha = \phi^* d\alpha \tag{B.89}$$

for all $\alpha \in \Gamma\Lambda^p\mathbf{M}$.

Proof of B.3.8. First show for $f \in \Gamma\Lambda^0\mathbf{M}$

$$\begin{aligned} \phi^* df(V) &= df(\phi_* V) \\ &= \phi_* V(f) \\ &= V(f\circ\phi) \\ &= V(\phi^* f) \\ &= d\phi^* f(V) \end{aligned}$$

Now show for $\omega = gdf \in \Gamma\Lambda^1\mathbf{M}$ where $g, f \in \Gamma\Lambda^0\mathbf{M}$

$$\begin{aligned}
\phi^* d(gdf) &= \phi^*(dg \wedge df) \\
&= (\phi^* dg) \wedge (\phi^* df) \\
&= d\phi^* g \wedge \phi^* df \\
&= d(\phi^* g \phi^* df) \\
&= d\phi^*(gdf)
\end{aligned}$$

proof for a general 1-form $\nu = \nu_i dx^i$ follows by linearity.

Now, assuming true for $\alpha \in \Gamma\Lambda^p\mathbf{M}$

$$\begin{aligned}
\phi^* d(\nu \wedge \alpha) &= \phi^*[d\nu \wedge \alpha - \nu \wedge d\alpha] \\
&= \phi^*(d\nu) \wedge \phi^*\alpha - \phi^*\nu \wedge \phi^* d\alpha \\
&= d\phi^*\nu \wedge \phi^*\alpha - \phi^*\nu \wedge d\phi^*\alpha \\
&= d[\phi^*\nu \wedge \phi^*\alpha] \\
&= d\phi^*(\nu \wedge \alpha)
\end{aligned}$$

Hence by induction the relation holds for all p-forms. □

Lemma B.3.9. *The internal contraction of a pullback with respect to a vector field V, is equal to the pullback of the internal contraction with respect to the pushforward of V*

$$i_V \phi^* \alpha = \phi^*(i_{\phi_* V}\alpha) \tag{B.90}$$

Proof of B.3.9. (by induction)

Trivial for 0-forms. First prove for 1-form $\omega = gdf \quad \in \Gamma\Lambda^1\mathbf{N}$

$$\begin{aligned} i_V(\phi^* gdf) &= i_V(\phi^* g \phi^* df) \\ &= \phi^* g i_V \phi^* df \\ &= \phi^* g(\phi^* df).V \end{aligned}$$

Noticing that $\phi^* df(V) = df(\phi_* V)$, and remembering that the pullback of a number doesn't change the number, ie

$$\begin{aligned} df(\phi_* V) &= \phi^*(df.\phi_* V) \\ &= \phi^*(i_{\phi_* V} df) \end{aligned}$$

We can therefore write

$$\begin{aligned} i_V(\phi^* gdf) &= \phi^* g \phi^*(i_{\phi_* V} df) \\ &= \phi^*(g i_{\phi_* V} df) \\ &= \phi^*[i_{\phi_* V}(gdf)] \\ &= \phi^*(i_{\phi_* V} \nu) \end{aligned}$$

proof for a general 1-form $\nu = \nu_i dx^i$ follows by linearity.

Now, assuming the relation holds for $\alpha \epsilon \Gamma\Lambda^p\mathbf{N}$, we have

$$\begin{aligned} i_V \phi^*(\nu \wedge \alpha) &= i_V(\phi^* \nu \wedge \phi^* \alpha) \\ &= i_V \phi^* \nu \wedge \phi^* \alpha - \phi^* \nu \wedge \phi^* \alpha \\ &= \phi^*(i_{\phi_* V} \nu) \wedge \phi^* \alpha - \phi^* \nu \wedge \phi^*(i_{\phi_* V} \alpha) \\ &= \phi^*(i_{\phi_* V}(\nu \wedge \alpha)) \end{aligned}$$

Thus we have proved by induction that the relation must hold for all $(p+1)$ forms, and therefore for any general form $\beta \in \Gamma\Lambda^q\mathbf{N}$. $\square$

Curves

Definition B.3.10. A smooth parameterized curve $C(s)$ on a manifold $\mathbf{M}$ is a smooth map from an open interval $I \subset \mathbb{R}$ to $\mathbf{M}$,

$$C : I \to \mathbf{M}, \qquad s \mapsto C(s). \tag{B.91}$$

If y^a are local coordinates on $\mathbf{M}$ then we use the notation

$$y^a \circ C(s) = C^a(s), \tag{B.92}$$

thus if $C(s_0) = x$ is any point on the image of C then

$$y^a(x) = C^a(s_0). \tag{B.93}$$

The tangent vector to C at x is

$$\dot{C}|_x \in T_x\mathbf{M}, \qquad \dot{C}|_x = C_*\Big(\frac{\partial}{\partial s}\Big)\Big|_{s_0} \tag{B.94}$$

For any $f \in \mathcal{F}(\mathbf{M})$

$$C_*\Big(\frac{\partial}{\partial s}\Big)\Big|_{s_0}(f) = \frac{\partial}{\partial s}(C^* f)\Big|_{s_0} = \frac{\partial}{\partial s}(f \circ C(s_0)) = \frac{\partial f}{\partial y^a}\frac{\partial C^a}{\partial s}\Big|_{s_0}, \tag{B.95}$$

hence

$$\dot{C}|_x = \dot{C}^a \frac{\partial}{\partial y^a}\Big|_{s_0} = \frac{\partial C^a}{\partial s}\Big|_{s_0}\frac{\partial}{\partial y^a}. \tag{B.96}$$

There is an induced vector field $\dot{C} \in \Gamma\mathbf{TM}$ where $\dot{C}|_x$ is the tangent vector at x for all $x \in C(s)$.

B.4 Integration of p-forms

Definition B.4.1. Let σ be a diffeomorphism from the submanifold $\Sigma \subset \mathbf{M}$ of dimension n to the differentiable manifold $\mathbf{M}$ of dimension m.

$$\sigma : \Sigma \hookrightarrow \mathbf{M} \tag{B.97}$$

If y^a are local coordinates on $\mathbf{M}$ at $\sigma(x)$ then σ^* acting on the local basis of 1-forms dy^a is given by

$$\sigma^* dy^a = d(y^a \circ \sigma) \tag{B.98}$$

and for any $f \in \mathcal{F}(\mathbf{M})$

$$\sigma^*(f dy^a) = (f \circ \sigma) d(y^a \circ \sigma) \tag{B.99}$$

If we define a local coordinate system for Σ at $x \in \Sigma$ by

$$z^a = y^a \circ \sigma \tag{B.100}$$

then

$$\sigma^*(f dy^0 \wedge dy^1 \wedge .. \wedge dy^m) = (f \circ \sigma) dz^0 \wedge dz^1 \wedge .. \wedge dz^n \tag{B.101}$$

Definition B.4.2. If m-form $\alpha \in \Gamma\Lambda^m\mathbf{M}$ has compact support then so does the n-form $\sigma^*\alpha \in \Gamma\Lambda^n\Sigma$, and

$$\int_{\mathbf{M}} \alpha = \int_{\Sigma} \sigma^* \alpha. \tag{B.102}$$

Theorem B.4.3. *If Σ is an oriented differential manifold of dimension n, with* *boundary* $\partial\Sigma$ *of dimension* $(n-1)$ *then*

$$\int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha, \tag{B.103}$$

for all $\alpha \in \Gamma\Lambda^{n-1}\Sigma$ *with compact support. This theorem is often called the generalized* Stokes' *theorem.*

Theorem B.4.4. *Let* t *be a choice of coordinate on a manifold* $\mathbf{M}$ *such that* $\frac{\partial}{\partial t}$ *is Killing and let* t *foliate* $\mathbf{M}$ *into surfaces* Σ_t *. Then for* $\alpha \in \Gamma\Lambda^p\mathbf{M}$

$$\frac{d}{dt}\int_{\Sigma_t} \alpha = \int_{\Sigma_t} \mathcal{L}_{\frac{\partial}{\partial t}}\alpha, \tag{B.104}$$

and thus

$$\int_{\mathbf{M}(t_1,t_0)} \alpha = \int_{t=t_0}^{t_1} dt \int_{\Sigma_t} i_{\partial_t}\alpha \tag{B.105}$$

where $\mathbf{M}(t_1, t_0)$ *is a submanifold of* $\mathbf{M}$ *with range of* t *between* t_0 *and* t_1.

Appendix C

Distributional p-forms

C.1 Definitions

The space of C^∞ functions with compact support is called the space of test functions. We extend this notion to the space of test p-forms.

Definition C.1.1. Let $\mathbf{M}$ be a differential manifold of dimension m. The space of test p-forms on $\mathbf{M}$ is denoted $\Gamma_0\Lambda^p\mathbf{M}$,

$$\Gamma_0\Lambda^p\mathbf{M} = \{\varphi \in \Gamma\Lambda^p\mathbf{M} |\ \varphi \ \text{has compact support}\}. \tag{C.1}$$

Definition C.1.2. The space of p-form distributions $\Gamma_D\Lambda^p\mathbf{M}$ is the vector space dual to the space of test $(m-p)$-forms $\Gamma_0\Lambda^{m-p}\mathbf{M}$,

$$\Gamma_D\Lambda^p\mathbf{M} \times \Gamma_0\Lambda^{m-p}\mathbf{M} \to \mathbb{R}, \qquad (\Psi, \varphi) \mapsto \Psi[\varphi] \in \mathbb{R}, \tag{C.2}$$

which satisfies

$$\Psi[\lambda\varphi + \psi] = \lambda\Psi[\varphi] + \Psi[\psi], \tag{C.3}$$

for $\lambda \in \mathbb{R}$, $\varphi, \psi \in \Gamma_0\Lambda^{m-p}\mathbf{M}$ and $\Psi \in \Gamma_D\Lambda^p\mathbf{M}$.

Definition C.1.3. The subspace of $\Gamma_D\Lambda^p\mathbf{M}$ comprising **piecewise continuous**

p-forms is the space of **regular distributions**. The action of a regular p-form distribution ψ^D on an $(m-p)$-test form φ is given by the integral

$$\psi^D[\varphi] = \int_{\mathbf{M}} \varphi \wedge \psi \tag{C.4}$$

for any $\varphi \in \Gamma_0 \Lambda^{m-p}\mathbf{M}$ and where $\psi \in \Gamma \Lambda^p \mathbf{M}$ is piecewise continuous. We say that ψ^D is the p-form distribution associated with the p-form ψ.

Definition C.1.4. The **exterior derivative** of a p-form distribution is defined as:

$$d : \Gamma_D \Lambda^p \mathbf{M} \to \Gamma_D \Lambda^{p+1} \mathbf{M}, \quad \Psi \mapsto d\Psi \tag{C.5}$$

and satisfies

$$d\Psi[\varphi] = -\Psi[d\varphi^\eta] \tag{C.6}$$

For any $\varphi \in \Gamma_0 \Lambda^{m-(p+1)} \mathbf{M}$

Lemma C.1.5. *If* $\mathbf{M}$ *has no boundary then for any regular distribution* $\psi^D \in \Gamma_D \Lambda^p \mathbf{M}$

$$d\psi^D[\varphi] = (d\psi)^D[\varphi] \tag{C.7}$$

Proof of C.1.5.

$$\begin{aligned}
d\psi^D[\varphi] &= -\int_{\mathbf{M}} d\varphi^\eta \wedge \psi \\
&= \int_{\mathbf{M}} \varphi \wedge d\psi - \int_{\mathbf{M}} d(\varphi^\eta \wedge \psi) \\
&= \int_{\mathbf{M}} \varphi \wedge d\psi - \int_{\partial\mathbf{M}} (\varphi^\eta \wedge \psi) \\
&= \int_{\mathbf{M}} \varphi \wedge d\psi \\
&= (d\psi)^D[\varphi]
\end{aligned}$$

□

C.2 Criteria for regular distributions in N-U coordinates

Theorem C.2.1. *Let the 1-form $\alpha \in \Gamma\Lambda^1(\mathcal{M}\backslash C)$ be represented in Newman-Unti coordinates by*

$$\alpha = \alpha_i dz^i, \qquad \textit{where} \quad z^0 = \tau, \quad z^1 = R, \quad z^2 = \theta, \quad z^3 = \phi, \tag{C.8}$$

and where the functions $\alpha_i = \alpha_i(\tau, R, \theta, \phi)$ are polynomials in R and are singular on the worldline. Let the most divergent terms in the polynomial functions α_i be denoted

$$\hat{\alpha}_i = \frac{\alpha_i'(\tau, \theta, \phi)}{R^{\beta_i}}. \tag{C.9}$$

where $\alpha_i'(\tau, \theta, \phi)$ are bounded and β_i are positive constants. The distribution $\alpha^D \in \Gamma_D\Lambda^1\mathcal{M}$, where

$$\alpha^D[\varphi] = \int_{\mathcal{M}} \varphi \wedge \alpha \qquad \textit{is finite for all} \quad \varphi \in \Gamma_0\Lambda^3\mathcal{M}, \tag{C.10}$$

is well defined providing the four constants β_i satisfy

$$\beta_0 < 3, \quad \beta_1 < 2, \quad \beta_2 < 2, \quad \beta_3 < 3. \tag{C.11}$$

Proof of C.2.1. An arbitrary test 3-form $\varphi \in \Gamma_0\Lambda^3\mathcal{M}$ is given in Minkowski coordinates by $\varphi = \varphi_{ijk}dy^i \wedge dy^j \wedge dy^k$. Applying a coordinate transformation such that $\varphi = \hat{\varphi}_{ijk}dz^i \wedge dz^j \wedge dz^k$ where $\{z^i\}$ are NU coordinates yields the following form for the coefficients $\hat{\varphi}_{ijk}$,

$$\begin{aligned} \hat{\varphi}_{123} &= R^2\mathrm{Y}^2_{123}, \\ \hat{\varphi}_{012} &= R\mathrm{Y}^1_{012}, \\ \hat{\varphi}_{013} &= R\mathrm{Y}^1_{013}, \\ \text{and} \quad \hat{\varphi}_{023} &= R^2\mathrm{Y}^2_{023} + R^3\mathrm{Y}^3_{023}. \end{aligned} \tag{C.12}$$

Here the functions Y^l_{ijk} depend on the test functions φ_{ijk}, sines and cosines of θ and ϕ, and the functions $\dot{C}_i$ and $\ddot{C}_i$. The key result is that they are bounded functions of τ, θ and ϕ.

