Identical delays in the nonautonomous Kuramoto model

Miraslau L. Barabash[1,3*], Spase Petkoski[2†], Aneta Stefanovska[1‡]

[1]Department of Physics, Lancaster University, Lancaster LA1 4YB, United Kingdom

[2]Institut de Neurosciences des Systemes, Aix-Marseille University, Marseille, France

[3]Department of Solid State Physics, Physics Faculty, Belarusian State University, Nezavisimosty Av., 4, 220030, Minsk, Belarus

*m.barabash@lancaster.ac.uk, †spase.petkoski@univ-amu.fr, ‡aneta@lancaster.ac.uk

Problem Formulation

Signal transmission speed is inevitably finite. In the Kuramoto model it causes time delays between the initial signal and its arrival. The generalizations that have been proposed to describe the effect of delays are only applicable to the autonomous case. To avoid this limitation, the method of deterministic time-dependent (TD) parameters was recently introduced [1].

Here and in Ref. [2] we explore the effects of constant identical time delays in the couplings of the Kuramoto model of equally coupled, nonidentical oscillators with explicitly time-varying parameters.

Theoretical basics

Time-variability establishes two distinct models ($\varepsilon = const$)

A: **forcing strengths are proportional to the natural frequencies, i.e.** $\widetilde{\omega}(t) = \omega[1+\varepsilon x(t)]$;

B: **coupling strength undergoes time-dependent bias:** $\tilde{K} = K(1+\varepsilon x(t))$.

Dynamic equations for the ensemble of oscillators are

$$\text{A:}\quad \dot{\theta}_i = \widetilde{\omega}_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t-\tau)-\theta_i(t)), \qquad (1)$$

$$\text{B:}\quad \dot{\theta}_i = \omega_i + \frac{\widetilde{K}}{N}\sum_{j=1}^{N}\sin(\theta_j(t-\tau)-\theta_i(t)), \qquad (2)$$

where θ_i is the phase of the i^{th} oscillator, τ is a time delay, $\widetilde{\omega}_i$ are the TD natural frequencies and the TD coupling strength is given by $\widetilde{K}$. The complex order parameter is introduced by $z(t) = r(t)e^{i\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}$. The thermodynamic limit $N \to \infty$ is assumed.

Dynamic equations for the order parameter in model A:

$$\dot{r} = -\gamma|1+\varepsilon x|r - \frac{K}{2}\left[r^2-1\right] r_{t-\tau}\cos\Delta\psi, \qquad (3a)$$

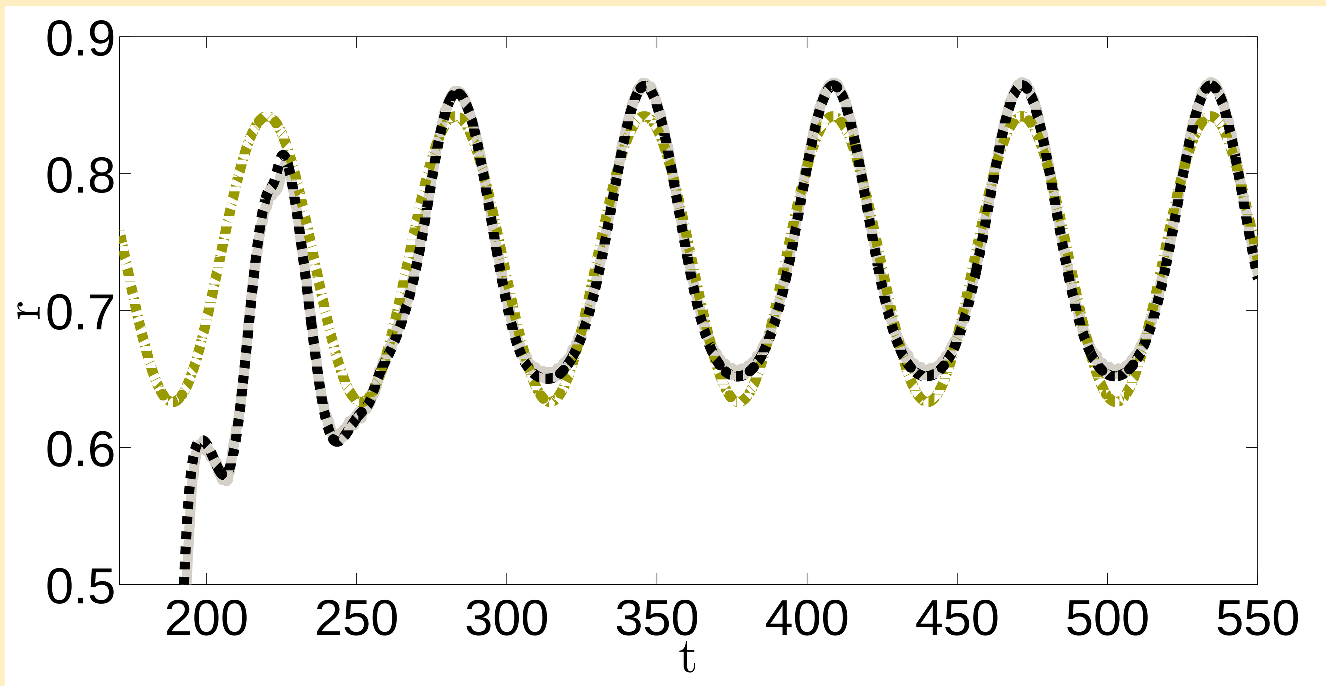
$$\dot{\psi} = \hat{\omega}|1+\varepsilon x| - \frac{K}{2}\left[r^2+1\right]\frac{r_{t-\tau}}{r}\sin\Delta\psi, \qquad (3b)$$

