Irreducible Components of the Restricted Nilpotent Commuting Variety of G_2, F_4 and E_6 in Good Characteristic

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Abstract

Let $\mathcal{N}_1$ denote the restricted nullcone of the Lie algebra $\mathfrak{g}$ of a simple algebraic group in characteristic $p > 0$, i.e. the set of $x \in \mathfrak{g}$ such that $x^{[p]} = 0$. For representatives $e_1, \ldots, e_n$ of the nilpotent orbits of $\mathfrak{g}$ we find the irreducible components of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$ for $\mathfrak{g} = G_2$ and F_4 in good characteristic p. We do the same for $\mathfrak{g} = E_6$ with the exception of three nilpotent orbits. We use this information to determine the irreducible components of the restricted nilpotent commuting variety $\mathcal{C}_1^{nil}(\mathfrak{g}) = \{(x, y) \in \mathcal{N}_1 \times \mathcal{N}_1 : [x, y] = 0\}$ for $\mathfrak{g} = G_2$ and F_4. We do the same for $\mathfrak{g} = E_6$ with the exception of when $p = 7$ where we describe $\mathcal{C}_1^{nil}(\mathfrak{g})$ as the union of an irreducible set of dimension 78 and one of dimension 76 which may or may not be an irreducible component.

This is to declare that this thesis is my own work, and has not been submitted in substantially the same form for the award of a higher degree elsewhere.

Contents

Chapter 0

Introduction

Over the last 30 years, support varieties have been a strong theme of research literature on representation theory of finite groups, Lie algebras and finite group schemes . For background on support varieties for finite groups see [Ben98] and see [Far12] for group schemes. There have been several major applications of support varieties including Premet's proof of the Kac-Weisfeiler conjecture [Pre95].

Let G be a reductive algebraic group over an algebraically closed field k of characteristic $p > 0$ with Lie algebra $\mathfrak{g}$. Denote by $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$ and let $\mathcal{N}_1$ be the set of elements $x \in \mathfrak{g}$ such that $x^{[p]} = 0$. This is the *restricted nullcone* of $\mathfrak{g}$ which is a Zariski closed subset of $\mathcal{N}$. The representation theory of G is captured by its Frobenius kernels. For GL_n the Frobenius morphisms are given by $F_r : GL_n \to GL_n$ which sends $(x_{i,j}) \mapsto (x_{i,j}^{p^r})$. The r-th Frobenius kernel of G is $G_r = \{M : F_r(M) = I\}$. For more details on Frobenius kernels see [Jan03]. It was shown in [SFB97] that for G the support variety of the trivial module over the r-th Frobenius kernel is isomorphic to

$$\mathcal{C}_{r-1}^{nil}(\mathfrak{g}) = \{(x_1, \ldots, x_r) \in \mathcal{N}_1 \times \cdots \times \mathcal{N}_1 \ : \ [x_i, x_j] = 0\}$$

A lot is known about the case $r = 1$. For example the dimension and nilpotent orbits of $\mathcal{C}_0^{nil}(\mathfrak{g}) = \mathcal{N}_1$ are known, see [CLNP03]. However very little is known when $r \geq 2$.

It was proved in [MT55] and [Ger61] that the set of all pairs of commuting $n \times n$ matrices over an algebraically closed field is an irreducible variety. This is a special case of the *commuting variety of* $\mathfrak{g}$ given by

$$\mathcal{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} : [x, y] = 0\}$$

It was shown by Richardson in [Ric79] that when $char(k) = 0$ the commuting variety $\mathcal{C}(\mathfrak{g})$ is irreducible. This was extended to good positive characteristic by Levy in [Lev02], under certain mild conditions on G.

The *nilpotent commuting variety* for a Lie algebra is given by

$$\mathcal{C}^{nil}(\mathfrak{g}) = \{(x, y) \in \mathcal{N} \times \mathcal{N} : [x, y] = 0\}.$$

It was shown in [Bar01] that $\mathcal{C}^{nil}(\mathfrak{sl}_n)$ is irreducible for $char(k) = 0$ and $char(k) > n$. A more general result was established for an arbitrary reductive algebraic group G (under some mild conditions) in [Pre03a]. This showed that $\mathcal{C}^{nil}(\mathfrak{g})$ is equidimensional, i.e that the irreducible components of $\mathcal{C}^{nil}(\mathfrak{g})$ all have the same dimension. Specifically, Premet showed that for a nilpotent element e, with centralizer $\mathfrak{g}^e$, the set $\mathfrak{g}^e \cap \mathcal{N}$ is irreducible and the irreducible components of $\mathcal{C}^{nil}(\mathfrak{g})$ are given by $\mathcal{C}(\mathcal{O}_e) = \overline{G \cdot (e, \mathfrak{g}^e \cap \mathcal{N})}$ for distinguished elements e.
When the characteristic p is greater than or equal to the Coxeter number h of $\mathfrak{g}$ then $\mathcal{N} = \mathcal{N}_1$, hence $\mathcal{C}^{nil}(\mathfrak{g}) = \mathcal{C}_1^{nil}(\mathfrak{g})$. Therefore by Premet's work the irreducible components of $\mathcal{C}_1^{nil}(\mathfrak{g})$ are known for p large enough. It is also known that when $p = 2$ then $\mathcal{C}_1^{nil}(\mathfrak{sl}_n)$ is equidimensional and its irreducible components are found in [Lev07].
The aim of this thesis is to consider the irreducible components of the *restricted nilpotent commuting variety* $\mathcal{C}_1^{nil}(\mathfrak{g})$. In particular we consider when $\mathfrak{g}$ is an exceptional type Lie algebra. Unlike the classical types, there are only finitely many cases to consider so a computational approach can help to obtain a complete answer. Also since the exceptional types greatly differ from $\mathfrak{sl}_n$, then considering these cases may giver a broader picture. Specifically we consider the following two questions for $\mathfrak{g} = G_2$, F_4 and E_6:
Question 1 *Find the irreducible components of* $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$*, where* $\mathcal{O}_{e_1}, \ldots, \mathcal{O}_{e_n}$ *are the nilpotent orbits of* $\mathfrak{g}$*.*
Question 2 *Find the irreducible components of the restricted nilpotent commuting variety* $\mathcal{C}_1^{nil}(\mathfrak{g})$*.*
The irreducible components found by answering the first question allow us to determine some of the components of $\mathcal{C}_1^{nil}(\mathfrak{g})$ and therefore help to answer the second question.
In Chapter 1 we start with an introduction to simple Lie algebras in characteristic zero. This is followed by a discussion of nilpotent orbits which includes the Jacobson-Morozov Theorem. This allows us to embed any nilpotent element of a simple Lie algebra into a triple $\{e, f, h\}$ satisfying the relations of the standard basis of $\mathfrak{sl}_2$. Then we present the details of three methods to classify nilpotent orbits, namely by partition types, weighted Dynkin diagrams and via the Bala-Carter Theorem. This lays the groundwork for the more complicated situation in positive characteristic.
In the positive characteristic case $\mathfrak{sl}_2$-triples are less helpful. Instead we define an associated cocharacter which is in some way analogous to the element h in an $\mathfrak{sl}_2$-triple. This is presented in Chapter 2 along with the classification of nilpotent orbits in positive characteristic. These classifications are important for answering Question 1.
Chapter 3 gives more details of some specific simple Lie algebras that are of particular interest. This includes some structural information for the simple Lie algebras G_2, F_4 and E_6 along with some classical Lie algebras which are helpful for our calculations. For each of these Lie algebras we examine some properties of their nilpotent orbits, including the Bala-Carter labels and the

Hasse diagrams of the nilpotent orbits.

In Chapter 4 we introduce some specialist topics which are necessary for our (partial) answer to Question 2, beginning with Lusztig-Spaltenstein induction. This is subsequently related to a description of the nilpotent commuting variety and some results from [Pre03a].

In Chapter 5 we present the research questions we wish to answer along with an outline of the basic methods we used to answer these questions. This includes some results which enable us to calculate the dimension of each of the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$. We then have all of the tools we require to answer Questions 1 and 2.

We automate part of the calculations to these questions by using [GAP12]. The details of the calculations for G_2 are presented in Chapter 6. This leads to the following result:

Result 1 *For $p = 5$ the variety $\mathcal{C}_1^{nil}(G_2)$ is irreducible of dimension $14 = \dim(\mathfrak{g})$ where*

$$\mathcal{C}_1^{nil}(G_2) = \mathcal{C}_1(G_2(a_1)).$$

The details of these calculations for F_4 are presented in Chapter 7. This leads to the following result:

Result 2 *The variety $\mathcal{C}_1^{nil}(F_4)$ is equidimensional of dimension $52 = \dim(\mathfrak{g})$ with respectively 1, 2, and 3 components given by*

$$\begin{aligned} p = 5: \quad & \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \\ p = 7: \quad & \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \cup \mathcal{C}_1(F_4(a_2)) \\ p = 11: \quad & \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \cup \mathcal{C}_1(F_4(a_2)) \cup \mathcal{C}_1(F_4(a_1)). \end{aligned}$$

In Chapter 8 we answer Question 1 for all the nilpotent orbits of E_6 with the exception of A_1, A_1^2 and A_1^3. Finally Chapter 9 presents most of the details of the calculations for answering Question 2 for E_6 with the exception of when $p = 7$. In this case we show $\mathcal{C}_1^{nil}(E_6) = \mathcal{C}(E_6(a_3)) \cup \mathcal{C}(D_4(a_1))$; however we do not know if $\mathcal{C}(D_4(a_1)) \subset \mathcal{C}(E_6(a_3))$. Therefore we have:

Result 3 *For $p = 5$ (resp. 11) the variety $\mathcal{C}_1^{nil}(E_6)$ is equidimensional of dimension 76 (resp. 78) with respectively 3 and 2 components.*

$$\begin{aligned} p = 5: \quad & \mathcal{C}_1^{nil}(E_6) = \overline{G \cdot (e, X_1)} \cup \overline{G \cdot (e, X_2)} \cup \mathcal{C}_1(D_4(a_1)); \\ p = 11: \quad & \mathcal{C}_1^{nil}(E_6) = \mathcal{C}_1(E_6(a_3)) \cup \mathcal{C}_1(E_6(a_1)). \end{aligned}$$

Here X_1 and X_2 are the two irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ for the nilpotent orbit $\mathcal{O}_e = A_4A_1$. For $p = 7$, $\mathcal{C}_1^{nil}(E_6)$ has one irreducible component of dimension 78 and perhaps one further component of dimension 76.

The results of all these calculations are summarised in Chapter 10 along with some suggestions for further work. A detailed description of the [GAP12] code used throughout is presented in the Appendix.

Notation

Throughout, G is an algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$.

- $k^\times$ is the multiplicative group of the field k.
- $Lie(G)$ is the Lie algebra of G, often denoted by $\mathfrak{g}$.
- $\mathfrak{Z}(G)$ is the centre of G.
- (A, B) is the group generated by the commutators $aba^{-1}b^{-1}$ for closed subgroups A and B in G and $a \in A$, $b \in B$. The commutator subgroup is closed and connected if either A or B are connected [Bor91, §2.3 Corollary]. In particular (G, G) is always closed.

Chapter 1

Simple Lie Algebras in Characteristic Zero

1.1 Preliminaries

In this chapter we discuss a few different methods for classifying nilpotent orbits in a simple Lie algebra. We assume basic knowledge of algebraic geometry, algebraic groups and Lie algebras. For more information on these areas refer to [Hum75] and [Hum72]. We start with some definitions and results about algebraic groups and Lie algebras and then go on to define nilpotent orbits in characteristic zero. Finally we discuss three methods for classifying these nilpotent orbits, namely via partitions, weighted Dynkin diagrams and the Bala-Carter Theorem.

Simple Algebraic Groups and Lie Algebras

Let G be an algebraic group over k with identity 1. A *morphism* of algebraic groups $\phi : G \to G'$ is a group homomorphism which is also a morphism of varieties. The *identity component* of G is the unique irreducible component that contains 1 [Hum75, p.53]. We denote this by G°. We say G is *connected* if $G = G^\circ$. The *derived series* of G is defined inductively by

$$\mathcal{D}^0 G = G,\ \mathcal{D}^{i+1} G = (\mathcal{D}^i G, \mathcal{D}^i G)$$

We say G is *solvable* if $\mathcal{D}^n(G) = \{1\}$ for some n. For all i, $\mathcal{D}^i(G)$ is a closed normal subgroup of G and is connected if G is connected. It can be shown that any connected algebraic group G contains a unique largest closed normal solvable subgroup [Hum75, Cor 7.4, Lemma 17.3(c)]. The identity component of this subgroup is known as the *radical of* G and denoted by $R(G)$. Then $R(G)$ is the largest connected normal solvable subgroup of G.

The set of unipotent elements in a connected solvable linear algebraic group is a closed connected normal subgroup, [Hum75, Thm 19.3]. Let $R_u(G)$ denote the set of unipotent elements in $R(G)$;

we call this the unipotent radical of G. It is easy to see that $R_u(G)$ is normal in G. The group G is *reductive* if $R_u(G) = \{1\}$. If G is reductive then the derived subgroup is semisimple [CM93, §1.2]. Throughout this chapter let G be a connected reductive algebraic group over $\mathbb{C}$.

We say that G is *simple* if it has no closed connected normal subgroups other than itself and $\{1\}$, and *semisimple* if the maximal connected solvable normal subgroup is $\{1\}$. Similarly a Lie algebra is *simple* if it has no non-zero proper ideals and *semisimple* if its unique maximal solvable ideal is zero. Therefore any simple Lie algebra is semisimple. It is shown in [Hum75, §13], that G is simple (resp. semisimple) if and only if $Lie(G)$ is simple (resp. semisimple). Note that this is not the case if $char(k) > 0$, which is discussed in Chapter 2.

Examples 1.1.1

Here are some examples of algebraic groups and their Lie algebras. All are simple, with the exception of the first case.

1. GL_n is the set of $n \times n$ matrices with non-zero determinant. Then $Lie(GL_n) = \mathfrak{gl}_n$ consists of all $n \times n$ matrices.
2. The algebraic group SL_n is the set of $n \times n$ matrices with determinant equal to 1. The Lie algebra $\mathfrak{sl}_n$ consists of matrices with zero trace.
3. $O_n = \{A \in GL_n : A^t A = I_n\}$ and $SO_n = \{A \in SL_n : A^t A = I_n\}$ are simple algebraic groups and $Lie(SO_n) = \mathfrak{so}_n = \{x \in \mathfrak{gl}_n : x^t = -x\}$ consists of skew symmetric matrices.
4. $Sp_{2n} = \{A \in GL_{2n} : A^t J_n A = J_n\}$ where $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ and $\mathfrak{sp}_{2n}$ is given by $\left\{\left(\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right) : A_i \in Mat_{n\times n}, A_1 = -A_4^t \text{ and } A_2, A_3 \text{ are symmetric}\right\}$.

These examples are the *classical groups*.

For an element $x \in \mathfrak{g}$, the *adjoint endomorphism* is the map

$$\begin{aligned} ad_x : \mathfrak{g} &\to \mathfrak{g} \\ y &\mapsto [x, y]. \end{aligned}$$

The *adjoint representation* of G is given by the homomorphism $Ad : G \to Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$ where $v \mapsto Ad_v$. Then we can define the map $Ad_v : \mathfrak{g} \to \mathfrak{g}$ where $y \mapsto Ad_v(y)$. In each example in 1.1.1 we have $Ad_v(y) = vyv^{-1}$. For a subset K in $\mathfrak{g}$ denote the centralizer of K in $\mathfrak{g}$ as $\mathfrak{g}^K = \{x \in \mathfrak{g} : [x, K] = 0\}$, similarly for a subset H in G the centralizer of H in G is given by $G^H = \{v \in G : vh = hv \ \ \forall h \in H\}$. If H is a closed subgroup of G then G^H is also a closed subgroup of G [Hum75, Cor 8.2]. For an element $x \in \mathfrak{g}$, we have $\mathfrak{g}^x = Lie(G^x)$ [Hum75, Thm 13.4]. From now on all subgroups are assumed to be closed unless otherwise specified.

A *Borel subgroup* of an algebraic group G is a maximal connected solvable subgroup of G. For example if $G = GL_n$, then

$$B = \left\{ \begin{pmatrix} b_{1,1} & \dots & \dots & b_{1,n} \\ & b_{2,2} & & \vdots \\ & & \ddots & \vdots \\ 0 & & & b_{n,n} \end{pmatrix} : \begin{array}{ll} b_{i,j} \in k & \text{for } 1 \le i \le j \le n \\ b_{i,i} \in k^{\times} & \text{for } 1 \le i \le n \end{array} \right\}$$

is a Borel subgroup. Any subgroup of G which is conjugate to a Borel subgroup is also a Borel subgroup. Conversely given a Borel subgroup B then any other Borel subgroup is conjugate to B [Hum75, Theorem 21.3]. A closed subgroup of G that contains a Borel subgroup is a *parabolic subgroup of* G. For a fixed Borel subgroup B then any parabolic subgroup in G is conjugate to one that contains B. A subgroup T of G is a *torus* if it is connected and contains only semisimple elements. We say T is a *maximal torus* if it is not properly contained in any other torus. In GL_n

$$T = \left\{ \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix} : t_i \in k^{\times} \right\}$$

is a maximal torus. A *Cartan subgroup* is a subgroup of the form G^T where T is a maximal torus. If G is reductive then $G^T = T$ [Hum75, Cor 26.2 A].

Now consider the Lie algebra $\mathfrak{g} = Lie(G)$. A *Borel subalgebra of* $\mathfrak{g}$ is a maximal solvable subalgebra. For example when $\mathfrak{g} = \mathfrak{gl}_n$ then

$$\mathfrak{b} = \left\{ \begin{pmatrix} b_{1,1} & \dots & \dots & b_{1,n} \\ & b_{2,2} & & \vdots \\ & & \ddots & \vdots \\ 0 & & & b_{n,n} \end{pmatrix} : b_{i,j} \in k \text{ for } 1 \le i \le j \le n \right\}$$

is a Borel subalgebra of $\mathfrak{g}$. As a consequence of our assumption on the characteristic, the Borel subalgebras are the subalgebras of the form $Lie(B)$ where B is a Borel subgroup of G. For a parabolic subgroup P of G then $\mathfrak{p} = Lie(P)$ is a *parabolic subalgebra of* $\mathfrak{g}$. If T is a torus in G then $\mathfrak{t} = Lie(T)$ is a *toral subalgebra* of $\mathfrak{g}$. For example if T is as above then

$$\mathfrak{t} = \left\{ \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} : a_i \in k \right\}$$

is a toral subalgebra of $\mathfrak{gl}_n$. A *Cartan subalgebra* of $\mathfrak{g}$ is $\mathfrak{h}_T = Lie(G^T)$ where T is a maximal torus of G. When G is reductive $Lie(G^T) = \mathfrak{g}^T$ therefore in this case $\mathfrak{h}_T = Lie(T)$ [Bor91, Prop 9.1].

Roots of a Simple Lie Algebras

For the remainder of this chapter let $\mathfrak{g}$ be a simple Lie algebra. We may assume that $\mathfrak{g} = Lie(G)$ for some complex Lie group G.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with dual space $\mathfrak{h}^*$. Then for an element $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}.$$

The *roots of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$ are the set of non-zero $\alpha \in \mathfrak{h}^*$ where $\mathfrak{g}_\alpha \neq \{0\}$. We say $\mathfrak{g}_\alpha$ is a *root space of* α *in* $\mathfrak{g}$. The *root system* of $\mathfrak{g}$ is the set Φ of roots of $\mathfrak{g}$, then

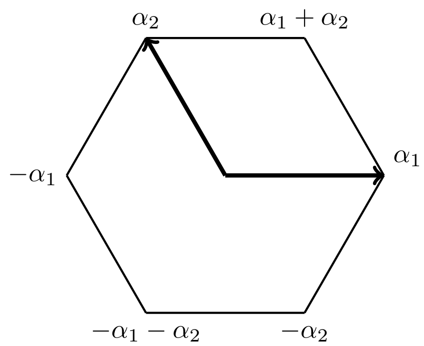
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

A *basis* of Φ is a subset Δ where each root $\beta \in \Phi$ can be written uniquely as $\beta = \sum\limits_{\alpha \in \Delta} k_\alpha \alpha$ where $k_\alpha \in \mathbb{Z}$ such that either all the k_α are non-negative or non-positive. A basis always exists and any two bases are conjugate by the action of the normalizer $N_G(T)$ of T, where $N_G(H) = \{x \in G : x^{-1}Hx = H\}$ for a subgroup H of G. The elements in Δ are the *simple roots* of $\mathfrak{g}$. We denote by Φ^+ all the *positive roots* of $\mathfrak{g}$, i.e. all the roots $\beta \in \Phi$ such that $k_\alpha \geq 0$ for all $\alpha \in \Delta$. The *height* of a root (relative to Δ) is given by $ht(\beta) = \sum\limits_{\alpha \in \Delta} k_\alpha$. We can relate roots in Φ to elements in $\mathfrak{g}$. For a root α in Φ^+ we can choose $e_\alpha \in \mathfrak{g}_\alpha$, and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\{e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]\}$ satisfy the relations of the standard basis of $\mathfrak{sl}_2$. We can write $e_{-\alpha}$ and f_α interchangeably. Throughout, the elements e_α, f_α and h_α are given in their natural upper-triangular, lower-triangular, and diagonal forms respectively.

Example 1.1.2

Let $\mathfrak{g} = \mathfrak{sl}_3$, $\Delta = \{\alpha_1, \alpha_2\}$, $\Phi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -(\alpha_1 + \alpha_2)\}$ and $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$.

We can represent these roots via the following diagram.



For a root $\alpha_i \in \Phi$, we can find elements e_{α_i}, f_{α_i}, $h_{\alpha_i} \in \mathfrak{sl}_3$ with the conditions described above. In this case $\mathfrak{g}_{\alpha_1} = \left\{ \begin{pmatrix} 0 & a & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} : a \in k \right\}$ therefore let $e_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$, here we are using the convention that blank entries in a matrix are zero. We use this convention throughout. For all $\alpha \in \Phi^+$ we have

$$e_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \qquad e_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \qquad e_{\alpha_1+\alpha_2} = [e_{\alpha_1}, e_{\alpha_2}] = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

$$f_{\alpha_1} = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 0 & 0 \end{pmatrix} \qquad f_{\alpha_2} = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 1 & 0 \end{pmatrix} \qquad f_{\alpha_1+\alpha_2} = [f_{\alpha_2}, f_{\alpha_1}] = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}$$

$$h_{\alpha_1} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad h_{\alpha_2} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad h_{\alpha_1+\alpha_2} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

The *Killing form* on $\mathfrak{g}$ is the bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$ defined by $\kappa(x, y) = Tr((ad_x)(ad_y))$ where $x, y \in \mathfrak{g}$ and where ad_x is the adjoint endomorphism of x.

Example 1.1.3

Let $\mathfrak{g} = \mathfrak{sl}_2$ with a basis given by

$$e = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}$$

Then

$$ad_e(e) = 0 \qquad ad_e(h) = \begin{pmatrix} 0 & -2 \\ & 0 \end{pmatrix} = -2e \qquad ad_e(f) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = h$$

Therefore we can represent ad_e relative to the basis $\{e, h, f\}$ so $ad_e = \left(\begin{smallmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$; similarly $ad_f = \left(\begin{smallmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{smallmatrix}\right)$. So $\kappa(e, f) = Tr((ad_e)(ad_f)) = Tr\left(\begin{smallmatrix} 2 & & \\ & 2 & \\ & & 0 \end{smallmatrix}\right) = 4$

The following results can be found in [Hum72, §5.1].

Theorem 1.1.4 *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate i.e. if $\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$ then $x = 0$.*

In fact κ is a symmetric bilinear form and is G-invariant which means for any $v \in G$ we have $\kappa(Ad_v(x), Ad_v(y)) = \kappa(x, y)$.

For any $\alpha \in \mathfrak{h}^*$ there exists a unique element $t_\alpha \in \mathfrak{h}$ such that $\alpha(h) = \kappa(h, t_\alpha)$ for all $h \in \mathfrak{h}$. Then for $\alpha, \beta \in \mathfrak{h}^*$ let $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$. We say $\alpha \in \Phi$ has *length* $\|\alpha\| = \sqrt{(\alpha, \alpha)}$. For a simple Lie algebra there are only two possible root lengths [Hum72, Lemma 10.4 C]. Therefore we can split Φ into *long roots* and *short roots*. If there is only one root length then by convention we say all the roots are long.

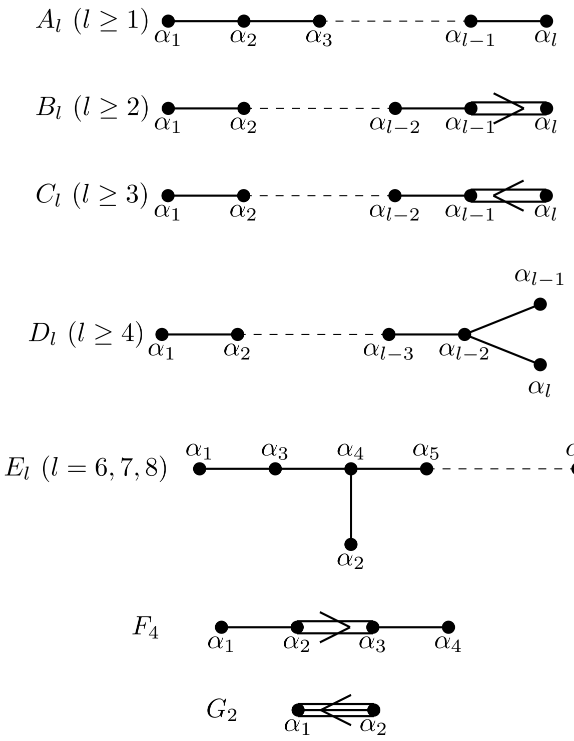


Figure 1.1: The Dynkin diagrams for all the simple Lie algebras

For two roots α, $\beta \in \Phi$ let

$$\langle \beta, \alpha \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

If α and β are linearly independent the *α-string (or α-chain) through β* is the maximal sequence of roots of the form $\beta - r\alpha, \ldots, \beta, \ldots, \beta + q\alpha$ for non-negative integers r and q where $r - q = \langle \beta, \alpha \rangle$ [Hum72, §9.4].

Each simple Lie algebra $\mathfrak{g}$ can be represented by a Dynkin diagram. This graph contains a node corresponding to each simple root α_i of $\mathfrak{g}$. The nodes α_i and α_j are joined by $\langle \alpha_i, \alpha_j \rangle \cdot \langle \alpha_j, \alpha_i \rangle$ number of edges. Finally if two roots of different lengths have a connecting edge then we mark that edge with an arrow pointing to the short root. The Dynkin diagrams for all the simple Lie algebras are given in Figure 1.1.

Example 1.1.5

In reference to Example 1.1.1, we have that SL_n is of type A_{n-1}, Sp_{2n} is of type C_n and SO_n is of type $D_{\frac{n}{2}}$ if n is even and of type $B_{\frac{n-1}{2}}$ if n is odd.

For a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, the simple roots $\{\alpha_1, \ldots, \alpha_n\}$ form a basis of $\mathfrak{h}^*$. There is another basis, the *fundamental basis*, $\{\omega_1, \ldots, \omega_n\}$ of $\mathfrak{h}^*$ such that

$$\omega_i(h_{\alpha_j}) = \langle \omega_i, \alpha_j \rangle = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

We define a *Chevalley basis* of $\mathfrak{g}$ to be a basis $\{e_\beta : \beta \in \Phi\} \cup \{h_{\alpha_i} : i = 1, \ldots, n\}$ such that

(i) $[h_{\alpha_i}, h_{\alpha_j}] = 0$ for all $1 \leq i, j \leq n$,

(ii) $[h_{\alpha_i}, e_\beta] = \langle \beta, \alpha_i \rangle e_\beta = \beta(h_{\alpha_i}) e_\beta$ for all $1 \leq i \leq n$ and $\beta \in \Phi$ i.e. $e_\beta \in \mathfrak{g}_\beta$,

(iii) $[e_\beta, e_{-\beta}] = h_\beta = \sum\limits_{j=1}^{m} \langle \omega_j, \beta \rangle h_{\alpha_i}$,

(iv) If α, β are linearly independent roots and $\beta - r\alpha, \ldots, \beta + q\alpha$ the α-string through β then

$$[e_\alpha, e_\beta] = \begin{cases} 0 \text{ if } q = 0 \\ \pm(r+1) e_{\alpha+\beta} \text{ if } \alpha + \beta \in \Phi \end{cases}$$

For a simple Lie algebra $\mathfrak{g}$ a Chevalley basis always exists, this is shown in [Hum72, §25].

Example 1.1.6

Let $\mathfrak{g} = \mathfrak{sl}_3$ and $\Delta = \{\alpha_1, \alpha_2\}$. Then $\mathfrak{g}$ has Chevalley basis

$$\{h_{\alpha_1}, h_{\alpha_2}, e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_1+\alpha_2}, f_{\alpha_1}, f_{\alpha_2}, f_{\alpha_1+\alpha_2}\}$$

where $e_{\alpha_i}, f_{\alpha_i}$ and h_{α_i} are given in Example 1.1.2.

We are particularly interested in the exceptional type Lie algebras. Below is a table which states the dimension, the number of positive roots and the highest root of each of these Lie algebras.

Lie Algebra	Dimension	Number of Positive Roots	Highest Root
G_2	14	6	$3\alpha_1 + 2\alpha_2$
F_4	52	24	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
E_6	78	36	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	133	63	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	248	120	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$

For the exceptional Lie algebras we can label the element $e_{a_1\alpha_1+\cdots+a_n\alpha_n} \in \Phi^+$ by e subscripted with the Dynkin diagram of $\mathfrak{g}$ with the node corresponding to α_i labelled a_i. For example the highest root in E_6 can be expressed as $e_{\substack{12321\\2}}$. To express the negative roots we replace an e with an f, for example $f_{\substack{12321\\2}}$.

Weyl Groups

Let G be an algebraic group with maximal torus T where $\mathfrak{h} = Lie(T)$ is a Cartan subalgebra of $\mathfrak{g} = Lie(G)$. The *Weyl group* is given by

$$\mathcal{W} = \frac{N_G(T)}{T}$$

The following result is a consequence of [Hum75, Cor 16.3].

Theorem 1.1.7 *The Weyl group is finite.*

The restriction $\kappa(\cdot,\cdot)|_{\mathfrak{h}\times\mathfrak{h}}$ is $\mathcal{W}$-invariant which means $\kappa(w(h_1), w(h_2)) = \kappa(h_1, h_2)$ for $w \in \mathcal{W}$ and $h_1, h_2 \in \mathfrak{h}$. Since this restriction is non-degenerate it induces an isomorphism

$$\mathfrak{h} \to \mathfrak{h}^*$$
$$h \mapsto (h, -)$$

This isomorphism still holds when restricting to $\mathfrak{h}_{\mathbb{R}} = \{h \in \mathfrak{h} : \alpha_i(h) \in \mathbb{R}\}$. In particular the map $\mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}}^*$ sends $h \mapsto (h, -)$ where $(h, h') \in \mathbb{R}$ for any $h, h' \in \mathfrak{h}_{\mathbb{R}}$. By this isomorphism, $w \in \mathcal{W}$ acts on $\mathfrak{h}^*$ by

$$(w \cdot \lambda)(h) = \lambda(w^{-1}(h)) \text{ for } \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}$$

The isomorphism $\pi : \mathfrak{h} \to \mathfrak{h}^*$ satisfies

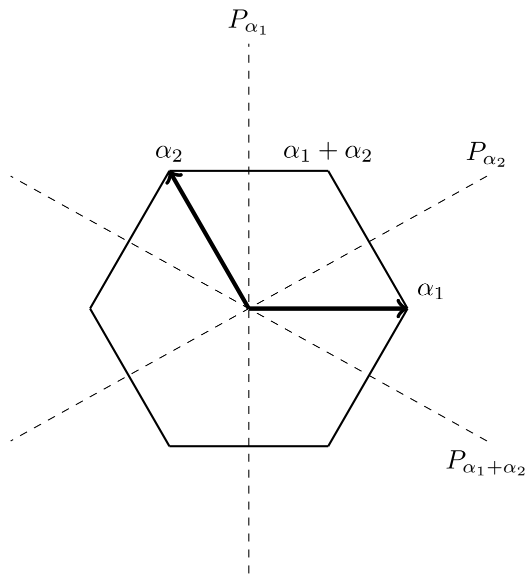
$$w(h) \mapsto (w(h), -) = (h, w^{-1}(-))$$

since $(w(h), w(x)) = (h, x)$ so $(w(h), x) = (w(h), w(w^{-1}x)) = (h, w^{-1}(x))$. Therefore π is also $\mathcal{W}$-invariant.

The Weyl group $\mathcal{W}$ is generated by elements s_α. When $\mathcal{W}$ acts on $\mathfrak{h}_{\mathbb{R}}^*$ then these elements are given by $s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha$ where $s_\alpha(\lambda)$ sends α to $-\alpha$ and fixes $\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \lambda(h_\alpha) = 0\}$. Similarly if $\mathcal{W}$ acts on $\mathfrak{h}$ then $s_\alpha(h) = h - \alpha(h)h_\alpha$ sends h_α to $-h_\alpha$ and fixes the set $\{h \in \mathfrak{h}_{\mathbb{R}} : \alpha(h) = 0\}$. For more details see [Hum75, §27.1]

Example 1.1.8

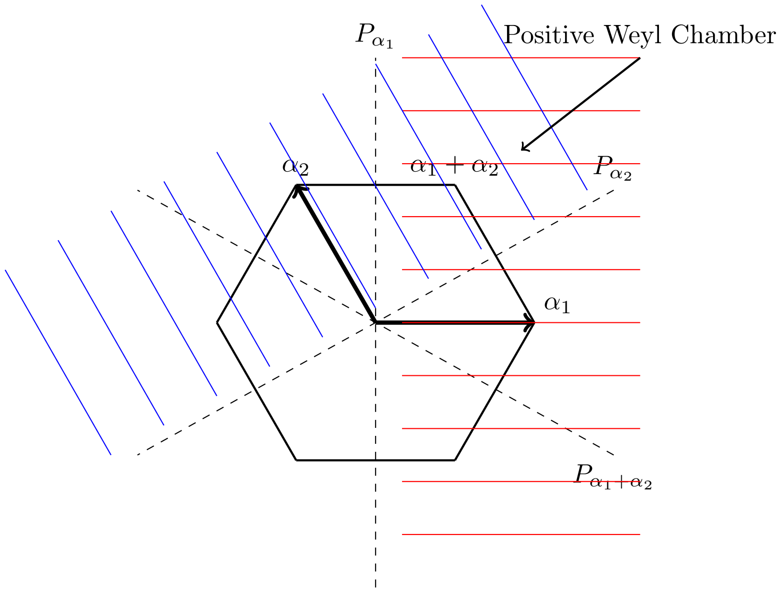
Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$. The element s_α on $\mathfrak{h}_{\mathbb{R}}^*$ fixes the hyperplane P_α given by $\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \lambda(h_\alpha) = 0\}$. This can be represented by



As can be seen in the diagram, $\mathcal{W}$ creates finitely many regions of $\mathfrak{h}_{\mathbb{R}}^*$ which are called *Weyl chambers*. These chambers are permuted transitively by the action of $\mathcal{W}$ on $\mathfrak{g}$. The *positive Weyl chamber* is the chamber where $\lambda(h_{\alpha_i}) > 0$ for all simple roots α_i.

Example 1.1.9

For the case when $\mathfrak{g} = \mathfrak{sl}_3$, the positive Weyl chamber is represented by



The horizontally hashed area is where $\lambda(h_{\alpha_1}) > 0$ and the diagonally hashed area is where $\lambda(h_{\alpha_2}) > 0$. Therefore the Weyl chamber with both hashes is the positive Weyl chamber.

Similarly the Weyl group partitions $\mathfrak{h}_{\mathbb{R}}$ into finitely many Weyl chambers. Then the positive Weyl chamber is given by $h \in h_{\mathbb{R}}$ such that $\alpha(h) > 0$ for all simple roots α.

Theorem 1.1.10 *[Hum75, §10.4] Every element in $h \in \mathfrak{h}_{\mathbb{R}}$ is conjugate by the Weyl group to a unique element in the closure of the positive Weyl chamber.*

Highest Weight Modules

Let V be a finite dimensional $\mathfrak{g}$-module for a Lie algebra $\mathfrak{g}$. Then V is *simple* if $V \neq \{0\}$ and has no proper non-zero submodules.

Let $\mathfrak{g} = \mathfrak{sl}_2$ with the usual basis $\{e, f, h\}$. Then $\mathfrak{h} = kh$ is a Cartan subalgebra of $\mathfrak{g}$. A linear map $\mu : \mathfrak{h} \to k$, in the dual space $\mathfrak{h}^*$, is completely defined by $\mu(h)$ since $\mu(\xi h) = \xi\mu(h)$. Therefore we can express μ as $\mu(h)\omega$ where $\omega \in \mathfrak{h}^*$ and $\omega(h) = 1$. Let V be a simple finite dimensional $\mathfrak{sl}_2$-module and, for any $\mu \in \mathfrak{h}^*$, let $V(\mu, h) = \{v \in V : h.v = \mu(h)v\}$. Then μ is a *weight* and $V(\mu, h)$ is a *weight space* of V if $V(\mu, h) \neq 0$. Now V can be expressed as

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V(\mu, h)$$

In fact since $\mu = \xi\omega$ for some $\xi \in k$ then

$$V = \bigoplus_{\xi \in k} V(\xi, h)$$

where $V(\xi, h) = \{v \in V : h.v = \xi v\}$. Therefore we refer to ξ as a weight of V if $V(\xi, h) \neq 0$. Any non-zero vector $v \in V(\xi, h)$ such that $e.v = 0$ is a *maximal vector* of weight ξ.

Lemma 1.1.11 *If $v \in V(\xi, h)$ then $e.v \in V(\xi + 2, h)$ and $f.v \in V(\xi - 2, h)$*

Proof. Consider the following

$$\begin{aligned} h.(e.v) &= [h, e].v + e.h.v \\ &= 2e.v + ae.v \\ &= (\xi + 2)e.v \end{aligned}$$

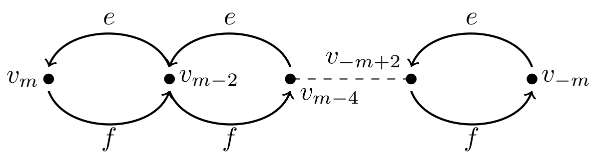
Therefore $e.v \in V(\xi + 2, h)$. Similarly $f.v \in V(\xi - 2, h)$ ■

Since V is finite dimensional there exists an n such that $V(n, h) \neq 0$ and $V(n + 2, h) = 0$. The following two results are presented in [Hum72, §7.2].

Theorem 1.1.12 *Let V be an irreducible module for $\mathfrak{g} = \mathfrak{sl}_2$ such that* $\dim(V) = m + 1$. *Then V is a direct sum of the weight spaces $V(m, h)$, $V(m - 2, h)$, $\ldots$, $V(-m + 2, h)$, $V(-m, h)$ and* $\dim(V(i, h)) = 1$ *for $i = m, m - 2, \ldots, -m$. Also V has a unique maximal vector v (up to non-zero scalar multiples), this vector has weight m.*

Theorem 1.1.13 *For $\mathfrak{g} = \mathfrak{sl}_2$ there exists exactly one irreducible $\mathfrak{sl}_2$-module of each possible dimension $m + 1$, $m \geq 0$ (up to isomorphism)*

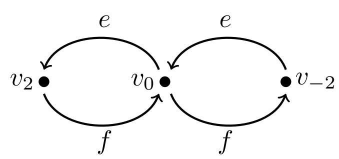
An irreducible module V for $\mathfrak{sl}_2$ of dimension $m + 1$ can be represented pictorially as follows.



We denote the unique (up to isomorphism) irreducible $\mathfrak{sl}_2$-modules of dimension $m+1$ as $L(m\omega)$ with maximal vector v_m of weight m. Then we call v_m the *highest weight vector* of $L(m\omega)$ and m the *highest weight*.

Examples 1.1.14

1. The $\mathfrak{sl}_2$-module $L(2\omega)$ has highest weight vector v_2. Moreover, there is an isomorphism of $L(2\omega)$ with the adjoint representation sending v_2 to e.



2. $L(\omega) \cong k^2$ has highest weight vector $v_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

$$\begin{tikzcd} v_1 \bullet \arrow[r, bend right, "f"'] & \bullet v_{-1} \arrow[l, bend right, "e"'] \end{tikzcd}$$

For a $\mathfrak{sl}_2$-module V let $S^n(V)$ be the *symmetric tensor* where $S^n(V)$ is the subspace of the n^{th} power tensor $V^{\otimes n}$ which contains all elements $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ such that $v_1 \otimes v_2 \otimes \cdots \otimes v_n = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$ for every permutation σ of $\{1, 2, \ldots, n\}$. Then for any $g \in \mathfrak{g}$,

$$\begin{aligned} g(v_1 \otimes \cdots \otimes v_n) = (g \cdot v_1) \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes (g \cdot v_2) \otimes v_3 \otimes \cdots \otimes v_n \\ + \cdots + v_1 \otimes \cdots \otimes v_{n-1} \otimes (g \cdot v_n) \end{aligned}$$

Theorem 1.1.15 $S^n(L(\omega)) = L(n\omega)$

Proof. The module $L(\omega)$ of $\mathfrak{sl}_2$ has basis $\{v_1, v_{-1}\}$ where $v_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $v_{-1} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ with the action of $\mathfrak{sl}_2$ by (left) matrix multiplication. Then $S^n(L(\omega))$ is spanned by $v_1^{\otimes n}$, $v_1^{\otimes n-1} \otimes v_{-1}$, $v_1^{\otimes n-2} \otimes v_{-1}^{\otimes 2}$, $\ldots$, $v_{-1}^{\otimes n}$. Then consider $h = \left(\begin{smallmatrix}1 & 0\\0 & -1\end{smallmatrix}\right) \in \mathfrak{g}$,

$$\begin{aligned} h(v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m}) =& (h \cdot v_1) \otimes v_1^{\otimes n-m-1} \otimes v_{-1}^{\otimes m} + \cdots + v_1^{\otimes m-n-1} \otimes (h \cdot v_1) \otimes v_{-1}^{\otimes m} + \\ & v_1^{\otimes n-m} \otimes (h \cdot v_{-1}) \otimes v_{-1}^{\otimes m-1} + \cdots + v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m-1} (h \cdot v_{-1}) \\ =& (n-m)(v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m}) - m(v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m}) \\ =& (n-2m)(v_1^{\otimes n-m} \otimes v_{-1}^{\otimes m}) \end{aligned}$$

Then $S^n(L(\omega))$ has weights $n, n-2, \ldots, -n$. Therefore by Theorem 1.1.13 the result holds. ■

We shall now consider the general case. Let $\mathfrak{g}$ be a simple Lie algebra with root system Φ with basis $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and Cartan subalgebra $\mathfrak{h}$. Let V be a finite dimensional $\mathfrak{g}$-module and let $V_\mu = \{v \in V : h.v = \mu(h)v \ \forall h \in \mathfrak{h}\}$. Then we have $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$. When $V_\mu \neq 0$ we say V_μ is a *weight space* and μ is a *weight* of V. A non-zero vector $v \in V_\mu$ is a *maximal vector* of weight μ if $x.v = 0$ for all $x \in \mathfrak{g}_{\alpha_i}$, $\alpha_i \in \Delta$. The $\mathfrak{g}$-module V has at least one maximal vector v.
For a weight μ of V and $\alpha_i \in \Delta$ the *α-string through* μ is the maximal sequence of weights

$$\mu - r\alpha_i, \ldots, \mu, \ldots, \mu + q\alpha_i \text{ where } r - q = \langle \mu, \alpha_i \rangle.$$

Let V be an irreducible $\mathfrak{g}$-module with maximal vector v of weight μ. Then every maximal vector of V has weight μ. We say that μ is the *highest weight* of V. The submodule of V generated by a maximal vector of v is equal to V. A proof of the following result is presented in [Hum72, §21.1].

Theorem 1.1.16 *If V is a finite dimensional irreducible $\mathfrak{g}$-module of highest weight μ then $\mu(h_{\alpha_i})$ is a non-negative integer. In fact for any weight μ of V then $\mu(h_{\alpha_i}) = \langle \mu, \alpha_i \rangle \in \mathbb{N}_0$.*

Let $L(\mu)$ be the irreducible $\mathfrak{g}$-module with maximal vector v_μ in the weight space $L(\mu)_\mu$. A module of a Lie algebra $\mathfrak{g}$ is *faithful* if the corresponding map $\mathfrak{g} \to \mathfrak{gl}(V)$ is injective. Every non-trivial module of a simple Lie algebra is faithful. We can describe the minimal (dimensional) faithful modules for the exceptional types. Specifically $L(\omega_1)$ is the minimal faithful module of G_2 and has dimension 7. For F_4 this module is $L(\omega_4)$ of dimension 26. For E_6 we have $L(\omega_1)$ or $L(\omega_6)$ both of dimension 27. The minimal faithful module of E_7 is $L(\omega_7)$ of dimension 56 and finally the minimal faithful module of E_8 is $L(\omega_8)$ of dimension 248, this is the adjoint module of E_8.

1.2 Nilpotent Orbits

Let $\mathfrak{g} = Lie(G)$ be a simple Lie algebra. Then an element $e \in \mathfrak{g}$ is *ad-nilpotent* if the map $ad_e : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent endomorphism, specifically if $(ad_e)^m = 0$ for some $m > 0$. For example an element $x \in \mathfrak{sl}_n$ is nilpotent if and only if the m-th matrix power $x^m = 0$ for some integer m. The set of nilpotent elements of $\mathfrak{g}$, denoted $\mathcal{N}(\mathfrak{g})$ or simply $\mathcal{N}$, is invariant under the adjoint action of G. For a nilpotent element $e \in \mathfrak{g}$ the *nilpotent orbit of* e is

$$\mathcal{O}_e = \{Ad_y(e) : y \in G\}$$

Theorem 1.2.1 *Let $\mathfrak{g}$ be a simple Lie algebra. Then there are finitely many nilpotent orbits in $\mathfrak{g}$.*

Proof. When $\mathfrak{g} = \mathfrak{gl}_n$ then any nilpotent element is conjugate to an element in Jordan normal form which has Jordan blocks of the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Nilpotent elements with the same Jordan normal form (up to re-ordering the blocks) are in the same nilpotent orbit. Since there are finitely many possible Jordan normal forms of this type then there are finitely many nilpotent orbits of $\mathfrak{gl}_n$. The method for the other classical types are similar, see §1.3. For the exceptional types we can use the Bala-Carter theorem to find all the nilpotent orbits of $\mathfrak{g}$, this is described in §1.5. For more details see [Ric67, Theorem 8.2]. ■

For a Lie algebra $\mathfrak{g}$ then $\{e, f, h\} \subset \mathfrak{g}$ is an $\mathfrak{sl}_2$-*triple* if

$$[h, e] = 2e, \quad [h, f] = -2f, \quad \text{and} \quad [e, f] = h.$$

Theorem 1.2.2 (Jacobson-Morozov Theorem) *Let $\mathfrak{g}$ be a simple Lie algebra and let $e \in \mathfrak{g}$ be a nilpotent element. Then there is an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ in $\mathfrak{g}$.*

Proof. The following method constructs an $\mathfrak{sl}_2$-triple for a given nilpotent element e in $\mathfrak{gl}_n$. Suppose e has a single Jordan block of size n. Then we can form an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ where

$$f = \begin{pmatrix} 0 & & & & \\ \mu_1 & 0 & & & \\ & \mu_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \mu_{n-1} & 0 \end{pmatrix}, \quad h = \begin{pmatrix} n-1 & & & & 0 \\ & n-3 & & & \\ & & n-5 & & \\ & & & \ddots & \\ 0 & & & & -(n-1) \end{pmatrix}$$

where $\mu_i = i(n-1)$ for $i \in \{1, \dots, n-1\}$.

If e has multiple Jordan blocks $e_1, \dots e_r$ where e_i is a $\lambda_i \times \lambda_i$ Jordan block, then e has the form

$$e = \begin{pmatrix} e_1 & & & \\ & e_2 & & \\ & & \ddots & \\ & & & e_r \end{pmatrix}$$

The block diagonal matrices

$$f = \begin{pmatrix} f_1 & & & \\ & f_2 & & \\ & & \ddots & \\ & & & f_r \end{pmatrix} \text{ and } h = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_r \end{pmatrix}$$

form an $\mathfrak{sl}_2$-triple if $\{e_i, f_i, h_i\}$ is an $\mathfrak{sl}_2$-triple for each i. Therefore every nilpotent element of $\mathfrak{gl}_n$ can be embedded in an $\mathfrak{sl}_2$-triple by Theorem 1.2.5 given below. A similar construction can be found for the other Lie algebras. For more details see [Car85, Theorem 5.3.2]. ∎

Example 1.2.3

For $\mathfrak{g} = \mathfrak{gl}_5$ and $e = \left(\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & \multicolumn{2}{c}{0} \\ & & 0 & & \\ \hline & & & 0 & 1 \\ & 0 & & & 0 \end{array}\right)$ we can form an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ as follows:

$$f = \left(\begin{array}{ccc|cc} 0 & & & & \\ 2 & 0 & & \multicolumn{2}{c}{0} \\ & 2 & 0 & & \\ \hline & & & 0 & \\ & 0 & & 1 & 0 \end{array}\right), \quad h = \left(\begin{array}{ccc|cc} 2 & & & & \\ & 0 & & & \\ & & -2 & & \\ \hline & & & 1 & \\ & & & & -1 \end{array}\right).$$

For a nilpotent element $e \in \mathfrak{g}$ with $\mathfrak{sl}_2$-triple $\{e, f, h\}$ and for $\xi \in k$, let

$$\mathfrak{g}(\xi, h) = \{x \in \mathfrak{g} : [h, x] = \xi x\}.$$

Clearly $e \in \mathfrak{g}(2, h)$ and by the Jacobi identity we have $[\mathfrak{g}(\xi), \mathfrak{g}(\xi')] \subset \mathfrak{g}(\xi + \xi')$. The following theorem means we can express $\mathfrak{g}$ as the direct sum of these eigenspaces. For more details refer to [CM93, §3.4].

Theorem 1.2.4 *Let $\mathfrak{g}$ be a simple Lie algebra and let $\{e, f, h\} \subset \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple. Then*

$$\mathfrak{g} = \bigoplus_{\xi \in \mathbb{Z}} \mathfrak{g}(\xi, h)$$
$$\mathfrak{g}^e = \bigoplus_{\xi \in \mathbb{N}_0} \mathfrak{g}^e(\xi, h)$$

Where $\mathfrak{g}^e(\xi, h) = \mathfrak{g}^e \cap \mathfrak{g}(\xi, h)$.

Theorem 1.2.5 *Let $\mathfrak{g}$ be a simple Lie algebra and let $e \in \mathfrak{g}$ be a nilpotent element. Then any two $\mathfrak{sl}_2$-triples containing e are G^e-conjugate.*

Proof. Suppose there are two $\mathfrak{sl}_2$-triples $\{e, f, h\}$ and $\{e, f', h'\}$. Then

$$[h' - h, e] = [h', e] - [h, e] = 0$$

So $h' - h \in \mathfrak{g}^e$. Also $[e, f' - f] = h' - h$ so $h' - h \in [e, \mathfrak{g}]$, therefore by [CM93, Lemma 3.4.7] there exists an $x \in G^e$ such that $x \cdot h = h'$ and $x \cdot e = e$. Similarly $x \cdot f - f' \in \mathfrak{g}^e$ since $[e, x \cdot f - f'] = [x \cdot e, x \cdot f] - [e, f'] = x \cdot h - h' = 0$. Then

$$\begin{aligned}
[h', x \cdot f - f'] &= [h', x \cdot f] - [h', f'] \\
&= x \cdot [h, f] - [h', f'] \\
&= -2(x \cdot f - f')
\end{aligned}$$

By Theorem 1.2.4 we must have $x \cdot f - f' = 0$. Therefore $\{e, f, h\}$ and $\{e, f', h'\}$ are G^e-conjugate. ■

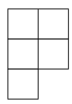
1.3 Partitions

As we have seen in the previous section a nilpotent orbit in $\mathfrak{gl}_n$ is completely determined by the sizes of its Jordan blocks. Therefore we can associate a partition of n with each nilpotent orbit and vice-versa where a *partition of* n is a sequence of integers $[\lambda_1, \lambda_2, \ldots, \lambda_m]$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$. Since a classical simple group G embeds in some GL_n there is a map from the set of nilpotent orbits in $\mathfrak{g}$ to partitions of n. This map turns out to be injective if $G = SL_n, Sp_{2n}$ or O_n.
If $\mathfrak{g} = \mathfrak{sl}_n$ then the map between nilpotent orbits of $\mathfrak{sl}_n$ and partitions of n is bijective. For $\mathfrak{sp}_{2n}$ the map from nilpotent orbits to partitions of $2n$ is injective but not surjective. The partitions of $2n$ in the image of this map are those in which odd parts occur with even multiplicity. For example $[2, 1, 1]$ corresponds to a nilpotent orbit in $\mathfrak{sp}_4$ but $[3, 1]$ does not.

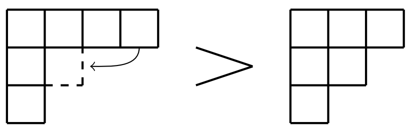
Finally if $\mathfrak{g} = \mathfrak{so}_n$ then the partitions of n in the image of this map are those where even parts occur with even multiplicity. For example $[4, 4]$ corresponds to a nilpotent orbit in $\mathfrak{so}_8$ but $[6, 2]$ does not. However for $\mathfrak{so}_{2n}$ the *very even partitions*, those which only have even parts, correspond to two nilpotent orbits for the action of SO_{2n} but only a single O_{2n}-orbit. For example for $\mathfrak{so}_8$ there are ten partitions of 8 where the even parts occur with even multiplicity. The very even partitions of 8 are $[4, 4]$ and $[2, 2, 2, 2]$, so $\mathfrak{so}_8$ has twelve nilpotent orbits. For these statements above we refer to [CM93].
Given two partitions $[p_1, \ldots, p_r]$ and $[q_1, \ldots, q_s]$ of n. If $s \geq r$ then let $p_{r+1}, \ldots, p_s = 0$ or vice versa. Then we define $[p_1, \ldots, p_r] \geq [q_1, \ldots, q_s]$ if

$$
\begin{aligned}
p_1 &\geq q_1 \\
p_1 + p_2 &\geq q_1 + q_2 \\
&\vdots \\
p_1 + p_2 + \cdots + p_s &\geq q_1 + q_2 + \cdots + q_s
\end{aligned}
$$

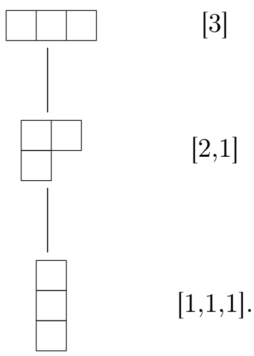
This is known as the *dominance ordering* on partitions. A partition $[\lambda_1, \ldots, \lambda_m]$ of n can be represented by a Young diagram. A *Young diagram* is an arrangement of blocks with λ_i blocks in the i-th row. For example the partition $[2, 2, 1]$ for $n = 5$ has the following Young diagram.



