Operators Associated with Soft and Hard Spectral Edges from Unitary Ensembles

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Abstract. Using Hankel operators and shift-invariant subspaces on Hilbert space, this paper develops the theory of the integrable operators associated with soft and hard edges of eigenvalue distributions of random matrices. Such Tracy–Widom operators are realized as controllability operators for linear systems, and are reproducing kernels for weighted Hardy spaces, known as Sonine spaces. Periodic solutions of Hill's equation give a new family of Tracy–Widom type operators. This paper identifies a pair of unitary groups that satisfy the von Neumann–Weyl anti-commutation relations and leave invariant the subspaces of L^2 that are the ranges of projections given by the Tracy–Widom operators for the soft edge of the Gaussian unitary ensemble and hard edge of the Jacobi ensemble.

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1. Introduction

This paper concerns the spectral theory and invariant subspaces of operators that arise in random matrix theory, particularly the soft and hard edges that occur on the limiting eigenvalue distributions of the Gaussian and Jacobi unitary ensembles. Tracy and Widom [35, 36, 37] introduced various operators to describe the soft edge of the spectrum of the Gaussian unitary ensemble; that is, the eigenvalues near to the supremum of the support of the equilibrium distribution. Here we develop this theory in a systematic manner to show that Tracy and Widom's calculations are instances of more general results on Hankel operators, and introduce new settings where the theory applies.

Definition (GUE) Let $x_{j,k}$ and $y_{j,k}$ $(1 \leq j \leq k \leq n)$ be a family of mutually independent Gaussian $N(0, 1/n)$ random variables. We let X_n be the $n \times n$ complex Hermitian matrix that has entries $[X_n]_{jk} = (x_{j,k} + iy_{j,k})/\sqrt{2}$ for $j < k$, $[X_n]_{jj} = x_{j,j}$ for $1 \leq j \leq n$ and $[X_n]_{kj} = (x_{j,k} - iy_{j,k})/\sqrt{2}$ for $j < k$. We define the *Gaussian unitary ensemble* to be the probability measure $\sigma_n^{(2)}$ on the $n \times n$ complex Hermitian matrices such that a random matrix X_n under $\sigma_n^{(2)}$ has entries with this joint distribution. The probability measure $\sigma_n^{(2)}$ is called *unitary* since $\sigma_n^{(2)}$ is invariant under the natural action $X_n \mapsto UX_nU^\dagger$ by elements U of the group of $n \times n$ complex unitary matrices.

Bulk of the spectrum. The eigenvalues of X_n are real and may be ordered as $\lambda_1 \leq \ldots \leq \lambda_n$. For each $\varepsilon > 0$, and bounded and continuous $f : \mathbf{R} \to \mathbf{R}$ we have

$$\sigma_n^{(2)}\Big\{X_n : \Big|\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j) - \frac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}dx\Big| > \varepsilon\Big\} \to 0 \qquad (n \to \infty). \tag{1.1}$$

So we say that the *bulk of the spectrum* consists of those eigenvalues in $[-2,2]$; see [28, p. 93]. To describe the distribution of neighbouring eigenvalues within small subintervals of $[-2,2]$, we let D_t be the operator on $L^2(\mathbf{R})$ that has the (Dirichlet) sine kernel

$$D_t(x,y) = \frac{\sin t\pi x\, \cos t\pi y - \cos t\pi x\, \sin t\pi y}{\pi(x-y)}. \tag{1.2}$$

Now let $\mathbf{I}_S$ be the indicator function of a set S, and let $P_{(\alpha,\beta)}$ be the orthogonal projection on $L^2(\mathbf{R})$ given by $P_{(\alpha,\beta)}f(x) = \mathbf{I}_{(\alpha,\beta)}(x)f(x)$; we write $P_+ = P_{(0,\infty)}$ and $P_- = P_{(-\infty,0)}$.

Let $\mathrm{B}_{\sigma_n^{(2)}}(k;\alpha,\beta)$ be the probability with respect to $\sigma_n^{(2)}$ that $(\alpha/n, \beta/n)$ includes exactly k eigenvalues of X_n. Mehta and Gaudin [28, A10, (5.3.10)] showed that

$$\mathrm{B}_{\sigma_n^{(2)}}(k;-\alpha,\alpha) \to \frac{(-1)^k}{k!}\Big(\frac{d^k}{dt^k}\Big)_{t=1} \det\Big[I - tP_{(-\alpha,\alpha)}D_1P_{(-\alpha,\alpha)}\Big] \qquad (n \to \infty). \tag{1.3}$$

This determinant can alternatively be expressed in terms of the operator $\Psi_a : L^2[-a,a] \to L^2$ that has kernel $\Psi_a(x,y) = e^{ixy}\mathbf{I}_{[-a,a]}(y)/\sqrt{2\pi}$ and satisfies $\Psi_a\Psi_a^\dagger = D_{a/\pi}$.

Soft edge of the spectrum. The points ± 2 are said to be *soft edges* since for each $n < \infty$, the eigenvalues can lie outside the bulk of the spectrum $[-2,2]$ with positive probability with respect to $\sigma_n^{(2)}$. Now we present an asymptotic formula for this probability. The Airy function $\mathrm{Ai}(x)$, as defined by the oscillatory integral

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(zt+t^3/3)}\, dt, \tag{1.4}$$

satisfies the Airy differential equation $y'' - xy = 0$; see [34, p. 18]. Let $W_{1/3}$ be the integral operator on $L^2(\mathbf{R})$ defined by the *Airy kernel*

$$W_{1/3}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}. \tag{1.5}$$

We scale the eigenvalues of X_n by introducing $\xi_j = n^{2/3}\big(\lambda_j - 2\big)$, and let $\mathrm{E}_{\sigma_n^{(2)}}(k;\xi;\alpha,\beta)$ be the probability with respect to $\sigma_n^{(2)}$ that (α,β) contains exactly k of the ξ_j $(j = 1,\ldots,n)$;

see [28, p. 116, A7]. Aubrun [3] proved that the operator $W^{\alpha,\beta}_{1/3} = P_{(\alpha,\beta)}W_{1/3}P_{(\alpha,\beta)}$ on $L^2(\mathbf{R}_+)$ is of trace class for $0<\alpha<\beta\leq\infty$, and

$$\mathrm{E}_{\sigma_n^{(2)}}(k;\xi;\alpha,\beta)\to\frac{(-1)^k}{k!}\Big(\frac{d^k}{dt^k}\Big)_{t=1}\det\big(I-tW^{\alpha,\beta}_{1/3}\big)\qquad (n\to\infty). \tag{1.6}$$

The compression of $W^{\alpha,\infty}_{1/3}$ to $L^2(\alpha,\infty)$ may be identified, under the change of variables $s\mapsto\alpha+s$, with $\Gamma^2_{(\alpha)}$ where the Hankel integral operator $\Gamma_{(\alpha)}$ on $L^2[0,\infty)$ satisfies

$$\Gamma_{(\alpha)}f(s)=\int_0^\infty \mathrm{Ai}(\alpha+s+t)f(t)\,dt\qquad (f\in L^2(0,\infty)). \tag{1.7}$$

For compact operators S and T on Hilbert space, the spectrum of ST equals the spectrum of TS; hence the spectrum of $P_{(\alpha,\beta)}\Gamma^2_{(0)}P_{(\alpha,\beta)}$ equals the spectrum of $\Gamma_{(0)}P_{(\alpha,\beta)}\Gamma_{(0)}$, so

$$\det(I-tP_{(\alpha,\infty)}W_{1/3}P_{(\alpha,\infty)})=\det(I-t\Gamma^2_{(\alpha)}). \tag{1.8}$$

Edge distributions and KdV. For $0\leq t\leq 1$ let $w(x;t)$ be the unique solution to the Painlevé II equation $w''=2w^3+xw$ that satisfies $w(x;t)\asymp-\sqrt{t}\mathrm{Ai}(x)$ as $x\to\infty$. By the theory of inverse scattering for the concentric Korteweg–de Vries equation, this solution is given by the Fredholm determinant

$$w(x;t)^2=-\frac{\partial^2}{\partial x^2}\log\det(I-t\Gamma^2_{(x)}); \tag{1.9}$$

see [1, 14, pp. 86, 174]. The Tracy–Widom distribution is $\det(I-\Gamma^2_{(x)})$; see [35].

Definition (Jacobi Ensemble) For n be a positive integer, we let

$$\Delta^n=\{(x_j)_{j=1}^n\in\mathbf{R}^n:-1\leq x_1\leq\ldots\leq x_n\leq 1\}$$

and let $\beta>0$, $\nu,\gamma>-1/2$. Then there exists $Z_n<\infty$, which depends upon these constants, such that

$$\mu_n^{(\beta)}(dx)=\frac{1}{Z_n}\prod_{j=1}^n(1+x_j)^{\beta\gamma}(1-x_j)^{\beta\nu}\prod_{1\leq j<k\leq n}(x_k-x_j)^\beta\,dx_1\ldots dx_n \tag{1.10}$$

determines a probability measure on Δ^n. We define the *Jacobi ensemble* of order n with parameters $\nu,\gamma>-1/2$ at inverse temperature $\beta>0$ to be the probability measure $\mu_n^{(\beta)}$. When $\beta=2$, one can regard the $(x_j)_{j=1}^n$ as the ordered eigenvalues of some $n\times n$ Hermitian matrix which is random under a suitable probability measure.

Hard edges. For $0 > \gamma, \nu > -1/2$, the Jacobi ensemble is said to have *hard edges* at ± 1, since the x_j lie in $(-1,1)$ with probability one with respect to $\mu_n^{(\beta)}$ and the density $d\mu_n^{(\beta)}/dx$ diverges to infinity as $x_1 \to (-1)+$ or $x_n \to (+1)-$.

Let J_ν be the Bessel function of order $\nu > -1/2$. Forrester [16] considered the integral operator $F^{a,b}$ on $L^2((0,1), dx)$ with kernel

$$F^{a,b}(x,y) = \mathbf{I}_{(a,b)}(x) \frac{J_\nu(\sqrt{x})\sqrt{y}J'_\nu(\sqrt{y}) - \sqrt{x}J'_\nu(\sqrt{x})J_\nu(\sqrt{y})}{2(x-y)} \mathbf{I}_{(a,b)}(y) \tag{1.11}$$

and conjectured that $F^{a,b}$ determines the limiting distribution of scaled eigenvalues from the Jacobi ensemble near to the hard edge. Using the orthogonal polynomial technique, Forrester and Rains [17] have verified the cases of $\beta = 1, 2$ and 4, following earlier work by Borodin [5] and Dueñez. We introduce the scaled eigenvalues ξ_j by $x_j = \cos \xi_j/\sqrt{n}$, to ensure that the mean spacing of the ξ_j is of order $O(1)$ near to the hard edge at $x_j \approx 1$. One can show that

$$\mu_n^{(2)}[(a,b) \quad \text{contains no} \quad \xi_j] \to \det(I - F^{a,b}) \qquad (n \to \infty). \tag{1.12}$$

For subsequent analysis we change variables by writing $x = e^{-2\xi}$ and $y = e^{-2\eta}$ so that $\xi, \eta \in (0, \infty)$ for $x, y \in (0,1)$. Let G_ℓ be the unitary integral operator on $L^2(\mathbf{R})$ that has kernel $e^{-\ell-\xi-\eta}J_\nu(e^{-\ell-\xi-\eta})$; let $Q_\ell = G_\ell P_+ G_\ell$ ($\ell \in \mathbf{R}$), which gives a strongly continuous family of orthogonal projections. The operator $\Phi_\ell = P_+ G_\ell P_+$ on $L^2(0,\infty)$ is Hilbert–Schmidt, and when $0 < a < 1$ and $\alpha = -(1/2)\log a$ satisfies

$$\det(I - tF^{0,a}) = \det(I - t\Phi^2_{(\alpha)}). \tag{1.13}$$

Linear systems and integrable operators. The operators $W^{\alpha,\beta}_{1/3}$ and $F^{a,b}$ arise via the following theorem, which we prove in section 2. Let R be the reversal map $Rf(x) = f(-x)$, let $f^*(z) = \overline{f(\bar{z})}$; further, $T^\dagger$ denotes the adjoint of T. For $\varepsilon > 0$, let $\Omega : \mathbf{C} \setminus (-\infty, -\varepsilon] \to M_2(\mathbf{C})$ be an analytic matrix function that satisfies

$$\Omega(x) = \Omega(x)^\dagger \qquad (-\varepsilon < x < \infty), \tag{1.14}$$

and

$$\frac{\Omega(z) - \Omega(z)^\dagger}{2i} \geq 0 \qquad (\Im z > 0); \tag{1.15}$$

so that $\langle \Omega(z)\xi, \xi \rangle$ is a Loewner's mapping function for each $\xi \in \mathbf{C}^2$ as in [19, p. 541]. Then there exist analytic functions $\alpha, \beta, \gamma : \mathbf{C} \setminus (-\infty, -\varepsilon] \to \mathbf{C}$ such that

$$\Omega(z) = -\begin{bmatrix} \gamma(z) & \alpha(z) \\ \alpha^*(z) & \beta(z) \end{bmatrix}, \tag{1.16}$$

where by Schwarz's reflection principle $\beta^*(z) = \beta(z)$ and $\gamma^*(z) = \gamma(z)$. We further suppose that $\alpha^*(z) = \alpha(z)$; so that, $\Omega(x)$ is real symmetric for $x \in (-\varepsilon, \infty)$.

