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Finite and Infinitesimal Rigidity with Polyhedral Norms

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Abstract We characterise finite and infinitesimal rigidity for bar-joint frameworks in $\mathbb{R}^d$ with respect to polyhedral norms (i.e. norms with closed unit ball $\mathcal{P}$, a convex d-dimensional polytope). Infinitesimal and continuous rigidity are shown to be equivalent for finite frameworks in $\mathbb{R}^d$ which are well-positioned with respect to $\mathcal{P}$. An edge-labelling determined by the facets of the unit ball and placement of the framework is used to characterise infinitesimal rigidity in $\mathbb{R}^d$ in terms of monochrome spanning trees. An analogue of Laman's theorem is obtained for all polyhedral norms on $\mathbb{R}^2$.

Keywords Bar-joint framework · Infinitesimally rigid · Laman's theorem · Polyhedral norm

Mathematics Subject Classification 52C25 · 52A21 · 52B12

1 Introduction

A bar-joint framework in $\mathbb{R}^d$ is a pair (G, p) consisting of a simple undirected graph $G = (V(G), E(G))$ (i.e. no loops or multiple edges) and a placement $p : V(G) \to \mathbb{R}^d$ of the vertices such that p_v and p_w are distinct whenever vw is an edge of G. The graph

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G may be either finite or infinite. Given a norm on $\mathbb{R}^d$ we are interested in determining when a given framework can be continuously and non-trivially deformed without altering the lengths of the bars. A well-developed rigidity theory exists in the Euclidean setting for finite bar-joint frameworks (and their variants), which stems from classical results of Cauchy [6], Maxwell [17], Alexandrov [1] and Laman [14]. Of particular relevance is Laman's landmark characterisation for generic minimally infinitesimally rigid finite bar-joint frameworks in the Euclidean plane. Asimow and Roth proved the equivalence of finite and infinitesimal rigidity for regular bar-joint frameworks in two key papers [2,3]. A modern treatment can be found in works of Graver et al. [9] and Whiteley [24,26]. More recently, significant progress has been made in topics such as global rigidity [7,8,11] and the rigidity of periodic frameworks [5,16, 20,21] in addition to newly emerging themes such as symmetric frameworks [22] and frameworks supported on surfaces [19]. In this article, we consider rigidity properties of both finite and infinite bar-joint frameworks (G, p) in $\mathbb{R}^d$ with respect to polyhedral norms. A norm on $\mathbb{R}^d$ is polyhedral (or a block norm) if the closed unit ball $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$ is the convex hull of a finite set of points. Such norms form an important class as they are computationally easy to use and are dense in the set of all norms on $\mathbb{R}^d$. While classical rigidity theory is strongly linked to statics, it has also provided valuable new connections between different areas of pure mathematics and this latter property is one of the emerging features of non-Euclidean rigidity theory. In particular, the rigidity theory obtained with polyhedral norms is distinctly different from the Euclidean setting in admitting new edge-labelling and spanning tree methods. There are potential applications of this theory to physical networks with inherent directional constraints, or to abstract networks with a suitable notion of distance imposed. Non-Euclidean norms, and in particular polyhedral norms, have been applied in this way to optimisation problems in location modelling (see the industry which has resulted from [23]) and, more recently, machine learning with submodular functions [4]. A study of rigidity with respect to the classical non-Euclidean ℓ^p norms was initiated in [12] for finite bar-joint frameworks and further developed for infinite bar-joint frameworks in [13]. Among these norms the ℓ^1 and ℓ^∞ norms are simple examples of polyhedral norms and so the results obtained here extend some of the results of [12].

In Sect. 2, we provide the relevant background material on polyhedral norms and finite and infinitesimal rigidity. In Sect. 3, we establish the role of support functionals in determining the space of infinitesimal flexes of a bar-joint framework (Theorem 5). We then distinguish between general bar-joint frameworks and those which are well-positioned with respect to the unit ball. The well-positioned placements of a finite graph are open and dense in the set of all placements, and we show that finite and infinitesimal rigidity are equivalent for these bar-joint frameworks (Theorem 7). We then introduce the rigidity matrix for a general finite bar-joint framework, the non-zero entries of which are derived from extreme points of the polar set of the unit ball. In Sect. 4, we apply an edge-labelling to G which is induced by the placement of each bar in $\mathbb{R}^d$ relative to the facets of the unit ball. With this edge-labelling we identify necessary conditions for infinitesimal rigidity and obtain a sufficient condition for a subframework to be relatively infinitesimally rigid (Proposition 12). We then characterise the infinitesimally rigid bar-joint frameworks with d induced framework colours as those which contain monochrome spanning trees of each framework colour

(Theorem 13). This result holds for both finite and infinite bar-joint frameworks and does not require the framework to be well-positioned. In Sect. 5, we apply the spanning tree characterisation to show that certain graph moves preserve minimal infinitesimal rigidity for any polyhedral norm on $\mathbb{R}^2$. We then show that in two dimensions a finite graph has a well-positioned minimally infinitesimally rigid placement if and only if it satisfies the counting conditions $|E(G)| = 2|V(G)| - 2$ and $|E(H)| \leq 2|V(H)| - 2$ for all subgraphs H (Theorem 23). This is an analogue of Laman's theorem [14] which characterises the finite graphs with minimally infinitesimally rigid generic placements in the Euclidean plane as those which satisfy the counting conditions $|E(G)| = 2|V(G)| - 3$ and $|E(H)| \leq 2|V(H)| - 3$ for subgraphs H with at least two vertices. Many of the results obtained hold equally well for both finite and infinite bar-joint frameworks.

2 Preliminaries

Let $\mathcal{P}$ be a convex symmetric d-dimensional polytope in $\mathbb{R}^d$ where $d \geq 2$. Following [10] we say that a proper face of $\mathcal{P}$ is a subset of the form $\mathcal{P} \cap H$, where H is a supporting hyperplane for $\mathcal{P}$. A facet of $\mathcal{P}$ is a proper face which is maximal with respect to inclusion. The set of extreme points (vertices) of $\mathcal{P}$ is denoted by $\text{ext}(\mathcal{P})$. The polar set of $\mathcal{P}$, denoted by $\mathcal{P}^\triangle$, is also a convex symmetric d-dimensional polytope in $\mathbb{R}^d$:

$$\mathcal{P}^\triangle = \{y \in \mathbb{R}^d : x \cdot y \leq 1 \text{ for all } x \in \mathcal{P}\}. \tag{1}$$

Moreover, there exists a bijective map which assigns to each facet F of $\mathcal{P}$ a unique extreme point $\hat{F}$ of $\mathcal{P}^\triangle$ such that

$$F = \{x \in \mathcal{P} : x \cdot \hat{F} = 1\}. \tag{2}$$

The polar set of $\mathcal{P}^\triangle$ is $\mathcal{P}$.

The Minkowski functional (or gauge) for $\mathcal{P}$ defines a norm on $\mathbb{R}^d$,

$$\|x\|_\mathcal{P} = \inf\{\lambda \geq 0 : x \in \lambda\mathcal{P}\}.$$

This is what is known as a polyhedral norm or a block norm. The dual norm of $\|\cdot\|_\mathcal{P}$ is also a polyhedral norm and is determined by the polar set $\mathcal{P}^\triangle$,

$$\|y\|^*_\mathcal{P} = \max_{x\in\mathcal{P}} x \cdot y = \inf\{\lambda \geq 0 : y \in \lambda\mathcal{P}^\triangle\} = \|y\|_{\mathcal{P}^\triangle}.$$

In general, a linear functional on a convex polytope will achieve its maximum value at some extreme point of the polytope and so the polyhedral norm $\|\cdot\|_\mathcal{P}$ is characterised by

$$\|x\|_{\mathcal{P}} = \|x\|_{\mathcal{P}}^{**} = \|x\|_{\mathcal{P}^\triangle}^{*} = \max_{y \in \mathcal{P}^\triangle} x \cdot y = \max_{y \in \text{ext}(\mathcal{P}^\triangle)} x \cdot y. \tag{3}$$

A point $x \in \mathbb{R}^d$ belongs to the conical hull cone(F) of a facet F if $x = \sum_{j=1}^{n} \lambda_j x_j$ for some non-negative scalars λ_j and some finite collection $x_1, x_2 \ldots, x_n \in F$. By formulas (1), (2) and (3) the following equivalence holds:

$$x \in \text{cone}(F) \quad \Leftrightarrow \quad \|x\|_{\mathcal{P}} = x \cdot \hat{F}. \tag{4}$$

Each isometry of the normed space $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ is affine (by the Mazur–Ulam theorem) and hence is a composition of a linear isometry and a translation. A linear isometry must leave invariant the finite set of extreme points of $\mathcal{P}$ and is completely determined by its action on any d linearly independent extreme points. Thus there exist only finitely many linear isometries on $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$.