For two partitions λ and μ, then $\lambda \geq \mu$ if the Young diagram of μ can be obtained from that of λ by moving some blocks downwards. For example $[4, 1, 1] \geq [3, 2, 1]$ since



The ordering of partitions of n can be represented via a Hasse diagram in which the relation $\lambda > \mu$ is represented by a series of edges connecting λ downwards to μ. For example the partitions of 3 are $[3] > [2, 1] > [1, 1, 1]$. Therefore the corresponding Hasse diagram is



Let e and e' be nilpotent elements with corresponding partitions λ and μ. Then $\lambda \geq \mu$ if and only if $\overline{G.e} \supset \overline{G.e'}$ [CM93, Theorem 6.2.5]. Therefore the closure of a nilpotent orbit is contained in the closure of the nilpotent orbit which is connected to it above in the Hasse diagram.

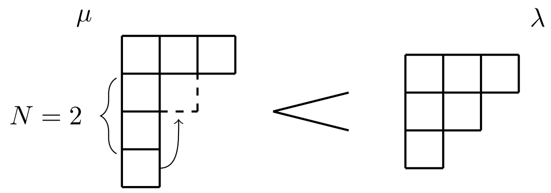
For the classical Lie algebras the dimension of a nilpotent orbit can be calculated using the corresponding partition. Consider the orbit $\mathcal{O}_e$ in $\mathfrak{g}$ which corresponds to the partition $[\lambda_1, \lambda_2, \ldots, \lambda_m]$ of n. Let r_i be the number of rows of length i in the Young diagram and let s_i be the number of rows of length greater than or equal to i. Then it is shown in [CM93] that

$$\dim(\mathcal{O}_e) = \begin{cases} n^2 - \sum_i s_i^2 & \text{if } \mathfrak{g} = \mathfrak{sl}_n \\ 2n^2 + n - \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i\ odd} r_i & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1} \\ 2n^2 + n - \frac{1}{2}\sum_i s_i^2 - \frac{1}{2}\sum_{i\ odd} r_i & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n} \\ 2n^2 - n - \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i\ odd} r_i & \text{if } \mathfrak{g} = \mathfrak{so}_{2n} \end{cases}$$

Example 1.3.1

Consider the orbit corresponding to $[4, 1^2]$ in $\mathfrak{sp}_6$. Then $r_1 = 2$, $r_2 = r_3 = 0$ and $r_4 = 1$ so $\sum\limits_{i\ odd} r_i = 2$. Also $s_1 = 3$ and $s_2 = s_3 = s_4 = 1$ so $\sum\limits_i s_i^2 = 3^2 + 1 + 1 + 1 = 12$. Therefore the dimension of this orbit is $2n^2 + n - \frac{1}{2}\sum\limits_i s_i^2 - \frac{1}{2}\sum\limits_{i\ odd} r_i = 18 + 3 - 6 - 1 = 14$.

Alternatively for $\mathfrak{g} = \mathfrak{sl}_n$ we can inductively calculate the dimensions of the nilpotent orbits by considering Young diagrams. Consider two Young diagrams λ and μ which are adjacent in the dominance ordering such that $\lambda > \mu$. Then let N be the number of rows a block moves in transforming λ to μ. Then the dimension of the nilpotent orbit corresponding to λ is the dimension of the orbit corresponding to μ plus $2N$. This can be seen from the fact that we have replaced an adjacent pair s_i and $s_{i+1} = s_i - N - 1$ of μ by $s_i - 1$ and $s_i - N$. For example



In this case the dimension of the nilpotent orbit corresponding to μ is 18. Therefore $dim(\lambda)$ is equal to $dim(\mu) + 2N = 22$.

1.4 Weighted Dynkin Diagrams

Another method for classifying nilpotent orbits in $\mathfrak{g}$ is by a unique labelling of the nodes of the corresponding Dynkin diagram.

In the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$, the set of diagonal matrices with trace zero forms a Cartan subalgebra of $\mathfrak{g}$. Let this subalgebra be denoted by $\mathfrak{h}$. Then for a given nilpotent element e there is an $\mathfrak{sl}_2$-triple where we may assume, after conjugating if necessary, that $h \in \mathfrak{h}$. Let $\widetilde{h}$ have the same diagonal entries as h but re-ordered so that they decrease from top left to bottom right. The *weighted Dynkin diagram* is produced by labelling the node of the Dynkin diagram of $\mathfrak{sl}_n$ corresponding to α_i by $\alpha_i(\widetilde{h}) = \lambda_i$ (where $[\widetilde{h}, e_{\alpha_i}] = \lambda_i e_{\alpha_i}$).

Example 1.4.1

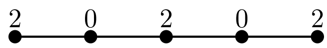
Let $\mathfrak{g} = \mathfrak{sl}_6$ and consider the nilpotent element e with partition $[4, 2]$. Then we can form an $\mathfrak{sl}_2$-triple with h of the form

$$h = \left(\begin{array}{cccc|cc} 3 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & -3 & & \\ \hline & & & & 1 & \\ & & & & & -1 \end{array}\right) \text{ so } \tilde{h} = \begin{pmatrix} 3 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -3 \end{pmatrix}.$$

Then

$$\begin{aligned} [\widetilde{h}, e_{\alpha_1}] &= (3-1)e_{\alpha_1} = 2e_{\alpha_1} \\ [\widetilde{h}, e_{\alpha_2}] &= (1-1)e_{\alpha_2} = 0e_{\alpha_2} \\ &\vdots \\ [\widetilde{h}, e_{\alpha_5}] &= (-1-(-3))e_{\alpha_5} = 2e_{\alpha_5} \end{aligned}$$

Therefore the weighted Dynkin diagram corresponding to the nilpotent orbit e is



For an arbitrary simple Lie algebra $\mathfrak{g}$ we have a choice of Cartan subalgebra $\mathfrak{h}$. For a nilpotent element e there is an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ such that, after conjugation if necessary, $h \in \mathfrak{h}$. Then we can apply $w \in \mathcal{W}$ to h giving an element $\widetilde{h}$ in the closure of the positive Weyl chamber, i.e. such that $\alpha_i(\widetilde{h}) \geq 0$ for all i. Then the node on the Dynkin diagram of $\mathfrak{g}$ corresponding to α_i is labelled by $\alpha_i(\widetilde{h})$. A proof for the next result can be found in [BC76a].

Theorem 1.4.2 *Let e, e' be nilpotent elements in a simple Lie algebra $\mathfrak{g}$. Then the weighted Dynkin diagrams of e and e' are the same if and only if e and e' are conjugate. Therefore there is a unique weighted Dynkin diagram for each nilpotent orbit in $\mathfrak{g}$.*

It was shown by Dynkin that $\alpha_i(\tilde{h}) = \{0, 1, 2\}$ (see [Car85, Proposition 5.6.6]). However not every possible way of labelling a Dynkin diagram with $0, 1, 2$ corresponds to a nilpotent orbit.

1.5 Bala-Carter Theorem

The final method for classifying nilpotent orbits is via the Bala-Carter theorem. This method utilizes distinguished nilpotent orbits. A nilpotent element e in a simple Lie algebra $\mathfrak{g}$ is *distinguished* if e does not commute with any non-zero semisimple element of $\mathfrak{g}$. For example a nilpotent element $e \in \mathfrak{sl}_n$ is distinguished if it is regular (an $n \times n$ matrix A is regular if $\dim(\mathfrak{g}^A) = n - 1$).

Let $\mathfrak{h}^T$ be a Cartan subalgebra of $\mathfrak{g}$ corresponding to a maximal torus T of G. Let Δ be a basis for the root system Φ of $\mathfrak{g}$ and for a subset $I \subset \Delta$ let $\Phi_I = \mathbb{Z}I \cap \Phi$. Now let $\mathfrak{p}_I$ be

$$\mathfrak{p}_I = \mathfrak{h}_T \oplus \sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

Clearly $\mathfrak{p}_I \supset \mathfrak{b} = \mathfrak{h}_T \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. The subalgebras $\mathfrak{p}_I$ with $I \subset \Delta$ are the *standard parabolic subalgebras.*

Example 1.5.1

Let $\mathfrak{g} = \mathfrak{gl}_3$, $\Delta = \{\alpha_1, \alpha_2\}$ and $\mathfrak{h}_T = \left\{ t = \left(\begin{smallmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{smallmatrix} \right) : t_i \in k \right\}$. Then $[t, e_{\alpha_1}] = (t_1 - t_2)e_{\alpha_1}$ so $\alpha_1(t) = t_1 - t_2$ and similarly $\alpha_2(t) = t_2 - t_3$. For $I = \{\alpha_1\}$ then $\Phi_I = \{\alpha_1, -\alpha_1\}$ so $\mathfrak{p}_I = \mathfrak{h}_T \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$. This gives

$$\mathfrak{p}_I = \left\{ \begin{pmatrix} t_1 & a & b \\ d & t_2 & c \\ 0 & 0 & t_3 \end{pmatrix} : t_i, a, b, c, d \in k \right\}$$

Proposition 1.5.2 *[CM93, Lemma 3.8.1] Let G be a simple algebraic group and let $\mathfrak{g} = Lie(G)$. Then any parabolic subalgebra of $\mathfrak{g}$ is conjugate to at least one standard parabolic $\mathfrak{p}_I$ for some $I \subset \Delta$.*

Let $\mathfrak{g}$ be a Lie algebra with basis Δ and let $I \subset \Delta$. Then the *standard Levi subalgebra* and *unipotent radical* of $\mathfrak{p}_I$ are respectively

$$\mathfrak{l}_I = \mathfrak{h}_T \oplus \sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$$
$$\mathfrak{u}_I = \sum_{\alpha \in \Phi^+ \setminus \Phi_I} \mathfrak{g}_\alpha.$$

Then $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$. Consider the $\mathfrak{p}_I$ in Example 1.5.1 then $\mathfrak{l}_I = \left\{ \left(\begin{smallmatrix} t_1 & a & 0 \\ d & t_2 & 0 \\ 0 & 0 & t_3 \end{smallmatrix} \right) : a, d, t_i \in k \right\}$ and $\mathfrak{u}_I = \left\{ \left(\begin{smallmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{smallmatrix} \right) : b, c \in k \right\}$. Then $\mathfrak{l}_I \cong \mathfrak{gl}_2 \oplus k$ is reductive. We can now state the first Bala-Carter Theorem from [BC76b, Theorem 6.1].

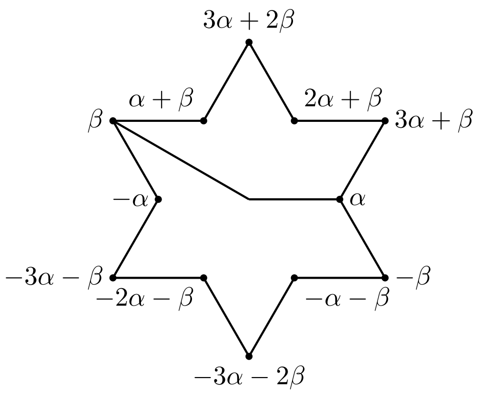
Theorem 1.5.3 (Bala-Carter Theorem I) *Any nilpotent element of a Lie algebra $\mathfrak{g}$ is conjugate to a distinguished nilpotent element of some standard Levi subalgebra of $\mathfrak{g}$.*

For example consider $\mathfrak{g} = \mathfrak{gl}_3$ as above; then the nilpotent element $\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ is distinguished in $\mathfrak{l}_{\{\alpha_1\}}$.

For $I \subset \Delta$ there is an associated grading of $\mathfrak{g}$. Let $\mathfrak{g}(I;0) = \mathfrak{l}_I$ and let $\mathfrak{g}(I;2m)$ be spanned by all $\mathfrak{g}_\alpha$ where $\alpha = \sum_i a_i \alpha_i$ and $\sum\limits_{\alpha_i \in \Delta \setminus I} a_i = m$.

Example 1.5.4

Let $\mathfrak{g} = G_2$, $\Delta = \{\alpha, \beta\}$ and let $I = \{\alpha\}$. Then $\mathfrak{g}_\alpha \subset \mathfrak{g}(I;0)$ and $\mathfrak{g}_\beta \subset \mathfrak{g}(I;2)$. Therefore $\mathfrak{g}_{\alpha+\beta}$ has degree 2, $\mathfrak{g}_{3\alpha+2\beta}$ has degree 4 etc. We can represent the roots of G_2 as in Figure 1.2.



Then the grading corresponding to $I = \{\alpha\}$ can be represented as

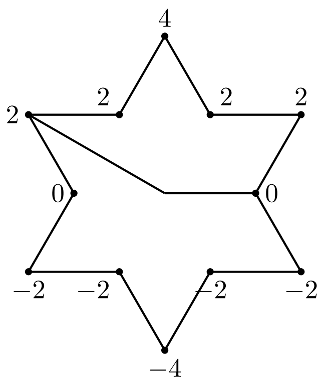


Figure 1.2: Positive Roots of G_2

Since $\mathfrak{g} = \bigoplus\limits_{i\in\mathbb{Z}} \mathfrak{g}(I;i)$ and $[\mathfrak{g}(I;i), \mathfrak{g}(I;j)] \subset \mathfrak{g}(I;i+j)$, then $\mathfrak{p}_I = \sum_{i\geq 0} \mathfrak{g}(I;i)$, $\mathfrak{l}_I = \mathfrak{g}(I;0)$ and $\mathfrak{u}_I = \sum_{I;i>0} \mathfrak{g}(i)$. A nilpotent orbit $\mathcal{O}_e$ is distinguished in $\mathfrak{g}$ if the only Levi subalgebra of $\mathfrak{g}$ containing e is $\mathfrak{g}$ itself. Equivalently, e is distinguished if it does not commute with any non-zero semisimple elements of $[\mathfrak{g}, \mathfrak{g}]$. For a simple Lie algebra $\mathfrak{g}$, $\mathfrak{p}_I$ is a *distinguished parabolic* if $\dim \mathfrak{g}(I;2) = \dim \mathfrak{g}(I;0)$ in the grading corresponding to I. The second Bala-Carter Theorem from [BC76a] is as follows.

Theorem 1.5.5 (Bala-Carter Theorem II) *Let $\mathfrak{g}$ be a simple Lie algebra. Then there is a bijective map between distinguished nilpotent orbits in $\mathfrak{g}$ and distinguished parabolic subalgebras of $\mathfrak{g}$ up to conjugacy given by*

$$\mathfrak{p}_I \mapsto G \cdot e_I$$

where e_I is a nilpotent element contained in the dense P_I orbit on $\mathfrak{u}_I$.

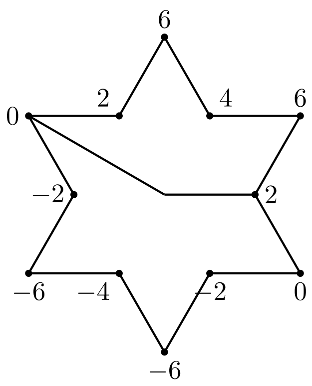
Example 1.5.6

When $\mathfrak{g} = G_2$, there are four standard parabolics. These are $\mathfrak{g}$, $\mathfrak{p}_\alpha$, $\mathfrak{p}_\beta$ and $\mathfrak{p}_\emptyset = \mathfrak{b}$. For the proper parabolic subgroups the regular nilpotent orbits of $\mathfrak{l}_\alpha$, $\mathfrak{l}_\beta$ and $\mathfrak{l}_\emptyset$ correspond to three nilpotent orbits in G_2 which we denote A_1, $\widetilde{A_1}$ and 0 respectively. Therefore all that remains is to find the distinguished parabolic subalgebras of G_2. Now $\mathfrak{g}$ is not a distinguished parabolic because $\dim \mathfrak{g}(\{\alpha,\beta\};2) = 0$ and $\dim \mathfrak{g}(\{\alpha,\beta\};0) = \dim(\mathfrak{g})$. Alternatively $\mathfrak{b}$ is a distinguished parabolic because $\mathfrak{g}(\emptyset;2) = \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$. In fact for any semisimple $\mathfrak{g}$ we always have that $\mathfrak{b}$ is a distinguished parabolic and $\mathfrak{g}$ is not. Therefore it only remains to check $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$.

First consider $\mathfrak{p}_\alpha$ then by the previous example we have

$$\mathfrak{g}(\{\alpha\};0) = \mathfrak{h}_T \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$
$$\mathfrak{g}(\{\alpha\};2) = \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+\beta}.$$

Therefore $\dim \mathfrak{g}(\{\alpha\};2) = \dim \mathfrak{g}(\{\alpha\};0) = 4$ so $\mathfrak{p}_\alpha$ is distinguished. Finally for $\mathfrak{p}_\beta$ the associated grading is indicated in the following diagram:

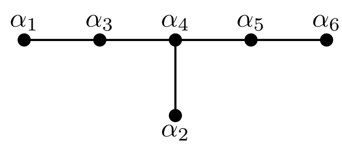


So

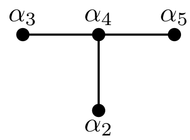
$$\mathfrak{g}(\{\beta\};2) = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta}$$
$$\mathfrak{g}(\{\beta\};0) = \mathfrak{h}_T \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_\beta$$

Since $\dim \mathfrak{g}(\{\beta\};0) = 4$ and $\dim \mathfrak{g}(\{\beta\};2) = 2$ then $\mathfrak{p}_\beta$ is not distinguished. Therefore G_2 has two distinguished nilpotent orbits. The Borel subalgebra $\mathfrak{b}$ always corresponds to the regular nilpotent orbit in $\mathfrak{g}$. The non-regular distinguished nilpotent orbit in G_2 associated to $\mathfrak{p}_\alpha$ is labelled $G_2(a_1)$ and has representative $e_\alpha + e_{2\alpha+\beta}$. Therefore G_2 has five nilpotent orbits.

This gives us another method for labelling nilpotent orbits in $\mathfrak{g}$, by considering $I \subset \Delta$. For example if $\mathfrak{g} = E_6$ with $\Delta = \{\alpha_1, \ldots, \alpha_6\}$. The Dynkin diagram of $\mathfrak{g}$ is



Now $I = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ is of type D_4 which can be seen by the sub-diagram of the Dynkin diagram



The nilpotent orbit corresponding to the regular nilpotent orbit of $\mathfrak{l}_I$ is labelled D_4. The orbit which is subregular in the Levi subalgebra of type D_4 is denoted $D_4(a_1)$. (In $\mathfrak{so}_8$ the regular orbit has partition $[7,1]$ and the subregular has partition $[5,3]$). Similarly for $\mathfrak{g} = F_4$ the regular nilpotent elements in $\mathfrak{l}_{\{\alpha_1,\alpha_2\}}$ and $\mathfrak{l}_{\{\alpha_3,\alpha_4\}}$ correspond to different nilpotent orbits in F_4. Since both are of type A_2 then the label of the orbit corresponding to $\mathfrak{l}_{\{\alpha_1,\alpha_2\}}$ is denoted as A_2 and $\mathfrak{l}_{\{\alpha_3,\alpha_4\}}$ as $\widetilde{A_2}$, since $\{\alpha_1,\alpha_2\}$ are long roots and $\{\alpha_3,\alpha_4\}$ are short roots.
For example F_4 has the following standard Levi subalgebras up to conjugacy (with the number of distinguished orbits in parentheses):

$$\begin{aligned}
&\emptyset(1)\\
&A_1(1), \widetilde{A_1}(1)\\
&A_2(1), A_1 + \widetilde{A_1}(1), \widetilde{A_2}(1), B_2(1)\\
&B_3(1), A_2 + \widetilde{A_1}(1), A_1 + \widetilde{A_2}(1), C_3(2)\\
&F_4(4)
\end{aligned}$$

Therefore there are 16 nilpotent orbits in F_4.

Chapter 2

Classification of Nilpotent Orbits in Positive Characteristic

2.1 Preliminaries

In this chapter we discuss the positive characteristic analogues of the classifications of nilpotent orbits which were shown in the previous chapter. This is followed by an introduction to transverse slices.

Let G be a connected reductive algebraic group over an algebraically closed field k of characteristic $p > 0$. Then let $\mathfrak{g} = Lie(G)$. The examples of simple algebraic groups in characteristic zero are still simple in positive characteristic p. However, the Lie algebra of a simple algebraic group need not be simple. The Lie algebra $\mathfrak{sl}_n$ is not simple when $p|n$; all other classical Lie algebras are simple if $p > 2$. Similarly the exceptional type Lie algebras are simple if $p > 3$. Also there are new simple Lie algebras that arise which we do not consider (see [BW84]). Most of Section 1.1 still holds in positive characteristic, however some definitions need to change.

The definitions of a Borel subgroup, parabolic subgroup and maximal torus are the same as the definitions given in Chapter 1. However a *Borel subalgebra* of $\mathfrak{g}$ is defined to be a subalgebra $\mathfrak{b}$ where $\mathfrak{b} = Lie(B)$ for some Borel subgroup B of G. Unlike when the characteristic is zero, a Borel subalgebra is not necessarily a maximal solvable subalgebra of $\mathfrak{g}$. A standard example is a Borel subalgebra in $\mathfrak{sl}_2$ when $p = 2$. A *parabolic subalgebra* $\mathfrak{p}$ and *torus* $\mathfrak{t}$ of $\mathfrak{g}$ are defined similarly. Specifically $\mathfrak{p} = Lie(P)$ (resp. $\mathfrak{t} = Lie(T)$) for some parabolic subgroup P of G (resp. for some maximal torus T of G). All of the statements about conjugacy of Borel subgroups and Cartan subalgebras now hold.

Roots of Simple Lie Algebras

In positive characteristic the root system is generally defined in relation to a maximal torus T in G. Let $X(T)$ be the set of morphisms $\lambda : T \to k^\times$. In $X(T)$ we use additive notation, therefore for $\lambda, \mu \in X(T)$ then $(\lambda + \mu)(t) = \lambda(t)\mu(t)$ for $t \in T$. Now let $\mathfrak{g}_\lambda = \{g \in \mathfrak{g} : Ad_t(g) = \lambda(t)g \ \forall t \in T\}$. Then the *root system* Φ of $\mathfrak{g}$ is the set of non-trivial $\alpha \in X(T)$ such that $\mathfrak{g}_\alpha$ is not trivial. Any element $\alpha \in \Phi$ is a *root* of $\mathfrak{g}$. Then we have that

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \text{ where } \mathfrak{t} = Lie(T)$$

When G is semisimple and simply connected (defined in the next section) then a Chevalley basis $\{e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} : \alpha_i \in \Phi\}$ still exists and is defined in the same way as for characteristic zero [BGP09, §2.2]. A proof of the following result is presented in [Hum75, §26.3].

Theorem 2.1.1 *Let G be a reductive group with a maximal torus T and let $\alpha \in \Phi$.*

(i) There exists a unique connected T-stable subgroup U_α of G with $Lie(U_\alpha) = \mathfrak{g}_\alpha$.

(ii) G is generated by T and the subgroups U_α, where α runs over all elements of Φ.

For a root $\alpha \in \Phi$ there exists an isomorphism $\mathcal{E}_\alpha : k \to U_\alpha$ such that $t(\mathcal{E}_\alpha(\xi))t^{-1} = \mathcal{E}_\alpha(\alpha(t)\xi)$ and $d\mathcal{E}_\alpha|_0(1) = e_\alpha$ where $d\mathcal{E}_\alpha|_0(1)$ denotes the differential of the morphism $\mathcal{E}_\alpha$ at 0 evaluated at 1. If we consider $G = SL_n$ then $\mathcal{E}_\alpha(\xi) = I + \xi e_\alpha$ for $\xi \in k$.

The definition of the Weyl group is the same as in characteristic zero, specifically $\mathcal{W} = N_G(T)/T$. Let n_α be the map given by $\mathcal{E}_\alpha(1)\mathcal{E}_{-\alpha}(-1)\mathcal{E}_\alpha(1)$, which is an element of $N_G(T)$. This is a representative of the element s_α in $\mathcal{W}$ which reflects elements in the hyperplane P_α, in particular $s_\alpha = n_\alpha T$. To see this observe that the subgroup of G generated by $U_\alpha, U_{-\alpha}$ is isomorphic to either SL_2 or PGL_2. Then the result is given by the following matrix calculation.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Highest Weight Modules

Let G be a reductive algebraic group with a maximal torus T and simple roots $\Delta = \{\alpha_1, \dots, \alpha_m\}$. The roots $\alpha \in \Phi$ give rise to *coroots* $\alpha^\vee : k^\times \to T$ such that $d\alpha^\vee|_1(1) = h_\alpha$. Let $\Phi^\vee$ be the set of coroots of G.

Example 2.1.2

Let $G = SL_3$ with maximal torus $T = \left\{ \begin{pmatrix} t & & 0 \\ & s & \\ 0 & & (st)^{-1} \end{pmatrix} : s, t \in k^\times \right\}$ and simple roots $\alpha_1, \alpha_2 \in \Phi$. Then

$$\alpha_1^\vee(t) = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{pmatrix} \qquad \alpha_2^\vee(t) = \begin{pmatrix} 1 & & \\ & t & \\ & & t^{-1} \end{pmatrix}$$

Now let $Y(T)$ be the set of morphisms $\phi : k^\times \to T$ then $a_1\alpha_1^\vee + \cdots + a_m\alpha_m^\vee \in Y(T)$ for $a_i \in \mathbb{Z}$ where $(a_1\alpha_1^\vee + \cdots + a_m\alpha_m^\vee)(t) = \alpha_1^\vee(t)^{a_1}\alpha_2^\vee(t)^{a_2}\dots\alpha_m^\vee(t)^{a_m}$. If all the elements of $Y(T)$ have this form then $Y(T) = \mathbb{Z}\Phi^\vee$ and so G is (semisimple and) *simply connected.*

Examples 2.1.3

1. Let $G = SL_2$ with maximal torus $T = \left\{\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right) : t \in k^\times\right\}$ and $\alpha^\vee = \left(\begin{smallmatrix} t & \\ & t^{-1}\end{smallmatrix}\right)$. Now $\mathbb{Z} \cong Y(T)$ where

$$a \mapsto \begin{pmatrix} t^a & 0 \\ 0 & t^{-a}\end{pmatrix}.$$

   We have $Y(T) = \mathbb{Z}\alpha^\vee$ where α is the unique positive root, hence G is simply connected.

2. Let $G = SO_3$ with maximal torus $T = \left\{\begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1}\end{pmatrix} : t \in k^\times\right\}$. Now $\alpha^\vee(t) = \begin{pmatrix} t^2 & & \\ & 1 & \\ & & t^{-2}\end{pmatrix}$ so $Y(T) = \mathbb{Z}\left(\frac{1}{2}\alpha^\vee\right)$. Therefore SO_3 is not simply connected.

A finite dimensional G-module V is *rational* if $\rho : G \to GL(V)$ is a morphism of algebraic groups. From now on all G-modules are assumed to be rational. Then for a finite dimensional G-module V

$$V = \bigoplus_{\mu \in X(T)} V_\mu \text{ where } V_\mu = \{v \in V : \rho(t)v = \mu(t)v \ \ \forall t \in T\}$$

If V_μ is non-trivial then μ is a *weight* of V and V_μ is a *weight space*. For every $\alpha_i \in \Delta$, let $\omega_i \in X(T)$ be such that $\omega_i(\alpha_j^\vee) = \delta_{ij}$ Then every weight $\mu \in X(T)$ we can express as

$$\mu = \langle \mu, \alpha_1^\vee\rangle\omega_1 + \langle \mu, \alpha_2^\vee\rangle\omega_2 + \cdots + \langle \mu, \alpha_m^\vee\rangle\omega_m$$

A weight μ is *dominant* if $\langle \mu, \alpha_i^\vee\rangle \geq 0$ for all i.

A non-zero vector $v \in V_\mu$ for some weight μ is a *maximal vector of weight* μ if it is fixed by all $\rho(U_{\alpha_i})$. Given a Borel subgroup B of G, if V is irreducible then there is a unique B-stable 1-dimensional subspace spanned by a maximal vector with dominant weight μ. We say μ is the *highest weight of* V. All the other weights of V are of the form $\mu - \sum c_i\alpha_i$ for $\alpha_i \in \Phi^+, c_i \in \mathbb{Z}^+$. An irreducible G-module V' of highest weight μ' is isomorphic to V if and only if $\mu = \mu'$. For every dominant weight $\mu \in X(T)$ there exists an irreducible G-module with highest weight μ. For more details and other standard results refer to [Hum75, §31].

Let $L(\mu)$ denote the unique irreducible G-module with a maximal vector v_μ in the weight space $L(\mu)_\mu$. Any rational G-module is also a $\mathfrak{g}$-module by differentiation of the morphism $G \to GL(V)$. However an irreducible G-module is not necessarily irreducible as a $\mathfrak{g}$-module. This leads to the next result, for details see [Jan03, Chap II §3.15].

Theorem 2.1.4 *For a G-module $L(\mu)$ then $L(\mu)|_{\mathfrak{g}}$ is simple if $\langle \mu, \alpha_i^\vee\rangle \in \{0, \dots, p-1\}$ for $1 \leq i \leq m$.*

Example 2.1.5

For $n < p$, the $\mathfrak{sl}_2$-module $L(n\omega)$ is simple as a $\mathfrak{g}$-module.

$$L(n\omega) = S^n(V) = \langle w_1^{\otimes m} \otimes w_{-1}^{\otimes n-m} : 0 \leq m \leq n \rangle \text{ where } w_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \text{ and } w_{-1} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right).$$

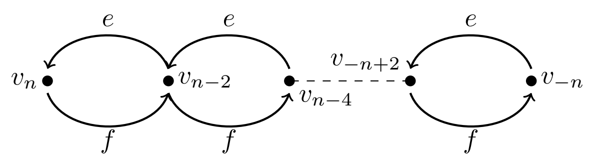
Then

$$\begin{aligned} e \cdot (w_1^{\otimes m} \otimes w_{-1}^{\otimes(n-m)}) &= (e \cdot w_1) \otimes w_1^{\otimes(m-1)} + \cdots + w_1^{\otimes m} \otimes w_{-1}^{\otimes(n-m-1)} \otimes (e \cdot w_{-1}) \\ &= (n-m)(w_1^{\otimes(m+1)} \otimes w_{-1}^{\otimes(n-m-1)}) \end{aligned}$$

Therefore if we let $v_{2m-n} = w_1^{\otimes m} \otimes w_{-1}^{\otimes(n-m)}$; then $L(n\omega) = \langle v_n, v_{n-2}, \ldots, v_{-n} \rangle$ where

$$e \cdot v_{2m-n} = (n-m)v_{2m-n+2} \qquad\qquad f \cdot v_{2m-n} = m v_{2m-n-2}.$$

We can depict this by the following diagram



If $n \geq p$ then $L(n\omega)$ is not simple over $\mathfrak{g}$. For example if $n = p$ then $e \cdot v_{-p} = 0$ so kv_{-p} is a (trivial) submodule of $L(p\omega)$

Let V be an irreducible G-module with highest weight μ. The length of the α-chain of weights in V given by $\{\mu, \mu-\alpha, \mu-2\alpha, \ldots\}$ is $\langle \mu, \alpha^\vee \rangle$. For example if $\mu = \omega_i$ is the highest weight of V then $\mu - \alpha_i$ is a weight of V but $\mu - 2\alpha$ is not and neither is $\mu - \alpha_j$ for $i \neq j$. For example $L(p\omega)$ is 2-dimensional and consists of just v_p and v_{-p}.
The following lemma utilizes the exponential map given by the usual power series $exp(x) = \sum\limits_0^\infty \frac{x^n}{n!}$. Note that for positive characteristic p, this map is not defined if $x^p \neq 0$.

Lemma 2.1.6 *Let $G = SL_n$ with highest weight module $L(r\omega) = S^r(L(\omega))$ for $r < p$. Then $Ad_{\mathcal{E}_\alpha}(\xi) = exp(ad(\xi e_\alpha))$ for $\xi \in k^\times$.*

Proof. In this case $\mathcal{E}_\beta(\xi) = (I + \xi e_\beta)$. Then for $u_1 \otimes \cdots \otimes u_r \in S^r(L(\omega))$ we have

$$\begin{aligned} \mathcal{E}_\beta(\xi) \cdot (u_1 \otimes \cdots \otimes u_r) =& (I + \xi e_\beta)u_1 \otimes \cdots \otimes (I + \xi e_\beta)u_r \\ =& (u_1 \otimes \cdots \otimes u_r) + \xi e_\beta \cdot (u_1 \otimes \cdots \otimes u_r) + \frac{\xi^2}{2} e_\beta \cdot (e_\beta \cdot (u_1 \otimes \cdots \otimes u_r)) + \\ & \cdots + \frac{\xi^r}{r!} e_\beta \cdot (e_\beta \cdot (\ldots (e_\beta \cdot (u_1 \otimes \cdots \otimes u_r)) \ldots)) \\ =& exp(\xi e_\beta) \cdot (u_1 \otimes \cdots \otimes u_r) \end{aligned}$$

Since $\frac{\xi^{r+1}}{r+1!} e_\beta \cdot (e_\beta \cdot (\ldots (e_\beta \cdot (u_1 \otimes \cdots \otimes u_r)) \ldots)) = 0$. ■

If $p > 3$ then the exceptional type Lie algebras $\mathfrak{g}$ are simple. Then every non-trivial module of $\mathfrak{g}$ is faithful. We can form modules which correspond to the minimal faithful modules in characteristic zero. Although we call these the minimal faithful modules it is not clear if these modules have the smallest dimension. If we form the modules over $\mathbb{Z}$, then the root elements act via integer matrices. We can now reduce these modules mod p. Once the action of roots elements is determined, this can then be extended by linearity and the bracket.

2.2 Nilpotent Orbits

The *p-operation* on a Lie algebra $\mathfrak{g}$ is a map $x \mapsto x^{[p]}$ satisfying

i) $ad_{x^{[p]}} = (ad_x)^p$ for $x \in \mathfrak{g}$

ii) $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ for $x \in \mathfrak{g},\ \lambda \in k$

iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=i}^{p-1} \frac{s_i(x,y)}{i}$ for $x, y \in \mathfrak{g}$

where $s_i(x,y)$ is the coefficient of t^{i-1} in the expression $ad(tx+y)^{p-1}(x)$. For example when $\mathfrak{g}$ is a classical Lie algebra then the p-operation is the p-th power of matrices. We denote by $\mathcal{N}_1$ the subset of $\mathfrak{g}$ of elements satisfying $x^{[p]} = 0$. The iterated p-th power $(((x^{[p]})^{[p]})\dots)^{[p]}$ is denoted $x^{[p^i]}$. By definition x is *nilpotent* if $x^{[p^n]}$ for some $n > 0$. Note that if $x \in \mathcal{N}_1$ then x is nilpotent. A nilpotent element e is *distinguished* if $(G^e)^\circ$ contains no non-trivial semisimple elements. Here a non-trivial semisimple element is a semisimple element not contained in $\mathfrak{Z}(G)$.
For the remainder of this chapter we consider the classifications of nilpotent orbits of simple Lie algebras $\mathfrak{g}$ over an algebraically closed field of characteristic p. There are a few primes p for which the following classifications do not hold. Let $\mathfrak{g}$ be the simple Lie algebra of a simple group G with simple roots $\Delta = \{\alpha_1, \dots \alpha_n\}$, and highest root $\hat{\alpha} = \sum_{i=1}^{n} m_i \alpha_i$ for $m_i \in k$. A prime p is *bad* if p divides some m_i, otherwise p is *good* [SS70].
The bad primes for the simple groups are as follows:

Lie Algebra $\mathfrak{g}$	Bad Primes p
A_n	No bad primes
Classical types (except A_n)	2
Exceptional types (except E_8)	2,3
E_8	2,3,5

If p is good then the partition classification of nilpotent orbits given in the previous chapter still holds for the classical types. Note that any p is good for $\mathfrak{sl}_n$ and for the other classical types $p > 2$ is good. The following is from [NPV02, Theorem 6.3.1].

Theorem 2.2.1 *Let p be a good prime and let G be a reductive group. Then $\mathcal{N}_1$ is irreducible.*

Let G be simple but not of type SL_n and let p be good. Then for $x \in G$ we have $Lie(G^x) = \mathfrak{g}^x$ [SS70]. The following classifications of nilpotent orbits do not hold when p is bad, therefore for the remainder of the chapter we assume that p is good.

2.3 Weighted Dynkin diagrams

For the weighted Dynkin diagram classification to hold we require a few more conditions on G. Let G be a reductive algebraic group and $\mathfrak{g} = Lie(G)$. We require that the derived subgroup of G is simply connected and that there exists a non-degenerate symmetric bilinear G-equivariant form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$. We assume these hold throughout this section. In positive characteristic embedding nilpotent elements into $\mathfrak{sl}_2$-triples is less helpful. This is because a nilpotent element of $\mathfrak{g}$ need not lie in $\mathcal{N}_1$. In the case where p is greater than the Coxeter number then we have $\mathcal{N}_1 = \mathcal{N}$, otherwise we need to find an alternative. We recall that the *Coxeter number* is $(1 + \sum m_i)$ where the highest root is given by $\sum m_i \alpha_i$ for simple roots α_i.
A *cocharacter* is a morphism $\lambda : k^\times \to G$. Let

$$\mathfrak{g}(\lambda; i) = \{x \in \mathfrak{g} : Ad_{\lambda(t)}(x) = t^i x \; \forall t \in k^\times\}.$$

For an element $e \in \mathfrak{g}$, a cocharacter λ is an *associated cocharacter for e* if

(i) $e \in \mathfrak{g}(\lambda; 2)$

(ii) $\mathfrak{g}^e \subset \bigoplus\limits_{i \geq 0} \mathfrak{g}(\lambda; i)$

(iii) There exists a Levi subgroup L of G such that $\lambda(k^\times) \subset (L, L)$ and e is a distinguished nilpotent element of $Lie(L)$.

An associated cocharacter of a nilpotent element e is in some ways analogous to h in an $\mathfrak{sl}_2$-triple containing e. The following result was proved by Pommerening in [Pom77] and [Pom80]. A uniform proof is also given in [Pre03b, Theorem A].

Theorem 2.3.1 *Let $\mathfrak{g} = Lie(G)$ with the above assumptions on G. Then any nilpotent element $e \in \mathfrak{g}$ has an associated cocharacter. Any two such cocharacters are G^e-conjugate.*

Any parabolic subgroup of G can be decomposed as $P = L \cdot U$ where L is a Levi subgroup and U is the unipotent radical. There is a parabolic subgroup $P(\lambda) = L(\lambda)U(\lambda)$ where $U(\lambda)$ is the unique connected T-stable unipotent subgroup of G such that $L(\lambda) = G^\lambda$, $Lie(U(\lambda)) = \sum\limits_{i>0} \mathfrak{g}(\lambda; i)$ and $Lie(P(\lambda)) = \mathfrak{p}(\lambda) = \sum\limits_{i \geq 0} \mathfrak{g}(\lambda; i)$. The following two results are from [Pre03b, Theorem A].

Theorem 2.3.2 *Let G be a reductive algebraic group such that the above assumptions hold. Then for any nilpotent element $e \in \mathfrak{g}$ and any associated cocharacter λ we have $G^e = C \ltimes U$ where $C = L(\lambda) \cap G^e$ and $U = U(\lambda) \cap G^e$.*

Lemma 2.3.3 *Let G be a reductive algebraic group and $\mathfrak{g} = Lie(G)$. Then for $e \in \mathfrak{g}$, with associated cocharacter λ*

$$\sum_{i\geq 2}\mathfrak{g}(\lambda;i) = \overline{P(\lambda)\cdot e}$$

For any nilpotent element e of $\mathfrak{g}$ we have $\mathfrak{g}^e = \mathfrak{g}^e(\lambda;0) \oplus \bigoplus_{i\geq 1}\mathfrak{g}^e(\lambda;i)$. Let the *reductive part* of $\mathfrak{g}^e$ be $\mathfrak{c} = \mathfrak{g}^e(\lambda;0)$ and the *unipotent part* of $\mathfrak{g}^e$ be $\mathfrak{u}^e = \bigoplus_{i\geq 1}\mathfrak{g}^e(\lambda;i)$; then $\mathfrak{g}^e = \mathfrak{c} + \mathfrak{u}^e$. With our assumption on G we have $\mathfrak{c} = Lie(C)$ and $\mathfrak{u}^e = Lie(U)$. Thus if e is distinguished then $\mathfrak{c} = \mathfrak{z}(\mathfrak{g})$.

Lemma 2.3.4 *Let e be a distinguished nilpotent element in the Lie algebra of a simple group. If $e \in \mathcal{N}_1$ then $\mathfrak{g}^e \subset \mathcal{N}_1$.*

Lemma 2.3.5 *Let $\mathcal{O}_e$ be a nilpotent orbit of $\mathfrak{g}$. If $c, c' \in \mathfrak{c}\cap\mathcal{N}_1$ are such that $\overline{C\cdot c} \supset C\cdot c'$ and $c + \mathfrak{u}^e \subset \mathcal{N}_1$, then $\overline{C\cdot(c+\mathfrak{u}^e)} \supset C\cdot(c'+\mathfrak{u}^e)$.*

Proof. If $C\cdot c' \subset \overline{C\cdot c}$ and $c + \mathfrak{u}^e \subset \mathcal{N}_1$ then $c' + \mathfrak{u}^e \subset \mathcal{N}_1$. Since $\mathfrak{u}^e$ is C-stable then $\overline{C\cdot c'} + \mathfrak{u}^e \subset \overline{C\cdot c} + \mathfrak{u}^e$. ■

Consider a homomorphism $\rho : SL_2 \to G$ where $d\rho : \mathfrak{sl}_2 \to \mathfrak{g}$ and $d\rho\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) = e$. We say ρ is an *optimal* SL_2-homomorphism if

$$\rho\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix} = \lambda(t) \quad \forall t \in k^\times$$

for some associated cocharacter λ of e [McN05]. The image of an optimal homomorphism is called a *good* SL_2 subgroup of G (see [McN05] and [Sei00]). A proof for the following proposition can be found in [McN05, Prop 33] and [Sei00, Theorem 1.1].

Proposition 2.3.6 *Let $e \in \mathcal{N}_1$ with associated cocharacter λ. There is an optimal homomorphism ρ for e where $\rho\left(\begin{smallmatrix}t&0\\0&t^{-1}\end{smallmatrix}\right) = \lambda(t)$.*

For an element $e \in \mathcal{N}_1$ with associated cocharacter λ we have that $\mathfrak{g} = \sum_{i\in\mathbb{Z}}\mathfrak{g}(\lambda;i)$. Each $v_i \in \mathfrak{g}^e(\lambda;i)$ generates an SL_2-submodule $L(i\omega)$. If $i < p$ then this submodule is irreducible by Theorem 2.1.4. If all the weights of λ on $\mathfrak{g}^e$ are less than p then $\mathfrak{g} = \bigoplus_{i=0}^{p-1} L(i\omega)^{n_i}$ where $\dim(\mathfrak{g}^e(\lambda;i)) = n_i$.

Let e be a nilpotent element of $\mathfrak{g}$ with associated cocharacter $\lambda : k^\times \to T$. Then we associate a weight $\langle\lambda,\alpha_i\rangle$ to each $\alpha_i \in \Delta$, where

$$Ad_{\lambda(t)}(e_{\alpha_i}) = t^{\langle\lambda,\alpha_i\rangle}e_{\alpha_i} \text{ for } t \in k^\times$$

After conjugating by some element of $\mathcal{W}$ we may assume $\langle\lambda,\alpha_i\rangle \geq 0$ for all $\alpha_i \in \Delta$ by Theorem 1.1.10. Then the weighted Dynkin diagram corresponding to e is given by the Dynkin diagram of $\mathfrak{g}$ where the node corresponding to α_i is labelled with $\langle\lambda,\alpha_i\rangle$. As before the weighted Dynkin diagrams of nilpotent elements e and e' are the same if and only if e and e' are conjugate. Therefore there is a unique weighted Dynkin diagram for each nilpotent orbit in $\mathfrak{g}$.

2.4 Generalization of the Bala-Carter Theorem

The Bala-Carter theorem holds in positive characteristic p as well as in characteristic zero. When $p \gg 0$ and $p = 0$ the proof presented in [BC76b] holds. For p good, this is shown in [Pom80] and a uniform proof is given in [Pre03b]. There is also a generalization of the Bala-Carter theorem which can be helpful for describing representatives of nilpotent orbits. This generalization was shown in [Som98, Thm 13] for $char(k) = 0$ and in [MS03] or [Pre03b, Thm 3.7] for $char(k) = p$. For this we require the same assumptions on G as mentioned at the beginning of Section 2.3.

A subgroup H of G is a *pseudo Levi subgroup* if $H = (G^s)^o$ for some semisimple element $s \in G$. Similarly a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *pseudo Levi subalgebra* if $\mathfrak{h} = \mathfrak{g}^s$. Since $Lie((G^s)^o) = \mathfrak{g}^s$ [Bor91, §9] then the pseudo Levi subalgebras in $\mathfrak{g}$ correspond to pseudo Levi subgroups of G.

Let Φ be the root system of $\mathfrak{g}$ with basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ and let $\hat{\alpha}$ be the highest root in Φ^+. Then we denote $\widetilde{\Delta} = \Delta \cup \{\alpha_0\}$ where $\alpha_0 = -\hat{\alpha}$. Let J be a proper subset of $\{0, 1, \ldots, l\}$ and let Φ_J be the set of all roots in Φ of the form $\sum\limits_{i \in J} a_i \alpha_i$ for $a_i \in \mathbb{Z}$. For a torus T of G then $\mathfrak{l}_J$ is a *standard pseudo Levi subalgebra* of $\mathfrak{g}$ where

$$\mathfrak{l}_J = \mathfrak{t} \oplus \sum_{\alpha \in \Phi_J} \mathfrak{g}_\alpha$$

Then we define a subgroup L_J of G generated by T and U_α for $\alpha \in \Phi_J$, where $Lie(L_J) = \mathfrak{l}_J$. Then L_J is a *standard pseudo-Levi subgroup* of G. For details of the next proposition see [Pre03b, Proposition 3.1].

Proposition 2.4.1 *A subgroup $H \subset G$ is a pseudo Levi subgroup of G if and only if it is G-conjugate to a standard pseudo Levi subgroup.*

For a nilpotent element e in $\mathfrak{g}$ the *component group* of e is $A(e) = G^e/((G^e)^\circ \mathfrak{Z}(G))$.

Theorem 2.4.2 (Generalization of Bala-Carter Theorem) *There is a bijection between G-conjugacy classes of pairs (L, e) where L is a pseudo Levi subgroup of G and e is a distinguished nilpotent element in $\mathfrak{l} = Lie(L)$, and G-conjugacy classes (e, D) where e is a nilpotent element in $\mathfrak{g}$ and D is a conjugacy class in $A(e)$.*

Example 2.4.3

Consider $\mathfrak{g} = \mathfrak{sp}_6$ and nilpotent orbit $\mathcal{O}_{[4,2]}$. This orbit has Bala-Carter labelling $C_3(a_1)$. However we can consider the following representative e of $\mathcal{O}_{[4,2]}$:

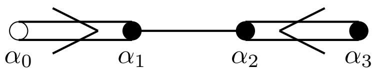
$$e = \begin{pmatrix} 0 & & & & & 1 \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & -1 & \\ & & & & 0 & \\ 0 & & & & & 0 \end{pmatrix}.$$

Then the inner block corresponds to a distinguished nilpotent element in $\mathfrak{sp}_4$. The outer corners form a block which corresponds to a distinguished element in $\mathfrak{sl}_2$. Therefore e is distinguished in the pseudo-Levi subalgebra $\mathfrak{sp}_4 \oplus \mathfrak{sl}_2$, of type $C_2 \times A_1$.

Let $\mathfrak{g}$ be a simple Lie algebra with highest root $\hat{\alpha}$. The *affine Dynkin diagram* of $\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{g}$ with an extra node attached corresponding to α_0. This node is attached to the α_i where $-\hat{\alpha}+\alpha_i$ is a root. (In type A_n, α_0 is attached to both α_1 and α_n. In other types it is only connected to one α_i). If this α_i is a short root element then we draw a double bond between α_0 and α_i.

Example 2.4.4

The highest root of $\mathfrak{sp}_6$ is $\hat{\alpha} = 2\alpha_1 + 2\alpha_2 + \alpha_3$. Then the affine Dynkin diagram of $\mathfrak{sp}_6$ is



This is because $-(\alpha_0+\alpha_1)$ and $-(\alpha_0+2\alpha_1)$ are roots in $\mathfrak{sp}_6$. The arrow from α_0 is present because α_0 is long but α_1 is not. Then by Example 2.4.3 we can represent the nilpotent orbit $\mathcal{O}_{[4,2]}$ by the sub-diagram

The regular orbit in $A_1 \times C_2$ can be represented by the element $e_{\alpha_o} + e_{\alpha_2} + e_{\alpha_3}$.

2.5 Transverse Slices

Throughout this section let X be an affine variety with coordinate ring $k[X]$ where k is an algebraically closed field of arbitrary characteristic. For a multiplicative set U of $k[X]$, the *localization of* $k[X]$ *at* U, denoted $U^{-1}k[X]$, is the set of equivalence classes f/g for $f \in k[X]$ and $g \in U$, where $f_1/g_1 \sim f_2/g_2$ if and only if there exists an $u \in U$ such that $u(f_1g_2 - f_2g_1) = 0$ (see [Lan93, Chap II §4]). For a point $x \in X$ the set $U_{X,x} = \{g \in k[X] : g(x) \neq 0\}$ is a multiplicative set of $k[X]$ and the *local ring of* X *at* x is the localization $\mathcal{O}_{X,x} = U_{X,x}^{-1}k[X]$. There is a unique maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ which consists of all elements which vanish at x. For example let $X = \mathbb{A}_1$ with coordinate ring $k[t]$ then the local ring of X at 0 is given by $\mathcal{O}_{X,0} = \{f(t)/g(t) : g(0) \neq 0\}$ and $\mathfrak{m}_0 = \{f(t)/g(t) : g(0) \neq 0, f(0) = 0\}$. The dimension of a local ring is the Krull dimension, i.e. $\dim(\mathcal{O}_{X,x})$ is the largest r such that there exists a chain $p_0 \subset p_1 \subset \cdots \subset p_r$ of prime ideals of $\mathcal{O}_{X,x}$.

Proposition 2.5.1 *Let X be an affine variety with point $x \in X$. Then $\dim(\mathcal{O}_{X,x}) = \dim_x(X)$, where $\dim_x(X)$ is the largest dimension of an irreducible component of X passing through x.*

Proof. Consider the prime ideal $p = \{g \in k[X] : g(x) = 0\}$ of $k[X]$, then by [AM69, Cor 3.13] there is a one-to-one correspondence between chains of prime ideals $q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_r$ of $\mathcal{O}_{X,x}$ and chains $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_r$ of prime ideals of $k[X]$ which contain p. Therefore $\dim \mathcal{O}_{X,x}$ is equal to the maximum length of such a chain of prime ideals of $k[X]$ containing p. There is also a one-to-one correspondence between chains $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_r$ of prime ideals of $k[X]$ which contain p and chains $X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_r$ of irreducible closed subsets of X which contain x. Therefore $\dim(\mathcal{O}_{X,x}) = \dim_x(X)$. ■

Consider the sequence

$$\cdots \to \mathcal{O}_{X,x}/\mathfrak{m}_x^3 \to \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \to \mathcal{O}_{X,x}/\mathfrak{m}_x \cong k$$

The isomorphism $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong k$ is a consequence of the weak Nullstellensatz. In particular since $\mathcal{O}_{X,x}$ is a finitely generated k-algebra and $\mathfrak{m}_x$ is a maximal ideal then $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is a field which is a finitely generated k-algebra. Hence by the Weak Nullstellensatz $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is an algebraic extension of k. Since k is algebraically closed then $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong k$ (see [AM69, Cor 7.10]). In the case where $X = \mathbb{A}_n$ and $x = (0, 0, \ldots, 0)$, every element of $\mathcal{O}_{X,x}/\mathfrak{m}_x^r$ can be represented by a polynomial truncated at rth-degree. The *completion of* $\mathcal{O}_{X,x}$ is given by the limit of this sequence. This is denoted by

$$\widehat{\mathcal{O}_{X,x}} = \lim_{\leftarrow} (\mathcal{O}_{X,x}/\mathfrak{m}_x^r)$$

In particular an element of $\widehat{\mathcal{O}_{X,x}}$ is given by a sequence $(f_1, f_2, f_3, \ldots)$ where $f_r \in \mathfrak{m}_x^r$ and $f_r = f_{r-1} \mod \mathfrak{m}_x^{r-1}$. In the case $X = \mathbb{A}_1$ and $x = 0$ the completion $\widehat{\mathcal{O}_{X,x}}$ is isomorphic to the formal power series denoted $k[[t]]$. By [AM69, Cor 11.19] we have the following proposition.

Proposition 2.5.2 *Let $\mathcal{O}_{X,x}$ be the local ring of $x \in X$. Then* $\dim(\mathcal{O}_{X,x}) = \dim(\widehat{\mathcal{O}_{X,x}})$.

We say $\mathcal{O}_{X,x}$ is *regular* if $\dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(\mathcal{O}_{X,x})$. A point x is *smooth* in X if $\mathcal{O}_{X,x}$ is regular, otherwise we say x is *singular*. An affine variety X is *smooth* if every point in X is smooth.

Example 2.5.3

Consider the irreducible affine variety $X \subset \mathbb{A}^2$ defined by the polynomial $y^2 = x^2(x+1)$, as depicted in Figure 2.1. Then the point $p = (-1, 0)$ in X is smooth. To see this first note that $\dim(\mathcal{O}_{X,p}) = \dim(X) = 1$ by Proposition 2.5.1. Now $\mathfrak{m}_p$ and $\mathfrak{m}_p^2$ are generated by $(x+1)$, y and $(x+1)^2$, y^2, $(x+1)y$ respectively. Since $x^2(x+1) \equiv y^2 \equiv 0 \mod \mathfrak{m}_p^2$ then

$$x + 1 \equiv x + 1 - x^2(x+1) = (1+x)^2(1-x) \equiv 0 \mod \mathfrak{m}_p^2$$

Therefore $\dim(\mathfrak{m}_p/\mathfrak{m}_p^2) = 1$. The point $q = (0, 0)$ is singular since $\dim(\mathfrak{m}_q/\mathfrak{m}_q^2) = 2$. In fact X is smooth at every point except at $(0, 0)$.

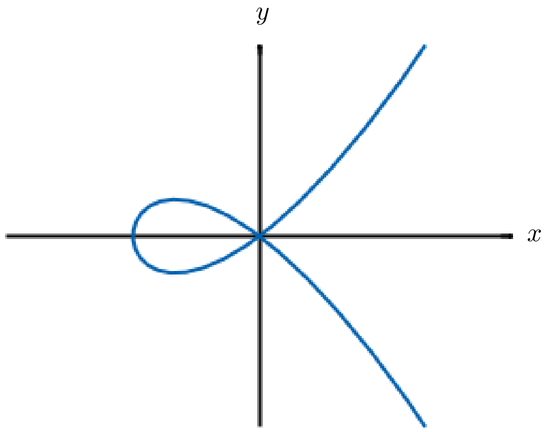


Figure 2.1: $X = \{(x, y) : y^2 = x^2(x + 1)\}$

A proof for the following results is presented in [ZS60, Chap VIII, §11 and §12].

