Spatially Dispersive Inhomogeneous Dielectric Wire Media with Periodic Structure

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Abstract— Dielectric wire media are modelled both numerically using CST Microwave studio and analytically as media with spatially dispersion. In the latter case this leads to a differential equation that can be solved in terms of Mathieu functions. A periodic variation in the radii of the wires is considered as a method for shaping the propagating mode shape. A profile is proposed which gives a flatter shaped mode. Such a mode would increase the acceleration of particles in a particle accelerator.

1. INTRODUCTION

In this article we model the electromagnetic response to a dielectric wire grid with periodic variation in the radii of the wires (see Figure 1). Being periodic in all three dimensions, such a medium can be modelled into a standard 3D electromagnetic simulator by considering its unit cell wrapped into periodic boundary conditions. In this article we also model it as one dimensional spatially dispersive medium with a periodic inhomogeneity. Being spatially dispersive requires that the permittivity ϵ depends on k, whereas being inhomogeneous requires that the permittivity also depends on x. However k and x are Fourier conjugate variables so we need to consider what it means for a function to depends simultaneously on both k and x. In [1–3] we solve this by working in the x domain and considering the permittivity relation to be a differential equation in x, with parameters that depend on x. An alternative interpretation is in terms of a susceptibility kernel [4, 5].

Wire medium has many applications. The advantage here is that we can approximate it by a one dimensional epsilon near zero (ENZ) medium [6, 7]. As such it supports a purely electric longitudinal mode. Such modes may be used to accelerate particles. Although in general the phase velocity of such a mode is not equal to the speed of light. Therefore one may require drift tubes in order to provide positive acceleration.

By altering the shape of the wires, along the longitudinal direction, one may change the mode shape. This would enable one to flatten the shape of the mode from the usual sinusoidal shape. This would have definite advantages in that for the same power, the beam will experience enhanced acceleration. By contrast in signal transmission one may desire a higher peak for a given total energy.

In Section 2 we summarise the results of [1], when applied to longitudinal modes in a dielectric wire media. In this case the differential equation for the polarization corresponds to the Mathieu equations and therefore the modes can be written in terms of Mathieu functions. In Section 3 we compare these results to numerical simulations. We show that for a dielectric wire, the electric field is primarily in the longitudinal direction and that the average magnetic field is very low and therefore one may consider it to be modelled well by a one dimensional longitudinal wire.

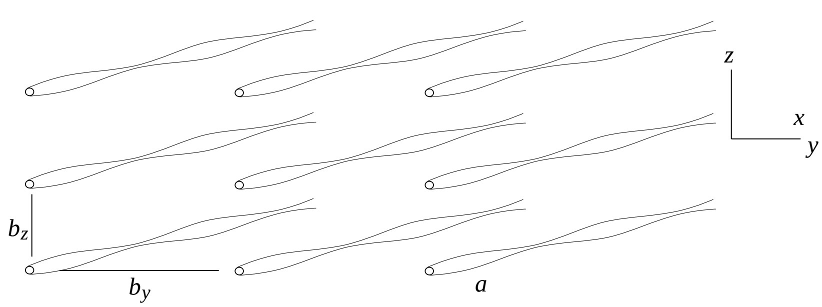

Figure 1: Wire medium with a periodic variation in the radius of the wires. The inter wire spacing are (b_y, b_z) and the period of the longitudinal variation is a.

2. WIRE MEDIA

Since the electric and polarization fields are longitudinal and the magnetic field vanishes, i.e., $\mathbf{E} = E(t,x)\mathbf{e}_1$, $\mathbf{P} = P(t,x)\mathbf{e}_1$ and $\mathbf{B} = \mathbf{0}$, then Maxwell's equations are automatically satisfied if

$$\epsilon_0 E + P = 0 \tag{1}$$

i.e., $\mathbf{D} = 0$, thus we are looking for epsilon near zero (ENZ) media. When the medium is homogeneous we will use an empirical model of the permittivity via

$$\widetilde{P}(\omega, k) = \frac{-\epsilon_0 k_{\mathrm{p}}^2}{\omega^2 - \beta^2 k^2}\, \widetilde{E}(\omega, k) \tag{2}$$

where ω is the temporal frequency, k is the wave number, β is the limiting phase velocity and k_p is the "plasma frequency". The Fourier transform of $P(t,x)$ with respect to t and x is given by

$$\widetilde{P}(\omega, k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(\omega t + kx)} P(t,x)\,dt\,dx \tag{3}$$

The plasma frequency will be a function of the inter wire spacing $(b_y,\, b_z)$, the thickness of the wires r and the dielectric permittivity of the wires ϵ_{wire}. The denominator on the right hand side of (2) is motivated by the conducting wire medium [8, 9] and is valid when $r \ll a$.