A continuous rigid motion of a normed space $(\mathbb{R}^d, \|\cdot\|)$ is a family of continuous paths,

$$\alpha_x : (-\delta, \delta) \to \mathbb{R}^d, \quad x \in \mathbb{R}^d,$$

with the property that $\alpha_x(0) = x$ and for every pair $x, y \in \mathbb{R}^d$ the distance $\|\alpha_x(t) - \alpha_y(t)\|$ remains constant for all values of t. In the case of a polyhedral norm $\|\cdot\|_{\mathcal{P}}$, if δ is sufficiently small, then the isometries $\Gamma_t : x \mapsto \alpha_x(t)$ are necessarily translational since by continuity the linear part must equal the identity transformation. Thus we may assume that a continuous rigid motion of $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ is a family of continuous paths of the form

$$\alpha_x(t) = x + c(t), \quad x \in \mathbb{R}^d,$$

for some continuous function $c : (-\delta, \delta) \to \mathbb{R}^d$ (cf. [13, Lemma 6.2]).

An infinitesimal rigid motion of a normed space $(\mathbb{R}^d, \|\cdot\|)$ is a vector field on $\mathbb{R}^d$ which arises from the velocity vectors of a continuous rigid motion. For a polyhedral norm $\|\cdot\|_{\mathcal{P}}$, since the continuous rigid motions are of translational type, the infinitesimal rigid motions of $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ are precisely the constant maps

$$\gamma : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto a,$$

for some $a \in \mathbb{R}^d$ (cf. [12, Lemma 2.3]).

Let (G, p) be a (finite or infinite) bar-joint framework in a normed vector space $(\mathbb{R}^d, \|\cdot\|)$. A *continuous (or finite) flex* of (G, p) is a family of continuous paths

$$\alpha_v : (-\delta, \delta) \to \mathbb{R}^d, \quad v \in V(G),$$

such that $\alpha_v(0) = p_v$ for each vertex $v \in V(G)$ and $\|\alpha_v(t) - \alpha_w(t)\| = \|p_v - p_w\|$ for all $|t| < \delta$ and each edge $vw \in E(G)$. A continuous flex of (G, p) is regarded as trivial if it arises as the restriction of a continuous rigid motion of $(\mathbb{R}^d, \|\cdot\|)$ to $p(V(G))$. If every continuous flex of (G, p) is trivial then we say that (G, p) is *continuously rigid*.

An *infinitesimal flex* of a (finite or infinite) bar-joint framework (G, p) in a normed space $(\mathbb{R}^d, \|\cdot\|)$ is a map $u : V(G) \to \mathbb{R}^d$, $v \mapsto u_v$ which satisfies

$$\|(p_v + tu_v) - (p_w + tu_w)\| - \|p_v - p_w\| = o(t) \quad \text{as } t \to 0 \tag{5}$$

for each edge $vw \in E(G)$. We will denote the collection of infinitesimal flexes of (G, p) by $\mathcal{F}(G, p)$. An infinitesimal flex of (G, p) is regarded as trivial if it arises as the restriction of an infinitesimal rigid motion of $(\mathbb{R}^d, \|\cdot\|)$ to $p(V(G))$. In other words, in the case of a polyhedral norm, an infinitesimal flex of (G, p) is trivial if and only if it is constant. A bar-joint framework is *infinitesimally rigid* if every infinitesimal flex of (G, p) is trivial. Regarding $\mathcal{F}(G, p)$ as a real vector space with component-wise addition and scalar multiplication, the trivial infinitesimal flexes of (G, p) form a d-dimensional subspace $\mathcal{T}(G, p)$ of $\mathcal{F}(G, p)$.

The interior of a subset $A \subset \mathbb{R}^d$ will be denoted by A°.

3 Support Functionals and Rigidity

In this section, we begin by highlighting the connection between the infinitesimal flex condition (5) for a general norm on $\mathbb{R}^d$ and support functionals on the normed space $(\mathbb{R}^d, \|\cdot\|)$. We then characterise the space of infinitesimal flexes for a general (finite or infinite) bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ in terms of support functionals and prove the equivalence of finite and infinitesimal rigidity for finite bar-joint frameworks which are well-positioned in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. Following this, we describe the rigidity matrix for general finite bar-joint frameworks in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ and compute an example.

3.1 Support Functionals

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^d$, and denote by B the closed unit ball in $(\mathbb{R}^d, \|\cdot\|)$. A linear functional $f : \mathbb{R}^d \to \mathbb{R}$ is a support functional for a point $x_0 \in \mathbb{R}^d$ if $f(x_0) = \|x_0\|^2$ and $\|f\|^* = \|x_0\|$. Equivalently, f is a support functional for x_0 if the hyperplane

$$H = \{x \in \mathbb{R}^d : f(x) = \|x_0\|\}$$

is a supporting hyperplane for B which contains $\frac{x_0}{\|x_0\|}$.

Lemma 1 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. If $f : \mathbb{R}^d \to \mathbb{R}$ is a support functional for x_0, then*

$$f(y) \leq \|x_0\| \frac{\|x_0 + ty\| - \|x_0\|}{t} \quad \textit{for all } t > 0$$

and

$$f(y) \geq \|x_0\| \frac{\|x_0 + ty\| - \|x_0\|}{t} \quad \textit{for all } t < 0$$

for all $y \in \mathbb{R}^d$.

Proof Since f is linear and $f(x_0) = \|x_0\|^2$, we have for all $y \in \mathbb{R}^d$,

$$f(y) = \frac{1}{t}\big(f(x_0 + ty) - \|x_0\|^2\big).$$

If $t > 0$, then since $f(x) \leq \|x_0\|\|x\|$ for all $x \in \mathbb{R}^d$ we have

$$f(y) \leq \|x_0\| \frac{\|x_0 + ty\| - \|x_0\|}{t}.$$

If $t < 0$, then applying the above inequality

$$f(y) = -f(-y) \geq -\|x_0\| \frac{\|x_0 - t(-y)\| - \|x_0\|}{-t} = \|x_0\| \frac{\|x_0 + ty\| - \|x_0\|}{t}.$$

□

Let (G, p) be a (finite or infinite) bar-joint framework in $(\mathbb{R}^d, \|\cdot\|)$, and fix an orientation for each edge $vw \in E(G)$. We denote by $\mathrm{supp}(vw)$ the set of all support functionals for $p_v - p_w$. (The choice of orientation on the edges of G is for convenience only and has no bearing on the results that follow. Alternatively, we could avoid choosing an orientation by defining $\mathrm{supp}(vw)$ to be the set of all linear functionals which are support functionals for either $p_v - p_w$ or $p_w - p_v$.)

Proposition 2 *If (G, p) is a (finite or infinite) bar-joint framework in $(\mathbb{R}^d, \|\cdot\|)$ and $u : V(G) \to \mathbb{R}^d$ is an infinitesimal flex of (G, p), then*

$$u_v - u_w \in \bigcap_{f \in \mathrm{supp}(vw)} \ker f$$

for each edge $vw \in E(G)$.