Theorem 2.5.4 *Let X be a variety and let the local ring $\mathcal{O}_{X,x}$ for $x \in X$ have dimension n.*

(i) $\mathcal{O}_{X,x}$ is regular if and only if $\widehat{\mathcal{O}_{X,x}}$ is regular

(ii) $\widehat{\mathcal{O}_{X,x}}$ is regular if and only if $\widehat{\mathcal{O}_{X,x}} \cong k[[x_1, \ldots x_n]]$ where $k[[x_1, \ldots, x_n]]$ denotes the formal power series ring.

Proposition 2.5.5 *Let $x \in X$ and $y \in Y$ be points in varieties X and Y, and let $f : X \to Y$ be a morphism such that $f(x) = y$. Then there exists a homomorphism $\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{Y,y}}$.*

Proof. There exists a homomorphism $f_* : k[Y] \to k[X]$, $g \mapsto g \circ f$. Now consider the multiplicative set $U_{Y,y} = \{g \in k[Y] : g(y) \neq 0\}$ of $k[Y]$, then $g \in U_{Y,y}$ if and only if $f_*(g) \in U_{X,x} = \{h \in k[X] : h(x) \neq 0\}$. Therefore we get a homomorphism given by

$$\mathcal{O}_{Y,y} = U_{Y,y}^{-1} k[Y] \to f_*(U_{Y,y})^{-1} k[X] \to U_{X,x}^{-1} k[X] = \mathcal{O}_{X,x}$$

This homomorphism maps $\mathfrak{m}_y$ to $\mathfrak{m}_x$ so the homomorphism $\mathcal{O}_{Y,y}/\mathfrak{m}_y^r \to \mathcal{O}_{X,x}/\mathfrak{m}_x^r$ is well defined. Therefore we get the following diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathcal{O}_{Y,y}/\mathfrak{m}_y^r \arrow[r] \arrow[d] & \mathcal{O}_{Y,y}/\mathfrak{m}_y^{r-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & \mathcal{O}_{Y,y}/\mathfrak{m}_y \cong k \arrow[d] \\
\ldots \arrow[r] & \mathcal{O}_{X,x}/\mathfrak{m}_x^r \arrow[r] & \mathcal{O}_{X,x}/\mathfrak{m}_x^{r-1} \arrow[r] & \ldots \arrow[r] & \mathcal{O}_{X,x}/\mathfrak{m}_x \cong k
\end{tikzcd}$$

So we have the morphism $\widehat{\mathcal{O}_{Y,y}} \to \mathcal{O}_{X,x}/\mathfrak{m}_x^r$ for all r such that the following diagram commutes.

$$\begin{tikzcd}
\widehat{\mathcal{O}_{Y,y}} \arrow[d] \arrow[dr] & \\
\mathcal{O}_{X,x}/\mathfrak{m}_x^r \arrow[r] & \mathcal{O}_{X,x}/\mathfrak{m}_x^{r-1}
\end{tikzcd}$$

Therefore by the universal property of the limit there exists a morphism $\widehat{\mathcal{O}_{Y,y}} \to \widehat{\mathcal{O}_{X,x}}$. ■

Let $f : X \to Y$ be a morphism of affine varieties such that for points $x \in X$ and $y \in Y$ we have $f(x) = y$. Then (X, x) and (Y, y) are *locally analytically isomorphic* if the homomorphism induced by f as described in Proposition 2.5.5 is an isomorphism $\widehat{\mathcal{O}_{Y,y}} \cong \widehat{\mathcal{O}_{X,x}}$. Suppose that $\dim(X) = n$ and $\dim(Y) = n - r$. Then f is *smooth of relative dimension* r if the comorphism $f^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ can be extended to an isomorphism

$$\widehat{\mathcal{O}_{Y,y}}[[t_1, \dots, t_r]] \cong \widehat{\mathcal{O}_{X,x}}$$

Examples 2.5.6

1. If X and Y are both smooth varieties then any morphism $f : X \to Y$ is smooth.
2. Let X be as in Example 2.5.3 and let $Y \subset \mathbb{A}^3$ be defined by the polynomial $y^2 = x^2(x+1)$ as depicted in Figure 2.2. Then the morphism $f : Y \to X$ where $(x, y, z) \mapsto (x, y)$ is smooth at all points.

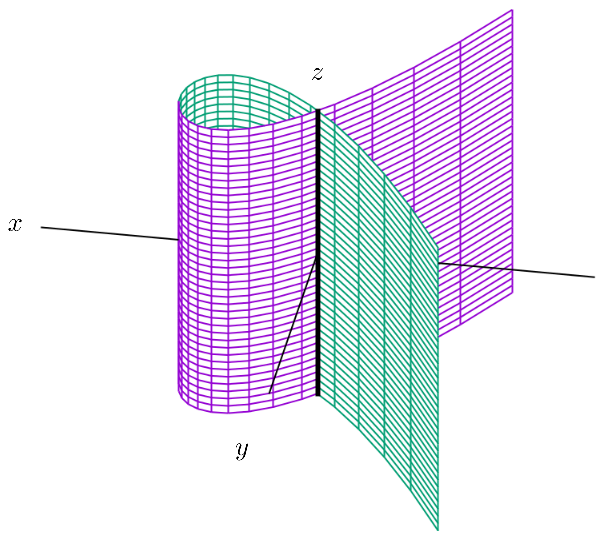


Figure 2.2: $Y = \{(x, y, z) : y^2 = x^2(x+1)\}$

Let G be an algebraic group which acts on X. A subvariety S of X is *locally closed* if it is the intersection of an open set with a closed set. A *transverse slice in* X *(to* $G \cdot x$*) at* $x \in X$ is a locally closed subvariety S of X such that

(i) $x \in S$,

(ii) the morphism $G \times S \to X$ where $(g, s) \mapsto g \cdot s$ is smooth at (e, x) where e is the identity of G,

(iii) the dimension of S is minimal given (i) and (ii) (see [Slo80]).

Let e be a nilpotent element of $\mathfrak{g} = Lie(G)$ and suppose that $Lie(G^e) = \mathfrak{g}^e$. This holds when $char(k) = 0$ and G is simple, or when $char(k)$ is good and G is simple but not of type SL_n.

Consider the map $\phi : G \to \mathfrak{g}$ where $v \mapsto Ad_v(e)$. Then $d\phi_1 : \mathfrak{g} \to T_e(\mathfrak{g})$ sends $y \mapsto e + [y, e]$ (note that $T_x(X)$ denotes the tangent space of X at x). Let E denote the image of $d\phi_1$ so $E = e + [\mathfrak{g}, e]$. Now let $\mathfrak{v}$ be some linear (C-stable) complement to E in $T_e(\mathfrak{g}) = e + \mathfrak{g}$. So

$$[\mathfrak{g}, e] \oplus \mathfrak{v} = \mathfrak{g} \qquad \text{and} \qquad [\mathfrak{g}, e] \cap \mathfrak{v} = \{0\} \tag{2.1}$$

Let S be the preimage of $\mathfrak{v}$ under the map $\pi : \mathfrak{g} \to T_e(\mathfrak{g})$ which sends $y \mapsto e + y$. Then, by the proof of [Slo80, Chap III, §5.1, Lemma 1], $S = e + \mathfrak{v}$ is a transverse slice of e in $\mathfrak{g}$.
Let $\mathfrak{g}$ be a Lie algebra of characteristic zero; then for a nilpotent element e of $\mathfrak{g}$ there exists an $\mathfrak{sl}_2$-triple $\{e, f, h\}$. The *Slodowy slice at* e given by $S_e = e + \mathfrak{g}^f$ is a transverse slice of e in $\mathfrak{g}$ [Slo80, §7.4]. In the case when $char(k) > 0$ then $e \in \mathcal{N}_1$ has an associated cocharacter λ with good SL_2-subgroup $\{e, f, h\}$ of G. If the weights of λ are between $-(p-1)$ and $p-1$ then the Slodowy slice $S_e = e + \mathfrak{g}^f$ is a transverse slice.

Example 2.5.7

Let $\mathfrak{g} = \mathfrak{sl}_3$ with $p \geq 3$. Consider $e = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ which is contained in $\mathcal{N}_1$. Now e has associated cocharacter $\lambda(t) = \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix}$ which has weights $\{-2, -1, 0, 1, 2\}$. Therefore there is a good SL_2 subgroup of G given by $\{e, f, h\}$ where $f = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}$. The Slodowy slice $S_e = e + \mathfrak{g}^f$ is a transverse slice when $p \geq 3$ where

$$S_e = \left\{ \begin{pmatrix} a & 0 & 1 \\ c & -2a & 0 \\ d & c & a \end{pmatrix} : a, b, c, d \in k \right\}$$

A proof of the following proposition is presented in [Slo80, §5 Lemma 2].

Proposition 2.5.8 *Let G be an algebraic group which acts on a variety X. Let S_x be a transverse slice at x in X and let Y be a closed G-stable subvariety of X such that $x \in Y$. Then $S_x \cap Y$ is a transverse slice at x in Y.*

Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$ and let e be contained in $\mathcal{O}$ with transverse slice S_e in $\mathfrak{g}$. For a nilpotent orbit $\mathcal{O}'$ such that $\mathcal{O} < \mathcal{O}'$ we have $e \in \overline{\mathcal{O}'}$. Then by Proposition 2.5.8, $S_e \cap \overline{\mathcal{O}'}$ is a transverse slice of e at $\overline{\mathcal{O}'}$.

Example 2.5.9

In Example 2.5.7, e is contained in $\mathcal{O}_{[2,1]}$, therefore a transverse slice of e in $\overline{\mathcal{O}_{[3]}}$ is given by

$$S_e \cap \overline{\mathcal{O}_{[3]}} = S_e \cap \mathcal{N} = \left\{ \begin{pmatrix} a & 0 & 1 \\ b & -2a & 0 \\ -3a^2 & c & a \end{pmatrix} : 4a^3 + bc = 0 \right\}$$

Lemma 2.5.10 *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$ with $e \in \mathcal{O}$. Let S_e be the Slodowy slice of e in $\mathfrak{g}$ which is also transverse. Then $S_e \cap \mathcal{O} = \{e\}$.*

Proof. Let e have associated cocharacter λ with good SL_2 subgroup $\{e, f, h\}$ of G such that $f \in \mathfrak{g}(-2; \lambda)$. Consider the element

$$e + x_0 + x_{-1} + x_{-2} + \cdots \in (e + \mathfrak{g}^f) \cap \mathcal{O}$$

where $x_{-i} \in \mathfrak{g}(-i; \lambda)$. There is a scaling action on S_e which preserves the intersection with every nilpotent orbit, this is given by the following two steps:

$$\begin{aligned} Ad_{\lambda(t)}(e + x_0 + x_{-1} + x_{-2} + \dots) &= t^2 e + x_0 + t^{-1} x_{-1} + t^{-2} x_{-2} + \cdots \in (t^2 e + \mathfrak{g}^f) \cap \mathcal{O} \\ t^{-2}(t^2 e + x_0 + t^{-1} x_{-1} + t^{-2} x_{-2} + \dots) &= e + t^{-2} x_0 + t^{-3} x_{-1} + \cdots \in (e + \mathfrak{g}^f) \cap \mathcal{O} \end{aligned}$$

So we obtain a 1-dimensional subset of elements belonging to $S_e \cap \mathcal{O}$. By [Slo80, Chap 5, Remark 2] we get

$$\dim_e(S_e \cap \overline{\mathcal{O}}) \leq \dim_e(\overline{\mathcal{O}}) - \dim(G \cdot e)$$

So if $\mathcal{O} = G \cdot e$ then $S_e \cap \overline{\mathcal{O}}$ is finite. Therefore we cannot have anything in $S_e \cap \overline{\mathcal{O}}$ of the form $e + x_0 + x_{-1} + \dots$ with $(x_0, x_{-1}, \dots) \neq (0, 0, \dots)$. ■

Corollary 2.5.11 *Let $\mathcal{O} < \mathcal{O}'$ be nilpotent orbits in $\mathfrak{g}$ such that there is no orbit $\mathcal{O}''$ where $\mathcal{O} < \mathcal{O}'' < \mathcal{O}'$, and let $e \in \mathcal{O}$. Then all elements of $S_e \cap \overline{\mathcal{O}'}$ are contained in $\mathcal{O}'$ except e.*

Proof. Let $x \in S_e \cap \overline{\mathcal{O}'}$ such that $x \neq e$. Then $G \cdot x \subset G \cdot \overline{\mathcal{O}'}$ and so $\overline{G \cdot x} \subset \overline{\mathcal{O}'}$. Similarly, by the scaling action described in Lemma 2.5.10, we have $e \in \overline{G \cdot x}$, therefore $\mathcal{O} \subset \overline{G \cdot x}$. By proof of Lemma 2.5.10 we have $\dim(\overline{G \cdot x}) > \dim(\overline{G \cdot e})$ therefore $x \in \mathcal{O}'$. ■

Hence in Example 2.5.9 all the elements in $S_e \cap \mathcal{N}$ such that $(a, b, c) \neq (0, 0, 0)$ are contained in $\mathcal{O}_{[3]}$. This property means that transverse slices can help to describe points of $\mathcal{O}'$ which are close to e. This is utilized in [FJLS15].

We can parametrize the Slodowy slice in Example 2.5.9. We have $S_e \cap \mathcal{N} \cong k[a, b, c]/(4a^3 + bc)$. There is a surjective map from this ring to $k[st, -4s^3, t^3]$ by sending $a \mapsto st$, $b \mapsto -4s^3$ and $c \mapsto t^3$. This map is well-defined since the images of a, b and c satisfy the polynomial $4a^3 + bc = 0$. Because this polynomial is irreducible then both of these rings are integral domains. Since these rings also have the same dimension they are isomorphic. A similar process of parametrization can often be applied to transverse slices making it easier to describe their elements.

Chapter 3

Special Cases

3.1 Classical Types

In this chapter we consider in detail a few simple Lie algebras that are helpful for subsequent calculations. Throughout this chapter we assume that $char(k) = p$ is good. We start by looking at $\mathfrak{sl}_2$, $\mathfrak{sl}_3$, $\mathfrak{sl}_4$ and $\mathfrak{sl}_6$. Below are the Hasse diagrams of the nilpotent orbits of these Lie algebras labelled by the corresponding partition along with the dimension of each orbit.

$$\begin{tikzcd}[row sep=small]
\mathcal{O}_{[2]} \arrow[d, dash] & \dim(\mathcal{O}_{[2]}) = 2 \\
\mathcal{O}_{[1^2]} & \dim(\mathcal{O}_{[1^2]}) = 0
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
\mathcal{O}_{[3]} \arrow[d, dash] & \dim(\mathcal{O}_{[3]}) = 6 \\
\mathcal{O}_{[2,1]} \arrow[d, dash] & \dim(\mathcal{O}_{[2,1]}) = 4 \\
\mathcal{O}_{[1^3]} & \dim(\mathcal{O}_{[1^3]}) = 0
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
\mathcal{O}_{[4]} \arrow[d, dash] & \dim(\mathcal{O}_{[4]}) = 12 \\
\mathcal{O}_{[3,1]} \arrow[d, dash] & \dim(\mathcal{O}_{[3,1]}) = 10 \\
\mathcal{O}_{[2^2]} \arrow[d, dash] & \dim(\mathcal{O}_{[2^2]}) = 8 \\
\mathcal{O}_{[2,1^2]} \arrow[d, dash] & \dim(\mathcal{O}_{[2,1^2]}) = 6 \\
\mathcal{O}_{[1^4]} & \dim(\mathcal{O}_{[1^4]}) = 0
\end{tikzcd}$$

Figure 3.1: Hasse Diagram of Nilpotent Orbits of $\mathfrak{sl}_2$, $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ respectively.

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathcal{O}_{[6]} \arrow[d, dash] & & \dim(\mathcal{O}_{[6]}) = 30 \\
& \mathcal{O}_{[5,1]} \arrow[d, dash] & & \dim(\mathcal{O}_{[5,1]}) = 28 \\
& \mathcal{O}_{[4,2]} \arrow[dl, dash] \arrow[dr, dash] & & \dim(\mathcal{O}_{[4,2]}) = 26 \\
\mathcal{O}_{[3,3]} \arrow[dr, dash] & & \mathcal{O}_{[4,1^2]} \arrow[dl, dash] & \dim(\mathcal{O}_{[3,3]}) = \dim(\mathcal{O}_{[4,1^2]}) = 24 \\
& \mathcal{O}_{[3,2,1]} \arrow[dl, dash] \arrow[dr, dash] & & \dim(\mathcal{O}_{[3,2,1]}) = 22 \\
\mathcal{O}_{[3,1^3]} \arrow[dr, dash] & & \mathcal{O}_{[2^3]} \arrow[dl, dash] & \dim(\mathcal{O}_{[3,1^3]}) = \dim(\mathcal{O}_{[6]}) = 18 \\
& {[2^2,1^2]} \arrow[d, dash] & & \dim(\mathcal{O}_{[2^2,1^1]}) = 16 \\
& {[2,1^4]} \arrow[d, dash] & & \dim(\mathcal{O}_{[2,1^4]}) = 10 \\
& \mathcal{O}_{[1^6]} & & \dim(\mathcal{O}_{[1^6]}) = 0
\end{tikzcd}$$

Figure 3.2: Hasse Diagram of Nilpotent Orbits of $\mathfrak{sl}_6$

Consider the nilpotent orbit $\mathcal{O}_e$ of $\mathfrak{sl}_\mathfrak{n}$ with partition type $[\lambda_1, \dots, \lambda_m]$, then $\mathcal{O}_e$ has Bala-Carter label $A_{\lambda_1 - 1} \times \cdots \times A_{\lambda_m - 1}$, for more details see [Pan99, §3].

The other classical cases we are interested in are $\mathfrak{so}_5$, $\mathfrak{so}_7$ and $\mathfrak{sp}_6$. Below are the Hasse diagrams with each nilpotent orbit labelled by the corresponding partition type and Bala-Carter label. Alongside this is also the dimension of each nilpotent orbit.

$$
\begin{array}{ccc}
B_2 & \mathcal{O}_{[5]} & \dim(\mathcal{O}_{[5]}) = 8 \\
 & | & \\
\widetilde{A_1} & \mathcal{O}_{[3,1^2]} & \dim(\mathcal{O}_{[3,1^2]}) = 6 \\
 & | & \\
A_1 & \mathcal{O}_{[2^2,1]} & \dim(\mathcal{O}_{[2^2,1]}) = 4 \\
 & | & \\
0 & \mathcal{O}_{[1^5]} & \dim(\mathcal{O}_{[1^5]}) = 0
\end{array}
\qquad
\begin{array}{ccc}
B_3 & \mathcal{O}_{[7]} & \dim(\mathcal{O}_{[7]}) = 18 \\
 & | & \\
B_2 & \mathcal{O}_{[5,1^2]} & \dim(\mathcal{O}_{[5,1^2]}) = 16 \\
 & | & \\
A_2 & \mathcal{O}_{[3^2,1]} & \dim(\mathcal{O}_{[3^2,1]}) = 14 \\
 & | & \\
A_1 + \widetilde{A_1} & \mathcal{O}_{[3,2^2]} & \dim(\mathcal{O}_{[3,2^2]}) = 12 \\
 & | & \\
\widetilde{A_1} & \mathcal{O}_{[3,1^4]} & \dim(\mathcal{O}_{[3,1^4]}) = 10 \\
 & | & \\
A_1 & \mathcal{O}_{[2^2,1^3]} & \dim(\mathcal{O}_{[2^2,1^3]}) = 8 \\
 & | & \\
0 & \mathcal{O}_{[1^7]} & \dim(\mathcal{O}_{[1^7]}) = 0
\end{array}
$$

Figure 3.3: Hasse Diagram of Nilpotent Orbits of $\mathfrak{so}_5$ and $\mathfrak{so}_7$ respectively

$$
\begin{array}{cccccc}
 & C_3 & & \mathcal{O}_{[6]} & & \dim(\mathcal{O}_{[6]}) = 18 \\
 & & & | & & \\
 & C_3(a_1) & & \mathcal{O}_{[4,2]} & & \dim(\mathcal{O}_{[4,2]}) = 16 \\
 & & \diagup & & \diagdown & \\
C_2 & \mathcal{O}_{[4,1^2]} & & & \mathcal{O}_{[3^2]} \quad \widetilde{A_2} & \dim(\mathcal{O}_{[4,1^2]}) = \dim(\mathcal{O}_{[3^2]}) = 14 \\
 & & \diagdown & & \diagup & \\
 & \widetilde{A_1} \times A_1 & & \mathcal{O}_{[2^3]} & & \dim(\mathcal{O}_{[2^3]}) = 12 \\
 & & & | & & \\
 & \widetilde{A_1} & & \mathcal{O}_{[2^2,1^1]} & & \dim(\mathcal{O}_{[2^2,1^2]}) = 10 \\
 & & & | & & \\
 & A_1 & & \mathcal{O}_{[2,1^4]} & & \dim(\mathcal{O}_{[2,1^4]}) = 6 \\
 & & & | & & \\
 & 0 & & \mathcal{O}_{[1^6]} & & \dim(\mathcal{O}_{[1^6]}) = 0
\end{array}
$$

Figure 3.4: Hasse Diagram of Nilpotent Orbits of $\mathfrak{sp}_6$

If a nilpotent orbit $\mathcal{O}_e$ of $\mathfrak{sp}_{2n}$ has a partition of distinct even parts then $\mathcal{O}_e$ is distinguished. If it is the regular orbit then its Bala-Carter label is C_n. The first example of a distinguished non-regular orbit is $[4, 2]$ in $\mathfrak{sp}_6$ which has label $C_3(a_1)$. Otherwise the partition of $\mathcal{O}_e$ has a pair of elements $[\lambda_i, \lambda_i]$ which are equal. For each such pair there is a component $\widetilde{A}_{\lambda_i - 1}$ in the Bala-Carter label. Removing these pairs leaves a partition with at most one even part. The final part of the Bala-Carter labelling is C_i in which this new partition is distinguished (or A_1 if we are left with a single part of length 2). The method is same for a nilpotent orbit in $\mathfrak{so}_n$ with the pairs of elements $[\lambda_i, \lambda_i]$ in the partition corresponding to an A_{λ_i-1} component instead of $\widetilde{A}_{\lambda_i-1}$. (These results are presented in [Pan99, §3]).

We are interested in some highest weight modules of G. For $G = SL_2$ the highest weight module $L(n\omega)$ has dimension $n+1$. When $G = SL_n$ then $L(\omega_i) = \Lambda^i(k^n)$ where Λ^i represents the alternating product. Therefore the highest weight module $L(\omega_2) = \Lambda^2(k^4)$ for SL_4 and so has dimension 6. Both $L(\omega_1)$ and $L(\omega_3)$ have dimension 4. For SL_6 we are interested in $L(\omega_3) = \Lambda^3(k^6)$ which has dimension 20. Also SL_3 has an 8 dimensional highest weight module given by $L(\omega_1+\omega_2)$. Similarly we are interested in $L(\omega_2)$ and $L(\omega_1)$ of SO_5 which have dimensions 4 and 5 receptively. Finally $L(\omega_1)$ of SO_7 has dimension 8 and $L(\omega_3)$ of Sp_6 has dimension 14.

3.2 Exceptional Types

The main focus is the exceptional Lie algebras G_2, F_4 and E_6. There are five nilpotent orbits of G_2, two of which are distinguished. Similarly F_4 has sixteen nilpotent orbits, four of which are distinguished. Finally E_6 has twenty two orbits, three of which are distinguished.

Let ρ be a minimal faithful representation of $\mathfrak{g}$. For G_2, F_4 and E_6 respectively, ρ has dimension 7, 26 and 27 respectively. For a nilpotent element e, the sizes of the Jordan blocks of $\rho(e)$ are calculated by considering the successive powers of $\rho(e)$. An element $x \in \mathfrak{g}$ is contained in $\mathcal{O}_e$ if and only if the Jordan blocks of $\rho(x)$ are the same size as those of $\rho(e)$. The following result is shown in [CLNP03, §4.4].

Theorem 3.2.1 *For $\mathfrak{g} = G_2$, F_4 and respectively E_6 we have $\mathcal{N}_1(G_2) = \overline{\mathcal{O}_{G_2(a_1)}}$ when $p = 5$ and*

$$\mathcal{N}_1(F_4) = \begin{cases} \overline{\mathcal{O}_{F_4(a_1)}} & \textit{when } p = 11 \\ \overline{\mathcal{O}_{F_4(a_2)}} & \textit{when } p = 7 \\ \overline{\mathcal{O}_{F_4(a_3)}} & \textit{when } p = 5 \end{cases} \qquad \textit{and} \qquad \mathcal{N}_1(E_6) = \begin{cases} \overline{\mathcal{O}_{E_6(a_1)}} & \textit{when } p = 11 \\ \overline{\mathcal{O}_{E_6(a_3)}} & \textit{when } p = 7 \\ \overline{\mathcal{O}_{A_4+A_1}} & \textit{when } p = 5 \end{cases}$$

The Tables 3.2, 3.4 and 3.6 give details of the non-zero nilpotent orbits of G_2, F_4 and E_6. These are labelled using the Bala-Carter labelling which is described in Section 1.5. The tables also include the weighted Dynkin diagram of each orbit $\mathcal{O}_e$ and the sizes of the Jordan blocks of $\rho(e)$. These tables are followed by the corresponding Hasse diagram of the nilpotent orbits.

Orbit	Weighted Dynkin diagram	Size of Jordan blocks in Characteristic p	
		$p=5$	$p \geq 7$
G_2	2 2	-	$[7]$
$G_2(a_1)$	0 2	$[3^2, 1]$	
$\widetilde{A_1}$	1 0	$[3, 2^2]$	
A_1	0 1	$[2^2, 1^3]$	

Table 3.2: Nilpotent Orbits of G_2

$$\begin{array}{c} G_2 \\ | \\ G_2(a_1) \\ | \\ \widetilde{A_1} \\ | \\ A_1 \\ | \\ 0 \end{array}$$

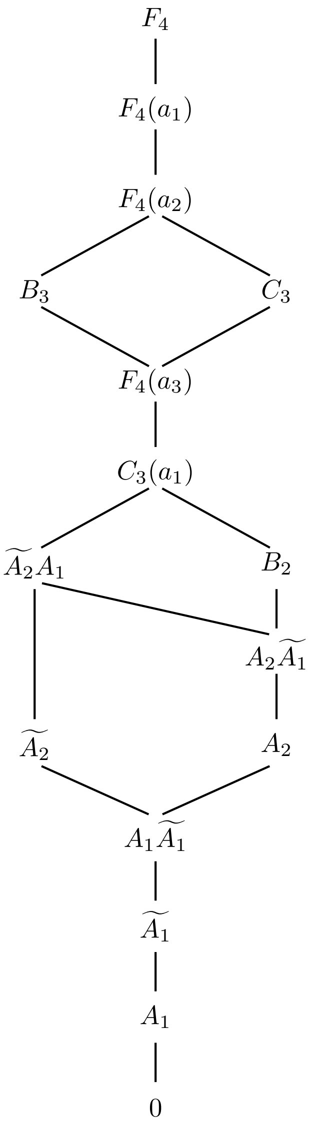
Figure 3.5: Hasse Diagram of Nilpotent Orbits of G_2

Orbits	Weighted Dynkin Diagram	Size of Jordan blocks in Characteristic p				
		$p=5$	$p=7$	$p=11$	$p=13$	$p \geq 17$
F_4	2 2 2 2	-	-	-	$[13^2]$	$[17, 9]$
$F_4(a_1)$	2 2 0 2	-	-	$[11, 9, 5, 1]$		
$F_4(a_2)$	0 2 0 2	-	$[7^3, 5]$	$[9, 7, 5^2]$		
B_3	2 2 0 0	-	$[7^3, 1^5]$			
C_3	1 0 1 2	-	$[7^2, 6^2]$	$[9, 6^2, 5]$		
$F_4(a_3)$	0 2 0 0	$[5^3, 3^3, 1^2]$				
$C_3(a_1)$	1 0 1 0	$[5^2, 4^2, 3, 2^2, 1]$				
$\widetilde{A_2}A_1$	0 1 0 1	$[5, 4^2, 3^2, 2^2]$				
B_2	2 0 0 1	$[5, 4^4, 1^5]$				
$A_2\widetilde{A_1}$	0 0 1 0	$[4^2, 3^3, 2^4, 1]$				
$\widetilde{A_2}$	0 0 0 2	$[5, 3^7]$				
A_2	2 0 0 0	$[3^6, 1^8]$				
$A_1\widetilde{A_1}$	0 1 0 0	$[3^3, 2^6, 1^5]$				
$\widetilde{A_1}$	0 0 0 1	$[3, 2^8, 1^7]$				
A_1	1 0 0 0	$[2^6, 1^{14}]$				

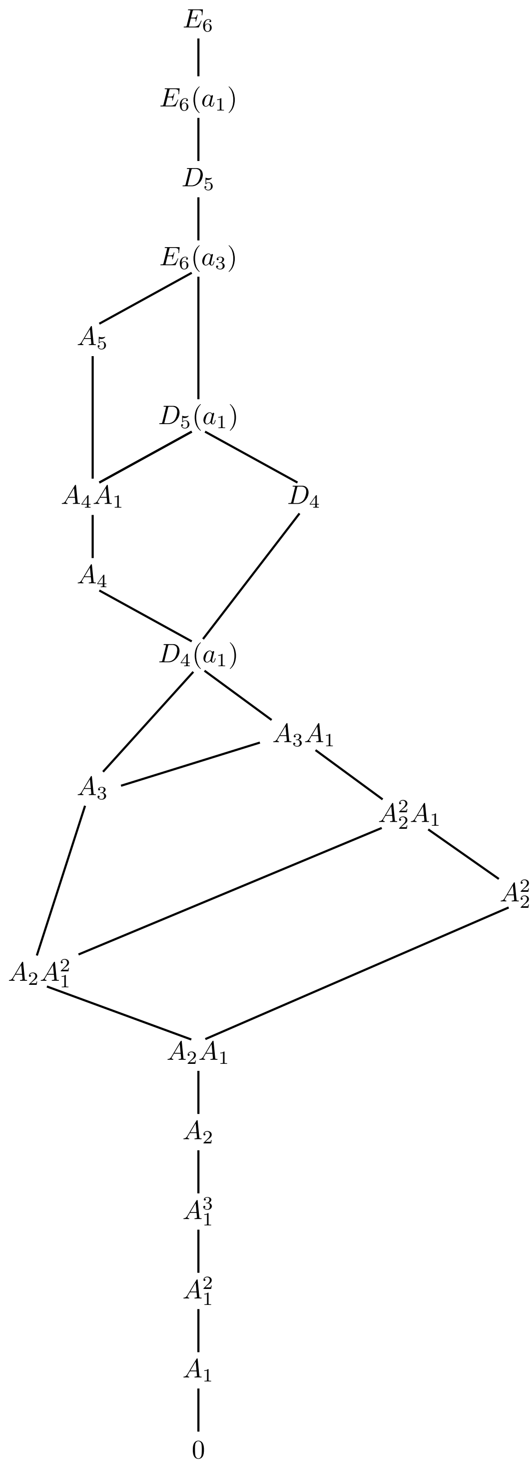
Table 3.4: Table of Nilpotent Orbits of F_4

Orbits	Weighted Dynkin Diagram	Size of Jordan blocks in Characteristic p				
		$p=5$	$p=7$	$p=11$	$p=13$	$p\geq 17$
E_6	$\begin{smallmatrix}2&2&2&2&2\\&&2&&\end{smallmatrix}$	-	-	-	$[13^2,1]$	$[17,9,1]$
$E_6(a_1)$	$\begin{smallmatrix}2&2&0&2&2\\&&2&&\end{smallmatrix}$	-	$[13,7^2]$	$[11^2,5]$	$[13,9,5]$	
D_5	$\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}$	-	-	$[11,9,5,1^2]$		
$E_6(a_3)$	$\begin{smallmatrix}2&0&2&0&2\\&&0&&\end{smallmatrix}$	-	$[7^3,5,1]$	$[9,7,5^2,1]$		
$D_5(a_1)$	$\begin{smallmatrix}1&1&0&1&1\\&&2&&\end{smallmatrix}$	-	$[7^3,3,2,1]$	$[8,7,6,3,2,1]$		
A_5	$\begin{smallmatrix}2&1&0&1&2\\&&1&&\end{smallmatrix}$	-	$[7^2,6^2,1]$	$[9,6^2,5,1]$		
A_4A_1	$\begin{smallmatrix}1&1&0&1&1\\&&1&&\end{smallmatrix}$	$[5^5,2]$	$[7,6,5,4,3,2]$			
D_4	$\begin{smallmatrix}0&0&2&0&0\\&&2&&\end{smallmatrix}$	-	$[7^3,1^6]$			
A_4	$\begin{smallmatrix}2&0&0&0&2\\&&2&&\end{smallmatrix}$	$[5^5,1]$	$[7,5^3,3,1^2]$			
$D_4(a_1)$	$\begin{smallmatrix}0&0&2&0&0\\&&0&&\end{smallmatrix}$	$[5^3,3^3,1^3]$				
A_3A_1	$\begin{smallmatrix}0&1&0&1&0\\&&1&&\end{smallmatrix}$	$[5^2,4^2,3,2^2,1^2]$				
$A_2^2A_1$	$\begin{smallmatrix}1&0&1&0&1\\&&0&&\end{smallmatrix}$	$[5,4^2,3^3,2^2,1]$				
A_3	$\begin{smallmatrix}1&0&0&0&1\\&&2&&\end{smallmatrix}$	$[5,4^4,1^6]$				
$A_2A_1^2$	$\begin{smallmatrix}0&1&0&1&0\\&&0&&\end{smallmatrix}$	$[4^2,3^3,2^4,1^2]$				
A_2^2	$\begin{smallmatrix}2&0&0&0&2\\&&0&&\end{smallmatrix}$	$[5,3^7,1]$				
A_2A_1	$\begin{smallmatrix}1&0&0&0&1\\&&1&&\end{smallmatrix}$	$[4,3^4,2^4,1^3]$				
A_2	$\begin{smallmatrix}0&0&0&0&0\\&&2&&\end{smallmatrix}$	$[3^6,1^9]$				
A_1^3	$\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}$	$[3^3,2^6,1^6]$				
A_1^2	$\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}$	$[3,2^8,1^8]$				
A_1	$\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}$	$[2^6,1^{15}]$				

Table 3.6: Table of Nilpotent Orbits of E_6



Hasse Diagram of Nilpotent Orbits of F_4



Hasse Diagram of Nilpotent Orbits of E_6

Chapter 4

The Nilpotent Commuting Variety and Induced Nilpotent Orbits

In this chapter we discuss Lusztig-Spaltenstein induction, which induces a nilpotent orbit in a Lie algebra $\mathfrak{g}$ from a nilpotent orbit of a Levi subalgebra of $\mathfrak{g}$. This is followed by a description of the nilpotent commuting variety including some results from [Pre03a] which utilize induced orbits. Throughout this chapter we assume that G be a connected reductive algebraic group over an algebraically closed field k of good characteristic p.

4.1 Lusztig-Spaltenstein Induction

Let $\mathfrak{p}$ be a parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ is a Levi subalgebra and $\mathfrak{u}$ is the unipotent radical. For a nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ in $\mathfrak{l}$ there is a unique nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ such that $\mathcal{O} \cap (\mathcal{O}_{\mathfrak{l}} + \mathfrak{u})$ is dense in $\mathcal{O}_{\mathfrak{l}} + \mathfrak{u}$. This orbit is the *induced orbit* and denoted $Ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$. This process is called *Lusztig-Spaltenstein* induction. This procedure was introduced by [LS79] for unipotent orbits in G with parabolic subgroup $P = L \ltimes U$. This is equivalent to our description in good characteristic by considering the homeomorphism $\mathfrak{U}(G) \to \mathcal{N}(\mathfrak{g})$ given in [SS70, Thm 3.12] where $\mathfrak{U}(G)$ is the set of unipotent elements in G. For characteristic zero a proof is presented in [CM93, Theorem 7.1.1]. The following result is from [LS79, Theorem 2.2]

Theorem 4.1.1 *Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ and $\mathfrak{p}' = \mathfrak{l} \oplus \mathfrak{u}'$ be two parabolic subalgebras of $\mathfrak{g}$ with the same Levi subalgebra $\mathfrak{l}$. For a nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ of $\mathfrak{l}$ we have $Ind_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = Ind_{\mathfrak{p}'}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$.*

Not all the orbits of $\mathfrak{g}$ can be induced from proper Levi subalgebras. Those that cannot are called *rigid orbits*. The following gives a sufficient condition for an orbit to be induced, see [LS79, Proposition 1.9].

Proposition 4.1.2 *Let $\mathfrak{g}$ be a simple Lie algebra with root system $\Delta = \{\alpha_1, \dots \alpha_n\}$ and nilpotent orbit $\mathcal{O}$ with weighted Dynkin diagram D. Let the vertices $\alpha_{i_1}, \dots \alpha_{i_s}$ of D be the those labelled with a 2 and let $I = \{1, \dots, n\} \backslash \{i_1, \dots, i_s\}$. Then $\mathcal{O} = Ind_{\mathfrak{p}_I}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}_I})$ for some nilpotent orbit $\mathcal{O}_{\mathfrak{l}_I}$ of $\mathfrak{l}_I$. Let D' be the subdiagram of D which only contains the nodes corresponding to I. If D' corresponds to a nilpotent orbit $\mathcal{O}'$ in $\mathfrak{l}$ then $\mathcal{O} = Ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$.*

Suppose $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are Levi subalgebra of $\mathfrak{g}$ such that $\mathfrak{l}_1 \subset \mathfrak{l}_2$ then $Ind_{\mathfrak{l}_2}^{\mathfrak{g}}(Ind_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_1})) = Ind_{\mathfrak{l}_1}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}_1})$. This is shown in [LS79, §1.7].

Example 4.1.3

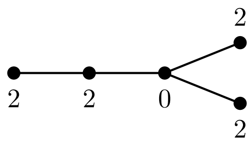
Let $\mathfrak{g} = D_5$ with maximal torus $\mathfrak{t}$ and simple roots $\Delta = \{\alpha_1, \dots, \alpha_5\}$ and positive roots Φ^+. Consider the parabolic subalgebra $\mathfrak{p}_{\alpha_3}$, then $\mathfrak{p}_{\alpha_3} = \mathfrak{l}_{\alpha_3} + \mathfrak{u}_{\alpha_3}$ where

$$\begin{aligned} \mathfrak{l}_{\alpha_3} &= \mathfrak{t} \oplus \mathfrak{g}_{\pm\alpha_3} \\ \mathfrak{u}_{\alpha_3} &= \sum_{\alpha \in \Phi^+ \backslash \{\alpha_3\}} \mathfrak{g}_\alpha \end{aligned}$$

Now let $\mathcal{O}_{\mathfrak{l}}$ be the zero nilpotent orbit in $\mathfrak{l}$. The induced orbit of $\mathcal{O}_{\mathfrak{l}}$ is an orbit $\mathcal{O}$ of $\mathfrak{g}$ such that $\mathcal{O} \cap \mathfrak{u}_{\alpha_3}$ is dense in $\mathfrak{u}_{\alpha_3}$.

Consider the orbit $\mathcal{O}_{D_5(a_1)}$ of $\mathfrak{g}$ then there exists an $e \in \mathcal{O}_{D_5(a_1)}$ such that there is an associated cocharacter λ where $\langle \lambda, \alpha_3 \rangle = 0$ and $\langle \lambda, \alpha_i \rangle = 2$ when $i \neq 3$. Now $\overline{G(\lambda, 0) \cdot e} = \mathfrak{g}(\lambda; 2)$ and since $\mathfrak{p}_{\alpha_3} = \sum_{i \geq 0} \mathfrak{g}(\lambda; i)$ then $\overline{P_{\alpha_3} \cdot e} = \sum_{i \geq 2} \mathfrak{g}(\lambda; i) = \mathfrak{u}_{\alpha_3}$. Therefore $\mathcal{O}_{D_5(a_1)} \cap \mathfrak{u}_{\alpha_3}$ is dense in $\mathfrak{u}_{\alpha_3}$. So $Ind_{\mathfrak{l}_{\alpha_3}}^{\mathfrak{g}}(\{0\}) = \mathcal{O}_{D_5(a_1)}$.

This can also be seen by considering the weighted Dynkin diagram of $D_5(a_1)$ which is



Then the subdiagram given by removing the nodes labelled by 2 leaves the single node α_3 which is labelled by 0. This corresponds to the zero orbit in $\mathfrak{l}_{\alpha_3}$.

4.2 Nilpotent Commuting Varieties

Let G be a connected reductive algebraic group with Lie algebra $\mathfrak{g} = Lie(G)$. The *nilpotent commuting variety* of $\mathfrak{g}$ is

$$\mathcal{C}^{nil}(\mathfrak{g}) = \{(x, y) \in \mathcal{N} \times \mathcal{N} : [x, y] = 0\}$$

It was proved in [Pre03a] that $\mathcal{C}^{nil}(\mathfrak{g})$ is equidimensional where *equidimensional* means that all the irreducible components have the same dimension. For a nilpotent orbit $\mathcal{O}_e$ of $\mathfrak{g}$ let $\mathcal{C}(\mathcal{O}_e) = \overline{G \cdot (e, \mathfrak{g}^e \cap \mathcal{N})}$.

Theorem 4.2.1 *[Pre03a] Let $e_1, \ldots, e_r$ be representatives of the distinguished nilpotent orbits of $\mathfrak{g}$. Then the sets $\mathcal{C}(e_i)$ are pairwise distinct and all have dimension equal to* $\dim(G, G)$ *and*

$$\mathcal{C}^{nil}(\mathfrak{g}) = \mathcal{C}(e_1) \cup \cdots \cup \mathcal{C}(e_r)$$

Suppose $\mathfrak{g}$ is a simple Lie algebra with simple roots $\Delta = \{\alpha_1, \ldots, \alpha_m\}$. A nilpotent element e in $\mathfrak{g}$ is *almost distinguished* if $\mathfrak{g}^e(0)$ is a torus. Therefore all distinguished elements are also almost distinguished. For example a representative of the nilpotent orbit $D_4(a_1)$ of E_6 is almost distinguished.

For a subset I in $\{1, \ldots, m\}$, let $\mathfrak{l}_I$ (resp. $\mathfrak{p}_I$) be the standard Levi (resp. parabolic) subalgebra of $\mathfrak{g}$ corresponding to I. For a subset $J \subset I$ let $\mathfrak{p}_{I,J}$ be the standard parabolic subalgebra of $\mathfrak{l}_I$ associated to J. Let $\lambda_{I,J}$ be a cocharacter contained in $\sum_{i \in I} \mathbb{Z}\alpha_i^\vee$ such that $\alpha_i(\lambda_{I,J}(t)) = 1$ when $i \in J$ and $\alpha_i(\lambda_{I,J}(t)) = t^2$ when $i \in I \backslash J$. Denote by λ_J the cocharacter $\lambda_{I,J}$ where $I = \{1, \ldots, m\}$.

Let e be an almost distinguished element. We may assume that e is distinguished in a standard Levi subalgebra $\mathfrak{l}_I$ for some subset I of $\{1, \ldots, m\}$ by Theorem 1.4.2 and that λ_I is an associated cocharacter for e. Let $\widetilde{e}$ be an element of $\mathfrak{p}_I$ in $\mathfrak{g}(\lambda_I; 2)$ such that $[\mathfrak{p}_I, \widetilde{e}] = \mathfrak{u}_I$, this is a *Richardson element* of $\mathfrak{p}_J$. We may assume that $\widetilde{e}$ is contained in $e + \sum\limits_{\alpha \in \Phi \backslash \Phi_I^+} \mathfrak{g}_\alpha = e + \mathfrak{u}_I$ and so $\widetilde{e}$ is in $Ind_{\mathfrak{l}_I}^{\mathfrak{g}}(L \cdot e)$.

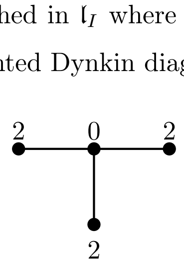
This is equivalent to extending the weighted Dynkin diagram of e in $\mathfrak{l}_I$ to a weighted Dynkin diagram for $\mathfrak{g}$ with the new nodes labelled by 2. We can let $\widetilde{e}$ be a representative to the nilpotent orbit in $\mathfrak{g}$ given by this extended Dynkin diagram. It can be observed that $\widetilde{e}$ is always a distinguished element of $\mathfrak{g}$. The following proposition is shown by combining [Pre03a, Prop 3.6] and the proof of [Pre03a, Theorem 3.7].

Proposition 4.2.2 *Let e be an almost distinguished element in $\mathfrak{g}$ and define $\widetilde{e}$ as above. Let $\widetilde{e_1}, \ldots, \widetilde{e_q}$ be representatives of distinguished nilpotent orbits in $\mathfrak{g}$ such that $G \cdot \widetilde{e_i} \subset \overline{G \cdot \widetilde{e}}$. Then*

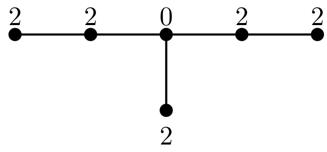
$$\mathcal{C}(e) \subset \bigcup_{1 \le i \le q} \mathcal{C}(\widetilde{e_i})$$

Example 4.2.3

Consider the case when $\mathfrak{g} = E_6$. A representative e of $\mathcal{O}_{D_4(a_1)}$ is almost distinguished. We may assume that e is distinguished in $\mathfrak{l}_I$ where $I = \{2, 3, 4, 5\}$. Now e is subregular in $\mathfrak{l}_I$ and so has corresponding weighted Dynkin diagram



Therefore let $J = \{4\}$ and the extended weighted Dynkin diagram is



This weighted Dynkin diagram corresponds to the nilpotent orbit $E_6(a_1)$ in E_6, therefore let $\widetilde{e}$ be a representative of $E_6(a_1)$. Then Proposition 4.2.2 gives

$$\mathcal{C}(D_4(a_1)) \subset \mathcal{C}(E_6(a_1)) \cup \mathcal{C}(E_6(a_3)).$$

Chapter 5

Research Question and Methodology

In this chapter we state the two questions we wish to answer and give an outline of the methods we use to answer them. Throughout let G be a reductive algebraic group over an algebraically closed field k of characteristic p and let $\mathfrak{g} = Lie(G)$.

5.1 Research Questions

For p good the commuting variety $\mathcal{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} : [x, y] = 0\}$ is irreducible. This was shown in [Ric79] for $p = 0$ and later extended to all good characteristic in [Lev02]. As was discussed in Section 4.2, the irreducible components of the nilpotent commuting variety were found in [Pre03a] for good characteristic p. The aim of this thesis is to consider a similar question for the *restricted nilpotent commuting variety* given by

$$\mathcal{C}_1^{nil}(\mathfrak{g}) = \{(x, y) \in \mathcal{N}_1 \times \mathcal{N}_1 : [x, y] = 0\}.$$

First we consider the following lemma.

Lemma 5.1.1 *Suppose $\mathcal{O}_{e_1}, \ldots, \mathcal{O}_{e_n}$ are the nilpotent orbits of $\mathfrak{g}$ contained in $\mathcal{N}_1$ with representatives $e_1, \ldots, e_n$. Let the irreducible components of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$ be $X_i^{(1)}, \ldots, X_i^{(n_i)}$. Then*

$$\mathcal{C}_1^{nil}(\mathfrak{g}) = \bigcup_{i,j} \overline{G \cdot (e_i, X_i^{(j)})} \text{ for } i = 1, \ldots, m, \ \ j = 1, \ldots, n_i.$$

Proof. Consider $e \in \mathcal{N}_1$ then clearly $(e, \mathfrak{g}^e \cap \mathcal{N}_1) \subset \mathcal{C}_1^{nil}(\mathfrak{g})$, therefore $\overline{G \cdot (e, \mathfrak{g}^e \cap \mathcal{N}_1)} \subset \mathcal{C}_1^{nil}(\mathfrak{g})$. We can express the restricted nilpotent commuting variety by the following union

$$\mathcal{C}_1^{nil}(\mathfrak{g}) = \bigcup_{1 \leq i \leq n} \overline{G \cdot (e_i, \mathfrak{g}^e \cap \mathcal{N}_1)}$$

Hence the result holds. ■

Thus every irreducible component of $\mathcal{C}_1^{nil}(\mathfrak{g})$ is of the form $\overline{G \cdot (e_i, X_i^{(j)})}$ for some i, j. Our aim is to answer the following two questions.

Question 1 *For $\mathfrak{g} = G_2$, F_4 and E_6 with nilpotent orbits $\mathcal{O}_{e_1}, \dots, \mathcal{O}_{e_n}$ and p good, find the irreducible components of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$.*

Question 2 *Find the irreducible components of $\mathcal{C}_1^{nil}(\mathfrak{g})$ for $\mathfrak{g} = G_2$, F_4 and E_6 and p good.*

When $p \geq h$, where h is the Coxeter number, then $\mathcal{N}_1 = \mathcal{N}$ and so $\mathfrak{g}^e \cap \mathcal{N}_1 = \mathfrak{g} \cap \mathcal{N}$ is irreducible [Pre03a]. Under this condition on the characteristic we have $\mathcal{C}_1^{nil}(\mathfrak{g}) = \mathcal{C}^{nil}(\mathfrak{g})$ and therefore the irreducible components of $\mathcal{C}_1^{nil}(\mathfrak{g})$ are given by Theorem 4.2.1. Since the Coxeter number for G_2 is 6 and p good implies that $p > 3$ then only $p = 5$ needs to be considered for Questions 1 and 2. The Coxeter number for both F_4 and E_6 is 12 therefore we need to consider $p = 5$, 7 and 11. The following two sections give an outline of the method we used to answer these questions. The details of the computations for answering these questions for $\mathfrak{g} = G_2$, F_4 and E_6 are given in Chapters 6 to 9.

5.2 Methodology for Question 2

The irreducible components found in Question 1 give us the form of the components of $C_1^{nil}(\mathfrak{g})$ as given by Lemma 5.1.1. Then all that remains is to eliminate some of the sets $\overline{G \cdot (e_i, X_i^{(j)})}$. To achieve this the following result from [Lev07, Lemma 1.1] is useful.

Proposition 5.2.1 *Let $\overline{G \cdot (e_i, X_i^{(j)})}$ be an irreducible component of $\mathcal{C}_1^{nil}(\mathfrak{g})$ for some irreducible component $X_i^{(j)}$ of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$. Then $X_i^{(j)} \subset \overline{(G \cdot e_i)}$.*

Therefore some components can be eliminated by finding elements in $X_i^{(j)}$ that are not contained in $\overline{G \cdot e_i}$. We do this by computing the Jordan normal form of the minimal faithful representation of an element in $X_i^{(j)}$ to show it is in a nilpotent orbit which is not contained in $\overline{G \cdot e_i}$. For the remaining components we need to check whether they are contained in any other component using transverse slice arguments and Proposition 4.2.2.
If $\mathfrak{g}^e \cap \mathcal{N}_1$ is irreducible for a nilpotent orbit $\mathcal{O}_e$ then let $\mathcal{C}_1(\mathcal{O}_e) = \overline{G \cdot (e, \mathfrak{g}^e \cap \mathcal{N}_1)}$. Now $\dim(\mathcal{C}_1(\mathcal{O}_e)) \leq \dim(\mathfrak{g})$ and equality holds when e is distinguished. This is because when e is distinguished then $\overline{G \cdot (e, \mathfrak{g}^e \cap \mathcal{N}_1)}$ is irreducible of dimension $\dim(G, G) - rank(G^e) = \dim(\mathfrak{g})$. Therefore, when e is distinguished, $\mathcal{C}_1(\mathcal{O}_e)$ is an irreducible component of $\mathcal{C}_1^{nil}(\mathfrak{g})$. Note that $rank(G^e) = rank(C)$ and $rank(C)$ is the dimension of maximal torus in C.

5.3 Methodology for Question 1

To answer Question 1 let e be a representative of a nilpotent orbit of $\mathfrak{g}$. Then by Theorem 2.3.2 there exists an associated cocharacter λ of e and

$$\mathfrak{g}^e = \mathfrak{c} \oplus \mathfrak{g}^e(\lambda; 1) \oplus \mathfrak{g}^e(\lambda; 2) \oplus \dots$$

where $\mathfrak{c}$ is the reductive part of $\mathfrak{g}^e$ and is isomorphic to a direct sum of Lie algebras. For an element $c + x_1 + x_2 + \dots$ of $\mathfrak{g}^e$ to be contained in $\mathcal{N}_1$ then c must belong in $\mathcal{N}_1(\mathfrak{c})$. Therefore let $c_1, \dots, c_n$ be representatives of the nilpotent orbits of $\mathfrak{c}$. Then define $\mathcal{M}_i^{(j)}$ to be the irreducible components of $(c_i + \mathfrak{g}^e(\lambda; > 0)) \cap \mathcal{N}_1$. Every irreducible component of $\mathfrak{g}^e \cap \mathcal{N}_1$ is of the form $\widetilde{\mathcal{M}_i}^{(j)} = \overline{C \cdot \mathcal{M}_i^{(j)}}$ for some j and $C = G^e \cap G^\lambda$. This holds by the following two results, for detail see [Hum75, §1.3 and §1.4]

Proposition 5.3.1 *A product of two irreducible affine varieties is irreducible.*

Proposition 5.3.2 *Let X be a topological space. Then a subspace Y of X is irreducible if and only if its closure $\overline{Y}$ is irreducible. Also* $\dim(Y) = \dim(\overline{Y})$.

Therefore $C \times X$ is irreducible if X is irreducible. Now $C \cdot X$ is the image of the map $C \times X \to \mathfrak{g}^e$ and therefore is irreducible since the image of a morphism from an irreducible variety is irreducible [Hum75, §1.3]. Once the $\mathcal{M}_i^{(j)}$ have been found, all that remains is to establish the inclusions $\widetilde{\mathcal{M}_i}^{(j)} \subset \widetilde{\mathcal{M}_{i'}}^{(j')}$.

To do this we consider [LT11] which gives a complete description of $\mathfrak{g}^e$ in terms of the grading $\mathfrak{g}^e(\lambda, i)$ for a given associated cocharacter λ for e. The cocharacter λ is presented diagrammatically in [LT11] by a Dynkin diagram with the node corresponding to α_i labelled by the λ-weight of α_i. From now on we denote $\mathfrak{g}(\lambda, i) = \mathfrak{g}(i)$. Also [LT11] specifies the precise structure of $\mathfrak{c}$ via a system of simple root elements. Similarly the maximal weight vectors in $\mathfrak{g}^e(i)$ for the action of $\mathfrak{c}$ are specified. This allows one to construct bases $u_1, \dots, u_s$ for each of the $\mathfrak{g}^e(i)$. Therefore every element in $\mathfrak{g}^e$ can be expressed as the following finite sum for some $a_i, b_i, \dots \in k$

$$\underbrace{c}_{\mathfrak{c}} + \underbrace{a_1 u_1 + \dots + a_s u_s}_{\mathfrak{g}^e(1)} + \underbrace{b_1 v_1 + \dots + b_t v_t}_{\mathfrak{g}^e(2)} + \dots \in \mathfrak{g}^e \text{ for all } a_i, b_i, \dots \in k,\ c \in \mathfrak{c}$$

Consider the minimal faithful representation ρ of $\mathfrak{g}$ as calculated by [GAP12]. Let $\gamma_i = \rho(c_i)$, $U_i = \rho(u_i)$, etc. and let $\mathcal{M}_i = \{\gamma_i + a_1 U_1 + \dots + a_p U_p + b_1 V_1 + \dots + b_q V_q + \dots : a_i, b_i \dots \in k\}$. Testing when an element $M_i \in \mathcal{M}_i$ satisfies $M_i^p = 0$ gives polynomial conditions on the coefficients $a_i, b_i, \dots \in k$. These polynomial conditions can be found using the [GAP12] code presented in the Appendix. Now let $\widetilde{\mathcal{M}_i} = \overline{C \cdot \{\gamma_i + a_1 U_1 + \dots : M_i^p = 0\}}$ which is contained in $\mathcal{N}_1$. We can now determine the irreducible components $\widetilde{\mathcal{M}_i^{(1)}}, \widetilde{\mathcal{M}_i^{(2)}}, \dots$ of $\widetilde{\mathcal{M}_i}$. Every irreducible component of $\mathfrak{g}^e \cap \mathcal{N}_1$ is equal to one of the sets $\widetilde{\mathcal{M}_i^{(j)}}$. We just need to establish inclusions $\widetilde{M_i^{(j)}} \subset \widetilde{M_{i'}^{(j')}}$ which are calculated case by case. The [GAP12] code used throughout this thesis, including the set up of each nilpotent orbit and the tests to verify the code is correct, can be found in the Lancaster University repository.