Theorem 1.1. *Suppose that A and B are bounded and continuous real functions in $L^2(0,\infty)$ such that $A(x) \to 0$ and $B(x) \to 0$ as $x \to \infty$, and*

$$\frac{d}{dx}\begin{bmatrix} A(x) \\ B(x) \end{bmatrix} = \begin{bmatrix} \alpha(x) & \beta(x) \\ -\gamma(x) & -\alpha(x) \end{bmatrix} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}. \tag{1.17}$$

Then there exists a separable Hilbert space K and $\phi \in L^2((0,\infty); K)$ such that the Hankel operator $\Gamma_\phi : L^2((0,\infty); K) \to L^2(0,\infty)$ is bounded, where

$$\Gamma_\phi g(s) = \int_0^\infty \langle g(t), \phi_{t+s} \rangle_K \, dt \qquad (g \in L^2((0,\infty); K)), \tag{1.18}$$

and $W = \Gamma_\phi \Gamma_\phi^\dagger$ has kernel

$$W(x,y) = \frac{A(x)B(y) - A(y)B(x)}{x-y} = \int_0^\infty \langle \phi_{x+u}, \phi_{y+u} \rangle_K \, du \quad (x, y > 0). \tag{1.19}$$

Theorem 1.1 gives a sufficient condition for W to be the square of a self-adjoint Hankel integral operator by exhibiting the operators involved in Megretskiĭ, Peller and Treil's realization via linear systems, as in [27 p. 245, 30]. Spectral information follows.

Spectral characterization of self-adjoint Hankel operators

Let Γ be a bounded and self-adjoint operator on separable Hilbert space H such that Γ is equivalent to multiplication by λ on the direct integral of Hilbert spaces $H = \int_\oplus H(\lambda)\, \mu(d\lambda)$ where μ is the spectral measure and $\dim H(\lambda) = \nu(\lambda)$ with $\nu(\lambda) \in \{1, 2, \dots\} \cup \{\infty\}$. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition. Then by [27], Γ is unitarily equivalent to a Hankel operator if and only if:

(C1) the nullspace of Γ is zero or infinite-dimensional;

(C2) Γ is not invertible;

(C3) $|\nu(\lambda) - \nu(-\lambda)| \leq 2$ for μ_a-almost all λ, and $|\nu(\lambda) - \nu(-\lambda)| \leq 1$ for μ_s-almost all λ.

Evidently $W = \Gamma^2$ also satisfies $(C1)$ and $(C2)$, while in Propositions 2.3 and 3.2 we deduce further information about the spectrum of W. In section 3, we recall how $\det(I - tW)$ is related to the solutions of Marchenko integral equations.

Hankel operators and invariant subspaces

Burnol proposed that the theory of random matrices should be expressed in terms of Sonine spaces [9, p 692; 10]. As we show in section 4, kernels such as W arise as reproducing kernels for weighted Hardy spaces on the upper half-plane $\mathbf{C}_+ = \{z : \Im z > 0\}$ as in [2, 8]. The classical Hardy space H^2 consists of the analytic functions F on $\mathbf{C}_+$ such that $\sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^2\,dx < \infty$, and we identify such a function with its L^2 boundary values. The Fourier transform is $\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx/\sqrt{2\pi}$. Given $u \in L^\infty$, M_u is the multiplication operator $f \mapsto uf$, and the bounded linear operator $\sqrt{2\pi}\mathcal{F}^\dagger M_u \mathcal{F}^\dagger$ is the Hankel operator Γ_u on $L^2(\mathbf{R}_+)$ with symbol u that has distributional kernel $\phi(x+y) = \mathcal{F}^\dagger u(x+y)$ as in [30].

Definition. Let $(V_t)_{t\geq 0}$ be a C_0 (strongly continuous) semigroup of isometric linear operators on an infinite-dimensional separable Hilbert space H, and let K be a closed linear subspace of H. Then K is *invariant* for $(V_t)_{t\geq 0}$ when $V_t K \subseteq K$ for $t \geq 0$, and *simply invariant* when moreover $\cap_{t\geq 0} V_t K = \{0\}$. Let $(U_t)_{t\in\mathbf{R}}$ be a C_0 group of unitary operators on H. Then K is *doubly invariant* for (U_t) when $U_t K \subseteq K$ for all $t \in \mathbf{R}$.

Let $T_t = e^{-itD}$ $(t \in \mathbf{R})$ be the unitary translation group on $L^2(\mathbf{R})$, where $D = -i\frac{\partial}{\partial x}$, and let U be any unitary on $L^2(\mathbf{R})$ such that $U = T_t U T_t$ for all $t > 0$. Then $R_{(\alpha)} = U^\dagger P_{(\alpha,\infty)} U$ is an orthogonal projection such that the nullspace of $R_{(\alpha)}$ is invariant under $(T_t)_{t>0}$. Further, $\Gamma_{(\alpha)} = P_+ T_{-\alpha} U P_+$ is a Hankel operator such that $W_{(\alpha)} = \Gamma_{(\alpha)}^\dagger \Gamma_{(\alpha)}$ satisfies $W_{(\alpha)} = P_+ R_{(\alpha)} P_+$; the nullspace of $W_{(\alpha)}$ is likewise invariant under $(T_t)_{t>0}$, and the closure of the range of $W_{(\alpha)}$ is invariant under the backward translations $(P_+ T_{-t} P_+)_{t>0}$. Such operators appear in the determinants (1.6) and (1.13). Here $\Gamma_{(\alpha)}$ describes the relative positions of the range of $R_{(\alpha)}$ and $L^2(0,\infty)$. By analogy with prediction theory, we call the range of $R_{(\alpha)}$ the *future subspace*; for comparison, $R_+ = \mathcal{F}^\dagger P_+ \mathcal{F}$ and $R_- = \mathcal{F}^\dagger P_- \mathcal{F}$ are the Riesz projections on L^2 that have images H^2 and $\overline{H^2}$ respectively.

The following table describes analogy between the subspaces and operators in the various cases.

	Classical	Bulk	Soft edge	Hard edge
Future projection	$\mathcal{F}^\dagger P_+ \mathcal{F}$	$\mathcal{F}^\dagger P_{(-a,a)} \mathcal{F}$	$Re^{-iD^3/3} P_+ e^{iD^3/3} R$	$G_\ell P_+ G_\ell$
Future space	H^2	$W_{1/3} L^2$	$D_{a/\pi} L^2$	$Q_\ell L^2$
Subspace position		$e^{i2ax} H^2 \subset H^2$	$e^{itx^3} H^2 \cap H^2 = \{0\}$	$u_\nu \overline{H^2} \cap H^2 \neq \{0\}$
Painlevé equation		σ-$\mathrm{P_V}$	$\mathrm{P_{II}}$	$\mathrm{P_{III}}$
Hankel operator		Ψ_a	$\Gamma_{(0)}$	Φ_ℓ

Position of the Invariant Subspaces

To describe the translation-invariant subspaces, we take Fourier transforms. We recall the shift operators $S_s : f(x) \mapsto e^{isx} f(x)$ as in [21, 23 p. 114]; note that $S_s = \mathcal{F}^\dagger T_s \mathcal{F}$ for $s \in \mathbf{R}$. For simplicity, we write $e^{isx} H^2 = \{e^{isx} f(x) : f \in H^2\}$.

By the Beurling–Lax theorem, a closed linear subspace $\mathcal{T}$ of $L^2(\mathbf{R})$ is simply invariant for $(S_s)_{s\geq 0}$, if and only if there exists a unimodular measurable function u such that

$\mathcal{T} = uH^2$; such a u is uniquely determined up to a unimodular constant factor. For the soft-edge ensemble in section 5 and the hard-edge ensemble in section 6, we start by making unitary transformations to identify u and to determine the relative positions of uH^2 and H^2, namely the nullspace of $R_{(\alpha)}$ and $L^2(0,\infty)$ after transformation. In the case of the hard-edge ensemble, we obtain the subspaces H^2 and $u_\nu H^2$, where

$$u_\nu(x) = 2^{ix}\frac{\Gamma((1+\nu+ix)/2)}{\Gamma((1+\nu-ix)/2)}; \tag{1.20}$$

due to a remarkable identity of Sonine [33], the subspaces are not in general position.

The closure of the range of $W_{(\alpha)}$ is invariant under backward translations, and hence its Fourier image $\mathcal{F}W_{(\alpha)}L^2$ is invariant under the backward shifts. For the bulk of the spectrum, the shifts operate as unitaries on $L^2[-a,a]$ and we obtain the space $K = e^{-iax}H^2 \ominus e^{iax}H^2$ which has $D_{a/\pi}$ as its reproducing kernel for each $a>0$. Generally, either $uH^2\cap H^2 = \{0\}$ or there exist inner functions v and w, uniquely determined up to unimodular constant factors, such that $u = v\bar{w}$, $uH^2 \cap H^2 = vH^2$ and $vwH^2 = vH^2\cap wH^2$. For the soft-edge and hard-edge ensembles, we find $uH^2\cap H^2 = \{0\}$, so we factorize $u(z) = E^*(z)/E(z)$ where E is a meromorphic function on $\mathbf{C}$ that has no zeros. Following de Branges's version of Beurling's theory [8], we introduce the weighted Hardy space EH^2 and show that $W : EH^2 \to EH^2$ is unitarily equivalent to $\Gamma^\dagger_{u^*}\Gamma_{u^*}$ and hence that W is the reproducing kernel of some weighted Hardy spaces of analytic functions inside $\mathbf{C}_+$.

Weyl relations and families of invariant subspaces

Definition. A *Weyl pair* (U_s, V_t) consists of a pair of C_0 unitary groups $(U_s)_{s\in\mathbf{R}}$ and $(V_t)_{t\in\mathbf{R}}$ on H that satisfy $U_sV_t = e^{ist}V_tU_s$ for all $s,t\in\mathbf{R}$.

The shifts $(S_s)_{s\in\mathbf{R}}$ and the translations $(T_t)_{t\in\mathbf{R}}$ give a Weyl pair on L^2; moreover, this is the unique representation of the Weyl relations of multiplicity one on L^2, up to unitary equivalence; see [38]. Katavolos and Power [21] obtained a description of the invariant subspaces for a Weyl pair of multiplicity one.

For the soft-edge ensemble, we show in section 5 that the appropriate Weyl pair consists of e^{isD} and the Schrödinger group $e^{it(D^2+x)}$ where $D = -i\partial/\partial x$. In section 6 we introduce for the Jacobi ensemble an appropriate Weyl pair for the subspaces $Q_\ell L^2$. Borodin *et al.* have emphasized eigenfunction equations in their analysis of integrable kernels in [6]; they refer to *bispectral* properties of kernels. When the kernel of a Hankel operator satisfies an eigenvalue equation, the operator satisfies an intertwining relation with respect to a suitable Weyl pair as in the proof of Theorem 5.4.