Proof Let $vw \in E(G)$ and suppose f is a support functional for $p_v - p_w$. Applying Lemma 1 with $x_0 = p_v - p_w$ and $y = u_v - u_w$, we have

$$\lim_{t \to 0^-} \frac{\|x_0 + ty\| - \|x_0\|}{t} \leq \frac{f(y)}{\|x_0\|} \leq \lim_{t \to 0^+} \frac{\|x_0 + ty\| - \|x_0\|}{t}.$$

Since u is an infinitesimal flex of (G, p), $\lim_{t \to 0} \frac{1}{t}(\|x_0 + ty\| - \|x_0\|) = 0$ and so $f(y) = 0$. □

Let $\|\cdot\|_{\mathcal{P}}$ be a polyhedral norm on $\mathbb{R}^d$. For each facet F of $\mathcal{P}$, denote by φ_F the linear functional

$$\varphi_F : \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto x \cdot \hat{F}.$$

Lemma 3 *Let $\|\cdot\|_{\mathcal{P}}$ be a polyhedral norm on $\mathbb{R}^d$, let F be a facet of $\mathcal{P}$ and let $x_0 \in \mathbb{R}^d$. Then $x_0 \in \operatorname{cone}(F)$ if and only if the linear functional*

$$\varphi_{F,x_0} : \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto \|x_0\|_{\mathcal{P}}\, \varphi_F(x),$$

is a support functional for x_0.

Proof If $x_0 \in \operatorname{cone}(F)$, then by formula (4) $\varphi_{F,x_0}(x_0) = \|x_0\|^2_{\mathcal{P}}$. By (1), we have $\varphi_{F,x_0}(x) \leq \|x_0\|_{\mathcal{P}}$ for each $x \in \mathcal{P}$, and it follows that φ_{F,x_0} is a support functional for x_0. Conversely, if $x_0 \notin \operatorname{cone}(F)$, then by (4) $\varphi_{F,x_0}(x_0) < \|x_0\|^2_{\mathcal{P}}$ and so φ_{F,x_0} is not a support functional for x_0. □

For each oriented edge $vw \in E(G)$, we denote by $\operatorname{supp}_{\Phi}(vw)$ the set of all linear functionals φ_F which are support functionals for $\frac{p_v - p_w}{\|p_v - p_w\|_{\mathcal{P}}}$.

Proposition 4 *Let (G, p) be a finite bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. If a mapping $u : V(G) \to \mathbb{R}^d$ satisfies*

$$u_v - u_w \in \bigcap_{\varphi_F \in \operatorname{supp}_{\Phi}(vw)} \ker \varphi_F$$

for each edge $vw \in E(G)$, then there exists $\delta > 0$ such that the family

$$\alpha_v : (-\delta, \delta) \to \mathbb{R}^d, \quad \alpha_v(t) = p_v + t u_v,$$

is a finite flex of (G, p).

Proof Let $vw \in E(G)$ and write $x_0 = p_v - p_w$ and $u_0 = u_v - u_w$. If φ_F is a support functional for $\frac{x_0}{\|x_0\|_{\mathcal{P}}}$, then by the hypothesis $\varphi_F(u_0) = 0$. By Lemma 3, x_0 is contained in the conical hull of the facet F. Applying formulas (3) and (4),

$$\|x_0\|_{\mathcal{P}} = \max_{y \in \operatorname{ext}(\mathcal{P}^{\triangle})} x_0 \cdot y = x_0 \cdot \hat{F}.$$

By continuity, there exists $\delta_{vw} > 0$ such that for all $|t| < \delta_{vw}$

$$\begin{aligned} \|x_0 + t u_0\|_{\mathcal{P}} &= \max_{y \in \operatorname{ext}(\mathcal{P}^{\triangle})} (x_0 + t u_0) \cdot y \\ &= (x_0 + t u_0) \cdot \hat{F} \\ &= \|x_0\|_{\mathcal{P}} + t\, \varphi_F(u_0) \\ &= \|x_0\|_{\mathcal{P}}. \end{aligned}$$

Since G is a finite graph, the result holds with $\delta = \min_{vw \in E(G)} \delta_{vw} > 0$. □

The following is a characterisation of the space of infinitesimal flexes of a general bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$.

Theorem 5 *Let* (G, p) *be a (finite or infinite) bar-joint framework in* $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. *Then a mapping* $u : V(G) \to \mathbb{R}^d$ *is an infinitesimal flex of* (G, p) *if and only if*

$$u_v - u_w \in \bigcap_{\varphi_F \in \mathrm{supp}_{\Phi}(vw)} \ker \varphi_F$$

for each edge $vw \in E(G)$.

Proof If u is an infinitesimal flex of (G, p), then the result follows from Proposition 2. For the converse, let $vw \in E(G)$ and write $x_0 = p_v - p_w$ and $u_0 = u_v - u_w$. Applying the argument in the proof of Proposition 4, there exists $\delta_{vw} > 0$ with $\|x_0 + tu_0\|_{\mathcal{P}} = \|x_0\|_{\mathcal{P}}$ for all $|t| < \delta_{vw}$. Hence u is an infinitesimal flex of (G, p). □

3.2 Equivalence of Finite and Infinitesimal Rigidity

A placement of a simple graph G in $\mathbb{R}^d$ is a map $p : V(G) \to \mathbb{R}^d$ for which $p_v \neq p_w$ whenever $vw \in E(G)$. A placement $p : V(G) \to \mathbb{R}^d$ is *well-positioned* with respect to a polyhedral norm on $\mathbb{R}^d$ if $p_v - p_w$ is contained in the conical hull of exactly one facet of the unit ball $\mathcal{P}$ for each edge $vw \in E(G)$. We denote this unique facet by F_{vw}. In the following discussion, G is a finite graph and each placement is identified with a point $p = (p_v)_{v \in V(G)}$ in the product space $\prod_{v \in V(G)} \mathbb{R}^d$ which we regard as having the usual topology. The set of all well-positioned placements of G in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ is an open and dense subset of this product space. The *configuration space* for a bar-joint framework (G, p) is defined as

$$V(G, p) = \left\{ x \in \prod_{v \in V(G)} \mathbb{R}^d : \|x_v - x_w\|_{\mathcal{P}} = \|p_v - p_w\|_{\mathcal{P}} \quad \text{for all } vw \in E(G) \right\}.$$

Proposition 6 *Let* (G, p) *be a finite and well-positioned bar-joint framework in* $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ *with* $p_v - p_w \in \mathrm{cone}(F_{vw})$ *for each* $vw \in E(G)$. *Then there exists a neighbourhood* U *of* p *in* $\prod_{v \in V(G)} \mathbb{R}^d$ *such that*

(i) *if* $x \in U$, *then* $x_v - x_w \in \mathrm{cone}(F_{vw})$ *for each edge* $vw \in E(G)$,
(ii) (G, x) *is a well-positioned bar-joint framework for each* $x \in U$ *and*
(iii) $V(G, p) \cap U = \{x \in U : \varphi_{F_{vw}}(x_v - x_w) = \varphi_{F_{vw}}(p_v - p_w)$ *for all* $vw \in E(G)\}$.

In particular, $V(G, p) \cap U = (p + \mathcal{F}(G, p)) \cap U$.

Proof Let $vw \in E(G)$ be an oriented edge and consider the continuous map

$$T_{vw} : \prod_{v' \in V(G)} \mathbb{R}^d \to \mathbb{R}^d, \quad (x_{v'})_{v' \in V(G)} \mapsto x_v - x_w.$$

Since (G, p) is well-positioned, $p_v - p_w$ is an interior point of the conical hull of a unique facet F_{vw} of $\mathcal{P}$. The preimage $T_{vw}^{-1}(\mathrm{cone}(F_{vw})^\circ)$ is an open neighbourhood of

p. Since G is a finite graph, the intersection

$$U = \bigcap_{vw \in E(G)} T_{vw}^{-1}(\text{cone}(F_{vw})^\circ)$$

is an open neighbourhood of p which satisfies (i), (ii) and (iii).

Since (G, p) is well-positioned, by Lemma 3, there is exactly one support functional in $\text{supp}_\Phi(vw)$ for each edge vw and this functional is given by $\varphi_{F_{vw}}$. If $x \in U$, then define $u = (u_v)_{v \in V(G)}$ by setting $u_v = x_v - p_v$ for each $v \in V(G)$. By (iii), $x \in V(G, p) \cap U$ if and only if $x \in U$ and

$$\varphi_{F_{vw}}(u_v - u_w) = \varphi_{F_{vw}}(x_v - x_w) - \varphi_{F_{vw}}(p_v - p_w) = 0$$

for each edge $vw \in E(G)$. By Theorem 5, the latter identity is equivalent to the condition that u is an infinitesimal flex of (G, p). Thus $x \in V(G, p) \cap U$ if and only if $x \in U$ and $x - p \in \mathcal{F}(G, p)$. □

We now prove the equivalence of continuous rigidity and infinitesimal rigidity for finite well-positioned bar-joint frameworks.