To calculate the dimension of each irreducible component $\mathfrak{g}^e \cap \mathcal{N}_1$ we can use the following results, see [Sha72, Ch1 §6 and Thm 7]

Proposition 5.3.3 *If X, Y are irreducible then* $\dim(X \times Y) = \dim(X) + \dim(Y)$.

A *fibre* of a morphism $\Phi : X \to Y$ is the closed set $\Phi^{-1}(y)$ for some $y \in Y$. If X is irreducible then we say Φ is *dominant* if $\Phi(X)$ is dense in Y.

Theorem 5.3.4 *Let $\Phi : X \to Y$ be a morphism of irreducible varieties such that $\Phi(X) = Y$. Then* $\dim(Y) \leq \dim(X)$ *and*

(i) $\dim(\Phi^{-1}(y)) \geq \dim(X) - \dim(Y)$ *for every $y \in Y$*

(ii) in Y there exists a non-empty open set W such that $\dim(\Phi^{-1}(y)) = \dim(X) - \dim(Y)$ *for $y \in W$.*

Lemma 5.3.5 *Let G be a reductive algebraic group and let $e \in \mathcal{N}_1$; then by Theorem 2.3.2, $G^e = C \ltimes U^e$. Let $c_i \in \mathcal{N}_1(\mathfrak{c})$ and $V = \{u \in \mathfrak{u}^e : c_i + u \in \mathcal{N}_1\}$ where $\mathfrak{c} = Lie(C)$ and $\mathfrak{u}^e = Lie(U^e)$. Finally let X be an irreducible component of V. Then*

$$\dim(C \cdot (c_i + X)) = \dim(X) + \dim(C \cdot c_i)$$
$$\Rightarrow \dim(C \cdot (c_i + X)) = \dim(X) + (\dim(\mathfrak{c}) - \dim(\mathfrak{c}^{c_i}))$$

Proof. To see this first note that since X is a component of V then it must also be a component of $C^{c_i} \cdot X$. Moreover, all component of $C^{c_i} \cdot X$ must be translates of X therefore $\dim(C^{c_i} \cdot X)$ is equal to $\dim(X)$. Now consider the following commuting diagram where $\pi(g \cdot (c_i + x)) = g \cdot c_i$ for $g \in C$ and $x \in X$.

$$\begin{tikzcd}
\mathfrak{c} \oplus X \arrow[d, two heads] & C \cdot (c_i + X) \arrow[l, hook'] \arrow[d, two heads, "\pi"] \\
\mathfrak{c} & C \cdot c_i \arrow[l, hook']
\end{tikzcd}$$

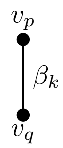
Now $\pi(g \cdot (c_i + z)) = c_i \Leftrightarrow g \in C^{c_i}$, therefore $\pi^{-1}(c_i) = c_i + (C^{c_i} \cdot X)$. Since $\dim(C^{c_i} \cdot X) = \dim(X)$ then $\dim(\pi^{-1}(c_i)) = \dim(X)$. Moreover $\pi^{-1}(g \cdot c_i) = g \cdot \pi^{-1}(c_i)$ for any $g \in C$, therefore $\dim(\pi^{-1}(g \cdot c_i)) = \dim(X)$ for all $g \in C^{c_i}$. By Theorem 5.3.4 there is a subset W which is open in $C \cdot c_i$ such that $\dim(\pi^{-1}(w)) = \dim(C \cdot (c_i + X)) - \dim(C \cdot c_i)$. Now let $w = g \cdot c_i \in W$; then

$$\dim(\pi^{-1}(g \cdot c_i)) = \dim(C \cdot (c_i + X)) - \dim(C \cdot c_i)$$
$$\Rightarrow \dim(X) = \dim(C \cdot (c_i + X)) - \dim(C \cdot c_i)$$
$$\Rightarrow \dim(C \cdot (c_i + X)) = \dim(X) + \dim(C \cdot c_i)$$

■

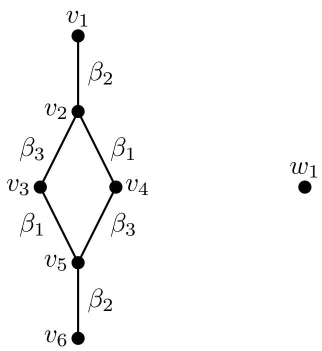
By the definition of $C = G^\lambda \cap G^e$, it is clear that $\mathfrak{g}^e(i)$ is C-stable. A description of the submodules of $\mathfrak{g}^e(i)$ with respect to the action of C is given in [LT11]. These submodules are always irreducible for G_2, F_4 and E_6; although this is not the case for E_7 and E_8. One example

of when $\mathfrak{g}^e(i)$ does not decompose into irreducible submodules is when e is contained in the nilpotent orbit $A_3A_2A_1$ of E_6 when the $char(k) = 5$. For more details see [LT11, §8].
We can represent the action of C on the positive part of $\mathfrak{g}^e$ diagrammatically. An irreducible submodule is represented by a connected graph with the highest weight vectors at the top. Let C have simple roots $\beta_1, \ldots, \beta_n$. Then [LT11] specifies the highest weight vectors of the submodules of $\mathfrak{g}^e(i)$ for the action of C. Then if two elements v_p, v_q in a submodule are such that $v_p = [e_{\beta_k}, v_q]$ then this can be represented by



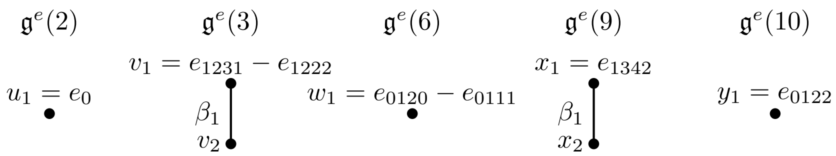
Example 5.3.6

Consider the $\widetilde{A_1}$ orbit of F_4 which has representative $e = e_{0001}$. The reductive part $\mathfrak{c} = \mathfrak{sl}_4$ has three simple roots where $e_{\beta_1} = e_{1000}$, $e_{\beta_2} = e_{0100}$ and $e_{\beta_3} = e_{1242}$. Then $\mathfrak{g}^e(2)$ has two submodules with maximal weight vectors $v_1 = e_{1222}$ and $w_1 = e_{0001}$. Then these submodules can be represented pictorially via



Example 5.3.7

Let $\mathfrak{g} = F_4$ with characteristic $p = 11$. Consider the nilpotent orbit denoted C_3 which has representative $e = e_{0001} + e_{0010} + e_{0100}$. Then [LT11] tells us that $\mathfrak{c} = \mathfrak{sl}_2$ with simple root $e_{\beta_1} = e_{2342}$ and gives us the elements of $\mathfrak{g}^e(i)$ as follows.



Now $\mathfrak{sl}_2$ has one non-zero nilpotent orbit $\mathcal{O}_{[2]}$ with representative $c_1 = e_{\beta_1}$. Then we have

$$\mathcal{M}_0 = a_1u_1 + b_1v_1 + b_2v_2 + c_1w_1 + d_1x_1 + d_2x_2 + g_1y_1$$

$$\mathcal{M}_1 = \mathcal{M}_0 + e_{\beta_1}$$

for $a_i, b_i, \dots \in k$. By considering M_0^{11} and M_1^{11} we can show that $\mathcal{M}_0 \subset \mathcal{N}_1$ and $\mathcal{M}_1 \subset \mathcal{N}_1$. In this case we have $\widetilde{\mathcal{M}_1} \cong \mathcal{N}(\mathfrak{sl}_2) \times \widetilde{\mathcal{M}_0}$; however in general the relationship is more complicated. In fact in this case $\mathfrak{g}^e \cap \mathcal{N}_1 = \mathfrak{g}^e \cap \mathcal{N}$ and by Lemma 2.3.4 it is irreducible. Now $\dim(\widetilde{\mathcal{M}_0}) = \dim(\mathcal{M}_0) = 7$ and since the dimension of the nilpotent orbit corresponding to $[2]$ in $\mathfrak{sl}_2$ is 2 then $\dim(\widetilde{\mathcal{M}_1}) = 7 + 2 = 9$. Therefore $\mathfrak{g} \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_1}$ is irreducible of dimension 9.

Chapter 6

G_2 Results

In this chapter we give the details of the computations to answer Questions 1 and 2 for G_2. Each section presents the details of a nilpotent orbit $\mathcal{O}_e$ of G_2. For each nilpotent orbit a representative e, as given by [LT11], is stated along with the form of $\mathfrak{c}$ and its simple root elements e_{β_l}. For the cocharacter λ, as given by [LT11], the highest weight vectors of $\mathfrak{g}^e(\lambda; j) = \mathfrak{g}^e(j)$ for the action of C on the positive part of $\mathfrak{g}^e$ are stated. Then we construct bases $u_1, \ldots, u_s$ for each $\mathfrak{g}^e(j)$. We represent this diagrammatically where an irreducible submodule is represented by a connected graph with the highest weight vectors at the top.

For each nilpotent orbit $\mathcal{O}_{c_i}$ of $\mathfrak{c}$, we give an explicit description of each $\mathcal{M}_i$ as discussed in Chapter 5. We define $\mathcal{M}_i = \{\gamma_i + a_1U_1 + \cdots + a_pU_p : a_i \in k\}$ where $\gamma_i = \rho(c_i)$ and $U_i = \rho(u_i)$ for the minimal faithful representation ρ of $\mathfrak{g}$. We denote the set corresponding to the zero orbit in $\mathfrak{c}$ as $\mathcal{M}_0$, therefore $\mathcal{M}_i = \gamma_i + \mathcal{M}_0$ for each nilpotent orbit $\mathcal{O}_{c_i}$ in $\mathfrak{c}$. We denote the representative of the regular orbit of $\mathfrak{c}$ as c_1.

For $M_i \in \mathcal{M}_i$, we also present the conditions on the coefficients given by $M_i^p = 0$ which are calculated using [GAP12]. Then $\widetilde{\mathcal{M}_i} = \overline{C \cdot \{\gamma_i + a_1U_1 + \ldots a_qU_q \cdots : M_i^p = 0\}}$. At the end of each section the arguments to determine the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are presented. The chapter concludes by presenting the arguments for determining the irreducible components of $\mathcal{C}_1^{nil}(G_2)$.

6.1 Orbits G_2 and $G_2(a_1)$

Since these orbits are distinguished, then by Corollary 2.3.4, if $e \in \mathcal{N}_1$ then $\mathfrak{g}^e \subset \mathcal{N}_1$ and so $\mathfrak{g}^e \cap \mathcal{N}_1 = \mathfrak{g}^e$. For these orbits $\mathfrak{c}$ is trivial therefore $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_0}$ and so $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component $\widetilde{\mathcal{M}_0}$.

Below is a table which contains a representative for each orbit, the characteristics p for which $e \in \mathcal{N}_1$ and the dimension of $\widetilde{\mathcal{M}_0}$. A basis for $\mathfrak{u}^e$ is not stated but can be found in [LT11].

Orbit	Representative e	Characteristic p	Dimension of $\widetilde{\mathcal{M}_0}$
$G_2(a_1)$	$e_{01} + e_{31}$	$p \geq 5$	4
G_2	$e_{10} + e_{01}$	$p \geq 7$	2

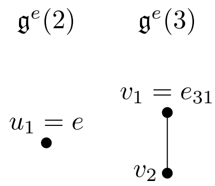
6.2 Orbit $\widetilde{A_1}$

For this orbit $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component $\widetilde{\mathcal{M}_1}$.

$e = e_{10}$

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = e_{32}$



$M_0 = a_1U_1 + b_1V_1 + b_2V_2$

$M_1 = e_{\beta_1} + M_0$

$$M_1^5 = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 2 + 3 = 5$$
$$M_0^5 = 0 \quad \dim(\widetilde{\mathcal{M}_0}) = 3$$

For any $c_i, c_j \in \mathfrak{c} \cap \mathcal{N}_1$ such that $c_j + \mathfrak{u}^e \in \mathcal{N}_1$, Lemma 2.3.5 states that if $C \cdot c_i \subset \overline{C \cdot c_j}$ then $C \cdot (c_i + \mathfrak{u}^e) \subset \overline{C \cdot (c_j + \mathfrak{u}^e)}$. Since $\mathcal{M}_1$ corresponds to the unique maximal orbit in $\mathcal{N}_1(\mathfrak{c})$ and $e_{\beta_1} + \mathfrak{u}^e \subset \mathcal{N}_1$ then by Lemma 2.3.5 we have $\mathfrak{u}^e \subset \overline{e_{\beta_1} + \mathfrak{u}^e}$. Therefore $\mathcal{M}_0 \subset \widetilde{\mathcal{M}_1}$.

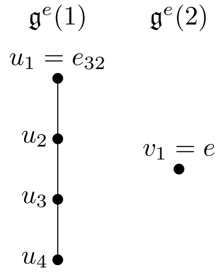
6.3 Orbit A_1

For this orbit we have $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_1}$.

$e = e_{01}$

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = e_{21}$



$M_0 = a_1U_1 + \cdots + a_4U_4 + b_1V_1$

$M_1 = e_{\beta_1} + M_0$

$$\begin{aligned} M_1^5 = 0 \Leftrightarrow a_4 = 0 \quad & \dim(\widetilde{\mathcal{M}_1}) = 2 - 1 + 5 = 6 \\ M_0^5 = 0 \quad & \dim(\widetilde{\mathcal{M}_0}) = 5 \end{aligned}$$

To show $\mathcal{M}_0 \subset \widetilde{\mathcal{M}_1}$ we first note that, by identifying C with SL_2, the cocharacter $\beta_1^\vee : k^\times \to C$, is given by $\beta_1^\vee(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. Now consider the set $e_{\beta_1} + \{u \in \mathfrak{u}^e : a_4 = 0\} \subset \widetilde{\mathcal{M}_1}$ then $Ad_{\beta_1^\vee(t)}(e_{\beta_1} + \{u \in \mathfrak{u}^e : a_4 = 0\}) \subset \widetilde{\mathcal{M}_1}$. By considering the action of $\beta_1^\vee$ on $\mathfrak{g}^e(1)$, which is a 4-dimensional irreducible C-module and so is isomorphic to $S^3(k^2)$, we get:

$$\begin{aligned} \overline{t^2 e_{\beta_1} + \{u \in \mathfrak{u}^e : a_4 = 0\}} &\subset \widetilde{\mathcal{M}_1} \\ \Rightarrow \{u \in \mathfrak{u}^e : a_4 = 0\} &\subset \widetilde{\mathcal{M}_1} \end{aligned}$$

We can denote an element of $\{u \in \mathfrak{u}^e : a_4 = 0\}$ by a series of column vectors as follows:

$$\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix}, b_1 \right)$$

We can view k^n as the symmetric tensor $S^{n-1}(k^2)$. Let $\omega_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\omega_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Then we consider $\mathfrak{g}(1)$ as $S^3(k^2)$. To do this we identify u_1 (respectively u_2, u_3 and u_4) with the symmetric tensor $\omega_1 \otimes \omega_1 \otimes \omega_1$ (respectively $\omega_1 \otimes \omega_1 \otimes \omega_2$, $\omega_1 \otimes \omega_2 \otimes \omega_2$ and $\omega_2 \otimes \omega_2 \otimes \omega_2$). This may require scaling the u_i by some (possibly different) factors. Now consider the element $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \in C$, we want to calculate $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \cdot \omega_1 = \omega_1 + \lambda\omega_2$ and $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \cdot \omega_2 = \omega_2$ then

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} (\omega_1 \otimes \omega_1 \otimes \omega_1) &= (\omega_1 + \lambda\omega_2) \otimes (\omega_1 + \lambda\omega_2) \otimes (\omega_1 + \lambda\omega_2) \\
&= (\omega_1 \otimes \omega_1 \otimes \omega_1) + 3\lambda(\omega_1 \otimes \omega_1 \otimes \omega_2) + \\
&\qquad 3\lambda^2(\omega_1 \otimes \omega_2 \otimes \omega_2) + \lambda^3(\omega_2 \otimes \omega_2 \otimes \omega_2) \\
\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} (\omega_1 \otimes \omega_1 \otimes \omega_2) &= (\omega_1 + \lambda\omega_2) \otimes (\omega_1 + \lambda\omega_2) \otimes \omega_2 \\
&= (\omega_1 \otimes \omega_1 \otimes \omega_2) + 2\lambda(\omega_1 \otimes \omega_2 \otimes \omega_2) + \lambda^2(\omega_2 \otimes \omega_2 \otimes \omega_2) \\
\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} (\omega_1 \otimes \omega_2 \otimes \omega_2) &= (\omega_1 + \lambda\omega_2) \otimes \omega_2 \otimes \omega_2 \\
&= (\omega_1 \otimes \omega_2 \otimes \omega_2) + \lambda(\omega_2 \otimes \omega_2 \otimes \omega_2) \\
\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} (\omega_2 \otimes \omega_2 \otimes \omega_2) &= \omega_2 \otimes \omega_2 \otimes \omega_2
\end{aligned}$$

Therefore $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ 3\lambda a_1 + a_2 \\ 3\lambda^2 a_1 + 2\lambda a_2 + a_3 \\ \lambda^3 a_1 + \lambda^2 a_2 + \lambda a_3 \end{pmatrix}$.

Hence

$$\left(\begin{pmatrix} a_1 \\ 3\lambda a_1 + a_2 \\ 3\lambda^2 a_1 + 2\lambda a_2 + a_3 \\ \lambda^3 a_1 + \lambda^2 a_2 + \lambda a_3 \end{pmatrix}, b_1 \right) \subset \widetilde{M_1}$$

For all but finitely many a_1', a_2', a_3' and a_4' in k we can find λ, a_1, a_2 and a_3 such that $a_1' = a_1$, $a_2' = 3\lambda a_1 + a_2$ etc. Hence $\widetilde{\mathcal{M}_1}$ contains a dense subset of $\mathcal{M}_0$, so $\mathcal{M}_0 \subset \widetilde{\mathcal{M}_1}$.

6.4 Irreducible Components of $C_1^{nil}(G_2)$

In this section we calculate the irreducible components of $\mathcal{C}_1^{nil}(G_2)$. If $\mathcal{O}_e \subset \mathcal{N}_1$ is distinguished then $\mathcal{C}_1(\mathcal{O}_e)$ is an irreducible component of $\mathcal{C}_1^{nil}(\mathfrak{g})$. For the remaining orbits $\mathcal{O}_{e_i}$ of G_2 there is an element in each irreducible component of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$ that is not contained in $\overline{G \cdot e_i}$. In each case $\mathfrak{g}^e \cap \mathcal{N}_1$ is irreducible. To show that an element $e' \in \mathfrak{g}^e \cap \mathcal{N}_1$ is not contained in $\overline{(G \cdot e)}$ we find its Jordan normal form. This is done by considering the rank of successive powers of its 7-dimensional representation. For the orbit A_1 the element $e_{01} + e_{21}$ in $\mathfrak{g}^e \cap \mathcal{N}_1$ has Jordan normal form $[3^2, 1]$ and so is contained in orbit $G_2(a_1)$. Similarly for $\widetilde{A_1}$ the element $e_{10} + e_{32}$ also has Jordan normal form $[3^2, 1]$ and so also contained in $G_2(a_1)$. Therefore in both cases $\mathfrak{g}^e \cap \mathcal{N}_1 \not\subset \overline{G \cdot e}$ and so by Proposition 5.2.1 the irreducible components of $\mathcal{C}_1^{nil}(G_2)$ are given by

$$p = 5: \;\; \mathcal{C}_1^{nil}(G_2) = \mathcal{C}_1(G_2(a_1)).$$

Chapter 7

F_4 Results

In this chapter we give the details of the computations to answer Questions 1 and 2 for F_4. We group the nilpotent orbits of F_4 into sections; for each orbit in a given section the arguments used to find the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are similar.

For each nilpotent orbit a representative e, as given by [LT11], is stated along with the form of $\mathfrak{c}$ and its simple root elements e_{β_l}. For the cocharacter λ, as given by [LT11], the highest weight vectors of $\mathfrak{g}^e(\lambda; j) = \mathfrak{g}^e(j)$ for the action of C on the positive part of $\mathfrak{g}^e$ are stated. We represent this diagrammatically where an irreducible submodule is represented by a connected graph with the highest weight vectors at the top. There is also be an explicit description of each $\mathcal{M}_i$ as discussed in Chapter 5. At the end of each section the arguments to determine the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are presented.

The chapter concludes by presenting the arguments for determining the irreducible components of $\mathcal{C}_1^{nil}(F_4)$ for each characteristic $p = 5, 7, 11$.

7.1 Orbits F_4, $F_4(a_1)$, $F_4(a_2)$ and $F_4(a_3)$

First we consider the distinguished orbits of F_4. Since each of these orbits is distinguished, then by Corollary 2.3.4 if $e \in \mathcal{N}_1$, then $\mathfrak{g}^e \subset \mathcal{N}_1$ and so $\mathfrak{g}^e \cap \mathcal{N}_1 = \mathfrak{g}^e$. For these orbits $\mathfrak{c}$ is trivial therefore $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_0}$ and so $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component.

Below is a table which contains a representative for each orbit, the characteristics p for which $e \in \mathcal{N}_1$ and the dimension of $\widetilde{\mathcal{M}_0}$. A basis for $\mathfrak{u}^e$ is not stated but can be found in [LT11].

Orbit	Representative e	Characteristic p	Dimension of $\widetilde{\mathcal{M}_0}$
$F_4(a_3)$	$e_{0100} + e_{1120} + e_{1111} + e_{0121}$	$p \geq 5$	12
$F_4(a_2)$	$e_{1110} + e_{0001} + e_{0120} + e_{0100}$	$p \geq 7$	8
$F_4(a_1)$	$e_{0100} + e_{1000} + e_{0120} + e_{0001}$	$p \geq 11$	6
F_4	$e_{1000} + e_{0100} + e_{0010} + e_{0001}$	$p \geq 13$	4

7.2 Orbits A_2, $\widetilde{A_2}$, B_2, $C_3(a_1)$, B_3 and C_3

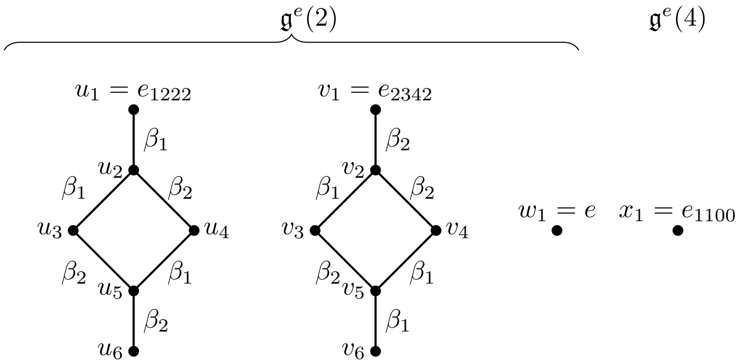
For each of these orbits $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component with the exception of A_2 for $p = 5$ and $\widetilde{A_2}$ for $p = 7$. The $\widetilde{A_2}$ case for $p = 7$ is considered in Section 7.4. In the A_2 case for $p = 5$, the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_2}$ (see below for details). Otherwise the method to show that $\widetilde{\mathcal{M}_i} \subset \widetilde{\mathcal{M}_j}$ is the same and is considered at the end of this section.

Orbit A_2

$e = e_{1000} + e_{0100},\ f = 2f_{1000} + 2f_{0100}$

$\mathfrak{c} \cong \mathfrak{sl}_3$

$e_{\beta_1} = e_{0001},\ e_{\beta_2} = e_{1231},\ e_{\beta_3} = [e_{\beta_1}, e_{\beta_2}]$.



$M_0 = a_1U_1 + \cdots + a_6U_6 + b_1V_1 + \cdots + b_6V_6 + c_1W_1 + d_1X_1$

$M_1 = e_{\beta_1} + e_{\beta_2} + M_0$

$M_2 = e_{\beta_3} + M_0$

Characteristic $p = 5$:

$$M_1^5 = 0 \Leftrightarrow a_6 = b_6 = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 6 - 2 + 14 = 18$$
$$M_2^5 = 0 \quad \dim(\widetilde{\mathcal{M}_2}) = 4 + 14 = 18$$

Characteristic $p \geq 7$:

$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 6 + 14 = 20$$

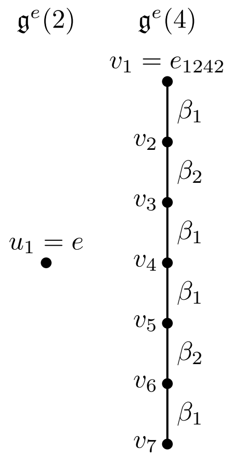
For $p = 5$ we have $\dim(\widetilde{\mathcal{M}_1}) = \dim(\widetilde{\mathcal{M}_2})$ and clearly $\widetilde{\mathcal{M}_1} \neq \widetilde{\mathcal{M}_2}$. Therefore $\mathfrak{g}^e \cap \mathcal{N}_1$ has two irreducible components $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_2}$.

Orbit $\widetilde{A_2}$

$e = e_{0010} + e_{0001},\ f = 2f_{0010} + 2f_{0001}$

$\mathfrak{c} \cong G_2$

$e_{\beta_1} = e_{0111} - e_{0120},\ e_{\beta_2} = e_{1000}$



$M_0 = a_1U_1 + b_1V_1 + b_2V_2 + b_3V_3 + b_4V_4 + b_5V_5 + b_6V_6 + b_7V_7$

Nilpotent Orbit in $\mathfrak{c}$	Representative e of nilpotent orbit	M_i label of $e + M_0$
G_2	$e_{\beta_1} + e_{\beta_2}$	M_1
$G_2(a_1)$	$e_{\beta_2} + e_{3\beta_1+\beta_2}$	M_2
$\widetilde{A_1}$	e_{β_1}	M_3
A_1	e_{β_2}	M_4

Characteristic $p = 5$:

Since $\mathcal{N}(G_2) = \overline{\mathcal{O}_{G_2(a_1)}}$ for $p = 5$, we do not consider the regular orbit.

$$M_2^5 = 0 \quad \dim(\widetilde{\mathcal{M}_2}) = 10 + 8 = 18$$

Characteristic $p = 7$:

$$M_1^7 = 0 \Leftrightarrow b_7 = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 12 - 1 + 8 = 19$$
$$M_2^5 = 0 \qquad\qquad \dim(\widetilde{\mathcal{M}_2}) = 10 + 8 = 18$$

Characteristic $p = 11$:

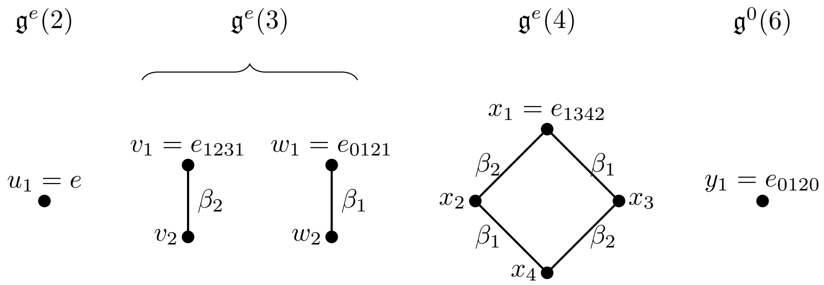
$$M_1^{11} = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 12 + 8 = 20$$

Orbit B_2

$e = e_{0100} + e_{0010}$, $f = 4f_{0100} + 3f_{0010}$

$\mathfrak{c} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$

$e_{\beta_1} = e_{0122}$, $e_{\beta_2} = e_{2342}$



$M_0 = a_1U_1 + b_1V_1 + b_2V_2 + c_1W_1 + c_2W_2 + d_1X_1 + \cdots + d_4X_4 + g_1Y_1$

$M_1 = e_{\beta_1} + e_{\beta_2} + M_0$

$M_2 = e_{\beta_1} + M_0$

$M_3 = e_{\beta_2} + M_0$

Characteristic $p \geq 5$:

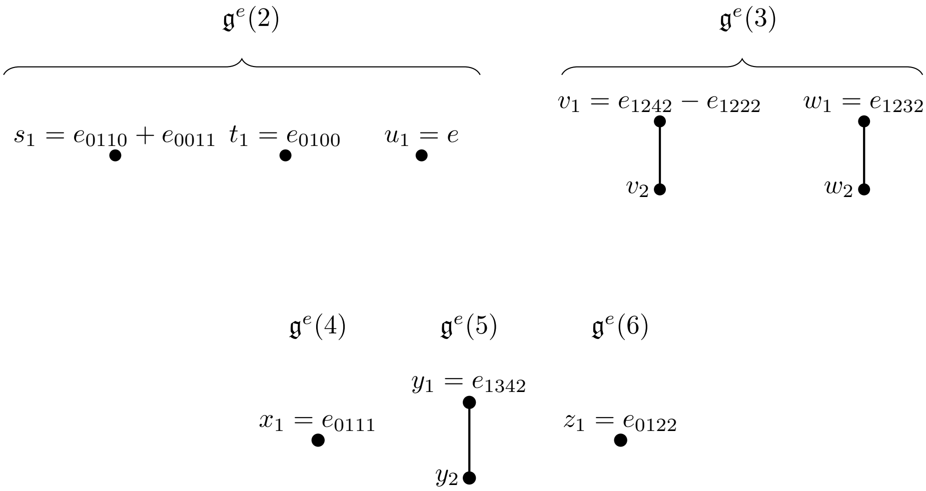
$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 2 + 2 + 10 = 14$$

Orbit $C_3(a_1)$

$e = e_{0001} + e_{0120} + e_{0100},\ f = 3f_{0001} + 4f_{0120} + f_{0100}$

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = e_{2342}$



$M_0 = a_1S_1 + b_1T_1 + c_1U_1 + d_1V_1 + d_2V_2 + g_1W_1 + g_2W_2 + i_1X_1 + j_1Y_1 + j_2Y_2 + k_1Z_1$

$M_1 = e_{\beta_1} + M_0$

Characteristic $p \geq 5$:

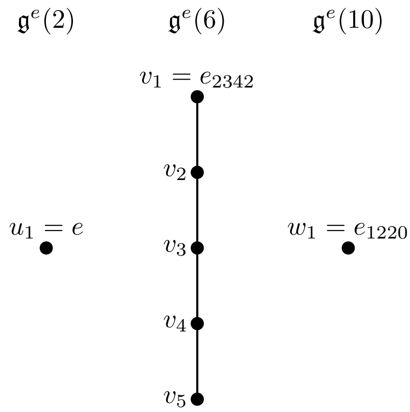
$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 11 + 2 = 13$$

Orbit B_3

$e = e_{1000} + e_{0100} + e_{0010},\ f = 6f_{1000} + 10f_{0100} + 6f_{0010}$

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = e_{1111} - e_{0121}$



$M_0 = a_1U_1 + b_1V_1 + \cdots + b_5V_5 + c_1W_1$

$M_1 = e_{\beta_1} + M_0$

Characteristic $p \geq 7$:

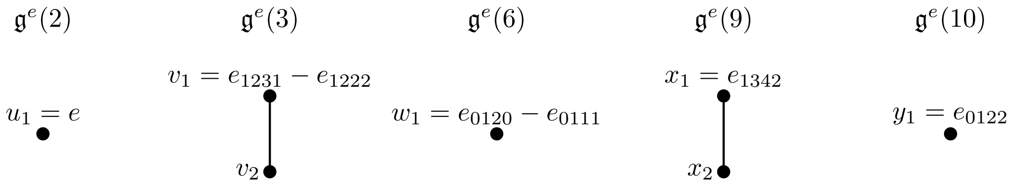
$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 2 + 7 = 9$$

Orbit C_3

$e = e_{0001} + e_{0010} + e_{0100}$, $f = 5f_{0001} + 8f_{0010} + 9f_{0100}$

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = e_{2342}$



$M_0 = a_1U_1 + b_1V_1 + b_2V_2 + c_1W_1 + d_1X_1 + d_2X_2 + g_1Y_1$

$M_1 = e_{\beta_1} + M_0$

Characteristic $p \geq 7$:

$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 2 + 7 = 9$$

In each of these cases we have $\mathcal{M}_j = c_j + \mathfrak{u}^e \subset \mathcal{N}_1$ for each j. If $C \cdot c_i \subset \overline{C \cdot c_j}$ then by Lemma 2.3.5 we have $C \cdot (c_i + \mathfrak{u}^e) \subset \overline{C \cdot (c_j + \mathfrak{u}^e)}$ i.e. $\widetilde{\mathcal{M}_i} \subset \widetilde{\mathcal{M}_j}$. In particular since $\mathcal{M}_1$ corresponds to the unique maximal orbit in $\mathcal{N}_1(\mathfrak{c})$, (with the exception of $\widetilde{A_2}$ for $p = 5$ where $\mathcal{M}_2$ is the maximal orbit), it follows that in each case $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_1}$, except for $\widetilde{A_2}$ for $p = 5$ where $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_2}$.

7.3 Orbits $A_2\widetilde{A_1}$ and $\widetilde{A_2}A_1$

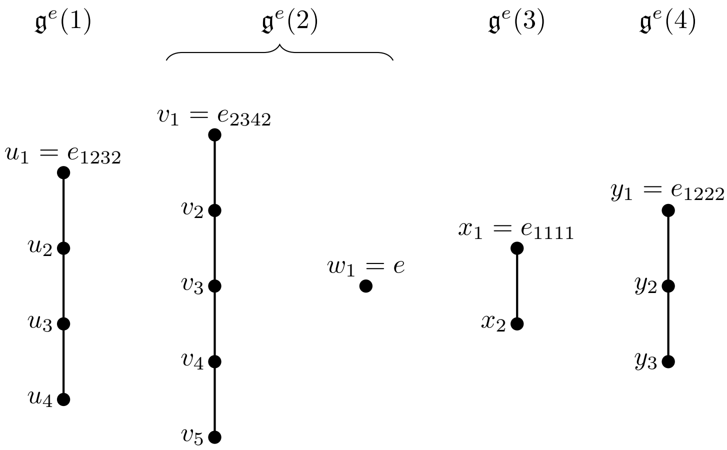
For these orbits we have $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_1}$ with the exception of $A_2\widetilde{A_1}$ for $p = 5, 7$ where $\mathfrak{g}^e \cap \mathcal{N}_1$ has two irreducible components $\widetilde{\mathcal{M}_0}$ and $\widetilde{\mathcal{M}_1}$. For the cases $\widetilde{A_2}A_1$ for $p = 11$ and $A_2\widetilde{A_1}$ for $p = 5, 7$ the arguments are the same as in the previous section. For the remaining cases the required results can be shown using the same argument as in Section 6.3.

Orbit $A_2\widetilde{A_1}$

$e = e_{1000} + e_{0100} + e_{0001}$, $f = 2f_{1000} + 2f_{0100} + f_{0001}$

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = 2e_{0122} + e_{1220} - e_{1121}$



$M_0 = a_1U_1 + \cdots + a_4U_4 + b_1V_1 + \cdots + b_5V_5 + c_1W_1 + d_1X_1 + d_2X_2 + g_1Y_1 + \cdots + g_3Y_3$

$M_1 = e_{\beta_1} + M_0$

Characteristic $p = 5, 7$:

$$\begin{array}{ll} M_1^p = 0 \Leftrightarrow a_4 = 0,\ b_5 = a_3^2 & \dim(\widetilde{\mathcal{M}_1}) = 2 - 2 + 15 = 15 \\ M_0^p = 0 & \dim(\widetilde{\mathcal{M}_0}) = 15 \end{array}$$

Characteristic $p = 11$:

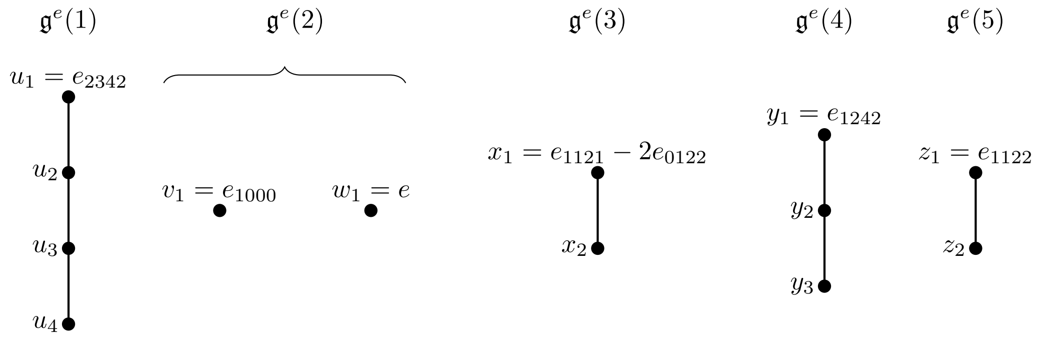
$$\begin{array}{ll} M_1^{11} = 0 \Leftrightarrow a_4 = 0 & \dim(\widetilde{\mathcal{M}_1}) = 2 - 1 + 15 = 16 \\ M_0^p = 0 & \dim(\widetilde{\mathcal{M}_0}) = 15 \end{array}$$

Orbit $\widetilde{A_2}A_1$

$e = e_{0010} + e_{0001} + e_{1000}$, $f = 2f_{0010} + 2f_{0001} + f_{1000}$.

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = e_{1222} - e_{1231}$



$M_0 = a_1U_1 + \cdots + a_4U_4 + b_1V_1 + c_1W_1 + d_1X_1 + d_2X_2 + g_1Y_1 + \cdots + g_3Y_3 + i_1Z_1 + i_2Z_2$

$M_1 = e_{\beta_1} + M_0$

Characteristic $p = 5$:

$$\begin{array}{ll} M_1^5 = 0 \Leftrightarrow a_4 = 0 & \dim(\widetilde{\mathcal{M}_1}) = 2 - 1 + 13 = 14 \\ M_0^5 = 0 & \dim(\widetilde{\mathcal{M}_0}) = 13 \end{array}$$

Characteristic $p = 7$:

$$\begin{array}{ll} M_1^7 = 0 \Leftrightarrow a_4 = 0 \text{ or } d_2 = 0 & \widetilde{\mathcal{M}_1} \text{ has two irreducible components of dimension 2-1+13=14} \\ M_0^5 = 0 & \dim(\widetilde{\mathcal{M}_0}) = 13 \end{array}$$

Characteristic $p = 11$:

$$M_1^{11} = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 2 + 13 = 15$$

For $\widetilde{A_2}A_1$ for $p = 7$ we have that $\widetilde{\mathcal{M}_1}$ is the union of two irreducible components X_1 and X_2 of dimension 14, where

$$X_1 = \overline{C \cdot (e_{\beta_1} + \{u \in \mathfrak{u}^e : a_4 = 0\})}$$
$$X_2 = \overline{C \cdot (e_{\beta_1} + \{u \in \mathfrak{u}^e : d_2 = 0\})}$$

By the argument in Section 6.3 we can show $\widetilde{\mathcal{M}_0} \subset X_1$, and therefore $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_1}$. Hence $\mathfrak{g}^e \cap \mathcal{N}_1$ has two irreducible components X_1 and X_2.

7.4 Orbits $A_1\widetilde{A_1}$, $\widetilde{A_2}$ when $p = 7$, A_1 and $\widetilde{A_1}$.

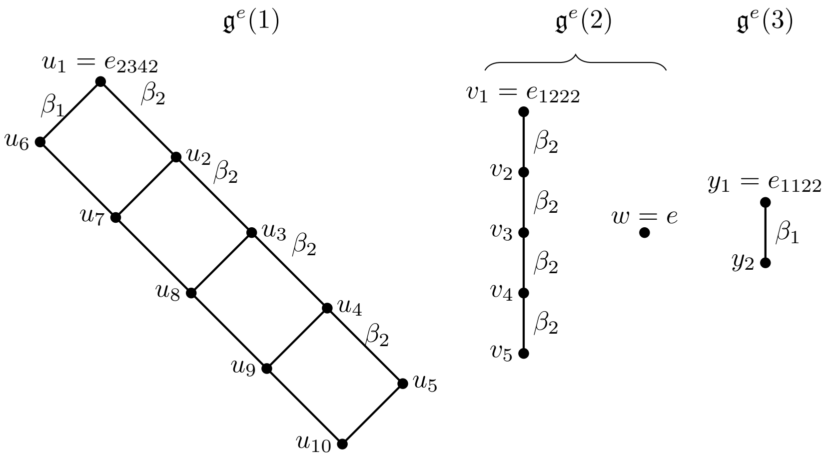
For these remaining orbits each case is considered separately.

Orbit $A_1\widetilde{A_1}$

$e = e_{1000} + e_{0001}$, $f = f_{1000} + f_{0001}$

$\mathfrak{c} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$

$e_{\beta_1} = e_{1242}$, $e_{\beta_2} = e_{1110} - e_{0111}$.



$$M_0 = a_1U_1 + \cdots + a_{10}U_{10} + b_1V_1 + \cdots + b_5V_5 + c_1W_1 + d_1Y_1 + d_2Y_2$$
$$M_1 = e_{\beta_1} + e_{\beta_2} + M_0$$
$$M_2 = e_{\beta_2} + M_0$$
$$M_3 = e_{\beta_1} + M_0.$$

Characteristic $p = 5$:

$$\begin{array}{ll} M_1^5 = 0 \Leftrightarrow a_{10} = 0,\ a_5 = a_9,\ b_5 = a_3a_5 + 4a_4a_8 + 4a_8^2 & \dim(\widetilde{\mathcal{M}_1}) = 2 + 2 - 3 + 18 = 19 \\ M_2^5 = 0 \Leftrightarrow a_{10} = a_5 = 0,\ b_5 = a_3a_9 - a_4a_8 & \dim(\widetilde{\mathcal{M}_2}) = 2 - 3 + 18 = 17 \\ M_3^5 = 0 \Leftrightarrow 4a_6a_8a_{10} + a_6a_9^2 + a_7^2a_{10} + 3a_7a_8a_9 + a_8^3 = 0 & \dim(\widetilde{\mathcal{M}_3}) = 2 - 1 + 18 = 19 \\ M_0^5 = 0 & \dim(\widetilde{\mathcal{M}_0}) = 18 \end{array}$$

Characteristic $p = 7, 11$:

$$\begin{array}{ll} M_1^p = 0 \Leftrightarrow a_{10} = 0 & \dim(\widetilde{\mathcal{M}_1}) = 2 + 2 - 1 + 18 = 21 \\ M_2^p = 0 & \dim(\widetilde{\mathcal{M}_2}) = 2 + 18 = 20 \\ M_3^p = 0 & \dim(\widetilde{\mathcal{M}_3}) = 2 + 18 = 20 \end{array}$$

Characteristic $p = 5$

Since $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_3}$ have the same dimension we only need to check if $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_3}$. Note that $\widetilde{\mathcal{M}_3}$ is irreducible as $\mathcal{M}_3$ is a hypersurface determined by an irreducible polynomial in $k[a_1, \ldots, a_{10}]$. This polynomial is irreducible because it is linear in a_{10} and the coefficient of a_{10} has no common factors with the constant term. An element in C which is contained in the copy of SL_2 with root element e_{β_1} (resp. e_{β_2}) is subscripted by e_{β_1} (resp. e_{β_2}).
Firstly consider elements of $\widetilde{\mathcal{M}_1}$ which have the form

$$e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_5 = a_9,\ a_{10} = 0,\ b_5 = a_3a_5 - a_4a_8 - a_8^2\}.$$

Then applying $Ad_{\beta_1^\vee(t)}$ gives:

$$t^2e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_5 = t^2a_9,\ ta_{10} = 0,\ b_5 = a_3a_9 - a_4a_8 - t^2a_8^2\} \subset \widetilde{\mathcal{M}_1} \quad \forall t \neq 0$$

Therefore it we take the formal limit as $t \to 0$ we get

$$e_{\beta_2} + \{u \in \mathfrak{u}^e : a_5 = a_{10} = 0, b_5 = a_3a_9 - a_4a_8\} \subset \widetilde{\mathcal{M}_1}$$
$$\Rightarrow \widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$$

To show that $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_3}$ let $P(a_1, \ldots, a_{10}) = 4a_6a_8a_{10} + a_6a_9^2 + a_7^2a_{10} + 3a_7a_8a_9 + a_8^3$. Consider elements in $\widetilde{\mathcal{M}_3}$ of the form $e_{\beta_1} + \{u \in \mathfrak{u}^e : P(a_1, \ldots, a_{10}) = 0\}$. Then applying $Ad_{\beta_1^\vee(t)}$ gives

$$t^2 e_{\beta_1} + \{u \in \mathfrak{u}^e : t^3 P(a_1, \ldots, a_{10}) = 0\} \subset \widetilde{\mathcal{M}_3}$$
$$\Rightarrow \{u \in \mathfrak{u}^e : P(a_1, \ldots, a_{10}) = 0\} \subset \widetilde{\mathcal{M}_3}$$

Let this set be X. Then X is an irreducible subset of $\mathcal{M}_0$ of codimension 1. Then the dimension of the set $\left\{ Ad_{\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)_{e_{\beta_1}}}(X) : \lambda \in k \right\}$ is strictly greater than the dimension of X. Therefore its closure is equal to $\widetilde{\mathcal{M}_0}$ and so $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_3}$. Therefore $\mathfrak{g}^e \cap \mathcal{N}_1$ has two irreducible components $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_3}$, both of dimension 19.

Characteristic $p = 7, 11$

We just need to show that $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_1}$. The set $e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_{10} = 0\}$ is contained in $\widetilde{\mathcal{M}_1}$. We can denote elements of $\mathfrak{g}(1)$ via $\left(\begin{smallmatrix} a_1 & a_2 & \cdots & a_5 \\ a_6 & a_7 & \cdots & a_{10} \end{smallmatrix}\right)$. For $e_{\beta_1} + e_{\beta_2} + \left(\begin{smallmatrix} a_1 & \cdots & a_5 \\ a_6 & \cdots & a_{10} \end{smallmatrix}\right)$ to be in $\mathcal{N}_1$ we require

$$(ad_{e_{\beta_1}})(ad_{e_{\beta_2}})^4 \begin{pmatrix} a_1 & \ldots & a_5 \\ a_6 & \ldots & a_{10} \end{pmatrix} = 0.$$

Therefore for any non-zero nilpotent element $x \in \mathcal{O}_{e_{\beta_1}}$ the condition for $x + e_{\beta_2} + \left(\begin{smallmatrix} a_1 & \cdots & a_5 \\ a_6 & \cdots & a_{10} \end{smallmatrix}\right) \in \mathcal{N}_1$ is given by

$$(ad_x)(ad_{e_{\beta_2}})^4 \begin{pmatrix} a_1 & \ldots & a_5 \\ a_6 & \ldots & a_{10} \end{pmatrix} = 0.$$

Now

$$(ad_{e_{\beta_2}})^4 \begin{pmatrix} a_1 & \ldots & a_5 \\ a_6 & \ldots & a_{10} \end{pmatrix} = \begin{pmatrix} 24a_5 & 0 & \ldots & 0 \\ 24a_{10} & 0 & \ldots & 0 \end{pmatrix}$$

So let $x \in \mathcal{O}_{e_{\beta_1}}$ be of the form $x = \left(\begin{smallmatrix} a_5a_{10} & -a_5^2 \\ a_{10}^2 & -a_5a_{10} \end{smallmatrix}\right)$. Then

$$ad_x \begin{pmatrix} 24a_5 & 0 & \ldots & 0 \\ 24a_{10} & 0 & \ldots & 0 \end{pmatrix} = 0.$$

Therefore since ξx is also in $\mathcal{O}_{e_{\beta_1}}$ then $\xi x + e_{\beta_2} + \left(\begin{smallmatrix} a_1 & \cdots & a_5 \\ a_6 & \cdots & a_{10} \end{smallmatrix}\right)$ is contained in $\widetilde{\mathcal{M}_1}$ for any $\xi \neq 0$. Taking the closure gives $\mathcal{M}_2 \subset \widetilde{\mathcal{M}_1}$.
To show that $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_1}$ first we consider

$$Ad_{\beta_2^\vee(t)}(e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_{14} = 0\}) \subset \widetilde{\mathcal{M}_1}$$
$$\Rightarrow e_{\beta_1} + t^2 e_{\beta_2} + \{u \in \mathfrak{u}^e : t^4 a_{10} = 0\} \subset \widetilde{\mathcal{M}_1}$$
$$\Rightarrow e_{\beta_1} + \{u \in \mathfrak{u}^e : a_{10} = 0\} \subset \widetilde{\mathcal{M}_1}$$

Let this set be X. Then X is an irreducible subset of M_3 of codimension 1. Then the dimension of the set $\left\{ Ad_{\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)_{e_{\beta_2}}}(X) : \lambda \in k \right\}$ is strictly greater than the dimension of X. Therefore its closure is equal to $\widetilde{\mathcal{M}_3}$ and so $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_1}$. Therefore $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component of dimension 21.

Orbit $\widetilde{A_2}$ with $p = 7$

The details of this orbit are presented in section 7.4. By the methods used in Section 7.2 we can see that $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_4} \subset \widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_2}$, then $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component of dimension 20. All that remains to be shown is $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$. To do this we consider a transverse slice to $e' = e_{\beta_2} + e_{3\beta_1+\beta_2}$ in $\mathfrak{c}$. By embedding $\mathfrak{c} = \mathfrak{g}_2$ in $\mathfrak{so}_7$ via the action on the 7$-$dimensional module we get

$$e' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & -1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} \quad f' = \begin{pmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ 0 & -2 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & \\ 2 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 2 & 0 & \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the Slodowy slice to e' in $\mathfrak{g}_2$ is (by a computer calculation):

$$\mathcal{A} = (\mathfrak{c}^{f'} + e') \cap \mathcal{N}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 1 \\ c & -b & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 & 1 & 0 \\ d & 0 & c & -2b & 0 & 0 & 0 \\ 0 & d & 0 & -2c & b & 0 & 0 \end{pmatrix} : d^2 = 4(b^3 + c^3) \right\}$$

We can now parametrize $\mathcal{A}$. There is a surjective map given by

$$\frac{k[b,c,d]}{(d^2 - 4(b^3+c^3))} \to k[3s^4 + 6s^2t^2 - t^4, 3s^4 - 6s^2t^2 - t^4, st(3s^4 + t^4)]$$

which sends $b \mapsto 3s^4 + 6s^2t^2 - t^4$ etc. To see this we just have to check that these polynomials in s, t satisfy the equation $d^2 - 4(b^3 + c^3) = 0$. Since the two k-algebras are integral domains of the same Krull dimension, the kernel of this map must be trivial and so the rings are isomorphic. Therefore let $A_{s,t}$ be the element of $\mathcal{A}$ with $b = \frac{1}{4}(3s^4 + 6s^2t^2 - t^4)$, $c = -\frac{1}{4}(3s^4 - 6s^2t^2 - t^4)$ and $d = \frac{3}{2}st(3s^4 + t^4)$ for $s, t \in k$. For $(s,t) \neq (0,0)$ we can show that $A_{s,t}$ is contained in $\mathcal{O}_{G_2}$ by considering its Jordan normal form. Hence $A_{s,t}$ is conjugate to $e_{\beta_1} + e_{\beta_2}$ for all $(s,t) \neq (0,0)$. The reductive part $\mathfrak{c}$ of $\mathfrak{g}^e$ acts on $\mathfrak{g}^e(4)$. For $x \in \mathfrak{g}^e(4)$ the condition $b_7 = 0$ is equivalent to

$(ad(e_{\beta_1}+e_{\beta_2}))^6(x)=0$. By considering $v=(b_1,b_2,\dots,b_7)^t$ as a vector this condition becomes $\rho(e_{\beta_1}+e_{\beta_2})^6(v)=0$, where ρ is the representation of $\mathfrak{c}$ on $\mathfrak{g}^e(4)$.

Therefore we have

$$A_{s,t}+\{u\in\mathfrak{u}^e : A_{s,t}^6(v)=0,\ s,t\in k,\ (s,t)\neq(0,0)\}\subset\widetilde{\mathcal{M}_1}$$

Then by a computation using [GAP12], the condition $A_{s,t}^6(v)=0$ gives us the following polynomial

$$\begin{aligned}P_{s,t}:(s^2+2st+3t^2)^2(s^2+5st+3t^2)^2b_1+(s^2+st+3t^2)^2(s^2-st+3t^2)b_2\\ -st(s+2t)(s+5t)(s^2+4t^2)b_4+(s^2+st+3t^2)(s^2-st+3t^2)b_6\\ +(s^2+2st+3t^2)(s^2+5st+3t^2)b_7=0\end{aligned}$$

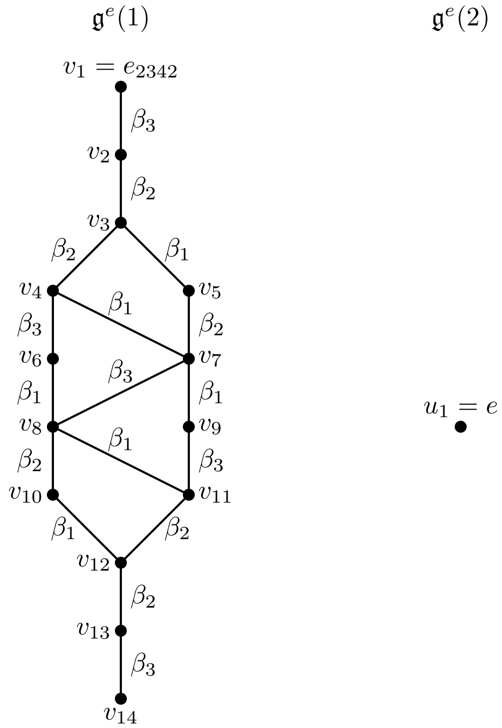
Now $P_{s,t}$ is an irreducible polynomial in $k[s,t,a_1,b_1,\dots,b_7]$ since $P_{s,t}$ is linear in the b_i's and their coefficients have no common factors. Therefore the set $X=\{(s,t,a_1,b_1,\dots,b_7):P_{s,t}=0\}$ is an irreducible hypersurface in $\mathbb{A}^{10}$ of dimension 9. Now $\{(0,0,a_1,b_1,\dots,b_7)\}\subset X$ is a closed subset of codimension 1 and so $X\backslash\{(0,0,a_1,b_1,\dots,b_7)$ is a non-empty open subset of X. Since all non-empty open subsets of an irreducible variety are dense then $\overline{X\backslash\{(0,0,a_1,b_1,\dots,b_7)\}}=X$. Therefore $A_{0,0}+X\subset\widetilde{\mathcal{M}_1}$ and so $\widetilde{\mathcal{M}_2}\subset\widetilde{\mathcal{M}_1}$.