In section 7 we extend some of these ideas to a new context, namely the Mathieu functions, which are related to the spheroidal wave functions from [28, p.99]. Here the

KdV equation is 2π-periodic and associated with flows on an infinite-dimensional torus. The results illustrate the scope of the theory of Tracy–Widom operators.

2. Kernels from differential equations and Hankel operators

In this section we prove Theorem 1.1; thus we extend some results concerning Tracy–Widom operators, which are already known in specific cases from [11, 35, 36, 37], and we set them in the general context of linear systems, as in [30, Chapter 11]. Here $\mathrm{B}(H)$, c^2 and c^1 respectively denote the bounded, Hilbert–Schmidt and trace-class linear operators on Hilbert space H, and $T \geq 0$ means that $T \in \mathrm{B}(H)$ is self-adjoint and positive semi-definite.

Lemma 2.1. *Suppose that A and B are bounded, measurable and real functions. Then*

$$W(x,y) = \frac{A(x)B(y) - A(y)B(x)}{x-y} \tag{2.1}$$

defines a self-adjoint and bounded linear operator on $L^2(\mathbf{R})$.

Proof. The Hilbert transform $-i(R_+ - R_-)$ has kernel $1/\pi(x-y)$ and defines a bounded operator on $L^2(\mathbf{R})$; likewise M_A and M_B are bounded, so W is bounded. See also [11], where W is treated as a particular kind of *integrable operator*.

□

Proof of Theorem 1.1. The aim is to find a Hankel operator Γ_ϕ such that $W = \Gamma_\phi \Gamma_\phi^\dagger$ and our technique is to consider a Lyapunov equation [30, p. 502]. We take the usual sesquilinear inner product on $\mathbf{C}^2$ and write

$$A(x)B(y) - A(y)B(x) = \Big\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \begin{bmatrix} A(y) \\ B(y) \end{bmatrix} \Big\rangle, \tag{2.2}$$

and deduce from the differential equation (1.17) that

$$\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) \frac{A(x)B(y) - A(y)B(x)}{x-y} = -\Big\langle \frac{\Omega(x) - \Omega(y)}{x-y} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \begin{bmatrix} A(y) \\ B(y) \end{bmatrix} \Big\rangle. \tag{2.3}$$

By Loewner's theorem [19, p. 541], there exist constant self-adjoint matrices $\Omega_1 \geq 0$ and Ω_0 in $M_2(\mathbf{C})$, and a $M_2(\mathbf{C})$–valued Radon measure ω on (ε, ∞) such that $\omega(a,b) \geq 0$ for $\varepsilon < a < b$ and $\int \|\omega(du)\|/u^2 < \infty$ such that

$$\Omega(z) = \Omega_1 z + \Omega_0 + \int_\varepsilon^\infty \Big(\frac{u}{1+u^2} - \frac{1}{u+z}\Big)\, \omega(du). \tag{2.4}$$

Now

$$\frac{\Omega(x) - \Omega(y)}{x-y} = \Omega_1 + \int_\varepsilon^\infty \frac{1}{(u+x)(u+y)} \omega(du); \tag{2.5}$$

so we introduce the total variation measure $\nu(du) = \|\omega(du)\|_{M_2(\mathbf{C})}$, a Borel-measurable function $w : (\varepsilon, \infty) \to M_2(\mathbf{C})$ such that $\|w(u)\|_{M_2(\mathbf{C})} \leq 1$ and $\omega(du) = w(u)^\dagger w(u)\nu(du)$, and the operator square root $\sqrt{\Omega_1} \geq 0$. Next, we introduce the Hilbert space $K = \mathbf{C}^2 \oplus L^2((0,\infty), d\nu; \mathbf{C}^2)$ and for each $x > 0$ the vector $\phi_x \in K$ by

$$\phi_x(u) = \sqrt{\Omega_1} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix} \oplus \frac{1}{u+x} w(u) \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}; \tag{2.6}$$

then the norm satisfies

$$\begin{aligned} \|\phi_x\|_K^2 &= \Big\| \sqrt{\Omega_1} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix} \Big\|_{\mathbf{C}^2}^2 + \int_\varepsilon^\infty \frac{1}{(u+x)^2} \Big\| w(u) \begin{bmatrix} A(x) \\ B(x) \end{bmatrix} \Big\|_{\mathbf{C}^2}^2 \nu(du) \\ &\leq \Big(\|\Omega_1\|_{M_2(\mathbf{C})} + \int_\varepsilon^\infty \frac{\nu(du)}{u^2} \Big) \Big\| \begin{bmatrix} A(x) \\ B(x) \end{bmatrix} \Big\|_{\mathbf{C}^2}^2. \end{aligned} \tag{2.7}$$

Consequently, $\int_0^\infty \|\phi_x\|_K^2 \, dx < \infty$ holds since $A(x)$ and $B(x)$ belong to $L^2(0,\infty)$, and so $\phi \in L^2((0,\infty); K)$. Further, by (2.5) the vectors satisfy

$$\begin{aligned} \langle \phi_x, \phi_y \rangle_K &= \Big\langle \Omega_1 \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \begin{bmatrix} A(y) \\ B(y) \end{bmatrix} \Big\rangle + \int_\varepsilon^\infty \frac{1}{(u+x)(u+y)} \Big\langle \omega(du) \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \begin{bmatrix} A(y) \\ B(y) \end{bmatrix} \Big\rangle \\ &= \Big\langle \frac{\Omega(x) - \Omega(y)}{x-y} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \begin{bmatrix} A(y) \\ B(y) \end{bmatrix} \Big\rangle. \end{aligned} \tag{2.8}$$

Now from the equations (2.3) and (2.8), we have

$$\frac{A(x)B(y) - A(y)B(x)}{x-y} = \int_0^\infty \langle \phi_{x+u}, \phi_{y+u} \rangle_K \, du + g(x-y) \tag{2.9}$$

where g is some differentiable function; but the left-hand side and the integral converge to zero as $x \to \infty$ or $y \to \infty$; hence $g = 0$. We deduce that the right-hand side of identity (1.19) holds.

By Lemma 2.1, the kernel W of (1.19) defines a bounded linear operator on $L^2(0,\infty)$, and we shall identify W with the operator

$$W = \int_0^\infty T_t^\dagger \Phi \Phi^\dagger T_t \, dt, \tag{2.10}$$

which is known as the *controllability Gramian* [30, p. 469], where $T_t : f(x) \mapsto f(x-t)$ is translation on $L^2(0,\infty)$ and $\Phi \in \mathrm{B}(K, L^2(0,\infty))$ is the operator $\Phi\xi = \langle \xi, \phi \rangle_K \in L^2(0,\infty)$.

Evidently $T_t^\dagger \Phi \Phi^\dagger T_t \geq 0$, and we shall prove that (2.10) converges in the weak operator topology and has kernel $W(x,y)$. The backward translations $(T_t^\dagger)_{t \geq 0}$ form a C_0 contraction semigroup on $L^2(0,\infty)$ which satisfies the stability property $\|T_t^\dagger f\|_{L^2(0,\infty)} \to 0$ as $t \to \infty$.

Since $\phi \in L^2((0,\infty); K)$, the operator Φ is c^2 and one can deduce that $\|\Phi^\dagger T_t f\|_K \to 0$ as $t \to \infty$. So the integrand of (2.10) is strongly continuous and converges to 0 as $t \to \infty$.

Now for $f, g \in L^2(0,\infty)$ the definitions at once give us

$$\begin{aligned}\langle T_t^\dagger \Phi\Phi^\dagger T_t f, g\rangle_{L^2(0,\infty)} &= \langle \Phi^\dagger T_t f, \Phi^\dagger T_t g\rangle_K \\ &= \Big\langle \int_t^\infty f(x-t)\phi_x(u)\,dx, \int_t^\infty g(y-t)\phi_y(u)\,dy\Big\rangle_K \\ &= \int_0^\infty\int_0^\infty \langle \phi_{x+t}, \phi_{y+t}\rangle_K f(x)\bar g(y)\,dxdy, \end{aligned} \tag{2.11}$$

and by integrating we obtain expressions for $\langle Wf, g\rangle_{L^2(0,\infty)}$, namely

$$\int_0^\infty \langle T_t^\dagger \Phi\Phi^\dagger T_t f, g\rangle_{L^2(0,\infty)}\,dt = \int_0^\infty\int_0^\infty\int_0^\infty \langle \phi_{x+t}, \phi_{y+t}\rangle_K f(x)\bar g(y)\,dxdydt. \tag{2.12}$$

Finally, we deduce from (2.12) that $W = \Gamma_\phi \Gamma_\phi^\dagger$, and hence Γ_ϕ defines a bounded linear operator by Lemma 2.1.

□

Corollary 2.2. *Suppose moreover that $\Omega(z) = \Omega_1 z + \Omega_0$ where $\Omega_1 \geq 0$ is a real symmetric matrix of rank one. Then there exists an entire function ϕ with $\phi \in L^2((0,\infty); \mathbf{R})$ such that the Hankel operator Γ_ϕ with kernel $\phi(x+y)$ satisfies $W = \Gamma_\phi^2$; hence*

$$\frac{A(x)B(y) - A(y)B(x)}{x-y} = \int_0^\infty \phi(x+t)\phi(y+t)\,dt. \tag{2.13}$$

Proof. The differential equation (1.17) has coefficients which are entire functions by [18, p. 177], so the solution involves entire functions $A(z)$ and $B(z)$. Let $\lambda > 0$ be the non-zero eigenvalue of Ω_1, and let $\mathrm{col}\,[\cos\theta, \sin\theta]$ be a corresponding eigenvector. Then (2.8) simplifies to the identity

$$\Big\langle \Omega_1 \begin{bmatrix} A(x) \\ B(x)\end{bmatrix}, \begin{bmatrix} A(y) \\ B(y)\end{bmatrix}\Big\rangle = \lambda\big(A(x)\cos\theta + B(x)\sin\theta\big)\big(A(y)\cos\theta + B(y)\sin\theta\big); \tag{2.14}$$

so we can take $K = \mathbf{R}$ and $\phi(z) = \sqrt{\lambda}(A(z)\cos\theta + B(z)\sin\theta)$ so that ϕ is also entire, and the restriction of ϕ to $(0,\infty)$ satisfies (2.13).

□

Proposition 2.3. *Suppose that $W = \Gamma_\phi^2$ where Γ_ϕ is a self-adjoint and bounded Hankel operator, and that λ is an eigenvalue of W with multiplicity $m < \infty$.*

(i) If m is odd, then $\pm\sqrt{\lambda}$ are eigenvalues of Γ_ϕ with multiplicities that differ by one.

(ii) If m is even, then $\pm\sqrt{\lambda}$ are eigenvalues of Γ_ϕ with equal multiplicities.

Proof. This follows immediately from (C3) in the introduction and [27, Theorem 1].

3. Determinants and the Marchenko integral equation

In this section we show how the conclusion of Corollary 2.2 enables us to calculate a determinant as in (1.3), (1.6) and (1.12). We shall not use the differential equation (1.17), but we impose a slightly stronger integrability hypotheses on ϕ to ensure that Fredholm determinants exist.

Lemma 3.1. *Let $\phi : (0,\infty) \to \mathbf{R}$ be continuous and such that $\int_0^\infty u\phi(u)^2\, du \le 1$. Then*

$$W(x,y) = \int_0^\infty \phi(x+t)\phi(t+y)\, dt \tag{3.1}$$

is the kernel of a trace-class operator on $L^2(0,\infty)$ such that, when $|\kappa|<1$,

$$K(x,z) - \kappa^2 \int_x^\infty K(x,y)W(y,z)\, dy = \kappa W(x,z) \tag{3.2}$$

has a solution $K(x,z)$, which is a trace-class kernel, such that

$$\frac{\partial}{\partial x} \log\det(I - \kappa^2 P_{(x,\infty)} W P_{(x,\infty)}) = \kappa K(x,x) \qquad (x>0). \tag{3.3}$$

Proof. See [14, p. 56] for a discussion of Marchenko's integral equation.