Theorem 7 *Let (G, p) be a finite well-positioned bar-joint framework in a normed space $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$, where $\|\cdot\|_{\mathcal{P}}$ is a polyhedral norm. Then the following statements are equivalent:*

(i) (G, p) *is continuously rigid.*
(ii) (G, p) *is infinitesimally rigid.*

Proof (i) $\Rightarrow$ (ii). If $u = (u_v)_{v \in V(G)} \in \mathcal{F}(G, p)$ is an infinitesimal flex of (G, p), then by Theorem 5 and Proposition 4, the family

$$\alpha_v : (-\varepsilon, \varepsilon) \to \mathbb{R}^d, \quad \alpha_v(t) = p_v + t u_v, \quad v \in V(G),$$

is a finite flex of (G, p) for some $\varepsilon > 0$. Since (G, p) is continuously rigid, this finite flex must be trivial. Thus there exist $\delta > 0$ and a continuous path $c : (-\delta, \delta) \to \mathbb{R}^d$ such that $\alpha_v(t) = p_v + c(t)$ for all $|t| < \delta$ and all $v \in V(G)$. Now $u_v = \alpha_v'(0) = c'(0)$ for all $v \in V(G)$ and so u is a constant, and hence trivial, infinitesimal flex of (G, p). We conclude that (G, p) is infinitesimally rigid.

(ii) $\Rightarrow$ (i). If (G, p) has a finite flex given by the family

$$\alpha_v : (-\varepsilon, \varepsilon) \to \mathbb{R}^d, \quad v \in V(G),$$

then consider the continuous path

$$\alpha : (\varepsilon, \varepsilon) \to V(G, p), \quad t \mapsto (\alpha_v(t))_{v \in V(G)}.$$

By Proposition 6, $V(G, p) \cap U = (p + \mathcal{F}(G, p)) \cap U$ for some neighbourhood U of p. Since $\alpha(0) = p$, there exists $\delta > 0$ such that $\alpha(t) \in V(G, p) \cap U$ for all $|t| < \delta$. Choose $t_0 \in (-\delta, \delta)$ and define

$$u : V(G) \to \mathbb{R}^d, \quad u_v = \alpha_v(t_0) - p_v.$$

Then $u = \alpha(t_0) - p \in \mathcal{F}(G, p)$ is an infinitesimal flex of (G, p). Since (G, p) is infinitesimally rigid, u must be a trivial infinitesimal flex. Hence $u_v = c(t_0)$ for all $v \in V(G)$ and some $c(t_0) \in \mathbb{R}^d$. Apply the same argument to show that for each $|t| < \delta$ there exists $c(t)$ such that $\alpha_v(t) = p_v + c(t)$ for all $v \in V(G)$. Note that $c : (-\delta, \delta) \to \mathbb{R}^d$ is continuous and so $\{\alpha_v : v \in V(G)\}$ is a trivial finite flex of (G, p). We conclude that (G, p) is continuously rigid. □

The non-equivalence of finite and infinitesimal rigidity for general finite bar-joint frameworks in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ is demonstrated in Example 9.

3.3 The Rigidity Matrix

We define the *rigidity matrix* $R_{\mathcal{P}}(G, p)$ for a finite bar-joint framework (G, p) in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ as follows: Fix an ordering of the vertices $V(G)$ and edges $E(G)$ and choose an orientation on the edges of G. For each vertex v, assign d columns in the rigidity matrix and label these columns $p_{v,1}, \dots, p_{v,d}$. For each directed edge $vw \in E(G)$ and each facet F with $p_v - p_w \in \operatorname{cone}(F)$, assign a row in the rigidity matrix and label this row by (vw, F). The entries for the row (vw, F) are given by

$$\begin{bmatrix} 0 & \cdots & 0 & \overset{p_{v,1}}{\hat{F}_1} & \overset{\cdots}{\cdots} & \overset{p_{v,d}}{\hat{F}_d} & 0 & \cdots & 0 & \overset{p_{w,1}}{-\hat{F}_1} & \overset{\cdots}{\cdots} & \overset{p_{w,d}}{-\hat{F}_d} & 0 & \cdots & 0 \end{bmatrix}, \tag{6}$$

where $p_v - p_w \in \operatorname{cone}(F)$ and $\hat{F} = (\hat{F}_1, \dots, \hat{F}_d) \in \mathbb{R}^d$. If (G, p) is well-positioned, then the rigidity matrix has size $|E(G)| \times d|V(G)|$.

Proposition 8 *Let (G, p) be a finite bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. Then*

(i) $\mathcal{F}(G, p) \cong \ker R_{\mathcal{P}}(G, p)$.
(ii) (G, p) *is infinitesimally rigid if and only if* $\operatorname{rank}\ R_{\mathcal{P}}(G, p) = d|V(G)| - d$.

Proof The system of equations in Theorem 5 is expressed by the matrix equation $R_{\mathcal{P}}(G, p)u^{\mathrm{T}} = 0$ where we identify $u : V(G) \to \mathbb{R}^d$ with a row vector $(u_{v_1}, \dots, u_{v_n}) \in \mathbb{R}^{d|V(G)|}$. Thus $\mathcal{F}(G, p) \cong \ker R_{\mathcal{P}}(G, p)$. The space of trivial infinitesimal flexes of (G, p) has dimension d and so in general we have

$$\operatorname{rank}\ R_{\mathcal{P}}(G, p) \le d|V(G)| - d$$

with equality if and only if (G, p) is infinitesimally rigid. □

If F is a facet of $\mathcal{P}$ and $y_1, y_2, \dots, y_d \in \operatorname{ext}(\mathcal{P})$ are extreme points of $\mathcal{P}$ which are contained in F, then for each column vector y_k we compute $[1 \cdots 1]\, A^{-1}\, y_k = 1$, where $A = [y_1 \cdots y_d] \in M^{d\times d}(\mathbb{R})$. Hence,

$$\hat{F} = [1 \cdots 1]A^{-1}. \tag{7}$$

Moreover, if $y_1, y_2, \ldots, y_d$ are pairwise orthogonal, then

$$A^{-1} = \left[\frac{y_1}{\|y_1\|_2^2} \cdots \frac{y_d}{\|y_d\|_2^2} \right]^{\mathrm{T}}$$

and so

$$\hat{F} = \sum_{j=1}^{d} \frac{y_j}{\|y_j\|_2^2}, \tag{8}$$

where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^d$.

Example 9 Let $\mathcal{P}$ be a crosspolytope in $\mathbb{R}^d$ with $2d$ many extreme points $\operatorname{ext}(\mathcal{P}) = \{\pm e_k : k = 1, \ldots, d\}$, where $e_1, e_2, \ldots, e_d$ is the usual basis in $\mathbb{R}^d$. Then each facet F contains d pairwise orthogonal extreme points $y_1, y_2, \ldots, y_d$ each of Euclidean norm 1. By (8), $\hat{F} = \sum_{j=1}^{d} y_j$ and the resulting polyhedral norm is the 1-norm

$$\|x\|_{\mathcal{P}} = \max_{y \in \operatorname{ext}(\mathcal{P}^{\triangle})} x \cdot y = \sum_{i=1}^{d} |x_i| = \|x\|_1.$$

Consider for example the placements of the complete graph K_2 in $(\mathbb{R}^2, \|\cdot\|_1)$ illustrated in Fig. 1. The polytope $\mathcal{P}$ is indicated on the left with facets labelled F_1 and F_2. The extreme points of the polar set $\mathcal{P}^{\triangle}$ which correspond to these facets are $\hat{F}_1 = e_1 + e_2 = (1, 1)$ and $\hat{F}_2 = e_1 - e_2 = (1, -1)$. The first placement is well-positioned with respect to $\mathcal{P}$ and the rigidity matrix is

$$\begin{array}{c} \\ (vw, F_1) \end{array} \begin{array}{c} \begin{matrix} p_{v,1} & p_{v,2} & p_{w,1} & p_{w,2} \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 1 & -1 & -1 \end{array} \right] \end{array}.$$

Evidently, this bar-joint framework has a non-trivial infinitesimal flex. The second placement is not well-positioned and the rigidity matrix is

$$\begin{array}{c} \\ (vw, F_1) \\ (vw, F_2) \end{array} \begin{array}{c} \begin{matrix} p_{v,1} & p_{v,2} & p_{w,1} & p_{w,2} \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right] \end{array}.$$

As the rigidity matrix has rank 2, this bar-joint framework is infinitesimally rigid in $(\mathbb{R}^2, \|\cdot\|_1)$, but continuously flexible.