Orbit A_1

$e=e_{1000},\ f=f_{1000}$

$\mathfrak{c}\cong\mathfrak{sp}_6$

$e_{\beta_1}=e_{0010},\ e_{\beta_2}=e_{0001},\ e_{\beta_3}=e_{1220}$.



$M_0 = a_1V_1 + a_2V_2 + \cdots + a_{14}V_{14} + b_1U_1$

Nilpotent Orbit in $\mathfrak{c}$	Representative e of nilpotent orbit	M_i label $e + M_0$
$[6]$	$e_{\beta_1} + e_{\beta_2} + e_{\beta_3}$	M_1
$[4,2]$	$e_{\beta_2} + e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3}$	M_2
$[4,1^2]$	$e_{\beta_2} + e_{\beta_3}$	M_3
$[3^2]$	$e_{\beta_1} + e_{\beta_2}$	M_4
$[2^3]$	$e_{\beta_1} + e_{\beta_3}$	M_5
$[2^2,1^2]$	$e_{\beta_1+2\beta_2+\beta_3}$	M_6
$[2,1^4]$	$e_{2\beta_1+2\beta_2+\beta_3}$	M_7

Characteristic $p = 5$:

Since $\mathcal{N}_1(\mathfrak{sp}_6) = \overline{\mathcal{O}_{[4,2]}}$ we do not consider the regular orbit for $p = 5$.

$$\begin{aligned}
&M_2^5 = 0 \Rightarrow a_{14} = a_{10} = 0,\ a_{13} = 2a_6 && \dim(\widetilde{\mathcal{M}_2}) = 16 - 3 + 15 = 28\\
&M_3^5 = 0 \Rightarrow a_7 = a_{10} = a_{14} = 0 && \dim(\widetilde{\mathcal{M}_3}) = 14 - 3 + 15 = 26\\
&M_4^5 = 0 \Rightarrow a_9 = a_{13} = 0 && \dim(\widetilde{\mathcal{M}_4}) = 14 - 2 + 15 = 27\\
&M_5^5 = 0 \Rightarrow a_{11} = 0 && \dim(\widetilde{\mathcal{M}_5}) = 12 - 1 + 15 = 26\\
&M_6^5 = 0 && \dim(\widetilde{\mathcal{M}_6}) = 10 + 15 = 25
\end{aligned}$$

Characteristic $p = 7$:

$$\begin{aligned}
&M_1^7 = 0 \Rightarrow a_{14} = 0,\ a_{11} = 4a_{10} && \dim(\widetilde{\mathcal{M}_1}) = 18 - 2 + 15 = 31\\
&M_2^7 = 0 \Rightarrow a_{14} = 0 && \dim(\widetilde{\mathcal{M}_2}) = 16 - 1 + 15 = 30\\
&M_3^7 = 0 && \dim(\widetilde{\mathcal{M}_3}) = 14 + 15 = 29\\
&M_4^7 = 0 && \dim(\widetilde{\mathcal{M}_4}) = 14 + 15 = 29
\end{aligned}$$

Characteristic $p = 11$:

$$\begin{aligned}
&M_1^{11} = 0 \Rightarrow a_{14} = 0 && \dim(\widetilde{\mathcal{M}_1}) = 18 - 1 + 15 = 32\\
&M_2^{11} = 0 && \dim(\widetilde{\mathcal{M}_2}) = 16 + 15 = 31
\end{aligned}$$

Characteristic $p = 5$

We claim that $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_2}$. To see this we need to show that $\widetilde{\mathcal{M}_6} \subset \widetilde{\mathcal{M}_5} \subset \widetilde{\mathcal{M}_4} \subset \widetilde{\mathcal{M}_2}$ and $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_2}$. The inclusion $\widetilde{\mathcal{M}_7} \subset \widetilde{\mathcal{M}_6}$ holds by the same methods as demonstrated in Section 7.2.

To show that $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_2}$ consider

$$e_{\beta_2} + e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3} + \{u \in \mathfrak{u}^e : a_{10} = a_{14} = 0, a_{13} = 2a_6\} \subset \widetilde{\mathcal{M}_2}.$$

Now let $\beta^\vee(t) = \beta_1^\vee(t)\beta_2^\vee(t)\beta_3^\vee(t)$ for $t \neq 0$. By considering γ-chains of weights in $\mathfrak{g}^e(1) = L(\omega_3)$ where $\gamma = 2\beta_1 + 2\beta_2 + \beta_3$ we get

$$\begin{aligned}
&Ad_{\beta^\vee(t)}(e_{\beta_2} + e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3} + \{u \in \mathfrak{u}^e : a_{10} = a_{14} = 0, a_6 = 3a_{13}\}) \subset \widetilde{\mathcal{M}_2}\\
&\quad\Rightarrow e_{\beta_2} + e_{\beta_3} + t^2 e_{2\beta_1+2\beta_2+\beta_3} + \{u \in \mathfrak{u}^e : a_{10} = a_{14} = 0, a_6 = 3t^2 a_{13}\} \subset \widetilde{\mathcal{M}_2}
\end{aligned}$$

By taking the limit as $t \to 0$ we get

$$\begin{aligned} e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_6 = a_{10} = a_{14} = 0\} \subset \widetilde{\mathcal{M}_2} \\ \Rightarrow \mathcal{M}_3 \subset \widetilde{\mathcal{M}_2} \end{aligned}$$

To show that $\widetilde{\mathcal{M}_4} \subset \widetilde{\mathcal{M}_2}$ we consider a transverse slice of $e' = e_{\beta_1} + e_{\beta_2}$ in $\mathfrak{c}$. Specifically, we consider $e' + \mathfrak{c}^{f'}$. With respect to the standard representation of elements of $\mathfrak{sp}_6$ as six-by-six matrices, we have

$$e' = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & -1 & \\ & & & & 0 & -1 \\ & & & & & 0 \end{pmatrix}, \quad f' = \begin{pmatrix} 0 & & & & & \\ 2 & 0 & & & & \\ & 2 & 0 & & & \\ & & & 0 & & \\ & & & -2 & 0 & \\ & & & & -2 & 0 \end{pmatrix} \in \mathfrak{c} \cong \mathfrak{sp}_6$$

A GAP computation shows that:

$$\mathcal{A} = (e' + \mathfrak{c}^{f'}) \cap \mathcal{N}_1 = \left\{ \begin{pmatrix} a & 1 & 0 & d & 0 & 0 \\ 0 & a & 1 & 0 & -d & 0 \\ 0 & 0 & a & 0 & 0 & d \\ b & 0 & 0 & -a & -1 & 0 \\ 0 & -b & 0 & 0 & -a & -1 \\ 0 & 0 & b & 0 & 0 & -a \end{pmatrix} : a^2 + db = 0 \right\}$$

We get an isomorphism of $k[a,b,d]/(2a^2 + 2db)$ with $k[st, -t^2, s^2]$ since $a^2 + db$ is irreducible. Therefore let $A_{s,t}$ be the element in $\mathcal{A}$ with $a = st$, $d = s^2$ and $b = -t^2$ for $(s,t) \neq (0,0)$, so

$$A_{s,t} = \begin{pmatrix} st & 1 & 0 & s^2 & 0 & 0 \\ 0 & st & 1 & 0 & -s^2 & 0 \\ 0 & 0 & st & 0 & 0 & s^2 \\ -t^2 & 0 & 0 & -st & -1 & 0 \\ 0 & t^2 & 0 & 0 & -st & -1 \\ 0 & 0 & -t^2 & 0 & 0 & -st \end{pmatrix}$$

Note that the calculation of the transverse slice and parametrization are independent of characteristic greater than or equal to 5.

The reductive part $\mathfrak{c}$ acts on $\mathfrak{g}^e(1)$, so for $x \in \mathfrak{g}^e(1)$, the conditions $a_{14} = a_{10} = 0$ and $a_{13} = 2a_6$ are equivalent to

$$ad(e_{\beta_2} + e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3})^4(x) = 0$$

$$[ad(e_{\beta_2} + e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3})^3(x), x] = 0$$

Then considering $v = (a_1, a_2, \dots, a_{14})^t$ as a vector these conditions become

$$\rho(e_{\beta_2} + e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3})^4(v) = 0$$
$$\langle \rho(e_{\beta_2} + e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3})^3(v), v\rangle = 0$$

where $\langle \cdot\,, \cdot\rangle$ is the unique alternating C-equivariant form on $\mathfrak{g}^e(1) = k^{14}$. Here ρ is the representation of $\mathfrak{c}$ on $\mathfrak{g}^e(1)$. Let $T = \rho(A_{s,t})$; then by a [GAP12] calculation $T^4(v) = 0$ implies

$$a_9 = s^2a_{12} - sta_7 \tag{7.1}$$
$$a_{13} = sta_{12} - t^2a_7 \tag{7.2}$$

Now after substituting (7.1) and (7.2) then

$$T^3(v) = \begin{pmatrix} s^6a_{14} + 2s^5ta_{10} + 4s^5ta_{11} + 3s^4t^2a_4 + 2s^4t^2a_5 + 2s^3t^3a_1 \\ 2s^2a_{10} + 2s^2a_{11} + sta_4 + 3sta_5 + a_7 \\ 4s^2a_{12} + sta_7 \\ s^5ta_{14} + 2s^4t^2a_{10} + 4s^4t^2a_{11} + 3s^3t^3a_4 + 2s^3t^3a_5 + 2s^2t^4a_1 \\ 3s^5ta_{14} + s^4t^2a_{10} + 2s^4t^2a_{11} + 4s^3t^3a_4 + s^3t^3a_5s^2t^4a_1 \\ 2sta_{10} + 2sta_{11} + t^2a_4 + 3t^2a_5 + a_{12} \\ 0 \\ 4sta_{12} + t^2a_7 \\ 0 \\ 2s^4t^2a_{14} + s^3t^3a_{10} + 2s^3t^3a_{11} + 4s^2t^4a_4 + s^2t^4a_5st^5a_1 \\ 0 \\ 0 \\ 3s^3t^3a_{14} + s^2t^4a_{10} + 2s^2t^4a_{11} + 4st^5a_4 + st^5a_5 + t^6a_1 \end{pmatrix}$$

So $\langle T^3(u), u\rangle = 0$ implies

$$\begin{aligned} &4s^6a_{14}^2 + 4s^5ta_{10}a_{14} + 2s^5ta_{11}a_{14} + 4s^4t^2a_5a_{14} + 2s^4t^2a_{10}^2 + s^4t^2a_{10}a_{14} + 4s^4t^2a_{11}^2 \\ &+ s^3t^3a_1a_{14} + 2s^3t^3a_4a_{11} + 4s^3t^3a_5a_{10} + s^3t^3a_5a_{11} + 3s^2t^4a_1a_{10} + 4s^2t^4a_1a_{11} + s^2t^4a_4^2 \\ &\quad + 2s^2t^4a_4a_5 + 2s^2t^4a_5^2 + 3st^5a_1a_{14} + 3st^5a_1a_5 + t^6a_1^2 + s^2a_{12}^2 + 3sta_7a_{12} + t^2a_7^2 = 0 \end{aligned}$$

Letting $t = \xi s$ gives $a_{13} = \xi s^2a_{12} - \xi s^2a_7$, $a_9 = s^2a_{12} - \xi s^2a_7$ and

$$s^2a_{12}^2 - 2\xi s^2a_7a_{12} + \xi^2s^2a_7^2 + s^6(4a_{14}^2 + 4\xi a_{10}a_{14} + \dots) = 0$$
$$\Rightarrow a_{12}^2 - 2\xi a_7a_{12} + \xi^2a_7^2 + s^4(4a_14^2 + 4\xi a_{10}a_{14} + \dots) = 0$$

Therefore

$$A_{s,\xi s} + \{u \in \mathfrak{u}^e : a_{13} = \xi s^2a_{12} - \xi s^2a_7, a_9 = s^2a_{12} - \xi s^2a_7,$$
$$(a_{12} - \xi a_7)^2 + s^4(4a_14^2 + 4\xi a_{10}a_{14} + \dots) = 0\} \subset \widetilde{\mathcal{M}_2}$$

Taking the limit at $s \to 0$ we get

$$A_{0,0} + \{u \in \mathfrak{u}^e : a_9 = a_{13} = 0, a_{12} = \xi a_7\} \subset \widetilde{\mathcal{M}_2}$$

Therefore as ξ varies then a_{12} can take any value as long as a_7 does not equal zero. Hence

$$\begin{aligned} A_{0,0} + \{u \in \mathfrak{u}^e : a_9 = a_{13} = 0\} &\subset \widetilde{\mathcal{M}_2} \\ \Rightarrow \widetilde{\mathcal{M}_4} &\subset \widetilde{\mathcal{M}_2} \end{aligned}$$

Next we show that $\widetilde{\mathcal{M}_5} \subset \widetilde{\mathcal{M}_4}$. It is straightforward to see by looking at the weight graph of $L(\omega_3)$ that if e' is of type $[3^2]$ then the condition $e'+x \in \mathcal{N}_1$ for $x \in \mathfrak{g}^e(1)$ is equivalent to $ad(e')^4(x) = 0$. In particular, if we have $e' = e_{\beta_1+\beta_2+\beta_3} + e_{2\beta_2+\beta_3}$ then this holds for $x = a_1u_1 + \dots$ if and only if $a_{14} = 0$. Now

$$e' = \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix} \; f' = \begin{pmatrix} 0 & 0 \\ I_3 & 0 \end{pmatrix}$$

then by a computer calculation a transverse slice of e' in $\mathfrak{c}$ is given by

$$\mathcal{A} = (\mathfrak{c}^{f'} + e') \cap \mathcal{N}_1 = \left\{ \begin{pmatrix} a & b & 0 & 1 & 0 & 0 \\ d & 0 & -b & 0 & 1 & 0 \\ 0 & -d & -a & 0 & 0 & 1 \\ 3bd & 2ab & 3b^2 & a & b & 0 \\ 2ad & 4bd & 2ab & d & 0 & -b \\ 3d^2 & 2ad & 3bd & 0 & -d & -a \end{pmatrix} : a^2 + 2bd = 0 \right\}$$

We get an isomorphism of $k[a, b, d]/(a^2+2bd)$ with $k[st, \frac{1}{2}s^2, -t^2]$ since $a^2+2bd = 0$ is irreducible. Therefore let $A_{s,t}$ be the element in $\mathcal{A}$ with $a = st$, $b = \frac{1}{2}s^2$, $d = -t^2$ and $(s,t) \neq (0,0)$ giving

$$A_{s,t} = \begin{pmatrix} st & \frac{1}{2}s^2 & 0 & 1 & 0 & 0 \\ -t^2 & 0 & -\frac{1}{2}s^2 & 0 & 1 & 0 \\ 0 & t^2 & -st & 0 & 0 & 1 \\ -\frac{3}{2}s^2t^2 & s^3t & \frac{3}{4}s^4 & st & \frac{1}{2}s^2 & 0 \\ -2st^3 & -2s^2t^2 & s^3t & -t^2 & 0 & -\frac{1}{2}s^2 \\ 3t^4 & -2st^3 & -\frac{3}{2}s^2t^2 & 0 & t^2 & -st \end{pmatrix}$$

Let ρ be the representation of $L(\omega_3) \cong k^{14}$ and let $T_{s,t} = \rho(A_{s,t})$ and $u = (a_1, a_2, \dots, a_{14})^t$. Now the condition $ad(e')^4(x) = 0$ is equivalent to $T^4(u) = 0$. A computer calculation using [GAP12] shows that $T^4(u) = 0$ if and only if

$$P_1 : t^4a_7 + 2st^3a_9 + s^2t^2a_{10} + 4s^2t^2a_{11} + 4s^3ta_{12} + 4s^4a_{13} = 0$$

$$\begin{aligned} P_2 : 3st^5a_2 + 4s^2t^4a_3 + 2s^3t^3a_4 + 3s^3t^3a_5 + 3s^4t^2a_6 + t^4a_7 + 2s^5ta_8 + st^3a_9 \\ + 3s^3ta_{12} + s^4a_{13} + 2sta_{14} = 0 \end{aligned}$$

By rearranging and letting $t = \xi s$ for some $\xi \in k^\times$ considered as a constant, gives

$$\begin{aligned} Q_{1,\xi}:\ a_{13} &= \xi^4 a_7 + 2\xi^3 a_9 + \xi^2 a_{10} + 4\xi^2 a_{11} + 4\xi a_{12} \\ Q_{2,s,\xi}:\ a_{14} &= 2s^2(\xi^3 a_7 + 2\xi a_9 + \xi a_{10} + 4\xi a_{11} + 4a_{12} + 3a_{12} + \xi a_9 + \xi^3 a_7) \\ &\qquad + 2s^4(2a_8 + 3\xi a_6 + 3\xi^2 a_5 + 2\xi^2 a_4 + 4\xi^3 a_3 + 3\xi a_2). \end{aligned}$$

Therefore

$$\begin{aligned} A_{s,\xi s} + \{u \in \mathfrak{u}^e : Q_{1,\xi} = Q_{2,s,\xi} = 0\} &\subset \widetilde{\mathcal{M}_4} \\ \Rightarrow A_{0,0} + \{u \in \mathfrak{u}^e : a_{14} = 0, a_{13} = \xi^4 a_7 + \cdots : \xi \in k^\times\} &\subset \widetilde{\mathcal{M}_4} \end{aligned}$$

As ξ varies, a_{13} can take on any value (assuming $a_7, a_9, \ldots, a_{12}$ are not all zero) and so taking the closure we obtain $\mathcal{M}_5 \subset \widetilde{\mathcal{M}_4}$.

Next we want to show that $\widetilde{\mathcal{M}_6} \subset \widetilde{\mathcal{M}_5}$. Elements of the form $e_{\beta_1} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{11} = 0\}$ are contained in $\widetilde{\mathcal{M}_3}$. Then

$$\begin{aligned} Ad_{\beta_3^\vee(t)}(e_{\beta_1} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{11} = 0\}) = e_{\beta_1} + t^2 e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{11} = 0\} &\subset \widetilde{\mathcal{M}_5} \\ \Rightarrow e_{\beta_1} + \{u \in \mathfrak{u}^e : a_{11} = 0\} &\subset \widetilde{\mathcal{M}_5} \text{ as } t \to 0 \end{aligned}$$

Now let $\xi \in k$ and consider

$$Ad_{\mathcal{E}_{-\beta_3}(\xi)}(e_{\beta_1} + \{u \in \mathfrak{u}^e : a_{11} = 0\}) = e_{\beta_1} + \{u \in \mathfrak{u}^e : a_{11} = \xi a_9\} \subset \widetilde{\mathcal{M}_5}$$

Then as ξ varies a_{11} can take any value (assuming a_9 is not zero). Therefore $e_{\beta_1} + \mathfrak{u}^e \subset \widetilde{\mathcal{M}_1}$. Since $e_{\beta_1} \in \mathcal{O}_{e_{\beta_1+2\beta_2+\beta_3}}$ then $\widetilde{\mathcal{M}_6} \subset \widetilde{\mathcal{M}_5}$.

Characteristic $p = 7$

In this case we claim that $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_1}$ which requires us to show that $\widetilde{\mathcal{M}_3}$ and $\widetilde{\mathcal{M}_2}$ are contained in $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_4}$ is contained in $\widetilde{\mathcal{M}_2}$. The other inclusions can be shown using similar methods to Section 7.2.

To show that $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$ consider a transverse slice for $e' = 3e_{\beta_2} + 4e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3}$ in $\mathfrak{c}$. We consider this element of the subregular orbit in $\mathfrak{c}$ instead of the representative e to make the calculation slightly easier. Let

$$e' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad f' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{c} \cong \mathfrak{sp}_6$$

Then by a computer calculation we get:

$$\mathcal{A} = (\mathfrak{c}^{f'} + e') \cap \mathcal{N}_1 = \left\{ \begin{pmatrix} 0 & a & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & c & 0 & 4 & 0 & 0 \\ a & 0 & c & 0 & -3 & 0 \\ j & \frac{7}{9}a^2 + \frac{91}{36}c^2 & 0 & -c & 0 & -a \\ -10c & j & a & 0 & 0 & 0 \end{pmatrix} : 160a^2c + 250c^3 + 2j^9 = 0 \right\}$$

We get an isomorphism of $k[a,c,j]/(160a^2c + 250c^3 + 2j^9)$ with $k[\frac{1}{4}s^2t^2, -\frac{2}{5}(s^4-t^4), \frac{2}{3}st(s^t+t^4)]$ since $-a^2c - 2c^3 + 2j^2$ is irreducible. Therefore let $A_{s,t}$ be the element in $\mathcal{A}$ with $c = s^2t^2$, $a = 2(s^4 - t^4)$ and $j = 3st(s^4+t^4)$ for $s, t \in k$ and $(s,t) \neq (0,0)$.

$$A_{s,t} = \begin{pmatrix} 0 & \frac{1}{4}(s^4-t^4) & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & -\frac{2}{5}s^2t^2 & 0 & 4 & 0 & 0 \\ \frac{1}{4}(s^4-t^4) & 0 & -\frac{2}{5}s^2t^2 & 0 & -3 & 0 \\ \frac{2}{3}st(s^4+t^4) & \frac{7}{144}(s^4-t^4) + \frac{91}{225}s^2t^2 & 0 & \frac{2}{5}s^2t^2 & 0 & -\frac{1}{4}(s^4-t^4) \\ 4s^2t^2 & \frac{2}{3}st(s^4+t^4) & \frac{1}{4}(s^4-t^4) & 0 & 0 & 0 \end{pmatrix}$$

Note that the computation of the intersection $(\mathfrak{c}^{f'} + e') \cap \mathcal{N}_1$ and parametrization are independent of characteristic greater than or equal to 7 although some coefficients disappear modulo 7.
The conditions $a_{14} = 0$ and $a_{11} = 4a_{10}$ are easily seen to be equivalent to the condition $ad(e_{\beta_1} + e_{\beta_2} + e_{\beta_3})^6(x) = 0$ for $x \in \mathfrak{g}^e(1)$. By considering $(a_1, \ldots, a_{14})^t$ as a vector this condition becomes $\rho(e')^6(u) = 0$ where ρ is the representative of $\mathfrak{c}$ of $\mathfrak{g}^e(1)$. Using [GAP12], the condition $T^6(u) = 0$, where $T = \rho(A_{s,t})$, gives the polynomials

$$\begin{aligned} P_1 : (4s^9t^2 + 3s^5t^6 + 5st^{10})a_1 + (2s^8t + s^4t^5 + 4t^9)a_2 + (2s^7 + 5s^3t^4)a_3 + 3s^6t^3a_4 \\ + (2s^2t^3)a_5 + (s^5t^2 + 6st^6)a_6 + (3st^2)a_7 + (4s^8t + 3s^4t^5)a_8 \\ + (5s^4t + 2t^5)a_9 + (s^7 + s^3t^4)a_{10} + 3s^3a_{11} + st^2a_{13} + 5ta_{14} = 0 \end{aligned}$$

$$\begin{aligned} P_2 : (2s^{10}t + 4s^6t^5 + 3s^2t^9)a_1 + (4s^9 + s^5t^4 + 2st^8)a_2 + (2s^4t^3 + 5t^7)a_3 + 4s^3t^6a_4 \\ + 2s^3t^2a_5 + (6s^6t + s^2t^5)a_6 + 4s^2ta_7 + (3s^5t^4 + 4st^8)a_8 + (5s^5 + 2st^4)a_9 \\ + (6s^4t^3 + 6t^7)a_{10} + 3t^3a_{11} + 6s^2ta_{13} + 5sa_{14} = 0 \end{aligned}$$

Let $Q_1 = \frac{stP_1 + P_2}{s^2 - t^2}$ and $Q_2 = \frac{stP_1 - P_2}{s^2 + t^2}$ and set $t = \xi s$. By considering $\frac{Q_1 - Q_2}{s^2}$ we get

$$a_{14} = -\frac{1}{5}s^2(6\xi a_{13} + 3\xi^3 a_{11} + (6s^4\xi^7 + 6s^4\xi^3)a_{10} + \ldots)$$

Similarly by considering $\frac{\xi^2(Q_1 - Q_2) - (Q_1 + Q_2)}{s^4\xi}$ we get

$$a_{13} = -\frac{1}{5\xi^2}(\xi a_7 + 3\xi^4 a_{11} + 4a_{11} + s^4((6\xi^8 + 5\xi^4)a_{10} + (4s^2\xi^9 + 3s^2\xi)a_8 + \ldots))$$

Therefore

$$\overline{T_{s,\xi s} + \{u \in \mathfrak{u}^e : a_{14}, a_{13} \text{ are as above }\}} \subset \widetilde{\mathcal{M}_1} \quad \forall \xi \neq 0$$

As ξ varies a_{13} can take any value (assuming a_{11} and a_7 are not both zero), therefore taking the closure we obtain $T_{0,0} + \{u \in \mathfrak{u}^e : a_{14} = 0\} \subset \widetilde{\mathcal{M}_1}$ and so $\mathcal{M}_2 \subset \widetilde{\mathcal{M}_1}$.
Next we want to show that $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_2}$. By conjugating $e_{\beta_2}+e_{\beta_3}+e_{2\beta_1+2\beta_2+\beta_3}+\{u \in \mathfrak{u}^e : a_{14} = 0\}$ by $(2\beta_1+2\beta_2+\beta_3)^\vee(t)$, as we did for $p = 5$, we can show that $e_{\beta_2}+e_{\beta_3}+\{u \in \mathfrak{u}^e : a_{14} = 0\} \subset \widetilde{\mathcal{M}_2}$. Then for $\xi \in k^\times$

$$\begin{aligned} Ad_{\mathcal{E}_{-2\beta_1-2\beta_2-\beta_3}(\xi)}(e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{14} = 0\}) &\subset \widetilde{\mathcal{M}_2} \text{ for any } \xi \in k^\times \\ \Rightarrow e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{14} = 2\xi a_6\} &\subset \widetilde{\mathcal{M}_2} \text{ for any } \xi \\ \Rightarrow \widetilde{\mathcal{M}_3} &\subset \widetilde{\mathcal{M}_2} \end{aligned}$$

Finally we need to show that $\widetilde{\mathcal{M}_4} \subset \widetilde{\mathcal{M}_2}$. To do this we need to consider the transverse slice of $e' = e_{\beta_1} + e_{\beta_2}$ which is given in the $p = 5$ case. Now it is easy to see that the condition $a_{14} = 0$ is equivalent to $ad(e_{\beta_2}+e_{\beta_3}+e_{2\beta_1+2\beta_2+\beta_3})^5(x) = 0$. Therefore for $v = (a_1, \ldots, a_{14})^t$ this condition is equivalent to $\rho(e_{\beta_2} + e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3})^5(v) = 0$ where ρ is the representation of $\mathfrak{c}$ on $\mathfrak{g}^e(1)$. Therefore we need to consider $T^5(v) = 0$, for $T = \rho(A_{s,t})$, which gives

$$2ta_8 = sa_{13}$$

Therefore for $\xi \in k^\times$

$$\begin{aligned} A_{s,t} + \{u \in \mathfrak{u}^e : 2ta_8 = sa_{13}\} &\subset \widetilde{\mathcal{M}_2} \\ \Rightarrow A_{s,\xi s} + \{u \in \mathfrak{u}^e : 2\xi sa_8 = sa_{13}\} &\subset \widetilde{\mathcal{M}_2} \quad \text{by } t = \xi s \\ \Rightarrow A_{s,\xi s} + \{u \in \mathfrak{u}^e : 2\xi a_8 = a_{13}\} &\subset \widetilde{\mathcal{M}_2} \end{aligned}$$

Then as ξ varies a_{13} can take any value as long as a_8 does not equal zero. Therefore taking the closure gives $\mathcal{M}_4 \subset \widetilde{\mathcal{M}_2}$.

Characteristic $p = 11$

In this case we only need to show $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$ since the other inclusions can be shown by the same method as in Section 7.2, then $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component of dimension 32. To do this we find the transverse slice of $e' = 3e_{\beta_2} + 4e_{\beta_3} + e_{2\beta_1+2\beta_2+\beta_3}$ in $\mathfrak{c}$. The computation of this slice from the $p = 7$ case is also valid here. The condition that $a_{14} = 0$ implies that $T^9_{s,t}(u) = 0$ which gives the condition

$$\begin{aligned} P_{s,t} : st(4s^{12} + s^8t^4 - s^4t^8 + 7t^{12})a_1 + 8(s^4 + t^4)(s^4 - t^4)^2a_2 + 7s^3t^3(s^4 - t^4)a_3 \\ + 3s^2t^2(s^4 + t^4)(s^4 - t^4)a_4 + 8s^2t^2(s^4 + t^4)a_5 + st(8t^8 + s^4t^4 + 8s^8)a_6 + 5st(s^4 - t^4)a_7 \\ + 5(s^4 + 2t^4)(s^4 + 5t^4)a_9 + s^3t^3a_{11} + s^2t^2a_{12} + 7st(s^4 - t^4)a_{13} + 3(s^4 + t^4)a_{14} = 0 \end{aligned}$$

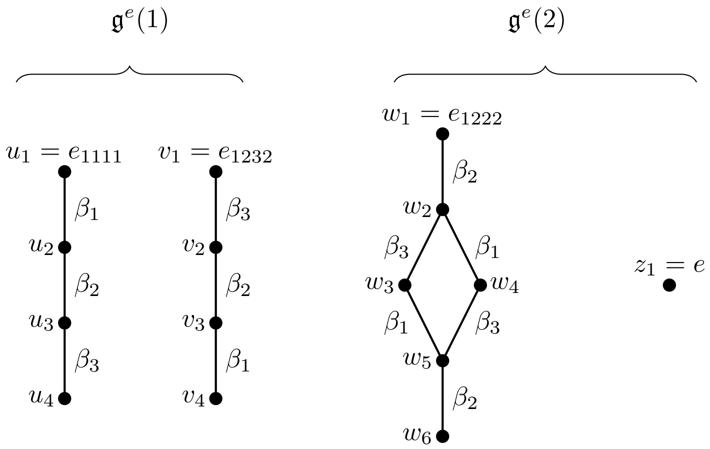
Now $P_{s,t}$ is an irreducible polynomial in $k[s, t, a_1, \ldots, a_{14}]$ as it is linear in the a_i's and its coefficients have no common factors. Therefore $X = \{(s, t, a_1, \ldots, a_{14}) : P_{s,t} = 0\}$ is an irreducible hypersurface with dimension 15. The subset $(0, 0, a_1, \ldots, a_{14})$ has codimension 1 in X, so its complement is dense in X. Then it follows that $A_{0,0} + \mathfrak{u}^e = \mathcal{M}_2$ is contained in $\widetilde{\mathcal{M}_1}$.

Orbit $\widetilde{A_1}$

$e = e_{0001},\ f = f_{0001}$

$\mathfrak{c} \cong \mathfrak{sl}_4$

$e_{\beta_1} = e_{1000},\ e_{\beta_2} = e_{0100},\ e_{\beta_3} = e_{1242}.$



$M_0 = a_1 U_1 + \cdots + a_4 U_4 + b_1 V_1 + \cdots + b_4 V_4 + c_1 W_1 + \cdots + c_6 W_6 + d_1 Z_1$

$M_1 = e_{\beta_1} + e_{\beta_2} + e_{\beta_3} + M_0$

$M_2 = e_{\beta_1} + e_{\beta_2} + M_0$

$M_3 = e_{\beta_1} + e_{\beta_3} + M_0$

$M_4 = e_{\beta_1} + M_0$

Characteristic $p = 5$:

$$\begin{array}{ll} M_1^5 = 0 \Rightarrow b_4 = a_4 = 0,\ c_6 = a_3^2 + b_3^2 & \dim(\widetilde{\mathcal{M}_1}) = 12 - 3 + 15 = 24 \\ M_2^5 = 0 \Rightarrow a_3 = b_4 = 0 & \dim(\widetilde{\mathcal{M}_2}) = 10 - 2 + 15 = 23 \\ M_3^5 = 0 & \dim(\widetilde{\mathcal{M}_3}) = 8 + 15 = 23 \end{array}$$

Characteristic $p = 7$:

$$\begin{array}{ll} M_1^7 = 0 \Rightarrow a_4^2 + b_4^2 = 0 & \dim(\widetilde{\mathcal{M}_1}) = 12 - 1 + 15 = 26 \\ M_2^7 = 0 & \dim(\widetilde{\mathcal{M}_2}) = 10 + 15 = 25 \end{array}$$

Characteristic $p = 11$:

$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 12 + 15 = 27$$

By similar methods as demonstrated in Section 7.2 we can show that $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_1}$ when $p = 11$.

Characteristic $p = 5$

We have $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_4} \subset \widetilde{\mathcal{M}_3}$ by the same method in Section 7.2. Therefore we only need to show $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$. Then $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component of dimension 24.
To show $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$ then we consider the transverse slice of $e' = e_{\beta_1} + e_{\beta_2}$ where

$$e' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \quad f' = \begin{pmatrix} 0 & & & \\ 2 & 0 & & \\ 0 & 2 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then by a computer calculation

$$\mathcal{A} = (e' + \mathfrak{c}^{f'}) \cap \mathcal{N}_1 = \left\{ \begin{pmatrix} j & 1 & 0 & 0 \\ -3j^2 & j & 1 & 0 \\ -3j^2 & 20j^3 & j & k \\ l & 0 & 0 & -3j \end{pmatrix} : 81j^4 + kl = 0 \in k \right\}$$

We get an isomorphism of $k[j,k,l]/(81j^4 + kl)$ with $k[st, -81s^4, t^4]$ since $81j^4 + kl = 0$ is irreducible. Therefore let $A_{s,t}$ be the element in $\mathcal{A}$ with $j = st$, $k = -81s^4$ and $l = t^4$ for $(s,t) \neq (0,0)$.

$$A_{s,t} = \left\{ \begin{pmatrix} st & 1 & 0 & 0 \\ -3s^2t^2 & st & 1 & 0 \\ 20s^3t^3 & -3s^2t^2 & st & -81s^4 \\ t^4 & 0 & 0 & -3st \end{pmatrix} s,t \in k \right\}$$

Let $\epsilon = e_{\beta_1} + e_{\beta_2} + e_{\beta_3}$; for any $u, v \in \mathfrak{g}^e(1)$ and $w \in \mathfrak{g}^e(2)$, the conditions $a_4 = b_4 = 0$ and $c_6 = a_3^2 + b_3^2$ are equivalent to

$$\begin{aligned} &ad(\epsilon)^3(u) = 0 \\ &ad(\epsilon)^3(v) = 0 \\ &\frac{1}{2} ad(\epsilon)^4(w) = [ad(\epsilon)^2(u), ad(\epsilon)(u)] + [ad(\epsilon)^2(v), ad(\epsilon)(v)] \end{aligned}$$

Consider $u = (a_1, a_2, a_3, a_4)^t$, $v = (b_4, b_3, b_2, b_1)^t$ and $w = (c_1, \ldots, c_6)^t$ as vectors, the $\mathfrak{sl}_4$ acts on the left of u so $ad(\epsilon)(u) = \epsilon \cdot u$. Similarly $\mathfrak{sl}_4$ acts on the right of v given by $ad(\epsilon)(v) = v(-\epsilon)$. The conditions above, on replacing ϵ by $A_{s,t}$, are equivalent to

$$A_{s,t}^3(u) = 0 \tag{7.3}$$

$$(v)(-A_{s,t}^3) = 0 \tag{7.4}$$

$$\frac{1}{2} T_{s,t}^4(w) = A_{s,t}^2(u) \wedge A_{s,t}(u) + (v) A_{s,t}^2 \wedge (v)(-A_{s,t}) \tag{7.5}$$

where $T_{s,t} = \Lambda^2(A_{s,t})$. A computer calculation using [GAP12] shows that the conditions (7.3) and (7.4) hold if and only if

$$4s^2t^3a_1 - st^2a_2 + ta_3 - 27s^3a_4 = 0$$
$$3sb_4 - 3s^2tb_3 + 12s^3t^3b_2 + t^3b_1 = 0$$

If we let $a_4 = tx_4$ and $b_1 = sx_1$ these become

$$a_3 = -4s^2t^2a_1 + sta_2 + 27s^3x_4$$
$$3b_4 = 3stb_3 - 12s^2t^2b_2 - t^3x_1$$

Next we find the condition which is implied by equation (7.5). First let U be the irreducible module of $\mathfrak{g}^e(1)$ with highest weight ω_1, and let the other module of $\mathfrak{g}^e(1)$ be V. Then the basis for Λ^2U and Λ^2V respectively are given by the diagrams below.

$$\begin{tikzcd}
 & u_1 \wedge u_2 \arrow[d, dash, "f_{\beta_2}"] & \\
 & u_1 \wedge u_3 \arrow[dl, dash, "f_{\beta_3}"'] \arrow[dr, dash, "f_{\beta_1}"] & \\
u_1 \wedge u_4 \arrow[dr, dash, "f_{\beta_1}"'] & & u_2 \wedge u_3 \arrow[dl, dash, "f_{\beta_3}"] \\
 & u_2 \wedge u_4 \arrow[d, dash, "f_{\beta_2}"] & \\
 & u_3 \wedge u_4 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v_1 \wedge v_2 \arrow[d, dash, "-f_{\beta_2}"] & \\
 & -v_1 \wedge v_3 \arrow[dl, dash, "-f_{\beta_3}"'] \arrow[dr, dash, "-f_{\beta_1}"] & \\
v_2 \wedge v_3 \arrow[dr, dash, "-f_{\beta_1}"'] & & v_1 \wedge v_4 \arrow[dl, dash, "-f_{\beta_3}"] \\
 & -v_2 \wedge v_4 \arrow[d, dash, "-f_{\beta_2}"] & \\
 & v_3 \wedge v_4 &
\end{tikzcd}$$

Hence $vA_{s,t}^2 \wedge v(-A_{s,t})$ and $A_{s,t}^2u \wedge A_{s,t}u$ are respectively

$$\begin{aligned}(-2st^2a_1 + ta_2 + 9s^2x_4)^2(-s^2u_1 \wedge u_2 + 2s^3tu_1 \wedge u_3 - t^2u_1 \wedge u_4 + 2s^4t^2u_2 \wedge u_3 \\ - 2st^3u_2 \wedge u_4 + 2s^2t^4u_3 \wedge u_4)\end{aligned}$$

$$\begin{aligned}(t^2x_1 + 18s^2tb_2 - 9sb_3)^2(-s^2u_1 \wedge u_2 + 2s^3tu_1 \wedge u_3 - t^2u_1 \wedge u_4 + 2s^4t^2u_2 \wedge u_3 \\ + 3st^3u_2 \wedge u_4 + 2s^2t^4u_3 \wedge u_4)\end{aligned}$$

Finally we need to calculate the matrix $T_{s,t}$ for $A_{s,t}$ acting on w. Let W be the irreducible 6-dimensional module of $\mathfrak{g}^e(2)$. Then we can consider W as Λ^2U with basis given by: $w_1 = u_1 \wedge u_2$, $w_2 = u_1 \wedge u_3$, $w_3 = u_1 \wedge u_4$, $w_4 = u_2 \wedge u_3$, $w_5 = u_2 \wedge u_4$ and $w_6 = u_3 \wedge u_4$. Then

$$\begin{aligned}
A_{s,t} \cdot w_1 &= (A_{s,t} \cdot u_1) \wedge u_2 + u_1 \wedge (A_{s,t} \cdot u_2) \\
&= (stu_1 + 2s^2t^2u_2 + t^4u_4) \wedge u_2 + u_1 \wedge (u_1 + stu_2 + 2s^2t^2u_3) \\
&= 2stw_1 + 2s^2t^2w_2 - t^4w_5 \\
A_{s,t} \cdot w_2 &= w_1 + 2stw_2 + 2s^2t^2w_4 - t^4w_6 \\
A_{s,t} \cdot w_3 &= -s^4w_2 + 3stw_3 + 2s^2t^2w_6 \\
A_{s,t} \cdot w_4 &= w_2 + 2stw_4
\end{aligned}$$

$$A_{s,t} \cdot w_5 = w_3 - s^4 w_4 + 3stw_5 + 2s^2t^2 w_6$$
$$A_{s,t} \cdot w_6 = w_5 + 3stw_6$$

So let

$$T_{s,t} = \begin{pmatrix} 2st & 1 & 0 & 0 & 0 & 0 \\ 2s^2t^2 & 2st & -s^4 & 1 & 0 & 0 \\ 0 & 0 & 3st & 0 & 1 & 0 \\ 0 & 2s^2t^2 & 0 & 2st & -s^4 & 0 \\ -t^4 & 0 & 2s^2t^2 & 0 & 3st & 1 \\ 0 & -t^4 & 0 & 0 & 2s^2t^2 & 3st \end{pmatrix}$$

Then

$$\begin{aligned} T_{st}^4(w) = {} & (2s^2t^4c_1 + 2st^3c_2 + 2s^4t^2c_3 + 4t^2c_4 + 3s^3tc_5 + 4s^2c_6) \\ & (2s^2w_1 + s^3tw_2 + 2t^2w_3 + s^4t^2w_4 + 4st^3w_5 + s^2t^4w_6) \end{aligned}$$

Therefore the condition (7.5) gives

$$\begin{aligned} P_{s,t} : {} & 2s^2t^4c_1 + 2st^3c_2 + 2s^4t^2c_3 + 4t^2c_4 + 3s^3tc_5 + 4s^2c_6 = \\ & - (t^2x_1 + 18s^2tb_2 - 9sb_3)^2 - (-2st^2a_1 + ta_2 + 9s^3x_4)^2 \end{aligned}$$

Letting $t = \xi s$ gives the conditions

$$\begin{aligned} & a_3 = s^2(-4\xi^2s^2a_1 + \xi a_2 + 27s^3x_4) \\ & 3b_4 = \xi s^2(3b_3 - 12s^2\xi b_2 - \xi^2 sx_1) \\ & \frac{1}{s^2}P_{s,\xi s} : 4c_6 = -4\xi^2c_4 - 81b_3^2 - \xi^2a_2^2 + s^2(-2\xi^4s^4c_1 - 2\xi^2s^2c_2 + \dots) \end{aligned}$$

Therefore

$$\begin{aligned} A_{s,\xi s} + \{u \in \mathfrak{u}^e : {} & a_3 = s^2(-4\xi^2s^2a_1 + \xi a_2 + 27s^3x_4), \\ & 3b_4 = \xi s^2(3b_3 - 12s^2\xi b_2 - \xi^2 sx_1), P_{s,\xi s}\} \subset \widetilde{\mathcal{M}_1} \end{aligned}$$

Taking the limit at $s \to 0$ gives

$$A_{0,0} + \{u \in \mathfrak{u}^e : a_3 = b_4 = 0, 4c_6 = -4\xi^2c_4 - 81b_3^2 - \xi^2a_2^2\} \subset \widetilde{\mathcal{M}_1}$$

As ξ varies then c_6 can take any value, (assuming that c_4 and a_2 are not both zero). Therefore taking the closure we obtain

$$\begin{aligned} A_{0,0} + \{u \in \mathfrak{u}^e : a_3 = b_4 = 0\} & \subset \widetilde{\mathcal{M}_1} \\ \Rightarrow \widetilde{\mathcal{M}_2} & \subset \widetilde{\mathcal{M}_1} \end{aligned}$$

Finally we want to show that $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_1}$. To do this let

$$e' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, f' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then $(f' + \mathfrak{c}^{e'}) = \{\left(\begin{smallmatrix} A & B \\ I & A \end{smallmatrix}\right) : A, B \in SL_2\}$ and so

$$(f' + \mathfrak{c}^{e'}) \cap \mathcal{N} = \left\{ \begin{pmatrix} A & B \\ I & A \end{pmatrix} : \begin{matrix} det(B) = -3(detA)^2, trace(B) = 2det(A), \\ [A,[A,B]] = -4(detA)(B + A^2) \end{matrix} \right\}$$

Now

$$\mathcal{A} = \left\{ \begin{pmatrix} x & 0 & -x^2 & 4x^4/y \\ 0 & -x & y & -x^2 \\ 1 & 0 & x & 0 \\ 0 & 1 & 0 & -x \end{pmatrix} x, y \neq 0 \in k \right\} \subset (f' + \mathfrak{c}^{e'}) \cap \mathcal{N}$$

Then $(f' + \mathfrak{c}^{e'}) \cap \mathcal{N} = \overline{\left\{ Ad_\lambda \mathcal{A} : \lambda = \left(\begin{smallmatrix} g & 0 \\ 0 & g \end{smallmatrix}\right), g \in SL_2 \right\}}$ since the conditions to define $(f' + \mathfrak{c}^{e'}) \cap \mathcal{N}$ are invariant under $Ad(G)$. When $x \neq 0$ then it is easy to check that the elements of this set are contained in $\mathcal{O}_{reg}(\mathfrak{sl}_4)$. Let $A_{s,t}$ be the element in $\mathcal{A}$ with $x = st$ and $y = 2s^4$ where $s \neq 0$ giving

$$A_{s,t} = \begin{pmatrix} st & 0 & -s^2t^2 & 2t^4 \\ 0 & -st & 2s^4 & -s^2t^2 \\ 1 & 0 & st & 0 \\ 0 & 1 & 0 & -st \end{pmatrix}$$

As in the previous calculation, the conditions $a_4 = b_4 = 0$ and $c_6 = a_3^2 + b_3^2$ are equivalent to

$$ad(e)^3(u) = 0 \tag{7.6}$$

$$ad(e)^3(v) = 0 \tag{7.7}$$

$$\frac{1}{2}ad(e)^4(w) = ad(e)^2(v) \wedge ad(e) + ad(e)^2(u) \wedge ad(e)(v). \tag{7.8}$$

for $e = e_{\beta_1} + e_{\beta_2} + e_{\beta_3}$. Therefore the conditions $A^3_{s,t}(u) = 0$ and $(v)A^3_{s,t} = 0$ give respectively

$$a_1 = \frac{-t^2}{s^2}a_2 + sta_3 - \frac{t^3}{s}a_4$$
$$b_1 = \frac{t^3}{s}b_4 - stb_3 - \frac{t^2}{s^3}b_2$$

Now by the same method as the previous argument we can find the matrix $T_{s,t}$ for $A_{s,t}$ acting on w.

$$T_{s,t} = \begin{pmatrix} 0 & 2s^4 & -s^2t^2 & s^2t^2 & -2t^4 & 0 \\ 0 & 2st & 0 & 0 & 0 & -2t^4 \\ 1 & 0 & 0 & 0 & 0 & -s^2t^2 \\ -1 & 0 & 0 & 0 & -2st & 2s^4 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

Therefore

$$T^4_{s,t}(w) = 8(-stc_1 + s^4c_2 + t^4c_5 - s^3t^3c_6)(s^3t^3w_1 + t^4w_2 + s^4w_5 + stw_6)$$

Finally $T^2_{s,t}(u) \wedge T_{s,t}(u)$ and $(v)T^2_{s,t} \wedge (v)T_{s,t}$ are as follows:

$$\begin{aligned}
&-4(s^3a_3 - ta_2)^2(st^3u_1 \wedge u_2 + \frac{t^4}{s^2}u_1 \wedge u_3 + s^3u_2 \wedge u_4 + \frac{t}{s}u_3 \wedge u_4)\\
&-4(s^3b_3 + tb_2)^2(st^3u_1 \wedge u_2 + \frac{t^4}{s^2}u_1 \wedge u_3 + s^2u_2 \wedge u_4 + \frac{t}{s}u_3 \wedge u_4)
\end{aligned}$$

So the condition (7.8) is equivalent to

$$P_{s,t} : s^3t^3(-stc_1 + s^4c_2 + t^4c_5 - s^3t^3c_6) = -st^3(s^3a_3 - ta_2)^2 - st^3(s^3b_3 + tb_2)^2$$

Let $t = \xi s$; then we get

$$\begin{aligned}
&a_1 = -\xi^2a_2 + \xi s^2a_3 - \xi^3s^2a_4\\
&b_1 = -\xi^2sb_4 - \xi s^2b_3 - \xi b_2\\
&P_{s,\xi s} : \xi s^2c_1 + s^4c_2 + \xi^4s^4c_5 - \xi^3s^6c_6 = -(s^2a_3 - \xi a_2)^2 - (s^2b_3 + \xi b_2)^2
\end{aligned}$$

Therefore

$$\begin{aligned}
As, \xi s + \{u \in \mathfrak{u}^e : a_1 = -\xi^2a_2 + \xi s^2a_3 - \xi^3s^2a_4, b_1 = -\xi^2sb_4 - \xi s^2b_3 - \xi b_2, P_{s,\xi s}\} \subset \widetilde{\mathcal{M}_1}\\
\Rightarrow A_{0,0} + \{u \in \mathfrak{u}^e : a_1 = -\xi^2a_2, b_1 = -\xi^2b_2, a_2^2 = -b_2^2\} \subset \widetilde{\mathcal{M}_1}\\
\Rightarrow Ad\begin{pmatrix} g & 0\\ 0 & g\end{pmatrix}\big(A_{0,0} + \{u \in \mathfrak{u}^e : a_1 = -\xi^2a_2, b_1 = -\xi^2b_2, b_2 = 2a_2\}\big) \subset \widetilde{\mathcal{M}_1}
\end{aligned}$$

for $g \in SL_2$. Since $\left\{\left(\begin{smallmatrix} g & 0\\ 0 & g\end{smallmatrix}\right) : g \in SL_2\right\}$ centralizes $A_{0,0} = f'$ then to show $\mathcal{M}_3 \subset \widetilde{\mathcal{M}_1}$ it suffices to show that

$$\overline{\left\{Ad\begin{pmatrix} 0 & g\\ g & 0\end{pmatrix} : g \in SL_2\right\} \cdot \{u \in \mathfrak{u}^e : a_1 = -\xi^2a_2, b_1 = -\xi^2b_2, b_2 = 2a_2\}} = \mathfrak{u}^e$$

Let $x = a_2$, $y = -\xi a_2$ and let π be the map $\pi : SL_2 \times k^2 \to k^2 \times (k^2)^*$ which sends $(g, \left(\begin{smallmatrix} x\\ y\end{smallmatrix}\right))$ to $\big(g\left(\begin{smallmatrix} x\\ y\end{smallmatrix}\right), 2\left(\begin{smallmatrix} y & x\end{smallmatrix}\right)g^{-1}\big)$, where $(k^2)^*$ represents the dual of k^2. Then the fibre

$$\pi^{-1}\left(\begin{pmatrix} x\\ y\end{pmatrix}, 2\begin{pmatrix} y & x\end{pmatrix}\right) = \left\{\begin{pmatrix} \alpha & 0\\ 0 & 1/\alpha\end{pmatrix}, \begin{pmatrix} 1/\alpha x\\ \alpha y\end{pmatrix}\right\}$$

The dimension of this fibre is 1 so by Theorem 5.3.4 the dimension of the image of π is greater than or equal to $5 - 1 = 4$. Since $k^2 \times (k^2)^*$ has dimension 4 then the dimension of the image of π equals 4.

As we did in Section 6.3, we can represent an element in $\mathfrak{u}^e$ such that $a_1 = -\xi^2a_2$, $b_1 = -\xi^2b_2$

and $b_2 = 2a_2$ by a series of matrices as follows:

$$\begin{pmatrix} -\xi^2 a_2 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \begin{pmatrix} b_4 & b_3 & 2a_2 & -2\xi^2 a_2 \end{pmatrix}, \begin{pmatrix} & c_1 & \\ & c_2 & \\ c_3 & & c_4 \\ & c_5 & \\ & c_6 & \end{pmatrix}$$

Therefore

$$\begin{aligned} &Ad\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\left(\begin{pmatrix} -\xi^2 a_2 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \begin{pmatrix} b_4 & b_3 & 2a_2 & -2\xi^2 a_2 \end{pmatrix}, \begin{pmatrix} & c_1 & \\ & c_2 & \\ c_3 & & c_4 \\ & c_5 & \\ & c_6 & \end{pmatrix}\right) \\ &= \begin{pmatrix} g\begin{pmatrix} -\xi^2 s \\ a_2 \end{pmatrix} \\ g\begin{pmatrix} a_3 \\ a_4 \end{pmatrix} \end{pmatrix}, \left(\begin{pmatrix} b_4 & b_3 \end{pmatrix} g^{-1} \quad 2\begin{pmatrix} a_2 & -\xi^2 a_2 \end{pmatrix} g^{-1} \right) + \\ &\qquad Ad\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} & c_1 & \\ & c_2 & \\ c_3 & & c_4 \\ & c_5 & \\ & c_6 & \end{pmatrix} \end{aligned}$$

Then by the fibre argument this has dimension 14 and so is equal to $\mathfrak{u}^e$.

Characteristic $p = 7$

Now $e_{\beta_1} + e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_4^2 + b_4^2 = 0\} \subset \widetilde{\mathcal{M}_1}$ and let $\beta^\vee(t) = \beta_1^\vee(t)\beta_2^\vee(t^2)\beta_3^\vee(t^3)$. Then $Ad_{\beta^\vee(t)}$ gives

$$\begin{aligned} &e_{\beta_1} + e_{\beta_2} + t^2 e_{\beta_3} + \{u \in \mathfrak{u}^e : t^6 a_4^2 + t^2 b_4^2 = 0\} \subset \widetilde{\mathcal{M}_1} \\ \Rightarrow\ &e_{\beta_1} + e_{\beta_2} + t^2 e_{\beta_3} + \{u \in \mathfrak{u}^e : t^4 a_4^2 + b_4^2 = 0\} \subset \widetilde{\mathcal{M}_1} \end{aligned}$$

Taking the limit at $t \to 0$ gives $e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : b_4 = 0\} \subset \widetilde{\mathcal{M}_1}$. Then for $\xi \in k^\times$, $Ad_{\mathcal{E}_{\beta_2+\beta_3}(\xi)}$ gives

$$e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : b_4 = \xi b_3\} \subset \widetilde{\mathcal{M}_1}$$

Then by taking the closure we get $\mathcal{M}_2 \subset \widetilde{\mathcal{M}_1}$.

7.5 Irreducible Components of $\mathcal{C}_1^{nil}(F_4)$

In this section we calculate the irreducible components of $\mathcal{C}_1^{nil}(F_4)$. Above we have calculated the irreducible components $X_i^{(j)}$ of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$ for each nilpotent orbit $\mathcal{O}_{e_i}$ of F_4. Then by Lemma 5.1.1 we have

$$\mathcal{C}_1^{nil}(\mathfrak{g}) = \bigcup \overline{G \cdot (e_i, X_i^{(j)})} \text{ for } i = 1, \dots, m,\ j = 1, \dots, n_i.$$

By Proposition 5.2.1, a necessary condition for $\overline{G \cdot (e_i, X_i^{(j)})}$ to be an irreducible component of $\mathcal{C}_1^{nil}(F_4)$ is that $X_i^{(j)} \subset \overline{(G \cdot e_i)}$. Now if $\mathcal{O}_{e_i} \subset \mathcal{N}_1$ is distinguished then $\mathcal{C}_1(\mathcal{O}_{e_i})$ is an irreducible component of $\mathcal{C}_1^{nil}(\mathfrak{g})$. For the remaining orbits $\mathcal{O}_{e_i}$ of F_4 we can verify computationally that there is an element in each irreducible component of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$ that is not contained in $\overline{G \cdot e_i}$. Therefore the irreducible components of $\mathcal{C}_1^{nil}(F_4)$ are given by

$$\begin{aligned}
p = 5: &\quad \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \\
p = 7: &\quad \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \cup \mathcal{C}_1(F_4(a_2)) \\
p = 11: &\quad \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \cup \mathcal{C}_1(F_4(a_2)) \cup \mathcal{C}_1(F_4(a_1))
\end{aligned}$$

An element in each irreducible component of $\mathfrak{g}^e \cap \mathcal{N}_1$ which is not contained in $\overline{G \cdot e}$ is specified in Table 7.1. To show that an element is not contained in $\overline{(G \cdot e)}$ we found its Jordan normal form. This was done by considering the rank of successive powers of its 26-dimensional representation.