□

Proposition 3.2. *Suppose further that ϕ is an entire function such that $\int_0^\infty u|\phi(z+u)|^2\, du < \infty$ for each $z \in \mathbf{C}$, and let $\Gamma_{(z)}$ be the Hankel operator on $L^2(0,\infty)$ that has kernel $\phi(z+s+t)$.*

(i) Then the singular numbers satisfy $s_j(P_{(x,\infty)} W P_{(x,\infty)}) = s_j(\Gamma_{(x)})^2$, and they decrease with increasing $x>0$ for $j=1,2,\dots$.

(ii) The function

$$\psi(z) = \frac{d}{dz} \log\det\big(I - \kappa^2 \Gamma_{(z)}^2\big) \tag{3.4}$$

is meromorphic on $\mathbf{C}$ and satisfies $\psi(x) = \kappa K(x,x)$ for $x \in (0,\infty)$ and $|\kappa|<1$.

(iii) Suppose moreover that $\phi(x) \ge 0$ for all $x \ge 0$. Then $\Gamma_{(0)}$ has a unique positive unit eigenvector in $L^2(0,\infty)$ that corresponds to the eigenvalue $\|\Gamma_{(0)}\| = \|W\|^{1/2}$.

Proof. We have $P_{(x,\infty)} W P_{(x,\infty)} = P_{(x,\infty)} \Gamma_{(0)} \Gamma_{(0)}^\dagger P_{(x,\infty)}$ and $s_j(P_{(x,\infty)}\Gamma_{(0)}) = s_j(\Gamma_{(x)})$ since $\Gamma_{(x)}^\dagger = \Gamma_{(0)}^\dagger T_x$ and there is a unique scale of singular numbers for operators on Hilbert space. So the identity in (i) follows, and the characterization of s_j via approximation numbers shows that these expressions decrease with increasing x; see [19, p. 134, 30, p.705].

(ii) Since $\Gamma_{(z)} \in c^2$ for each z, the spectrum of $\Gamma_{(z)}$ consists of 0 together with non-zero eigenvalues (λ_j), as listed according to algebraic multiplicity, such that $\sum_{j=1}^{\infty} |\lambda_j|^2 < \infty$.

By Morera's theorem, $z \mapsto \Gamma_{(z)}$ defines an entire function with values in c^2, and hence $\det\big(I - \kappa^2\Gamma_{(z)}^2\big)$ defines an entire function. The formula (3.4) defines an analytic function, except at those isolated points where the determinant vanishes, and these give rise to poles. The real poles occur at x_j such that $s_j(\Gamma_{(x_j)})^2 = 1/\kappa^2$, and since $|\kappa|\|\Gamma_{(x)}\|_{B(H)} < 1$ for $x > 0$, there are no poles on $(0, \infty)$.

(iii) Under the stated condition, we have $\langle Wf, f\rangle = \|\Gamma_\phi f\|^2$ and $\|\Gamma_\phi\| = \sup\{\langle \Gamma_\phi f, f\rangle : \|f\|_{L^2} \leq 1\}$; so by positivity and compactness the supremum is attained by some $f \geq 0$. By analyticity, ϕ can vanish only at isolated points and hence $\Gamma_{(0)} f(x) > 0$ for all $x > 0$. Hence $\Gamma_{(0)}$ has eigenvalue $\|\Gamma_{(0)}\|$ with multiplicity one by [39, p. 326].

□

Example. (i) Proposition 3.2 applies in particular to $\phi(z) = \mathrm{Ai}(z)$, as in (1.4) and section 5. In this case, $w = \frac{d}{dx} K(x, x)$ satisfies the Painlevé II equation as in [20, p. 344]; so that $w'' = xw + w^3$. Further, by [14, p. 173]

$$u(x,t) = \frac{2}{(12t)^{2/3}} w\Big(\frac{x}{(12t)^{2/3}}\Big)^2$$

satisfies the concentric Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} + \frac{u}{2t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{3.5}$$

(ii) Likewise the sine kernel (1.2) gives rise to the σ form of $\mathrm{P_V}$, whereas the hard-edge ensemble gives rise to the $\mathrm{P_{III}}$ equation as in [35, 36, 37, 16].

4. Reproducing kernels and the bulk of the spectrum

In this section we provide sufficient conditions for kernels W in (1.19) to be reproducing kernels for weighted Hardy spaces, and verify these for the sine kernel (1.2). Let E be a meromorphic and zero-free function on $\mathbf{C}$ and let $E^*(z) = \overline{E(\bar z)}$, which has similar properties. We also introduce the meromorphic functions $A(z) = (E(z) + E^*(z))/2$ and $B(z) = (E^*(z) - E(z))/(2i)$, which have $A(x)$ and $B(x)$ real for real x. (In some cases A and B satisfy (1.17), but we shall not use this in section 4.)

Let EH^2 be the weighted Hardy space of meromorphic functions g on $\mathbf{C}_+$ such that g/E belongs to the usual Hardy space H^2, and with the inner product

$$\langle g_1, g_2\rangle_{EH^2} = \langle g_1/E, g_2/E\rangle_{H^2} = \int_{-\infty}^{\infty} g_1(t)\bar g_2(t)\frac{dt}{|E(t)|^2}. \tag{4.1}$$

Similarly we can introduce E^*H^2. When $\zeta \in \mathbf{C}_+$ is not a pole of E, the linear functional $g \mapsto g(\zeta)$ is bounded on EH^2, and hence given by $g(\zeta) = \langle g, k_\zeta \rangle_{EH^2}$, where the reproducing kernel is

$$k_\zeta(z) = \frac{E(z)\overline{E(\zeta)}}{2\pi i(\bar{\zeta} - z)}. \tag{4.2}$$

We introduce $\mathcal{D}$ as the domain consisting of points $z \in \mathbf{C}_+$, that are not poles of E or E^*. Let $u(z) = E^*(z)/E(z)$, which is meromorphic on $\mathbf{C}$ and unimodular on the real line; let $M_u : EH^2 \to E^*H^2$ be the isometry $M_u f = uf$; let $\tau_{u^*} : H^2 \to H^2$ be the Toeplitz operator $\tau_{u^*} = R_+ M_{u^*} R_+$; finally, let $\Gamma_{u^*} : H^2 \to \overline{H^2}$ be the Hankel operator $\Gamma_{u^*} = R_- M_{u^*} R_+$.

Theorem 4.1. *(i) The operator $W : EH^2 \to EH^2$ that has kernel*

$$W(z, w) = \frac{A(z)B(\bar{w}) - B(z)A(\bar{w})}{\pi(\bar{w} - z)} \qquad (z, w \in \mathcal{D}) \tag{4.3}$$

is unitarily equivalent to $\Gamma_{u^*}^\dagger \Gamma_{u^*}$.

(ii) There exists a unique Hilbert space $H(W)$ of analytic functions on $\mathcal{D}$ such that $W(z, w)$ is the reproducing kernel for $H(W)$.

(iii) Suppose that τ_{u^*} has a non-zero nullspace K. Then Γ_{u^*} restricts to an isometry $K \to \overline{H^2}$, so $Wf = f$ for all f in some non-zero subspace of EH^2.

Proof. (i) First, one checks that

$$W(z, w) = \frac{E^*(z)\overline{E^*(w)} - E(z)\overline{E(w)}}{2\pi i(z - \bar{w})} \qquad (z, w \in \mathcal{D}). \tag{4.4}$$

Then we write

$$\int_{-\infty}^{\infty} \frac{E^*(z)E(t) - E(z)\overline{E(t)}}{2\pi i(z - t)} \frac{f(t)\,dt}{E(t)\overline{E(t)}} = \frac{E(z)}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)/E(t)}{t - z}\,dt - \frac{E^*(z)}{2\pi i} \int_{-\infty}^{\infty} \frac{u^*(t)f(t)/E(t)}{t - z}\,dt, \tag{4.5}$$

and hence by Cauchy's integral formula we have

$$Wf(z) = E\big(f/E - M_u R_+ M_{u^*}(f/E)\big) = E M_u R_- M_{u^*}(f/E). \tag{4.6}$$

The map $V : EH^2 \to H^2 : f \mapsto f/E$ is a unitary equivalence with adjoint $V^\dagger : g \mapsto Eg$, and $\Gamma_{u^*}^\dagger \Gamma_{u^*} : H^2 \to H^2$ reduces to $\Gamma_{u^*}^\dagger \Gamma_{u^*} = R_+ M_u R_- M_{u^*} R_+$, so $\langle Wf, g \rangle_{EH^2} = \langle V^\dagger \Gamma_{u^*}^\dagger \Gamma_{u^*} V f, g \rangle_{H^2}$ for all $f, g \in EH^2$.

(ii) By (i), W is a positive operator on EH^2, so the kernel $W(z,w)$ is of positive type on $\mathcal{D}$; further, $z \mapsto W(z,w)$ and $w \mapsto W(z,\bar{w})$ are analytic on $\mathcal{D}$. Hence we can apply [2, Theorem 2.3.5] to obtain the Hilbert space of analytic functions such that $W(z,w)$ is the reproducing kernel.

(iii) By [30, p. 89], we have $\tau_{u*}^{\dagger}\tau_{u^*} = I - \Gamma_{u^*}^{\dagger}\Gamma_{u^*}$, which leads directly to the identity $K = \{g \in H^2 : \|\Gamma_{u^*}g\| = \|g\|\}$, so $Wf = f$ for all $f \in V^{\dagger}K$.

□

Corollary 4.2. *Suppose that u belongs to H^∞ so that E^*H^2 is a closed linear subspace of EH^2, and let $K = EH^2 \ominus E^*H^2$ be the orthogonal complement of the range of $M_u : EH^2 \to EH^2$. Then K equals $H(W)$ and has reproducing kernel $K_w(z) = W(z,w)$.*

Proof. We observe that

$$\frac{E^*(z)\overline{E^*(w)}}{2\pi i(z-\bar{w})} = u(z)\frac{E(z)\overline{E^*(w)}}{2\pi i(z-\bar{w})} \tag{4.7}$$

lies in the range of M_u; so for $g \in K$ the proof of Theorem 4.1(i) simplifies to give

$$\langle g, K_w \rangle_{EH^2} = \langle g, k_w \rangle_{EH^2} = g(w) \qquad (w \in \mathcal{D}). \tag{4.8}$$

□

Bulk of the spectrum. Thus when u is an inner function we can identify $H(W)$ explicitly as a subspace of EH^2 that is invariant under the backward shifts. In particular, by taking the entire function $E(z) = e^{-iaz}$, we find $u(z) = e^{2iaz}$ and the reproducing kernel for $K = EH^2 \ominus E^*H^2$ to be

$$K_w(z) = \frac{\sin a(z-\bar{w})}{\pi(z-\bar{w})}, \tag{4.9}$$

as in the sine kernel $D_{a/\pi}(z,w)$ of (1.2). Here we have $EH^2 = \mathcal{F}^*L^2[-a,\infty)$, and $\Psi_a = \mathcal{F}^{\dagger}|L^2[-a,a]$ gives a unitary isomorphism $L^2[-a,a] \to K$ with $\Psi_a\Psi_a^{\dagger} = D_{a/\pi}$. The Hankel operator Γ_{u^*} is isometric on $H^2 \ominus e^{2iax}H^2 \simeq K$.

Let (δ_t) $(t \in \mathbf{R})$ be the unitary dilatation group on $L^2(\mathbf{R})$ with $\delta_t f(x) = e^{t/2}f(e^t x)$. In [22], Katavolos and Power characterize the lattice of closed linear subspaces of L^2 that are simply invariant for both $(S_s)_{s\geq 0}$ and $(\delta_s)_{s\geq 0}$.