Combining (1) and (2) we obtain the dispersion relation

$$\omega^2 - \beta^2 k^2 = k_{\mathrm{p}}^2 \tag{4}$$

By using CST Microwave Studio for a range of k with fixed a we can test the general form of the equation. In addition varying a we can get the dependence of k_{p} on a.

The simplest method to include an inhomogeneity in the permittivity is to let the plasma frequency k_{p} to depend on position x, that is $k_{\mathrm{p}} = k_{\mathrm{p}}(x)$. Since k_{p} depends on the radius r of the wires the easiest way to achieve this it to let the radius of the wires vary whilst keeping the inter wire spacing $(b_y,\, b_z)$ constant. In order to extend the permittivity (2) to include inhomogeneous medium we work in the frequency-time domain. Thus (2) becomes a differential equation for $\hat{P}(\omega, x)$, which using (1) we may write as a differential equation[1] for $\hat{P}(\omega, x)$

$$\frac{\beta^2}{(2\pi)^2}\frac{\partial^2 \hat{P}}{\partial x^2} + \omega^2 \hat{P} = k_{\mathrm{p}}(x)^2 \hat{P} \tag{5}$$

where

$$\hat{P}(\omega, x) = \int_{-\infty}^{\infty} e^{-2\pi i\omega t} P(t,x)\,dt \tag{6}$$

The simplest modification to make the system inhomogeneous and periodic is to let the plasma frequency vary with x:

$$k_{\mathrm{p}}^2 = k_0^2 - 2\Lambda\cos(2\pi x/a) \tag{7}$$

As stated in the introduction, this can be constructed by varying $r(x)$ periodically.

From (5) this gives the Mathieu equation

$$\frac{\beta^2}{(2\pi)^2}\frac{\partial^2 \hat{P}}{\partial x^2} + \left(\omega^2 - k_0^2 + 2\Lambda\cos\left(2\pi x/a\right)\right)\hat{P} = 0 \tag{8}$$

From Floquet's theorem the solution can be written

$$\hat{P}(\omega, x) = e^{2\pi i\kappa x/a}\mathcal{P}(\omega, x) \tag{9}$$

[1] Note that we work with the polarization P instead of the electric field so that we can directly compare the results with [1]. Clearly one may use (1) to convert into E.

where $\mathcal{P}(\omega, x)$ is periodic in x, i.e., $\mathcal{P}(\omega, x+1) = \mathcal{P}(\omega, x)$. For each κ in the range $0 \leq \kappa < a$ there exist an infinite number of values

$$\omega_n^2 - k_0^2 = A_n\left(\kappa, \Lambda/\beta^2\right) \tag{10}$$

such that (9) is a solution to (8) with the periodicity of $\mathcal{P}(\omega, x)$. One may therefore regard (8) as a dispersion relation. The solution to (8) is given in terms of the Mathieu Function F.

$$\hat{P}(\omega_n, x) = F\left(\frac{4a^2(\omega_n^2 - k_0^2)}{\beta^2}, \frac{4a^2\Lambda}{\beta^2}, \frac{\pi x}{a}\right) \tag{11}$$

In the case when $\Lambda = 0$ and hence $k_{\text{p}} = k_0$ then (5) reduces to the simple harmonic oscillator and has solution $\hat{E}(\omega, x) = e^{2\pi k x}$. From (9) we may set $\mathcal{P}(\omega, x) = e^{2\pi i n x/a}$ where $n \in \mathbb{Z}$, so that (4) becomes the dispersion relation, for the unperturbed frequency ω_n,

$$\Omega_n = \sqrt{k_0^2 + \beta^2(\kappa+n)^2/a^2} \tag{12}$$

and hence

$$A_n(\kappa, 0) = \beta^2(\kappa+n)^2/a^2 \tag{13}$$

In [1] we look the approximation for small Λ. In order to make the translation we observe that

$$f_q(\omega) = \omega^2 - k_0^2 - \beta^2(q+\kappa)^2/a^2 \tag{14}$$

and hence $f_n(\Omega_n) = 0$. In this case

$$\mathcal{F}_q = f_q\left(\Omega_n\right) = k_0^2 - \frac{\beta^2(n+\kappa)^2}{a^2} - \left(k_0^2 - \frac{\beta^2(q+\kappa)^2}{a^2}\right) = \frac{\beta^2}{a^2}(n-q)\left(n+q+2\kappa\right) \tag{15}$$