Orbit	Characteristic	Irreducible Component X	Element x in X	JNF of x	Nilpotent Orbit which contains x
A_1	5	$\widetilde{\mathcal{M}_2}$	$e_{\beta_2}+e_{\beta_3}+e_{2\beta_1+2\beta_2+\beta_3}$	$[5^2,4^2,3,2^2,1]$	$C_3(a_1)$
	7	$\widetilde{\mathcal{M}_1}$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_3}$	$[7^2,6^2]$	C_3
	≥ 11	$\widetilde{\mathcal{M}_1}$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_3}$	$[9,6^2,5]$	C_3
$\widetilde{A_1}$	≥ 5	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[3^3,2^6,1^7]$	$A_1\widetilde{A_1}$
$A_1\widetilde{A_1}$	≥ 5	$\widetilde{\mathcal{M}_1}$	$e_{\beta_1}+e_{\beta_2}$	$[5,4^2,3^3,2^2]$	$\widetilde{A_2}A_1$
		$\widetilde{\mathcal{M}_3}$	$e_{\beta_1}+e_{1100}$	$[3^6,1^8]$	A_2
A_2	≥ 5	$\widetilde{\mathcal{M}_1}$	$e_{\beta_1}+e_{\beta_2}$	$[5,3^7]$	$\widetilde{A_2}$
		$\widetilde{\mathcal{M}_2}$	$e_{\beta_1}+e_{1110}$	$[5,3^7]$	$\widetilde{A_2}$
$\widetilde{A_2}$	5	$\widetilde{\mathcal{M}_2}$	$e_0+e_{\beta_2}+e_{3\beta_1+\beta_2}$	$[5^3,3^3,1^2]$	$F_4(a_3)$
	≥ 7	$\widetilde{\mathcal{M}_1}$	$e_{\beta_1}+e_{\beta_2}$	$[7^3,1^5]$	B_3
$A_2\widetilde{A_1}$	≥ 5	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[5^3,3^3,1^2]$	$F_4(a_3)$
		$\widetilde{\mathcal{M}_0}$	$e_0+e_{1110}+e_{0111}$	$[5^2,4^2,3,2^2,1]$	$C_3(a_1)$
B_2	≥ 5	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}+e_{\beta_2}$	$[5^3,3^3,1^2]$	$F_4(a_3)$
$\widetilde{A_2}A_1$	$5, 11$	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[5^3,3^3,1^2]$	$F_4(a_3)$
	7	$X_1, X_2 \in \widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[5^3,3^3,1^2]$	$F_4(a_3)$
$C_3(a_1)$	≥ 5	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[5^3,3^3,1^2]$	$F_4(a_3)$
C_3	7	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[7^3,5]$	$F_4(a_2)$
	≥ 11	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[9,7,5^2]$	$F_4(a_2)$
B_3	7	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[7^3,5]$	$F_4(a_2)$
	≥ 11	$\widetilde{\mathcal{M}_1}$	$e_0+e_{\beta_1}$	$[9,7,5^2]$	$F_4(a_2)$

Table 7.1: Example of an element in each irreducible component of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$ which is not contained in $\overline{(G \cdot e_i)}$

Chapter 8

E_6 Results

In this chapter we give the details of the computations for answering Question 1 for E_6 (with the exception of the last few cases). As in Chapter 7, we group the nilpotent orbits of E_6 into sections where for each orbit in a given section the arguments for finding the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are similar.

8.1 Orbits $E_6(a_3)$, $E_6(a_1)$ and E_6

These orbits are all distinguished therefore by Corollary 2.3.4 we have $\mathfrak{g}^e \subset \mathcal{N}_1$ when $e \in \mathcal{N}_1$. Since $\mathfrak{c}$ is trivial then $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_0}$. Below is a table with contains a representative for each orbit, the characteristic for which $e \in \mathcal{N}_1$ and the dimension of $\widetilde{\mathcal{M}_0}$. A basis for $\mathfrak{u}^e$ is not stated but can be found in [LT11].

Orbit	Representative e	Characteristic p	Dimension of $\widetilde{\mathcal{M}_0}$
$E_6(a_3)$	$e_{\substack{01100\\1}} + e_{\substack{10000\\0}} + e_{\substack{01110\\0}}$ $+ e_{\substack{00001\\0}} + e_{\substack{00110\\1}} + e_{\substack{00100\\0}}$	$p \geq 7$	12
$E_6(a_1)$	$e_{\substack{10000\\0}} + e_{\substack{00001\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}}$ $+ e_{\substack{00110\\0}} + e_{\substack{01100\\0}} + e_{\substack{00000\\1}}$	$p \geq 11$	8
E_6	$e_{\substack{10000\\0}} + e_{\substack{00000\\1}} + e_{\substack{01000\\0}}$ $+ e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$	$p \geq 13$	6

8.2 Orbits $D_4(a_1)$, $D_5(a_1)$, D_5 and A_4A_1

In each of these cases it is clear that $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_0}$. In all of these cases $\mathcal{M}_0$ is irreducible with the exception of A_4A_1 in characteristic 5 and 7. In fact these orbits are *almost distinguished* which means that $\mathfrak{c}$ is a torus.

Orbit $D_4(a_1)$

$e = e_{\substack{01000\\0}} + e_{\substack{00100\\1}} + e_{\substack{00110\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$
$\mathfrak{c} \cong k^2$

$\mathfrak{g}^e(2)$: $v_1 = f_{\substack{11111\\1}} - f_{\substack{11211\\1}}$, $v_2 = e_{\substack{12221\\1}} + e_{\substack{12321\\1}}$, $v_3 = f_{\substack{00011\\0}} - f_{\substack{00111\\0}}$,
$v_4 = e_{\substack{01111\\1}} - e_{\substack{01211\\1}}$, $v_5 = f_{\substack{11100\\0}}$, $v_6 = e_{\substack{11110\\1}}$,
$v_7 = 2e_{\substack{00000\\1}} - e_{\substack{00100\\1}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}}$, $v_8 = e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$, $v_9 = e$
$\mathfrak{g}^e(4)$: $v_{10} = f_{\substack{11111\\0}}$, $v_{11} = e_{\substack{12321\\2}}$, $v_{12} = f_{\substack{00001\\0}}$, $v_{13} = e_{\substack{01221\\1}}$, $v_{14} = f_{\substack{10000\\0}}$,
$v_{15} = e_{\substack{12210\\1}}$, $v_{16} = 2e_{\substack{00110\\1}} + e_{\substack{01100\\1}} - e_{\substack{01110\\0}}$
$\mathfrak{g}^e(6)$: $v_{17} = e_{\substack{01110\\1}}$, $v_{18} = e_{\substack{01210\\1}}$

$M_0 = a_1V_1 + \cdots + a_{18}V_{18}$

Characteristic $p \geq 5$:

$$M_0^p = 0 \quad \dim(\widetilde{\mathcal{M}_0}) = 18$$

Orbit A_4A_1

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00001\\0}}$
$\mathfrak{c} \cong k$

$\mathfrak{g}^e(1)$: $v_1 = f_{\substack{01110\\1}} + f_{\substack{00111\\1}} - f_{\substack{01111\\0}} - 2f_{\substack{11110\\0}}$, $v_2 = e_{\substack{01111\\1}} - e_{\substack{01210\\1}} - e_{\substack{11110\\1}} - 2e_{\substack{11111\\0}}$
$\mathfrak{g}^e(2)$: $v_3 = e_{\substack{00001\\0}}$, $v_4 = e$
$\mathfrak{g}^e(3)$: $v_5 = f_{\substack{01110\\0}} - f_{\substack{00110\\1}}$, $v_6 = e_{\substack{01211\\1}} + e_{\substack{11111\\1}}$
$\mathfrak{g}^e(4)$: $v_7 = f_{\substack{01221\\1}}$, $v_8 = e_{\substack{12321\\2}}$, $v_9 = e_{\substack{11000\\0}} + e_{\substack{01100\\0}} - e_{\substack{00100\\1}}$
$\mathfrak{g}^e(5)$: $v_{10} = f_{\substack{00110\\0}} + f_{\substack{00011\\0}}$, $v_{11} = e_{\substack{11211\\1}} + e_{\substack{12210\\1}}$
$\mathfrak{g}^e(6)$: $v_{12} = e_{\substack{11100\\0}} - e_{\substack{01100\\1}}$
$\mathfrak{g}^e(7)$: $v_{13} = f_{\substack{00010\\0}}$, $v_{14} = e_{\substack{12211\\1}}$
$\mathfrak{g}^e(8)$: $v_{15} = e_{\substack{11100\\1}}$

$M_0 = a_1V_1 + \cdots + a_{15}V_{15}$

Characteristic $p = 5, 7$;

$M_0^p = 0 \Rightarrow a_1 = 0$ or $a_2 = 0$ $\quad \widetilde{\mathcal{M}_0}$ has two irreducible components of dimension 14

Characteristic $p \geq 11$:

$$M_0^p = 0 \quad \dim(\widetilde{\mathcal{M}_0}) = 15$$

Orbit $D_5(a_1)$

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\1}} + e_{\substack{00110\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$

$\mathfrak{c} \cong k$

$\mathfrak{g}^e(1)$: $v_1 = f_{\substack{11111\\0}} - f_{\substack{01211\\1}}$, $v_2 = e_{\substack{11111\\1}} + e_{\substack{01221\\1}}$

$\mathfrak{g}^e(2)$: $v_3 = e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$, $v_4 = e$

$\mathfrak{g}^e(4)$: $v_5 = e_{\substack{11100\\0}} + e_{\substack{01110\\0}} - e_{\substack{01100\\1}} - 2e_{\substack{00110\\1}}$

$\mathfrak{g}^e(5)$: $v_6 = f_{\substack{00011\\0}} - f_{\substack{00111\\0}}$, $v_7 = e_{\substack{12221\\1}} + e_{\substack{12321\\1}}$

$\mathfrak{g}^e(6)$: $v_8 = e_{\substack{11100\\1}} - e_{\substack{11110\\0}} + 2e_{\substack{01110\\1}}$, $v_9 = e_{\substack{11100\\1}} + e_{\substack{01110\\1}} - e_{\substack{01210\\1}}$

$\mathfrak{g}^e(7)$: $v_{10} = f_{\substack{00001\\0}}$, $v_{11} = e_{\substack{12321\\2}}$

$\mathfrak{g}^e(8)$: $v_{12} = e_{\substack{11110\\1}}$

$\mathfrak{g}^e(10)$: $v_{13} = e_{\substack{12210\\1}}$

$M_0 = a_1 V_1 + \cdots + a_{13} V_{13}$

Characteristic $p \geq 7$:

$$M_0^p = 0 \quad \dim(\widetilde{\mathcal{M}_0}) = 13$$

Orbit D_5

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$

$\mathfrak{c} \cong k$

$\mathfrak{g}^e(2)$: $v_1 = e$

$\mathfrak{g}^e(4)$: $v_2 = f_{\substack{01111\\0}} - f_{\substack{00111\\1}}$, $v_3 = e_{\substack{12211\\1}} + e_{\substack{11221\\1}}$

$\mathfrak{g}^e(6)$: $v_4 = e_{\substack{01100\\1}} - e_{\substack{01110\\0}} - e_{\substack{11100\\0}} + 2e_{\substack{00110\\1}}$

$\mathfrak{g}^e(8)$: $v_5 = e_{\substack{11110\\0}} + e_{\substack{11100\\1}}$

$\mathfrak{g}^e(10)$: $v_6 = f_{\substack{00001\\0}}$ $v_7 = e_{\substack{12321\\2}}$ $v_8 = e_{\substack{11110\\1}} + e_{\substack{01210\\1}}$

$\mathfrak{g}^e(14)$: $v_9 = e_{\substack{12210\\1}}$

$M_0 = a_1 V_1 + \cdots + a_9 V_9$

Characteristic $p \geq 11$:

$$M_0^p = 0 \quad \dim(\widetilde{\mathcal{M}_0}) = 9$$

8.3 Orbits A_3, D_4 and A_5

All of these orbits require similar arguments as those in Section 7.2. In each case $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_1}$ is irreducible with the exception of A_3 when $p = 5$ where $\mathfrak{g}^e \cap \mathcal{N}_1$ has three irreducible components of dimension 21.

Orbit A_3

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}}$

$\mathfrak{c} \cong \mathfrak{so}_5 \oplus k$

$e_{\beta_1} = e_{\substack{00001\\0}}\,,\ e_{\beta_2} = e_{\substack{11110\\1}} + e_{\substack{01210\\1}}$

$$\overbrace{\quad t_1 = e \quad}^{\mathfrak{g}^e(2)} \qquad \overbrace{\begin{tikzcd}[row sep=small] u_1 = e_{\substack{11100\\1}} \arrow[d, dash, "\beta_2"] & v_1 = e_{\substack{12321\\1}} \arrow[d, dash, "\beta_2"] \\ u_2 \arrow[d, dash, "\beta_1"] & v_2 \arrow[d, dash, "\beta_1"] \\ u_3 \arrow[d, dash, "\beta_2"] & v_3 \arrow[d, dash, "\beta_2"] \\ u_4 & v_4 \end{tikzcd}}^{\mathfrak{g}^e(3)} \qquad \overbrace{\begin{tikzcd}[row sep=small] w_1 = e_{\substack{12211\\1}} \arrow[d, dash, "\beta_1"] \\ w_2 \arrow[d, dash, "\beta_2"] \\ w_3 \arrow[d, dash, "\beta_2"] \\ w_4 \arrow[d, dash, "\beta_1"] \\ w_5 \end{tikzcd}}^{\mathfrak{g}^e(4)} \qquad \overbrace{\quad x_1 = e_{\substack{11100\\0}} \quad}^{\mathfrak{g}^e(6)}$$

$M_0 = a_1T_1 + b_1U_1 + \cdots + b_4U_4 + c_1V_1 + \cdots + c_4V_4 + d_1W_1 + \cdots + d_5W_5 + g_1X_1$

$M_1 = e_{\beta_1} + e_{\beta_2} + M_0$

$M_2 = e_{\beta_2} + M_0$

$M_3 = e_{\beta_1} + M_0$

Characteristic $p = 5$:

$M_1^5 = 0 \Rightarrow d_5 = 0$ and ($b_4 = 0$ or $c_4 = 0$) $\qquad$ $\widetilde{\mathcal{M}_1}$ has two irreducible components of dimension 8-2+15=21

$M_2^5 = 0 \qquad \dim(\widetilde{\mathcal{M}_2}) = 6 + 15 = 21$

Characteristic $p \geq 7$:

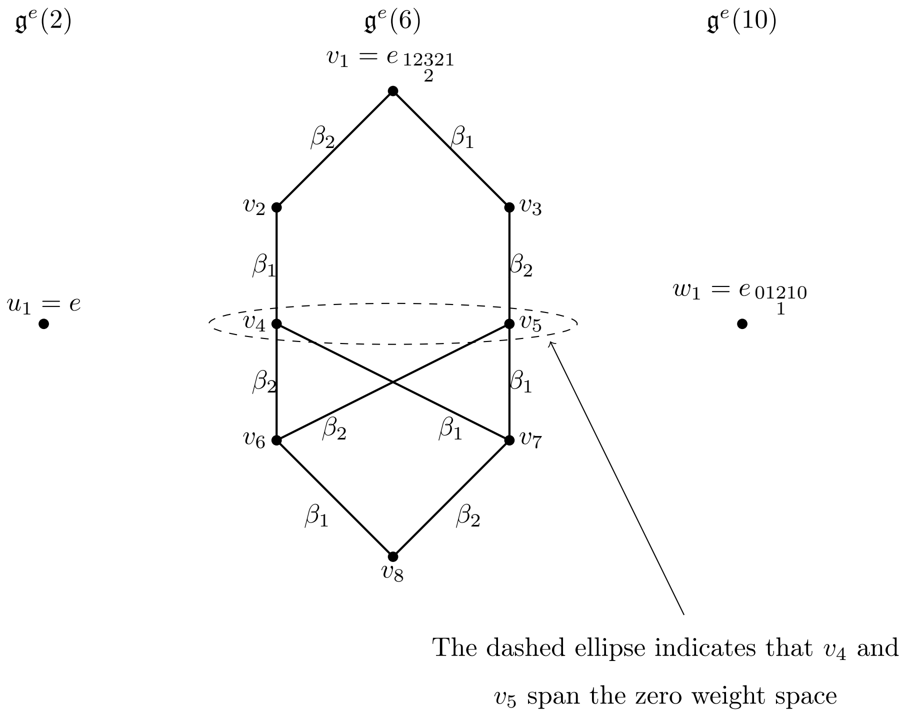
$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 8 + 15 = 23$$

Orbit D_4

$e = e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}} + e_{\substack{00010\\0}}$

$\mathfrak{c} \cong \mathfrak{sl}_3$

$e_{\beta_1} = e_{\substack{11100\\1}} + e_{\substack{11110\\0}}\,,\ e_{\beta_2} = e_{\substack{00111\\1}} - e_{\substack{01111\\0}}\,,\ e_{\beta_1+\beta_2} = [e_{\beta_1}, e_{\beta_2}]$



$M_0 = a_1U_1 + b_1V_1 + \cdots + b_8V_8 + c_1W_1$

$M_1 = e_{\beta_1} + e_{\beta_2} + M_0$

$M_2 = e_{\beta_1+\beta_2} + M_0$

Characteristic $p \geq 7$:

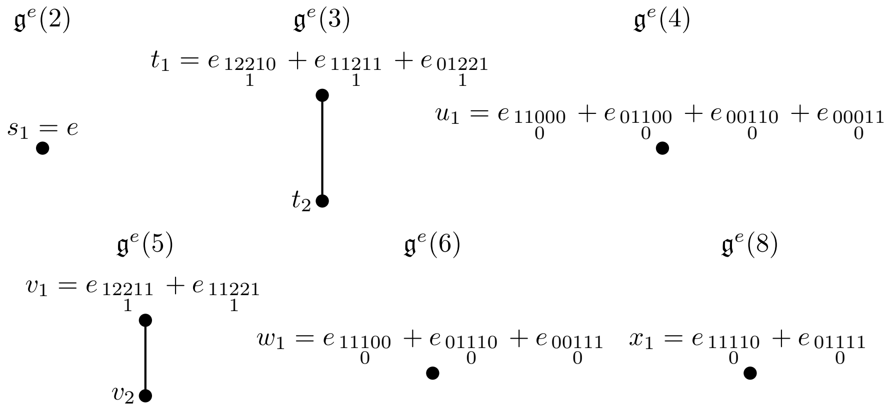
$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 6 + 10 = 16$$

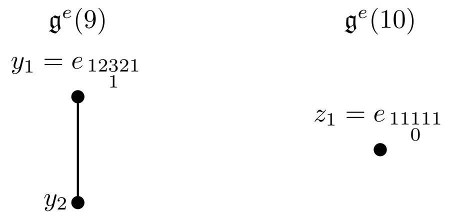
Orbit A_5

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = e_{\substack{12321\\2}}$





$$M_0 = a_1S_1 + b_1T_1 + b_2T_2 + c_1U_1 + d_1V_1 + d_2V_2 + g_1W_1 + h_1X_1 + i_1Y_1 + i_2Y_2 + j_1Z_1$$
$$M_1 = e_{\beta_1} + M_0$$

Characteristic $p \geq 7$:

$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 2 + 11 = 13$$

8.4 Orbits $A_2^2A_1$, A_3A_1, A_4 and $A_2A_1^2$

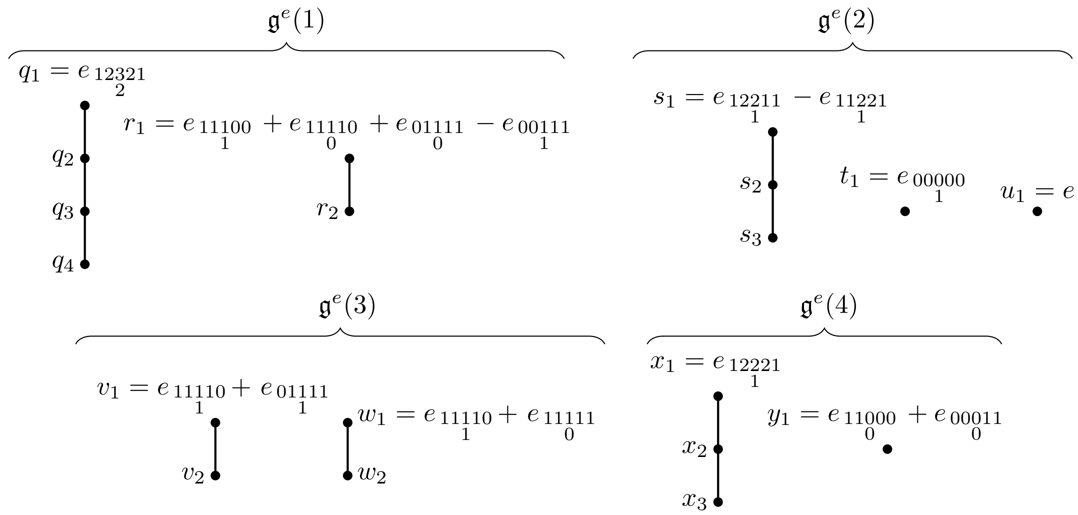
The arguments for $A_2A_1^2$, $A_2^2A_1$ for $p = 7$, and A_4, A_3A_1 for $p \geq 7$ are the same as in Section 7.2. Otherwise we use similar methods to those in Section 6.3 with the exception of $A_2A_1^2$, A_3A_1 and $A_2^2A_1$ when $p = 5$. In the $A_2A_1^2$ case, $\widetilde{\mathcal{M}_1}$ is a union of three irreducible components of dimension 22 and $\widetilde{\mathcal{M}_0}$ has dimension 24. Clearly $\widetilde{\mathcal{M}_0} \not\subset \widetilde{\mathcal{M}_1}$ and therefore $\mathfrak{g}^e \cap \mathcal{N}_1$ has 4 irreducible components. This is the first example we have found for which $\mathfrak{g}^e \cap \mathcal{N}_1$ is not equidimensional. For the A_3A_1 case, $\mathfrak{g}^e \cap \mathcal{N}_1$ has three components, two of dimension 18 and one of dimension 19. The final case $A_2^2A_1$ is considered in Section 8.6.

Orbit $A_2^2A_1$

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}} + e_{\substack{00000\\1}}$

$\mathfrak{c} \cong \mathfrak{sl}_2$

$e_{\beta_1} = e_{\substack{12210\\1}} + e_{\substack{11211\\1}} + e_{\substack{01221\\1}}$



$$\begin{tikzcd} \overbrace{z_1 = e_{\substack{11111\\1}}}^{\mathfrak{g}^e(5)} \arrow[d, dash] \\ z_2 \end{tikzcd}$$

$$\begin{aligned} M_0 = {} & a_1Q_1 + \cdots + a_4Q_4 + b_1R_1 + b_2R_2 + c_1S_1 + c_2S_2 + c_3S_3 + d_1T_1 + g_1U_1 + h_1V_1 + h_2V_2 + \\ & i_1W_1 + i_2W_2 + j_1X_1 + j_2X_2 + j_3X_3 + k_1Y_1 + l_1Z_1 + l_2Z_2 \\ M_1 = {} & e_{\beta_1} + M_0 \end{aligned}$$

Characteristic $p = 5$:

$$\begin{aligned} M_0^5 = 0 \Rightarrow & 2a_1^2a_4b_2^2 + a_1a_2a_3b_2^2 + 2a_1a_2a_4b_1b_2 + 2a_1a_3^2b_1b_2 + a_1a_3a_4b_1^2 - a_1b_1b_2^3 + \\ & 2a_2^3b_2^2 + 3a_2^2a_3b_1b_2 + a_2^2a_4b_1^2 - a_2a_3^2b_1^2 + a_2b_1^2b_2^2 - a_3b_1^3b_2 + a_4b_1^4 = 0, \\ & -a_1a_2a_4b_2^2 - a_1a_3^2b_2^2 + 3a_1a_3a_4b_1b_2 + 3a_1a_4^2b_1^2 - a_1b_2^4 + a_2^2a_3b_2^2 + \\ & 3a_2^2a_4b_1b_2 + 2a_2a_3^2b_1b_2 - a_2a_3a_4b_1^2 + a_2b_1b_2^3 + 3a_3^3b_1^2 - a_3b_1^2b_2^2 + a_4b_1^3b_2 = 0 \\ M_1^5 = 0 \Rightarrow & a_4 = b_2 = 0,\ c_3 = -a_3b_1 \end{aligned}$$

In this case $\widetilde{\mathcal{M}_0}$ has four components of dimension 19 (see §8.6) and $\widetilde{\mathcal{M}_1}$ has dimension 2-3+21=20. This case is considered in Section 8.6.
Characteristic $p = 7$:

$$\begin{aligned} & M_0^7 = 0 \\ & M_1^7 = 0 \Rightarrow b_2 = 0 \text{ and either } a_4 = 0 \text{ or } 5a_2a_4b_1^2 + a_3^2b_1^2 + 5a_3b_1c_3 + 2a_4b_1c_2 - a_4i_2 + c_3^2 = 0 \end{aligned}$$

Now $\dim(\widetilde{\mathcal{M}_0}) = 21$ and $\widetilde{\mathcal{M}_1}$ has two irreducible components of dimension 2-2+21=21.
The final polynomial is irreducible in $k[a_1, \ldots, a_4, b_1, b_2, \ldots, l_1, l_2]$ because it is linear in a_4 and the coefficients of a_4 has no common factors with the constant term. Therefore $\mathfrak{g}^e \cap \mathcal{N}_1$ has three irreducible components of dimension 21.
Characteristic $p = 11$:

$$\begin{aligned} & M_0^{11} = 0 && \dim(\mathcal{M}_0) = 21 \\ & M_1^{11} = 0 \Rightarrow b_2 = 0 \text{ or } a_4 = 0 && \widetilde{\mathcal{M}_1} \text{ has two irreducible components of dimension 2-1+21=22} \end{aligned}$$

In this case we only need to show that $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_1}$. This can be done with similar methods to Section 6.3. Specifically

$$\begin{aligned} & Ad_{\beta_1^\vee(t)}(e_{\beta_1} + \{u \in \mathfrak{u}^e : b_2 = 0\}) \subset \widetilde{\mathcal{M}_1} \\ & \Rightarrow t^2e_{\beta_1} + \{u \in \mathfrak{u}^e : tb_2 = 0\} \subset \widetilde{\mathcal{M}_1} \end{aligned}$$

Then by taking the limit as $t \to 0$ we get

$$
\begin{aligned}
\{u \in \mathfrak{u}^e : b_2 = 0\} &\subset \widetilde{\mathcal{M}_1} \\
\Rightarrow Ad_{\mathcal{E}_{-\beta_1}(\xi)}\{u \in \mathfrak{u}^e : b_2 = 0\} &\subset \widetilde{\mathcal{M}_1} \text{ for } \xi \in k^\times \\
\Rightarrow \{u \in \mathfrak{u}^e : b_2 = \xi b_1\} &\subset \widetilde{\mathcal{M}_1}
\end{aligned}
$$

As ξ varies, b_2 can take any value (assuming that b_1 is not zero). Therefore taking the closure gives $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_1}$ so $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component of dimension 20.

Orbit A_3A_1

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$

$\mathfrak{c} \cong \mathfrak{sl}_2 \oplus k$

$e_{\beta_1} = e_{\substack{12321\\2}}$

$\overbrace{\quad l_1 = e_{\substack{11111\\1}} + e_{\substack{01211\\1}} \quad}^{\mathfrak{g}^e(1)}$

l_2

$\overbrace{\quad n_1 = f_{\substack{00110\\0}} + f_{\substack{00011\\0}} \qquad p_1 = e_{\substack{01111\\0}} + e_{\substack{11110\\0}} \qquad q_1 = e_{\substack{00001\\0}} \qquad r_1 = e \quad}^{\mathfrak{g}^e(2)}$

$\overbrace{\quad s_1 = e_{\substack{11100\\1}} \qquad t_1 = e_{\substack{11211\\1}} + e_{\substack{12210\\1}} \qquad u_1 = e_{\substack{12321\\1}} \quad}^{\mathfrak{g}^e(3)}$

$s_2 \qquad t_2 \qquad u_2$

$\overbrace{\quad v_1 = f_{\substack{00010\\0}} \qquad w_1 = e_{\substack{11111\\0}} \qquad x_1 = e_{\substack{11000\\0}} + e_{\substack{01100\\0}} \quad}^{\mathfrak{g}^e(4)}$

$\overbrace{\quad y_1 = e_{\substack{12211\\1}} \quad}^{\mathfrak{g}^e(5)} \qquad \overbrace{\quad z_1 = e_{\substack{11100\\0}} \quad}^{\mathfrak{g}^e(6)}$

y_2

$$M_0 = a_1L_1 + a_2L_2 + b_1N_1 + c_1P_1 + d_1Q_1 + g_1R_1 + h_1S_1 + h_1S_2 + i_1T_1 + i_2T_2 + j_1U_1 + j_2U_2 + k_1V_1 + l_1W_1 + m_1X_1 + n_1Y_1 + n_2Y_2 + p_1Z_1$$
$$M_1 = e_{\beta_1} + M_0$$

Characteristic $p = 5$:

$M_0^p = 0 \qquad \dim(\widetilde{\mathcal{M}_0}) = 18$

$M_1^p = 0 \Rightarrow (a_2 = 0)$ or $(i_2 = b_1 = 0)$ or $(i_2 = c_1 = 0)$

In this case $\widetilde{\mathcal{M}_1}$ has three components, two of dimension 18 and one of dimension 19. Let $\widetilde{\mathcal{M}_1^{(1)}}$ be the component of dimension 19. Then by the same argument as $A_2^2A_1$ for $p = 11$ we can show $\mathcal{M}_0 \subset \widetilde{\mathcal{M}_1^{(1)}}$. Therefore $\mathfrak{g}^e \cap \mathcal{N}_1$ has three irreducible components.

Characteristic $p \geq 7$:

$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 2 + 18 = 20$$

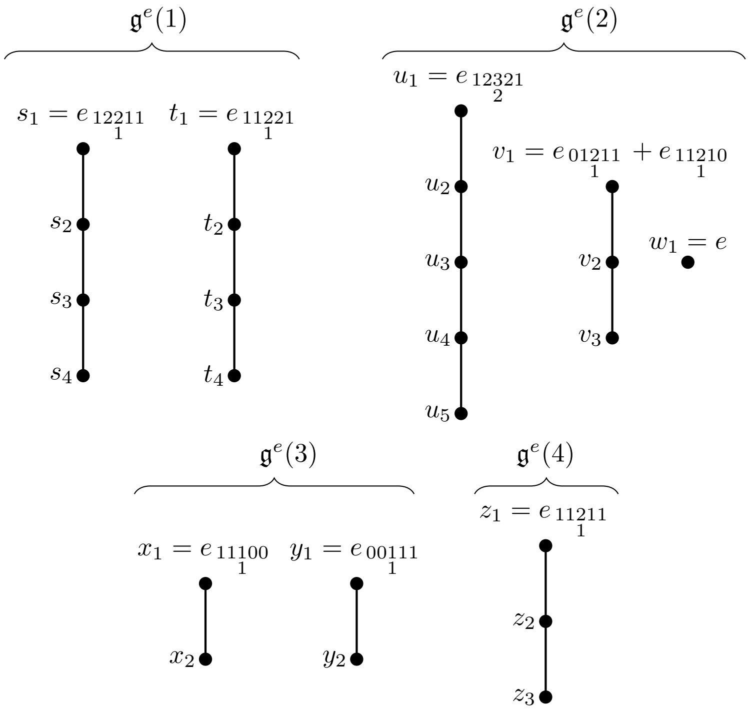
In this case we can show $\mathcal{M}_0 \subset \widetilde{\mathcal{M}_1}$ by the same argument as presented in Section 7.2. Therefore $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component of dimension 20.

Orbit $A_2A_1^2$

$e = e_{\substack{00000\\1}} + e_{\substack{00100\\0}} + e_{\substack{10000\\0}} + e_{\substack{00001\\0}}$

$\mathfrak{c} \cong \mathfrak{sl}_2 \oplus k$

$e_{\beta_1} = 2e_{\substack{11111\\0}} + e_{\substack{01210\\1}} + e_{\substack{11110\\1}} - e_{\substack{01111\\1}}$



$$\begin{aligned} M_0 = \ & a_1S_1 + \cdots + a_3S_3 + b_1T_1 + \cdots + b_3T_3 + c_1U_1 + \cdots + c_5U_5 + d_1V_1 + \cdots + d_3V_3 + g_1W_1 + \\ & h_1X_1 + h_2X_2 + i_1Y_1 + i_2Y_2 + j_1Z_1 + \cdots + j_3Z_3 \\ M_1 = \ & e_{\beta_1} + M_0 \end{aligned}$$

Characteristic $p = 5$:

$$\begin{aligned} & M_0^5 = 0 \\ & M_1^5 = 0 \Rightarrow (a_4 = b_4 = 0,\ c_5 = 4a_3b_3,\ d_3 = 3a_2b_3 - 3a_3b_2) \text{ or} \\ & \qquad (a_4 = 0,\ \ c_5 = 3a_2b_4 + 4a_3b_3,\ \ d_3 = 4a_1b_4 + 3a_2b_3 + 2a_3b_2, \\ & \qquad h_2 = 3a_1^2b_4 + 3a_1a_3b_2 + a_2^2b_2 + a_1c_4 + a_2c_3 + a_2d_2 + a_3c_2 + 4a_3d_1) \text{ or} \\ & \qquad (b_4 = 0,\ \ c_5 = 4a_3b_3 + 3a_4b_2,\ \ d_3 = 3a_2b_3 + 2a_3b_2 + a_4b_1, \\ & \qquad i_2 = 2a_2b_1b_3 + 4a_2b_2^2 + 2a_4b_1^2 + 4b_1c_4 + 4b_2c_3 + b_2d_2 + 4b_3c_2 + 4b_3d_1) \end{aligned}$$

In this case we have $\dim(\widetilde{\mathcal{M}_0}) = 24$ and $\widetilde{\mathcal{M}_1}$ has three irreducible components of dimension 2-4+24=22.

Characteristic $p = 7$:

$$\begin{aligned} & M_1^7 = 0 \Rightarrow (b_4 = 0 \text{ and } c_5 = -a_3b_3 + 4a_4b_2) \text{ or } (a_4 = 0 \text{ and } c_5 = 4a_2b_4 - a_3b_3) \\ & M_0^7 = 0 \end{aligned}$$

Then $\dim(\mathcal{M}_0) = 24$ and $\widetilde{\mathcal{M}_1}$ has two irreducible components of dimension of 2-2+24=24.

Characteristic $p = 11$:

$$\begin{aligned} & M_1^{11} = 0 \Rightarrow a_4 = 0 \text{ or } b_4 = 0 \quad \widetilde{\mathcal{M}_1} \text{ has two irreducible components of dimension 2-1+24=25} \\ & M_0^{11} = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dim(\mathcal{M}_0) = 24 \end{aligned}$$

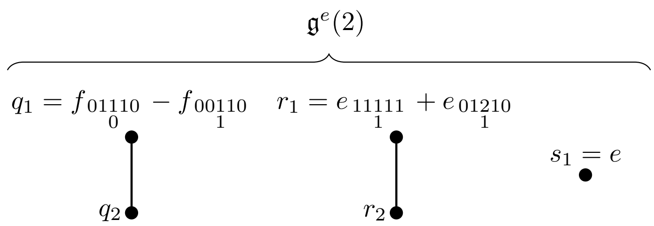
In the cases when $p = 5, 7$ then $\mathfrak{g}^e \cap \mathcal{N}_1 = \widetilde{\mathcal{M}_1} \cup \widetilde{\mathcal{M}_0}$. For $p = 11$ we can show that $\mathcal{M}_0 \subset \widetilde{\mathcal{M}_1}$ by the same method as for $A_2^2A_1$.

Orbit A_4

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00000\\1}}$

$\mathfrak{c} \cong \mathfrak{sl}_2 \oplus k$

$e_{\beta_1} = e_{\substack{00001\\0}}$



$$\overbrace{t_1 = f_{\substack{01221\\1}} \quad u_1 = e_{\substack{12321\\2}} \quad v_1 = e_{\substack{11000\\0}} + e_{\substack{01100\\0}} - e_{\substack{00100\\1}}}^{\mathfrak{g}^e(4)}$$

$$\overbrace{w_1 = f_{\substack{00010\\0}} \quad x_1 = e_{\substack{12211\\1}} \quad y_1 = e_{\substack{11100\\0}} - e_{\substack{01100\\1}}}^{\mathfrak{g}(6)} \qquad \overbrace{z_1 = e_{\substack{11100\\1}}}^{\mathfrak{g}(8)}$$

$$w_2 \qquad x_2$$

$M_0 = a_1Q_1 + a_2Q_2 + b_1R_1 + b_2R_2 + c_1S_1 + d_1T_1 + g_1U_1 + h_1V_1 + i_1W_1 + i_2W_2 + j_1X_1 + j_2X_2 + k_1Y_1 + l_1Z_1$

$M_1 = e_{\beta_1} + M_0$

Characteristic $p = 5$:

$M_0^5 = 0$	$\dim(\widetilde{\mathcal{M}_0}) = 14$
$M_1^5 = 0 \Rightarrow a_2 = 0$ or $b_2 = 0$	$\widetilde{\mathcal{M}_1}$ has two irreducible components of dimension 2-1+14=15

Characteristic $p \geq 7$:

$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 2 + 14 = 16$$

When $p = 5$ the method to show $\mathcal{M}_0 \subset \widetilde{\mathcal{M}_1}$ is the same as for $A_2^2A_1$ for $p = 11$. Whereas when $p \geq 7$ then the argument required is the same as that in Section 7.2.

8.5 Orbits A_2A_1 and A_2

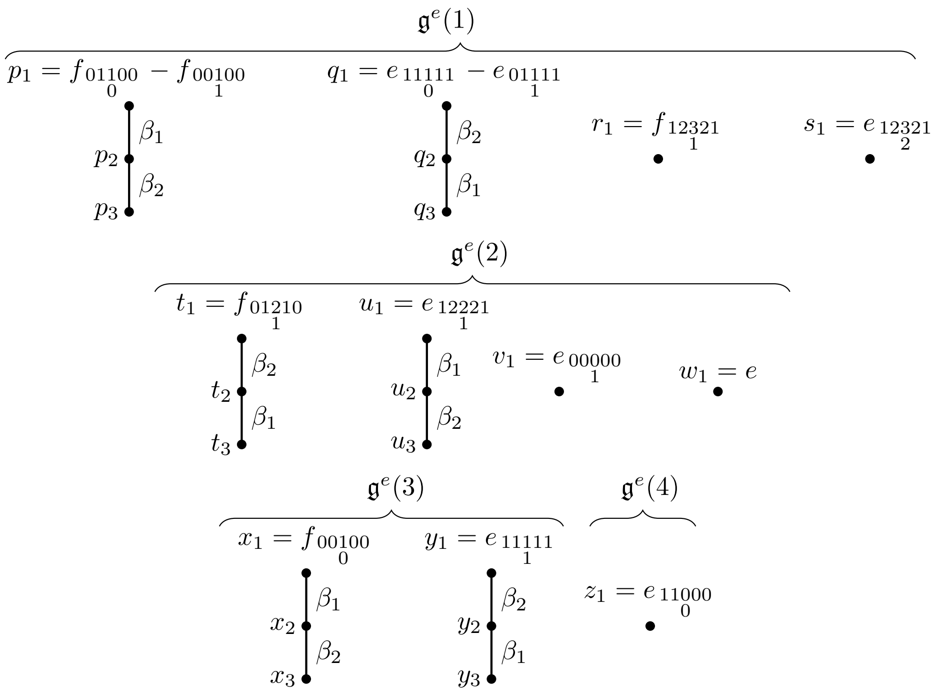
For both of these orbits when $p \geq 7$ the methods are similar to those in Section 7.2. Otherwise the methods are very similar and are considered below.

Orbit A_2A_1

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00000\\1}}$

$\mathfrak{c} \cong \mathfrak{sl}_3 \oplus k$

$e_{\beta_1} = e_{\substack{00010\\0}}, \ e_{\beta_2} = e_{\substack{00001\\0}}, \ e_{\beta_1+\beta_2} = [e_{\beta_1}, e_{\beta_2}]$



$$\begin{aligned}
M_0 = {} & a_1P_1 + \cdots + a_3P_3 + b_1Q_1 + \cdots + b_3Q_3 + c_1R_1 + d_1S_1 + g_1T_1 + \cdots + g_3T_3 + h_1U_1 + \\
& \cdots + h_3U_3 + i_1V_1 + j_1W_1 + k_1X_1 + \cdots + k_3X_3 + l_1Y_1 + \cdots + l_3Y_3 + m_1Z_1 \\
M_1 = {} & e_{\beta_1} + e_{\beta_2} + M_0 \\
M_2 = {} & e_{\beta_1+\beta_2} + M_0
\end{aligned}$$

Characteristic $p = 5$:

$$\begin{array}{ll}
M_0^5 = 0 & \dim(\widetilde{\mathcal{M}_0}) = 23 \\
M_1^5 = 0 \Rightarrow b_3 = a_3 = 0 & \dim(\widetilde{\mathcal{M}_1}) = 6 - 2 + 23 = 27 \\
M_2^5 = 0 \Rightarrow b_3 = 0 \text{ or } a_3 = 0 & \widetilde{\mathcal{M}_2} \text{ has two irreducible components of dimension 4-1+23=26}
\end{array}$$

Characteristic $p = 7$:

$$\begin{aligned}
& M_1^7 = 0 \Rightarrow (a_3 = b_3 = 0) \text{ or } (a_3 = c_1 = 0) \text{ or } (b_3 = d_1 = 0) \\
& M_2^7 = 0
\end{aligned}$$

Here $\widetilde{\mathcal{M}_1}$ has three irreducible components of dimension 6-2+23=27 and $\dim(\widetilde{\mathcal{M}_2}) = 4+23 = 27$.

Characteristic $p = 11$:

$$M_1^{11} = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 6 + 23 = 29$$

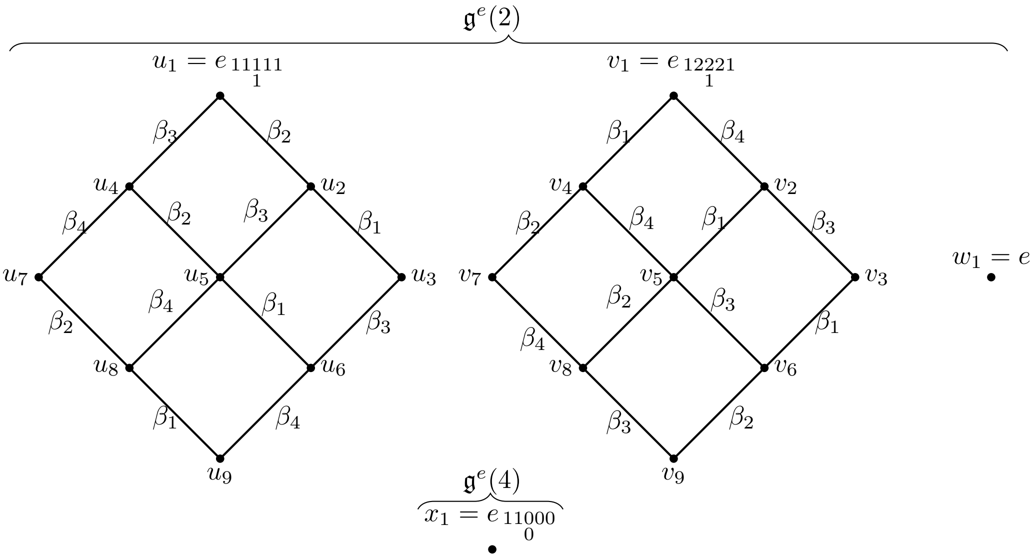
For $p = 7$, $\mathfrak{g}^e \cap \mathcal{N}_1$ has 4 irreducible components of dimension 27 and for $p = 11$ it has one irreducible component of dimension 29. The case when $p = 5$ is considered below.

Orbit A_2

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}}$

$\mathfrak{c} \cong \mathfrak{sl}_3 \oplus \mathfrak{sl}_3$

$e_{\beta_1} = e_{\substack{00010\\0}},\ e_{\beta_2} = e_{\substack{00001\\0}},\ e_3 = e_{\substack{00000\\1}},\ e_4 = e_{\substack{12321\\1}},\ e_{\beta_1+\beta_2} = [e_{\beta_1}, e_{\beta_2}],\ e_{\beta_3+\beta_4} = [e_{\beta_3}, e_{\beta_4}]$



$M_0 = a_1U_1 + \cdots + a_9U_9 + b_1V_1 + \cdots + b_9V_9 + c_1W_1 + d_1X_1$

Nilpotent Orbits of $\mathfrak{c}$	Representative e of nilpotent orbit	M_i label of $e + M_0$
$[3]+[3]$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_3}+e_{\beta_4}$	M_1
$[3]+[2,1]$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_3+\beta_4}$	M_2
$[2,1]+[3]$	$e_{\beta_1+\beta_2}+e_{\beta_3}+e_{\beta_4}$	M_3
$[3]+[1^3]$	$e_{\beta_1}+e_{\beta_2}$	M_4
$[1^3]+[3]$	$e_{\beta_3}+e_{\beta_4}$	M_5
$[2,1]+[2,1]$	$e_{\beta_1+\beta_2}+e_{\beta_3+\beta_4}$	M_6
$[2,1]+[1^3]$	$e_{\beta_1+\beta_2}$	M_7
$[1^3]+[2,1]$	$e_{\beta_3+\beta_4}$	M_8

Characteristic $p = 5$:

$M_1^5 = 0 \Rightarrow a_9 = b_9 = 0$	$\dim(\widetilde{\mathcal{M}_1}) = 6+6-2+20 = 30$
$M_2^5 = 0 \Rightarrow a_9 = 0$ or $b_9 = 0$	$\widetilde{\mathcal{M}_2}$ has two irreducible components of dimension 6+4-1+20=29
$M_3^5 = 0 \Rightarrow a_9 = 0$ or $b_9 = 0$	$\widetilde{\mathcal{M}_3}$ has two irreducible components of dimension 4+6-1+20=29
$M_4^5 = 0$	$\dim(\widetilde{\mathcal{M}_4}) = 6+20 = 26$
$M_5^5 = 0$	$\dim(\widetilde{\mathcal{M}_5}) = 6+20 = 26$
$M_6^5 = 0$	$\dim(\widetilde{\mathcal{M}_6}) = 4+4+20 = 28$

Characteristic $p \geq 7$:

$$M_1^p = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 6+6+20 = 32$$

When $p \geq 7$ the $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component of dimension 32.

Characteristic $p = 5$:

We start with A_1A_2 for $p = 5$; we need to show that $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_2}$. The set $\mathcal{M}_2$ is the union of two sets X_1 and X_2 where

$$X_1 = e_{\beta_1+\beta_2} + \{u \in \mathfrak{u}^e : b_3 = 0\}$$
$$X_2 = e_{\beta_1+\beta_2} + \{u \in \mathfrak{u}^e : a_3 = 0\}$$

To show that $X_1 \subset \widetilde{\mathcal{M}_1}$ consider $e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_3 = b_3 = 0\} \subset \widetilde{\mathcal{M}_1}$ and consider $\beta^\vee(t) = \beta_1^\vee(t)\beta_2^\vee(t^2)$. Then

$$Ad_{\beta^\vee(t)}(e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_3 = b_3 = 0\}) \subset \widetilde{\mathcal{M}_1}$$
$$\Rightarrow e_{\beta_1} + t^3 e_{\beta_2} + \{u \in \mathfrak{u}^e : a_3 = b_3 = 0\} \subset \widetilde{\mathcal{M}_1}$$

Taking the limit as $t \to 0$ gives

$$e_{\beta_1} + \{u \in \mathfrak{u}^e : a_3 = b_3 = 0\} \subset \widetilde{\mathcal{M}_1}$$

As we did in Section 6.3, we can represent an element in $e_{\beta_1} + \{u \in \mathfrak{u}^e : a_3 = b_3 = 0\}$ by a series of matrices as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \left(\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 & b_1 \end{pmatrix}, \ldots \right)$$

Now consider $n_{\beta_2} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right) \in C = SL_3$. Then

$$Ad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \left(\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 & b_1 \end{pmatrix}, \ldots \right) \right) \subset \widetilde{\mathcal{M}_1}$$
$$\Rightarrow \begin{pmatrix} 0 & 0 & -1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \left(\begin{pmatrix} a_1 \\ 0 \\ -a_2 \end{pmatrix}, \begin{pmatrix} 0 & b_1 & -b_2 \end{pmatrix}, \ldots \right) \subset \widetilde{\mathcal{M}_1}$$

For $\xi \in k$ then conjugating by $\mathcal{E}_{\beta_2}(\xi)$ gives

$$Ad \begin{pmatrix} 1 & 0 & 0 \\ & 1 & \xi \\ & & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 & -1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \left(\begin{pmatrix} a_1 \\ 0 \\ -a_2 \end{pmatrix}, \begin{pmatrix} 0 & b_1 & -b_2 \end{pmatrix}, \ldots \right) \right) \subset \widetilde{\mathcal{M}_1}$$
$$\Rightarrow \begin{pmatrix} 0 & 0 & -1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \left(\begin{pmatrix} a_1 \\ -\xi a_2 \\ -a_2 \end{pmatrix}, \begin{pmatrix} 0 & b_1 & -(\xi b_1 + b_2) \end{pmatrix}, \ldots \right) \subset \widetilde{\mathcal{M}_1}$$

As ξ varies then ξa_2 can take any value as long as a_2 does not equal zero. Therefore taking the closure gives $X_1 \subset \widetilde{\mathcal{M}_1}$.

For A_2A_1 all that remains is to show that $\mathcal{M}_0 \subset \widetilde{\mathcal{M}_2}$ and $X_2 \subset \widetilde{\mathcal{M}_1}$.

For the A_2 orbit, elements in C or $\mathfrak{c}$ corresponding to the copy of SL_3 with root elements e_{β_1} and e_{β_2} is subscripted with a 1 otherwise it is subscripted with a 2. Both $\mathcal{M}_2$ and $\mathcal{M}_3$ have two irreducible components, one with $a_9 = 0$ and one with $b_9 = 0$. These are denoted as $\mathcal{M}_2(a_9)$ and $\mathcal{M}_2(b_9)$ (resp. $\mathcal{M}_3(a_9)$ and $\mathcal{M}_3(b_9)$). For this orbit we have $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_8}$, $\widetilde{\mathcal{M}_7} \subset \widetilde{\mathcal{M}_6}$ and $\widetilde{\mathcal{M}_8} \subset \widetilde{\mathcal{M}_6}$ by the arguments in Section 7.2. We still need to show that $\mathcal{M}_4 \subset \widetilde{\mathcal{M}_2}$, $\mathcal{M}_6$ and $\mathcal{M}_5$ are contained in $\widetilde{\mathcal{M}_3}$ and $\mathcal{M}_2(a_9)$, $\mathcal{M}_2(b_9)$, $\mathcal{M}_3(a_9)$ and $\mathcal{M}_3(b_9)$ are contained in $\widetilde{\mathcal{M}_1}$.

All these remaining inclusions can be shown using a similar method $X_1 \subset \widetilde{\mathcal{M}_1}$. In each case we conjugate by a cocharacter then by 1 or 2 elements in C. These elements are presented in the following table for each inclusion argument.

Orbit	Inclusion	Cocharacter	Elements in Centralizer
	$X_1 \subset \widetilde{\mathcal{M}_1}$	$\beta_1^\vee(t)\beta_2^\vee(t^2)$	n_{β_2} and $\mathcal{E}_{\beta_2}(\xi)$
A_2A_1	$X_2 \subset \widetilde{\mathcal{M}_1}$	$\beta_1^\vee(t^2)\beta_2^\vee(t)$	n_{β_1} and $\mathcal{E}_{\beta_1}(\xi)$
	$\mathcal{M}_0 \subset \widetilde{\mathcal{M}_2}$	$Ad(\beta_1^\vee(t))$	$\mathcal{E}_{\beta_2(\xi)}$
	$\mathcal{M}_6 \subset \widetilde{\mathcal{M}_3}$	$\beta_3^\vee(t)\beta_4^\vee(t^2)$	n_{β_4} and $\mathcal{E}_{\beta_4}(\xi)$
	$\mathcal{M}_4 \subset \widetilde{\mathcal{M}_2}$	$\beta_3^\vee(t)$	$\mathcal{E}_{\beta_3}(\xi)$
	$\mathcal{M}_5 \subset \widetilde{\mathcal{M}_3}$	$\beta_1^\vee(t)$	$\mathcal{E}_{\beta_3}(\xi)$
A_2	$\mathcal{M}_2(a_9) \subset \widetilde{\mathcal{M}_1}$	$\beta_1^\vee(t)\beta_2^\vee(t^2)$	n_{β_2} and $\mathcal{E}_{\beta_2}(\xi)$
	$\mathcal{M}_2(b_9) \subset \widetilde{\mathcal{M}_1}$	$\beta_1^\vee(t^2)\beta_2^\vee(t)$	n_{β_1} and $\mathcal{E}_{\beta_1}(\xi)$
	$\mathcal{M}_3(a_9) \subset \widetilde{\mathcal{M}_1}$	$\beta_3^\vee(t)\beta_4^\vee(t^2)$	n_{β_4} and $\mathcal{E}_{\beta_4}(\xi)$
	$\mathcal{M}_3(b_9) \subset \widetilde{\mathcal{M}_1}$	$\beta_3(t^2)\beta_4(t)$	n_{β_3} and $\mathcal{E}_{\beta_3}(\xi)$

Therefore for each orbit when $p = 5$ then $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component.

8.6 Orbits $A_2^2A_1$ when $p = 5$ and A_2^2

Each of these cases is considered separately.