Proposition 4.3. *The closed linear subspace D_tL^2 is simply invariant for $(\delta_s)_{s\leq 0}$, doubly invariant for $(T_s)_{s\in\mathbf{R}}$ and invariant under R. Conversely, if K is any closed linear subspace of L^2 that is simply invariant for $(\delta_t)_{t\leq 0}$, doubly invariant for $(T_s)_{s\in\mathbf{R}}$ and invariant under R, then $K = D_aL^2$ for some $a > 0$.*

Proof. We have $\delta_{-s} = \mathcal{F}^\dagger \delta_s \mathcal{F}$ and $T_s = \mathcal{F}^\dagger S_{-s}\mathcal{F}$, so we shall characterize the subspaces $L^2[-\pi t, \pi t]$ under the operation of δ_s, S_s and R. Now $L^2[-t\pi, \pi t]$ is clearly doubly invariant for $(S_s)_{s\in \mathbf{R}}$, and $\delta_s L^2[-\pi t, \pi t] = L^2[-\pi t e^{-s}, \pi t e^{-s}]$; so $L^2[-\pi t, \pi t]$ is simply invariant for $(\delta_s)_{s\geq 0}$. Conversely, all closed linear subspaces $\hat K$ of L^2 that are simply invariant under $(\delta_s)_{s\geq 0}$ and doubly invariant under $(S_s)_{s\in\mathbf{R}}$ have the form $\hat K = L^2(-a, b)$ for some $a, b \in \mathbf{R}\cup\{\infty\}$ by a simple case of Beurling's theorem. When $\hat K$ is additionally invariant under R, we need to have $a = b$; hence $\hat K = L^2[-a, a]$.

5 Soft-edge operators and the Airy group

In this section, we consider the special case of Corollary 2.2 given by the system

$$\frac{d}{dx}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}, \tag{5.1}$$

which has solutions $A(x) = \mathrm{Ai}(x)$ and $B(x) = A'(x)$, so $\phi(z) = A(z)$ and $E(z) = A(z) - iB(z)$ are entire. We shall show that the hypotheses of Corollary 2.2 are satisfied, so we can introduce the Hankel operator with kernel $A(x+y)$ which satisfies Corollary 2.2 and Proposition 3.2. Then we shall consider the invariant subspaces for related operators.

With $D = -i\frac{\partial}{\partial x}$, the Airy group e^{itD^3} is a C_0 group of unitary operators on $L^2(\mathbf{R})$, as defined by

$$e^{itD^3} f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\xi^3 + i\xi x}\,\mathcal{F}f(\xi)\,d\xi. \tag{5.2}$$

Here J_t denotes the operator $e^{itD^3}R$ on $L^2(\mathbf{R})$, not a Bessel function, and we shall use a subscript t to indicate scaling of the space variables x and y with respect to time t.

Lemma 5.1. *The operator $J_t = e^{itD^3}R$ is self-adjoint with $J_t^2 = I$, and J_t as an integral operator on $L^2(\mathbf{R})$ has kernel*

$$\frac{1}{(3t)^{1/3}}\mathrm{Ai}\Big(\frac{x+y}{(3t)^{1/3}}\Big). \tag{5.3}$$

Proof. For any compactly supported and smooth function f we have

$$Re^{itD^3}Rf(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\xi^3 - i\xi x}\mathcal{F}f(\xi)\,d\xi = e^{-itD^3}f(x), \tag{5.4}$$

so $J_t^2 = I$. Further, the kernel of J_t is given by

$$\begin{aligned} e^{itD^3}Rf(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\xi^3 + i\xi x}\int_{-\infty}^{\infty} e^{i\xi y} f(y)\,dy\,d\xi \\ &= \int_{-\infty}^{\infty}\Big\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi^3 t + i\xi(x+y)}\,d\xi\Big\} f(y)\,dy \\ &= \int_{-\infty}^{\infty}\frac{1}{(3t)^{1/3}}\mathrm{Ai}\Big(\frac{x+y}{(3t)^{1/3}}\Big) f(y)\,dy. \end{aligned} \tag{5.5}$$

Since the Airy function on $\mathbf{R}$ is real-valued, it also follows that J_t is self-adjoint.

□

Most of the next result is essentially contained in [35, Lemma 2], but we include a proof for completeness.

Proposition 5.2. *(i) The operator*

$$W_t = e^{itD^3} P_- e^{-itD^3} = J_t P_+ J_t \tag{5.6}$$

on $L^2(\mathbf{R})$ *is an orthogonal projection and the range of* $\mathcal{F} W_t \mathcal{F}^\dagger$ *equals* $e^{it\xi^3} H^2$.

(ii) The Hankel operator $\Gamma_{0,t} = P_+ J_t P_+$ *has square* $\Gamma_{0,t}^2 = P_+ W_t P_+$.

(iii) The kernel of W_t *as an integral operator on* $L^2(\mathbf{R})$ *is*

$$W_t(x,y) = \frac{\mathrm{Ai}(x/(3t)^{1/3})\mathrm{Ai}'(y/(3t)^{1/3}) - \mathrm{Ai}'(x/(3t)^{1/3})\mathrm{Ai}(y/(3t)^{1/3})}{x-y}. \tag{5.7}$$

Proof. (i) By Lemma 5.1, we have $W_t^2 = J_t P_+ J_t^2 P_+ J_t = J_t P_+ J_t = W_t$, so that W_t is a projection; further $W_t^\dagger = W_t$. The range of W_t equals the range of $J_t P_+$.

If $f \in L^2(\mathbf{R}_+)$, then $\mathcal{F} f(\xi) = \bar{G}(\xi)$, where $G \in H^2$. Since e^{-itD^3} is unitary, we have $W_t L^2 = e^{itD^3} P_- e^{-itD^3} L^2 = e^{itD^3} P_- L^2$, and hence the image of WL^2 under the Fourier transform $\mathcal{F}$ is $\mathcal{F} W_t L^2 = \{ e^{it\xi^3} F(\xi) : F \in H^2 \}$.

(ii) We have $\Gamma_{0,t} = P_+ e^{itD^3} R P_+$ and hence

$$\begin{aligned}\Gamma_{0,t}^2 &= P_+ e^{itD^3} R P_+ e^{itD^3} R P_+ \\ &= P_+ e^{itD^3} R P_+ R R e^{itD^3} R P_+ \\ &= P_+ e^{itD^3} P_- e^{-itD^3} P_+ = P_+ W_t P_+. \end{aligned} \tag{5.8}$$

(iii) It also follows from Lemma 5.1 that the kernel function is

$$W_t(x,y) = \frac{1}{(3t)^{2/3}} \int_0^\infty \mathrm{Ai}\Big(\frac{x+u}{(3t)^{1/3}}\Big)\mathrm{Ai}\Big(\frac{u+y}{(3t)^{1/3}}\Big) du, \tag{5.9}$$

a formula which reduces to (5.7) on account of the identity

$$W_{1/3}(x,y) = \int_0^\infty \mathrm{Ai}(x+u)\mathrm{Ai}(u+y)\, du = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}. \tag{5.10}$$

This formula is presented by Tracy and Widom in [35], and follows from Corollary 2.2.

□

Definition. [13] A function $G \in H^2$ is said to be *cyclic* (for the backward shifts) when $\mathrm{span}\{S_t^\dagger G; t > 0\}$ is dense in H^2. Likewise, $f \in L^2(\mathbf{R}_+)$ is *cyclic* when $\mathrm{span}\{T_t^\dagger f : t > 0\}$ is dense in $L^2(\mathbf{R}_+)$; $g \in L^2(\mathbf{R}_-)$ is *cyclic* when $\mathrm{span}\{T_t^\dagger g : t < 0\}$ is dense in $L^2(\mathbf{R}_-)$.

Evidently $W_0 = P_-$, and the relative positions of the ranges of P_- and W_t are described in the following Proposition.

Proposition 5.3. *(i) For each $t \neq 0$, the subspaces $W_t L^2 \cap L^2(\mathbf{R}_-)$ and $(W_t L^2)^\perp \cap L^2(\mathbf{R}_+)$ equal $\{0\}$; while any non-zero vector in $W_t L^2 \cap L^2(\mathbf{R}_+)$ or $(W_t L^2)^\perp \cap L^2(\mathbf{R}_-)$ is cyclic.*

(ii) For each $t > 0$, the operator W_t on $L^2(\mathbf{R}_-) \oplus L^2(\mathbf{R}_+)$ has block matrix form

$$\begin{bmatrix} P_- W_t P_- & P_- W_t P_+ \\ P_+ W_t P_- & P_+ W_t P_+ \end{bmatrix} \in \begin{bmatrix} \mathrm{B} & c^2 \\ c^2 & c^1 \end{bmatrix}. \tag{5.11}$$

(iii) For any real t, the operators $P_+ W_t P_-$ and $P_- W_t P_+$ are Hilbert–Schmidt.

Proof. (i) First we check that $W_t L^2 \cap L^2(\mathbf{R}_-) = \{0\}$, or equivalently by Proposition 5.2(i) that $e^{it\xi^3} H^2 \cap H^2 = \{0\}$. Suppose that $F, G \in H^2$ are non-zero and satisfy $e^{it\xi^3} F(\xi) = G(\xi)$ for almost all $\xi \in \mathbf{R}$. Then $K(\zeta) = e^{it\zeta^3} F(\zeta) - G(\zeta)$ is an analytic function with zero boundary values at almost all points of $\mathbf{R}$; so by the Lusin–Privalov theorem, $K(\zeta)$ is identically zero on $\mathbf{C}_+$. Now by Szegö's Theorem [23, p. 108], the integrals

$$\int_{-\infty}^{\infty} \frac{\log |F(\xi + i\eta)|}{1 + \xi^2} d\xi \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\log |G(\xi + i\eta)|}{1 + \xi^2} d\xi \tag{5.12}$$

converge. But this contradicts the identity $e^{it\zeta^3} F(\zeta) = G(\zeta)$, since

$$t \int_{-\infty}^{\infty} \frac{\Im(\xi + i\eta)^3}{1 + \xi^2} d\xi \tag{5.13}$$

diverges for $\eta, t > 0$; so $F = G = 0$. Likewise the only solution of the equation $e^{it\xi^3} \overline{F(\xi)} = \overline{G(\xi)}$ with $F, G \in H^2$ is $F = G = 0$.

Next we prove that all non-zero vectors in $W_t L^2 \cap L^2(\mathbf{R}_+)$ are cyclic; the case of $(W_t L^2)^\perp \cap L^2(\mathbf{R}_-)$ is similar. Suppose that $G \neq 0$ is a non-cyclic vector in $H^2 \cap \overline{e^{it\xi^3} H^2}$; so that $\overline{G(\xi)} = e^{it\xi^3} F(\xi)$ for some $F \in H^2$, and where G is orthogonal to uH^2 for some inner function u. We have $u\overline{G} \in H^2$; so we introduce inner functions v and w, and an outer function θ, such that $u\overline{G} = v\theta$ and $F = w\theta$. Then, as in [13, Theorem 3.1.1],

$$e^{it\xi^3} = \frac{\overline{G}}{F} = \frac{v}{uw} \tag{5.15}$$

is a quotient of inner functions and hence is of finite Nevanlinna type, but the corresponding logarithmic integral (5.13) diverges, and we have a contradiction. (The author conjectures that $W_tL^2 \cap L^2(\mathbf{R}_+) = \{0\}$ so that W_tL^2 and $L^2(\mathbf{R}_+)$ are in general position, since any non-zero elements in the intersection of the subspaces would satisfy some implausible equations.)

(ii) The Hankel operator $\Gamma_{0,t} = P_+J_tP_+ = P_+e^{itD^3}RP_+$ has kernel

$$\frac{1}{(3t)^{1/3}}\mathbf{I}_{(0,\infty)}(x)\mathrm{Ai}\Big(\frac{x+y}{(3t)^{1/3}}\Big)\mathbf{I}_{(0,\infty)}(y), \tag{5.16}$$

which is of Hilbert–Schmidt type; see [30, p. 46] since we have the bounds from [15, p. 43]

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}\big(1+O(x^{-3/2})\big)\exp\Big(-\frac{2}{3}x^{3/2}\Big) \qquad (x \to \infty). \tag{5.17}$$

Hence the off-diagonal operators $P_-W_tP_+ = P_-J_t(P_+J_tP_+)$ and $P_+W_tP_- = (P_+J_tP_+)J_tP_-$ are Hilbert–Schmidt. For the bottom-right entry, we have a stronger conclusion, namely that $P_+W_tP_+ = (P_+J_tP_+)(P_+J_tP_+)$ is trace class, as in Proposition 3.2.