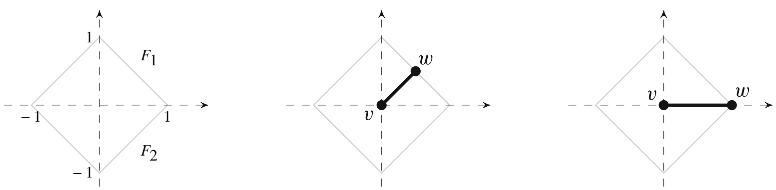

Fig. 1 An infinitesimally flexible and an infinitesimally rigid placement of K_2 in $(\mathbb{R}^2, \|\cdot\|_1)$

4 Edge-Labellings and Monochrome Subgraphs

In this section, we describe an edge-labelling on G which depends on the placement of the bar-joint framework (G, p) in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ relative to the facets of $\mathcal{P}$. We provide methods for identifying infinitesimally flexible frameworks and subframeworks which are relatively infinitesimally rigid. We then characterise infinitesimal rigidity for bar-joint frameworks with d framework colours in terms of the monochrome subgraphs induced by this edge-labelling.

4.1 Edge-Labellings

Let (G, p) be a general bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ (i.e. it is not assumed here that (G, p) is finite or well-positioned). Since $\mathcal{P}$ is symmetric in $\mathbb{R}^d$, if F is a facet of $\mathcal{P}$ then $-F$ is also a facet of $\mathcal{P}$. Denote by $\Phi(\mathcal{P})$ the collection of all pairs $[F] = \{F, -F\}$. For each edge $vw \in E(G)$, define

$$\Phi(vw) = \{[F] \in \Phi(\mathcal{P}) : p_v - p_w \in \text{cone}(F) \cup \text{cone}(-F)\}.$$

We refer to the elements of $\Phi(vw)$ as the *framework colours* of the edge vw. For example, if $p_v - p_w$ lies in the conical hull of exactly one facet of $\mathcal{P}$, then the edge vw has just one framework colour. If $p_v - p_w$ lies along a ray through an extreme point of $\mathcal{P}$, then vw has at least d distinct framework colours. By Lemma 3, $[F]$ is a framework colour for an edge vw if and only if either φ_F or $-\varphi_F$ is a support functional for $\frac{p_v - p_w}{\|p_v - p_w\|_{\mathcal{P}}}$.

For each vertex $v_0 \in V(G)$, denote by $\Phi(v_0)$ the collection of framework colours of all edges which are incident with v_0:

$$\Phi(v_0) = \bigcup_{v_0 w \in E(G)} \Phi(v_0 w).$$

Proposition 10 *If a (finite or infinite) bar-joint framework (G, p) is infinitesimally rigid in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$, then $|\Phi(v)| \geq d$ for each vertex $v \in V(G)$.*

Proof If $v_0 \in V(G)$ and $|\Phi(v_0)| < d$, then there exists non-zero

$$x \in \bigcap_{[F] \in \Phi(v_0)} \ker \varphi_F.$$

By Theorem 5, if $u : V(G) \to \mathbb{R}^d$ is defined by

$$u_v = \begin{cases} x & \text{if } v = v_0, \\ 0 & \text{if } v \neq v_0. \end{cases}$$

then u is a non-trivial infinitesimal flex of (G, p). □

We now consider the subgraphs of G which are spanned by edges possessing a particular framework colour. For each facet F of $\mathcal{P}$, define

$$E_F(G, p) = \{vw \in E(G) : [F] \in \Phi(vw)\}$$

and let G_F be the subgraph of G spanned by $E_F(G, p)$. We refer to G_F as a *monochrome subgraph* of G.

Denote by $\Phi(G, p)$ the collection of all framework colours of edges of G:

$$\Phi(G, p) = \bigcup_{vw \in E(G)} \Phi(vw).$$

We refer to the elements of $\Phi(G, p)$ as the *framework colours* of the bar-joint framework (G, p).

Proposition 11 *Let (G, p) be a (finite or infinite) bar-joint framework which is infinitesimally rigid in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. If C is a collection of framework colours of (G, p) with $|\Phi(G, p)\backslash C| < d$, then*

$$\bigcup_{[F] \in C} G_F$$

contains a spanning tree of G.

Proof Suppose that $\bigcup_{[F]\in C} G_F$ does not contain a spanning tree of G. Then there exists a partition $V(G) = V_1 \cup V_2$ for which there is no edge $v_1 v_2 \in E(G)$ with framework colour contained in C satisfying $v_1 \in V_1$ and $v_2 \in V_2$. Since $|\Phi(G, p)\backslash C| < d$, there exists non-zero

$$x \in \bigcap_{[F] \in \Phi(G,p)\backslash C} \ker \varphi_F.$$

By Theorem 5, if $u : V(G) \to \mathbb{R}^d$ is defined by

$$u_v = \begin{cases} x & \text{if } v \in V_1, \\ 0 & \text{if } v \in V_2, \end{cases}$$

then u is a non-trivial infinitesimal flex of (G, p). We conclude that $\bigcup_{[F]\in C} G_F$ contains a spanning tree of G. □

It is possible to construct examples which show that the converse to Proposition 11 does not hold in general. In Theorem 13, we show that a converse statement does hold under the additional assumption that $|\Phi(G, p)| = d$.

4.2 Edge-Labelled Paths and Relative Infinitesimal Rigidity

Let (G, p) be a finite bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ and, for each edge $vw \in E(G)$, let X_{vw} be the vector subspace of $\mathbb{R}^d$:

$$X_{vw} = \bigcap_{\varphi_F \in \mathrm{supp}_\Phi(vw)} \ker \varphi_F = \bigcap_{[F] \in \Phi(vw)} \ker \varphi_F.$$

If $\gamma = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ is a path in G from a vertex v_1 to a vertex v_n, then we define

$$X_\gamma = X_{v_1v_2} + X_{v_2v_3} + \cdots + X_{v_{n-1}v_n}.$$

For each pair of vertices $v, w \in V(G)$, denote by $\Gamma_G(v, w)$ the set of all paths γ in G from v to w.

A subframework of (G, p) is a bar-joint framework (H, p) obtained by restricting p to the vertex set of a subgraph H. We say that (H, p) is *relatively infinitesimally rigid* in (G, p) if the restriction of every infinitesimal flex of (G, p) to (H, p) is trivial.

Proposition 12 *Let (G, p) be a finite bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ and let (H, p) be a subframework of (G, p). If for each pair of vertices $v, w \in V(H)$*

$$\bigcap_{\gamma \in \Gamma_G(v,w)} X_\gamma = \{0\},$$

then (H, p) is relatively infinitesimally rigid in (G, p).

Proof Let $u \in \mathcal{F}(G, p)$ be an infinitesimal flex of (G, p) and let $v, w \in V(H)$. Suppose $\gamma \in \Gamma_G(v, w)$, where $\gamma = \{v_1v_2, \ldots, v_{n-1}v_n\}$ is a path in G with $v = v_1$ and $w = v_n$. Then by Theorem 5,

$$u_v - u_w = (u_{v_1} - u_{v_2}) + (u_{v_2} - u_{v_3}) + \cdots + (u_{v_{n-1}} - u_{v_n}) \in X_\gamma.$$

Since this holds for all paths in $\Gamma_G(v, w)$, the hypothesis implies that $u_v = u_w$. Applying this argument to every pair of vertices in H, we see that the restriction of u to $V(H)$ is constant and hence a trivial infinitesimal flex of (H, p). Thus (H, p) is relatively infinitesimally rigid in (G, p). □

4.3 Monochrome Spanning Subgraphs

Applying the results of the previous sections, we can now characterise the infinitesimally rigid bar-joint frameworks in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ which use exactly d framework colours.

