BANACH SPACES WHOSE ALGEBRA OF BOUNDED OPERATORS HAS THE INTEGERS AS THEIR K_0-GROUP

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Abstract. Let X and Y be Banach spaces such that the ideal of operators which factor through Y has codimension one in the Banach algebra $\mathscr{B}(X)$ of all bounded operators on X, and suppose that Y contains a complemented subspace which is isomorphic to $Y \oplus Y$ and that X is isomorphic to $X \oplus Z$ for every complemented subspace Z of Y. Then the K_0-group of $\mathscr{B}(X)$ is isomorphic to the additive group $\mathbb{Z}$ of integers.

A number of Banach spaces which satisfy the above conditions are identified. Notably, it follows that $K_0(\mathscr{B}(C([0,\omega_1]))) \cong \mathbb{Z}$, where $C([0,\omega_1])$ denotes the Banach space of scalar-valued, continuous functions defined on the compact Hausdorff space of ordinals not exceeding the first uncountable ordinal ω_1, endowed with the order topology.

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1. Introduction

The purpose of this note is to prove that, for certain Banach spaces X, the K_0-group of the Banach algebra $\mathscr{B}(X)$ of (bounded, linear) operators on X is isomorphic to the additive group $\mathbb{Z}$ of integers. More precisely, our main result, which will be proved in Section 3, is as follows.

Theorem 1.1. *Let X and Y be Banach spaces such that:*
- (i) *Y contains a complemented subspace which is isomorphic to $Y \oplus Y$;*
- (ii) *X is isomorphic to $X \oplus Z$ for every complemented subspace Z of Y; and*
- (iii) *the ideal $\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$ of operators on X that factor through Y has codimension one in $\mathscr{B}(X)$.*

Then the mapping

$$n \mapsto n \cdot [I_X]_0, \quad \mathbb{Z} \to K_0(\mathscr{B}(X)), \tag{1.1}$$

is an isomorphism of abelian groups, where $[I_X]_0$ denotes the K_0-class of the identity operator on X.

As a consequence, we shall deduce in Section 4 that the K_0-group of $\mathscr{B}(X)$ is isomorphic to $\mathbb{Z}$ for a number of Banach spaces X, including the following:

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(i) $X = C([0,\omega_1])$, the Banach space of scalar-valued, continuous functions defined on the compact Hausdorff space of ordinals not exceeding the first uncountable ordinal ω_1, endowed with the order topology;

(ii) $X = C(K)$ or $X = C(K)\oplus Y$, where K is the compact Hausdorff space constructed by the second-named author [13], assuming either the Continuum Hypothesis, or Martin's Axiom together with the negation of the Continuum Hypothesis, and where Y is a Banach space which is isomorphic to the ℓ_p- or c_0-direct sum of countably many copies of itself for some $p \in [1,\infty)$, Y contains a complemented subspace that is isomorphic to c_0, and no complemented subspace of Y is isomorphic to $C(K)$;

(iii) $X = X_{\mathrm{AH}} \oplus C_p$, where X_{AH} is Argyros and Haydon's Banach space which solves the scalar-plus-compact problem (see [2]), $p \in [1,\infty]$, and C_p is Johnson's p^{th} universal space through which all approximable operators factor (see [9]);

(iv) $X = W \oplus Y$, where W is the non-separable Banach space constructed by Shelah and Steprāns [19] such that every operator on W is a scalar multiple of the identity plus an operator with separable range, and Y is the ℓ_p-direct sum of a certain family of separable subspaces of W for some $p \in (1,\infty)$; see Example 4.7 for details.

We also obtain a couple of known results as consequences of Theorem 1.1, namely that $K_0(\mathscr{B}(X)) \cong \mathbb{Z}$ for $X = J_p$ or $X = J_p(\omega_1)$, where $p \in (1,\infty)$ and J_p denotes the p^{th} quasi-reflexive James space, while $J_p(\omega_1)$ denotes Edgar's long version of it (see [8] and [5], respectively).

It is known that the K_0-group of $\mathscr{B}(X)$ vanishes for most "classical" Banach spaces X, including every Banach space X which is primary and isomorphic to its square $X \oplus X$ (see [14, Proposition 2.3]). By contrast, the existence of an ideal of finite codimension in $\mathscr{B}(X)$ implies that the K_0-class $[I_X]_0$ of the identity operator on X is an element of infinite order in $K_0(\mathscr{B}(X))$, as we shall show in Remark 2.3, below. Theorem 1.1 can therefore be viewed as a minimality result for $K_0(\mathscr{B}(X))$: condition (iii) implies that $[I_X]_0$ has infinite order in $K_0(\mathscr{B}(X))$, and (1.1) states that this element generates the whole group.

2. Preliminaries

We shall begin by outlining the definition of the K_0-group of a unital ring $\mathscr{A}$; further details can be found in standard texts such as [3] and [18]. For $m, n \in \mathbb{N}$, we denote by $M_{m,n}(\mathscr{A})$ the additive group of $(m \times n)$-matrices over $\mathscr{A}$. We write $M_n(\mathscr{A})$ instead of $M_{n,n}(\mathscr{A})$; this is a unital ring. Define

$$\mathrm{IP}_n(\mathscr{A}) = \{P \in M_n(\mathscr{A}) : P^2 = P\}, \quad \text{the set of idempotent } (n \times n)\text{-matrices over } \mathscr{A}.$$

Given $P \in \mathrm{IP}_m(\mathscr{A})$ and $Q \in \mathrm{IP}_n(\mathscr{A})$, where $m, n \in \mathbb{N}$, we say that P and Q are *algebraically equivalent*, written $P \sim_0 Q$, if $P = AB$ and $Q = BA$ for some $A \in M_{m,n}(\mathscr{A})$ and $B \in M_{n,m}(\mathscr{A})$. This defines an equivalence relation $\sim_0$ on the set $\mathrm{IP}_\infty(\mathscr{A}) = \bigcup_{n\in\mathbb{N}} \mathrm{IP}_n(\mathscr{A})$, and the quotient $V(\mathscr{A}) = \mathrm{IP}_\infty(\mathscr{A})/{\sim_0}$ is an abelian semigroup with respect to the operation

$$\big([P]_V, [Q]_V\big) \mapsto \left[\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}\right]_V, \quad V(\mathscr{A}) \times V(\mathscr{A}) \to V(\mathscr{A}), \tag{2.1}$$

where $[P]_V$ denotes the equivalence class of $P \in \mathrm{IP}_\infty(\mathscr{A})$ in $V(\mathscr{A})$. The K_0-group of $\mathscr{A}$, denoted by $K_0(\mathscr{A})$, is now defined as the Grothendieck group of $V(\mathscr{A})$. The fundamental property of the Grothendieck group implies that we have the following *standard picture* of $K_0(\mathscr{A})$:

$$K_0(\mathscr{A}) = \big\{[P]_0 - [Q]_0 : P, Q \in \mathrm{IP}_\infty(\mathscr{A})\big\}, \tag{2.2}$$

where $[P]_0$ is the canonical image of $[P]_V$ in $K_0(\mathscr{A})$. This image can be described more explicitly as follows: for $P, Q \in \mathrm{IP}_\infty(\mathscr{A})$,

$$[P]_0 = [Q]_0 \iff \begin{pmatrix} P & 0 \\ 0 & 1_{M_n(\mathscr{A})} \end{pmatrix} \sim_0 \begin{pmatrix} Q & 0 \\ 0 & 1_{M_n(\mathscr{A})} \end{pmatrix} \text{ for some } n \in \mathbb{N}_0, \tag{2.3}$$

where $1_{M_n(\mathscr{A})}$ denotes the $(n \times n)$-identity matrix over $\mathscr{A}$.