(iii) When we replace $t \geq 0$ by $t \leq 0$, we need to switch the roles of P_+ and P_- in the previous discussion and we deduce that $P_-W_tP_+$ and $P_+W_tP_-$ are Hilbert–Schmidt, while $P_-W_tP_-$ is of trace class.

□

Theorem 5.4. *(i) The C_0 unitary groups S_s and $U_t = e^{-it(D-x^2)}$ satisfy the von Neumann–Weyl relations $S_sU_t = e^{ist}U_tS_s$ for $s,t \in \mathbf{R}$.*

(ii) For $\alpha \geq 0$ and real δ, the subspace $e^{ix^3/3-i\alpha x^2+i\delta x}H^2$ is simply invariant for $(S_s)_{s\geq 0}$ and $(U_t)_{t\geq 0}$. Conversely, if $\mathcal{T}$ is a non-zero simply invariant subspace for $(S_s)_{s\geq 0}$ and for $(U_s)_{s\geq 0}$, then $\mathcal{T} = e^{ix^3/3-i\alpha x^2+i\delta x}H^2$ for some $\alpha \geq 0$ and real δ.

Proof. (i) One can prove directly that the operators U_t defined by

$$U_tf(x) = e^{i(x^2t-xt^2+t^3/3)}f(x-t) \qquad (s,t \in \mathbf{R}) \tag{5.18}$$

define a C_0 unitary group on $L^2(\mathbf{R})$. Indeed, when f is differentiable, the function $g(x,t) = e^{i(x^3-(x-t)^3)/3}f(x-t)$ satisfies

$$\begin{aligned} \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} &= ix^2g, \\ g(x,0) &= f(x); \end{aligned} \tag{5.19}$$

and so $U_tf(x) = e^{-it(D-x^2)}f(x) = g(x,t)$ gives the unique solution of the initial value problem (5.19) and we recover (5.18) by the method of characteristics.

Let V be the unitary operator $V : f(x) \mapsto e^{ix^3/3} f(x)$ on $L^2(\mathbf{R})$, then clearly $S_s = V^\dagger S_s V$. The generator of the unitary group $V^\dagger U_t V$ equals

$$-iV^\dagger (D - x^2) V = -ie^{-ix^3/3}(D - x^2)e^{ix^3/3} = -\frac{\partial}{\partial x} = -iD; \qquad (5.20)$$

so by the uniqueness of groups with given generator we have $V^\dagger U_s V = e^{-isD} = T_s$ and hence $U_s = Ve^{-isD}V^\dagger = VT_sV^\dagger$. By conjugating the Weyl relations $T_sS_t = e^{-ist}S_tT_s$ for $(s, t \in \mathbf{R})$ by V, we can deduce (i).

(ii) Clearly any $\mathcal{T} = e^{ix^3/3 - i\alpha x^2 + i\delta x} H^2$ is simply invariant under $(S_s)_{s\geq 0}$, and we can use the preceding calculations to show that $\mathcal{T}$ is also invariant for $(U_s)_{s\geq 0}$. Indeed, for $g \in \mathcal{T}$ we can take $f \in H^2$ such that $g(x) = e^{ix^3/3 - i\alpha x^2 + i\delta x} f(x)$ and we have

$$\begin{aligned} U_s g &= U_s(e^{ix^3/3 - i\alpha x^2 + i\delta x} f) \\ &= VT_sV^\dagger V\{e^{-i\alpha x^2 + i\delta x} f\} \\ &= VT_s\{e^{-i\alpha x^2 + i\delta x} f\} \\ &= e^{2i\alpha sx - i\alpha s^2 - i\delta s} e^{ix^3/3 - i\alpha x^2 + i\delta x} f(x - s) \end{aligned} \qquad (5.21)$$

where $f(x - s)$ is an H^2 function; so $U_s g$ belongs to the subspace $e^{i2\alpha sx}\mathcal{T}$ of $\mathcal{T}$. This proves the forward implication.

To prove the converse, we take any $\mathcal{T}$ that is simply invariant as in the Theorem, and observe that $V^\dagger \mathcal{T}$ is simply invariant under $(S_s)_{s\geq 0}$ since $V^\dagger$ commutes with S_s, and $V^\dagger \mathcal{T}$ is also invariant under $(T_s)_{s\geq 0}$ since $T_s V^\dagger \mathcal{T} = V^\dagger U_s \mathcal{T} \subseteq V^\dagger \mathcal{T}$. By the Katavolos–Power Theorem [21], there exist $\alpha > 0$ and a real δ such that $V^\dagger \mathcal{T} = e^{-i\alpha x^2 + i\delta x} H^2$, and hence $\mathcal{T}$ has the required form.

6. Hard-edge operators and Sonine spaces

The formulæ (1.11) and (1.12) are derived from the theory of orthogonal polynomials in [5, 16]. In this section we show how to recover the kernel $F^{a,b}$ in (1.11) from the general theory of sections 2 and 4; thus we deduce information concerning the invariant subspaces of the associated operators. Let J_ν be the Bessel function of the first kind for real $\nu > -1/2$, and let

$$h(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (1 + 2ik) z^k}{2^{\nu+2k}\Gamma(\nu + k + 1)k!} = z^{-\nu/2} J_\nu(\sqrt{z}) + 2iz \frac{d}{dz}\Big(z^{-\nu/2} J_\nu(\sqrt{z})\Big) \qquad (6.1)$$

which is entire and of order 1/2 as in [20, p. 190]. Then $E(z) = 1/h(z)$ is a meromorphic function, with no zeros, such that

$$\frac{E^*(z)\overline{E^*(w)} - E(z)\overline{E(w)}}{2\pi i(z - \bar{w})}$$

$$= \Big(\frac{J_\nu(z^{1/2})\bar w^{1/2}J'_\nu(\bar w^{1/2}) - z^{1/2}J'_\nu(z^{1/2})J_\nu(\bar w^{1/2})}{\pi(z-\bar w)}\Big)\Big(\frac{E(z)E^*(z)\overline{E^*(w)E(w)}}{z^{\nu/2}\bar w^{\nu/2}}\Big). \tag{6.2}$$

We recognise the first factor on the right-hand side from (1.11), and the left-hand side from (4.4); but Corollary 4.2 does not apply directly to $E^*(z)/E(z)$; so we introduce operators that correspond to these kernels indirectly by means of the Hankel transform as in [32, p. 298]. The Hankel transform of $f \in L^2((0,\infty), ydy)$ is

$$\mathcal{H}_\nu\big(f(y); x\big) = \int_0^\infty J_\nu(xy)f(y)\, ydy. \tag{6.3}$$

On $L^2((0,\infty), xdx)$ we introduce the unitary dilatation group $(\tilde\delta_t)$ by $\tilde\delta_t g(x) = e^t g(e^t x)$ and the unitary operator $U : L^2((0,\infty), xdx) \to L^2(\mathbf{R})$ by $Ug(\xi) = e^{-\xi}g(e^{-\xi})$ such that $U^\dagger T_t U = \tilde\delta_t$.

Lemma 6.1. *Let G_ℓ be the integral operator on $L^2(\mathbf{R})$ that has kernel function*

$$e^{-\ell-\xi-\eta}J_\nu(e^{-\ell-\xi-\eta}). \tag{6.4}$$

Then G_ℓ is a self-adjoint and unitary operator such that $G_\ell^2 = I$, and $G_\ell T_t = T_{-t}G_\ell$.

Proof. From the shape of the integral kernel, the identity $G_\ell U = T_{-\ell}U\mathcal{H}_\nu$. is evident. Further, Hankel's inversion formula leads to the identity $\mathcal{H}_\nu^2 = I$, whence to

$$G_\ell U U^\dagger G_\ell = T_{-\ell}U\mathcal{H}_\nu\mathcal{H}_\nu U^\dagger T_\ell = I. \tag{6.5}$$

The identity $G_\ell T_t = T_{-t}G_\ell$ is evident from the definitions, and by (6.5) is equivalent to the scaling property $\mathcal{H}_\nu\tilde\delta_t = \tilde\delta_{-t}\mathcal{H}_\nu$ of the Hankel transform as in [32, p. 299].

□

The following result on position of subspaces contrasts with Proposition 3.2(iii) and Corollary 4.2. Here Γ denotes Euler's gamma function.

Theorem 6.2. *(i) The operator $Q_\ell = G_\ell P_+ G_\ell$ on $L^2(\mathbf{R})$ is an orthogonal projection.*

(ii) The range of $\mathcal{F}Q_\ell\mathcal{F}^\dagger$ equals $e^{i\ell x}u_\nu H^2$, where the meromorphic function

$$u_\nu(z) = 2^{iz}\frac{\Gamma((1+\nu+iz)/2)}{\Gamma((1+\nu-iz)/2)} \tag{6.6}$$

is analytic on $\{z : \Im z < 0\}$, and unimodular and continuous on $\mathbf{R}$.

(iii) Whereas $u_\nu^* H^2 \cap H^2 = \{0\}$, for $\nu > 0$ the subspace $K = (u_\nu\overline{H^2}) \cap H^2$ is non-zero, and $\Gamma_{u_\nu^*} : H^2 \to \overline{H^2}$ restricts to an isometry $K \to \overline{H^2}$.

Proof. (i) This follows directly from Lemma 6.1.

(ii) Our aim is to show that the range of the orthogonal projection $\mathcal{F}^\dagger Q_\ell \mathcal{F}$ is simply invariant under $(S_\lambda)_{\lambda>0}$. By Plancherel's theorem, we have

$$S_\lambda \mathcal{F} Q_\ell L^2 = \mathcal{F} T_{-\lambda} G_\ell P_+ L^2 = \mathcal{F} G_0 T_{\lambda+\ell} P_+ L^2, \tag{6.7}$$

where $T_{\lambda+\ell}P_+L^2 = L^2(\lambda+\ell,\infty) \subseteq L^2(\ell,\infty)$ and $\cap_{\lambda>0} L^2(\lambda,\infty) = \{0\}$. Consequently by Beurling's theorem, there exists a unimodular and measurable function u_ν such that $\mathcal{F}Q_0L^2 = u_\nu H^2$, and u_ν is unique up to a unimodular constant factor. One can easily deduce that $\mathcal{F}Q_\ell L^2 = e^{i\ell x} u_\nu H^2$.

The Fourier conjugate of Q_ℓ is $\mathcal{F}Q_\ell\mathcal{F}^\dagger = \mathcal{F}G_\ell\mathcal{F}^\dagger\mathcal{F}P_+\mathcal{F}^\dagger\mathcal{F}G_\ell\mathcal{F}^\dagger$, wherein we recognise $\mathcal{F}P_+\mathcal{F}^\dagger$ as $R_- : L^2 \to \overline{H^2}$. To determine the range of $\mathcal{F}Q_\ell\mathcal{F}^\dagger$, or equivalently the subspace $\mathcal{F}G_\ell L^2(0,\infty)$, we write

$$\mathcal{F}G_\ell f(x) = \int_{-\infty}^{\infty} e^{-ix\xi} \int_0^\infty e^{-\ell-\xi-\eta} J_\nu(e^{-\ell-\xi-\eta}) f(\eta) d\eta \frac{d\xi}{\sqrt{2\pi}}$$

for $f \in L^2(0,\infty)$, and then reduce this integral by simple transformations to

$$\mathcal{F}G_\ell f(x) = e^{ix\ell}\big(\mathcal{F}^\dagger f(x)\big) \int_{-\infty}^{\infty} e^{-(1+\nu+ix)\xi} e^{\nu\xi} J_\nu(e^{-\xi})\, d\xi. \tag{6.8}$$

The substitution $y = e^{-\xi}$ reduces the final integral in (6.8) to a standard Mellin transform [32, p. 263], and we identify u_ν in the resulting expression

$$\mathcal{F}G_\ell f(x) = e^{ix\ell} \frac{2^{ix}\Gamma((1+\nu+ix)/2)}{\Gamma((1+\nu-ix)/2)} \mathcal{F}^\dagger f(x). \tag{6.9}$$

(iii) Let $E_\nu(z) = e^{-iz\log\sqrt{2}}\Gamma((1+\nu-iz)/2)$; so that, E_ν is meromorphic and zero-free with simple poles at $-i-\nu i-2ki$ for $k=0,1,\dots$, and $u_\nu(z) = E_\nu^*(z)/E_\nu(z)$ has simple zeros at $z_k = -i-\nu i-2ki$ for $k=0,1,\dots$ and simple poles at $i+\nu i+2ki$ for $k=0,1,\dots$. The function $u_\nu(z)$ is analytic in the lower half-plane, but does not define a bounded analytic function on $\{z : \Im z < 0\}$ since the series $\sum_{k=0}^\infty \Im z_k/(1+|z_k|^2)$ diverges, violating Blaschke's condition for the zeros of a non-trivial function in H^∞ or H^2 as in [23, p. 92]. Hence the equations $h_1(z) = u_\nu^*(z)h_2(z)$ with $h_1, h_2 \in H^2$ has only the trivial solution $h_1 = h_2 = 0$; so $u_\nu^* H^2 \cap H^2 = \{0\}$.