Theorem 13 *Let (G, p) be a (finite or infinite) bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ and suppose that $|\Phi(G, p)| = d$. Then the following statements are equivalent:*

(i) (G, p) *is infinitesimally rigid.*
(ii) G_F *contains a spanning tree of* G *for each* $[F] \in \Phi(G, p)$.

Proof The implication (i) $\Rightarrow$ (ii) follows from Proposition 11. To prove (ii) $\Rightarrow$ (i), let $u \in \mathcal{F}(G, p)$. If $v, w \in V(G)$, then for each framework colour $[F] \in \Phi(G, p)$ there exists a path in G_F from v to w. Hence

$$\bigcap_{\gamma \in \Gamma_G(v,w)} X_\gamma \subseteq \bigcap_{[F] \in \Phi(G,p)} \ker \varphi_F = \{0\}$$

and, by Proposition 12, $u_v = u_w$. Applying this argument to all pairs $v, w \in V(G)$, we see that u is a trivial infinitesimal flex and so (G, p) is infinitesimally rigid. □

A bar-joint framework (G, p) is *minimally infinitesimally rigid* in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ if it is infinitesimally rigid and every subframework obtained by removing a single edge from G is infinitesimally flexible.

Corollary 14 *Let* (G, p) *be a (finite or infinite) bar-joint framework in* $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ *and suppose that* $|\Phi(G, p)| = d$. *If* G_F *is a spanning tree in* G *for each* $[F] \in \Phi(G, p)$, *then* (G, p) *is minimally infinitesimally rigid.*

Proof By Theorem 13, (G, p) is infinitesimally rigid. If any edge vw is removed from G, then G_F is no longer a spanning tree for some $[F] \in \Phi(G, p)$. By Theorem 13, the subframework $(G\backslash\{vw\}, p)$ is not infinitesimally rigid and so we conclude that (G, p) is minimally infinitesimally rigid. □

There exist bar-joint frameworks which show that the converse statement to Corollary 14 does not hold in full generality. In the following corollary, the converse is established for bar-joint frameworks that are well-positioned.

Corollary 15 *Let* (G, p) *be a (finite or infinite) well-positioned bar-joint framework in* $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ *and suppose that* $|\Phi(G, p)| = d$. *Then the following statements are equivalent:*

(i) (G, p) *is minimally infinitesimally rigid.*
(ii) G_F *is a spanning tree in* G *for each* $[F] \in \Phi(G, p)$.

Proof (i) $\Rightarrow$ (ii). Let $[F] \in \Phi(G, p)$. If (G, p) is minimally infinitesimally rigid, then by Theorem 13 the monochrome subgraph G_F contains a spanning tree of G. Suppose vw is an edge of G which is contained in G_F. Since (G, p) is minimally infinitesimally rigid, $(G\backslash\{vw\}, p)$ is infinitesimally flexible. Since (G, p) is well-positioned, vw is contained in exactly one monochrome subgraph of G and so G_F is the only monochrome subgraph which is altered by removing the edge vw from G. By Theorem 13, $G_F\backslash\{vw\}$ does not contain a spanning tree of G. We conclude that G_F is a spanning tree of G. The implication (ii) $\Rightarrow$ (i) is proved in Corollary 14. □

5 An Analogue of Laman's Theorem

In this section, we address the problem of whether there exists a combinatorial description of the class of graphs for which a minimally infinitesimally rigid placement exists in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. We restrict our attention to finite bar-joint frameworks and prove that in two dimensions such a characterisation exists (Theorem 23). This result is analogous to Laman's theorem [14] for bar-joint frameworks in the Euclidean plane and extends [12, Thm. 4.6] which holds in the case where $\mathcal{P}$ is a quadrilateral.

5.1 Regular Placements

Let $\omega(G, \mathbb{R}^d, \mathcal{P})$ denote the set of all well-positioned placements of a finite simple graph G in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. A bar-joint framework (G, p) is *regular in* $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ if the function

$$\omega(G, \mathbb{R}^d, \mathcal{P}) \to \{1, 2, \dots, d|V(G)| - d\}, \quad x \mapsto \text{rank } R_{\mathcal{P}}(G, x)$$

achieves its maximum value at p.

Lemma 16 *Let G be a finite simple graph.*

(i) *The set of placements of G in* $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ *which are both well-positioned and regular is an open set in* $\prod_{v\in V(G)} \mathbb{R}^d$.
(ii) *The set of placements of G in* $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ *which are well-positioned and not regular is an open set in* $\prod_{v\in V(G)} \mathbb{R}^d$.

Proof Let p be a well-positioned placement of G and let U be an open neighbourhood of p as in the statement of Proposition 6. The matrix-valued function $x \mapsto R_{\mathcal{P}}(G, x)$ is constant on U and so either (G, x) is regular for all $x \in U$ or (G, x) is not regular for all $x \in U$. □

A finite simple graph G is (*minimally*) *rigid* in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$ if there exists a well-positioned placement of G which is (minimally) infinitesimally rigid.

Example 17 The complete graph K_4 is minimally rigid in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$ for every polyhedral norm $\|\cdot\|_{\mathcal{P}}$. To see this, let $F_1, F_2, \dots, F_n$ be the facets of $\mathcal{P}$ and let $x_0 \in \text{ext}(\mathcal{P})$ be any extreme point of $\mathcal{P}$. Then x_0 is contained in exactly two facets, F_1 and F_2 say. Choose a point x_1 in the relative interior of F_1 and a point x_2 in the relative interior of F_2. Then by formulas (3) and (4),

$$\max_{k\neq 1} (x_1 \cdot \hat{F}_k) < \|x_1\|_{\mathcal{P}} = x_1 \cdot \hat{F}_1 = 1, \tag{9}$$

$$\max_{k\neq 2} (x_2 \cdot \hat{F}_k) < \|x_2\|_{\mathcal{P}} = x_2 \cdot \hat{F}_2 = 1. \tag{10}$$

Since $(x_0 \cdot \hat{F}_1) = (x_0 \cdot \hat{F}_2) = \|x_0\|_{\mathcal{P}} = 1$, if x_1 and x_2 are chosen to lie in a sufficiently small neighbourhood of x_0 then by continuity we may assume

$$x_1 \cdot \hat{F}_2 = \max_{k \neq 1} (x_1 \cdot \hat{F}_k) > 0, \tag{11}$$

$$x_2 \cdot \hat{F}_1 = \max_{k \neq 2} (x_2 \cdot \hat{F}_k) > 0. \tag{12}$$

We may also assume without loss of generality that

$$x_1 \cdot \hat{F}_2 = x_2 \cdot \hat{F}_1. \tag{13}$$

Define a placement $p : V(K_4) \to \mathbb{R}^2$ by setting

$$p_{v_0} = (0,0), \quad p_{v_1} = x_1, \quad p_{v_2} = (1-\varepsilon)x_2, \quad p_{v_3} = x_1 + (1+\varepsilon)x_2,$$

where $0 < \varepsilon < 1$. The edges v_0v_1, v_0v_2 and v_1v_3 have framework colours

$$\Phi(v_0v_1) = [F_1], \quad \Phi(v_0v_2) = [F_2], \quad \Phi(v_1v_3) = [F_2].$$

To determine the framework colours for the remaining edges, we will apply the above identities together with formulas (3) and (4). Consider the edge v_2v_3. If $k \neq 1$ and ε is sufficiently small, then applying (9)

$$(p_{v_3} - p_{v_2}) \cdot \hat{F}_k = (x_1 \cdot \hat{F}_k) + 2\varepsilon\,(x_2 \cdot \hat{F}_k) < 1.$$