We are interested in the boundedness of $\alpha^D[\varphi]$ therefore it is sufficient to show that $\int_{\mathcal{M}} \varphi \wedge \hat{\alpha}_i dz^i$ is bounded for all $\varphi \in \Gamma_0\Lambda^3\mathcal{M}$. In component form we have

$$\left| \int_{\mathcal{M}} \varphi \wedge \hat{\alpha}_i dz^i \right| = \left| \int_{\mathcal{M}} (-\hat{\varphi}_{123}\hat{\alpha}_0 + \hat{\varphi}_{023}\hat{\alpha}_1 - \hat{\varphi}_{013}\hat{\alpha}_2 + \hat{\varphi}_{012}\hat{\alpha}_3) dz^{0123} \right|,$$

where $dz^{0123} = dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3$.

Substituting the relations C.12 yields

$$\begin{aligned} \left| \int_{\mathcal{M}} \varphi \wedge \hat{\alpha}_i dz^i \right| =& \left| \int_{\mathcal{M}} R^2\mathrm{Y}^2_{123}\hat{\alpha}_0 dz^{0123} \right| + \left| \int_{\mathcal{M}} (R^2\mathrm{Y}^2_{023} + R^3\mathrm{Y}^3_{023})\hat{\alpha}_1 dz^{0123} \right| \\ &+ \left| \int_{\mathcal{M}} R\mathrm{Y}^1_{013}\hat{\alpha}_2 dz^{0123} \right| + \left| \int_{\mathcal{M}} R\mathrm{Y}^1_{012}\hat{\alpha}_3 dz^{0123} \right| \end{aligned} \tag{C.13}$$

Substituting C.9 and separating with respect to R-dependence yields

$$\begin{aligned}\left|\int_{\mathcal{M}} \varphi\wedge\hat{\alpha}_i dz^i\right| \leq &\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha_0'\mathrm{Y}^2_{123}dz^{023}\right)\right|\int_0^\epsilon \frac{R^2}{R^{\beta_0}}dz^1\\
&+\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha_1'(\mathrm{Y}^2_{023}+\mathrm{Y}^3_{023})dz^{023}\right)\right|\int_0^\epsilon \frac{R^2}{R^{\beta_1}}dz^1\\
&+\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha_2'\mathrm{Y}^1_{013}dz^{023}\right)\right|\int_0^\epsilon \frac{R}{R^{\beta_2}}dz^1\\
&+\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha_3'\mathrm{Y}^1_{012}dz^{023}\right)\right|\int_0^\epsilon \frac{R}{R^{\beta_3}}dz^1\end{aligned} \tag{C.14}$$

We now consider the integrals w.r.t $z^1 = R$. The standard integral

$$\int_0^\epsilon \frac{R^\gamma}{R^\beta}dR = \left[\frac{R^{1+\gamma-\beta}}{1+\gamma-\beta}\right]_0^\epsilon \tag{C.15}$$

where $\epsilon \in \mathbb{R}^+$, is bounded in the limit $\epsilon \to 0$ providing $\beta < 1+\gamma$. Comparison with the powers in C.14 yields the conditions C.11. □

Theorem C.2.2. *Let the 2-form $\alpha \in \Gamma\Lambda^2(\mathcal{M}\backslash C)$ be represented in Newman-Unti coordinates by*

$$\alpha = \alpha_{ij}dz^i\wedge dz^j, \qquad \textit{where} \quad z^0=\tau, \quad z^1=R, \quad z^2=\theta, \quad z^3=\phi, \tag{C.16}$$

and where the functions $\alpha_{ij} = \alpha_{ij}(\tau, R, \theta, \phi)$ are polynomials in R and are singular on the worldline. Let the most divergent terms in the functions α_{ij} be denoted

$$\hat{\alpha}_{ij} = \frac{\alpha_{ij}'(\tau,\theta,\phi)}{R^{\beta_{ij}}}. \tag{C.17}$$

where $\alpha_{ij}'(\tau,\theta,\phi)$ are bounded. and β_{ij} are positive constants. The distribution $\alpha^D \in \Gamma_D\Lambda^2\mathcal{M}$, where

$$\alpha^D[\varphi] = \int_{\mathcal{M}}\varphi\wedge\alpha \qquad \textit{is finite for all} \quad \varphi\in\Gamma_0\Lambda^2\mathcal{M}, \tag{C.18}$$

is well defined providing the six constants β_{ij} satisfy

$$\begin{aligned} &\beta_{01} < 1, \quad \beta_{12} < 2, \quad \beta_{13} < 2, \\ &\beta_{02} < 2, \quad \beta_{03} < 2, \quad \beta_{23} < 3. \end{aligned} \tag{C.19}$$

Proof of C.2.2. An arbitrary test 2-form $\phi \in \Gamma_0\Lambda^2\mathcal{M}$ is given by

$$\varphi = \varphi_{ij} dy^i \wedge dy^j \tag{C.20}$$

Applying a coordinate transformation such that $\varphi = \hat{\varphi}_{ij} dz^i \wedge dz^j$ where $\{z^i\}$ are NU coordinates yields the following form for the coefficients $\hat{\varphi}_{ij}$,

$$\begin{aligned} \hat{\varphi}_{12} =& R\mathrm{Y}^1_{12}, \\ \hat{\varphi}_{13} =& R\mathrm{Y}^1_{13}, \\ \hat{\varphi}_{02} =& R\mathrm{Y}^1_{02} + R^2\mathrm{Y}^2_{02}, \\ \hat{\varphi}_{03} =& R\mathrm{Y}^1_{03} + R^2\mathrm{Y}^2_{03}, \\ \hat{\varphi}_{01} =& \mathrm{Y}^0_{01}, \\ \text{and} \quad \hat{\varphi}_{23} =& R^2\mathrm{Y}^2_{23}. \end{aligned} \tag{C.21}$$

Here as before the functions Y^l_{ij} are bounded functions of τ, θ and ϕ. We are interested in the boundedness of $\alpha^D[\varphi]$ therefore it is sufficient to show that $\int_{\mathcal{M}} \varphi \wedge \hat{\alpha}_{ij} dz^i \wedge dz^j$ is bounded for all $\varphi \in \Gamma_0\Lambda^2\mathcal{M}$. Hence

$$\begin{aligned} &\left| \int_{\mathcal{M}} \varphi \wedge \hat{\alpha}_{ij} dz^i \wedge dz^j \right| \\ &= \left| \int_{\mathcal{M}} (-\hat{\varphi}_{13}\hat{\alpha}_{02} + \hat{\varphi}_{12}\hat{\alpha}_{03} + \hat{\varphi}_{03}\hat{\alpha}_{12} - \hat{\varphi}_{02}\hat{\alpha}_{13} + \hat{\varphi}_{23}\hat{\alpha}_{01} + \hat{\varphi}_{01}\hat{\alpha}_{23}) dz^{0123} \right| \end{aligned} \tag{C.22}$$

Substituting the relations C.21 yields

$$\begin{aligned}\left|\int_{\mathcal{M}}\varphi\wedge\hat{\alpha}_{ij}dz^i\wedge dz^j\right| = &\left|\int_{\mathcal{M}} R\mathrm{Y}^1_{13}\hat{\alpha}_{02}dz^{0123}\right| + \left|\int_{\mathcal{M}} R\mathrm{Y}^1_{12}\hat{\alpha}_{03}dz^{0123}\right| \\ &+ \left|\int_{\mathcal{M}}(R\mathrm{Y}^1_{03}+R^2\mathrm{Y}^2_{03})\hat{\alpha}_{12}dz^{0123}\right| \\ &+ \left|\int_{\mathcal{M}}(R\mathrm{Y}^1_{02}+R^2\mathrm{Y}^2_{02})\hat{\alpha}_{13}dz^{0123}\right| \\ &+ \left|\int_{\mathcal{M}} R^2\mathrm{Y}^2_{23}\hat{\alpha}_{01}dz^{0123}\right| + \left|\int_{\mathcal{M}}\mathrm{Y}^0_{01}\hat{\alpha}_{23}dz^{0123}\right| \end{aligned} \tag{C.23}$$

Substituting C.17 and separating with respect to R-dependence yields

$$\begin{aligned}\left|\int_{\mathcal{M}}\hat{\alpha}_{ij}dz^i\wedge dz^j\wedge\varphi\right| \le &\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha'_{02}\mathrm{Y}^1_{13}dz^{023}\right)\right|\int_0^{\epsilon}\frac{R}{R^{\beta_{02}}}dz^1 \\ &+\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha'_{03}\mathrm{Y}^1_{12}dz^{023}\right)\right|\int_0^{\epsilon}\frac{R}{R^{\beta_{03}}}dz^1 \\ &+\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha'_{12}(\mathrm{Y}^1_{03}+\mathrm{Y}^2_{03})dz^{023}\right)\right|\int_0^{\epsilon}\frac{R}{R^{\beta_{12}}}dz^1 \\ &+\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha'_{13}(\mathrm{Y}^1_{02}+\mathrm{Y}^2_{02})dz^{023}\right)\right|\int_0^{\epsilon}\frac{R}{R^{\beta_{13}}}dz^1 \\ &+\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha'_{01}\mathrm{Y}^2_{23}dz^{023}\right)\right|\int_0^{\epsilon}\frac{R^2}{R^{\beta_{01}}}dz^1 \\ &+\left|\max\left(\int_{\tau=-\infty}^{\tau=\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\alpha'_{23}\mathrm{Y}^0_{01}dz^{023}\right)\right|\int_0^{\epsilon}\frac{1}{R^{\beta_{23}}}dz^1 \end{aligned} \tag{C.24}$$

Once again comparing the integrals with respect to $z^1 = R$ with the standard result C.15 yields the relations C.19. □

Appendix D

Dirac Geometry

D.1 Definitions

Definition D.1.1. Consider the region $N = \widetilde{N} \backslash C$ where $\widetilde{N} \subset \mathcal{M}$ is a local neighborhood of the worldline. For every field point $x \in N$ there is a unique point $\tau_D(x)$ at which the worldline crosses the plane of simultaneity according to an observer comoving with the charge at x.

$$C : \mathbb{R} \to \mathcal{M}, \quad \tau \mapsto C(\tau) \tag{D.1}$$

$$\tau_D : \mathcal{M} \to \mathbb{R}, \quad x \mapsto \tau_D(x) \tag{D.2}$$

Definition D.1.2. Dirac geometry uses a spacelike displacement vector $Y = x - C(\tau_D(x))$, which satisfies

$$g(Y, \dot{C}(\tau_D)) = 0, \qquad R_D^2 = g(Y, Y), \tag{D.3}$$

to associate a spacetime point with a point on the worldline. We observe that $R_D > 0$ is the magnitude of Y.

Definition D.1.3. We use the notation $C_D = C(\tau_D(x))$. The vector fields

$V_D, A_D, \dot{A}_D \in \Gamma \mathrm{T}N$ are defined as

$$V_D|_x = \dot{C}^j(\tau_D(x))\frac{\partial}{\partial y^j}, \quad A_D|_x = \ddot{C}^j(\tau_D(x))\frac{\partial}{\partial y^j} \quad \text{and} \quad \dot{A}_D|_x = \dddot{C}^j(\tau_D(x))\frac{\partial}{\partial y^j}, \tag{D.4}$$

Lemma D.1.4. *The exterior derivative of the Dirac time τ_D is given by*

$$d\tau_D = -\frac{\widetilde{V}_D}{g(Y, A_D) + 1}. \tag{D.5}$$

Proof of D.1.4.

It follows from definition D.1.2 that

$$\begin{aligned} 0 =& dg(Y, V_D), \\ =& dg(x, V_D) - dg(C_D, V_D), \\ =& \widetilde{V}_D + \big(g(x, A_D) + 1 - g(A_D, C_D)\big)d\tau_D, \\ =& \widetilde{V}_D + (g(Y, A_D) + 1)d\tau_D. \end{aligned} \tag{D.6}$$

□

Lemma D.1.5.

$$dR_D = \frac{\widetilde{Y}}{R_D} \tag{D.7}$$

Proof of D.1.5.

Let

$$\mathbf{x} \in \Gamma \mathrm{T}\mathcal{M}, \qquad \mathbf{x}|_x = x^a \frac{\partial}{\partial y^a} \qquad \text{and} \qquad \mathbf{C}_D \in \Gamma \mathrm{T}\mathcal{M}, \qquad \mathbf{C}_D|_x = C_D^a \frac{\partial}{\partial y^a}, \tag{D.8}$$

It follows from definition D.1.2 that

$$
\begin{aligned}
dR_D =& d\sqrt{g(Y,Y)} \\
=& \frac{1}{2\sqrt{g(Y,Y)}} dg(Y,Y)
\end{aligned} \tag{D.9}
$$

$$
\begin{aligned}
dg(Y,Y) =& dg(x - C_D, x - C_D) \\
=& dg(\mathbf{x},\mathbf{x}) + dg(\mathbf{C}_D,\mathbf{C}_D) - 2dg(\mathbf{x},\mathbf{C}_D)
\end{aligned} \tag{D.10}
$$

$$
\begin{aligned}
dg(\mathbf{x},\mathbf{x}) &= d(g_{ab}x^a x^b), \\
&= g_{ab}(dx^a)x^b + g_{ab}x^a(dx^b), \\
&= x_a dx^a + x_a dx^a,
\end{aligned} \tag{D.11}
$$

Now $dx^a = dy^a$ since $x = (y^0, y^1, y^2, y^3)$, therefore

$$
\begin{aligned}
dg(\mathbf{x},\mathbf{x}) &= 2x_a dy^a, \\
&= 2\widetilde{\mathbf{x}}.
\end{aligned} \tag{D.12}
$$

Also

$$
\begin{aligned}
dg(\mathbf{C}_D,\mathbf{C}_D) &= d(g_{ab}C_D^a C_D^b), \\
&= (dC_D^a)g_{ab}C_D^b + (dC_D^a)g_{ab}C_D^b, \\
&= 2C_{Da}V_D^a d\tau_D, \\
&= 2g(\mathbf{C}_D, V_D)d\tau_D,
\end{aligned} \tag{D.13}
$$

and

$$
\begin{aligned}
dg(\mathbf{C}_D,\mathbf{x}) &= d(g_{ab}x^a C_D^b), \\
&= g_{ab}(dx^a)C_D^b + g_{ab}x^a d(C_D^b), \\
&= C_{Da}dx^a + x_a d(C_D^a),
\end{aligned} \tag{D.14}
$$

where $d(C_D^a) = \dfrac{\partial(C_D^a)}{\partial\tau}d\tau = V_D^a d\tau$, therefore

$$\begin{aligned} dg(\mathbf{C}_D, \mathbf{x}) &= \widetilde{\mathbf{C}_D} + x_a V_D^a d\tau_D, \\ &= \widetilde{\mathbf{C}_D} + g(\mathbf{x}, V_D) d\tau_D. \end{aligned} \tag{D.15}$$

Thus

$$\begin{aligned} dR_D =& \frac{1}{2R_D}(dg(\mathbf{x}, \mathbf{x}) + dg(\mathbf{C}_D, \mathbf{C}_D) - 2dg(\mathbf{x}, \mathbf{C}_D)) \\ =& \frac{1}{R_D}\left(\widetilde{Y} + g(Y, V_D)d\tau_D\right) \end{aligned} \tag{D.16}$$

The definition D.3 yields

$$dR_D = \frac{\widetilde{Y}}{R_D} \tag{D.17}$$

□

Lemma D.1.6.

$$\begin{aligned} dg(Y, A_D) =& \widetilde{A}_D - \frac{\widetilde{V}_D g(Y, \dot{A}_D)}{g(Y, A_D) + 1} \\ dg(Y, \dot{A}_D) =& \widetilde{\dot{A}}_D - \frac{\widetilde{V}_D(g(Y, \ddot{A}_D) + g(A_D, A_D))}{g(Y, A_D) + 1} \\ dg(A_D, A_D) =& \frac{-2g(A_D, \dot{A}_D)\widetilde{V}_D}{g(Y, A_D) + 1} \end{aligned} \tag{D.18}$$

Definition D.1.7. We define the normalized vector field

$$n_D = \frac{Y}{R_D}, \qquad \text{where} \qquad g(n_D, n_D) = 1 \qquad \text{and} \qquad g(n_D, V_D) = 0. \tag{D.19}$$

D.2 The Liénard-Wiechert potential expressed in Dirac Geometry

Dirac geometry is not a natural choice to use to describe electromagnetic phenomena because all retarded (and advanced) quantities are given only as Taylor expansions around the Dirac time τ_D. The retarded stress form must be calculated as such an expansion. Below we give the advanced and retarded Liénard-Wiechert potentials.