We take $a>0$ and $\nu+1/2 > \lambda > 1/2$, and let

$$f(x) = a^{\nu-\lambda+3/2} x^{1/2-\nu}(x^2-a^2)^{(\lambda-1)/2} J_{\lambda-1}\big(a\sqrt{x^2-a^2}\big)\mathbf{I}_{(a,\infty)}(x),$$

with Hankel transform

$$g(t) = t^{1/2}\mathcal{H}_\nu\big(x^{-1/2}f(x);t\big).$$

Then by a result of Sonine [8 p. 301, 31 p. 75, 33 p. 38], both f and g are supported on (a, ∞), and we have

$$\int_a^\infty g(t)t^{-1/2+ix}\,dt = u_\nu(x)\int_a^\infty f(t)t^{-1/2-ix}\,dt \qquad (x \in \mathbf{R}). \tag{6.10}$$

Hence, when $a = 1$, there exist non-zero functions $h_1, h_2 \in H^2$ such that $h_2(x) = u_\nu(x)h_1^*(x)$, so $h_2 \in u_\nu\overline{H^2}$. Now we apply Theorem 4.1(iii) to deduce that $\Gamma_{u_\nu^*}|H^2 \cap u_\nu\overline{H^2}$ is an isometry.

□

Proposition 6.3. *(i) The Hankel operator $\Phi_\ell = P_+G_\ell P_+$ on $L^2(0,\infty)$ has $\Phi_\ell^2 = P_+Q_\ell P_+$.*

(ii) The operator Φ_ℓ on $L^2(0,\infty)$ is Hilbert–Schmidt, and each non-zero $f \in L^2((0,1), xdx)$ such that

$$\lambda f(x) = \int_0^1 J_\nu(\sqrt{sxy})f(y)\,dy \tag{6.11}$$

corresponds to an eigenfunction $g \in L^2(0,\infty)$ of Φ_ℓ with eigenvalue $\frac{1}{2}\lambda\sqrt{s}$.

(iii) The kernel of Q_ℓ as an integral operator on $L^2(\mathbf{R})$ is

$$\frac{e^{-\ell-\xi}J_\nu(e^{-\ell-\xi})e^{-2\ell-2\eta}J_\nu'(e^{-\ell-\eta}) - e^{-2\ell-2\xi}J_\nu'(e^{-\ell-\xi})e^{-\ell-\eta}J_\nu(e^{-\ell-\eta})}{e^{-2\ell-2\xi} - e^{-2\ell-2\eta}}. \tag{6.12}$$

(iv) $\det(I - zF^{0,a}) = \det(I - z\Phi_{(\alpha)}^2)$ *for* $\alpha = -(1/2)\log a$ *and* $a > 0$.

Proof. (i) For $t > 0$, we have the Hankel condition $\Phi_\ell T_t = T_t^\dagger \Phi_\ell$, where here $T_t : L^2(0,\infty) \to L^2(0,\infty)$. Then one uses Theorem 6.2(i).

(ii) The kernel function is clearly symmetric, real valued and square integrable, since

$$\int_0^\infty\int_0^\infty e^{-2(\ell+\eta+\xi)}J_\nu(e^{-(\ell+\eta+\xi)})^2\,d\xi d\eta = \int_0^\infty ue^{-2\ell-2u}J_\nu(e^{-\ell-u})^2\,du < \infty \tag{6.13}$$

due to the asymptotic formula $J_\nu(x) \asymp x^\nu/\Gamma(\nu+1)$ as $x \to 0+$. Hence Φ_ℓ gives a self-adjoint operator of Hilbert–Schmidt type. The operator U restricts to a unitary $L^2((0,1), xdx) \to L^2(0,\infty)$, and under this transformation the eigenfunction equations correspond via $g(\xi) = e^{-\xi}f(e^{-2\xi})$.

(iii) We use the method of proof of Theorem 1.1 to verify the stated formula for $Q_\ell = G_\ell P_+ G_\ell$, which is the square of a self-adjoint Hankel operator on $L^2(0,\infty)$. With $A(\xi) = e^{-\xi}J_\nu(e^{-\xi})$ and $B(\xi) = e^{-2\xi}J_\nu'(e^{-\xi})$, we have

$$\frac{d}{d\xi}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ (e^{-2\xi} - \nu^2) & -1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} \tag{6.14}$$

where

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ (e^{-2\xi} - \nu^2) & -1 \end{bmatrix} + \begin{bmatrix} -1 & (e^{-2\eta} - \nu^2) \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} e^{-2\eta} - e^{-2\xi} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \tag{6.15}$$

hence

$$\Bigl(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\Bigr) \frac{A(\xi)B(\eta) - A(\eta)B(\xi)}{e^{-2\xi} - e^{-2\eta}} = \Bigl(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\Bigr) \int_0^\infty A(\xi + u)A(\eta + u)\, du,$$

which leads to the required result. Alternatively, one can transform a formula in [32, p. 303].

(iv) The unitary equivalence between $L^2((0,1), dx)$ and $L^2(0,\infty)$ involves $g(x) \mapsto \sqrt{2}e^{-\xi} g(e^{-2\xi})$, so $F^{(0,1)}$ is unitarily equivalent to the operator that has kernel

$$2e^{-\xi-\eta} F^{(0,1)}(e^{-2\xi}, e^{-2\eta}) = \frac{e^{-\xi} J_\nu(e^{-\xi}) e^{-2\eta} J_\nu'(e^{-\eta}) - e^{-2\xi} J_\nu'(e^{-\xi}) e^{-\eta} J_\nu(e^{-\eta})}{e^{-2\xi} - e^{-2\eta}},$$

which we recognise as the kernel of $\Phi^2_{(0)}$. Comparing the spectra of the compressions to $L^2(0,a)$ and $L^2(\alpha,\infty)$, we deduce that

$$\det(I - zF^{0,a}) = \det(I - zP_{(\alpha,\infty)}\Phi^2_{(0)}P_{(\alpha,\infty)}) = \det(I - z\Phi_{(0)}P_{(\alpha,\infty)}\Phi_{(0)}). \tag{6.16}$$

Finally, $\Phi_{(0)}P_{(\alpha,\infty)}\Phi_{(0)}$ equals $\Phi^2_{(\alpha)}$ since they both have kernel

$$\int_\alpha^\infty e^{-\xi-u} J_\nu(e^{-\xi-u}) e^{-\eta-u} J_\nu(e^{-\eta-u})\, du. \tag{6.17}$$

□

Theorem 6.4. *Let L be the operator*

$$Lf(\xi) = -\frac{\partial}{\partial \xi}\Bigl(e^{2\xi}\frac{\partial f}{\partial \xi}\Bigr) + (\nu^2 - 1)e^{2\xi} f(\xi). \tag{6.18}$$

(i) Then L is an essentially self-adjoint and positive operator on $C_c^\infty(\mathbf{R})$ in $L^2(\mathbf{R})$, so that $V_t = e^{-it\ell}L^{-it/2}$ $(t \in \mathbf{R})$ defines a C_0 group of unitary operators on $L^2(\mathbf{R})$.

(ii) The unitary groups $(V_s)_{s\in\mathbf{R}}$ and $(T_t)_{t\in\mathbf{R}}$ satisfy $V_sT_t = e^{ist}T_tV_s$ for $s,t \in \mathbf{R}$.

(iii) The subspace $Q_\ell L^2$ is doubly invariant for $(V_s)_{s\in\mathbf{R}}$ and simply invariant for $(T_{-t})_{t\geq 0}$. Conversely, if K is a non-trivial closed linear subspace of L^2 that is simply invariant for $(T_{-t})_{t\geq 0}$ and doubly invariant for $(V_s)_{s\in\mathbf{R}}$, then $K = Q_\alpha L^2$ for some real α.

Proof. (i) The simplest way of proving that the operator L is self-adjoint is to compute its spectral resolution. By simple transformations of the Bessel equation [20, p. 171], we have

$$-e^{2\xi}\Big(\frac{\partial^2}{\partial\xi^2}+2\frac{\partial}{\partial\xi}+\nu^2-1\Big)\big(e^{-\xi-\ell-\eta}J_\nu(e^{-\xi-\ell-\eta})\big)=e^{-2\ell-2\eta}\big(e^{-\xi-\ell-\eta}J_\nu(e^{-\xi-\ell-\eta})\big), \quad (6.19)$$

so that $e^{-\xi-\ell-\eta}J_\nu(e^{-\xi-\ell-\eta})$ is an eigenfunction of L corresponding to the eigenvalue $e^{-2\ell-2\eta}>0$. By Hankel's inversion theorem [32, p. 299], the functions $\lambda yJ_\nu(\lambda xy)$ give a complete spectral family in $L^2((0,\infty);xdx)$, and the unitary transformation U takes $\lambda yJ_\nu(\lambda xy)$ to $e^{-\xi-\ell-\eta}J_\nu(e^{-\xi-\ell-\eta})$ after an obvious change of variable. By Stone's theorem, $(-i/2)\log L$ generates a C_0 unitary group $L^{-is/2}$.

(ii) We have the intertwining relation

$$\begin{aligned} V_sG_\ell f(\xi) &= e^{-is\ell}L^{-is/2}\int_{-\infty}^{\infty} e^{-\xi-\eta-\ell}J_\nu(e^{-\xi-\ell-\eta})f(\eta)\,d\eta \\ &= \int_{-\infty}^{\infty} e^{is\eta}e^{-\xi-\ell-\eta}J_\nu(e^{-\xi-\ell-\eta})f(\eta)\,d\eta \\ &= G_\ell S_s f(\xi); \end{aligned} \quad (6.20)$$

hence $G_\ell V_sG_\ell=S_s$. When we conjugate the relation $T_{-t}S_s=e^{ist}S_sT_{-t}$ by G_ℓ we obtain $G_\ell T_{-t}G_\ell G_\ell S_sG_\ell=e^{ist}G_\ell S_sG_\ell G_\ell T_{-t}G_\ell$ or $T_tV_s=e^{ist}V_sT_t$.

(iii) From earlier relations, we have

$$V_sQ_\ell=V_sG_\ell P_+G_\ell=G_\ell S_sP_+=G_\ell P_+S_s=G_\ell P_+G_\ell G_\ell S_s=Q_\ell G_\ell S_s, \quad (6.21)$$

which shows that the range of Q_ℓ is mapped onto itself by V_s; further

$$T_{-t}Q_\ell=T_{-t}G_\ell P_+G_\ell=G_\ell T_tP_+G_\ell=G_\ell P_{[t,\infty)}T_tG_\ell=G_\ell P_{[t,\infty)}G_\ell T_{-t}, \quad (6.22)$$

has range contained in the range of Q_ℓ for $t>0$, so $Q_\ell L^2$ is simply invariant.