Also by (9) and (12), we have

$$(p_{v_3} - p_{v_2}) \cdot \hat{F}_1 = (x_1 \cdot \hat{F}_1) + 2\varepsilon\,(x_2 \cdot \hat{F}_1) = 1 + 2\varepsilon\,(x_2 \cdot \hat{F}_1) > 1.$$

We conclude that F_1 is the unique facet of $\mathcal{P}$ for which $\|p_{v_3} - p_{v_2}\|_{\mathcal{P}} = (p_{v_3} - p_{v_2}) \cdot \hat{F}_1$ and so $p_{v_3} - p_{v_2} \in \operatorname{cone}(F_1)^\circ$. Thus $\Phi(v_2v_3) = [F_1]$. Consider the edge v_0v_3. Applying (10) and (11), for $k \neq 1, 2$ we have

$$(p_{v_3} - p_{v_0}) \cdot \hat{F}_k = (x_1 \cdot \hat{F}_k) + (1+\varepsilon)\,(x_2 \cdot \hat{F}_k) < (x_1 \cdot \hat{F}_2) + 1 + \varepsilon.$$

By applying (13),

$$(p_{v_3} - p_{v_0}) \cdot \hat{F}_1 = (x_1 \cdot \hat{F}_1) + (1+\varepsilon)(x_2 \cdot \hat{F}_1) < (x_1 \cdot \hat{F}_2) + 1 + \varepsilon$$

and by (10),

$$(p_{v_3} - p_{v_0}) \cdot \hat{F}_2 = (x_1 \cdot \hat{F}_2) + (1+\varepsilon)(x_2 \cdot \hat{F}_2) = (x_1 \cdot \hat{F}_2) + 1 + \varepsilon.$$

Hence F_2 is the unique facet of $\mathcal{P}$ for which $\|p_{v_3} - p_{v_0}\|_{\mathcal{P}} = (p_{v_3} - p_{v_0}) \cdot \hat{F}_2$. Thus $p_{v_3} - p_{v_0} \in \operatorname{cone}(F_2)^\circ$ and so $\Phi(v_0v_3) = [F_2]$. Finally, consider the edge v_1v_2.

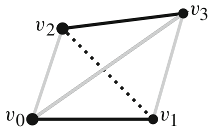

Fig. 2 A framework colouring for an infinitesimally rigid placement of K_4 in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$

Applying (13), we have

$$(p_{v_2} - p_{v_1}) \cdot \hat{F}_2 = (1-\varepsilon)(x_2 \cdot \hat{F}_2) - (x_1 \cdot \hat{F}_2) = 1 - \varepsilon - (x_2 \cdot \hat{F}_1)$$

and this value is positive provided ε is sufficiently small. By (9), we have

$$(p_{v_2} - p_{v_1}) \cdot (-\hat{F}_1) = -(1-\varepsilon)(x_2 \cdot \hat{F}_1) + (x_1 \cdot \hat{F}_1) = 1 + \varepsilon(x_2 \cdot \hat{F}_1) - (x_2 \cdot \hat{F}_1).$$

We conclude that $(p_{v_2} - p_{v_1}) \cdot (\pm\hat{F}_2) < \|p_{v_2} - p_{v_1}\|_{\mathcal{P}}$. Hence $p_{v_2} - p_{v_1} \notin \text{cone}(F_2)$. By making a small perturbation, we can assume that $p_{v_2} - p_{v_1}$ is contained in the conical hull of exactly one facet of $\mathcal{P}$ and so $\Phi(v_1v_2) = [F_k]$ for some $[F_k] \neq [F_2]$. Thus (G, p) is well-positioned. This framework colouring is illustrated in Fig. 2 with monochrome subgraphs G_{F_1} and G_{F_2} indicated in black and grey, respectively, and G_{F_k} indicated by the dotted line. Suppose $u \in \mathcal{F}(K_4, p)$. To show that u is a trivial infinitesimal flex, we apply the method of Proposition 12. The vertices v_0 and v_1 are joined by monochrome paths in both G_{F_1} and G_{F_2} and so $u_{v_0} = u_{v_1}$. Similarly, $u_{v_2} = u_{v_3}$. The vertices v_1 and v_2 are joined by monochrome paths in G_{F_2} and G_{F_k} and so $u_{v_1} = u_{v_2}$. Thus u is a constant and hence trivial infinitesimal flex of (K_4, p). We conclude that (K_4, p) and all regular and well-positioned placements of K_4 are infinitesimally rigid.

5.2 Counting Conditions

The Maxwell counting conditions [17] state that a finite minimally infinitesimally rigid bar-joint framework (G, p) in Euclidean space $\mathbb{R}^d$ must satisfy $|E(G)| = d|V(G)| - \binom{d+1}{2}$ with inequalities $|E(H)| \leq d|V(H)| - \binom{d+1}{2}$ for all subgraphs H containing at least d vertices. The following analogous statement holds for polyhedral norms.

Proposition 18 *Let (G, p) be a finite and well-positioned bar-joint framework in $(\mathbb{R}^d, \|\cdot\|_{\mathcal{P}})$. If (G, p) is minimally infinitesimally rigid, then*

(i) $|E(G)| = d|V(G)| - d$ *and*
(ii) $|E(H)| \leq d|V(H)| - d$ *for all subgraphs H of G.*

Proof If (G, p) is minimally infinitesimally rigid, then by Proposition 8 the rigidity matrix $R_{\mathcal{P}}(G, p)$ is independent and

$$|E(G)| = \text{rank } R_{\mathcal{P}}(G, p) = d|V(G)| - d.$$

The rigidity matrix for any subframework of (G, p) is also independent and so

$$|E(H)| = \text{rank } R_{\mathcal{P}}(H, p) \leq d|V(H)| - d$$

for all subgraphs H. □

A graph G is (d, d)*-tight* if it satisfies the counting conditions in the above proposition. The class of $(2, 2)$-tight graphs has the property that every member can be constructed from a single vertex by applying a sequence of finitely many allowable graph moves (see [18]). The allowable graph moves are:

1. The Henneberg 1-move (also called vertex addition, or 0-extension).
2. The Henneberg 2-move (also called edge splitting, or 1-extension).
3. The edge-to-K_3 move (also called vertex splitting).
4. The vertex-to-K_4 move.

A Henneberg 1-move $G \to G'$ adjoins a vertex v_0 to G together with two edges v_0v_1 and v_0v_2 where $v_1, v_2 \in V(G)$.

Proposition 19 *The Henneberg* 1*-move preserves infinitesimal rigidity for well-positioned bar-joint frameworks in* $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$.

Proof Suppose (G, p) is well-positioned and infinitesimally rigid and let $G \to G'$ be a Henneberg 1-move on the vertices $v_1, v_2 \in V(G)$. Choose distinct $[F_1], [F_2] \in \Phi(\mathcal{P})$ and define a placement p' of G' by $p'_v = p_v$ for all $v \in V(G)$ and

$$p'_{v_0} \in \left(p_{v_1} + \left(\text{cone}\,(F_1)^\circ \cup -\text{cone}\,(F_1)^\circ\right)\right) \cap \left(p_{v_2} + \left(\text{cone}\,(F_2)^\circ \cup -\text{cone}\,(F_2)^\circ\right)\right).$$

Then (G', p') is well-positioned and the edges v_0v_1 and v_0v_2 have framework colours $[F_1]$ and $[F_2]$, respectively. If $u \in \mathcal{F}(G', p')$, then the restriction of u to $V(G)$ is an infinitesimal flex of (G, p). This restriction must be trivial and hence constant. In particular, $u_{v_1} = u_{v_2}$. By Theorem 5, $\varphi_{F_1}(u_{v_0} - u_{v_1}) = 0$ and $\varphi_{F_2}(u_{v_0} - u_{v_1}) = \varphi_{F_2}(u_{v_0} - u_{v_2}) = 0$ and so $u_{v_0} = u_{v_1}$. We conclude that (G', p') is infinitesimally rigid. □

A Henneberg 2-move $G \to G'$ removes an edge v_1v_2 from G and adjoins a vertex v_0 together with three edges v_0v_1, v_0v_2 and v_0v_3.

Proposition 20 *The Henneberg* 2*-move preserves infinitesimal rigidity for well-positioned bar-joint frameworks in* $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$.