with $\Delta\psi = \psi - \psi(t-\tau)$, and $r_{t-\tau} = r(t-\tau)$. Model B reveals similar equations.

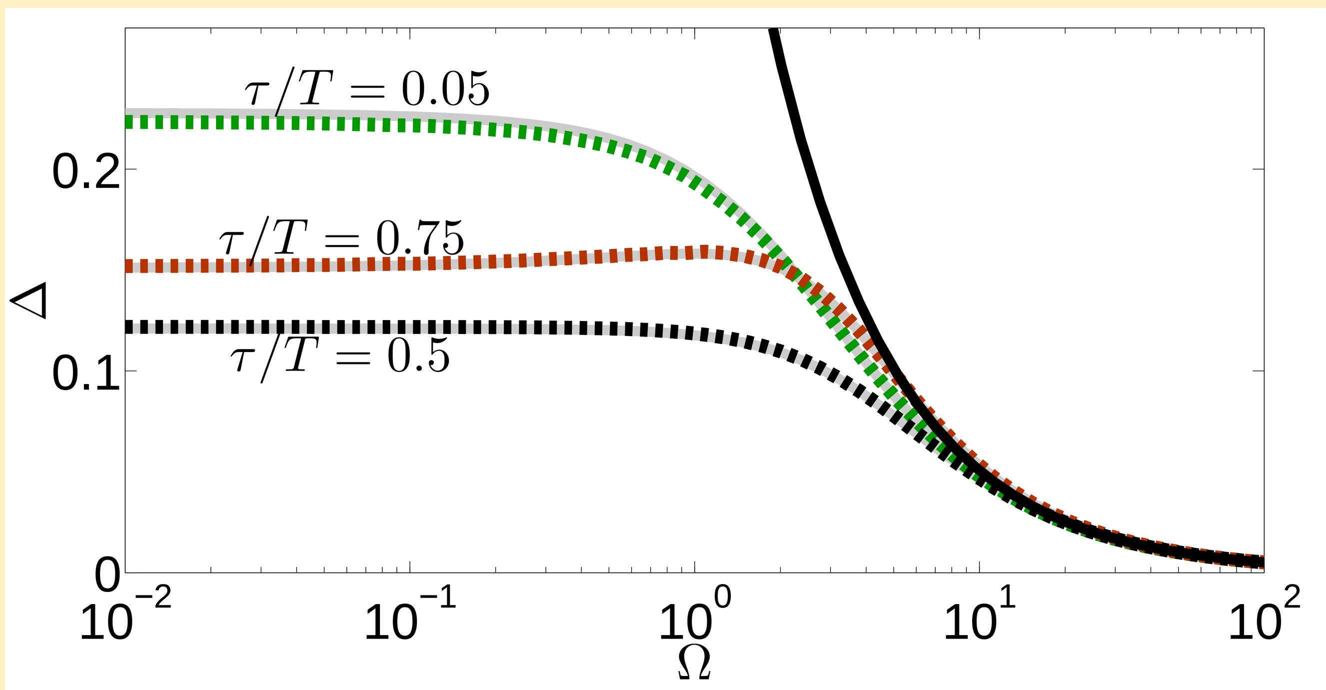
Numerical simulations

Model A with a simple cosine forcing

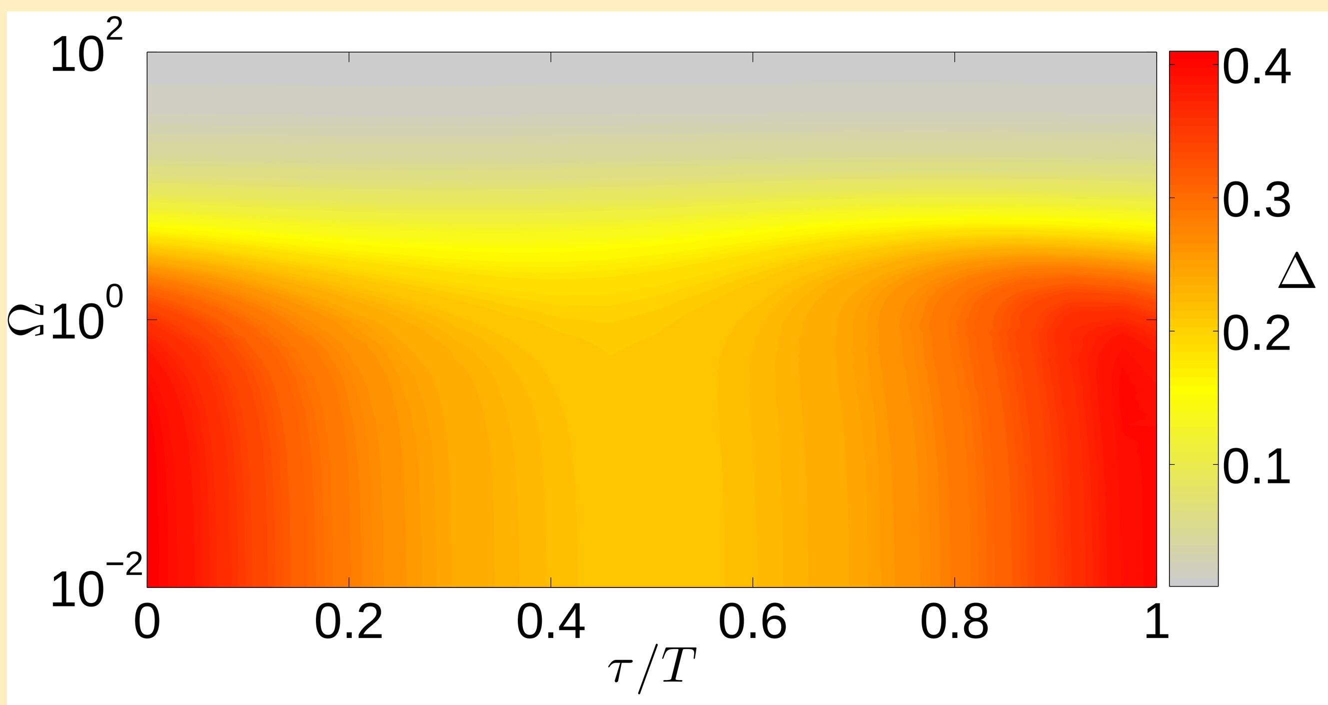
Let us denote the period of the forcing function as T, and redefine the parameters: $K/2\gamma \to \varkappa$, $t\gamma \to t$, $\Omega/\gamma \to \Omega$, $\hat{\omega}/\gamma \to \hat{\omega}$. The magnitude of the mean field oscillations, denoted by Δ, vanishes at high frequencies, while at low frequencies it is significant.