Let $n \in \mathbb{N}$, and suppose that $P, Q \in \mathrm{IP}_n(\mathscr{A})$ are *orthogonal*, in the sense that $PQ = 0 = QP$. Then $P + Q$ is idempotent, and the formula for addition in $K_0(\mathscr{A})$ takes the simple form

$$[P]_0 + [Q]_0 = [P + Q]_0. \tag{2.4}$$

We shall require one more basic property of K_0: given a ring homomorphism $\varphi\colon \mathscr{A} \to \mathscr{C}$ (where $\mathscr{C}$, like $\mathscr{A}$, is a unital ring, but φ need not be unital) and $n \in \mathbb{N}$, we can define a ring homomorphism $\varphi_n\colon M_n(\mathscr{A}) \to M_n(\mathscr{C})$ by entrywise application:

$$\varphi_n\big((A_{j,k})_{j,k=1}^n\big) = \big(\varphi(A_{j,k})\big)_{j,k=1}^n. \tag{2.5}$$

This induces a group homomorphism $K_0(\varphi)\colon K_0(\mathscr{A}) \to K_0(\mathscr{C})$ which satisfies

$$K_0(\varphi)([P]_0) = [\varphi_n(P)]_0 \qquad (n \in \mathbb{N},\ P \in \mathrm{IP}_n(\mathscr{A})). \tag{2.6}$$

Throughout Sections 1–4, except in Remark 3.2, all Banach spaces and algebras will be considered over a fixed scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, whereas in Remark 3.2 and Appendix A, we shall work with complex scalars only.

It is well known and elementary that the standard (unnormalized) trace

$$\mathrm{Tr}_n\colon\ (\lambda_{j,k})_{j,k=1}^n \mapsto \sum_{j=1}^n \lambda_{j,j}, \quad M_n(\mathbb{K}) \to \mathbb{K},$$

induces a group isomorphism $\tau\colon K_0(\mathbb{K}) \to \mathbb{Z}$ which satisfies

$$\tau([P]_0) = \mathrm{Tr}_n(P) \qquad (n \in \mathbb{N},\ P \in \mathrm{IP}_n(\mathbb{K})) \tag{2.7}$$

(see for instance [18, Example 3.3.2] for a proof for $\mathbb{K} = \mathbb{C}$; the proof for $\mathbb{K} = \mathbb{R}$ is similar).

The symbol $C(K)$ denotes the Banach space of scalar-valued, continuous functions defined on a compact Hausdorff space K. By an *operator*, we understand a bounded, linear mapping between Banach spaces. For Banach spaces X and Y, we write $\mathscr{B}(X, Y)$ for the Banach space of all operators from X into Y, and we identify $M_{m,n}(\mathscr{B}(X, Y))$ with the Banach space $\mathscr{B}(X^n, Y^m)$ of operators from X^n into Y^m, where X^n denotes the direct sum of n copies of X, equipped with the norm $\|(x_1, \ldots, x_n)\| = \max\{\|x_1\|, \ldots, \|x_n\|\}$ for $x_1, \ldots, x_n \in X$. We write $\mathscr{B}(X)$ instead of $\mathscr{B}(X, X)$; this is a unital Banach algebra. We denote by I_X the identity operator on X.

The following easy observation clarifies the meaning of the relation $\sim_0$ in this case.

Lemma 2.1. *Let X be a Banach space, and let $P, Q \in \mathrm{IP}_\infty(\mathscr{B}(X))$. Then $P \sim_0 Q$ if and only if the ranges of P and Q are isomorphic.*

We shall also require the following related result.

Lemma 2.2 ([16, Lemma 3.9(ii)])**.** *Let X and Y be Banach spaces, and let $S \in \mathscr{B}(X,Y)$ and $T \in \mathscr{B}(Y,X)$ be operators such that ST is idempotent. Then $TSTS$ is idempotent, and the ranges of ST and $TSTS$ are isomorphic.*

Given Banach spaces X, Y and Z, we define

$$\mathscr{G}_Y(X,Z) = \operatorname{span}\{TS : S \in \mathscr{B}(X,Y),\, T \in \mathscr{B}(Y,Z)\}.$$

This is an operator ideal in the sense of Pietsch provided that Y is non-zero. The 'span' is not necessary when Y contains a complemented subspace which is isomorphic to $Y \oplus Y$; in this case the set

$$\{TS : S \in \mathscr{B}(X,Y),\, T \in \mathscr{B}(Y,Z)\}$$

is automatically a linear subspace of $\mathscr{B}(X,Z)$. In line with standard practice, we write $\mathscr{G}_Y(X)$ instead of $\mathscr{G}_Y(X,X)$.

Remark 2.3. Let X be a Banach space for which $\mathscr{B}(X)$ contains a proper ideal $\mathscr{I}$ of finite codimension. Then, as mentioned in the Introduction, the element $[I_X]_0$ has infinite order in $K_0(\mathscr{B}(X))$. We shall now outline a simple proof of this fact. Assume towards a contradiction that $n \cdot [I_X]_0 = 0$ for some $n \in \mathbb{N}$. Then (2.1), (2.3) and Lemma 2.1 imply that the Banach spaces X^{m+n} and X^m are isomorphic for some $m \in \mathbb{N}$. Let $U \in \mathscr{B}(X^m, X^{m+n}) = M_{m+n,m}(\mathscr{B}(X))$ be an isomorphism. We can then define an algebra isomorphism $\varphi\colon M_{m+n}(\mathscr{B}(X)) \to M_m(\mathscr{B}(X))$ by $\varphi(T) = U^{-1}TU$ for each $T \in M_{m+n}(\mathscr{B}(X))$, and φ maps the ideal $M_{m+n}(\mathscr{I})$ of $M_{m+n}(\mathscr{B}(X))$ onto $M_m(\mathscr{I})$ because $\mathscr{I}$ is an ideal of $\mathscr{B}(X)$.

By elementary linear algebra, a linear bijection between two vector spaces maps a subspace of finite codimension $k \in \mathbb{N}$ in the domain onto a subspace of codimension k in the codomain. Consequently the codimension of $M_m(\mathscr{I})$ in $M_m(\mathscr{B}(X))$ is equal to the codimension of $M_{m+n}(\mathscr{I})$ in $M_{m+n}(\mathscr{B}(X))$. This, however, contradicts that a simple dimension count shows that the former codimension is jm^2, while the latter is $j(m+n)^2$, where $j \in \mathbb{N}$ denotes the codimension of $\mathscr{I}$ in $\mathscr{B}(X)$. Hence we conclude that $n \cdot [I_X]_0 \neq 0$ for each $n \in \mathbb{N}$, as required.

Note: the above result has another, perhaps more widely known, proof for complex scalars. It combines the isomorphism (2.7) with the classical theorems of Wedderburn and Frobenius that state that a simple, finite-dimensional, complex algebra is isomorphic to $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. The latter conclusion does not carry over to the real case, so this argument does not generalize easily.