Lemma D.2.1. *The difference $\delta_r = \tau_D - \tau_r$ is given in terms of R_D by*

$$\delta_r = R_D - \frac{1}{2}g(n,\ddot{C})R_D^2 + \big(\frac{3}{8}g(n_D,A_D)^2 + \frac{1}{6}g(n_D,\dot{A}_D) - \frac{1}{24}g(A_D,A_D)\big)R_D^3 + \mathcal{O}(R_D^4). \tag{D.20}$$

and the difference $\delta_a = \tau_a - \tau_D$ is given by

$$\delta_a = R_D - \frac{1}{2}g(n,\ddot{C})R_D^2 + \big(\frac{3}{8}g(n_D,A_D)^2 - \frac{1}{6}g(n_D,\dot{A}_D) - \frac{1}{24}g(A_D,A_D)\big)R_D^3 + \mathcal{O}(R_D^4). \tag{D.21}$$

Proof of D.2.1.

$$C(\tau_r) = C_D - V_D\delta_r + A_D\frac{\delta_r^2}{2} - \ddot{A}_D\frac{\delta_r^3}{6} + \dddot{A}_D\frac{\delta_r^4}{24} + \mathcal{O}(\delta_r^5) \tag{D.22}$$

and thus the null vector X is given by

$$\begin{aligned} X = x - C(\tau_r) &= x - C_D + V_D\delta_r - A_D\frac{\delta^2}{2} + \dot{A}_D\frac{\delta_r^3}{6} - \ddot{A}_D\frac{\delta_r^4}{24} + \mathcal{O}(\delta_r^5), \\ &= Y + V_D\delta_r - A_D\frac{\delta_r^2}{2} + \dot{A}_D\frac{\delta_r^3}{6} - \ddot{A}_D\frac{\delta_r^4}{24} + \mathcal{O}(\delta_r^5). \end{aligned} \tag{D.23}$$

Substituting (D.23) into the lightcone condition (1.61) gives

$$\begin{aligned} g(X,X) =& g(Y,Y) + 2g(Y,V_D)\delta_r - (1+g(Y,A_D))\delta_r^2 + \frac{1}{3}g(Y,\dot{A}_D)\delta_r^3, \\ & - \frac{1}{12}(g(Y,\ddot{A}_D) + g(A_D,A_D))\delta_r^4 + \mathcal{O}(\delta_r^5). \end{aligned} \tag{D.24}$$

Definition (D.1.2) and (D.1.7) yield

$$\begin{aligned} g(X,X) =& R_D^2 - (1+R_D g(n_D,A_D))\delta_r^2 + \frac{R_D}{3}g(n_D,\dot{A}_D)\delta_r^3 \\ & - \frac{1}{12}(R_D g(n_D,\ddot{A}_D) + g(A_D,A_D))\delta_r^4 + \mathcal{O}(\delta_r^5). \end{aligned} \tag{D.25}$$

We may solve this equation to obtain δ_r and δ_a in terms of R_D.

Let $\delta_r = a_1 R_D$, then equating coefficients of order R_D^2 yields

$$a_1^2 = 1. \tag{D.26}$$

We choose $\delta_r > 0$. Knowing that $R_D > 0$ it follows that $a_1 = +1$. Now let $\delta_r = R_D + a_2 R_D^2$ then then equating coefficients of order R_D^3 yields

$$\begin{aligned} 0 =& 2a_2 + g(n_D,\ddot{C}) \\ \Rightarrow \quad a_2 =& -\frac{g(n_D,\ddot{C})}{2} \end{aligned} \tag{D.27}$$

Let $\delta_r = R_D - \frac{g(n_D,\ddot{C})}{2}R_D^2 + a_3 R_D^3$, then then equating coefficients of order R_D^4 yields

$$a_3 = \frac{3}{8}g(n_D,A_D)^2 + \frac{1}{6}g(n_D,\dot{A}_D) - \frac{1}{24}g(A_D,A_D). \tag{D.28}$$

Thus to third order δ_r is given by (D.20).

A similar calculation may be performed in to obtain an expression for $\delta_a = \tau_a - \tau_D$ in terms of R_D. In this case all quantities on the left hand side are evaluated at the advanced time τ_a, so that instead of solving the retarded null condition $g(X,X) = 0$ we must solve the advanced null condition $g(W,W) = 0$.

Since $\tau_a - \tau_D$ is positive this means that all terms with odd powers of δ will have opposite sign to those in the retarded calculations. The resulting expression for δ_a is given by (D.21).

$\square$

Lemma D.2.2. *In terms of the Dirac time τ_D and the Dirac radius R_D the retarded Liénard-Wiechert potential is given by*

$$\begin{aligned} \mathrm{A}_r = & -\frac{V_D}{R_D} + \big(A_D + \frac{1}{2}g(n_D, A_D)V_D\big) \\ & + \Big(V_D\big(\frac{1}{8}g(A_D, A_D) - \frac{1}{8}g(n_D, A_D)^2 - \frac{1}{3}g(n_D, \dot{A}_D)\big) - \frac{1}{2}\dot{A}_D - \frac{1}{2}g(n_D, A_D)A_D\Big)R_D \\ & + \mathcal{O}(R_D^2), \end{aligned} \tag{D.29}$$

and the advanced Liénard-Wiechert potential is given by

$$\begin{aligned} \mathrm{A}_a = & \frac{V_D}{R_D} + \big(A_D - \frac{1}{2}g(n_D, A_D)V_D\big) \\ & + \Big(-V_D\big(\frac{1}{8}g(A_D, A_D) - \frac{1}{8}g(n_D, A_D)^2 + \frac{1}{3}g(n_D, \dot{A}_D)\big) + \frac{1}{2}\dot{A}_D - \frac{1}{2}g(n_D, A_D)A_D\Big)R_D \\ & + \mathcal{O}(R_D^2) \end{aligned} \tag{D.30}$$

Proof of D.2.2.

We evaluate the retarded Liénard-Wiechert potential as a series in R_D.

$$V = V_D - A_D\delta_r + \dot{A}_D\frac{\delta_r^2}{2} - \ddot{A}_D(\tau_D)\frac{\delta_r^3}{6} + \mathcal{O}(\tau^4) \tag{D.31}$$

Substituting (D.20) yields

$$\begin{aligned} V = & V_D - A_d R_D + \frac{1}{2}\big(\dot{A}_D + A_D g(n_D, A_D)\big)R_D^2 \\ & + \Big(A_D\big(\frac{3}{8}g(n_D, A_D)^2 + \frac{1}{6}g(n_D, \dot{A}_D) - \frac{1}{24}g(A_D, A_D)\big) - \frac{1}{6}\ddot{A}_D - \frac{1}{2}g(n_D, A_D)\dot{A}_D\Big)R_D^3 \\ & + \mathcal{O}(R_D^4) \end{aligned} \tag{D.32}$$

Also

$$\begin{aligned}g(X,V) =& -(g(Y,A_D)+1)\delta_r + g(Y,\dot{A}_D)\frac{\delta_r^2}{2} \\ &+ \big(g(A_D,A_D) - g(Y,\ddot{A}_D)\big)\frac{\delta_r^3}{6} + \mathcal{O}(\delta_r^4)\end{aligned} \tag{D.33}$$

Again substituting (D.20) yields

$$\begin{aligned}g(X,V) =& -R_D - \frac{1}{2}g(n_D,A_D)R_D^2 \\ &+ \Big(\frac{1}{8}g(n_D,A_D)^2 + \frac{1}{2}g(n_D,A_D) - \frac{1}{6}g(n_D,\dot{A}_D) + \frac{1}{24}g(A_D,A_D)\Big)R_D^3 + \mathcal{O}(R_D^4)\end{aligned} \tag{D.34}$$

Dividing (D.32) by (D.34) gives (D.29). Evaluating the advanced potential

$$\mathrm{A}_{\mathrm{adv}}|_x = \frac{\dot{C}(\tau_a)}{g(W,\dot{C}(\tau_a))} \tag{D.35}$$

using the same procedure leads to (D.30). □

The retarded and advanced Liénard-Wiechert fields $\mathrm{F}_{\mathrm{ret}}$ and $\mathrm{F}_{\mathrm{adv}}$ are obtained by taking the exterior derivative of $\mathrm{A}_{\mathrm{ret}}$ and $\mathrm{A}_{\mathrm{adv}}$ respectively. In 1938 Dirac [17] showed that the difference between the retarded and advanced fields is finite on the worldline and given by

$$\frac{1}{2}(\mathrm{F}_{\mathrm{ret}} - \mathrm{F}_{\mathrm{adv}}) = \frac{2}{3}(g(\ddot{C},\ddot{C})\widetilde{\dot{C}} - \widetilde{\dddot{C}}) \tag{D.36}$$

It is easily seen that taking the sum of expansions

$$\mathrm{F}_{\mathrm{ret}} = \frac{1}{2}(\mathrm{F}_{\mathrm{adv}} + \mathrm{F}_{\mathrm{ret}}) + \frac{1}{2}(\mathrm{F}_{\mathrm{adv}} - \mathrm{F}_{\mathrm{ret}}). \tag{D.37}$$

is equivalent to expanding $\mathrm{F}_{\mathrm{ret}}$ only. This point was emphasized by Infeld and Wallace [44], and later by Havas [45].

Appendix E

Adapted N-U coordinates $(\tau, \mathsf{r}, \theta, \phi)$

For the numerical investigation presented in chapter 7 we use a coordinate system $(\tau, \mathsf{r}, \theta, \phi)$ adapted from the Newman-Unti coordinates. This change in coordinates was initially motivated by our interest in the ultra-relativistic Liénard-Wiechert fields. The N-U coordinate system breaks down in the ultra-relativistic limit since $R = -g(X, V) = 0$ when V is null. In the new coordinate system the radial parameter is given by

$$\mathsf{r} = -\frac{R}{\alpha} = -g(X, \partial_{y^0}) \tag{E.1}$$

which remains non-zero in the ultra-relativistic limit. If (y^0, y^1, y^2, y^3) is the global Lorentzian coordinate chart then the coordinate transformation is given by

$$\begin{aligned}
y^0 &= C^0(\tau) + \mathsf{r} \\
y^1 &= C^1(\tau) + \mathsf{r}\sin(\theta)\cos(\phi) \\
y^2 &= C^2(\tau) + \mathsf{r}\sin(\theta)\sin(\phi) \\
y^3 &= C^3(\tau) + \mathsf{r}\cos(\theta).
\end{aligned} \tag{E.2}$$

Lemma E.0.3. *In terms of the new coordinates the vector fields* $X, V \in \Gamma\mathrm{T}(\mathcal{M}\backslash C)$

are given by

$$
\begin{aligned}
X &= \mathsf{r}\frac{\partial}{\partial \mathsf{r}} \\
V &= \frac{\partial}{\partial \tau}
\end{aligned}
\tag{E.3}
$$

Proof of E.0.3.

$$
\begin{aligned}
X &= \underline{x} - C(\tau) \\
&= \mathsf{r}\frac{\partial}{\partial y^0} + \mathsf{r}\sin(\theta)\cos(\phi)\frac{\partial}{\partial y^1} + \mathsf{r}\sin(\theta)\sin(\phi)\frac{\partial}{\partial y^2} + \mathsf{r}\cos\theta\frac{\partial}{\partial y^3} \\
&= \mathsf{r}\frac{\partial}{\partial \mathsf{r}} \\
\frac{\partial}{\partial \tau} &= \frac{\partial y^0}{\partial \tau}\frac{\partial}{\partial y^0} + \frac{\partial y^1}{\partial \tau}\frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial \tau}\frac{\partial}{\partial y^2} + \frac{\partial y^3}{\partial \tau}\frac{\partial}{\partial y^3} \\
&= \dot{C}^0(\tau)\frac{\partial}{\partial y^0} + \dot{C}^1(\tau)\frac{\partial}{\partial y^1} + \dot{C}^2(\tau)\frac{\partial}{\partial y^2} + \dot{C}^3(\tau)\frac{\partial}{\partial y^3} \\
&= \dot{C}^a(\tau)\frac{\partial}{\partial y^a} \\
&= \dot{C}(\tau) \\
&= V
\end{aligned}
$$

□

Lemma E.0.4. *The Minkowski metric $g \in \bigotimes^{[\mathbb{F},\mathbb{F}]}\mathbf{M}$ is given by*

$$
\begin{aligned}
g =& -c^2 d\tau \otimes d\tau + \mathsf{r}^2 d\theta \otimes d\theta + \mathsf{r}^2\sin^2\theta d\phi \otimes d\phi \\
&+ \alpha[d\tau \otimes d\mathsf{r} + d\mathsf{r} \otimes d\tau] + \mathsf{r}\alpha_\theta[d\tau \otimes d\theta + d\theta \otimes d\tau] + \mathsf{r}\alpha_\phi[d\tau \otimes d\phi \\
&+ d\phi \otimes d\tau]
\end{aligned}
\tag{E.4}
$$

and the inverse metric $g^{-1} \in \bigotimes^{[\mathbb{V},\mathbb{V}]} \mathbf{M}$ is given by

$$
\begin{aligned}
g^{-1} =& \frac{c^2 \sin^2(\theta) + \alpha_\theta^2 \sin^2(\theta) + \alpha_\phi^2}{\sin^2(\theta)\alpha^2} \Big(\frac{\partial}{\partial \mathsf{r}} \otimes \frac{\partial}{\partial \mathsf{r}} \Big) + \frac{1}{\mathsf{r}^2} \Big(\frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta} \Big) + \frac{1}{\mathsf{r}^2 \sin^2\theta} \Big(\frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial \phi} \Big) \\
&+ \frac{1}{\alpha} \Big(\frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial \mathsf{r}} + \frac{\partial}{\partial \mathsf{r}} \otimes \frac{\partial}{\partial \tau} \Big) - \frac{\alpha_\theta}{\alpha \mathsf{r}} \Big(\frac{\partial}{\partial \mathsf{r}} \otimes \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \mathsf{r}} \Big) \\
&- \frac{\alpha_\phi}{\alpha \mathsf{r} \sin^2(\theta)} \Big(\frac{\partial}{\partial \mathsf{r}} \otimes \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial \mathsf{r}} \Big)
\end{aligned} \tag{E.5}
$$