Orbit $A_2^2A_1$ when $p = 5$

Recall that in this case $\widetilde{\mathcal{M}_0}$ is the zero set of two complicated polynomials (which are stated in Section 8.4). By considering the prime decomposition of the ideal defining $\mathcal{M}_0$ we can show that $\mathcal{M}_0$ has four irreducible components. This was achieved using the MAGMA online calculator. The dimension of each of these components is 19 and they are defined as follows:

$$\begin{aligned}
\mathcal{M}_0^{(1)} =&\{u\in\mathfrak{u}^e : a_1^2a_4^2+a_1a_2a_3a_4+2a_1a_3^3+2a_2^3a_4+2a_2^2a_3^2=0,\\
&a_1a_3b_2+a_1a_4b_1+3a_2^2b_2+a_2a_3b_1=0,\\
&a_1a_4^2b_1+3a_2^2a_4b_2+4a_2a_3^2b_2+2a_3^3b_1=0, a_1a_4b_2+a_2a_3b_2+a_2a_4b_1+3a_3^2b_1=0,\\
&a_1b_2^2+a_2b_1b_2+2a_3b_1^2=0, a_2b_2^2+3a_3b_1b_2+3a_4b_1^2=0\}\\
\mathcal{M}_0^{(2)} =&\{u\in\mathfrak{u}^e : a_1a_3+3a_1b_2+3a_2^2+4a_2b_1+b_1^2=0, a_1a_4+a_2a_3+4a_2b_2+a_3b_1+4b_1b_2=0,\\
&a_2a_4+3a_3^2+a_3b_2+2a_4b_1+b_2^2=0\}\\
\mathcal{M}_0^{(3)} =&\{u\in\mathfrak{u}^e : a_1a_3+2a_1b_2+3a_2^2+a_2b_1+b_1^2=0, a_1a_4+a_2a_3+a_2b_2+4a_3b_1+4b_1b_2=0,\\
&a_2a_4+3a_3^2+4a_3b_2+3a_4b_1+b_2^2=0\}\\
\mathcal{M}_0^{(4)} =&\{u\in\mathfrak{u}^e : b_1=0, b_2=0\}
\end{aligned}$$

The conditions for $\mathcal{M}_1^5=0$ are $a_4=b_2=0$ and $c_3=-a_3b_1$, so

$$e_{\beta_1}+\{u\in\mathfrak{u}^e : a_4=b_2=0,\ c_3=-a_3b_1\}\subset\widetilde{\mathcal{M}_1}$$

Then considering $Ad_{\beta_1^\vee(t)}$ gives

$$\begin{aligned}
t^2e_{\beta_1}+\{u\in\mathfrak{u}^e : t^3a_4=tb_2=0,\ t^2c_3=-a_3b_1\}&\subset\widetilde{\mathcal{M}_1}\\
\Rightarrow\{u\in\mathfrak{u}^e : a_4=b_2=0,\ a_3b_1=0\}&\subset\widetilde{\mathcal{M}_1}
\end{aligned}$$

Let $b_1=0$; by considering $Ad_{-\beta_1(\xi)}$ for $\xi\in k^\times$, we get

$$\begin{aligned}
\{u\in\mathfrak{u}^e : a_4=\xi^3a_1-\xi^2a_2+\xi a_3, b_1=b_2=0\}\subset\widetilde{\mathcal{M}_1}\ &\text{(by same method as §6.3)}\\
\Rightarrow\{u\in\mathfrak{u}^e : b_1=b_2=0\}=\mathcal{M}_0^{(4)}\subset\widetilde{\mathcal{M}_1}&
\end{aligned}$$

Alternatively if $a_3=0$ then

$$X_1=\{u\in\mathfrak{u}^e : a_3=a_4=b_2=0\}\subset\widetilde{\mathcal{M}_1}$$

Now $X_1\subset\mathcal{M}_0^{(1)}$ and it is easy to check that X_1 is not contained in $\mathcal{M}_0^{(2)},\mathcal{M}_0^{(3)}$ and $\mathcal{M}_0^{(4)}$. Since the set X_1 is not stabilized by C and $\dim(X_1)=18$ then $\dim(C\cdot X_1)>18$. Therefore $C\cdot X_1=\widetilde{\mathcal{M}_0^{(1)}}$ because $\dim(\mathcal{M}_0^{(1)})=19$, so $\widetilde{\mathcal{M}_0^{(1)}}\subset\widetilde{\mathcal{M}_1}$.
Now we show that $\mathcal{M}_0^{(2)}$ and $\mathcal{M}_0^{(3)}$ are not contained in $\widetilde{\mathcal{M}_1}$. Let the 4-dimension irreducible submodule of $\mathfrak{g}^e(1)$ be U and the 2-dimensional submodule be V. We can consider U as S^3V, where $u_1=\omega_1\otimes\omega_1\otimes\omega_1$, $u_2=\omega_1\otimes\omega_1\otimes\omega_2,\dots$ for $\omega_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\omega_2=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. If $e+u+v+\dots$ is contained in $\widetilde{\mathcal{M}_1}$ then the following conditions hold.

$$\begin{aligned}
u=\omega_1\otimes(a_1\omega_1\otimes\omega_1+a_2\omega_1\otimes\omega_2+a_3\omega_2\otimes\omega_2)\ \text{for some}\ a_1,a_2,a_3\in k\\
e\cdot v=0
\end{aligned}$$

Note that $e_{\beta_1} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_{\beta_1}\omega_1 = 0$.

We can parametrize the set of nilpotent elements in $\mathfrak{sl}_2$ by $e' = \left(\begin{smallmatrix} st & s^2 \\ -t^2 & -st \end{smallmatrix}\right)$. The non-zero elements are all conjugate to e_{β_1}. Therefore for $(s,t) \neq (0,0)$, $ad(e')(v) = 0$ if and only if $tb_1 + sb_2 = 0$, in which case v can be expressed as

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \xi \begin{pmatrix} s \\ -t \end{pmatrix} \quad \text{for some } \xi \in k$$

Similarly we require $u = \omega_1' \otimes (a_1\omega_1' \otimes \omega_1' + a_2\omega_1' \otimes \omega_2' + a_3\omega_2' \otimes \omega_2')$ where $e'\omega_1' = 0$. Therefore letting $\omega_1' = s\omega_1 - t\omega_2$ gives

$$u = (s\omega_1 - t\omega_2) \otimes (\mu_1\omega_1 \otimes \omega_1 + \mu_2\omega_1 \otimes \omega_2 + \mu_3\omega_2 \otimes \omega_2) \text{ for some } \mu_i \in k$$

Therefore we require

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \mu_1 s \\ \mu_2 s - t\mu_1 \\ s\mu_3 - t\mu_2 \\ -\mu_3 t \end{pmatrix}$$

$$\Rightarrow t^3a_1 + st^2a_2 + s^2ta_3 + s^3a_4 = 0$$

$$\Rightarrow -a_1b_2^3 + a_2b_1b_2^2 - a_3b_1^2b_2 + a_4b_1^3 = 0 \tag{8.1}$$

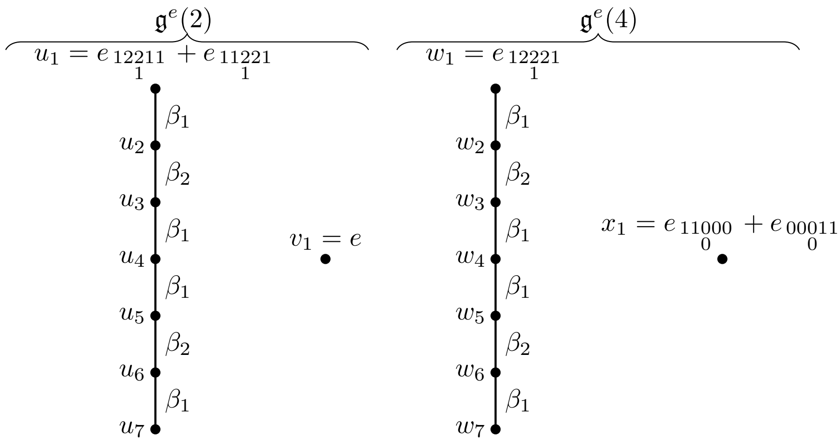
Therefore $e' + u + v \cdots \in \widetilde{\mathcal{M}_1}$ if equation (8.1) holds. By a MAGMA calculation we can show that (8.1) is not contained in the ideals generated by the by the polynomials defining $\mathcal{M}_0^{(2)}$ and $\mathcal{M}_0^{(3)}$. Since the ideals $\mathcal{M}_0^{(2)}$ and $\mathcal{M}_0^{(3)}$ are prime they cannot be contained in $\widetilde{\mathcal{M}_1}$. Hence $\mathfrak{g}^e \cap \mathcal{N}_1$ has three irreducible components namely $\widetilde{\mathcal{M}_1}$, $\mathcal{M}_0^{(2)}$ and $\mathcal{M}_0^{(3)}$ with dimensions 20, 19 and 19 respectively.

Orbit A_2^2

$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}}$

$\mathfrak{c} \cong \mathfrak{g}_2$

$e_{\beta_1} = e_{\substack{11100\\0}} + e_{\substack{01110\\0}} + e_{\substack{00111\\0}}$, $e_{\beta_2} = e_{\substack{00000\\1}}$, $e_{3\beta_1+\beta_2} = e_{\substack{12321\\1}}$



$M_0 = a_1U_1 + \cdots + a_7U_7 + b_1V_1 + c_1W_1 + \cdots + c_7W_7 + d_1X_1$

Nilpotent Orbit of $\mathfrak{c}$	Representative e of nilpotent orbit	M_i label of $e + M_0$
G_2	$e_{\beta_1} + e_{\beta_2}$	M_1
$G_2(a_1)$	$e_{\beta_2} + e_{3\beta_1+\beta_2}$	M_2
$\widetilde{A_1}$	$e_{2\beta_1+\beta_2}$	M_3
A_1	e_{β_2}	M_4

Characteristic $p = 5$:

Since $\mathcal{N}(\mathfrak{g}_2) = \overline{\mathcal{O}_{G_2(a_1)}}$ for $p = 5$ we do not consider the regular orbit.

$$\begin{aligned} &M_2^5 = 0 \Rightarrow a_6 = a_7 = 0 && \dim(\widetilde{\mathcal{M}_2}) = 10 - 2 + 16 = 24 \\ &M_3^5 = 0 \Rightarrow a_7 = 0 && \dim(\widetilde{\mathcal{M}_3}) = 8 - 1 + 16 = 23 \\ &M_4^5 = 0 && \dim(\widetilde{\mathcal{M}_4}) = 6 + 16 = 22 \end{aligned}$$

Characteristic $p = 7$:

$$\begin{aligned} &M_1^7 = 0 \Rightarrow a_7 = 0,\ c_7 = 2a_4a_6 + 6a_5^2 && \dim(\widetilde{\mathcal{M}_1}) = 12 - 2 + 16 = 26 \\ &M_2^7 = 0 && \dim(\widetilde{\mathcal{M}_2}) = 10 + 16 = 26 \end{aligned}$$

Characteristic $p = 11$:

$$M_1^{11} = 0 \Rightarrow a_7 = 0 \quad \dim(\widetilde{\mathcal{M}_1}) = 12 - 1 + 16 = 27$$

For $p = 7$ the methods to find the irreducible components are the same as Section 7.2, in this case $\mathfrak{g}^e \cap \mathcal{N}_1$ has two irreducible components of dimension 26. For $p = 11$ the method to show that $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component, is the same as in Section 7.4. Below are the details to show that $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component of dimension 24 when $p = 5$.

Characteristic $p = 5$

The inclusion $\widetilde{\mathcal{M}_0} \subset \widetilde{\mathcal{M}_4}$ holds by the same method as in Section 7.2. Therefore we need to show $\widetilde{\mathcal{M}_4} \subset \widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_2}$.

Let e' be a nilpotent element in the orbit $\mathcal{O}_{G_2(a_1)}$. Let $u \in \mathfrak{u}^e$; then for $e' + u$ to be contained in $\widetilde{\mathcal{M}_2}$ we require $[e', [e', u']] = 0$ where u' is the component of u in $\mathfrak{g}^e(2)$. To show $\mathcal{M}_3 \subset \widetilde{\mathcal{M}_2}$, we consider $M = e_{2\beta_1+\beta_2} + sth_{\beta_2} + s^2e_{\beta_2} - t^2f_{\beta_2}$. This is because for $(s,t) \neq (0,0)$, M is conjugate to $e_{2\beta_1+\beta_2} + e_{\beta_2} \in \mathcal{O}_{G_2(a_1)}$. Let ρ be the representation of $\mathfrak{c}$ on the highest weight module of G_2

of dimension 7. A [GAP12] computation gives

$$M_{s,t} = \rho(M) = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & st & -s^2 & 0 & -1 & 0 & 0 \\ 0 & t^2 & -st & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & st & s^2 & 0 \\ 0 & 0 & 0 & 0 & -t^2 & -st & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By considering $u = (u_1, \dots, u_7)^t$ as a vector then $M_{s,t}^2(u) = 0$ implies

$$a_7 = 0 \qquad\qquad ta_5 + sa_6 = 0$$

Then $M_{s,t} + \{u \in \mathfrak{u}^e : a_7 = 0, ta_5 + sa_6 = 0\} \subset \widetilde{\mathcal{M}_2}$. For fixed $a_5 = c$ and $a_6 = d$, let $t = -\xi d$ and $s = \xi c$ for some $\xi \in k^\times$. Then the condition $ta_5 + sa_6 = 0$ holds and

$$M_{\xi c, -\xi d} + \{u \in \mathfrak{u}^e : a_7 = 0\} \subset \widetilde{\mathcal{M}_2}$$

Therefore by taking the closure we get $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_2}$.

To show that $\mathcal{M}_4 \subset \widetilde{\mathcal{M}_3}$ firstly note that $f_{3\beta_1+2\beta_2} + \mathfrak{u}^e \subset \widetilde{\mathcal{M}_4}$. Let $f' = f_{3\beta_1+2\beta_2}$ and $e' = e_{3\beta_1+2\beta_2}$. Then consider the transverse slice:

$$\mathcal{A} = (f' + \mathfrak{c}^{e'}) \cap \mathcal{O}_{\widetilde{A_1}} = \left\{ \begin{pmatrix} st & s^2 & 2st^2 & 4s^2t & 2s^3 & 0 & 0 \\ -t^2 & -st & -2t^3 & -4st^2 & -2s^2t & 0 & 0 \\ 0 & 0 & 2st & 2s^2 & 0 & 2s^2t & 2s^3 \\ 0 & 0 & -t^2 & 0 & s^2 & -2st^2 & -2s^2t \\ 0 & 0 & 0 & -2t^2 & -2st & 2t^3 & 2st^2 \\ 1 & 0 & 0 & 0 & 0 & st & s^2 \\ 0 & 1 & 0 & 0 & 0 & -t^2 & -st \end{pmatrix} : s, t \in k \right\}$$

Let $u \in \mathfrak{u}^e$; then for $e' + u$ to be contained in $\widetilde{\mathcal{M}_3}$ we require $[e', [e', u']] = 0$ where u' is the component of $u \in \mathfrak{g}^e(2)$. Then for an element $A_{s,t}$ in $\mathcal{A}$ with $(s,t) \neq (0,0)$, $A_{s,t}^2(u) = 0$ implies

$$2ta_1 + 2sa_2 + 2t^2a_3 + 4sta_4 + 2s^2a_5 = 0$$
$$\Rightarrow 2\xi a_1 + 2a_2 + s(2\xi^2 a_3 + 4\xi^2 a_4 + 2a_5) = 0 \text{ for } t = \xi s, \xi \in k^\times$$

Therefore

$$A_{s,\xi s} + \{u \in \mathfrak{u}^e : 2\xi a_1 + 2a_2 + s(2\xi^2 a_3 + 4\xi^2 a_4 + 2a_5) = 0\} \subset \widetilde{\mathcal{M}_3}$$

Then by taking the limit as $s \to 0$ gives

$$A_{0,0} + \{u \in \mathfrak{u}^e : a_2 = -\xi a_1\} \subset \widetilde{\mathcal{M}_3}$$

As ξ varies, a_2 can take any value as long as a_1 is not zero. Therefore taking the closure gives $\mathcal{M}_4 \subset \widetilde{\mathcal{M}_3}$.

8.7 Orbits A_1^3, A_1^2 and A_1

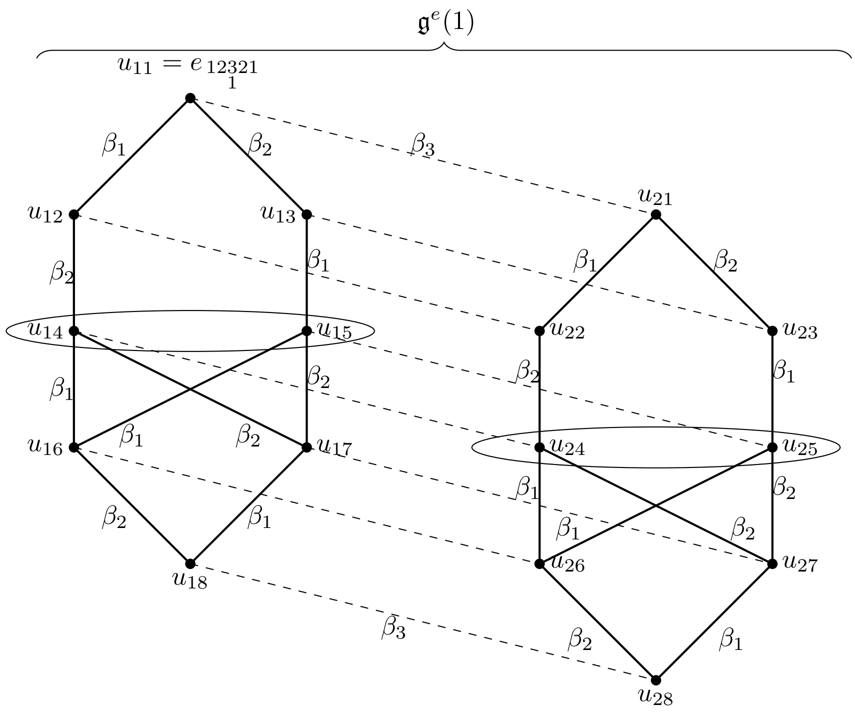
For these cases the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ have not been found, but we have expressed $\mathfrak{g}^e \cap \mathcal{N}_1$ as a union of possible irreducible components X_i. Some of the possible components have been eliminated, however the remaining cases are more complicated and the standard methods we have used throughout do not work. Due to time constraints we were unable to find alternative methods for these cases. Note that not establishing the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ for these orbits did not obstruct our work to find the irreducible components of $\mathcal{C}_1^{nil}(E_6)$. This is because each possible component of $\mathcal{C}_1^{nil}(E_6)$ corresponding to X_i can be eliminated using Proposition 5.2.1.

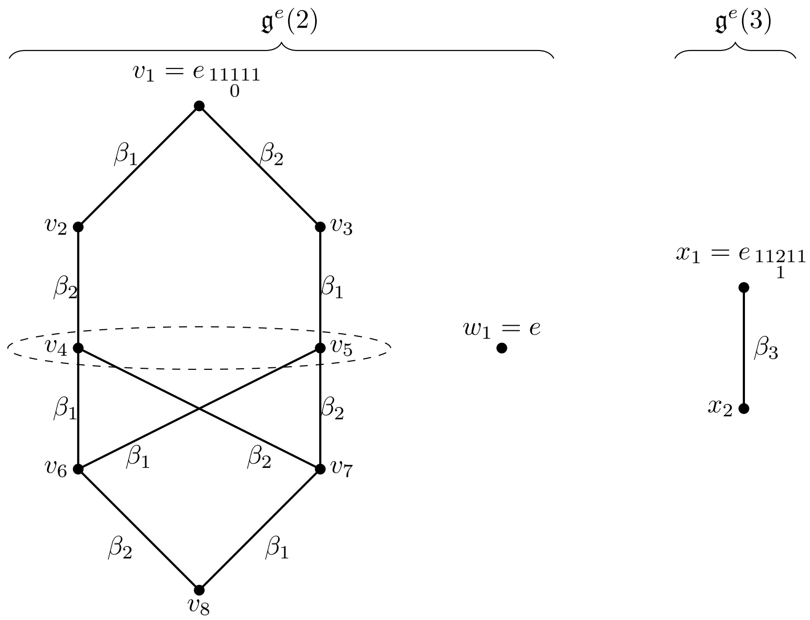
Orbits A_1^3

$e = e_{\substack{10000\\0}} + e_{\substack{00100\\0}} + e_{\substack{00001\\0}}$

$\mathfrak{c} \cong \mathfrak{sl}_3 \oplus \mathfrak{sl}_2$

$e_{\beta_1} = e_{\substack{11000\\0}} + e_{\substack{01100\\0}}\,,\ e_{\beta_2} = e_{\substack{00110\\0}} + e_{\substack{00011\\0}}\,,\ e_{\beta_3} = e_{\substack{12321\\2}}\,,\ e_{\beta_1+\beta_2} = [e_{\beta_1}, e_{\beta_2}]$





$$M_0 = a_1U_{11} + \cdots + a_8U_{18} + a_9U_{21} + \cdots + a_{16}U_{28} + b_1V_1 + \cdots + b_8V_8 + c_1W_1 + d_1X_1 + d_2X_2$$

Nilpotent Orbits of $\mathfrak{c}$	Representative e of nilpotent orbit	M_i label of $e + M_0$
$[3] + [2]$	$e_{\beta_1} + e_{\beta_2} + e_{\beta_3}$	M_1
$[2,1] + [2]$	$e_{\beta_1+\beta_2} + e_{\beta_3}$	M_2
$[3] + [1^2]$	$e_{\beta_1} + e_{\beta_2}$	M_3
$[2,1] + [1^2]$	$e_{\beta_1+\beta_2}$	M_4
$[1^3] + [2]$	e_{β_3}	M_5

Characteristic $p = 5$:

$M_0^5 = 0$

$M_1^5 = 0 \Rightarrow a_{28} = 0,\ a_{18} = a_{26} = a_{27},\ b_8 = a_{14}a_{18} + a_{15}a_{18} + 4a_{16}a_{24} + 4a_{17}a_{25} + a_{24}a_{25}$

$M_2^5 = 0 \Rightarrow a_{28} = 0$ then either

$(a_{26} = 0$ and either $a_{18} = -a_{25}$ or $(a_{24}+a_{25})(a_{24}-a_{18})+a_{27}(a_{16}-a_{22}) = 0)$ or

$(a_{27} = 0,$ and either $a_{18} = a_{24},$ or $(a_{25} + a_{24})(a_{25} + a_{18}) - a_{26}(a_{17} + a_{23}) = 0)$

$M_3^5 = 0 \Rightarrow a_{28} = a_{18} = 0,\ b_8 = a_{14}a_{26} + a_{15}a_{27} + 4a_{16}a_{24} + 4a_{17}a_{25}$

$M_4^5 = 0 \Rightarrow (a_{18} = a_{28} = 0)$ or

$(a_{18}a_{27} = a_{28}a_{17}$ and $a_{14}a_{28} + a_{15}a_{28} - a_{18}a_{24} - a_{18}a_{25} = a_{16}a_{27} - a_{17}a_{26})$ or

$(a_{18}a_{26} = a_{28}a_{16}$ and $a_{14}a_{28} + a_{15}a_{28} - a_{18}a_{24} - a_{18}a_{25} = a_{17}a_{26} - a_{16}a_{27})$

$M_5^5 = 0 \Rightarrow 3a_{21}a_{24}a_{28} + 3a_{21}a_{25}a_{28} + 2a_{21}a_{26}a_{27} + 2a_{22}a_{23}a_{28} + 3a_{22}a_{25}a_{27} + 3a_{23}a_{24}a_{26} + 2a_{24}^2a_{25} + 2a_{24}a_{25}^2 = 0$

Then we have $\dim(\widetilde{\mathcal{M}_0}) = 27$, $\dim(\widetilde{\mathcal{M}_1}) = 6 + 2 - 4 + 27 = 31$, $\widetilde{\mathcal{M}_2}$ has four irreducible components of dimension $4 + 2 - 3 + 27 = 30$, $\dim(\widetilde{\mathcal{M}_3}) = 6 - 3 + 27 = 30$, $\widetilde{\mathcal{M}_4}$ has three irreducible components of dimension $4-2+27=29$ and $\dim(\widetilde{\mathcal{M}_5})=2-1+27=28$.

Characteristic $p=7$:

$M_1^7=0 \Rightarrow a_{28}=0,\ a_{26}=a_{27}$	$\dim(\widetilde{\mathcal{M}_1})=6+2-2+27=33$
$M_2^7=0 \Rightarrow a_{28}=0$ or $2a_{24}a_{28}+2a_{25}a_{28}+5a_{26}a_{27}=0$	$\widetilde{\mathcal{M}_2}$ has two irreducible components of dimension 4+2-1+27=32
$M_3^7=0 \Rightarrow a_{18}(a_{26}-a_{27})-a_{28}(a_{16}-a_{17})=0$	$\dim(\widetilde{\mathcal{M}_3})=6-1+27=32$
$M_4^7=0$	$\dim(\widetilde{\mathcal{M}_4})=4+27=31$
$M_5^7=0$	$\dim(\widetilde{\mathcal{M}_5})=2+27=29$

Characteristic $p=11$:

$M_1^{11}=0 \Rightarrow a_{28}=0$	$\dim(\widetilde{\mathcal{M}_1})=6+2-1+27=34$
$M_2^{11}=0$	$\dim(\widetilde{\mathcal{M}_2})=4+2+27=33$
$M_3^{11}=0$	$\dim(\widetilde{\mathcal{M}_3})=6+27=33$

Characteristic $p=5$

We can show that $\mathcal{M}_3 \subset \widetilde{\mathcal{M}_1}$ by considering

$$e_{\beta_1}+e_{\beta_2}+e_{\beta_3}+\{u\in\mathfrak{u}^e : a_{28}=0, a_{18}=a_{26}=a_{27},$$
$$b_8=a_{14}a_{26}+a_{15}a_{26}+4a_{16}a_{24}+4a_{17}a_{25}+a_{24}a_{25}\}\subset\widetilde{\mathcal{M}_1}$$

Then applying $Ad_{\beta_3^\vee(t)}$ gives

$$e_{\beta_1}+e_{\beta_2}+t^2e_{\beta_3}+\{u\in\mathfrak{u}^e : ta_{28}=0, t^{-1}a_{18}=ta_{26}=ta_{27},$$
$$b_8=a_{14}a_{26}+a_{15}a_{26}+4a_{16}a_{24}+4a_{17}a_{25}+t^2a_{24}a_{25}\}\subset\widetilde{\mathcal{M}_1}$$

Taking the limit as $t\to 0$ gives

$$e_{\beta_1}+e_{\beta_2}+\{u\in\mathfrak{u}^e : a_{18}=a_{28}=0, a_{26}=a_{27},$$
$$b_8=a_{14}a_{26}=a_{15}a_{26}+4a_{16}a_{24}+4a_{17}a_{25}\}\subset\widetilde{\mathcal{M}_1}$$

Then $Ad_{\mathcal{E}_{-\beta_3}(\xi)}$ for $\xi\in k^\times$, gives

$$e_{\beta_1}+e_{\beta_2}+\{u\in\mathfrak{u}^e : a_{18}=a_{28}=0, a_{27}-\xi a_{17}=a_{26}-\xi a_{16},$$
$$b_8=a_{14}(a_{26}-\xi a_{16})+a_{15}(a_{26}-\xi a_{16})+4a_{16}(a_{24}-\xi a_{14})+4a_{17}(a_{25}-\xi a_{15})\}\subset\widetilde{\mathcal{M}_1}$$

$$\Rightarrow e_{\beta_1}+e_{\beta_2}+\{u\in\mathfrak{u}^e : a_{18}=a_{28}=0, a_{27}=a_{26}-\xi a_{16}+\xi a_{17},$$
$$b_8=a_{14}a_{26}+a_{15}a_{27}+4a_{16}a_{24}+4a_{17}a_{25}\}\subset\widetilde{\mathcal{M}_1}$$

$$\Rightarrow e_{\beta_1}+e_{\beta_2}+\{u\in\mathfrak{u}^e : a_{18}=a_{28}=0,$$
$$b_8=a_{14}a_{26}+a_{15}a_{27}+4a_{16}a_{24}+4a_{17}a_{25}\}\subset\widetilde{\mathcal{M}_1}$$

$$\Rightarrow \mathcal{M}_3\subset\widetilde{\mathcal{M}_1}$$

The possible components of $\mathfrak{g}^e\cap\mathcal{N}_1$ are $\widetilde{\mathcal{M}_0}$, $\widetilde{\mathcal{M}_5}$, $\widetilde{\mathcal{M}_1}$, the three components of $\widetilde{\mathcal{M}_4}$ and the four components of $\widetilde{\mathcal{M}_2}$.

Characteristic $p = 7$

The possible irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are $\widetilde{\mathcal{M}_3}$, $\widetilde{\mathcal{M}_1}$ and the two components of $\widetilde{\mathcal{M}_2}$. The arguments to show that $\widetilde{\mathcal{M}_5}$ and $\widetilde{\mathcal{M}_4}$ are not components are given below.
To show $\widetilde{\mathcal{M}_5} \subset \widetilde{\mathcal{M}_2}$ consider $e_{\beta_1+\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\}$ which is contained in $\widetilde{\mathcal{M}_2}$. Then $Ad_{\beta_1^\vee(t)}$ gives

$$\begin{aligned} te_{\beta_1+\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\} &\subset \widetilde{\mathcal{M}_2} \\ \Rightarrow e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\} &\subset \widetilde{\mathcal{M}_2} \\ Ad_{\mathcal{E}_{-\beta_1}(\xi)}(e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\}) &\subset \widetilde{\mathcal{M}_2} \\ \Rightarrow e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = \xi a_{27}\} &\subset \widetilde{\mathcal{M}_2} \\ \Rightarrow \widetilde{\mathcal{M}_5} &\subset \widetilde{\mathcal{M}_2} \end{aligned}$$

Similarly we can show that $\widetilde{\mathcal{M}_4} \subset \widetilde{\mathcal{M}_2}$ by considering $Ad_{\beta_3^\vee(t)}$ then $Ad_{\mathcal{E}_{-\beta_3}(\xi)}$.

Characteristic $p = 11$

Most of the inclusions can be shown using the same methods as Section 7.2 with the exception of $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$. Therefore $\mathfrak{g}^e \cap \mathcal{N}_1$ has one irreducible component. Firstly to show that $\widetilde{\mathcal{M}_3} \subset \widetilde{\mathcal{M}_1}$ consider

$$e_{\beta_1} + e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\} \subset \widetilde{\mathcal{M}_1}$$

Then applying $Ad_{\beta_3^\vee(t)}$ gives

$$\begin{aligned} e_{\beta_1} + e_{\beta_2} + t^2 e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\} &\subset \widetilde{\mathcal{M}_1} \\ \Rightarrow Ad_{\mathcal{E}_{-\beta_3}(\xi)}(e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_{28} = 0\} &\subset \widetilde{\mathcal{M}_1} \\ \Rightarrow e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_{28} = \xi a_{18}\} &\subset \widetilde{\mathcal{M}_1} \end{aligned}$$

As ξ varies a_{18} can take any value as long as a_{27} does not equal zero. Therefore taking the closure gives $\mathcal{M}_3 \subset \widetilde{\mathcal{M}_1}$.
Similarly to show that $\widetilde{\mathcal{M}_2} \subset \widetilde{\mathcal{M}_1}$ let $\beta^\vee(t) = \beta_1^\vee(t^2)\beta_2^\vee(t)$. Then

$$\begin{aligned} Ad_{\beta^\vee(t)}(e_{\beta_1} + e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\}) &\subset \widetilde{\mathcal{M}_1} \\ \Rightarrow t^3 e_{\beta_1} + e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\} &\subset \widetilde{\mathcal{M}_1} \\ \Rightarrow e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\} &\subset \widetilde{\mathcal{M}_1} \\ \Rightarrow Ad_{\mathcal{E}_{-\beta_1(\xi)}}(e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = 0\}) &\subset \widetilde{\mathcal{M}_1} \\ \Rightarrow e_{\beta_2} + e_{\beta_3} + \{u \in \mathfrak{u}^e : a_{28} = \xi a_{27}\} &\subset \widetilde{\mathcal{M}_1} \\ \Rightarrow e_{\beta_2} + e_{\beta_3} + \mathfrak{u}^e &\subset \widetilde{\mathcal{M}_1} \end{aligned}$$

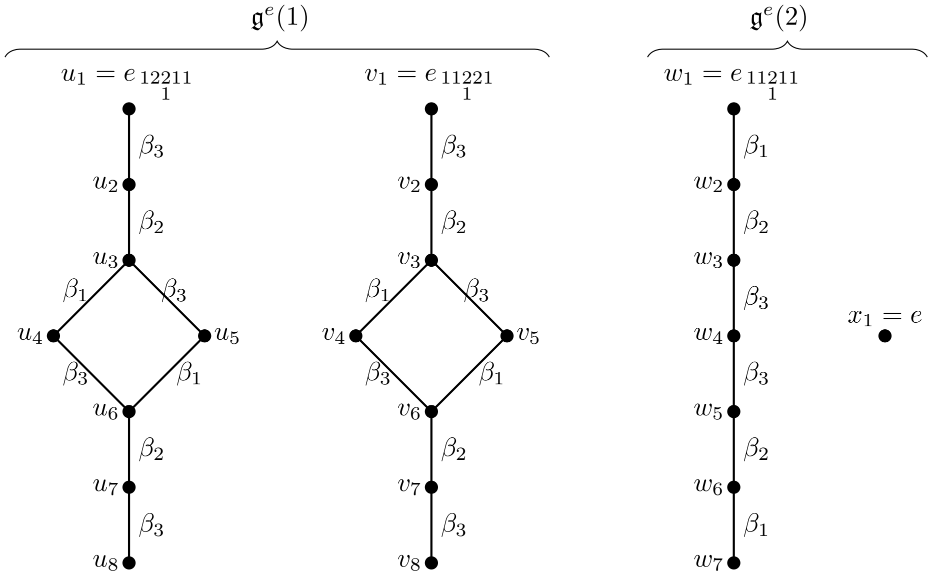
Applying the reflection $n_{\beta_1} \in SL_3 \subset C$ gives $\mathcal{M}_2 \subset \widetilde{\mathcal{M}_1}$.

Orbits A_1^2

$e = e_{\substack{10000\\0}} + e_{\substack{00001\\0}}$

$\mathfrak{c} = \mathfrak{so}_7 \oplus k$

$e_{\beta_1} = e_{\substack{00100\\0}}\,,\ e_{\beta_2} = e_{\substack{00000\\1}}\,,\ e_{\beta_3} = e_{\substack{11110\\0}} + e_{\substack{01111\\0}}$



$M_0 = a_1U_1 + \cdots + a_8U_8 + b_1V_1 + \cdots + b_8V_8 + c_1W_1 + \cdots + c_7W_7 + d_1X_1$

Nilpotent Orbit of $\mathfrak{c}^\circ$	Representative e of nilpotent orbit	M_i label of $e + M_0$
$[7]$	$e_{\beta_1} + e_{\beta_2} + e_{\beta_3}$	M_1
$[5, 1^2]$	$e_{\beta_2} + e_{\beta_3}$	M_2
$[3^2, 1]$	$e_{\beta_1} + e_{\beta_2}$	M_3
$[3, 2^2]$	e_{β_3}	M_4
$[3, 1^4]$	e_{β_2}	M_5
$[2^2, 1^3]$	e_{β_1}	M_6

Characteristic $p = 5$:

We do not consider the regular orbit since $\mathcal{N}_1(\mathfrak{so}_7) = \overline{\mathcal{O}_{[5,1^2]}}$ for $p = 5$.

$$\begin{aligned}
M_2^5 = 0 \Rightarrow\ & (a_8 = a_5 = 0,\ c_6 = 4a_2b_8 + a_3b_7 + 4a_6b_5 + a_7b_3) \text{ or}\\
& (b_8 = b_5 = 0,\ c_6 = a_3b_7 + 4a_5b_6 + a_7b_3 + 4a_8b_2)\\
M_3^5 = 0 \Rightarrow\ & (a_4 = 0 \text{ or } b_4 = 0) \text{ and } (a_7 = 0 \text{ or } b_7 = 0)\\
M_4^5 = 0 &
\end{aligned}$$

Here we have that $\widetilde{\mathcal{M}_2}$ has two irreducible components of dimension 16-3+24=37, $\widetilde{\mathcal{M}_3}$ has four irreducible components of dimension 14-2+24=36 and $\dim(\widetilde{\mathcal{M}_4}) = 12 + 24 = 36$.

Characteristic $p = 7$:

$$\begin{array}{ll} M_1^7 = 0 \Rightarrow a_8 = b_8 = 0,\ c_7 = a_4b_7 + 6a_6b_6 + a_7b_4 & \dim(\widetilde{\mathcal{M}_1}) = 18 - 3 + 24 = 39 \\ M_2^7 = 0 \Rightarrow a_5b_8 + a_8b_5 = 0 & \dim(\widetilde{\mathcal{M}_2}) = 16 - 1 + 24 = 39 \\ M_3^7 = 0 & \dim(\widetilde{\mathcal{M}_3}) = 14 + 24 = 38 \end{array}$$

Characteristic $p = 11$:

$$\begin{array}{ll} M_1^{11} = 0 \Rightarrow a_8 = 0 \text{ or } b_8 = 0 & \widetilde{\mathcal{M}_1} \text{ has two irreducible components of dimension 18-1+24=41} \\ M_2^{11} = 0 & \dim(\widetilde{\mathcal{M}_2}) = 16 + 24 = 40 \end{array}$$

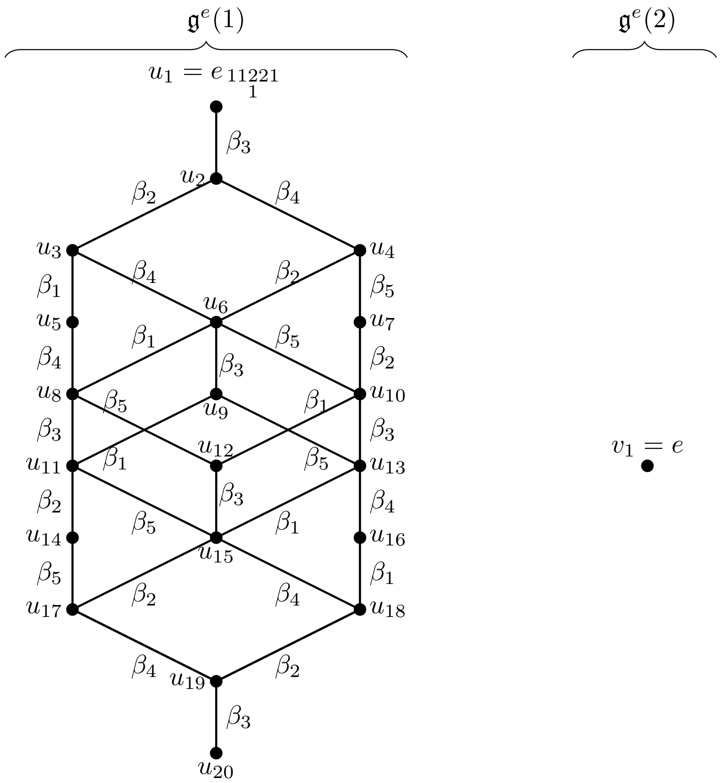
When $p = 5$ the possible irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are $\widetilde{\mathcal{M}_4}$, the four components of $\widetilde{\mathcal{M}_3}$ and the two components of $\widetilde{\mathcal{M}_2}$. Similarly the possible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ when $p = 7$ (resp. $p = 11$) are $\widetilde{\mathcal{M}_3}$, $\widetilde{\mathcal{M}_2}$ and $\widetilde{\mathcal{M}_1}$ (resp. the two components of $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_2}$). The other inclusions all hold by the argument in Section 7.2.

Orbit A_1

$e = e_{\substack{10000\\0}}$

$\mathfrak{c} = \mathfrak{sl}_6$

$e_{\beta_1} = e_{\substack{00000\\1}},\ e_{\beta_2} = e_{\substack{00100\\0}},\ e_{\beta_3} = e_{\substack{00010\\0}},\ e_{\beta_4} = e_{\substack{00001\\0}},\ e_{\beta_5} = e_{\substack{12210\\1}}$



$M_0 = a_1U_1 + \cdots + a_{20}U_{20} + b_1V_1$

Nilpotent Orbit of $\mathfrak{c}$	Representative e of nilpotent orbit	M_i label of $e+M_0$
$[6]$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_3}+e_{\beta_4}+e_{\beta_5}$	M_1
$[5,1]$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_3}+e_{\beta_4}$	M_2
$[4,2]$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_3}+e_{\beta_5}$	M_3
$[4,1^2]$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_3}$	M_4
$[3^2]$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_4}+e_{\beta_5}$	M_5
$[3,2,1]$	$e_{\beta_1}+e_{\beta_2}+e_{\beta_4}$	M_6
$[3,1^3]$	$e_{\beta_1}+e_{\beta_2}$	M_7
$[2^3]$	$e_{\beta_1}+e_{\beta_3}+e_{\beta_5}$	M_8
$[2^2,1^2]$	$e_{\beta_1}+e_{\beta_3}$	M_9
$[2,1^4]$	e_{β_1}	M_{10}

Characteristic $p=5$:

Since $\mathcal{N}_1(\mathfrak{sp}_6)=\overline{\mathcal{O}_{[5,1]}}$ for $p=5$ we do not consider the regular orbit.

$$\begin{aligned}
&M_2^5=0\Rightarrow a_{14}=a_{20}=0,\ a_8=a_9,\ a_{17}=a_{18} && \dim(\widetilde{\mathcal{M}_2})=28-4+21=45\\
&M_3^5=0\Rightarrow a_{17}=0,\ a_{14}=a_{15},\ \text{then either } a_5=0 \text{ or } a_{20}=0 &&\\
&M_4^5=0\Rightarrow a_{14}=a_{17}=0,\ \text{then either } a_5=0 \text{ or } a_{20}=0 &&\\
&M_5^5=0\Rightarrow a_{19}=a_{12}=0 && \dim(\widetilde{\mathcal{M}_5})=24-2+21=43\\
&M_6^5=0\Rightarrow a_8=0 \text{ or } a_{19}=0 && \dim(\widetilde{\mathcal{M}_6})=22-1+21=42\\
&M_7^5=0 && \dim(\widetilde{\mathcal{M}_7})=18+21=39\\
&M_8^5=0\Rightarrow a_{15}=0 && \dim(\widetilde{\mathcal{M}_8})=18-1+21=38
\end{aligned}$$

In this case $\widetilde{\mathcal{M}_3}$ has two irreducible components of dimension 26-3+21=44 and $\widetilde{\mathcal{M}_4}$ has two irreducible components of dimension 24-3+21=42.

Characteristic $p=7$:

$$\begin{aligned}
&M_1^7=0\Rightarrow a_{20}=0,\ a_{16}=6a_{14}+a_{15},\ a_{17}=a_{18} && \dim(\widetilde{\mathcal{M}_1})=30-3+21=48\\
&M_2^7=0\Rightarrow a_{14}=a_{20}=0 && \dim(\widetilde{\mathcal{M}_2})=28-2+21=47\\
&M_3^7=0\Rightarrow a_{17}=0 && \dim(\widetilde{\mathcal{M}_3})=26-1+21=46\\
&M_4^7=0 && \dim(\widetilde{\mathcal{M}_4})=24+21=45\\
&M_5^7=0 && \dim(\widetilde{\mathcal{M}_5})=24+21=45
\end{aligned}$$

Characteristic $p=11$:

$$\begin{aligned}
&M_1^{11}=0\Rightarrow a_{20}=0 && \dim(\widetilde{\mathcal{M}_1})=30-1+21=50\\
&M_2^{11}=0 && \dim(\widetilde{\mathcal{M}_2})=28+21=49
\end{aligned}$$

Characteristic $p=5$

The possible components of $\mathfrak{g}^e\cap\mathcal{N}_1$ are $\widetilde{\mathcal{M}_8}$, $\widetilde{\mathcal{M}_6}$, $\widetilde{\mathcal{M}_5}$, $\widetilde{\mathcal{M}_2}$, the two components of $\widetilde{\mathcal{M}_4}$ and the two components of $\widetilde{\mathcal{M}_3}$. The inclusions $\widetilde{\mathcal{M}_{10}}\subset\widetilde{\mathcal{M}_9}\subset\widetilde{\mathcal{M}_7}$ hold by the same argument in

Section 7.2. To show that $\widetilde{\mathcal{M}_7} \subset \widetilde{\mathcal{M}_6}$ consider $e_{\beta_1} + e_{\beta_2} + e_{\beta_4} + \{u \in \mathfrak{u}^e : a_{19} = 0\}$ which is contained in $\widetilde{\mathcal{M}_6}$. Then $Ad_{\beta_4^\vee(t)}$ gives

$$\begin{aligned} e_{\beta_1} + e_{\beta_2} + t^2 e_{\beta_4} + \{u \in \mathfrak{u}^e : a_{19} = 0\} \subset \widetilde{\mathcal{M}_6} \\ e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_{19} = 0\} \subset \widetilde{\mathcal{M}_6} \end{aligned}$$

Then $Ad_{\mathcal{E}_{-\beta_4}(\xi)}$ for $\xi \in k^\times$ gives

$$\begin{aligned} e_{\beta_1} + e_{\beta_2} + \{u \in \mathfrak{u}^e : a_{19} = \xi a_{17}\} \subset \widetilde{\mathcal{M}_6} \\ \Rightarrow \widetilde{\mathcal{M}_7} \subset \widetilde{\mathcal{M}_6} \end{aligned}$$

Characteristic $p = 7, 11$

When $p = 7$ the possible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are $\widetilde{\mathcal{M}_5}$, $\widetilde{\mathcal{M}_4}$, $\widetilde{\mathcal{M}_3}$, $\widetilde{\mathcal{M}_2}$ and $\widetilde{\mathcal{M}_1}$. Similarly when $p = 11$ the possible irreducible components are $\widetilde{\mathcal{M}_1}$ and $\widetilde{\mathcal{M}_2}$. The other components can be eliminated using the argument in Section 7.2.

Chapter 9

Irreducible Components of $\mathcal{C}_1^{nil}(E_6)$

In this chapter we calculate the irreducible components of $\mathcal{C}_1^{nil}(E_6)$ for $p = 5$ and 11. In the $p = 7$ case we show that $\mathcal{C}_1^{nil}(E_6) = \mathcal{C}_1(D_4(a_1)) \cup \mathcal{C}_1(E_6(a_3))$. However we do not know whether $\mathcal{C}_1(D_4(a_1)) \subset \mathcal{C}_1(E_6(a_1))$.

When $p = 5$ and $e \in A_4A_1$, the set $\mathfrak{g}^e \cap \mathcal{N}_1$ has two irreducible components X_1 and X_2. We then show that

$$\mathcal{C}_1^{nil}(E_6) = \overline{G \cdot (e, X_1)} \cup \overline{G \cdot (e, X_2)} \cup \mathcal{C}_1(D_4(a_1))$$

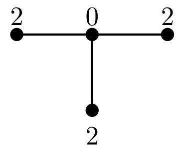
All of these components have dimension 76. Finally in the $p = 11$ case we have that $\mathcal{C}_1^{nil}(E_6)$ has two irreducible components, namely

$$\mathcal{C}_1^{nil}(E_6) = \mathcal{C}_1(E_6(a_3)) \cup \mathcal{C}_1(E_6(a_1))$$

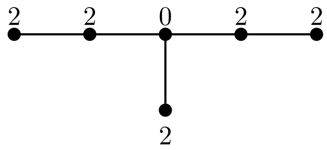
For all but three of the remaining orbits $\mathcal{O}_e$ in E_6 we can verify computationally that there is an element in each irreducible component X_i of $\mathfrak{g}^e \cap \mathcal{N}_1$ that is not contained in $\overline{G \cdot e}$. Therefore, by Proposition 5.2.1, $\overline{G \cdot (e, X_i)}$ is not an irreducible component of $\mathcal{C}_1^{nil}(E_6)$. These elements are presented in Table 9.2 at the end of the chapter. The three remaining orbits are D_5 and $D_4(a_1)$ for $p = 11$ and A_4A_1 for $p = 7$. For each of these orbits we show that they are contained in another component of $\mathcal{C}_1^{nil}(E_6)$ case by case.

9.1 Argument to show $\mathcal{C}_1(D_4(a_1)) \subset \mathcal{C}_1(E_6(a_1)) \cup \mathcal{C}_1(E_6(a_3))$ for $p = 11$

Note that for $p = 11$, $\mathcal{C}_1(D_4(a_1)) = \mathcal{C}(D_4(a_1))$. The same equality also holds for the distinguished orbits of E_6. Therefore we use Proposition 4.2.2 to show this inclusion. A representative e of orbit $D_4(a_1)$ in E_6 is almost distinguished. We may assume that e is distinguished in $\mathfrak{l}_I$ where $I = \{2, 3, 4, 5\}$. Then $\mathcal{O}_e$ is subregular in $\mathfrak{l}_I$ and has corresponding weighted Dynkin diagram



Therefore let $J = \{4\}$ and the extended weighted Dynkin diagram is



This weighted Dynkin diagram corresponds to the nilpotent orbit $E_6(a_1)$ in E_6, therefore let $\widetilde{e}$ be a representative of $E_6(a_1)$. Note that for $p = 11$, $\widetilde{e} \subset \mathcal{N}_1$. When $p = 5, 7$ then $\widetilde{e} \not\subset \mathcal{N}_1$, therefore this method only works for $p = 11$. Then by Theorem 4.2.2

$$\mathcal{C}(D_4(a_1)) \subset \mathcal{C}(E_6(a_1)) \cup \mathcal{C}(E_6(a_3))$$

Hence $\mathcal{C}_1(D_4(a_1))$ is not an irreducible component of $\mathcal{C}_1^{nil}(E_6)$.

9.2 Argument to show $\mathcal{C}_1(D_5) \subset \mathcal{C}_1(E_6(a_1))$ for $p = 11$

We want to show that $\mathcal{C}(\mathcal{O}_{D_5})$ is not an irreducible component of $\mathcal{C}_1^{nil}(E_6)$. To do this we show that $\mathcal{C}(\mathcal{O}_{D_5}) \subset \mathcal{C}(\mathcal{O}_{sr})$ i.e. $G \cdot (e_{D_5}, \mathfrak{g}^{e_{D_5}} \cap \mathcal{N}_1) \subset \overline{G \cdot (e_{sr}, \mathfrak{g}^{e_{sr}} \cap \mathcal{N}_1)}$ where $\mathcal{O}_{sr}$ is the subregular orbit of E_6. Since $\mathfrak{g}^e \cap \mathcal{N}_1 = \mathfrak{g}^e \cap \mathcal{N}$ for both $e = e_{D_5}$ and e_{sr} then this is the same as $G \cdot (e_{D_5}, \mathfrak{g}^{e_{D_5}} \cap \mathcal{N}) \subset \overline{G \cdot (e_{sr}, \mathfrak{g}^{e_{sr}})}$. To do this we consider a transverse slice to $\mathcal{O}_{D_5}$ at f', where f' is given by

$$\begin{aligned} f' &= 8f_{\alpha_1} + 14f_{\alpha_3} + 18f_{\alpha_4} + 10f_{\alpha_5} + 10f_{\alpha_2} \\ e' &= e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_2} \end{aligned}$$

We consider $f' + \mathfrak{g}^{e'}$ rather than $e' + \mathfrak{g}^{f'}$ because we have a known basis of $\mathfrak{g}^{e'}$ from [LT11]. The centralizer of e' as given by [LT11] has basis $h \in \mathfrak{g}(0)$ along with

$$\begin{aligned}
&h = 2h_{\alpha_1} + 3h_{\alpha_2} + 4h_{\alpha_3} + 6h_{\alpha_4} + 5h_{\alpha_5} + 4h_{\alpha_6} && \in \mathfrak{c} \\
&v_1 = e' && \in \mathfrak{g}(2) \\
&v_2 = f_{\substack{01111\\0}} - f_{\substack{00111\\1}}, \quad v_3 = e_{\substack{12211\\1}} + e_{\substack{11221\\1}} && \in \mathfrak{g}(4) \\
&v_4 = e_{\substack{01100\\1}} - e_{\substack{01110\\0}} - e_{\substack{11100\\0}} + 2e_{\substack{00110\\1}} && \in \mathfrak{g}(6) \\
&v_5 = e_{\substack{11110\\0}} + e_{\substack{11100\\1}} && \in \mathfrak{g}(8) \\
&v_6 = f_{\substack{00001\\0}} \quad v_7 = e_{\substack{12321\\2}} \quad v_8 = e_{\substack{11110\\1}} + e_{\substack{01210\\1}} && \in \mathfrak{g}(10) \\
&v_9 = e_{\substack{12210\\1}} && \in \mathfrak{g}(14)
\end{aligned}$$

Then the centralizer of f' is given by

$$
\begin{array}{ll}
h, \ \ f', & \in \mathfrak{g}(-2) \\
u_2 = 2e_{\substack{01111\\0}} - 5e_{\substack{00111\\1}}, \ \ u_3 = 5f_{\substack{12211\\1}} + 2f_{\substack{11221\\1}} & \in \mathfrak{g}(-4) \\
u_4 = 6f_{\substack{01100\\1}} + 5f_{\substack{01110\\0}} + 4f_{\substack{11100\\0}} - 4f_{\substack{00110\\1}} & \in \mathfrak{g}(-6) \\
u_5 = f_{\substack{11110\\0}} + f_{\substack{11100\\1}} & \in \mathfrak{g}(-8) \\
u_6 = f_{\substack{12321\\2}}, \ \ u_7 = e_{\substack{00001\\0}}, \ \ u_8 = 3f_{\substack{11110\\1}} + 4f_{\substack{01210\\1}} & \in \mathfrak{g}(-10) \\
u_9 = f_{\substack{12210\\1}} & \in \mathfrak{g}(-14)
\end{array}
.
$$

We cannot use the usual Slodowy slice $f' + \mathfrak{g}^{e'}$ because $\mathfrak{g}^{e'} \cap [f', \mathfrak{g}] \neq \{0\}$ (see below). Therefore we need to find an alternative linear space $V \in \mathfrak{g}$ of dimension 10 such that $V \cap [f', \mathfrak{g}] = \{0\}$. To do this consider the following basis of $\mathfrak{g}(8)$:

$$w_1 = v_5 \qquad w_2 = [f', v_6] \qquad w_3 = [f', v_8] \qquad w_4 = [f', v_7] \qquad w_5 = [f', [f', [f', v_9]]]/6$$

This basis for $\mathfrak{g}(8)$ generates a subspace of $\mathfrak{g}(6)$ i.e.

$$z_1 = -[f', w_1]/8 \quad z_2 = [f', w_2]/180 \quad z_3 = [f', w_3]/20 \quad z_4 = [f', w_4]/180 \quad z_5 = [f', w_5]/336$$

(We divide by the constants in order to make the corresponding elements more manageable). When $p = 11$ then $-4z_5 = v_4$ therefore $\mathfrak{g}^{e'} \cap [f', \mathfrak{g}] \neq \{0\}$. Therefore we let V be similar to $\mathfrak{g}^{e'}$ but replacing the element v_4. Consider the element $e_{\substack{11100\\0}} \in \mathfrak{g}(6) \backslash \langle w_1, w_2, w_3, w_4, w_5 \rangle$ where $[h, e_{\substack{11100\\0}}] = 6e_{\substack{11100\\0}}$. Therefore let

$$\mathcal{M} = \{f' + x_0 h + x_1 e' + x_2 v_2 + x_3 v_3 + x_4 e_{\substack{1\,1\,1\,0\,0\\0}} + x_5 v_5 + \cdots + x_9 v_9 : x_0, \ldots, x_9 \in k\}$$

We want to know when an element of $\mathcal{M}$ belongs to $\overline{\mathcal{O}_{E_6(a_1)}}$. Let M be the 27×27 matrix representation of an element of $\mathcal{M}$ with coordinates $x_0, \ldots, x_9$. Using [GAP12], we show:

$$Tr(M^2) = 0 \Rightarrow x_1 = x_0^2$$
$$Tr(M^5) = 0 \Rightarrow x_5 = 2x_0^5 - x_0 x_4$$

Then for M to be contained in $\mathcal{O}_{E_6(a_1)}$ we require $M^{11} = 0$. This holds if and only if

$$x_4 = x_6 = x_7 = x_8 = 0$$
$$x_9 = 4x_0^2 x_2 x_3$$
$$x_0^6 = x_2 x_3$$

Now let $x_0 = st$, $x_2 = s^6$, $x_3 = t^6$ for $s, t \neq 0$. Therefore the set $\mathcal{M} \cap \overline{\mathcal{O}_{E_6(a_1)}}$ is the set of all $M_{s,t}$ for $s, t \in k$, where

$$M_{s,t} = f' + sth + s^2t^2 e' + s^6 v_2 + t^6 v_3 + 2s^5 t^5 v_5 + 4s^8 t^8 v_9$$

It is easy to check in [GAP12] that $M_{s,t} \in \mathcal{O}_{sr}$ if $(s,t) \neq (0,0)$. We now want to provide some information about $\mathfrak{g}^{M_{s,t}}$. If $y = y_r + y_{r+2} + \cdots \in \mathfrak{g}^{M_{s,t}}$ (where y_i is the part of y with degree i) such that $[y, M_{s,t}] = 0$ then

$$[y, M_{s,t}] = \underbrace{[y_r, f']}_{\deg(r-2)} + \underbrace{[y_{r+2}, f'] + [y_r, sth]}_{\deg(r)} + \ldots$$

Therefore $y_r \in \mathfrak{g}^{f'}$. We want to show that $(f', \mathfrak{g}^{f'} \cap \mathcal{N}_1) \subset \overline{\{(e, \mathfrak{g}^e) : e \in E_6(a_1)\}}$. We show this by studying the lowest degree terms y_r of elements of $\mathfrak{g}^{M_{s,t}}$. In the discussion which follows we assume that $(s,t) \neq (0,0)$.