3. The proof of Theorem 1.1

Throughout this section (except in Remark 3.2), we shall suppose that X and Y are Banach spaces which satisfy conditions (i)–(iii) of Theorem 1.1. The third of these conditions implies that we can define a bounded, unital algebra homomorphism $\varphi\colon \mathscr{B}(X) \to \mathbb{K}$ by

$$\varphi(\lambda I_X + T) = \lambda \qquad (\lambda \in \mathbb{K},\, T \in \mathscr{G}_Y(X)). \tag{3.1}$$

Recall that we identify $M_n(\mathscr{B}(X))$ with $\mathscr{B}(X^n)$ for each $n \in \mathbb{N}$. Under this identification, we have $M_n(\mathscr{G}_Y(X)) = \mathscr{G}_Y(X^n)$ because $\mathscr{G}_Y$ is an operator ideal, and hence

$$\ker\varphi_n = M_n(\ker\varphi) = \mathscr{G}_Y(X^n) = \{TS : S \in \mathscr{B}(X^n, Y),\, T \in \mathscr{B}(Y, X^n)\}, \tag{3.2}$$

where the final equality follows from the fact that Y satisfies condition (i).

The following lemma is the key step in the proof of the surjectivity of the mapping (1.1).

Lemma 3.1. *Let $P \in \mathrm{IP}_n(\mathscr{B}(X))$ for some $n \in \mathbb{N}$, and set $k = \mathrm{Tr}_n \circ \varphi_n(P)$. Then*

$$[P]_0 = k \cdot [I_X]_0 \qquad \textit{in} \qquad K_0(\mathscr{B}(X)).$$

Proof. We shall first establish the result for $k = 0$. In this case we have $\varphi_n(P) = 0$ because the zero matrix is the only idempotent, scalar-valued matrix with trace zero, so that $P = TS$ for some operators $S\colon X^n \to Y$ and $T\colon Y \to X^n$ by (3.2). Lemma 2.2 then implies that the operator $Q = SPT \in \mathscr{B}(Y)$ is idempotent with $Q[Y] \cong P[X^n]$. Combining this with condition (ii), we obtain $X \oplus P[X^n] \cong X \oplus Q[Y] \cong X$; that is, the operators $\left(\begin{smallmatrix} I_X & 0 \\ 0 & P \end{smallmatrix}\right)$ and I_X have isomorphic ranges. Hence $[I_X]_0 + [P]_0 = [I_X]_0$ in $K_0(\mathscr{B}(X))$ by (2.1) and Lemma 2.1, so that $[P]_0 = 0$, as required.

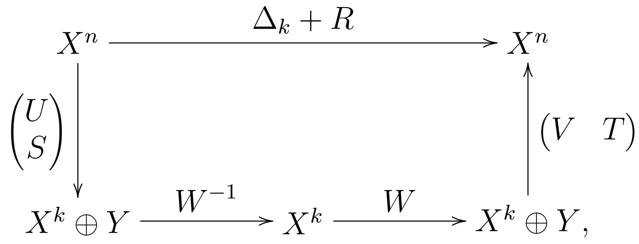
We shall next consider the case where $k = n$. Then $\mathrm{Tr}_n \circ \varphi_n(I_{X^n} - P) = 0$, so that $[I_{X^n} - P]_0 = 0$ by the result established in the first paragraph of the proof. Hence, by (2.4) and (2.1), we conclude that

$$[P]_0 = [P]_0 + [I_{X^n} - P]_0 = [I_{X^n}]_0 = n \cdot [I_X]_0 \qquad \text{in} \qquad K_0(\mathscr{B}(X)).$$

Finally, suppose that $k \in \{1, 2, \ldots, n-1\}$. Since $\varphi_n(P)$ is an idempotent, scalar-valued matrix, it is diagonalizable, so that there exists $R \in \ker\varphi_n$ such that $\Delta_k + R$ is idempotent and $P \sim_0 \Delta_k + R$, where

$$\Delta_k = \begin{pmatrix} I_{X^k} & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{IP}_n(\mathscr{B}(X)).$$

By (3.2), we can find operators $S\colon X^n \to Y$ and $T\colon Y \to X^n$ such that $R = TS$. Moreover, Δ_k has an obvious factorization as $\Delta_k = VU$, where $U\colon X^n \to X^k$ and $V\colon X^k \to X^n$ denote the projection onto the first k coordinates and the embedding into the first k coordinates, respectively. Condition (ii) implies that there exists an isomorphism $W\colon X^k \to X^k \oplus Y$, and we then have a commutative diagram



where the operators $\binom{U}{S}$ and $(V\ \ T)$ are given by $x \mapsto (Ux, Sx)$ and $(x, y) \mapsto Vx + Ty$, respectively. Hence Lemma 2.2 shows that the operator

$$Q = W^{-1}\begin{pmatrix} U \\ S \end{pmatrix}(\Delta_k + R)\begin{pmatrix} V & T \end{pmatrix} W \in \mathscr{B}(X^k)$$

is idempotent and the ranges of Q and $\Delta_k + R$ are isomorphic, so that $Q \sim_0 P$ by Lemma 2.1. The trace property implies that $\mathrm{Tr}_k \circ \varphi_k(Q) = \mathrm{Tr}_n \circ \varphi_n(\Delta_k + R) = k$, and therefore, as shown in the second paragraph of the proof, we have $k \cdot [I_X]_0 = [Q]_0 = [P]_0$, as required. □

Proof of Theorem 1.1. We shall show below that the group homomorphism

$$\tau \circ K_0(\varphi)\colon K_0(\mathscr{B}(X)) \to \mathbb{Z}, \tag{3.3}$$

where τ is the isomorphism given by (2.7), is an isomorphism. Since $\tau \circ K_0(\varphi)([I_X]_0) = 1$, which generates the group $\mathbb{Z}$, the mapping given by (1.1) is the inverse of this isomorphism, and hence the conclusion follows.

The surjectivity of the homomorphism (3.3) is immediate because, as observed above, its range contains the generator 1 of the group $\mathbb{Z}$.

To see that the homomorphism (3.3) is injective, suppose that $g \in \ker \tau \circ K_0(\varphi)$. By (2.2), we have $g = [P]_0 - [Q]_0$ for some $P \in \mathrm{IP}_m(\mathscr{B}(X))$ and $Q \in \mathrm{IP}_n(\mathscr{B}(X))$, where $m, n \in \mathbb{N}$. Using (2.6) and (2.7), we obtain

$$0 = \tau \circ K_0(\varphi)(g) = \mathrm{Tr}_m \circ \varphi_m(P) - \mathrm{Tr}_n \circ \varphi_n(Q).$$

This implies that $[P]_0 = [Q]_0$ by Lemma 3.1, so that $g = 0$. □

Remark 3.2. Let X be a Banach space for which $\mathscr{B}(X)$ contains a closed ideal of codimension one. In the light of Theorem 1.1, one may ask whether such an ideal is necessarily of the form $\mathscr{G}_Y(X)$ for some complemented subspace Y of X. To see that this need not be the case, suppose that X is *hereditarily indecomposable*, in the sense that X is infinite-dimensional and whenever a closed subspace W of X is decomposed into a direct sum of two closed subspaces A and B, either A or B is finite-dimensional. Gowers and Maurey [7] proved that such Banach spaces exist and that, in the case where $\mathbb{K} = \mathbb{C}$, the ideal $\mathscr{S}(X)$ of strictly singular operators has codimension one in $\mathscr{B}(X)$.