Where α is defined by (1.88) and α_θ and α_ϕ are the derivatives of α with respect to θ and ϕ respectively. Let $z^0 = \tau, \quad z^1 = \mathsf{r}, \quad z^2 = \theta, \quad z^3 = \phi$, then the matrices $G' = G'_{ab} = g(\partial_{z^a}, \partial_{z^b})$ and $G'^{-1} = G'^{-1}_{ab} = g^{-1}(dz^a, dz^b)$ are given by

$$
G' = g(\partial_{z^a}, \partial_{z^b}) = \begin{pmatrix} -c^2 & \alpha & \mathsf{r}\alpha_\theta & \mathsf{r}\alpha_\phi \\ \alpha & 0 & 0 & 0 \\ \mathsf{r}\alpha_\theta & 0 & \mathsf{r}^2 & 0 \\ \mathsf{r}\alpha_\phi & 0 & 0 & \mathsf{r}^2 \sin^2\theta \end{pmatrix}
$$

$$
G'^{-1} = \begin{pmatrix} 0 & \frac{1}{\alpha} & 0 & 0 \\ \frac{1}{\alpha} & \frac{c^2 \sin^2(\theta) + \alpha_\theta^2 \sin^2(\theta) + \alpha_\phi^2}{\sin^2(\theta)\alpha^2} & -\frac{\alpha_\theta}{\alpha \mathsf{r}} & -\frac{\alpha_\phi}{\alpha \mathsf{r} sin^2(\theta)} \\ 0 & -\frac{\alpha_\theta}{\alpha \mathsf{r}} & \frac{1}{\mathsf{r}^2} & 0 \\ 0 & -\frac{\alpha_\phi}{\alpha \mathsf{r} \sin^2(\theta)} & 0 & \frac{1}{\mathsf{r}^2 \sin^2\theta} \end{pmatrix}
$$

Proof of E.0.4.

$$
g = -dy^0 \otimes dy^0 + dy^1 \otimes dy^1 + dy^2 \otimes dy^2 + dy^3 \otimes dy^3 \tag{E.6}
$$

$dy^0 = \dot{C}^0(\tau)d\tau + d\mathsf{r}$

$dy^1 = \dot{C}^1(\tau)d\tau + \sin(\theta)\cos(\phi)d\mathsf{r} + \mathsf{r}\cos(\theta)\cos(\phi)d\theta - \mathsf{r}\sin(\theta)\sin(\phi)d\phi$

$dy^2 = \dot{C}^2\tau d\tau + \sin(\theta)\sin(\phi)d\mathsf{r} + \mathsf{r}\cos(\theta)\sin(\phi)d\theta + \mathsf{r}\sin(\theta)\cos(\phi)d\phi$

$dy^3 = \dot{C}^3\tau d\tau + \cos(\theta)d\mathsf{r} - \mathsf{r}\sin(\theta)d\theta$

Thus

$$
\begin{aligned}
g = &-c^2 d\tau \otimes d\tau + \mathsf{r}^2 d\theta \otimes d\theta + \mathsf{r}^2 \sin^2\theta d\phi \otimes d\phi \\
&+ (-\dot{C}^0 + \dot{C}^1 \sin\theta\cos\phi + \dot{C}^2 \sin\theta\sin\phi + \dot{C}^3\cos\theta)[d\tau \otimes d\mathsf{r} + d\mathsf{r} \otimes d\tau] \\
&+ (\dot{C}^1 \mathsf{r}\cos\theta\cos\phi + \dot{C}^2 \mathsf{r}\cos\theta\sin\phi - \dot{C}^3 \mathsf{r}\sin\theta)[d\tau \otimes d\theta + d\theta \otimes d\tau] \\
&+ (\dot{C}^2 \mathsf{r}\sin\theta\cos\phi - \dot{C}^1 \mathsf{r}\sin\theta\sin\phi)[d\tau \otimes d\phi + d\phi \otimes d\tau] \\
&= -c^2 d\tau \otimes d\tau + \mathsf{r}^2 d\theta \otimes d\theta + \mathsf{r}^2 \sin^2\theta) d\phi \otimes d\phi \\
&+ \alpha[d\tau \otimes d\mathsf{r} + d\mathsf{r} \otimes d\tau] + \mathsf{r}\alpha_\theta[d\tau \otimes d\theta + d\theta \otimes d\tau] + \mathsf{r}\alpha_\phi[d\tau \otimes d\phi + d\phi \otimes d\tau]
\end{aligned}
$$

g^{-1} follows from the matrix $(E.0.4)$ □

Corollary E.0.5.

$$
\begin{aligned}
\widetilde{d\tau} &= \frac{1}{\alpha}\partial_{\mathsf{r}} \\
\widetilde{d\mathsf{r}} &= \frac{c^2\sin^2(\theta) + \alpha_\theta^2 \sin^2(\theta) + \alpha_\phi^2}{\sin^2(\theta)\alpha^2}\partial_{\mathsf{r}} + \frac{1}{\alpha}\partial_\tau - \frac{\alpha_\theta}{\alpha\mathsf{r}}\partial_\theta - \frac{\alpha_\phi}{\alpha\mathsf{r} sin^2(\theta)}\partial\phi \\
\widetilde{d\theta} &= \frac{1}{\mathsf{r}^2}\partial_\theta - \frac{\alpha_\theta}{\alpha\mathsf{r}}\partial_{\mathsf{r}} \\
\widetilde{d\phi} &= \frac{1}{\mathsf{r}^2\sin^2\theta}\partial_\phi - \frac{\alpha_\phi}{\alpha\mathsf{r}\sin^2(\theta)}\partial_{\mathsf{r}}
\end{aligned}
\tag{E.7}
$$

Proof of E.0.5. follows from definition of g^{-1}. □

Lemma E.0.6. *The 1-forms $\widetilde{X}, \widetilde{V} \in \Gamma\Lambda^1\mathbf{M}$ are given by*

$$\widetilde{X} = \mathsf{r}\alpha d\tau \tag{E.8}$$

$$\widetilde{V} = -\epsilon^2 d\tau + \alpha d\mathsf{r} + \mathsf{r}\alpha_\theta d\theta + \mathsf{r}\alpha_\phi d\phi \tag{E.9}$$

Proof of E.0.6.

$$\begin{aligned}
\widetilde{X} &= \mathsf{r}g(\frac{\partial}{\partial \mathsf{r}}, -) \\
&= \mathsf{r}(-\dot{C}^0 + \dot{C}^1 \sin\theta\cos\phi + \dot{C}^2 \sin\theta\sin\phi + \dot{C}^3\cos\theta)d\tau \\
&= \mathsf{r}\alpha d\tau \\
\widetilde{V} &= g(\frac{\partial}{\partial \tau}, -) \\
&= -c^2 d\tau + \alpha d\mathsf{r} + \mathsf{r}\alpha_\theta d\theta + \mathsf{r}\alpha_\phi d\phi
\end{aligned}$$

□

Lemma E.0.7.

$$\star 1 = -\alpha \mathsf{r}^2 \sin\theta d\tau \wedge d\mathsf{r} \wedge d\theta \wedge d\phi \tag{E.10}$$

Proof of E.0.7.

$$\begin{aligned}
\star 1 &= \sqrt{|\det(g)|} d\tau \wedge d\mathsf{r} \wedge d\theta \wedge d\phi \\
&= \sqrt{|-\alpha^2 \mathsf{r}^4 \sin^2\theta|} d\tau \wedge d\mathsf{r} \wedge d\theta \wedge d\phi \\
&= -\alpha \mathsf{r}^2 \sin\theta d\tau \wedge d\mathsf{r} \wedge d\theta \wedge d\phi
\end{aligned}$$

□

Lemma E.0.8.

$$\widetilde{A} = \dot{\alpha} d\mathsf{r} + \mathsf{r}\dot{\alpha_\theta} d\theta + \mathsf{r}\dot{\alpha_\phi} d\phi \tag{E.11}$$

Proof of E.0.8.

$$
\begin{aligned}
\widetilde{A} &= \frac{dV_a}{d\tau} dy^a \\
&= -\ddot{C}^0(\tau) dy^0 + \ddot{C}^1(\tau) dy^1 + \ddot{C}^2(\tau) dy^2 + \ddot{C}^3(\tau) dy^3 \\
&= -\ddot{C}^0(\tau)\Big[\dot{C}^0(\tau) d\tau + d\mathsf{r}\Big] \\
&\quad + \ddot{C}^1(\tau)\Big[\dot{C}^1(\tau) d\tau + \sin(\theta)\cos(\phi) d\mathsf{r} + \mathsf{r}\cos(\theta)\cos(\phi) d\theta - \mathsf{r}\sin(\theta)\sin(\phi) d\phi\Big] \\
&\quad + \ddot{C}^2(\tau)\Big[\dot{C}^2(\tau) d\tau + \sin(\theta)\sin(\phi) d\mathsf{r} + \mathsf{r}\cos(\theta)\sin(\phi) d\theta + \mathsf{r}\sin(\theta)\cos(\phi) d\phi\Big] \\
&\quad + \ddot{C}^3(\tau)\Big[\dot{C}^3(\tau) d\tau + \cos(\theta) d\mathsf{r} - \mathsf{r}\sin(\theta) d\theta\Big] \\
&= g(A, V) d\tau + \dot{\alpha} d\mathsf{r} + \mathsf{r}\dot{\alpha_\theta} d\theta + \mathsf{r}\dot{\alpha_\phi} d\phi \\
&= \dot{\alpha} d\mathsf{r} + \mathsf{r}\dot{\alpha_\theta} d\theta + \mathsf{r}\dot{\alpha_\phi} d\phi
\end{aligned}
$$

□

Lemma E.0.9.

$$
\begin{aligned}
A = \frac{\dot{\alpha}}{\alpha}\partial_\tau &+ \left[\Big(\frac{c^2\dot{\alpha}}{\alpha^2}\Big) + \Big(\frac{\dot{\alpha}\alpha_\theta^2}{\alpha^2} - \frac{\alpha_\theta\dot{\alpha_\theta}}{\alpha}\Big) + \frac{1}{\sin^2(\theta)}\Big(\frac{\alpha_\phi^2\dot{\alpha}}{\alpha^2} - \frac{\alpha_\phi\dot{\alpha_\phi}}{\alpha}\Big)\right]\partial_\mathsf{r} \\
&+ \frac{1}{\mathsf{r}}\Big(\dot{\alpha_\theta} - \frac{\dot{\alpha}\alpha_\theta}{\alpha}\Big)\partial_\theta + \frac{1}{\mathsf{r}\sin^2(\theta)}\Big(\dot{\alpha_\phi} - \frac{\dot{\alpha}\alpha_\phi}{\alpha}\Big)\partial_\phi
\end{aligned}
\tag{E.12}
$$

Proof of E.0.9.

$$
\begin{aligned}
A &= g^{-1}(\widetilde{A}, -) \\
&= g(A,V) g^{-1}(d\tau, -) + \dot{\alpha} g^{-1}(d\mathsf{r}, -) + \mathsf{r}\dot{\alpha_\theta} g^{-1}(d\theta, -) + \mathsf{r}\dot{\alpha_\phi} g^{-1}(d\phi, -) \\
&= \frac{g(A,V)}{\alpha}\partial_\mathsf{r} + \dot{\alpha}\left(\frac{c^2\sin^2(\theta) + \alpha_\theta^2\sin^2(\theta) + \alpha_\phi^2}{\sin^2(\theta)\alpha^2}\partial_\mathsf{r} + \frac{1}{\alpha}\partial_\tau - \frac{\alpha_\theta}{\alpha\mathsf{r}}\partial_\theta - \frac{\alpha_\phi}{\alpha\mathsf{r}\sin^2(\theta)}\partial_\phi\right) \\
&+ \mathsf{r}\dot{\alpha_\theta}\Big(\frac{1}{\mathsf{r}^2}\partial_\theta - \frac{\alpha_\theta}{\alpha\mathsf{r}}\partial_\mathsf{r}\Big) + \mathsf{r}\dot{\alpha_\phi}\Big(\frac{1}{\mathsf{r}^2\sin^2(\theta)}\partial_\phi - \frac{\alpha_\phi}{\alpha\mathsf{r}\sin^2(\theta)}\partial_\mathsf{r}\Big) \\
&= \frac{\dot{\alpha}}{\alpha}\partial_\tau + \left[\Big(\frac{g(A,V)}{\alpha} + \frac{c^2\dot{\alpha}}{\alpha^2}\Big) + \Big(\frac{\dot{\alpha}\alpha_\theta^2}{\alpha^2} - \frac{\alpha_\theta\dot{\alpha_\theta}}{\alpha}\Big) + \frac{1}{\sin^2(\theta)}\Big(\frac{\alpha_\phi^2\dot{\alpha}}{\alpha^2} - \frac{\alpha_\phi\dot{\alpha_\phi}}{\alpha}\Big)\right]\partial_\mathsf{r} \\
&+ \frac{1}{\mathsf{r}}\Big(\dot{\alpha_\theta} - \frac{\dot{\alpha}\alpha_\theta}{\alpha}\Big)\partial_\theta + \frac{1}{\mathsf{r}\sin^2(\theta)}\Big(\dot{\alpha_\phi} - \frac{\dot{\alpha}\alpha_\phi}{\alpha}\Big)\partial_\phi
\end{aligned}
$$

□

Lemma E.0.10.

$$g(X,V) = \mathsf{r}\alpha \tag{E.13}$$

$$g(X,A) = \mathsf{r}\dot{\alpha} \tag{E.14}$$

Proof of E.0.10.

$$\begin{aligned}
g(X,V) &= g(\mathsf{r}\frac{\partial}{\partial \mathsf{r}}, \frac{\partial}{\partial \tau}) \\
&= \mathsf{r}g(\frac{\partial}{\partial \mathsf{r}}, \frac{\partial}{\partial \tau}) \\
&= \mathsf{r}(-\dot{C}^0 + \dot{C}^1 \sin\theta\cos\phi + \dot{C}^2 \sin\theta\sin\phi + \dot{C}^3\cos\theta) \\
&= \mathsf{r}\alpha
\end{aligned}$$