and $\mathcal{F}'_q = f'_q(\Omega_n) = 2\Omega_n$. In this case we see that

$$\omega_n = \Omega_n + \frac{\Lambda^2\, a^2}{2\Omega_n\, \beta^2}\left(\frac{1}{2n+2\kappa-1} - \frac{1}{2n+2\kappa+1}\right) + O\left(\Lambda^4\right) \tag{16}$$

we then take the spatial Fourier series

$$\mathcal{P}^n(x) = \sum_{m=-\infty}^{\infty} e^{2\pi m x} P_m^n$$

and calculate the Fourier the approximate value of P_m^n, given in Equation (33) of [1]. One of the key results of [1] is the observation that this approximation scheme brakes down in the case when $\kappa = 0$ or $\kappa = \frac{1}{2}$. This can be seen in the (15) above, where $\mathcal{F}_{-n} = 0$ if $\kappa = 0$ and $\mathcal{F}_{-n-1} = 0$ if $\kappa = \frac{1}{2}$. This is because the approximate modes couple. The existence of coupled modes is a new feature of spatial dispersion. See [1] for details.

The approximation scheme described in [1] is more general since it will deal with fourth order differential equations applicable to transverse modes. Indeed it can be seen that the method is appropriate for any order differential equation in x.

3. NUMERICAL RESULTS

The eigen mode solver of CST Microwave Studio is used to perform mode analysis of the metamaterial unit cell. The unit cell has dimensions in the transverse plane $(b_y, b_z) = (22.86\,\text{mm}, 10.16\,\text{mm})$. Inside was a uniform dielectric rod $\epsilon_{\text{wire}} = 1600\epsilon_0$, with square cross section of width between $w = 0.2\,\text{mm}$ to $w = 0.5\,\text{mm}$. This reproduced results in the literature [10]. This the cross section was uniform for these simulations, the result was independent of the x longitudinal period of the unit cell. However for computational reasons this was set to 10.16 mm. By looking at at the fields, in Figure 2 it is clear that, away from the wires, the electric field is in the longitudinal direction (x) and that the magnetic field H is in the transverse plane. Not only is magnetic field concentrated near the rod, but the magnetic field averaged over the transverse plane is zero. Therefore this dielectric wire is modelled well by looking at longitudinal electric modes in an epsilon near zero medium.

The dispersion (ω, k) curves for a variety of wire widths are given in Figure 3. It is clear that for thin wires there is a hyperbolic relation (4) whereas this relation breaks down for thicker wires at high frequency. Mapping the hyperbolic portions onto (4) we get the following values for the limiting phase velocity β and the unperturbed plasma frequency k_0^2.

w	β	$k_{\rm p}^2 = k_0^2$
0.2mm	$0.157c$	$257c^2\,\mathrm{m}^{-2}$
0.3 mm	$0.155c$	$157c^2\,\mathrm{m}^{-2}$
0.4 mm	$0.154c$	$101c^2\,\mathrm{m}^{-2}$
0.5 mm	$0.155c$	$97c^2\,\mathrm{m}^{-2}$

(17)

It is clear from the Table (17) that we can use a constant phase velocity $\beta = 0.155c$. The relationship between w and $k_{\rm p}^2$ may be approximated by a quadratic equation given by

$$k_{\rm p}(w)^2 \approx c^2 \left(2420w^2 - 2230w + 607\right) \tag{18}$$

3.1. Proposed Model of Periodic Structure

We wish to find a function for the width $w(x)$ such that the plasma frequency given by (18) becomes (7). For this we set the unperturbed plasma frequency $k_0^2 = 100c^2\,\mathrm{s}^{-2} = (2.997\,\mathrm{GHz})^2$ and $\beta = 0.155c$. We need to choose the period a to be longer than the inter wire spacing, 22.86 mm. A reasonable value is to set $a = 0.1\,\mathrm{m}$. The flattest mode shape can be achieved when the second