We shall now show that no complemented subspace Y of a hereditarily indecomposable Banach space X satisfies $\mathscr{S}(X) = \mathscr{G}_Y(X)$. Indeed, since X is indecomposable, a complemented subspace Y of X is either finite-dimensional or finite-codimensional. In the first case, each operator which factors through Y has finite rank, and so $\mathscr{G}_Y(X) \subsetneqq \mathscr{S}(X)$, while in the second case, any idempotent operator P on X with range Y factors through Y, but P is not strictly singular (since Y is infinite-dimensional and the restriction of P to Y is the identity), so that $\mathscr{G}_Y(X) \not\subseteq \mathscr{S}(X)$ (and consequently $\mathscr{G}_Y(X) = \mathscr{B}(X)$ by [16, Proposition 7.2]).

In fact, the conclusion of Theorem 1.1 fails for each complex, hereditarily indecomposable Banach space X: applying [14, Corollary 4.7] for $m = 1$, we see that $K_0(\mathscr{B}(X)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

4. Applications

Example 4.1. Assuming either the Continuum Hypothesis, or Martin's Axiom together with the negation of the Continuum Hypothesis, the second-named author [13] has constructed a scattered compact Hausdorff space K such that:

(1) the ideal $\mathscr{X}(C(K))$ of operators with separable range has codimension one in $\mathscr{B}(C(K))$;
(2) every separable subspace of $C(K)$ is contained in a subspace which is isomorphic to c_0;
(3) whenever $C(K)$ is decomposed into a direct sum of two closed, infinite-dimensional subspaces A and B, either $A \cong c_0$ and $B \cong C(K)$, or *vice versa.*

We claim that $X = C(K)$ and $Y = c_0$ satisfy conditions (i)–(iii) of Theorem 1.1. Indeed, (i) is clear, while condition (2), above, implies that $\mathscr{X}(C(K)) = \mathscr{G}_{c_0}(C(K))$ (as already observed in [10, Theorem 5.5]), and hence (iii) is satisfied by (1). Finally, to verify (ii), we observe that:

- $C(K)$ contains a complemented subspace which is isomorphic to c_0 because K is scattered (see, *e.g.*, [6, Theorem 12.30(iv)]), and consequently $C(K) \cong C(K) \oplus c_0$ by (3), above; and
- every complemented subspace Z of c_0 is either finite-dimensional or isomorphic to c_0, and therefore $c_0 \oplus Z \cong c_0$.

Hence $C(K) \oplus Z \cong C(K) \oplus c_0 \oplus Z \cong C(K) \oplus c_0 \cong C(K)$, as required. Thus we conclude that $K_0(\mathscr{B}(C(K))) \cong \mathbb{Z}$. It is not known whether a compact space with the same properties as K can be constructed within ZFC.

In order to facilitate further applications of Theorem 1.1, we shall show that certain standard properties of Banach spaces ensure that the first two conditions of Theorem 1.1 are satisfied.

Lemma 4.2. *Let X and Y be Banach spaces such that X is isomorphic to $X \oplus Y$, and suppose that Y satisfies (at least) one of the following two conditions:*

(1) *Y is isomorphic to the ℓ_p- or the c_0-direct sum of countably many copies of itself for some $p \in [1, \infty)$; or*
(2) *Y is primary and contains a complemented subspace which is isomorphic to $Y \oplus Y$.*

Then conditions (i)–(ii) *of Theorem* 1.1 *are satisfied.*

Proof. Condition (i) of Theorem 1.1 is clearly satisfied in both cases.

To verify condition (ii), suppose that Z is a complemented subspace of Y.

In case (1), we observe that $Y \cong Y \oplus Y$, so that Y contains a complemented subspace which is isomorphic to $Y \oplus Z$. On the other hand, $Y \oplus Z$ evidently contains a complemented subspace which is isomorphic to Y. Hence the Pełczyński decomposition method (as stated in [1, Theorem 2.23(b)], for instance) implies that $Y \cong Y \oplus Z$. Combining this with the assumption that $X \cong X \oplus Y$, we obtain

$$X \oplus Z \cong X \oplus Y \oplus Z \cong X \oplus Y \cong X, \tag{4.1}$$

as required.

In case (2), either $Y \cong Z$ or $Y \cong Y \oplus Z$ because Y is primary. In the first case, $X \oplus Z \cong X$ is immediate from the fact that $X \cong X \oplus Y$, and in the second case the calculation (4.1), above, applies to give this conclusion. □

Example 4.3. For an ordinal α, denote by $[0, \alpha]$ the compact Hausdorff space consisting of all ordinals not exceeding α, endowed with the order topology, and set $X = C([0, \omega_1])$ and $Y = \left(\bigoplus_{\alpha<\omega_1} C([0, \alpha])\right)_{c_0}$, where ω_1 denotes the first uncountable ordinal. Then:

- $X \cong X \oplus Y$ by [11, Lemma 2.14(iv) and Corollary 2.16];
- Y is isomorphic to the c_0-direct sum of countably many copies of itself by [11, Lemma 2.12], so that condition (1) of Lemma 4.2 is satisfied (in fact, condition (2) is also satisfied by [12, Corollary 1.3]);
- the ideal $\mathscr{G}_Y(X)$ has codimension one in $\mathscr{B}(X)$ by [11, Theorem 1.6].

Hence Lemma 4.2 and Theorem 1.1 apply, so that $K_0(\mathscr{B}(X)) \cong \mathbb{Z}$.

This example provided the original motivation behind Theorem 1.1. We have since learnt that a different approach may be possible for complex scalars, based on a result of Edelstein and Mityagin, which will also show that the K_1-group of $\mathscr{B}(C([0, \omega_1]))$ vanishes; we refer to Appendix A for details.

Example 4.4. Let $p \in (1, \infty)$, and let $X = J_p$ be the p^{th} quasi-reflexive James space, which is defined by $J_p = \{x \in c_0 : \|x\|_{J_p} < \infty\}$, where

$$\|x\|_{J_p} = \sup\left\{ \Big(\sum_{j=1}^{m} |x_{k_j} - x_{k_{j+1}}|^p\Big)^{\frac{1}{p}} : m, k_1, \ldots, k_{m+1} \in \mathbb{N},\ k_1 < k_2 < \cdots < k_{m+1} \right\} \in [0, \infty]$$

for each scalar sequence $x = (x_k)_{k\in\mathbb{N}}$. (This space was first considered for $p = 2$ by James [8] and later generalized to arbitrary p by Edelstein and Mityagin [4, p. 229].) Moreover, let $Y = \left(\bigoplus_{n\in\mathbb{N}} J_p^{(n)}\right)_{\ell_p}$, where $J_p^{(n)}$ denotes the n-dimensional subspace of J_p consisting of those elements which vanish from the $(n+1)^{\text{st}}$ coordinate onwards. Then $X \cong X \oplus Y$ and Y is isomorphic to the ℓ_p-direct sum of countably many copies of itself by [4, Lemmas 5–6]. (Note, however, that a key condition appears to be missing in the statement of [4, Lemma 5], namely that the sequence denoted by ν is unbounded.) Further, the ideal $\mathscr{G}_Y(X)$ has codimension one in $\mathscr{B}(X)$ by [17, Theorem 4.3], so that Lemma 4.2 and Theorem 1.1 show that $K_0(\mathscr{B}(J_p)) \cong \mathbb{Z}$. This reproves [15, Theorem 4.6], whose proof inspired our proof of Theorem 1.1, above.