For $g(X,A)$ the only relevant term in the metric is $\alpha d\mathsf{r} \otimes d\tau$, thus

$$\begin{aligned}
g(X,A) &= \mathsf{r}\frac{\dot{\alpha}}{\alpha} g(\partial_{\mathsf{r}}, \partial_\tau) \\
&= \mathsf{r}\dot{\alpha}
\end{aligned}$$

□

Lemma E.0.11.

$$\mathrm{A} = -\frac{q}{4\pi\epsilon_0}\Big(\frac{c^2}{\alpha \mathsf{r}} d\tau + \frac{1}{\mathsf{r}} d\mathsf{r} + \frac{\alpha_\theta}{\alpha} d\theta + \frac{\alpha_\phi}{\alpha} d\phi\Big) \tag{E.15}$$

$$\mathrm{F_R} = \frac{q}{4\pi\epsilon_0}\frac{(\alpha\dot{\alpha}_\theta - \dot{\alpha}\alpha_\theta) d\tau\wedge d\theta + (\alpha\dot{\alpha}_\phi - \dot{\alpha}\alpha_\phi) d\tau \wedge d\phi}{\alpha^2} \tag{E.16}$$

$$\mathrm{F_C} = -\frac{q}{4\pi\epsilon_0}c^2\Big(\frac{\alpha}{\mathsf{r}^2} d\tau\wedge d\mathsf{r} + \frac{\alpha_\theta}{\mathsf{r}\alpha^2} d\tau\wedge d\theta + \frac{\alpha_\phi}{\mathsf{r}\alpha^2} d\tau\wedge d\phi\Big) \tag{E.17}$$

Proof of E.0.11. (53) follows directly from (20) (47) (51)

$$\mathrm{F_R} = \frac{q}{4\pi\epsilon_0}\frac{g(X,V)\tilde{X}\wedge\widetilde{A} - g(X,A)\tilde{X}\wedge\widetilde{V}}{g(X,V)^3}$$

Using the relations:

$$\begin{aligned}
\tilde{X} \wedge \widetilde{V} &= \Big(\mathsf{r}\alpha d\tau\Big) \wedge \Big(-c^2 d\tau + \alpha d\mathsf{r} + \mathsf{r}\alpha_\theta d\theta + \mathsf{r}\alpha_\phi d\phi\Big) \\
&= \mathsf{r}\alpha^2 d\tau \wedge d\mathsf{r} + \mathsf{r}^2\alpha\alpha_\theta d\tau \wedge d\theta + \mathsf{r}^2\alpha\alpha_\phi d\tau \wedge d\phi \qquad \text{(E.18)} \\
\tilde{X} \wedge \widetilde{A} &= \Big(\mathsf{r}\alpha d\tau\Big) \wedge \Big(g(A,V)d\tau + \dot{\alpha} d\mathsf{r} + \mathsf{r}\dot{\alpha_\theta} d\theta + \mathsf{r}\dot{\alpha_\phi} d\phi\Big) \\
&= \mathsf{r}\alpha\dot{\alpha} d\tau \wedge d\mathsf{r} + \mathsf{r}^2\alpha\dot{\alpha_\theta} d\tau \wedge d\theta + \mathsf{r}^2\alpha\dot{\alpha_\phi} d\tau \wedge d\phi \qquad \text{(E.19)}
\end{aligned}$$

along with (51)(52)gives

$$\begin{aligned}
\mathrm{F_R} &= \frac{q}{4\pi\epsilon_0}\frac{\alpha\mathsf{r}(\mathsf{r}\alpha\dot{\alpha} d\tau \wedge d\mathsf{r} + \mathsf{r}^2\alpha\dot{\alpha_\theta} d\tau \wedge d\theta + \mathsf{r}^2\alpha\dot{\alpha_\phi} d\tau \wedge d\phi)}{(\alpha\mathsf{r})^3} \\
&- \frac{q}{4\pi\epsilon_0}\frac{\mathsf{r}\dot{\alpha}(\mathsf{r}\alpha^2 d\tau \wedge d\mathsf{r} + \mathsf{r}^2\alpha\alpha_\theta d\tau \wedge d\theta + \mathsf{r}^2\alpha\alpha_\phi d\tau \wedge d\phi)}{(\alpha\mathsf{r})^3} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{\alpha^2}\Big((\alpha\dot{\alpha_\theta} - \dot{\alpha}\alpha_\theta) d\tau \wedge d\theta + (\alpha\dot{\alpha_\phi} - \dot{\alpha}\alpha_\phi) d\tau \wedge d\phi\Big)
\end{aligned}$$

$$\begin{aligned}
\mathrm{F_C} &= -\frac{q}{4\pi\epsilon_0}\frac{c^2\tilde{X} \wedge \widetilde{V}}{g(X,V)^3} \\
&= -\frac{q}{4\pi\epsilon_0}\frac{c^2(\mathsf{r}\alpha^2 d\tau \wedge d\mathsf{r} + \mathsf{r}^2\alpha\alpha_\theta d\tau \wedge d\theta + \mathsf{r}^2\alpha\alpha_\phi d\tau \wedge d\phi)}{(\alpha\mathsf{r})^3} \\
&= -\frac{q}{4\pi\epsilon_0}c^2\Big(\frac{1}{\alpha\mathsf{r}^2} d\tau \wedge d\mathsf{r} + \frac{\alpha_\theta}{\mathsf{r}\alpha^2} d\tau \wedge d\theta + \frac{\alpha_\phi}{\mathsf{r}\alpha^2} d\tau \wedge d\phi\Big)
\end{aligned}$$

□

Lemma E.0.12.

$$\star\,\mathrm{F_R} = \frac{q}{4\pi\epsilon_0}\Big(\frac{\alpha_\phi\dot{\alpha} - \alpha\dot{\alpha_\phi}}{\alpha^2\sin(\theta)} d\tau \wedge d\theta - \frac{\sin(\theta)(\alpha_\theta\dot{\alpha} - \alpha\dot{\alpha_\theta})}{\alpha^2} d\tau \wedge d\phi\Big) \qquad \text{(E.20)}$$

$$\star\,\mathrm{F_C} = \frac{q}{4\pi\epsilon_0}\frac{c^2\sin(\theta)}{\alpha^2} d\theta \wedge d\phi \qquad \text{(E.21)}$$

Proof of E.0.12.

$$\begin{aligned}
\star(\tilde{X} \wedge \widetilde{V}) &= i_V i_X \star 1 \\
&= \mathsf{r} i_{\partial_\tau} i_{\partial_\mathsf{r}} (\alpha \mathsf{r}^2 \sin\theta d\tau \wedge d\mathsf{r} \wedge d\theta \wedge d\phi) \\
&= \mathsf{r}^3 \alpha \sin(\theta) d\theta \wedge d\tau \qquad \text{(E.22)} \\
\star(\tilde{X} \wedge \widetilde{A}) &= i_A i_X \star 1 \\
&= \mathsf{r}^2 \sin(\theta)(\alpha\dot{\alpha}_\theta - \alpha_\theta\dot{\alpha}) d\tau \wedge d\phi - \frac{\mathsf{r}^2(\alpha\dot{\alpha}_\phi - \alpha_\phi\dot{\alpha})}{\sin(\theta)} d\tau \wedge d\theta \\
&\quad - \mathsf{r}^3 \dot{\alpha} \sin(\theta) d\theta \wedge d\phi \qquad \text{(E.23)}
\end{aligned}$$

therefore

$$\begin{aligned}
\star \mathrm{F_C} &= \frac{q}{4\pi\epsilon_0} \frac{-c^2 \star (\tilde{X} \wedge \widetilde{V})}{g(X,V)^3} \\
&= \frac{q}{4\pi\epsilon_0} \frac{c^2 \sin\theta}{\alpha^2} d\theta \wedge d\phi \\
\star \mathrm{F_R} &= \frac{q}{4\pi\epsilon_0} \frac{g(X,V) \star (\tilde{X} \wedge \widetilde{A}) - g(X,A) \star (\tilde{X} \wedge \widetilde{V})}{g(X,V)^3} \\
&= \frac{q}{4\pi\epsilon_0} \Big(\frac{\alpha_\phi \dot{\alpha} - \alpha\dot{\alpha}_\phi}{\alpha^2 \sin(\theta)} d\tau \wedge d\theta - \frac{\sin(\theta)(\alpha_\theta\dot{\alpha} - \alpha\dot{\alpha}_\theta)}{\alpha^2} d\tau \wedge d\phi \Big)
\end{aligned}$$

□

Lemma E.0.13. *The couloumbic and radiative terms of the 1-forms* $\widetilde{\mathrm{E}}$ *and* $\widetilde{\mathrm{B}}$ *take the form*

$$\begin{aligned}
\widetilde{\mathrm{E}}_\mathrm{C} =& \frac{q}{4\pi\epsilon_0} c^2 \Big(\frac{\alpha_\phi}{\mathsf{r}\alpha^3} d\phi + \frac{1}{\alpha^2 \mathsf{r}^2} d\mathsf{r} + \frac{\alpha_\theta}{\mathsf{r}\alpha^3} d\theta + \frac{\dot{C}^0 + \alpha}{\alpha^2 \mathsf{r}^2} d\tau \\
&+ \frac{\alpha_\theta^2}{\mathsf{r}^2\alpha^3} d\tau + \frac{\alpha_\phi^2}{\mathsf{r}^2 \alpha^3 \sin^2(\theta)} d\tau \Big) \\
\widetilde{\mathrm{E}}_\mathrm{R} =& \frac{q}{4\pi\epsilon_0} \Bigg(\frac{\dot{\alpha}\alpha_\phi - \alpha\dot{\alpha}_\phi}{\alpha^3} d\phi - \frac{\alpha\dot{\alpha}_\theta - \dot{\alpha}\alpha_\theta}{\alpha^3} d\theta \\
&- \Big(\frac{\alpha_\theta(\alpha\dot{\alpha}_\theta - \dot{\alpha}\alpha_\theta)}{\mathsf{r}\alpha^3} - \frac{\alpha_\phi(\dot{\alpha}\alpha_\phi - \alpha\dot{\alpha}_\phi)}{\mathsf{r}\alpha^3 \sin^2(\theta)} \Big) d\tau \Bigg) \qquad \text{(E.24)}
\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{\mathrm{B}}_{\mathrm{C}} &= \frac{q}{4\pi\epsilon_0} c\Big(\frac{\alpha_\theta \sin(\theta)}{\mathsf{r}\alpha^3} d\phi + \frac{\dot{C}^1 \sin(\phi) - \dot{C}^2 \cos(\phi)}{\mathsf{r}\alpha^3} d\theta\Big) \\
&= \frac{q}{4\pi\epsilon_0} c \sin(\theta)\Big(\frac{\alpha_\theta}{\mathsf{r}\alpha^3} d\phi - \frac{\alpha_\phi}{\mathsf{r}\alpha^3 \sin^2(\theta)} d\theta\Big) \\
\widetilde{\mathrm{B}}_{\mathrm{R}} &= \frac{1}{c}\frac{q}{4\pi\epsilon_0} \sin(\theta)\Bigg(\frac{\dot{\alpha}\alpha_\theta - \alpha\dot{\alpha}_\theta}{\alpha^3} d\phi + \frac{\alpha\dot{\alpha}_\phi - \dot{\alpha}\alpha_\phi}{\alpha^3 \sin^2(\theta)} d\theta \\
&\quad + \Big(\frac{\alpha_\theta(\alpha\dot{\alpha}_\phi - \dot{\alpha}\alpha_\phi)}{\mathsf{r}\alpha^3 \sin^2(\theta)} + \frac{\alpha_\phi(\dot{\alpha}\alpha_\theta - \alpha\dot{\alpha}_\theta)}{\mathsf{r}\alpha^3 \sin^2(\theta)}\Big) d\tau\Bigg) \\
&= \frac{1}{c}\frac{q}{4\pi\epsilon_0} \sin(\theta)\Bigg(\frac{\dot{\alpha}\alpha_\theta - \alpha\dot{\alpha}_\theta}{\alpha^3} d\phi + \frac{\alpha\dot{\alpha}_\phi - \dot{\alpha}\alpha_\phi}{\alpha^3 \sin^2(\theta)} d\theta \\
&\quad + \alpha \frac{\alpha_\theta\dot{\alpha}_\phi - \alpha_\phi\dot{\alpha}_\theta}{\mathsf{r}\alpha^3 \sin^2(\theta)} d\tau\Bigg)
\end{aligned} \tag{E.25}
$$

Proof of E.0.13.

$$
\begin{aligned}
\frac{\partial}{\partial y^0} &= \frac{\partial \tau}{\partial y^0}\frac{\partial}{\partial \tau} + \frac{\partial \mathsf{r}}{\partial y^0}\frac{\partial}{\partial \mathsf{r}} + \frac{\partial \theta}{\partial y^0}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y^0}\frac{\partial}{\partial \phi} \\
&= \frac{c}{\alpha}\Big(-\frac{\partial}{\partial \tau} + (\dot{C}^0 + \alpha)\frac{\partial}{\partial \mathsf{r}} + \frac{\alpha_\theta}{r}\frac{\partial}{\partial \theta} + \frac{\alpha_\phi}{\mathsf{r}\sin^2(\theta)}\frac{\partial}{\partial \phi}\Big)
\end{aligned} \tag{E.26}
$$

Results follow from definitions (1.16) and (1.22). □

Appendix F

MAPLE Input for Part I

In this thesis the we use the mathematical software MAPLE to implement the computations which support the results presented in parts I and II. In principle there are other programming tools which could have be used, such as MATHEMATICA and MATLAB, each of which has its own advantages and disadvantages. In general, MATHEMATICA and MAPLE are more suited to symbolic computation, whereas MATLAB is more suited to numerical computation.

In part I of the thesis we require heavy use of symbolic computation. In particular we utilize the tools of differential geometry to manipulate tensors and differential forms. These tools were readily available to us in MANIFOLDS package [46] written by Robin Tucker and Charles Wang for use with MAPLE. There are similar packages available for use with other software, such as RICCI for use with MATHEMATICA, and Tensor Toolbox for use with MATLAB, however the availability of the MANIFOLDS package and supporting documentation was an important factor in deciding to use MAPLE instead of other possible programming tools. In addition, the procedural language of MAPLE was appealing to the author based on his experience with C^{++} and FORTRAN programming languages.

The calculations carried out for part II of the thesis are more numerical by nature, however rather than adopting a programming tool more suited to numerical calculations we decided it would be more economical to build on the code already written in MAPLE.