The evolution of the order parameter, Eq. (1) (solid grey), the low-dimensional dynamics, Eq. (3) (dashed black), and the approximation (4) (dot-dashed golden) for $\tau = 0.5\ T$, $\Omega = 0.1$.



A comparison between the approximation (5) (solid grey), the low-dimensional dynamics (Eq. (1), discontinuous curves), and the high-frequency limit (6) (solid black).

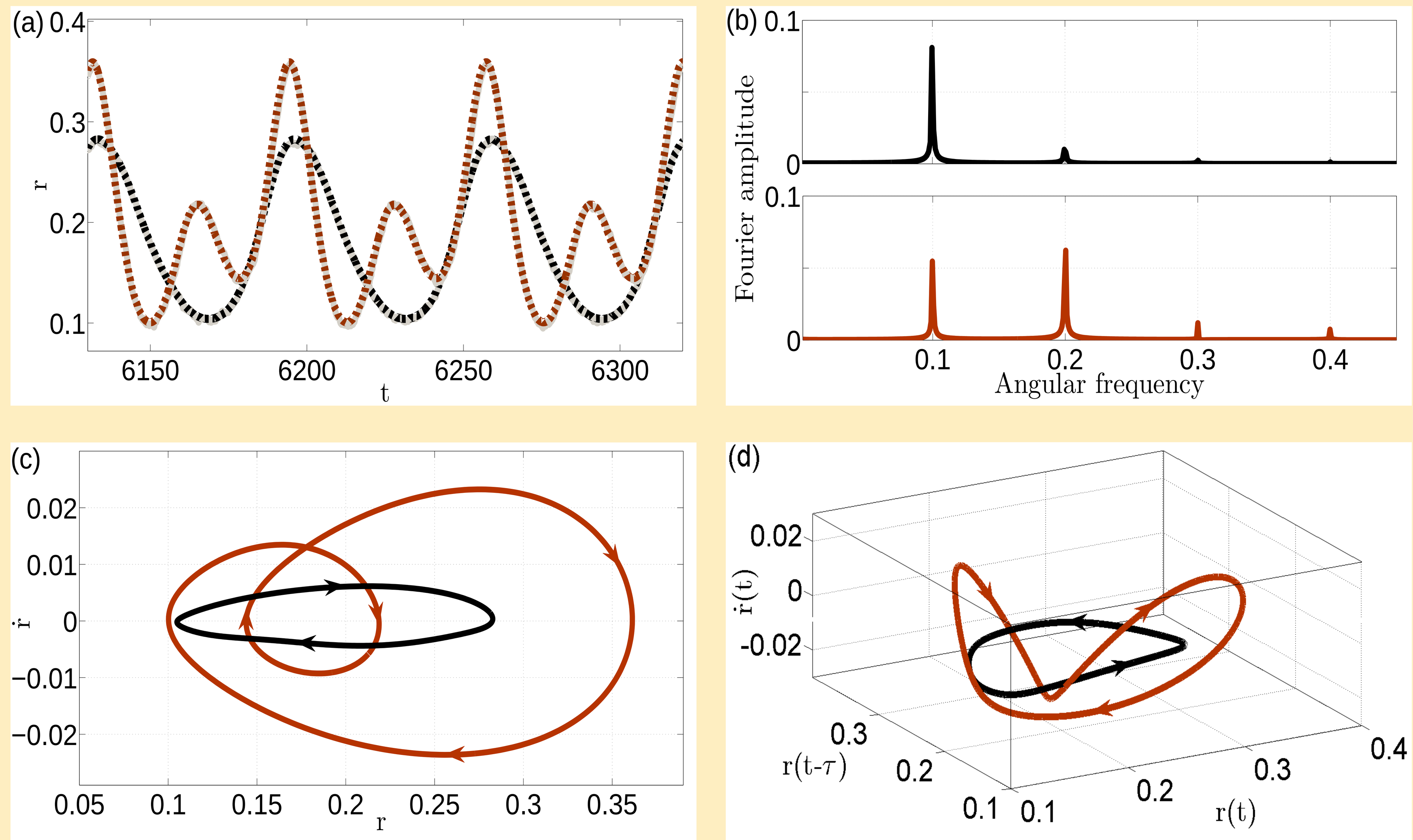


The mean-field magnitude Δ as a function of the ratio of τ/T and the forcing frequency Ω.