Kochanek and the first-named author have observed that the results obtained in [10, Section 3] ensure that the proof of [15, Theorem 4.6] carries over to *Edgar's long James space* $J_p(\omega_1)$, originally introduced in [5], so that $K_0(\mathscr{B}(J_p(\omega_1))) \cong \mathbb{Z}$ (see [10, Proposition 3.13]). Our results provide an explicit proof of this conclusion, using [10]. Indeed, let $X = J_p(\omega_1)$, and define $Y = \left(\bigoplus_{\alpha\in L} J_p(\alpha)\right)_{\ell_p}$, where L denotes the set of countably infinite limit ordinals and $J_p(\alpha)$ is the closed subspace of $J_p(\omega_1)$ spanned by the indicator functions of the ordinal intervals $[0, \beta]$ for $\beta < \alpha$. Then [10, Proposition 3.3, Lemma 3.4 and Theorem 3.7] show that $X \cong X \oplus Y$, that Y is isomorphic to the ℓ_p-direct sum of countably many copies of itself, and that the ideal $\mathscr{G}_Y(X)$ has codimension one in $\mathscr{B}(X)$, so that Lemma 4.2 and Theorem 1.1 apply.

Our final applications of Theorem 1.1 rely on the following general observation.

Lemma 4.5. *Suppose that $X = W \oplus Y$, where W and Y are Banach spaces such that:*

(1) *Y is isomorphic to the ℓ_p- or c_0-direct sum of countably many copies of itself for some $p \in [1,\infty)$; and*
(2) *the ideal $\mathscr{G}_Y(W)$ has codimension one in $\mathscr{B}(W)$.*

Then X and Y satisfy conditions (i)–(iii) *of Theorem* 1.1, *and hence $K_0(\mathscr{B}(X)) \cong \mathbb{Z}$.*

Proof. Condition (1) ensures that $Y \cong Y \oplus Y$, so that $X \cong X \oplus Y$, and Lemma 4.2 therefore shows that conditions (i)–(ii) of Theorem 1.1 are satisfied.

To verify condition (iii), we use the fact that each operator $T \in \mathscr{B}(X)$ can be represented as a (2×2)-matrix

$$T = \begin{pmatrix} T_{1,1}\colon W \to W & T_{1,2}\colon Y \to W \\ T_{2,1}\colon W \to Y & T_{2,2}\colon Y \to Y \end{pmatrix},$$

and T factors through Y if and only if $T_{j,k}$ does for each pair $j,k \in \{1,2\}$. Since $T_{j,k}$ trivially factors through Y for $(j,k) \neq (1,1)$, condition (2) shows that $\mathscr{G}_Y(X)$ has codimension one in $\mathscr{B}(X)$. □

Example 4.6. Let $W = X_{\mathrm{AH}}$ be Argyros and Haydon's Banach space which solves the scalar-plus-compact problem (see [2]), and, for some $p \in [1,\infty]$, let $Y = C_p$ be Johnson's p^{th} universal space with the property that all approximable operators factor through C_p (see [9]). Then, as noted in [9, p. 341], Y is (isometrically) isomorphic to either the ℓ_p-direct sum (for $p < \infty$) or the c_0-direct sum (for $p = \infty$) of countably many copies of itself.

Moreover, every compact operator T on W is approximable because W has a Schauder basis, and therefore T factors through Y by the fundamental property of Y (see [9, Theorem 1]). Hence we have $\mathscr{K}(W) \subseteq \mathscr{G}_Y(W)$. To show that these two ideals are equal, we assume the contrary. Then, as $\mathscr{K}(W)$ has codimension one in $\mathscr{B}(W)$, necessarily $I_W \in \mathscr{G}_Y(W)$, so that Lemma 2.2 implies that Y contains a complemented subspace which is isomorphic to W. However, as observed in [9, p. 341], every closed, infinite-dimensional subspace of Y contains a subspace which is isomorphic to ℓ_p (for $p < \infty$) or c_0 (for $p = \infty$), but no subspace of W is isomorphic to ℓ_p or c_0 because W is hereditarily indecomposable by [2, Theorem 8.11]. This contradiction proves that $\mathscr{G}_Y(W) = \mathscr{K}(W)$. In particular $\mathscr{G}_Y(W)$ has codimension one in $\mathscr{B}(W)$, so that Lemma 4.5 implies that $K_0(\mathscr{B}(X_{\mathrm{AH}} \oplus C_p)) \cong \mathbb{Z}$.

Example 4.7. Let W be the non-separable Banach space constructed by Shelah and Steprāns [19] such that the ideal $\mathscr{X}(W)$ of operators with separable range has codimension one in $\mathscr{B}(W)$, and choose a family $(Y_\gamma)_{\gamma\in\Gamma}$ of closed, separable subspaces of W such that:

(1) every closed, separable subspace of W is isomorphic to Y_γ for some $\gamma \in \Gamma$; and
(2) every subspace Y_β is repeated countably many times in the family $(Y_\gamma)_{\gamma\in\Gamma}$, in the sense that the set $\{\gamma \in \Gamma : Y_\gamma = Y_\beta\}$ is countably infinite for each $\beta \in \Gamma$.

Set $Y = \big(\bigoplus_{\gamma\in\Gamma} Y_\gamma\big)_{\ell_p}$ for some $p \in (1,\infty)$. Condition (2) ensures that Y is isomorphic to the ℓ_p-direct sum of countably many copies of itself.

We shall now proceed to show that $\mathscr{X}(W) = \mathscr{G}_Y(W)$. Indeed, for each $T \in \mathscr{X}(W)$, we can choose $\gamma \in \Gamma$ such that there is an isomorphism U of $\overline{T[W]}$ onto Y_γ. Let $\iota_\gamma \colon Y_\gamma \to Y$ and $\pi_\gamma \colon Y \to Y_\gamma$ denote the canonical γ^{th} coordinate embedding and projection, respectively. Then we have $T = SR$, where the operators R and S given by $R\colon w \mapsto \iota_\gamma UTw$, $W \to Y$, and $S\colon y \mapsto U^{-1}\pi_\gamma y$, $Y \to W$. This shows that $T \in \mathscr{G}_Y(W)$, and therefore the inclusion $\mathscr{X}(W) \subseteq \mathscr{G}_Y(W)$ holds.

On the other hand, the Banach space Y is weakly compactly generated, so that the same is true for each of its complemented subspaces. Wark [20, Proposition 2] has shown that W is not weakly compactly generated. Hence no complemented subspace of Y is isomorphic to W, so that $I_W \notin \mathscr{G}_Y(W)$ by Lemma 2.2. Thus we conclude that $\mathscr{G}_Y(W) = \mathscr{X}(W)$, and therefore Lemma 4.5 shows that $K_0(\mathscr{B}(W \oplus Y)) \cong \mathbb{Z}$.