The following script was written in MAPLE 15 and can be run with the packages *Plots*, *LinearAlgebra* and the additional package *Manifolds*[46] with tools for differential geometry.

```
# set up coordinate system
Manifoldsetup(M,[tau,R,theta,phi],[e,E,0],
map(x->simplify(x,symbolic),
[e[0]=d(tau),
e[1]=d(R),
e[2]=d(theta),
e[3]=d(phi)])):


Constants([epsilon, q, ep, b0, b1, b2, b3, a3, R0]);
Manfdomain(M, [a, ad, ath, aph, athd, aphd, adphph, adthth], [tau,
theta, phi]):
Manfdomain(M,[C0,C1,C2,C3,Cd0,Cd1,Cd2,Cd3,Cdd0,Cdd1,Cdd2,Cdd3],[tau])
:
g := (-1+2*R*ad/a)*d(tau) &X d(tau)
- (d(tau) &X d(R)+ d(R) &X d(tau))
+ (R^2/a^2)*(d(theta) &X d(theta))
+ (R^2/a^2)*sin(theta)^2 *d(phi) &X d(phi) :
Mancovmetric(M,g):
Manvol(M) := -(R^2/a^2)*sin(theta)*`&^`(e[0], e[1], e[2], e[3]) :
Basis1 := {d(tau),d(R),d(theta),d(phi)} :
Basis2 := {d(tau)&^d(R), d(tau)&^d(theta), d(tau)&^d(phi),
d(R)&^d(theta), d(R)&^d(phi), d(theta)&^d(phi)} :
Basis3 := {d(tau)&^d(R)&^d(theta), d(tau)&^d(R)&^d(phi),
d(tau)&^d(theta)&^d(phi), d(R)&^d(theta)&^d(phi)} :
Basis4 := {e(0) &^e(1) &^e(2), e(1) &^e(2) &^e(3),e(2) &^e(3)
&^e(0),e(3) &^e(0) &^e(1)}:
a_sublist:={diff(a,tau)=ad,diff(a,theta)=ath,diff(a,phi)=aph,
diff(ath,tau)=athd,diff(aph,tau)=aphd,
diff(ath,phi)=athph,diff(aph,theta)=aphth, diff(ad, theta)=athd,
diff(ad, phi)=aphd, diff(aph, phi)=aphph,
diff(ath, theta)=athth, diff(adph, phi)=adphph, diff(adth,
theta)=adthth}:
Cd_sublist := {diff(C0,tau)=Cd0,diff(C1,tau)=Cd1,
diff(C2,tau)=Cd2,diff(C3,tau)=Cd3} :
Cd_inv_sublist := {Cd0=diff(C0,tau),Cd1=diff(C1,tau),
Cd2=diff(C2,tau),Cd3=diff(C3,tau)} :
Cdd_sublist := {diff(C0,tau,tau)=Cdd0,diff(C1,tau,tau)=Cdd1,
diff(C2,tau,tau)=Cdd2,diff(C3,tau,tau)=Cdd3,
diff(Cd0,tau)=Cdd0,diff(Cd1,tau)=Cdd1,
diff(Cd2,tau)=Cdd2,diff(Cd3,tau)=Cdd3} :
Cddd_sublist := { diff(Cdd0,tau)=Cddd0,diff(Cdd1,tau)=Cddd1,
```

```
diff(Cdd2,tau)=Cddd2,diff(Cdd3,tau)=Cddd3} :
aa :=
-Cd0+Cd1*cos(phi)*sin(theta)+Cd2*sin(phi)*sin(theta)+Cd3*cos(theta)
:
aath := diff(aa,theta) :
aaph := diff(aa,phi) :
aad := subs(Cdd_sublist,diff(subs(Cd_inv_sublist,aa),tau)) :
aathd := subs(Cdd_sublist,diff(subs(Cd_inv_sublist,aath),tau)) :
aaphd := subs(Cdd_sublist,diff(subs(Cd_inv_sublist,aaph),tau)) :
aathth:=subs(Cdd_sublist,diff(subs(Cd_inv_sublist,aath),theta)) :
aaphph:=subs(Cdd_sublist,diff(subs(Cd_inv_sublist,aaph),phi)) :
aadphph:=subs(Cdd_sublist,diff(diff(subs(Cd_inv_sublist,aad),phi),
phi)) :
aadthth:=subs(Cdd_sublist,diff(diff(subs(Cd_inv_sublist,aad),theta),
theta)) :
aa_sublist:={a=aa, ath=aath, ad=aad, aph=aaph, athd=aathd,
aphd=aaphd, athth=aathth,
aphph=aaphph, adphph=aadphph, adthth=aadthth}:
x0 := C0-(R/a):
x1 := C1-(R/a)*sin(theta)*cos(phi):
x2 := C2-(R/a)*sin(theta)*sin(phi):
x3 := C3-(R/a)*cos(theta):
J := Matrix(4, 4):
for i from 0 to 3 do J[i+1, 1] := diff(x || i, tau):
J[i+1, 2] := diff(x || i, R):
J[i+1, 3] := diff(x || i, theta):
J[i+1, 4] := diff(x || i, phi) end do:
subs(a_sublist, J):
DetJ := simplify(Determinant(J)):
detJ := (R^2/a^2)*sin(theta) :
AdJ := simplify(eval(subs( a_sublist, Adjoint(J)))):
df_tau_0 :=AdJ[1, 1]/detJ :
df_tau_1 := AdJ[1, 2]/detJ :
df_tau_2 := AdJ[1, 3]/detJ :
df_tau_3 := AdJ[1, 4]/detJ :
df_R_0 := AdJ[2, 1]/detJ :
df_R_1 := AdJ[2, 2]/detJ :
df_R_2 := AdJ[2, 3]/detJ :
df_R_3 := AdJ[2, 4]/detJ :
df_theta_0 := AdJ[3, 1]/detJ :
df_theta_1 := AdJ[3, 2]/detJ :
df_theta_2 := AdJ[3, 3]/detJ :
df_theta_3 := AdJ[3, 4]/detJ :
df_phi_0 := AdJ[4, 1]/detJ :
df_phi_1 := AdJ[4, 2]/detJ :
df_phi_2 := AdJ[4, 3]/detJ :
```

```
df_phi_3 := AdJ[4, 4]/detJ :
PD_0:=df_tau_0*PD(tau)+df_R_0*PD(R) +df_theta_0*PD(theta)
+df_phi_0*PD(phi):
PD_1:=df_tau_1*PD(tau)+df_R_1*PD(R) +df_theta_1*PD(theta)
+df_phi_1*PD(phi):
PD_2:=df_tau_2*PD(tau)+df_R_2*PD(R) +df_theta_2*PD(theta)
+df_phi_2*PD(phi):
PD_3:=df_tau_3*PD(tau)+df_R_3*PD(R) +df_theta_3*PD(theta)
+df_phi_3*PD(phi):
VX := R*PD(R) ;
dualX := F2C(& (VX)) ;
VV := PD(tau)+VX*(ad/a) ;
dualV := collect(F2C(& (VV)), Basis1);
dualA :=collect(expand(R*(ad^2/a^2)*d(tau) + (-ad/a)*d(R)
+R*((ad*ath)/a^2-athd/a)*d(theta)+ R*((ad*aph)/a^2-aphd/a)*d(phi)),
Basis1);
VA :=collect(expand(F2C( & (dualA))), Basis6) ;
ALW := collect(expand(F2C(dualV/(-R))), Basis1) ;
FLW := collect(expand(subs(a_sublist, d(ALW))), Basis2) ;
starFLW:=collect(F2C(&i (&star(FLW))), Basis2);
stress:=proc(kill);
collect(subs(Cd_sublist,F2C(((ep/2)*((PD_||kill &i FLW) &^(&star
FLW)-(PD_||kill &i(&star FLW))&^FLW)))), Basis3);
end proc:
stress_0:=stress(0):
stress_1:=stress(1):
stress_2:=stress(2):
stress_3:=stress(3):
expansion_cdot_sublist:={epsilon=1, Cd0=1+(b0*tau^2/2)+O(tau^3),
Cd1=(b1*tau^2/2)+O(tau^3), Cd2=(b2*tau^2/2)+O(tau^3),
Cd3=a3*tau+(b3*tau^2/2)+O(tau^3), Cdd0=b0*tau+O(tau^2),
Cdd1=b1*tau+O(tau^2), Cdd2=b2*tau+O(tau^2),
Cdd3=a3+b3*tau+O(tau^2)};
S_k_cdot:=proc(sublist, kill)
local spl;
spl:=stress_||kill;
subs(sublist, subs(aa_sublist,
collect(expand(subs(Cd1*cos(phi)*sin(theta)
+Cd2*sin(phi)*sin(theta)+Cd3*cos(theta)=a+Cd0,
-Cd1*cos(phi)*sin(theta)-Cd2*sin(phi)*sin(theta)
-Cd3*cos(theta)=-a-Cd0,-Cd1*cos(phi)*cos(theta)
-Cd2*sin(phi)*cos(theta)+Cd3*sin(theta)=-ath,Cd1*cos(phi)*cos(theta)
+Cd2*sin(phi)*cos(theta)-Cd3*sin(theta)=ath,
-Cd1*sin(phi)+Cd2*cos(phi)=aph/sin(theta),Cd1*sin(phi)
-Cd2*cos(phi)=-aph/sin(theta), Cd_sublist,spl)), Basis3)));
end proc:
```

```
intgrd_0:= series(coeff(S_k_cdot(expansion_cdot_sublist, 0),
`&^`(d(tau), d(theta), d(phi))), tau=0):
intgrd_1:= series(coeff(S_k_cdot(expansion_cdot_sublist, 1),
`&^`(d(tau), d(theta), d(phi))), tau=0):
intgrd_2:= series(coeff(S_k_cdot(expansion_cdot_sublist, 2),
`&^`(d(tau), d(theta), d(phi))), tau=0):
intgrd_3:= series(coeff(S_k_cdot(expansion_cdot_sublist, 3),
`&^`(d(tau), d(theta), d(phi))), tau=0):

get_integrands:= proc();
print(t, intgrd_0);
print(x, intgrd_1);
print(y, intgrd_2);
print(z, intgrd_3);
end proc:

get_integrals:= proc(); print(t,
factor(simplify(int(int(int(intgrd_0, phi=0..2*Pi), theta=0..Pi),
tau))));
print(x, factor(simplify(int(int(int(intgrd_1, phi=0..2*Pi),
theta=0..Pi), tau))));
print(y, factor(simplify(int(int(int(intgrd_2, phi=0..2*Pi),
theta=0..Pi), tau))));
print(z, factor(simplify(int(int(int(intgrd_3, phi=0..2*Pi),
theta=0..Pi), tau))));
end proc:
get_integrals();
```

Comments

1-7 Set up the Newman-Unti coordinate system $(\tau, R, \theta, \phi) = (\texttt{tau}, \texttt{R}, \texttt{theta}, \texttt{phi})$

8-12 The global variables are defined. For $i = 0..3$ we use notation C^i = `Ci`, $\dot{C}^i$ = `Cdi`, $\ddot{C}^i$ = `Cddi`. Also α = `a`, $\dot{\alpha}$ = `ad`, α_θ = `ath`,α_ϕ = `aph`, $\dot{\alpha_\phi}$ = `aphd` etc. The constants a, b^i defining the comoving frame are given by `a` and `bi` respectively. Also `q` and `ep` are constants.

13-17 The metric (1.92) is input. This associates the manifold `M` with Minkowski space $\mathcal{M}$. The function `Mancovmetric(M, g)` identifies `g` as the metric on `M`. The *Manifolds* package will automatically give the inverse metric and the vector and covector bases on $\mathrm{T}\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$. Note that there is no factor of c^2 in the metric because we use dimensions such that $g(\ddot{C}, \ddot{C}) = -1$.

18 `Manvol(M)` defines the volume 4-form. Notice the negative orientation.

19-25 Define coordinate bases to simplify output

26-31 These lines define the relationships between α and its derivatives.

32-41 These lines define the relationships between the components of C, $\dot{C}$,$\ddot{C}$ and $\dddot{C}$.

42-58 The here we define the parameters `aa, aad, aath, aaph, aaphd...` which assign the coordinate representations to the variables `a, ad, ath, aph, aphd...`

59-62 The coordinate transformation from Newman-Unti (`tau, R,theta, phi`) to Minkowski coordinates (`x0, x1, x2, x3`).

63-70 We determine the Jacobian `J` and its determinant.

71-87 We calculate the partial derivatives of the Newman-Unti coordinates with respect to the Minkowski coordinates.

88-95 These lines define the Minkowski basis vectors PD_t=$\frac{\partial}{\partial x0}$,PD_x=$\frac{\partial}{\partial x1}$, PD_y=$\frac{\partial}{\partial x2}$, PD_z=$\frac{\partial}{\partial x3}$ in terms of Newman-Unti coordinates.

96-103 Defines the vectors X = `VX`, V = `VV`, and A = `VA` and their duals using (1.90) and (1.91) and (1.99).

104-106 The Liénard-Wiechert potential A = `ALW` is defined using (1.106). The 2-form field F = `FLW` may is calculated by taking the exterior derivative. This is included in the *Manifolds* package. The Hodge dual is also used to define $\star$F = `starFLW`

107-114 These lines define the four stress 3-forms S_K =`stress_i` for `i` = `0`, `1`, `2`, `3`.

115-119 Defines the expansion around the momentarily comoving frame

120-32 A procedure for substituting the expansion into either of the stress 3-forms and simplifying the resulting expression.

133-145 These lines provide the procedure `get_integrands` for obtaining the integrands.

147-156 These lines provide the procedure `get_integrals` for carry out the integration.

157 This calls the procedure `get_integrals`. The result is stated in (4.5).

Appendix G

MAPLE Input for Part II

For the numerical investigation in Part II we use MAPLE to perform many different calculations, integrals and plots for a wide range of input parameters. As a result I have many different files with variations on the code. With hindsight I would have liked to have kept all the code in one file, beautifully annotated and ready to reproduce any calculation. However coding in MAPLE is a skill which I have learnt throughout my PhD and the code I have written hasn't always been the most simple or the most elegant. In this section I present some of the most important code which has been used to obtain the results stated in chapter 7. Once again we use the packages *Plots*, *LinearAlgebra* and *Manifolds*[46].