Proof Suppose (G, p) is well-positioned and infinitesimally rigid and let $G \to G'$ be a Henneberg 2-move on the vertices $v_1, v_2, v_3 \in V(G)$ and the edge $v_1v_2 \in E(G)$. Let $[F_1]$ be the unique framework colour for the edge v_1v_2 and choose any $[F_2] \in \Phi(\mathcal{P})$ with $[F_2] \neq [F_1]$. Define a placement p' of G' by setting $p'_v = p_v$ for all $v \in V(G)$ and choosing p'_{v_0} to lie on the intersection of the line through p_{v_1} and p_{v_2} and the double cone $p_{v_3} + (\text{cone}(F_2)^\circ \cup -\text{cone}(F_2)^\circ)$. (If $p_{v_1}, p_{v_2}, p_{v_3}$ are collinear, then choose p'_{v_0} to lie in the intersection of this double cone and a small neighbourhood of p_{v_3}.) Then (G', p') is well-positioned. Both edges v_0v_1 and v_0v_2 have framework colour $[F_1]$ and the edge v_0v_3 has framework colour $[F_2]$. If $u \in \mathcal{F}(G', p')$, then by Theorem 5

$$\varphi_{F_1}(u_{v_1} - u_{v_2}) = \varphi_{F_1}(u_{v_1} - u_{v_0}) + \varphi_{F_1}(u_{v_0} - u_{v_2}) = 0.$$

Hence the restriction of u to $V(G)$ is an infinitesimal flex of (G, p) and must be trivial. In particular, $u_{v_1} = u_{v_3}$. Now $\varphi_{F_1}(u_{v_0} - u_{v_1}) = 0$ and $\varphi_{F_2}(u_{v_0} - u_{v_1}) = \varphi_{F_2}(u_{v_0} - u_{v_3}) = 0$ and so $u_{v_0} = u_{v_1}$. We conclude that u is a constant and hence trivial infinitesimal flex of (G', p'). □

Let v_1v_2 be an edge of G. An edge-to-K_3 move $G \to G'$ (on the edge v_1v_2 and the vertex v_1) is obtained in two steps: Firstly, adjoin a new vertex v_0 and two new edges v_0v_1 and v_0v_2 to G (creating a copy of K_3 with vertices v_0, v_1, v_2). Secondly, each edge v_1w of G which is incident with v_1 is either left unchanged or is removed and replaced with the edge v_0w.

Proposition 21 *The edge-to-K_3 move preserves infinitesimal rigidity for finite well-positioned bar-joint frameworks in* $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$.

Proof Suppose (G, p) is well-positioned and infinitesimally rigid and let $G \to G'$ be an edge-to-K_3 move on the vertex $v_1 \in V(G)$ and the edge $v_1v_2 \in E(G)$. Let $[F_1]$ be the unique framework colour for v_1v_2 and choose any $[F_2] \in \Phi(\mathcal{P})$ with $[F_2] \neq [F_1]$. Since v_1 has finite valence, there exists an open ball $B(p_{v_1}, r)$ such that if p_{v_1} is replaced with any point $x \in B(p_{v_1}, r)$, then the induced framework colouring of G is left unchanged. Define a placement p' of G' by setting $p'_v = p_v$ for all $v \in V(G)$ and choosing

$$p'_{v_0} \in (p_{v_1} + \text{cone}(F_2)^\circ) \cap B(p_{v_1}, r).$$

Then (G', p') is well-positioned. Suppose $u \in \mathcal{F}(G', p')$ is an infinitesimal flex of (G', p'). The framework colours for the edges v_0v_1 and v_0v_2 are $[F_2]$ and $[F_1]$, respectively. Thus there exists a path from v_0 to v_1 in the monochrome subgraph G'_{F_1} given by the edges v_1v_2, v_2v_0, and there exists a path from v_0 to v_1 in the monochrome subgraph G'_{F_2} given by the edge v_0v_1. By the relative rigidity method of Proposition 12, $u_{v_0} = u_{v_1}$. If an edge v_1w in G has framework colour $[F]$ induced by (G, p) and is replaced by v_0w in G', then the framework colour is unchanged. Thus applying Theorem 5,

$$\varphi_F(u_{v_1} - u_w) = \varphi_F(u_{v_1} - u_{v_0}) + \varphi_F(u_{v_0} - u_w) = 0,$$

and so the restriction of u to $V(G)$ is an infinitesimal flex of (G, p). This restriction is constant since (G, p) is infinitesimally rigid and so u is a trivial infinitesimal flex of (G', p'). □

A vertex-to-K_4 move $G \to G'$ replaces a vertex $v_0 \in V(G)$ with a copy of the complete graph K_4 by adjoining three new vertices v_1, v_2, v_3 and six edges v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_1v_3, v_2v_3. Each edge v_0w of G which is incident with v_0 may be left unchanged or replaced by one of v_1w, v_2w or v_3w.

Proposition 22 *The vertex-to-K_4 move preserves infinitesimal rigidity for finite well-positioned bar-joint frameworks in* $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$.

Proof Suppose (G, p) is well-positioned and infinitesimally rigid and let $G \to G'$ be a vertex-to-K_4 move on the vertex $v_0 \in V(G)$ which introduces new vertices v_1, v_2 and v_3. Since v_0 has finite valence, there exists an open ball $B(p_{v_0}, r)$ such that if p_{v_0} is replaced with any point $x \in B(p_{v_0}, r)$, then (G, x) and (G, p) induce the same framework colouring on G. Let $(K_4, \tilde{p})$ be the well-positioned and infinitesimally rigid placement of K_4 constructed in Example 17. Define a well-positioned placement p' of G' by setting $p'_v = p_v$ for all $v \in V(G)$ and

$$p'_{v_1} = p_{v_0} + \varepsilon\tilde{p}_{v_1}, \quad p'_{v_2} = p_{v_0} + \varepsilon\tilde{p}_{v_2}, \quad p'_{v_3} = p_{v_0} + \varepsilon\tilde{p}_{v_3},$$

where $\varepsilon > 0$ is chosen to be sufficiently small so that p'_{v_1}, p'_{v_2} and p'_{v_3} are all contained in $B(p_{v_0}, r)$. Suppose $u \in \mathcal{F}(G', p')$. By the argument in Example 17, the restriction of u to the vertices v_0, v_1, v_2, v_3 is constant. Thus if $v_0 w$ is an edge of G with framework colour $[F]$ which is replaced by $v_k w$ in G', then applying Theorem 5,

$$\varphi_F(u_{v_0} - u_w) = \varphi_F(u_{v_0} - u_{v_k}) + \varphi_F(u_{v_k} - u_w) = 0,$$

and so the restriction of u to $V(G)$ is an infinitesimal flex of (G, p). Since (G, p) is infinitesimally rigid, this restriction is constant, and we conclude that u is a trivial infinitesimal flex of (G', p'). □

We now show that the class of finite graphs which have minimally infinitesimally rigid well-positioned placements in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$ is precisely the class of $(2, 2)$-tight graphs. In particular, the existence of such a placement does not depend on the choice of polyhedral norm on $\mathbb{R}^2$.

Theorem 23 *Let G be a finite simple graph and let $\|\cdot\|_{\mathcal{P}}$ be a polyhedral norm on $\mathbb{R}^2$. The following statements are equivalent:*

(i) *G is minimally rigid in* $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$.
(ii) *G is* $(2, 2)$*-tight.*

Proof (i) ⇒ (ii). If G is minimally rigid, then there exists a placement p such that (G, p) is minimally infinitesimally rigid in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$ and the result follows from Proposition 18.

(ii) ⇒ (i). If G is $(2, 2)$-tight, then there exists a finite sequence of allowable graph moves, $K_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots \longrightarrow G$. Every placement of K_1 is certainly infinitesimally rigid. By Propositions 19–22, for each graph in the sequence there exists a well-positioned and infinitesimally rigid placement in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$. In particular, (G, p) is infinitesimally rigid for some well-positioned placement p. If a single edge is removed from G, then by Proposition 18 the resulting subframework is infinitesimally flexible. Hence (G, p) is minimally infinitesimally rigid in $(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$. □

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