Example 4.8. Assume either the Continuum Hypothesis, or Martin's Axiom together with the negation of the Continuum Hypothesis, and let $W = C(K)$, where K is the scattered compact Hausdoff space described in Example 4.1. Suppose that Y is a Banach space such that:

(1) Y is isomorphic to the ℓ_p- or c_0-direct sum of countably many copies of itself for some $p \in [1,\infty)$;
(2) Y contains a complemented subspace which is isomorphic to c_0; and
(3) no complemented subspace of Y is isomorphic to W.

Condition (3) ensures that the ideal $\mathscr{G}_Y(W)$ is proper, while condition (2) implies that it contains the ideal $\mathscr{G}_{c_0}(W)$, which has codimension one in $\mathscr{B}(W)$. Hence $\mathscr{G}_Y(W) = \mathscr{G}_{c_0}(W)$, so that $\mathscr{G}_Y(W)$ has codimension one in $\mathscr{B}(W)$. The conditions of Lemma 4.5 are therefore satisfied, and thus $K_0(\mathscr{B}(W \oplus Y)) \cong \mathbb{Z}$.

For instance, conditions (1)–(3), above, are satisfied for $Y = C(M)$, where M is any infinite, compact metric space.

Appendix A. An alternative approach based on homotopy

Edelstein and Mityagin stated in [4, Proposition 4] that the invertible group of the Banach algebra $\mathscr{B}(C([0,\omega_1])^n)$ is homotopy equivalent to the invertible group of scalar-valued $(n \times n)$-matrices for each $n \in \mathbb{N}$. The aim of this appendix is to apply this result in the complex case to show that the K_1-group of $\mathscr{B}(C([0,\omega_1]))$ vanishes, and to explain how a slightly stronger version of it can be used to reprove the conclusion of Example 4.3 that $K_0(\mathscr{B}(C([0,\omega_1]))) \cong \mathbb{Z}$. Our approach works for complex scalars only, so throughout this appendix we shall suppose that $\mathbb{K} = \mathbb{C}$.

The K_1-group will be the main object of interest, so we shall begin by defining it formally. In contrast to the purely ring-theoretic definition of K_0, topology plays a key role here. Let $\mathscr{A}$ be a complex, unital Banach algebra. For each $n \in \mathbb{N}$, we turn $M_n(\mathscr{A})$ into a Banach algebra by identifying it with its natural image in the Banach algebra $\mathscr{B}(\mathscr{A}^n)$ of operators acting on the direct sum of n copies of $\mathscr{A}$, where $\mathscr{A}^n$ is equipped with the norm $\|(A_1,\ldots,A_n)\| = \max\{\|A_1\|,\ldots,\|A_n\|\}$ for $A_1,\ldots,A_n \in \mathscr{A}$, as in Section 2.

Note. For a Banach space X, we have now equipped $M_n(\mathscr{B}(X))$ with two potentially different norms, one coming from its identification with $\mathscr{B}(X^n)$, the other arising from its embedding into $\mathscr{B}(\mathscr{B}(X)^n)$. Fortunately, these two norms are equal, as is easily checked.

Let $\operatorname{inv}_n(\mathscr{A})$ be the group of invertible elements of $M_n(\mathscr{A})$, and denote by $1_{M_n(\mathscr{A})}$ the $(n\times n)$-identity matrix over $\mathscr{A}$. Given $U \in \operatorname{inv}_m(\mathscr{A})$ and $V \in \operatorname{inv}_n(\mathscr{A})$, where $m,n\in\mathbb{N}$, we say that U and V are K_1-*equivalent*, written $U \sim_1 V$, if, for some integer $k \geqslant \max\{m,n\}$, there exists a continuous path $t\mapsto W_t$, $[0,1]\to \operatorname{inv}_k(\mathscr{A})$, such that

$$W_0 = \begin{pmatrix} U & 0 \\ 0 & 1_{M_{k-m}(\mathscr{A})}\end{pmatrix} \qquad \text{and} \qquad W_1 = \begin{pmatrix} V & 0 \\ 0 & 1_{M_{k-n}(\mathscr{A})}\end{pmatrix}. \tag{A.1}$$

This defines an equivalence relation $\sim_1$ on the set $\operatorname{inv}_\infty(\mathscr{A}) = \bigcup_{n\in\mathbb{N}} \operatorname{inv}_n(\mathscr{A})$, and the quotient

$$K_1(\mathscr{A}) = \operatorname{inv}_\infty(\mathscr{A})/{\sim_1}$$

is an abelian group with respect to the operation

$$\big([U]_1,[V]_1\big) \mapsto \left[\begin{pmatrix} U & 0 \\ 0 & V\end{pmatrix}\right]_1, \quad K_1(\mathscr{A})\times K_1(\mathscr{A}) \to K_1(\mathscr{A}),$$

where $[U]_1$ denotes the equivalence class of $U\in \operatorname{inv}_\infty(\mathscr{A})$ in $K_1(\mathscr{A})$.

We see immediately from these definitions that $K_1(\mathscr{A}) = \{0\}$ whenever $\mathscr{A}$ is a complex, unital Banach algebra for which $\operatorname{inv}_n(\mathscr{A})$ is path-connected for each $n\in\mathbb{N}$. This conclusion applies in particular to $\mathscr{A} = \mathscr{B}(C([0,\omega_1]))$ by the above-mentioned result [4, Proposition 4] of Edelstein and Mityagin because $\operatorname{inv}_n(\mathbb{C})$ is path-connected for each $n\in\mathbb{N}$.

Our next aim is to show that a much more general conclusion can be drawn whenever we have a homotopy equivalence between the groups of invertible matrices over two complex, unital Banach algebras, provided that these homotopies are induced by bounded, unital algebra homomorphisms; see Corollary A.2, below, for details. This result will rely on *Bott periodicity,* which is the statement that

$$K_0(\mathscr{A}) \cong K_1(\widetilde{S\mathscr{A}}) \tag{A.2}$$

for each complex, unital Banach algebra $\mathscr{A}$, where $\widetilde{S\mathscr{A}}$ denotes the *unitization* of the *suspension* of $\mathscr{A}$, that is,

$$\widetilde{S\mathscr{A}} = \{f\in C([0,1],\mathscr{A}) : f(0) = f(1) \in \mathbb{C}1_{\mathscr{A}}\};$$

here $C([0,1],\mathscr{A})$ denotes the Banach algebra of continuous, $\mathscr{A}$-valued functions defined on the unit interval $[0,1]$, and $1_{\mathscr{A}}$ is the multiplicative identity of $\mathscr{A}$. We may identify $M_n(C([0,1],\mathscr{A}))$ with $C([0,1],M_n(\mathscr{A}))$ for each $n\in\mathbb{N}$; under this identification, we have

$$\operatorname{inv}_n \widetilde{S\mathscr{A}} = \{f\in C([0,1],\operatorname{inv}_n\mathscr{A}) : f(0) = f(1) \in M_n(\mathbb{C}1_{\mathscr{A}})\}. \tag{A.3}$$

We shall require the following two homomorphisms associated with a bounded, unital algebra homomorphism $\varphi\colon \mathscr{A}\to\mathscr{C}$, where $\mathscr{A}$ and $\mathscr{C}$ are complex, unital Banach algebras.