G.1 Part 1 - Setup

```
# set up coordinate system
Manifoldsetup(M,[tau,r,theta,phi],[e,E,0],
map(x->simplify(x,symbolic),
[e[0]=d(tau),
e[1]=d(r),
e[2]=d(theta),
e[3]=d(phi)])):
```

```
Constants(epsilon, Lp, Rp, thetap, v, X0, Y0, Z0, q_e, ep, mu, c):
Manfdomain(M,gAV) :
Manfdomain(M,[a,ath,aph],[tau,theta,phi]) :
Manfdomain(M,[ad,athd,aphd, athph, aphth],[tau,theta,phi]) :
```

```
Manfdomain(M,[C0,C1,C2,C3,Cd0,Cd1,Cd2,Cd3,Cdd0,Cdd1,Cdd2,Cdd3],[tau])
:
Manfdomain(M,[rhat, cthhat, sthhat, cphhat, sphhat, T0], [tau]):
g := -c^2*d(tau) & X d(tau)
+ a*(d(tau) & X d(r)+ d(r) & X d(tau))
+ r*ath *(d(tau) & X d(theta)+ d(theta) & X d(tau))
+ r*aph*(d(tau) & X d(phi)+ d(phi) & X d(tau))
+ r^2*(d(theta) & X d(theta))
+ r^2*sin(theta)^2 *d(phi) & X d(phi) :
Mancovmetric(M,g):
G:=Manconmetric(M):
Cd_sublist := {diff(C0,tau)=Cd0,diff(C1,tau)=Cd1,
diff(C2,tau)=Cd2,diff(C3,tau)=Cd3} :
Cd_inv_sublist := {Cd0=diff(C0,tau),Cd1=diff(C1,tau),
Cd2=diff(C2,tau),Cd3=diff(C3,tau)} :
Cdd_sublist := {diff(C0,tau,tau)=Cdd0,diff(C1,tau,tau)=Cdd1,
diff(C2,tau,tau)=Cdd2,diff(C3,tau,tau)=Cdd3} :
aa :=
-Cd0+Cd1*cos(phi)*sin(theta)+Cd2*sin(phi)*sin(theta)+Cd3*cos(theta)
:
aath := diff(aa,theta) :
aaph := diff(aa,phi) :
aad := subs(Cdd_sublist,diff(subs(Cd_inv_sublist,aa),tau)) :
aathd := subs(Cdd_sublist,diff(subs(Cd_inv_sublist,aath),tau)) :
aaphd := subs(Cdd_sublist,diff(subs(Cd_inv_sublist,aaph),tau)) :
Basis1 := {d(tau),d(r),d(theta),d(phi)} :
Basis2 := {d(tau)& ^d(r), d(tau)& ^d(theta), d(tau)& ^d(phi), d(r)&
^d(theta), d(r)& ^d(phi), d(theta)& ^d(phi)} :
Basis3 := {d(tau)& ^d(r)& ^d(theta), d(tau)& ^d(r)& ^d(phi),
d(tau)& ^d(theta)& *d(phi), d(r)& ^d(theta)& ^d(phi)} :
VX := r*PD(r) :
VV := PD(tau) :
dualX := &  (VX) :
dualV := &  (VV):
dualA := subs(gAV *d(tau) + ad*d(r) + r*athd*d(theta) +
r*aphd*d(phi) ):
VA := &  (dualA) :
ALW := q_e*(dualV/(r*a)) :
FLW :=
collect(subs({diff(a,tau)=ad,diff(a,theta)=ath,diff(a,phi)=aph,
diff(ath,tau)=athd,diff(aph,tau)=aphd,
diff(ath,phi)=athph,diff(aph,theta)=athph},
simplify(F2C(d(ALW)))),Basis2) :
FLWc := subs(ad=0,athd=0,aphd=0,FLW) :
FLWr := collect(simplify(FLW - FLWc),Basis2) :
```

```
x0 := (C0+r)/c:
x1 := C1+r*sin(theta)*cos(phi):
x2 := C2+r*sin(theta)*sin(phi):
x3 := C3+r*cos(theta):
J := Matrix(4, 4):
J[1, 1]:=Cd0/c:
for i from 1 to 3 do J[i+1, 1] := Cd || i:
J[i+1, 2] := diff(x || i, r):
J[i+1, 3] := diff(x || i, theta):
J[i+1, 4] := diff(x || i, phi) end do:
J[1,2]:=diff(x0, r):J[1,3]:=diff(x0, theta):J[1,4]:=diff(x0, phi):
J:
DetJ := simplify(Determinant(J)):
detJ := -(1/c)*a*r^2*sin(theta) :
Manvol(M) :=-(1/c) a*r^2*sin(theta)*`&^`(e[0], e[1], e[2], e[3]) :
AdJ := simplify(Adjoint(J)):
df_tau_t := AdJ[1, 1]/detJ :
df_tau_x := AdJ[1, 2]/detJ :
df_tau_y := AdJ[1, 3]/detJ :
df_tau_z := AdJ[1, 4]/detJ :
#df_r_t := AdJ[2, 1]/detJ :
df_r_t := ((Cd0+a)*c)/a :
df_r_x := AdJ[2, 2]/detJ :
df_r_y := AdJ[2, 3]/detJ :
df_r_z := AdJ[2, 4]/detJ :
#df_theta_t := AdJ[3, 1]/detJ :
df_theta_t := (ath*c)/(r*a) :
df_theta_x := AdJ[3, 2]/detJ :
df_theta_y := AdJ[3, 3]/detJ :
df_theta_z := AdJ[3, 4]/detJ :
df_phi_t := AdJ[4, 1]/detJ :
df_phi_x := AdJ[4, 2]/detJ :
df_phi_y := AdJ[4, 3]/detJ :
df_phi_z := AdJ[4, 4]/detJ :
PD_t:=df_tau_t*PD(tau)+df_r_t*PD(r) +df_theta_t*PD(theta)
+df_phi_t*PD(phi):
PD_x:=df_tau_x*PD(tau)+df_r_x*PD(r) +df_theta_x*PD(theta)
+df_phi_x*PD(phi):
PD_y:=df_tau_y*PD(tau)+df_r_y*PD(r) +df_theta_y*PD(theta)
+df_phi_y*PD(phi):
PD_z:=df_tau_z*PD(tau)+df_r_z*PD(r) +df_theta_z*PD(theta)
+df_phi_z*PD(phi):
PDt_Fc:=PD_t &i FLWc:
PDt_starFc:=collect(subs(aph=aaph,Cd_sublist,F2C(PD_t &i
(&star(FLWc)))), Basis1,simplify):
Elec_c :=(1/c)*PD_t &i FLWc :
```

```
Mag_c :=(1/(c*c))*collect(subs(aph=aaph,Cd_sublist,F2C(PD_t &i
(&star(FLWc)))), Basis1,simplify) :
Elec_r := (1/c)*collect(PD_t &i FLWr,Basis1) :
Mag_r := (1/(c*c))*collect(PD_t &i F2C(&star(FLWr)),Basis1) :
Elec_cx := simplify(PD_x &i Elec_c) :
Elec_cy := simplify(PD_y &i Elec_c) :
Elec_cz := simplify(PD_z &i Elec_c) :
Elec_rx := simplify(PD_x &i Elec_r) :
Elec_ry := simplify(PD_y &i Elec_r) :
Elec_rz := simplify(PD_z &i Elec_r) :
Mag_cx := simplify(PD_x &i Mag_c) :
Mag_cy := simplify(PD_y &i Mag_c) :
Mag_cz := simplify(PD_z &i Mag_c) :
Mag_rx := simplify(PD_x &i Mag_r) :
Mag_ry := simplify(PD_y &i Mag_r) :
Mag_rz := simplify(PD_z &i Mag_r) :
Energy_res :=(1/2)*(ep*((Elec_cx+Elec_rx)^2+(Elec_cy+Elec_ry)^2
+(Elec_cz+Elec_rz)^2)+(1/mu)*((Mag_cx+Mag_rx)^2+(Mag_cy+Mag_ry)^2
+(Mag_cz+Mag_rz)^2)):
hat_ sublist := {T0=(sqrt((X0-C1)^2 + (Y0-C2)^2 + (Z0-C3)^2) +C0)/c,
rhat=sqrt((X0-C1)^2 + (Y0-C2)^2 + (Z0-C3)^2),
cthhat=(Z0-C3)/(sqrt((X0-C1)^2 + (Y0-C2)^2 + (Z0-C3)^2)),
sthhat=sqrt((X0-C1)^2+(Y0-C2)^2)/(sqrt((X0-C1)^2 + (Y0-C2)^2 +
(Z0-C3)^2)),
cphhat=(X0-C1)/(sqrt((X0-C1)^2+(Y0-C2)^2)),
sphhat=(Y0-C2)/(sqrt((X0-C1)^2+(Y0-C2)^2))}:
prehat_ subslist :=cos(theta)=cthhat,sin(theta)=sthhat,
cos(phi)=cphhat,sin(phi)=sphhat,r=rhat :
Curve3_ def := {
C0a=epsilon*gamma*tau,
C1a=epsilon*gamma*v*tau,
C2a=0,
C3a=0 } :
Curve2_ def := {
C0a=epsilon*gamma*tau,
C1a=Lp-epsilon*(Rp*sin((Lp/(epsilon*Rp))-(gamma*v*tau)/Rp)),
C2a=epsilon*Rp*(1-cos((Lp/(epsilon*Rp))-(gamma*v*tau)/Rp)),
C3a=0} :
Curve1_ def :={
C0a=epsilon*gamma*tau,
C1a=epsilon*(gamma*v*cos(thetap)*tau+Lp
-Rp*sin(thetap)+cos(thetap)*(thetap*Rp-Lp/epsilon)),
C2a=epsilon*(-gamma*v*sin(thetap)*tau +
Rp*(1-cos(thetap))-sin(thetap)*(thetap*Rp-Lp/epsilon)),
C3a=0} :
Curve3_ sublist := eval(subs(Diff=diff,eval(subs(Curve3_ def,
```

```
{
C0=C0a,Cd0=Diff(C0a,tau),Cdd0=Diff(C0a,tau,tau),
C1=C1a,Cd1=Diff(C1a,tau),Cdd1=Diff(C1a,tau,tau),
C2=C2a,Cd2=Diff(C2a,tau),Cdd2=Diff(C2a,tau,tau),
C3=C3a,Cd3=Diff(C3a,tau),Cdd3=Diff(C3a,tau,tau)}
)))) :
Curve2_ sublist := eval(subs(Diff=diff,eval(subs(Curve2_ def,
{
C0=C0a,Cd0=Diff(C0a,tau),Cdd0=Diff(C0a,tau,tau),
C1=C1a,Cd1=Diff(C1a,tau),Cdd1=Diff(C1a,tau,tau),
C2=C2a,Cd2=Diff(C2a,tau),Cdd2=Diff(C2a,tau,tau),
C3=C3a,Cd3=Diff(C3a,tau),Cdd3=Diff(C3a,tau,tau)}
)))) :
Curve1_ sublist := eval(subs(Diff=diff,eval(subs(Curve1_ def,
{
C0=C0a,Cd0=Diff(C0a,tau),Cdd0=Diff(C0a,tau,tau),
C1=C1a,Cd1=Diff(C1a,tau),Cdd1=Diff(C1a,tau,tau),
C2=C2a,Cd2=Diff(C2a,tau),Cdd2=Diff(C2a,tau,tau),
C3=C3a,Cd3=Diff(C3a,tau),Cdd3=Diff(C3a,tau,tau)}
)))) :
get_range3 := proc(Values_sublist)
local Taub ;
Taub := subs(Values_sublist,X0/(epsilon*gamma*v)) ;
0..Taub ;
end proc :
get_range2 := proc(Values_sublist)
local Taua ;
Taua := subs(Values_sublist,-Rp*thetap/(gamma*v)) ;
Taua..0 ;
end proc :
get_range1 := proc(Values_sublist)
local Taua ;
Taua := subs(Values_sublist,-Rp*thetap/(gamma*v)) ;
subs(Values_sublist,StartTau)..Taua ;
end proc :


Get_Fields := proc(Cnum,Values_sublist)
local Curve_sublist;
Curve_sublist := Curve||Cnum||_sublist ;
{
Elec_cx_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Elec_cx))))) , Elec_cy_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
```

```
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Elec_cy))))) ,
Elec_cz_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Elec_cz))))) ,
Elec_rx_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Elec_rx))))) ,
Elec_ry_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Elec_ry))))) ,
Elec_rz_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Elec_rz))))) ,
Mag_cx_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Mag_cx))))) ,
Mag_cy_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Mag_cy))))) ,
Mag_cz_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Mag_cz))))) ,
Mag_rx_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Mag_rx))))) ,
Mag_ry_res =
subs(Values_sublist,subs(Curve_sublist,
subs(hat_sublist,subs(prehat_subslist,
subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
Mag_ry))))) ,
Mag_rz_res =
```

```
243 subs(Values_sublist,subs(Curve_sublist,
244 subs(hat_sublist,subs(prehat_subslist,
245 subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
246 Mag_rz))))) ,
247 T0_res =
248 subs(Values_sublist,subs(Curve_sublist,
249 subs(hat_sublist,subs(prehat_subslist,
250 subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
251 T0))))) ,
252 C1_res =
253 subs(Values_sublist,subs(Curve_sublist,
254 subs(hat_sublist,subs(prehat_subslist,
255 subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
256 C1))))) ,
257 aa_res =
258 subs(Values_sublist,subs(Curve_sublist,
259 subs(hat_sublist,subs(prehat_subslist,
260 subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
261 a))))) ,
262 Energy_res =
263 subs(Values_sublist,subs(Curve_sublist,
264 subs(hat_sublist,subs(prehat_subslist,
265 subs(a=aa, ad=aad, ath=aath, athd=aathd, aph=aaph, aphd=aaphd,
266 sqrt((Elec_cx+Elec_rx)^2+(Elec_cy+Elec_ry)^2+(Elec_cz+Elec_rz)^2))))))
267 } ;
268 end proc:
```

Comments

1-98 This section of the code is almost identical to that of Part I with a few notable exceptions:

- We use the coordinate system given by (E.2) where $(\tau, \mathsf{r}, \theta, \phi)$=(`tau, r, theta, phi`).
- The metric is now given by (E.4).
- We We define $\mathrm{F_C}$ =`FLWc` by setting all components of acceleration to zero in `FLW`. We define $\mathrm{F_R}$ =`FLWr` as the difference `FLW-FLWc`.

99-106 Calculate $\mathrm{E_C}$ =`Elec_c`, $\mathrm{E_R}$ =`Elec_r`, $\mathrm{B_C}$ =`Mag_c` and $\mathrm{B_R}$ =`Mag_r`. This is easily done using (6.12).
107-118 Calculate the components of these vectors in the `x1, x2` and `x3` directions

by taking the internal contractions with respect to PD_x, PD_y and PD_z respectively.
119-121 Calculate the total energy of the electric field $||\mathrm{E}(\tau, \mathsf{r}, \theta, \phi)||^2$ =Energy_res.
122-130 Input the substitutions given by (7.4).
131-147 Define the three sections of the pre-bent path by inputting the components (7.1) according to (7.5). The labels for the axis in the code are different to those given in figure 5.5 due to the way I initially set up the trajectory. The axes x, y, z in the figure correspond to y, z, x in the code, and correspondingly the point $\boldsymbol{X} = (X_0, Y_0, Z_0)$ is given by (Y0, Z0, X0). The coordinate system is aligned so that instead of being located at the terminus of the bend as in the figure (5.5), the origin is located at the end of the small straight line section. As a result the parameter Lp is defined as the negative of the distance Z. We use notation R =Rp and Θ=thetap.
148-168 Define the three corresponding list of substitutions which will associate a field with a particular trajectory.
169-183 Calculate the ranges of τ =tau for each of the three sections of the path. The values Taua and Taub are the tau values at the start and the end of the bend respectively. The value StartTau is the initial value for tau.
184-186 The procedure get_fields uses the substitutions in the previous section to output the listed fields as functions of τ =tau for a given section of path and a given set of input parameters. The inputs are the number Cnum=1, 2 or 3 which tells maple which of the three sections of the path 131-147 we are considering, and a list of numerical inputs of the following format

```
Values_sublist0 :=
subs(gam=1000,{X0=0,Y0=0,Z0=1,epsilon=1,v=sqrt(1-1/gam^2),Lp=0,
Rp=1000, thetap=0.1, gamma=gam, StartTau=-20});
```

247-251 Notice the lab time $T_0(\tau)$ =T0_res and the total energy of the electric field $||\mathcal{E}(\tau, \mathsf{r}, \theta, \phi)||^2$ =Energy_res are also obtained as functions of tau.