Results for model B are qualitatively similar to those in model A (see [2] for details).

Echo effect

For low-frequency forcing in models A and B, one can observe that the mean-field has at least one more pair of local extrema than the external forcing, i.e. some kind of frequency-doubling appears.



Echo-like evolution ($\tau = T/2$, brown lines), and regular mode ($\tau = 0.2\ T$, black lines) in model A. (a) Low-dimensional behaviour and simulations for these time delays. (b) Frequency spectra. (c) Mean-field limit cycles. (d) Limit cycles $\dot{r}(r(t), r(t-\tau))$. Self-crossing of the limit cycles is removed in 3D.

Analytical treatment

First harmonic approximation: the evolution can be represented by a Fourier series with only the first two components:

$$r(t) = A\left(1+\eta\cos\left(\Omega t+\varphi\right)\right). \qquad (4)$$

Substituting (4) in Eq. (3a), expanding the obtained expression, and omitting higher harmonics yields

$$A = \sqrt{\frac{1-1/\varkappa\,(1+\varepsilon\eta\cos\varphi/2)}{1+\eta^2/2+\eta^2\cos\Omega\tau}}, \quad \tan\varphi = \frac{(A^2-1)\sin(\Omega\tau)-\Omega/\varkappa}{2A^2-(1-A^2)\cos[\Omega\tau]+1/\varkappa}, \qquad (5a)$$

$$\eta = -\frac{\varepsilon}{\varkappa\sqrt{\left[2A^2-(1-A^2)\cos\Omega\tau+1/\varkappa\right]^2+\left[(1-A^2)\sin\Omega\tau+\Omega/\varkappa\right]^2}}. \qquad (5b)$$

This system is solved by the method of successive approximations.

In the high frequency limit Eq. (5) yields

$$r_{\mathbf{A}fast}(t) = \sqrt{1-\frac{1}{\varkappa}}\left(1-\frac{\varepsilon}{\Omega}\sin\Omega t\right). \qquad (6)$$

That limit coincides with the results reported in Ref. [1] for systems without delays.

The formulas for model B emerge similarly [2].

Summary

We have established the impact of delays on a population of coupled phase oscillators affected by external forcing.

- Three independent time scales define the ensemble's dynamics: the original system, the external forcing and the value of the delays.
- The **echo effect** allows the identification of coupling delays in a harmonically forced network.
- The first harmonic approximation for these systems has been described.

Possible fields of application:

- Neurones, connected with finite length axons, and at the same time influenced by the cardiovascular system.
- The plasmodium of the slime mould, *Physarum polycephalum*,which is expected to manifest an echo effect under temperature variation.
- Heterogeneous networks with bimodal frequency distribution and time-periodic coupling.

Open questions:

- Time-varying bias of the natural frequencies.
- Influence of the delay-induced multistability on the dynamics.
- Behaviour of the systems with heterogeneous time-delay distributions [3].

Acknowledgements

M.L.B. expresses his gratitude to Prof Peter V.E. McClintock, Mrs Heather Willes, and Prof V.V. Uglov, who facilitated his current secondment to Lancaster University. The work was partially supported by the EPSRC (UK) and by the Slovenian Research Agency.

References

[1] S. Petkoski and A. Stefanovska, "Kuramoto model with time-varying parameters," *Phys. Rev. E*, vol. 86, p. 046212, Oct 2012.

[2] M. L. Barabash, S. Petkoski, and A. Stefanovska, "Homogeneous delays in the Kuramoto model with time-variable parameters," *Phys. Rev. E*, vol. 90, p. 052903, Nov 2014.

[3] W. S. Lee, E. Ott, and T. M. Antonsen, "Large coupled oscillator systems with heterogeneous interaction delays," *Phys. Rev. Lett.*, vol. 103, p. 044101, 2009.