First, in analogy with (2.6), we can define a group homomorphism $K_1(\varphi)\colon K_1(\mathscr{A}) \to K_1(\mathscr{C})$ by

$$K_1(\varphi)([U]_1) = [\varphi_n(U)]_1 \qquad (n \in \mathbb{N},\, U \in \operatorname{inv}_n \mathscr{A}), \tag{A.4}$$

where φ_n is given by (2.5), and secondly, we obtain a bounded, unital algebra homomorphism $\widetilde{S\varphi}\colon \widetilde{S\mathscr{A}} \to \widetilde{S\mathscr{C}}$ by the definition $\widetilde{S\varphi}(f) = \varphi \circ f$ for each $f \in \widetilde{S\mathscr{A}}$.

We can now state our key lemma; it is probably well known to experts, but since we have been unable to locate a precise reference to it, we include a proof.

Lemma A.1. *Let $\mathscr{A}$ and $\mathscr{C}$ be complex, unital Banach algebras, let $n \in \mathbb{N}$, and let $\varphi\colon \mathscr{A} \to \mathscr{C}$ and $\psi\colon \mathscr{C} \to \mathscr{A}$ be bounded, unital algebra homomorphisms such that the restriction to $\operatorname{inv}_n \mathscr{A}$ of the mapping $\psi_n \circ \varphi_n$ is homotopy equivalent to the identity mapping, in the sense that there exists a continuous mapping $F\colon [0,1] \times \operatorname{inv}_n \mathscr{A} \to \operatorname{inv}_n \mathscr{A}$ such that $F(0,U) = U$ and $F(1,U) = \psi_n \circ \varphi_n(U)$ for each $U \in \operatorname{inv}_n \mathscr{A}$. Then*

$$K_1(\widetilde{S\psi}) \circ K_1(\widetilde{S\varphi})([f]_1) = [f]_1 \qquad (f \in \operatorname{inv}_n \widetilde{S\mathscr{A}}).$$

Proof. Given $f \in \operatorname{inv}_n \widetilde{S\mathscr{A}}$, we define $g_t(r) = F(t, f(0))^{-1} F(t, f(r)) \in \operatorname{inv}_n \mathscr{A}$ for each pair $r, t \in [0,1]$, where F is chosen as above. An easy check using (A.3) shows that $g_t \in \operatorname{inv}_n \widetilde{S\mathscr{A}}$ for each $t \in [0,1]$. Moreover, the mapping $(r,t) \mapsto g_t(r)$, $[0,1]^2 \to \operatorname{inv}_n \mathscr{A}$, is continuous, and it is therefore uniformly continuous, so that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|g_t(r) - g_{t'}(r')\|_{M_n(\mathscr{A})} \leqslant \varepsilon$ whenever $r, r', t, t' \in [0,1]$ satisfy $\max\{|r - r'|, |t - t'|\} \leqslant \delta$. This implies that

$$\|g_t - g_{t'}\|_{C([0,1], M_n(\mathscr{A}))} = \sup_{r \in [0,1]} \|g_t(r) - g_{t'}(r)\|_{M_n(\mathscr{A})} \leqslant \varepsilon \qquad (t, t' \in [0,1],\, |t - t'| \leqslant \delta),$$

which shows that the mapping $t \mapsto g_t$, $[0,1] \to \operatorname{inv}_n \widetilde{S\mathscr{A}}$, is continuous. Hence we have

$$[f(0)^{-1} \cdot f]_1 = [f(0)^{-1} \cdot (\psi_n \circ \varphi_n \circ f)]_1 \quad \text{in} \quad K_1(\widetilde{S\mathscr{A}}) \tag{A.5}$$

because $g_0(r) = f(0)^{-1} f(r)$ and

$$g_1(r) = (\psi_n \circ \varphi_n)(f(0))^{-1} (\psi_n \circ \varphi_n)(f(r)) = f(0)^{-1} (\psi_n \circ \varphi_n \circ f)(r) \qquad (r \in [0,1]),$$

where we have used the fact that $\psi_n \circ \varphi_n(U) = U$ for each $U \in M_n(\mathbb{C}1_{\mathscr{A}})$.

Since $\operatorname{inv}_n(\mathbb{C}1_{\mathscr{A}})$ is homeomorphic to $\operatorname{inv}_n \mathbb{C}$, it is path-connected. We can therefore choose a continuous mapping $t \mapsto V_t$, $[0,1] \to \operatorname{inv}_n(\mathbb{C}1_{\mathscr{A}})$, such that $V_0 = 1_{M_n(\mathscr{A})}$ and $V_1 = f(0)^{-1}$. This implies that the mappings $t \mapsto V_t \cdot f$ and $t \mapsto V_t \cdot (\psi_n \circ \varphi_n \circ f)$ of $[0,1]$ into $\operatorname{inv}_n \widetilde{S\mathscr{A}}$ are continuous. They connect f with $f(0)^{-1} \cdot f$ and $\psi_n \circ \varphi_n \circ f$ with $f(0)^{-1} \cdot (\psi_n \circ \varphi_n \circ f)$, respectively. When combined with (A.5), this shows that

$$[f]_1 = [f(0)^{-1} \cdot f]_1 = [f(0)^{-1} \cdot (\psi_n \circ \varphi_n \circ f)]_1 = [\psi_n \circ \varphi_n \circ f]_1 = K_1(\widetilde{S\psi}) \circ K_1(\widetilde{S\varphi})([f]_1),$$

as required. ☐

Corollary A.2. *Let $\mathscr{A}$ and $\mathscr{C}$ be complex, unital Banach algebras, and suppose that there exist bounded, unital algebra homomorphisms $\varphi\colon \mathscr{A} \to \mathscr{C}$ and $\psi\colon \mathscr{C} \to \mathscr{A}$ such that the restrictions $\varphi_n\colon \operatorname{inv}_n \mathscr{A} \to \operatorname{inv}_n \mathscr{C}$ and $\psi_n\colon \operatorname{inv}_n \mathscr{C} \to \operatorname{inv}_n \mathscr{A}$ induce a homotopy equivalence* *for each $n \in \mathbb{N}$, in the sense that $\psi_n \circ \varphi_n$ is homotopy equivalent to the identity mapping on $\operatorname{inv}_n \mathscr{A}$ and $\varphi_n \circ \psi_n$ is homotopy equivalent to the identity mapping on $\operatorname{inv}_n \mathscr{C}$. Then*

$$K_0(\mathscr{A}) \cong K_0(\mathscr{C}) \qquad \text{and} \qquad K_1(\mathscr{A}) \cong K_1(\mathscr{C}). \tag{A.6}$$

Proof. Using the assumptions in tandem with Lemma A.1, we see that $K_1(\widetilde{S\varphi})$ is an isomorphism of $K_1(\widetilde{S\mathscr{A}})$ onto $K_1(\widetilde{S\mathscr{C}})$ with inverse $K_1(\widetilde{S\psi})$. Hence we have

$$K_0(\mathscr{A}) \cong K_1(\widetilde{S\mathscr{A}}) \cong K_1(\widetilde{S\mathscr{C}}) \cong K_0(\mathscr{C})$$

by two applications of Bott periodicity (A.2). This establishes the first part of (A.6).