G.2 Part 3 - minimize peak field

```
269 get_list3 := proc(Values_sublist)
270 local Taub ;
271 Taub := subs(Values_sublist,Lp/(epsilon*gamma*v)) ;
272 $(round(Taub)..0) ;
273 end proc :
274 get_list2 := proc(Values_sublist)
275 local Taua, Taub ;
276 Taub := subs(Values_sublist,Lp/(epsilon*gamma*v)) ;
277 Taua :=
278 subs(Values_sublist,Lp/(epsilon*gamma*v)-Rp*thetap/(gamma*v)) ;
279 $(round(Taua)..round(Taub) );
280 end proc :
281 get_list1 := proc(Values_sublist)
```

```
local Taua, Taub ;
Taub := subs(Values_sublist,Lp/(epsilon*gamma*v)) ;
Taua :=
subs(Values_sublist,Lp/(epsilon*gamma*v)-Rp*thetap/(gamma*v)) ;
$(round(subs(Values_sublist,StartTau))..round(Taua) )
end proc :
Max_ field := proc(Values_ sublist)
local Field1,Field2,Field3,taurng1,taurng2,taurng3, FUNCT1, FUNCT2,
FUNCT3,Taua, Taub,VAL1, VAL2, VAL3, i1, i2,i3, F1, FF1, F2, FF2;
Taub := evalf(subs(Values_ sublist,Lp/(epsilon*gamma*v))) ;
Taua := evalf(subs(Values_
sublist,Lp/(epsilon*gamma*v)-Rp*thetap/(gamma*v))) ;
Field1:=evalf(subs(Get_ Fields(1,Values_ sublist),Energy_ res));
Field2:=evalf(subs(Get_ Fields(2,Values_ sublist),Energy_ res));
Field3:=evalf(subs(Get_ Fields(3,Values_ sublist),Energy_ res));
taurng1 := get_ list1(Values_ sublist) ;
taurng2 := get_ list2(Values_ sublist) ;
taurng3 := get_ list3(Values_ sublist) ;
VAL1:=(abs(subs(Values_ sublist, StartTau))-(abs(round(Taua)))):
for i1 from 1 to VAL1 do:
for i2 from 1 to 100 do:
FUNCT1:= max(subs(tau=taurng1[i1], Field1),
subs(tau=taurng1[i1]-i2/100, Field1));
end do:
end do:
ARR:=Array(1..19):
#for i1 from 1 to (abs(round(Taua))) do:
for i3 from 1 to 19 do:
ARR[i3]:=(evalf(subs(tau=-i3*(0.05), Field2)));
FUNCT2:=max(ARR);
end do:
#end do:
max(FUNCT1, FUNCT2);
end proc :
Values_sublist1 :=
subs(gam=1000,X0=0,Y0=0,Z0=1,epsilon=1,v=sqrt(1-1/gam^2),
gamma=gam,Lp=0,Rp=Rpp, thetap=thetapp, StartTau=-20);
thetap_range:= 1/95, 1/90, 1/85, 1/80, 1/75, 1/70, 1/65, 1/60, 1/55,
1/50, 1/45, 1/40, 1/35, 1/30, 1/25, 1/20, 1/15, 1/10, 1/5, 1;
Rp_range:=500, 1000, 1500, 2000, 2500, 3000, 3500, 4000, 4500, 5000,
5500, 6000, 6500, 7000, 7500, 8000, 8500, 9000, 9500, 10000;
B:=Matrix(20, 20);
for i1 from 1 to 20 do:
for i2 from 1 to 20 do:
subs_LthetaR :=subs(thetapp=thetap_range[i1],Rpp=Rp_range[i2],
Values_sublist1);
B[i1, i2]:= Max_field(subs_LthetaR);
end do;
```

```
330 end do;
```

Comments

269-287 The procedures `get_list||Cnum` will round the values of `Taua` and `Taub` to the nearest integer and output the range of `tau` as a sequence of integers.

288-315 The procedure `Max_ field` will compare the peak value of `Energy_ res` for the initial straight line and the bend for a number of values of `tau`. The field given by the second straight line is negligible. The peak field for the straight segment is given by the local variable `FUNCT1` and the peak field for the bend is given by `FUNCT2`.

316-330 These lines of code will create a 20×20 matrix `B` whose elements are the peak fields corresponding to the given values of `thetap` and `Rp`. These values correspond to those given in table 7.1 and the resulting matrix was used to plot figure 7.1 using the MAPLE function *matrixplot*.

G.3 Part 4 - Plots

```
331 Plot_Field_tau := proc(Values_sublist,Field)
332 local Field1,Field2,Field3,taurng1,taurng2,taurng3;
333 Field1:=subs(Get_Fields(1,Values_sublist),Field) ;
334 Field2:=subs(Get_Fields(2,Values_sublist),Field) ;
335 Field3:=subs(Get_Fields(3,Values_sublist),Field) ;
336 taurng1 := get_range1(Values_sublist) ;
337 taurng2 := get_range2(Values_sublist) ;
338 taurng3 := get_range3(Values_sublist) ;
339 display(
340 plot(Field1,tau=taurng1,color=BLACK,_rest),
341 plot(Field2,tau=taurng2,color=RED,_rest, numpoints=1000),
342 plot(Field3,tau=taurng3,color=BLUE,_rest)
343 ):
344 end proc:
345 Plot_Field_T0 := proc(Values_sublist,Field)
346 local Field1,Field2,Field3,taurng1,taurng2,taurng3;
347 taurng1 := get_range1(Values_sublist) ;
348 taurng2 := get_range2(Values_sublist) ;
```

```
taurng3 := get_range3(Values_sublist) ;
Field1:=subs(Get_Fields(1,Values_sublist),[T0_res,Field,tau=taurng1])
;
Field2:=subs(Get_Fields(2,Values_sublist),[T0_res,Field,tau=taurng2])
;
Field3:=subs(Get_Fields(3,Values_sublist),[T0_res,Field,tau=taurng3])
;
display(
plot(Field1,color=BLACK,_rest),
plot(Field2,color=RED,_rest),
plot(Field3,color=BLUE,_rest)
):
end proc :

Values_sublist1 :=
subs(gam=1000,
{X0=0.005,Y0=0,Z0=0.0005,epsilon=1,v=sqrt(1-1/gam^2),gamma=gam,Lp=0,thetap=0.13
StartTau=-100
, c=3*10^(8), q_e=-1.80951262*10^(-8)});
Values_sublist2 :=
subs(gam=1000, {X0=0.005,Y0=0, Z0=0.0005,
epsilon=1,v=(sqrt(1-1/gam^2)),gamma=gam,Lp=0,thetap=0,Rp=0.5,
StartTau=-100, c=3*10^(8),
q_e=(-1.80951262*10^(-8))});

EEx:=subs(Get_Fields(1,Values_sublist2), Elec_cx_res+Elec_rx_res):
EEy:=subs(Get_Fields(1,Values_sublist2), Elec_cy_res+Elec_ry_res):
EEz:=subs(Get_Fields(1,Values_sublist2), Elec_cz_res+Elec_rz_res):
TT:=subs(Get_Fields(1,Values_sublist2),(T0_res)):

EEx1:=subs(Get_Fields(1,Values_sublist1),Elec_cx_res+Elec_rx_res):
EEy1:=subs(Get_Fields(1,Values_sublist1),Elec_cy_res+Elec_ry_res):
EEz1:=subs(Get_Fields(1,Values_sublist1),Elec_cz_res+Elec_rz_res):
TT1:=subs(Get_Fields(1,Values_sublist1),(T0_res)):

EEx2:=subs(Get_Fields(2,Values_sublist1),Elec_cx_res+Elec_rx_res):
EEy2:=subs(Get_Fields(2,Values_sublist1),Elec_cy_res+Elec_ry_res):
EEz2:=subs(Get_Fields(2,Values_sublist1),Elec_cz_res+Elec_rz_res):
TT2:=subs(Get_Fields(2,Values_sublist1),(T0_res)):

EEx3:=subs(Get_Fields(3,Values_sublist1),Elec_cx_res+Elec_rx_res):
EEy3:=subs(Get_Fields(3,Values_sublist1),Elec_cy_res+Elec_ry_res):
EEz3:=subs(Get_Fields(3,Values_sublist1),Elec_cz_res+Elec_rz_res):
TT3:=subs(Get_Fields(3,Values_sublist1),(T0_res)):

part1x:=plot([10^(12)*TT1, abs(EEx1),
tau=get_range1(Values_sublist1)], color=black, numpoints=10000):
part2x:=plot([10^(12)*TT2, abs(EEx2),
tau=get_range2(Values_sublist1)], color=red,resolution=600,
numpoints=50000):
part3x:=plot([10^(12)*TT3, abs(EEx3),
tau=get_range3(Values_sublist1)], color=blue, numpoints=10000):
```

```
395 part1y:=plot([10^(12)*TT1, abs(EEy1),
396 tau=get_range1(Values_sublist1)], color=black, numpoints=10000):
397 part2y:=plot([10^(12)*TT2, abs(EEy2),
398 tau=get_range2(Values_sublist1)], color=red,resolution=600,
399 numpoints=50000):
400 part3y:=plot([10^(12)*TT3, abs(EEy3),
401 tau=get_range3(Values_sublist1)], color=blue, numpoints=10000):
402 part1z:=plot([10^(12)*TT1, abs(EEz1),
403 tau=get_range1(Values_sublist1)], color=black, numpoints=10000):
404 part2z:=plot([10^(12)*TT2, abs(EEz2),
405 tau=get_range2(Values_sublist1)], color=red,resolution=600,
406 numpoints=50000):
407 part3z:=plot([10^(12)*TT3, abs(EEz3),
408 tau=get_range3(Values_sublist1)], color=blue, numpoints=10000):
409 Resize(display(part1x, part2x, part3x, axes=boxed,
410 view=[15.8..16.8, 0..8], axesfont=[TIMES, ROMAN, 20],
411 thickness=3));
412 Resize(display(part1y, part2y, part3y, axes=boxed,
413 view=[15.8..16.8, 0..8], axesfont=[TIMES, ROMAN, 20],
414 thickness=3));
415 Resize(display(part1z, part2z, part3z, axes=boxed,
416 view=[15.8..16.8, 0..8], axesfont=[TIMES, ROMAN, 20],
417 thickness=3));
```

Comments

331-334 This procedure will plot any field in the list `Get_fields` (or combination thereof) against `tau` for a given set of inputs. We can plot the field due to the straight trajectory by setting `thetap`$= 0$.

345-361 This procedure will plot any field in the list as a function of `T0`.

362-417 This will make the plots given in figure 7.3.

G.4 Part 5 - Convolution

```
418 rho_ box:= (t,a, b) -> 1/a*(Heaviside(t+a/2+b)-Heaviside(t-a/2+b));
419 plot(rho_ box(t,0.0005, 0),t=-0.01..0.01,title="box
420 distribution",colour=brown,axes=boxed);
421 rho_ Gauss:= (t, a, b) -> 1/(a*sqrt(2*Pi))*exp((-(t-b)^2)/(2*a^2));
422 plot(rho_ Gauss(t, 0.5, 0),t=-1..1,title="Gaussian
423 distribution",colour=brown,axes=boxed,numpoints=10000);
```

```
424 conv:=proc(PEAK, a, b, N,comp )
425 local t, i, E_seq, tau_seq,rho_seq,E0_seq, conv,sum1 ;
426 global convx1, convy1, convz1, convx2, convy2, convz2 ;
427 if PEAK=1 then
428 for i from 0 to (N-1) do
429 t:=16.6667;
430 #solve(a+((b-a)/N)*(i+1/2)=TT,tau);
431 #print("-----",tau_1_||i
432 :=solve(t-a-((b-a)/N)*(i+1/2)=10^(12)*TT,tau);
433 #print(tau_1_||i) ;
434 EEE_0_||i :=evalf(subs(tau=tau_1_||i, EE||comp));
435 EEE_1_||i :=EEE_0_||i*evalf(rho_Gauss(t-a-((b-a)/N)*(i+1/2), b-a,
436 t));
437 rho_1_||i:=rho_Gauss(t-a-((b-a)/N)*(i+1/2), b-a, t);
438 end do:
439 sum1:=add(EEE_1_||i, i=0..N-1);
440 conv||comp||PEAK:=sum1/add(evalf(rho_Gauss(t-a-((b-a)/N)*(i+1/2),
441 b-a, t)), i=0..N-1);
442 print(conv||comp||PEAK);
443 elif PEAK=2 then
444 for i from 0 to (N-1) do
445 t:=16.685;
446 tau_1_||i :=fsolve(t-a-((b-a)/N)*(i+1/2)=10^(12)*TT||PEAK,tau);
447 #print(tau_1_||i) ;
448 EEE_0_||i :=evalf(subs(tau=tau_1_||i, EE||comp||PEAK));
449 EEE_1_||i :=EEE_0_||i*evalf(rho_Gauss(t-a-((b-a)/N)*(i+1/2), b-a,
450 t));
451 rho_1_||i:=rho_Gauss(t-a-((b-a)/N)*(i+1/2), b-a,t);
452 end do:
453 sum1:=add(EEE_1_||i, i=0..N-1);
454 conv||comp||PEAK:=sum1/add(evalf(rho_Gauss(t-a-((b-a)/N)*(i+1/2),
455 b-a, t)), i=0..N-1);
456 print(conv||comp||PEAK);
457 end if:
458 end proc:
```

Comments

347-352 Defines the charge profile $\rho(\nu)$. We can use either a box profile or a Gaussian profile.

424-458 Procedure for calculating the convolution (6.52). The convolution has to be evaluated for the pre-bent path and for the straight path for a selection of different bunch lengths. We adopt a Gaussian form for ρ_{Lab} and define the bunch length as the full width at half maximum (FWHM). The results are given in table 7.2.

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