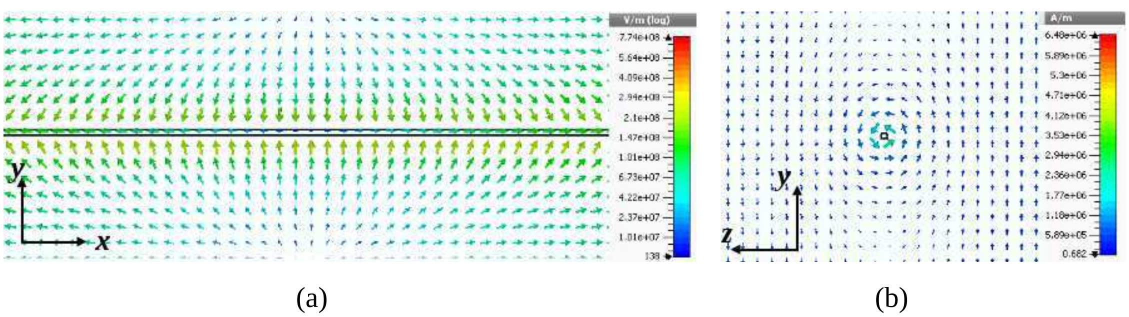

Figure 2: (a) The electric (in the (x, y) plane, and (b) magnetic field (in the (y, z) plane, for a square dielectric rod $\epsilon_{\rm wire} = 1600\epsilon_0$ (BST ceramic material) of width 0.4 mm. The frequency was 5.39 GHz, $k = 163.6\,\mathrm{m}^{-1}$, $a = 30\,\mathrm{mm}$, $(b_y, b_z) = (22.86\,\mathrm{mm}, 10.16\,\mathrm{mm})$.

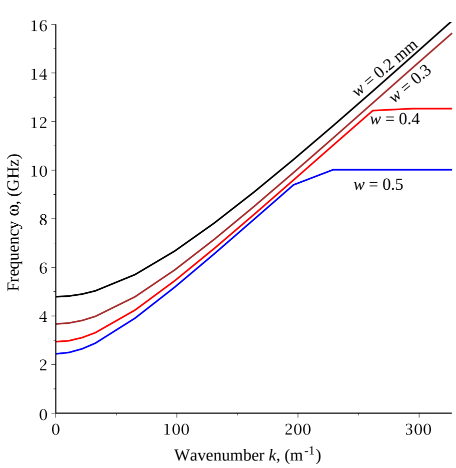

Figure 3: Dispersion relation ω versus k for a range of different widths w.

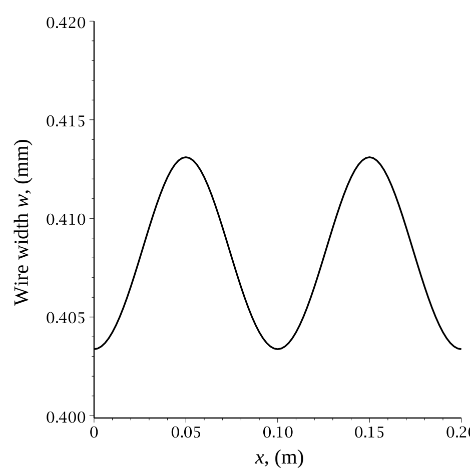

Figure 4: Proposed profile of dielectric wire to give a the periodic plasma frequency (7).

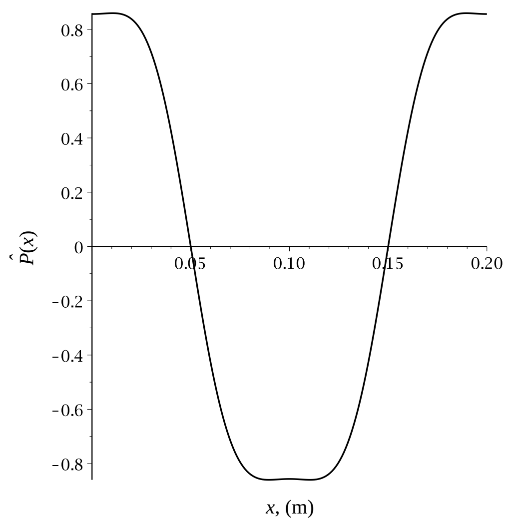

Figure 5: Mode shape of the polarization $\hat{P}(\omega, x)$ and the electric field $\hat{E}(\omega, x)$.

argument of F in (11) is equal to 1, i.e., $4a^2\Lambda/\beta^2 = 1$. This gives a value for $\Lambda = 0.618c^2\,\mathrm{m}^{-2} = (0.236\,\mathrm{GHz})^2$. Thus the with of the wires $w(x)$ varies between 0.404 mm and 0.413 mm its shape is given in Figure 4.

We also choose $\kappa = 0$. Placing these values into (10) and (11) we find that the a lowest mode has it's frequency only slightly altered $\omega_0 = 3.015\,\mathrm{GHz}$, The mode shape will then be given by Figure 5.

4. CONCLUSION

We have described how to shape the spatial modes of longitudinal mode by varying the width of a dielectric wire. We have also show how to model this in terms of a spatially dispersive media. There is clearly many exciting directions for this research. The next step will be to look at the modes numerically using CST studio or similar in order to compare the modes with the expected mode. If one wished to further shape the mode one could include a $\Lambda_1 \cos(4\pi x/a)$ in (8). The corresponding differential equation can then be solved using Heun functions. In addition, by removing one of the wires, one could create a photonic band gap structure which will maximise the fields in one region.

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