The second part is much simpler. Indeed, working straight from the definitions (A.1) and (A.4), we obtain

$$K_1(\psi) \circ K_1(\varphi)([U]_1) = [\psi_n \circ \varphi_n(U)]_1 = [U]_1 \qquad (n \in \mathbb{N},\, U \in \operatorname{inv}_n \mathscr{A})$$

because $\psi_n \circ \varphi_n$ is homotopy equivalent to the identity mapping on $\operatorname{inv}_n \mathscr{A}$. A similar argument shows that $K_1(\varphi) \circ K_1(\psi)$ is equal to the identity on $K_1(\mathscr{C})$, and $K_1(\varphi)$ is therefore an isomorphism of $K_1(\mathscr{A})$ onto $K_1(\mathscr{C})$ with inverse $K_1(\psi)$. □

We shall now combine this result with the work of Edelstein and Mityagin [4] to obtain alternative proofs of some previous conclusions.

Example A.3. Let $p \in (1, \infty)$. Edelstein and Mityagin [4, p. 225 and 229] identified a non-zero, multiplicative functional $\beta\colon \mathscr{B}(J_p) \to \mathbb{C}$ and proved that the mapping $\sigma_n \circ \beta_n$ is homotopy equivalent to the identity mapping on $\operatorname{inv}_n(\mathscr{B}(J_p))$ for each $n \in \mathbb{N}$, where $\sigma\colon \mathbb{C} \to \mathscr{B}(J_p)$ denotes the isometric, unital algebra homomorphism given by $\sigma(\lambda) = \lambda I_{J_p}$ for each $\lambda \in \mathbb{C}$. Clearly $\beta \circ \sigma = I_{\mathbb{C}}$, so that Corollary A.2 applies to show that

$$K_0(\mathscr{B}(J_p)) \cong K_0(\mathbb{C}) \cong \mathbb{Z} \qquad \text{and} \qquad K_1(\mathscr{B}(J_p)) \cong K_1(\mathbb{C}) = \{0\}.$$

This reproves [15, Theorem 4.6]; see also Example 4.4.

Example A.4. At the end of their paper, Edelstein and Mityagin [4, p. 230] stated that their constructions for the James spaces, on which Example A.3, above, was based, "can be carried out (with some modifications) also in the case of the Banach space $C([0, \omega_1])$". They then went on to identify a non-zero, multiplicative functional $\beta\colon \mathscr{B}(C([0, \omega_1])) \to \mathbb{C}$ before stating as their main conclusion that $\operatorname{inv}_n(\mathscr{B}(C([0, \omega_1])))$ is homotopy equivalent to $\operatorname{inv}_n(\mathbb{C})$ for each $n \in \mathbb{N}$. No explicit proof of this result is given, but a comparison with the authors' approach for the James spaces suggests that their intended strategy was to show that $\sigma_n \circ \beta_n$ is homotopy equivalent to the identity mapping on $\operatorname{inv}_n \mathscr{B}(C([0, \omega_1]))$ for each $n \in \mathbb{N}$, where $\sigma\colon \mathbb{C} \to \mathscr{B}(C([0, \omega_1]))$ is given by $\sigma(\lambda) = \lambda I_{C([0,\omega_1])}$ for each $\lambda \in \mathbb{C}$. If this is indeed the case, then Corollary A.2 would apply once again, showing that

$$K_0(\mathscr{B}(C([0, \omega_1]))) \cong K_0(\mathbb{C}) \cong \mathbb{Z} \qquad \text{and} \qquad K_1(\mathscr{B}(C([0, \omega_1]))) \cong K_1(\mathbb{C}) = \{0\},$$

and thus providing an alternative proof of the conclusion of Example 4.3.

Comparing the calculations of the K_0-groups of $\mathscr{B}(J_p)$ for $p \in (1, \infty)$ and $\mathscr{B}(C([0, \omega_1]))$ given in Examples 4.3–4.4 with those given in Examples A.3–A.4, above, we see some obvious advantages of the latter, namely that they are shorter and simultaneously lead to the determination of the K_1-groups; however, they also have some significant drawbacks:

- they do not apply to real scalars;
- they rely on some very heavy machinery, notably Bott periodicity, but also Edelstein and Mityagin's highly non-trivial results (which have not even been fully verified in the case of $C([0,\omega_1])$);
- they are entirely topological, despite the purely ring-theoretic nature of K_0.

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References

1. F. Albiac and N. J. Kalton, *Topics in Banach space theory,* Grad. Texts in Math. 233, Springer-Verlag, New York, 2006.
2. S. A. Argyros and R. G. Haydon, A hereditarily indecomposable $\mathscr{L}_\infty$-space that solves the scalar-plus-compact problem, *Acta Math.* **206** (2011), 1–54.
3. B. Blackadar, *K-theory for operator algebras,* M.S.R.I. Publications 5, 2nd ed., Cambridge University Press, Cambridge, 1998.
4. I. S. Edelstein and B. S. Mityagin, Homotopy type of linear groups of two classes of Banach spaces, *Funct. Anal. Appl.* **4** (1970), 221–231.
5. G. A. Edgar, A long James space, in: *Measure Theory,* Oberwolfach 1979, Lecture Notes in Math. 794, Springer-Verlag (1980), 31–37.
6. M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant and V. Zizler, *Functional analysis and infinite-dimensional geometry.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 8, Springer-Verlag, New York, 2001.
7. W. T. Gowers and B. Maurey, The unconditional basic sequence problem, *J. Amer. Math. Soc.* **6** (1993), 851–874.
8. R. C. James, Bases and reflexivity of Banach spaces, *Ann. of Math.* **52** (1950), 518–527.
9. W. B. Johnson, Factoring compact operators, *Israel J. Math.* **9** (1971), 337–345.
10. T. Kania and T. Kochanek, The ideal of weakly compactly generated operators acting on a Banach space, *J. Operator Theory* **71** (2014), 455–477.
11. T. Kania, P. Koszmider and N. J. Laustsen, A weak*-topological dichotomy with applications in operator theory, *Trans. London Math. Soc.* **1** (2014), 1–28.
12. T. Kania and N. J. Laustsen, Operators on two Banach spaces of continuous functions on locally compact spaces of ordinals, *Proc. Amer. Math. Soc.*, to appear; arXiv:1304.4951.
13. P. Koszmider, On decompositions of Banach spaces of continuous functions on Mrówka's spaces, *Proc. Amer. Math. Soc.* **133** (2005), 2137–2146.
14. N. J. Laustsen, K-theory for algebras of operators on Banach spaces, *J. London Math. Soc.* **59** (1999), 715–728.
15. N. J. Laustsen, K-theory for the Banach algebra of operators on James's quasi-reflexive Banach spaces, *K-Theory* **23** (2000), 115–127.

16. N. J. Laustsen, Maximal ideals in the algebra of operators on certain Banach spaces, *Proc. Edinburgh Math. Soc.* **45** (2002), 523–546.
17. N. J. Laustsen, Commutators of operators on Banach spaces, *J. Operator Theory* **48** (2002), 503–514.
18. M. Rørdam, F. Larsen and N. J. Laustsen, *An introduction to K-theory for C^*-algebras,* LMS Student Texts 49, Cambridge University Press, Cambridge, 2000.
19. S. Shelah and J. Steprāns, A Banach space on which there are few operators, *Proc. Amer. Math. Soc.* **104** (1988), 101–105.
20. H. M. Wark, A non-separable reflexive Banach space on which there are few operators, *J. London Math. Soc.* **64** (2001), 675–689.

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