Lemma 9.2.1 *For an element $y \in \mathfrak{g}^{M_{s,t}}$, h is not the lowest degree term of y.*

Proof. Suppose that $y = h + y_2 + y_4 + \cdots \in \mathfrak{g}^{M_{s,t}}$ where $y_i \in \mathfrak{g}(i)$. Then each part of $[M_{s,t}, y]$ with degree i must equal zero. The part of $[M_{s,t}, y]$ with degree zero is given by

$$[f', y_2] + [sth, h] = 0$$

Since $[sth, h] = 0$, then we must have that $[f', y_2] = 0$, specifically $y_2 \in \mathfrak{g}^{f'}$. Therefore, because $\mathfrak{g}^{f'} \subset \sum_{i \leq 0} \mathfrak{g}(i)$, we have $y_2 = 0$

Now the part of $[M_{s,t}, y]$ with degree 2 is given by

$$[f', y_4] + [sth, y_2] + [s^2t^2e_0, h] = 0$$

Since $[sth, y_2] = 0$ and $[s^2t^2e', h] = 0$ then by the same argument as above $y_4 = 0$.

Finally the part of degree 4 is given by

$$\begin{aligned} [f', y_6] + [sth, y_4] + [s^2t^2e', y_2] + [s^6v_2 + t^6v_3, h] &= 0 \\ [f', y_6] + s^6[v_2, h] + t^6[v_3, h] &= 0 \\ [f', y_6] + 3s^6v_2 - 3t^6v_3 &= 0 \\ \Rightarrow [f', y_6] &= -3(s^6v_2 - t^6v_3) \end{aligned}$$

Since $[f', \mathfrak{g}] \cap \langle h, e', v_2, v_3, e_{\substack{11100\\0}}, v_5, \ldots v_9 \rangle = \{0\}$, we cannot have $[f', y_6] = -3(s^6v_2 - t^6v_3)$ unless $s = t = 0$. So h cannot be the lowest degree term of an element in $\mathfrak{g}^{M_{s,t}}$. ∎

Lemma 9.2.2 *If the lowest degree term of $y \in \mathfrak{g}^{M_{s,t}}$ is $au_2 + bu_3$ then (a,b) is a multiple of (t^6, s^6).*

Proof. Let $y = au_2 + bu_3 + y_{-2} + y_0 + y_2 + \ldots$ for $a, b \in k$ with $(a,b) \neq (0,0)$. We want to know when $y \in \mathfrak{g}^{M_{s,t}}$. Therefore consider the part of $[M_{s,t}, y]$ with degree -4.

$$\begin{aligned} [f', y_{-2}] + [sth, au_2 + bu_3] &= 0 \\ [f', y_{-2}] + 3stau_2 - 3stbu_3 &= 0 \\ \Rightarrow [f', y_{-2}] &= 3st(bu_3 - au_2) \end{aligned}$$

Since $[f',[e',bu_3-au_2]]=4(bu_3-au_2)$ then $y_{-2}=\frac{3}{4}st[e',bu_3-au_2]+\xi$, where $\xi\in\mathfrak{g}^{f'}(-2)$. Since $\mathfrak{g}^{f'}(-2)=kf'$, we can assume after subtracting a multiple of $M_{s,t}$ that $\xi=0$. Therefore $y_{-2}=\frac{3}{4}st[e',bu_3-au_2]$.

Next we consider the part of $[M_{s,t},y]$ with degree -2.

$$\begin{aligned}[f',y_0]+[sth,y_{-2}]+[s^2t^2e',au_2+bu_3]&=0\\ [f',y_0]-\frac{9}{4}s^2t^2[e',bu_3+au_2]+[s^2t^2e',au_2+bu_3]&=0\\ [f',y_0]-\frac{5}{4}s^2t^2[e',au_2+bu_3]&=0\end{aligned}$$

Since au_2+bu_3 belongs to an irreducible highest weight module U for $\langle h',e',f'\rangle$ with highest weight 4 then we have

$$\begin{aligned}y_0&=\frac{1}{6}[e',\frac{5}{4}s^2t^2[e',au_2+bu_3]]+\lambda h\\ &=\frac{5}{24}s^2t^2[e',[e',au_2+bu_3]]+\lambda h\end{aligned}$$

Finally we consider the part of $[M_{s,t},y]$ with degree zero,

$$\begin{aligned}[f',y_2]+[sth,y_0]+[s^2t^2e',y_{-2}]+[s^6v_2+t^6v_3,au_2+bu_3]&=0\\ [f',y_2]+\frac{5}{8}s^3t^3[e',[e',bu_3-au_2]]+\frac{3}{4}s^3t^3[e',[e',bu_3-au_2]]+[s^6v_2+t^6v_3,au_2+bu_3]&=0\\ [f',y_2]+[s^6v_2+t^6v_3,au_2+bu_3]&=0\end{aligned}$$

Therefore $[s^6v_2+t^6v_3,au_2+bu_3]=as^6[v_2,u_2]+bt^6[v_3,u_3]\in[f',\mathfrak{g}(2)]$. Now by direct computation

$$\begin{aligned}[v_2,u_2]&=-7h_{\alpha_2}-5_{\alpha_3}-12(h_{\alpha_4}+h_{\alpha_5}+h_{\alpha_6})\\ [v_3,u_3]&=12(h_{\alpha_1}+h_{\alpha_2}+h_{\alpha_6})+19h_{\alpha_3}+17h_{\alpha_5}+24h_{\alpha_4}\end{aligned}$$

Now $h_{\alpha_1},h_{\alpha_2},h_{\alpha_3},h_{\alpha_4},h_{\alpha_5}\in[f',\mathfrak{g}]$ and so $[f',\mathfrak{g}]\cap\mathfrak{h}=\langle h_{\alpha_1},\dots,h_{\alpha_5}\rangle$. Therefore an element of $\mathfrak{h}$ belongs to $[f',\mathfrak{g}]$ if and only if the coefficient of h_{α_6} is zero. Therefore $[s^6v_2+t^6v_3,au_2+bu_3]$ can only belong to $[f',\mathfrak{g}(2)]$ if $12(as^6-bt^6)=0$. Therefore (a,b) is a multiple of (t^6,s^6). ■

Basis of $\mathfrak{g}^{M_{s,t}}$

Since $\dim(\mathfrak{g}^{f'})=\dim(\mathfrak{g}^{e'})=10$ and $\dim(\mathfrak{g}^{M_{s,t}})=\dim(\mathfrak{g}^{e_{E_6(a_1)}})=8$ then it follows that for any $y_i\in\mathfrak{g}^{f'}(i)$ with $i=-6,-8,-10,-14$ there exists an element of $\mathfrak{g}^{M_{s,t}}$ of the form $y_i+y_{i+2}+\dots$.

Therefore $\mathfrak{g}^{M_{s,t}}$ has a basis of the form

$$\begin{aligned}
X_1^{s,t} &= M_{s,t} = f' + \dots \\
X_2^{s,t} &= t^6 u_2 + s^6 u_3 + \dots \\
X_3^{s,t} &= u_4 + \dots \\
X_4^{s,t} &= u_5 + \dots \\
&\vdots \\
X_8^{s,t} &= u_9 + \dots
\end{aligned}$$

Note that it is a consequence of our description of $M_{s,t}$ that in the expression for $X_i^{s,t}$ when $i \neq 2$, all the higher degree terms are at least quadratic in s, t. In $X_2^{s,t}$ all higher degree terms are at least degree 8 in s, t.

For $a_1, \dots, a_8 \in k$ we want to show that $(f', a_1 f' + a_2 u_2 + a_3 u_3 + a_4 u_4 + \dots)$ is contained in $\overline{\bigcup_{(s,t)\neq(0,0)} (M_{s,t}, \mathfrak{g}^{M_{s,t}})}$. Now assume that $a_2 \neq 0$ and let $\mu \in k$ be such that $\mu^6 a_2 = a_3$. Consider $(M_{\mu t,t}, \mathfrak{g}^{M_{\mu t,t}})$ then

$$\begin{aligned}
X_1^{\mu t,t} &= f' + t^2(\dots) \\
X_2^{\mu t,t} &= t^6 u_2 + \mu^6 t^6 u_3 + t^8(\dots) \\
&\Rightarrow \frac{X_2^{\mu t,t}}{t^6} = u_2 + \mu^6 u_3 + t^2(\dots) \\
X_3^{\mu t,t} &= u_4 + t^2(\dots) \\
&\vdots
\end{aligned}$$

Then let Y be

$$\begin{aligned}
Y &= a_1 X_1^{\mu t,t} + \frac{a_2}{t^6} X_2^{\mu t,t} + a_4 X_3^{\mu t,t} + a_5 X_4^{\mu t,t} + \dots \\
&= a_1 f_0 + a_2 u_2 + a_2 \mu^6 u_3 + a_4 u_4 + \dots + t^2(\dots) \\
&= a_1 f_0 + a_2 u_2 + a_3 u_3 + a_4 u_4 + \dots + t^2(\dots)
\end{aligned}$$

Therefore the set $\{(M_{\mu t,t}, a_1 X_1^{\mu t,t} + \dots) : t \neq 0\}$ includes $(f', a_1 f' + a_2 u_2 + \dots)$ in its closure. So $(f', a_1 f' + \dots) \subset \mathcal{C}_1(E_6(a_1))$.

9.3 Argument to show $\mathcal{C}_1(A_4A_1) \subset \mathcal{C}_1(D_5(a_1))$ when $p = 7$

Now $\mathcal{C}_1(A_4A_1) = \overline{G \cdot (e', \mathfrak{g}^{e'} \cap \mathcal{N}_1)}$ has two components $X_1 = \overline{G \cdot (e', \{u \in \mathfrak{u}^e : a_2 = 0\})}$ and $X_2 = \overline{(e', \{u \in \mathfrak{u}^e : a_1 = 0\})}$. To show $X_1 \in \mathcal{C}_1(D_5(a_1))$ consider

$$\begin{aligned}
e' &= e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_2} + e_{\alpha_6} \\
f' &= 4f_{\alpha_1} + 6f_{\alpha_3} + 6f_{\alpha_4} + 4f_{\alpha_2} + f_{\alpha_6} \\
h &= 4h_{\alpha_1} + 6h_{\alpha_3} + 8h_{\alpha_3} + 12f_{\alpha_4} + 10h_{\alpha_5} + 4h_{\alpha_6}
\end{aligned}$$

Note that h, e', f' do not form an $\mathfrak{sl}_2$-triple. For $v_1 = f_{\substack{01110\\1}} + f_{\substack{00111\\1}} - f_{\substack{01111\\0}} - 2f_{\substack{11110\\0}}$ contained in $\mathfrak{g}^{e'}(1)$ and $v_7 = f_{\substack{01221\\1}}$ contained in $\mathfrak{g}^{e'}(4)$ let

$$u_1 = [f', v_1] \in \mathfrak{g}^{f'}(-1) \qquad\qquad u_7 = [f', [f', [f', [f', v_7]]]]/576 \in \mathfrak{g}^{f'}(-4)$$

$$M_t = e' + t^3 u_1 - 30t^6 u_7 \text{ for } t \in k^\times$$

We can show that M_t is contained in $\mathcal{O}_{D_5(a_1)}$ for any $t \in k^\times$.

Lemma 9.3.1 *For an element $y \in \mathfrak{g}^{M_t}$, h is not the highest degree term of y.*

Proof. Any element in the centralizer of M_t is of the form $y = y_i + y_{i-1} + \dots$ where $y_i \in \mathfrak{g}^{e'}(i)$. Suppose that $h + y_{-1} + y_{-2} \in \mathfrak{g}^{M_t}$. Then each part of $[y, M_t]$ with degree i must equal zero. Firstly the part of $[y, M_t]$ with degree 1 gives

$$[y_{-1}, e'] = 0 \Rightarrow y_{-1} = 0$$

Similarly the part of degree 0 part gives $y_{-2} = 0$. Finally the part of $[y, M_t]$ with degree -1 gives

$$\begin{aligned} &[y_{-3}, e'] + [h, u_1] = 0 \\ &\qquad \Rightarrow [e', y_{-3}] = -3t^3 u_1 \text{ since } [h, t^3 u_1] = -3t^3 u_1 \in \mathfrak{g}^{f'}(-1) \end{aligned}$$

Since $[e', \mathfrak{g}(-3)] \cap \mathfrak{g}^{f'}(-1) = \{0\}$ we cannot have $[e', y_{-3}] = -3t^3 u_1$ so h cannot be the highest degree term of an element on $\mathfrak{g}^{M_t}$. ■

Lemma 9.3.2 *For an element $y \in \mathfrak{g}^{M_t}$, v_2 is not the highest degree term of y.*

Proof. Suppose that $y = v_2 + y_0 + y_{-1} + \dots \in \mathfrak{g}^{M_t}$. Now the part of $[M_t, y]$ with degree 2 gives

$$[e', y_0] = 0 \Rightarrow y_0 = \xi h \text{ for some } \xi \in k$$

Similarly the part of $[M_t, y]$ with degree 1 gives

$$[e', y_{-1}] = 0 \Rightarrow y_{-1} = 0$$

Finally the degree 0 part of $[M_t, y]$ gives $[e', y_{-2}] = [v_2, t^3 u_1]$. However by inspection $[v_2, t^3 u_1]$ is not contained in $[e', \mathfrak{g}(-2)]$. Therefore v_2 cannot be the highest degree term of an element in $\mathfrak{g}^{M_t}$. ■

Basis of $\mathfrak{g}^{M_t}$

By a [GAP12] calculation we can show that $\mathfrak{g}^{M_t} \cap \mathcal{N}_1 = \mathfrak{g}^{M_t} \cap \mathcal{N}$ has basis of the form

$$
\begin{array}{lll}
V_1^t = v_1 & V_8^t = v_8 + t^3(\dots) & V_{13}^t = v_{13} \\
V_3^t = v_3 + t^3(\dots) & V_9^t = v_9 + t^3(\dots) & V_{14}^t = v_{14} + t^3(\dots) \\
V_4^t = v_4 + t^3(\dots) & V_{10}^t = v_{10} & V_{15}^t = v_{15} + t^3(\dots) \\
V_5^t = v_5 + t^3(\dots) & V_{11}^t = v_{11} + t^3(\dots) & \\
V_7^t = v_7 & V_{12}^t = v_{12} + t^3(\dots) &
\end{array}
$$

Now $(M_t, \mathfrak{g}^{M_t} \cap \mathcal{N}_1) \subset (e_{D_5(a_1)}, \mathfrak{g}^{e_{D_5(a_1)}} \cap \mathcal{N}_1)$ for all $t \neq 0$. So

$$(M_t, a_1V_1^t + a_3V_3^t + \cdots + a_{15}V_{15}^t) \subset G \cdot (e_{D_5(a_1)}, \mathfrak{g}^{e_{D_5(a_1)}} \cap \mathcal{N}_1) \text{ for all } a_i \in k$$

Taking the closure we obtain

$$
\begin{aligned}
(e', a_1v_1 + a_3v_3 + \cdots + a_{15}v_{15}) \subset G \cdot (e_{D_5(a_1)}, \mathfrak{g}^{e_{D_5(a_1)}} \cap \mathcal{N}_1) \text{ for all } a_i \in k \\
(e', \{u = \sum a_iv_i \in \mathfrak{u}^e : a_2 = a_6 = 0\}) \subset \left(e_{D_5(a_1)}, \mathfrak{g}^{e_{D_5(a_1)}} \cap \mathcal{N}_1\right)
\end{aligned}
$$

Now for $\xi \in k$ consider $exp(ad(\xi v_2)) \in G^{e'}$ then

$$
\begin{aligned}
exp(ad(\xi v_2))(e', \{u \in \mathfrak{u}^e : a_2 = a_6 = 0\}) \subset \overline{G \cdot (e_{D_5(a_1)}, \mathfrak{g}^{e_{D_5(a_1)}} \cap \mathcal{N}_1)} \\
\Rightarrow (e', \{u \in \mathfrak{u}^{e'} : a_2 = 0, a_6 = \xi a_3 - \frac{\xi^2}{2}a_1\}) \subset \overline{G \cdot (e_{D_5(a_1)}, \mathfrak{g}^{e_{D_5(a_1)}} \cap \mathcal{N}_1)}
\end{aligned}
$$

Therefore by taking the closure we get that X_1 is contained in $\mathcal{C}_1(D_5(a_1))$. By a similar argument using the element $M_t' = e' + t^3u_2 + 30t^6u_8$ we can show X_2 is also contained in $\mathcal{C}_1(D_5(a_1))$. Therefore $\mathcal{C}_1(A_4A_1)$ is not an irreducible component of $\mathcal{C}_1^{nil}(E_6)$.

Orbit	Characteristic	(Possible) Irreducible Component X	Element x in irreducible component	Nilpotent Orbit x is contained in
A_1	$p=5$	$\widetilde{M_2}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_5}+e_{\alpha_6}$	A_4A_1
		$X_1, X_2 \in \widetilde{M_3}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_5}+e_{\substack{12210\\1}}$	$D_4(a_1)$
		$X_1, X_2 \in \widetilde{M_4}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_5}$	A_3A_1
		$\widetilde{M_5}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_6}+e_{\substack{12210\\1}}$	$A_2^2A_1$
		$\widetilde{M_6}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_6}$	A_2A_1
		$\widetilde{M_8}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\substack{12210\\1}}$	$A_2A_1^2$
	$p \geq 7$	$\widetilde{M_1}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_5}+e_{\alpha_6}+e_{\substack{12210\\1}}$	$E_6(a_3)$
		$\widetilde{M_2}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_5}+e_{\alpha_6}$	A_4A_1
		$\widetilde{M_3}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_5}+e_{\substack{12210\\1}}$	$D_4(a_1)$
		$\widetilde{M_4}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_5}$	A_3A_1
		$\widetilde{M_5}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\alpha_6}+e_{\substack{12210\\1}}$	$A_2^2A_1$
A_1^2	$p=5$	$X_1, X_2 \in \widetilde{M_2}$	$e_0+e_{\alpha_2}+e_{\substack{11110\\0}}+e_{\substack{01111\\0}}$	$D_4(a_1)$
		$X_1, \ldots, X_4 \in \widetilde{M_3}$	$e_0+e_{\alpha_2}+e_{\alpha_4}$	$A_2A_1^2$
		$\widetilde{M_4}$	$e_0+e_{\substack{11110\\0}}+e_{\substack{01111\\0}}$	A_2
	$p=7$	$\widetilde{M_1}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\substack{11110\\0}}+e_{\substack{01111\\0}}$	$D_5(a_1)$
		$\widetilde{M_2}$	$e_0+e_{\alpha_2}+e_{\substack{11110\\0}}+e_{\substack{01111\\0}}$	$D_4(a_1)$
		$\widetilde{M_3}$	$e_0+e_{\alpha_2}+e_{\alpha_4}$	$A_2A_1^2$
	$p=11$	$X_1, X_2 \in \widetilde{M_1}$	$e_0+e_{\alpha_2}+e_{\alpha_4}+e_{\substack{11110\\0}}+e_{\substack{01111\\0}}$	$D_5(a_1)$
		$\widetilde{M_2}$	$e_0+e_{\alpha_2}+e_{\substack{11110\\0}}+e_{\substack{01111\\0}}$	$D_4(a_1)$

A_1^3	$p=5$	$\widetilde{M_1}$	$e_0 + e_{\substack{11000\\0}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}} + e_{\substack{00011\\0}} + e_{\substack{12321\\2}}$	$D_4(a_1)$
		$X_1, X_2 \in \widetilde{M_2}$	$e_0 + e_{\substack{11110\\0}} + e_{\substack{01111\\0}} + e_{\substack{12321\\2}}$	$A_2A_1^2$
		$\widetilde{M_4}$	$e_0 + e_{\substack{11110\\0}} + e_{\substack{01111\\0}}$	$A_2A_1^2$
		$\widetilde{M_5}$	$e_0 + e_{\substack{11110\\0}} + e_{\substack{01111\\0}}$	$A_2A_1^2$
	$p=7$	$\widetilde{M_1}$	$e_0 + e_{\substack{11000\\0}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}} + e_{\substack{00011\\0}} + e_{\substack{12321\\2}}$	$D_4(a_1)$
		$\widetilde{M_2}$	$e_0 + e_{\substack{11110\\0}} + e_{\substack{01111\\0}} + e_{\substack{12321\\2}}$	$A_2A_1^2$
		$\widetilde{M_3}$	$e_0 + e_{\substack{11000\\0}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}} + e_{\substack{00011\\0}}$	A_3A_1
	$p=11$	$\widetilde{M_1}$	$e_0 + e_{\substack{11000\\0}} + e_{\substack{01100\\0}} + e_{\substack{00110\\0}} + e_{\substack{00011\\0}} + e_{\substack{12321\\2}}$	$D_4(a_1)$
A_2	$p \geq 5$	$\widetilde{M_1}$	$e_0 + e_{\alpha_2} + e_{\alpha_5} + e_{\alpha_6} + e_{\substack{12321\\1}}$	$D_4(a_1)$
A_2A_1	$p=5,11$	$\widetilde{M_1}$	$e_0 + e_{\alpha_5} + e_{\alpha_6}$	A_3A_1
	$p=7$	$X_1, X_2, X_3 \in \widetilde{M_1}$	$e_0 + e_{\alpha_5} + e_{\alpha_6}$	A_3A_1
		$\widetilde{M_2}$	$e_0 + e_{\substack{00011\\0}}$	A_3A_1
A_2^2	$p=5$	$\widetilde{M_2}$	$e_0 + e_{\alpha_2} + e_{\substack{12321\\1}}$	$D_4(a_1)$
		$\widetilde{M_3}$	$e_0 + e_{\substack{11100\\0}} + e_{\substack{01110\\0}} + e_{\substack{00111\\0}}$	A_3A_1
		$\widetilde{M_4}$	$e_0 + e_{\alpha_2}$	$A_2^2A_1$
	$p=7$	$\widetilde{M_1}$	$e_0 + e_{\substack{11100\\0}} + e_{\substack{01110\\0}} + e_{\substack{00111\\0}} + e_{\alpha_2}$	$E_6(a_3)$
		$\widetilde{M_2}$	$e_0 + e_{\alpha_2} + e_{\substack{12321\\1}}$	$D_4(a_1)$
	$p=11$	$\widetilde{M_1}$	$e_0 + e_{\substack{11100\\0}} + e_{\substack{01110\\0}} + e_{\substack{00111\\0}} + e_{\alpha_2}$	$E_6(a_3)$

$A_2A_1^2$	$p=5$	$X_1, X_2, X_3 \in \widetilde{M_1}$ $\widetilde{M_0}$	$e_0 + 2e_{\substack{11111\\0}} + e_{\substack{01210\\1}} + e_{\substack{11110\\1}} - e_{\substack{01111\\1}}$ $e_0 + t_2 + s_2$	$D_4(a_1)$ A_3A_1
	$p=7$	$X_1, X_2 \in \widetilde{M_1}$ $\widetilde{M_0}$	$e_0 + 2e_{\substack{11111\\0}} + e_{\substack{01210\\1}} + e_{\substack{11110\\1}} - e_{\substack{01111\\1}}$ $e_0 + t_2 + s_2$	$D_4(a_1)$ A_3A_1
	$p=11$	$X_1, X_2 \in \widetilde{M_1}$	$e_0 + 2e_{\substack{11111\\0}} + e_{\substack{01210\\1}} + e_{\substack{11110\\1}} - e_{\substack{01111\\1}}$	$D_4(a_1)$
A_3	$p=5$	$X_1, X_2 \in \widetilde{M_1}$ $\widetilde{M_2}$	$e_0 + e_{\alpha_6} + e_{\substack{11110\\1}} + e_{\substack{01210\\1}}$ $e_0 + e_{\substack{11110\\1}} + e_{\substack{01210\\1}}$	A_4 $D_4(a_1)$
	$p \geq 7$	$\widetilde{M_1}$	$e_0 + e_{\alpha_6} + e_{\substack{11110\\1}} + e_{\substack{01210\\1}}$	A_4
$A_2^2A_1$	$p=5$	$\widetilde{M_1}$ $M_0^{(3)}, M_0^{(4)} \in \widetilde{M_0}$	$e_0 + e_{\substack{12210\\1}} + e_{\substack{11211\\1}} + e_{\substack{01221\\1}}$ $e_0 + q_1 + s_3$	$D_4(a_1)$ $D_4(a_1)$
	$p \geq 7$	$X_1, X_2 \in \widetilde{M_1}$ $\widetilde{M_0}$	$e_0 + e_{\substack{12210\\1}} + e_{\substack{11211\\1}} + e_{\substack{01221\\1}}$ $e_0 + q_1 + s_3$	$D_4(a_1)$ $D_4(a_1)$
A_3A_1	$p=5$	$X_1, X_2, X_3 \in \widetilde{M_1}$	$e_0 + e_{\substack{12321\\2}}$	$D_4(a_1)$
	$p \geq 7$	$\widetilde{M_1}$	$e_0 + e_{\substack{12321\\2}}$	$D_4(a_1)$
A_4	$p=5$	$X_1, X_2 \in \widetilde{M_1}$	$e_0 + e_{\alpha_6}$	A_4A_1
	$p \geq 7$	$\widetilde{M_1}$	$e_0 + e_{\alpha_6}$	A_4A_1
D_4	$p \geq 7$	$\widetilde{M_1}$	$e_0 + e_{\substack{11100\\1}} + e_{\substack{11110\\0}} + e_{\substack{00111\\1}} + e_{\substack{01111\\0}}$	$E_6(a_3)$
A_4A_1	$p=11$	$X_1, X_2 \in \widetilde{M_0}$	$e_0 + v_1 + v_2$	$E_6(a_1)$
A_5	$p \geq 7$	$\widetilde{M_1}$	$e_0 + e_{\substack{12321\\2}}$	$E_6(a_3)$
$D_5(a_1)$	$p \geq 7$	$\widetilde{M_0}$	$e_0 + v_1 + v_2 + v_5$	$E_6(a_3)$

Table 9.2: Example of an element in each irreducible component of $\mathfrak{g}^{e_i} \cap \mathcal{N}_1$ which is not contained in $\overline{(G \cdot e_i)}$

Chapter 10

Summary of Results

In this chapter we summarise the results of Questions 1 and 2 for $\mathfrak{g} = G_2$, F_4 and E_6. For each Lie algebra $\mathfrak{g}$ we state the number of irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ along with the dimension of each component in a table. This is followed by a description of the irreducible components of $\mathcal{C}_1^{nil}(\mathfrak{g})$.

10.1 G_2

For the case when $G = G_2$, we have found that $\mathfrak{g}^e \cap \mathcal{N}_1$ is always irreducible. This is highlighted by the table below.

Orbit	**Characteristic** p	**Number of Irreducible Components**	**Dimensions of Components**
$G_2(a_1)$	≥ 5	1	4
$\widetilde{A_1}$	≥ 5	1	5
A_1	≥ 5	1	9

Theorem 10.1.1 *Let $\mathfrak{g}$ be of type G_2 and let $p = 5$. Then the variety $\mathcal{C}_1^{nil}(G_2)$ is irreducible of dimension $14 = \dim(\mathfrak{g})$ where*

$$\mathcal{C}_1^{nil}(G_2) = \mathcal{C}_1(G_2(a_1))$$

10.2 F_4

When $G = F_4$ we have found that $\mathfrak{g}^e \cap \mathcal{N}_1$ is always equidimensional. In particular $\mathfrak{g}^e \cap \mathcal{N}_1$ is either irreducible or it had two components of the same dimension.

Orbit	**Characteristic** p	**Number of Irreducible Components**	**Dimensions of Components**
A_1	5	1	28
	7	1	31
	11	1	32
	≥ 13	1	33
$\widetilde{A_1}$	5	1	24
	7	1	26
	≥ 11	1	27
$A_1\widetilde{A_1}$	5	2	19,19
	7,11	1	21
	≥ 13	1	13
A_2	5	2	18,18
	≥ 7	1	20
$\widetilde{A_2}$	5	1	18
	7	1	19
	≥ 11	1	20
$A_2\widetilde{A_1}$	5,7	2	15,15
	11	1	16
	≥ 13	1	17
B_2	≥ 5	1	14
$\widetilde{A_2}A_1$	5	1	14
	7	2	14,14
	≥ 11	1	15
$C_3(a_1)$	≥ 5	1	13
$F_4(a_3)$	≥ 5	1	12
B_3	≥ 7	1	9
C_3	≥ 7	1	9
$F_4(a_2)$	≥ 7	1	8
$F_4(a_1)$	≥ 11	1	6
F_4	≥ 13	1	4

Theorem 10.2.1 *The variety $\mathcal{C}_1^{nil}(F_4)$ is equidimensional of dimension $52 = \dim(\mathfrak{g})$ with respectively 1, 2, 3 components given by*

$$
\begin{aligned}
p = 5: &\quad \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \\
p = 7: &\quad \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \cup \mathcal{C}_1(F_4(a_2)) \\
p = 11: &\quad \mathcal{C}_1^{nil}(F_4) = \mathcal{C}_1(F_4(a_3)) \cup \mathcal{C}_1(F_4(a_2)) \cup \mathcal{C}_1(F_4(a_1))
\end{aligned}
$$

10.3 E_6

In this case we have not found the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ for nilpotent orbits A_1, A_1^2 and A_1^3. The case when e is contained in the orbit $A_2A_1^2$ is the first example when $\mathfrak{g}^e \cap \mathcal{N}_1$ is not equidimensional. This is because $\mathfrak{c} = \mathfrak{sl}_2 \oplus k$ and the dimension of the component corresponding to the zero orbit has a higher dimension of the irreducible components corresponding to the orbit $\mathcal{O}_{[2]}$ in $\mathfrak{sl}_2$. There are two other cases when $\mathfrak{g}^e \cap \mathcal{N}_1$ is not equidimensional, namely orbits $A_2^2A_1$ and A_3A_1.

Orbit	**Characteristic** p	**Number of Irreducible Components**	**Dimensions of Components**
A_1	5	-	-
	7	-	-
	11	-	-
	≥ 13	1	51
A_1^2	5	-	-
	7	-	-
	11	-	-
	≥ 13	1	42
A_1^3	5	-	-
	7	-	-
	11	1	34
	≥ 13	1	35
A_2	5	1	30
	≥ 7	1	32
A_2A_1	5	1	27
	7	4	27, 27, 27, 27
	≥ 11	1	29

A_2^2	5	1	24
	7	2	26, 26
	11	1	27
	13	1	28
$A_2A_1^2$	5	3	24, 22, 22
	7	3	24, 24, 24
	11	2	25, 25
	≥ 13	1	26
A_3	5	3	21, 21, 21
	≥ 7	1	23
$A_2^2A_1$	5	3	20, 19, 19
	7	3	21, 21, 21
	11	2	22, 22
	≥ 13	1	23
A_3A_1	5	3	19,18,18
	≥ 7	1	20
$D_4(a_1)$	≥ 5	1	18
A_4	5	2	15, 15
	≥ 7	1	16
D_4	≥ 7	1	16
A_4A_1	5,7	2	14
	≥ 11	1	15
A_5	≥ 7	1	13
$D_5(a_1)$	≥ 7	1	13
$E_6(a_3)$	≥ 7	1	12
D_5	≥ 11	1	9
$E_6(a_1)$	≥ 11	1	8
E_6	≥ 13	1	6

Theorem 10.3.1 *For $p = 5$ (resp. 11) the variety $\mathcal{C}_1^{nil}(E_6)$ is equidimensional of dimension 76 (resp. 78) with respectively 3 and 2 components.*

$$p = 5: \quad \mathcal{C}_1^{nil}(E_6) = \overline{G \cdot (e, X_1)} \cup \overline{G \cdot (e, X_2)} \cup \mathcal{C}_1(D_4(a_1))$$

$$p = 11: \quad \mathcal{C}_1^{nil}(E_6) = \mathcal{C}_1(E_6(a_3)) \cup \mathcal{C}_1(E_6(a_1)).$$

Here X_1 and X_2 are the two irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ for the nilpotent orbit $\mathcal{O}_e = A_4A_1$.

When $p = 7$ we have $\mathcal{C}_1^{nil}(E_6) = \mathcal{C}_1(E_6(a_3)) \cup \mathcal{C}_1(D_4(a_1))$; however we do not know if the inclusion $\mathcal{C}_1(D_4(a_1)) \subset \mathcal{C}_1(E_6(a_3))$ holds.

10.4 Further Work

We have calculated the irreducible components of $\mathcal{C}_1^{nil}(E_6)$ for $p = 5$ and 11. Therefore the next step would be to find the irreducible components when $p = 7$. We have already shown that

$$\mathcal{C}_1^{nil}(E_6) = \mathcal{C}_1(D_4(a_1)) \cup \mathcal{C}_1(E_6(a_3))$$

Therefore we would need to establish whether $\mathcal{C}_1(D_4(a_1)) \subset \mathcal{C}_1(E_6(a_3))$. The method used in Section 9.1, which utilizes Theorem 4.2.2, does not work in this case since the induced orbit $E_6(a_1)$ is not contained in $\mathcal{N}_1$. For a transverse slice argument this would be the same as demonstrating that $\mathcal{C}_1(D_4(a_1)) \subset \mathcal{C}_1(D_5(a_1))$. The difference between the dimension of $D_5(a_1)$ and $D_4(a_1)$ is 6, which is larger than any calculations computed in this thesis. Hence the transverse slice has a more complex structure than others we have dealt with and therefore the methods we have used in other cases do not apply. I expect that this inclusion does hold and that $\mathcal{C}_1^{nil}(E_6)$ is irreducible of dimension 78 when the characteristic p is 7.

Another obvious extension to this work is to consider the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ for the nilpotent orbits A_1, A_1^2 and A_1^3 in E_6. The polynomials describing the components of $\mathfrak{g}^e \cap \mathcal{N}_1$ are more complex than the other cases we considered. The standard methods we have used throughout do not work and it is likely to be time consuming to establish these inclusions as it was for the orbit A_1 in F_4. Note that not establishing the irreducible components of $\mathfrak{g}^e \cap \mathcal{N}_1$ for these orbits did not obstruct our work to find the irreducible components of $\mathcal{C}_1^{nil}(E_6)$. This is because in each case we can express $\mathfrak{g}^e \cap \mathcal{N}_1$ as a union of possible irreducible components X_i. Then each possible component of $\mathcal{C}_1^{nil}(E_6)$ corresponding to X_i can be eliminated using Theorem 5.2.1. This is less time consuming than establishing each inclusion. For these orbits I expect that $\mathfrak{g}^e \cap \mathcal{N}_1$ is equidimensional. Specifically I expect that $\mathfrak{g}^e \cap \mathcal{N}_1$ is irreducible for the orbit A_1^3 of dimension 31, 33 and 34 for $p = 5, 7$ and 11 respectively. Similarly for A_1^2, I expect that $\mathfrak{g}^e \cap \mathcal{N}_1$ has two irreducible components both of dimension 37, 39 and 41 respectively. Also for A_1, I expect $\mathfrak{g}^e \cap \mathcal{N}_1$ has two components of dimension 45 when $p = 5$ and is irreducible of dimension 48 and 50 for $p = 7$ and $p = 11$ respectively.

Finally it would be interesting to consider Question 1 and 2 for E_7 and E_8. There are three main factors which means there is more work involved in these cases than there was for G_2, F_4 and E_6. The first is the number of nilpotent orbits that need to be considered. For E_7 there are 45 orbits and for E_8 there are 70. This is considerably more that the 16 and 21 of F_4 and E_6. Also there are nilpotent orbits in E_7 and E_8 where $\mathfrak{g}^e(i)$ does not decompose into irreducible submodules. Therefore a different method will be needed to tackle these cases.

The second factor is that the Coxeter number of E_7 and E_8 is larger than that of F_4 and E_6 at

18 and 30 respectively. Therefore instead of just considering the cases when the characteristic p is 5, 7 and 11 we would need to consider 5 different characteristics for E_7 and 7 for E_8. Finally the dimension of the minimal faithful representations of E_7 and E_8 are larger than the cases we considered. For E_7 the dimension is 58 and for E_8 it is 248. I expect that applying the same [GAP12] code used in this thesis to an orbit in E_8 will require more computing resources than was available for this thesis.

The most time consuming work for F_4 and E_6 was demonstrating inclusions of irreducible closed subsets when the standard strategies fail, for example establishing $\mathcal{C}_1^{nil}(D_5) \subset \mathcal{C}_1^{nil}(E_6(a_1))$ when $p = 11$. There is no reason to believe that E_7 and E_8 would not have any of these cases.

Appendix A

Appendix - GAP code

Here we present the [GAP12] code we have used in some of the calculations. The calculations in Chapters 6 to 8 frequently require us to find solutions to sets of polynomials, usually the entries of the p-th power of a matrix. Section A.2 explains how we have automated the process of solving a single polynomial. In Section A.3 we apply this set-up to a collection of polynomials. Before this we consider a method for inputting elements of a Lie algebra in Section A.1.

A.1 Elements of a Lie Algebra

The following code provides a method for inputting elements of a Lie algebra $\mathfrak{g} = F_4$. This code was originally written by Daniel Juteau. We thank him for allowing us to publish it. Executing $e([a,b,c,d])$ returns the element $e_{a\alpha_1+b\alpha_2+c\alpha_3+d\alpha_4}$ in the Chevalley basis, where the roots are labelled in the same order as given by the Dynkin diagrams in Figure 1.1. Similarly $f([a,b,c,d])$ (resp. $h([a,b,c,d])$ gives the corresponding negative root element (resp. corresponding element in the Cartan subalgebra).

The simple root elements of F_4 in [GAP12] are defined in a different order to those labelled in the Dynkin diagram in Figure 1.1. The following code defines the matrix J, which is used to rearrange the GAP ordering to match that of the Dynkin diagram.

We can do something similar for $\mathfrak{g} = G_2$ and E_6. For these cases the matrix J is not required.

```
R:=Rationals;
LF4 := SimpleLieAlgebra("F",4,R);
RF4 := RootSystem(LF4);
CF4 := CartanMatrix(RF4);
BF4 := ChevalleyBasis(LF4);
B:=Basis(LF4);

J:=[[0,0,0,1],[1,0,0,0],[0,0,1,0],[0,1,0,0]];

e := function(v)
```

```
 return BF4[1][Position(PositiveRoots(RF4),TransposedMat(CF4)*(J*v))];
end;

f := function(v)
 return BF4[2][Position(PositiveRoots(RF4),TransposedMat(CF4)*(J*v))];
end;

h := function(i)
 if i=1 then return BF4[3][2];
  elif i=2 then return BF4[3][4];
  elif i=3 then return BF4[3][3];
  elif i=4 then return BF4[3][1];
fi;
end;
```

A.2 Manipulating Polynomials

In this section we consider some methods for solving a single polynomial $P = 0$. It is assumed that all polynomials do not have a zero degree term.

Method 1 *If the polynomial is a single univariate monomial, i.e. of the form $ax^n = 0$, then this function returns $[x, 0]$ otherwise it returns false.*

```
FindIndeterminatesWhichEqualZero := function(P)
  if IsUnivariateMonomial(P) then
    return [IndeterminateOfUnivariateRationalFunction(P), P*0];
  fi;
  return false;
end;
```

Method 2 *This function checks if the polynomial $P = 0$ can be rearranged so that a single variable x can be expressed in terms of others, i.e $x = Q$ for some polynomial Q which does not contain the variable x. This is done by scanning over the linear univariate monomials of P, if one exists with a variable x which is not present in any other monomial of P we can rearrange P to find a value of x. If a value is found using this method it returns $[variable, value]$ (i.e. $[x, Q]$). Otherwise it returns false.*

```
FindIndeterminatesInTermsOfOtherIndeterminates := function(P)
  local monomials, univariateMonomials, nonUnivariateMonomials, monomial, i;
  monomials              := MonomialsOfPolynomial(P);
  univariateMonomials    := Filtered(monomials, IsUnivariateMonomialLinear);
  nonUnivariateMonomials := Filtered(monomials, x-> not IsUnivariateMonomialLinear(x));

  for monomial in univariateMonomials do
```

```
    i := IndeterminateOfUnivariateRationalFunction(monomial);
    if ForAll(nonUnivariateMonomials, m -> not IsIndeterminateContainedInMonomial(i, m)
        ) then
      return [i, (monomial - P) / CoefficientsOfUnivariatePolynomial(monomial)[2]];
    fi;
  od;
  return false;
end;
```

Next we discuss a few methods for finding the factors of a polynomial P.

Method 3 *This method checks if the monomials of a polynomial P have a common univariate factor, i.e. if $P = ax^nQ$ for some polynomial Q. If this is the case it returns a record containing these factors in the form rec(factor:= $[ax^n, Q]$). Otherwise it returns false. Note that if P has a single monomial it returns false.*

```
FindHighestCommonUnivariateFactorOfPolynomial := function(P)
  local gcd;
  gcd := Gcd(MonomialsOfPolynomial(P));
  if IsUnivariateMonomial(gcd) and Length(CoefficientsOfUnivariatePolynomial(gcd)) > 1
      then
    return rec( factors:= [gcd, P/gcd] );
  fi;
  return false;
end;
```

Method 4 *The following function identifies whether a polynomial P is factorizable i.e. whether P be expressed as $P = Q_1Q_2 \dots Q_n$ for some polynomials Q_i. If this is the case it returns a duplicate free list of these factors in the form rec(factors:= $[Q_1, Q_2, \dots, Q_n]$). Otherwise it returns false.*

```
FindFactorsOfPolynomial := function(P)
  local factors;
  factors:=Factors(P);
  if Length(factors) > 1 then
    return rec( factors:=DuplicateFreeList(factors) );
  fi;
  return false;
end;
```

Method 5 *If a polynomial P has a single monomial, then method 3 does not work. This function checks if a polynomial P has a single monomial and factorizes it via method 4.*

```
FindFactorsOfMonomial := function(p)
  if Length(MonomialsOfPolynomial(p)) = 1 then
    return FindFactorsOfPolynomial(p);
  fi;
  return false;
end;
```

If we have a polynomial expression Q for a variable x then this function substitutes this value for x into a polynomial P. We input the polynomial P and the substitution as $[x, Q]$. If P is the zero polynomial then it returns P otherwise it returns P evaluated at x.

```
ApplySubstitution:=function(polynomial, substitution)
  if not IsZero(polynomial) then
    return Value(polynomial, [substitution[1]], [substitution[2]]);
  fi;
  return polynomial;
end;
```

A.3 Solutions for a set of Polynomials

Now we present the code to solve the set of polynomials $P_1 = 0, \ldots, P_n = 0$ using the previous functions. This function scans over $P_1, \ldots, P_n$ and identifies any variables which we can find a substitution for. This function returns a list of the substitutions along with the polynomials $P_1, \ldots, P_n$ with these substitutions made.
The code makes use of recursion in order to be able to find all the valid substituions for the given set of polynomials. We input a list of polynomial $P_1, \ldots P_n$ and a list of substitutions of the form $[x, value]$. Note that the supplied list of polynomials should already been evaluated at the initial substitutions. Below is some pseudocode which outlines how this function works.

```
SubstituteIndeterminates(polynomials, substitutions)
  beginning:
  for each polynomial P_i in polynomials
    for each substitution_method in [method1..method5]
      result := substitution_method(P_i)
      if result.found_single_substitution
        substitutions := substitutions + result.substitution
        polynomials := evaluated(polynomials) # evaluated at found substitution
        goto beginning
      else if result.has_factors
        for each factors_found_in(result) replace P_i in polynomials with factor
          SubstituteIndeterminates(polynomials, substitutions)
   print polynomials + substitutions
```

The following is a [GAP12] implementation of the above pseudocode. This implementation adds an optimization to avoid reprocessing polynomials for which substitutions have been found. This is done by supplying the position i in polynomials to start processing from.

```
SubstituteIndeterminatesWithKnowledge:=function(polynomials, substitutions, i)
  local FindSubstitutionsUsingMethods, result;

  # Copy polynomials and substitutions so we can make modifications on these
  polynomials   := ShallowCopy(polynomials);
  substitutions := ShallowCopy(substitutions);

  # Use the four substitution methods in order to attempt to find values for
  # substitutions. This method either return false, indicating that it has
  # found a substitution but we are not finished. This function should be
  # called again to try and find more substitutions. Otherwise it returns a
  # list of results, stating the substitutions found and polynomials
  FindSubstitutionsUsingMethods := function()
    local p, res, subMethod, results, factor;
    for subMethod in [FindIndeterminatesWhichEqualZero,
                      FindIndeterminatesInTermsOfOtherIndeterminates,
                      FindFactorsOfMonomial,
                      FindHighestCommonUnivariateFactorOfPolynomial,
                      FindFactorsOfPolynomial] do
      for p in [i..Length(polynomials)] do
        # The zero polynomial will not yield a substitution
        if not IsZero(polynomials[p]) then
          # Use the current sub method to find a substitution
          res := subMethod(polynomials[p]);

          if IsRecord(res) then
            # We have identified that the current polynomial has factors which
            # may be used for finding more substitutions. What we can do is:
            #  - Replace the polynomial in polynomials with current factor
            #  - Perform a SubstituteIndeterminatesWithKnowledge with the
            #    modified polynomial list
            #  - Combine results and return
            results := [];                    # Start with an empty list
            for factor in res.factors do
              polynomials[p] := factor;       # Replace with the current factor
              # Perform a recursive call to substitute indeterminates
              Append(results, SubstituteIndeterminatesWithKnowledge(polynomials,
                  substitutions, p));
            od;
            return results;
          elif IsList(res) then
            Add(substitutions, res); # A single substition found. Save it
            #Sub back in to all the polynomials and substitution values
            polynomials   := List(polynomials,   p-> ApplySubstitution(p, res) );
            substitutions := List(substitutions, s-> [
                                            s[1], ApplySubstitution(s[2], res)]);
```

```
            return false;
          fi;
        fi;
      od;
    od;
    return [ rec( substitutions:=substitutions, polynomials:=polynomials ) ];
  end;

  # Keep calling FindSubstitutionsUsingMethods until we get a result
  repeat
    result := FindSubstitutionsUsingMethods();
    i := 1; # Reset the i to 1;
  until IsRecordCollection(result); # Have I got to the end?
  return result;
end;
```

The following function finds the solutions to a set of polynomials $P_1 = 0, \ldots, P_n = 0$ with no initial knowledge. It first sorts the polynomials $P_1, \ldots, P_n$ in increasing number of monomials so polynomials which are likely to be easier to factorize is considered first. This function returns a record with the substitutions and the polynomials $P_1, \ldots, P_n$ with these substitutions made.

```
SubstituteIndeterminates:=function(polynomials)
  SortBy(polynomials, p->Length(MonomialsOfPolynomial(p)));
  return SubstituteIndeterminatesWithKnowledge(polynomials, [], 1);
end;
```

Given a set of polynomials $P_1, \ldots, P_n$ and a number p, *MultipleFreeList* is a function which removes any polynomials which are a multiple m of another polynomial where m is contained in $\{1, 2, \ldots, p-1\}$.

```
MultipleFreeList := function(list, p)
  local multipleFree,i;
  multipleFree := [];
  for i in list do
    if not ForAny([1..p-1] * i, x-> x in multipleFree) then
       Add(multipleFree, i);
    fi;
  od;
  return multipleFree;
end;
```

Now let M be a matrix with polynomial entries defined over the finite field $GF(p)$. Note that if a matrix N is defined over the integers then to reduce N mod p let $M = N * One(GF(p))$. The following function gives the polynomial conditions for $M = 0$. This is done by producing a multiple free list of the elements in M then applying the function *SubstituteIndeterminates* defined above.

```
FindUniqueSolutionsOfPolynomialsInMInCharP := function(M, p)
```

```
  return DuplicateFreeList(SubstituteIndeterminates(MultipleFreeList(Flat(M),p)));
end;

# Given a list of solutions (such as those returned by SubstituteIndeterminates)
# factorize each of the polynomials in each of the solutions.
FactorizeUniqueSolutionPolynomials := function(solutions)
  local i;
  for i in solutions do
    i.polynomials := List(i.polynomials, Factors);
  od;
  return solutions;
end;
```

The following function does the same as *FindUniqueSolutionsOfPolynomialsInMCharP* but then the outputted polynomial conditions is factorized.

```
FindUniqueFactorizedSolutionsOfPolynomialsInMInCharP := function(M, p)
  return FactorizeUniqueSolutionPolynomials(
            FindUniqueSolutionsOfPolynomialsInMInCharP(M, p));
end;
```

Example A.3.1

Consider the orbit $A_2A_1^2$ in E_6 as discussed in Section 8.4. Then the following code finds the polynomial conditions for $M_1^7 = 0$.

```
#Set up for Lie Algebra E6, assuming already called function e,f,h as described
     at the beginning of the chapter
R:=Rationals;
LE6 := SimpleLieAlgebra("E",6,R);
RE6 := RootSystem(LE6);
CE6 := CartanMatrix(RE6);
BE6 := ChevalleyBasis(LE6);
B:=Basis(LE6);
V27:=HighestWeightModule(LE6,[0,0,0,0,0,1]);
B27:=Basis(V27);;

 #e0 is the representative of the orbit A_2A_1^2
 e0:=e([0,1,0,0,0,0])+e([0,0,0,1,0,0])+e([1,0,0,0,0,0])+e([0,0,0,0,0,1]);

 # This is the root element of the reductive part c of g^e where c=sl_2 +k
 e1:=2*e([1,0,1,1,1,1])+e([0,1,1,2,1,0])+e([1,1,1,1,1,0])-e([0,1,1,1,1,1]);
 f1:=f([1,0,1,1,1,1])+2*f([0,1,1,2,1,0])+f([1,1,1,1,1,0])-f([0,1,1,1,1,1]);

#Basis of g^e as given by Lawther and Testerman
 s1:=e([1,1,2,2,1,1]); s2:=s1*f1; s3:=(s2*f1)/2; s4:=(s3*f1)/3;
 t1:=e([1,1,1,2,2,1]); t2:=t1*f1; t3:=(t2*f1)/2; t4:=(t3*f1)/3;
 u1:=e([1,2,2,3,2,1]); u2:=u1*f1; u3:=(u2*f1)/2; u4:=(u3*f1)/3; u5:=(u4*f1)/4;
 v1:=e([0,1,1,2,1,1])+e([1,1,1,2,1,0]); v2:=v1*f1; v3:=(v2*f1)/2;
```

```
x1:=e([1,1,1,1,0,0]); x2:=x1*f1;
y1:=e([0,1,0,1,1,1]); y2:=y1*f1;
z1:=e([1,1,1,2,1,1]); z2:=z1*f1;  z3:=(z2*f1)/2;

#This finds representation \rho(x) for each element x where \rho is the
    minimal faithful representation
E0:=MatrixOfAction(B27,e0);
E1:=MatrixOfAction(B27,e1);
S1:=MatrixOfAction(B27,s1); S2:=MatrixOfAction(B27,s2);
S3:=MatrixOfAction(B27,s3); S4:=MatrixOfAction(B27,s4);
T1:=MatrixOfAction(B27,t1); T2:=MatrixOfAction(B27,t2);
T3:=MatrixOfAction(B27,t3); T4:=MatrixOfAction(B27,t4);
U1:=MatrixOfAction(B27,u1); U2:=MatrixOfAction(B27,u2);
U3:=MatrixOfAction(B27,u3); U4:=MatrixOfAction(B27,u4);
U5:=MatrixOfAction(B27,u5);
V1:=MatrixOfAction(B27,v1); V2:=MatrixOfAction(B27,v2);
V3:=MatrixOfAction(B27,v3);
X1:=MatrixOfAction(B27,x1); X2:=MatrixOfAction(B27,x2);
Y1:=MatrixOfAction(B27,y1); Y2:=MatrixOfAction(B27,y2);
Z1:=MatrixOfAction(B27,z1); Z2:=MatrixOfAction(B27,z2);
Z3:=MatrixOfAction(B27,z3);

#Define some indeterminates
R:=GF(7);

a:=X(R,"a"); b:=X(R,"b"); c:=X(R,"c"); d:=X(R,"d"); g:=X(R,"g"); h:=X(R,"h");
i:=X(R,"i"); j:=X(R,"j"); k:=X(R,"k"); l:=X(R,"l"); m:=X(R,"m"); n:=X(R,"n");
p:=X(R,"p"); q:=X(R,"q"); r:=X(R,"r"); s:=X(R,"s"); t:=X(R,"t"); u:=X(R,"u");
v:=X(R,"v"); w:=X(R,"w"); x:=X(R,"x"); y:=X(R,"y"); z:=X(R,"z"); A:=X(R,"A");

#Define M0 and M1
M0:=a*E0+b*S1+c*S2+d*S3+g*S4+h*T1+i*T2+j*T3+k*T4+l*U1+m*U2+n*U3+p*U4+q*U5
+r*V1+s*V2+t*V3+u*X1+v*X2+w*Y1+x*Y2+y*Z1+z*Z2+A*Z3;

M1:=E1+M0;

FindUniqueFactorizedSolutionsOfPolynomialsInMInCharP(M1^7,7);
```

This outputs the following

```
rec(
  polynomials   := [ [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ],
                     [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ] ],
  substitutions := [ [ k, 0*Z(7) ], [ q, -d*j+Z(7)^4*g*i ] ] ),
rec(
  polynomials   := [ [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ],
                     [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ] ],
  substitutions := [ [ j, 0*Z(7) ], [ k, 0*Z(7) ], [ q, Z(7)^4*g*i ] ] ),
```

```
rec(
  polynomials   := [ [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ],
                     [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ] ],
  substitutions := [ [ j, 0*Z(7) ], [ g, 0*Z(7) ], [ q, Z(7)^4*c*k ] ] ),
rec(
  polynomials   := [ [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ],
                     [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ], [ 0*Z(7) ] ],
  substitutions := [ [ g, 0*Z(7) ], [ q, Z(7)^4*c*k-d*j ] ] )
```

The first and last records gives that $M_1^7 = 0$ if ($k = 0$ and $q = -dj + 4gi$) or ($g = 0$ and $q = 4ck - dj$). The second and third records are equivalent to one of these cases. In the notation used in Section 8.4, this is equivalent to ($b_4 = 0$ and $c_5 = -a_3b_3 + 4a_4b_2$) or ($a_4 = 0$ and $c_5 = 4a_2b_4 - a_3b_3$).

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