To obtain the converse, we consider the Fourier transforms of the groups. On $e^{i\ell x}u_\nu H^2$, the unitary semigroups operate as

$$\hat{V}_s=\mathcal{F}V_s\mathcal{F}^\dagger : f(x)\mapsto e^{i\ell s}u_\nu(x)u_\nu(s-x)f(x-s) \qquad (s\in\mathbf{R}); \quad (6.23)$$

further, $\mathcal{F}T_{-t}\mathcal{F}^\dagger=S_t$. To verify (6.23), we recall the reversal map R by $Rf(x)=f(-x)$, and observe that $\mathcal{F}\mathcal{F}=R$ and $\mathcal{F}^\dagger\mathcal{F}^\dagger=R$. We have

$$\mathcal{F}V_s\mathcal{F}^\dagger=\mathcal{F}G_\ell S_sG_\ell\mathcal{F}^\dagger=S_\ell M_{u_\nu}\mathcal{F}^\dagger S_s\mathcal{F}\mathcal{F}^\dagger G_\ell\mathcal{F}^\dagger \quad (6.24)$$

so that $\mathcal{F}V_s\mathcal{F}^\dagger = S_\ell M_{u_\nu} T_s R S_\ell M_{u_\nu} R$. Using the von Neumann–Weyl relation for T_s and S_ℓ, one can easily simplify this expression to obtain $\hat{V}_s = \mathcal{F}V_s\mathcal{F}^\dagger = e^{i\ell s} M_{u_\nu} N_s T_s$, where $N_s f(x) = u_\nu(s-x)f(x)$. The functions u_ν satisfy $u_\nu(-x)u_\nu(x) = 1$ and

$$e^{i\ell s}u_\nu(x)u_\nu(s-x) = e^{i\ell s}2^{is}\frac{\Gamma((1+\nu+ix)/2)\Gamma((1+\nu+is-ix)/2)}{\Gamma((1+\nu-ix)/2)\Gamma((1+\nu+ix-is)/2)}. \tag{6.25}$$

Suppose that K is such an invariant subspace. Then by Beurling's theorem, there exists a unimodular and measurable function w such that $\mathcal{F}K = wH^2$; further, this w is uniquely determined up to a unimodular constant multiple. We apply $\mathcal{F}V_s\mathcal{F}^\dagger$ to this identity, and deduce by double invariance and (6.23) that

$$\{e^{i\ell s}u_\nu(x)u_\nu(s-x)w(x-s)f(x-s) : f \in H^2\} = wH^2; \tag{6.26}$$

so that,

$$e^{i\ell s}u_\nu(x)u_\nu(s-x)w(x-s) = c(s)w(x) \qquad (s \in \mathbf{R})$$

holds for some $c(s)$. We re-arrange this to $u_\nu(s-x)w(x-s) = u_\nu(-x)w(x)e^{-i\ell s}c(s)$, then solve to obtain $w(x) = e^{i\alpha x}u_\nu(x)$ for some $\alpha \in \mathbf{R}$; see [21] for details. Hence $\mathcal{F}K = e^{i\alpha x}u_\nu(x)H^2$, so $K = Q_\alpha L^2$ by Theorem 6.2(ii).

7. Mathieu functions and periodic potentials

In this section we construct Tracy–Widom type operators over the circle that are naturally related to random matrix ensembles and to the Korteweg–de Vries equation. The examples are based on Mathieu's equation, as in [20, p. 175], and go beyond the list in [37].

Definition. (Coulomb gas) Suppose that v is a real polynomial of degree $2m > 0$ that has positive leading term. For $\beta > 0$ and $n \in \mathbf{N}$, there exists $0 < Z < \infty$ such that

$$\sigma_n^{(v)}(dx) = Z^{-1}\exp\Bigl(-n\beta\sum_{j=1}^{n} v(x_j) + \beta\sum_{1\le j<k\le n}\log(x_j - x_k)\Bigr)dx_1\dots dx_n \tag{7.1}$$

defines a probability measure on $\{(x_j)_{j=1}^n : x_1 \le \dots \le x_n\}$. We then define the *joint distribution of the Coulomb gas* of n particles at inverse temperature β to be $\sigma_n^{(v)}$. For $\beta = 2$, $\sigma_n^{(v)}$ gives the joint eigenvalue distribution for matrices from the generalized unitary ensemble with potential v; see [28].

As $n \to \infty$, the x_j tend to accumulate near to the local minima of v. Boutet de Monvel *et al* [7] have shown that there exists a probability density function p_v of compact support S that is uniquely determined by the condition

$$v(x) \ge \int_S \log|x-y|\,p_v(y)\,dy + C \qquad (x \in \mathbf{R}) \tag{7.2}$$

for some constant C with equality on S. This p_v is called the *equilibrium distribution* of v, and by Theorem 1 of [7]

$$\sigma_n^{(v)}\Big\{(x_j)_{j=1}^n : \Big|\frac{1}{n}\sum_{j=1}^n f(x_j) - \int_S f(x)p_v(x)\,dx\Big| \geq \varepsilon\Big\} \to 0 \qquad (n\to\infty) \tag{7.3}$$

for all $\varepsilon > 0$ and bounded and continuous $f : \mathbf{R} \to \mathbf{R}$. For example, when $v(x) = x^2/4$ and $\beta = 2$, $\sigma_n^{(v)}$ gives the eigenvalue distribution of GUE and we obtain (1.1).

Generally by [12, p.408], there exist $k \leq m+1$ and $\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \ldots < \lambda_{2k-1}$ such that $S = \cup_{j=0}^{k-1}[\lambda_{2j}, \lambda_{2j+1}]$. Let $R(z) = \prod_{j=0}^{k-1}(z-\lambda_{2j})(z-\lambda_{2j+1})$ and choose branches of square roots so that $\sqrt{R(z)}$ is analytic on $\mathbf{C}\setminus S$, and $\sqrt{R(z)}/z^k = 1+O(1/z)$ as $|z| \to \infty$. Then by [29, p 252; 12]

$$p_v(x) = p.v.\frac{\sqrt{R(x)}}{\pi^2}\int_S \frac{v'(t)\,dt}{\sqrt{R(t)}(x-t)} \qquad (x \in S). \tag{7.4}$$

The Riemann surface for $\sqrt{R(z)}$ consists of a handlebody with $k-1$ handles, which is obtained by taking two copies of $\mathbf{C} \cup \{\infty\}$ and joining them crosswise along the cuts $[\lambda_{2j}, \lambda_{2j+1}]$. Hence it is natural to regard $[\lambda_{2j}, \lambda_{2j+1}]$ as the projection onto the real axis of a circle, and to consider the curve $w^2 = R(z)$.

We now introduce an analogous situation in which $k = \infty$, and introduce Tracy–Widom type operators in this context. Let Φ be the 2×2 fundamental solution matrix of Hill's equation with smooth π-periodic potential q, so that

$$\frac{d}{dx}\Phi = \begin{bmatrix} 0 & 1 \\ -(\lambda+q(x)) & 0 \end{bmatrix}\Phi, \qquad \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \tag{7.5}$$

then $\det\Phi = 1$, and the discriminant is $\Delta(\lambda) = \operatorname{trace}\Phi(\pi)$. When λ is real, evidently $\Delta(\lambda)^2 \geq 4$ if and only if the eigenvalues of $\Phi(\pi)$ are real, and $\Delta(\lambda)^2 = 4$ occurs if and only if Φ is periodic with period π or 2π. The periodic spectrum

$$\Lambda = \{\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \ldots < \lambda_n \nearrow \infty\} \tag{7.6}$$

consists of those real λ such that Hill's equation

$$y'' + (\lambda+q)y = 0 \tag{7.7}$$

has a non-trivial π or 2π-periodic solution as in [24, p. 11]. The discriminant satisfies

$$4 - \Delta^2(\lambda) = 4(\lambda-\lambda_0)\prod_{j=1}^\infty \frac{(\lambda_{2j-1}-\lambda)(\lambda_{2j}-\lambda)}{j^4}, \tag{7.8}$$

so $4 - \Delta^2(\lambda)$ is analogous to $R(\lambda)$.

Theorem 7.1. *(i) For each real α, there exists an infinite sequence of λ_n such that Hill's equation (7.7) with $q(x) = \alpha \cos 2x$ has a non-trivial 2π-periodic and real solution A_α.*

(ii) For such A_α, let W_α be the kernel

$$W_\alpha(x, y) = \frac{A_\alpha(x)A'_\alpha(y) - A'_\alpha(x)A_\alpha(y)}{\sin(x - y)}. \tag{7.9}$$

Then W_α is continuously differentiable and doubly periodic with period 2π. Further, W_α defines a self-adjoint and Hilbert–Schmidt operator on $L^2[0, 2\pi]$ and the eigenfunction corresponding to each non-zero simple eigenvalue of W_α is a 2π-periodic solution of (7.7).

Proof. (i) When $\alpha = 0$ and $\lambda = n^2$ with $n = 1, 2, \ldots$, we can take $A_0(x) = \sin nx$, and recover the kernel

$$W_0(x, y) = \frac{n \sin n(x - y)}{\sin(x - y)} \tag{7.10}$$

as in the circular ensemble from [28, p. 195]. (Observe that $W_\alpha(x, y) \to W_0(x, y)$ as $\alpha \to 0$.)

When $\alpha \neq 0$, there exists by Hochstadt's theorem [24, p. 40] an increasing sequence (λ'_n) which satisfies the estimates

$$\lambda'_{2n-1} = (2n - 1)^2 + \frac{\alpha^2}{32n^2} + o(n^{-2}), \qquad (n \to \infty) \tag{7.11}$$

$$0 < \lambda'_{2n} - \lambda'_{2n-1} = o(n^{-2}),$$

and such that, for each λ'_n, (7.7) has a non-trivial solution A_α, namely Mathieu's function of the first kind.

(ii) Evidently W_α is a real, symmetrical and continuous kernel, and hence determines a self-adjoint and Hilbert–Schmidt operator on $L^2[0, 2\pi]$.

By differentiating (7.9) and recalling the definition of A_α, one can easily deduce that

$$\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) W_\alpha(x, y) = -2\alpha(\sin x \cos y + \cos x \sin y) A_\alpha(x) A_\alpha(y), \tag{7.12}$$

an identity which is analogous to (2.3), so

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\Big) W_\alpha(x, y) = \alpha(\cos 2x - \cos 2y) W_\alpha(x, y). \tag{7.13}$$

For $\nu \neq 0$, any non-zero solution $f \in L^2[0, 2\pi]$ of the integral equation

$$\nu f(x) = \int_0^{2\pi} W_\alpha(x, y) f(y)\, dy \tag{7.14}$$

extends to define a twice continuously differentiable and 2π-periodic function on $\mathbf{R}$. Now $g(x) = f''(x) + \alpha \cos 2x f(x)$ also gives a 2π-periodic and continuous solution of (7.7); this follows from (7.12) by an integration-by-parts argument. By simplicity of the eigenvalue, we deduce that g is a constant multiple of f, and hence that f is a 2π-periodic solution of Mathieu's equation.

□

Remarks (i) Conversely, let $\mathcal{M}_\Lambda$ be the space of potentials q such that (7.7) has periodic spectrum equal to a given Λ. McKean, van Moerbeke and Trubowitz [25, 26] have shown that $\mathcal{M}_\Lambda$ can be considered as a torus

$$\mathcal{M}_\Lambda = \Big\{ \frac{1}{2}\Big(\Delta(x_j) + \sqrt{\Delta(x_j)^2 - 4}\Big)_{j=1}^{\infty} : \lambda_{2j-1} \leq x_j \leq \lambda_{2j}; j = 1, 2, \dots \Big\} \tag{7.15}$$

over the product over the intervals of instability $(\lambda_{2j-1}, \lambda_{2j})$ where $\Delta(\lambda)^2 < 4$ and that $\mathcal{M}_\Lambda$ is associated with the Jacobi manifold over the Riemann surface of $\sqrt{\Delta^2(\lambda) - 4}$. Hence $\mathcal{M}_\Lambda$ can have dimension $n = 0, 1, \dots, \infty$, equal to the number of simple zeros of $\Delta(\lambda)^2 - 4$. The periodic spectrum Λ is preserved by Hamiltonian flows; in particular, there is a 2π-periodic Korteweg–de Vries flow on $\mathcal{M}_\Lambda$ associated with

$$\frac{\partial u}{\partial t} = 3u \frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial^3 u}{\partial x^3}. \tag{7.16}$$

(ii) By Theorem 7.1, the potential $\alpha \cos 2x$ gives an infinite-dimensional $\mathcal{M}_\Lambda$ on which there are solutions to KdV that are 2π-periodic in the space variable and almost periodic in time [4, Appendix].

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