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Financial Frictions and the Volatility of Monetary Policy in a DSGE model

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Abstract

The paper investigates the impacts of the volatility of monetary policy on the economy in a DSGE model with financial frictions *à* la Bernanke, Gertler, and Gilchrist (1999). The model is estimated by the particle filter maximum likelihood estimator for the U.S. economy. Our results first show that a positive monetary volatility shock causes a contraction in economic activity: output, consumption, investment, hours, and real wages fall. Second, we argue that financial frictions amplify the effects of the shock via the financial accelerator mechanism. Third, we document that the size of the effects of the shock is relatively small mostly because of the counteracting response of monetary policy to the shock. Therefore, the impacts would be substantial if monetary policy was restrained to respond to changes in current conditions in the economy.

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1 Introduction

The role of financial frictions in business cycles has been attracting the interest of both academics and policy makers, especially after the recent financial crisis. The seminal work of Bernanke, Gertler and Gilchrist (1999) develops a framework combining nominal rigidities with an agency cost model and argues that endogenous developments in the credit market can significantly amplify and propagate shocks to the economy through the financial accelerator mechanism. The core of this mechanism lies at the negative relationship between the net worth of firms and the external premium demanded by lenders. With respect to monetary policy, their model shows that an unanticipated increase in the nominal interest rate decreases the demand for capital and therefore causes a fall in its price. The decline in the value of capital reduces entrepreneurs' net worth and thus leads to a higher external premium, which further lowers investment and output. Christensen and Dib (2008) and Christiano, Motto and Rostagno (2010), among others, provide quantitative evidence to support the financial accelerator and assert that financial frictions play a significant role in transmitting monetary policy disturbances to the real economy.[1]

This paper investigates further the interaction between financial frictions and monetary policy. However, our attention is directed to the impacts of changes in the volatility of monetary policy on the economy instead of those in its level. While the latter has been discussed extensively in the literature, including the papers above, only few studies have considered the former, as reviewed below. Shifts in the volatility of monetary policy are important because they relate to monetary policy uncertainty which has been a pivotal theme in policy discussions, especially after the recent financial crisis. For example, Hawks and doves at the Federal Reserve System have argued about the extent of quantitative easing and the appropriate monetary stance given opposing signals from core and headline inflation measures (Born and Peifer 2014). Furthermore, an increasing number of studies (for instance, Justiniano and Primiceri 2008, Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez 2010, Mumtaz and Zanetti 2013) have shown that the volatility of monetary policy shocks has changed substantially in the U.S. Specifically, it was large during the Great Inflation of the mid 1970s and early 1980s, became mild after the mid 1980s and increased significantly during the recent crisis (Mumtaz and Zanetti 2013).

In order to model changes in the volatility of shocks, the literature has proposed three alternatives: stochastic volatility, GARCH, and Markov regime switching models. A detailed comparison between these approaches is reported in Fernández-Villaverde and Rubio-Ramírez (2010). We use the first method following most of the literature on macroeconomics and volatility (for example, Born and Peifer 2014, Gilchrist, Sim and Zakrajšek 2014, Cesa-Bianchi and Fernandez-Corugedo 2014, Fernández-Villaverde et al. 2010, Arellano, Bai and Kehoe 2010, Justiniano and Primiceri 2008). Therefore, there are two types of shocks relating to monetary policy: one affects the level of the interest rate (first moment shocks or structural shocks or level shocks) and the other affects the standard deviation of the interest rate (second moment shocks or volatility shocks). Note

[1] Christensen and Dib (2008) estimate a dynamic stochastic general equilibrium model with the Bernanke-Gerltler-Gilchrist financial frictions for the U.S., while Christiano et al. (2010) consider both the Euro Area and the U.S.

that we assume the nominal interest rate to be the only instrument of monetary policy, as opposed to a monetary supply aggregate, as in line with Smets and Wouters (2007). This assumption appears to be reasonable to describe U.S. monetary policy (Clarida, Galí and Gertler 1999).

We then incorporate the stochastic volatility of monetary policy into a sticky-price DSGE model embedded with the financial frictions *à* la Bernanke et al. (1999). We also allow time-variation in the standard deviations of other structural innovations, including those of government spending innovations, investment-specific technology innovations, and technology innovations, in order to capture aggregate dynamics. The diverse sources of volatility in our paper are desirable as has been argued by the growing literature on the role of volatility in business fluctuations such as Sims and Zha (2006) and Justiniano and Primiceri (2008) among others. Moreover, Hamilton (2008) shows that even if the object of interest is in the conditional mean, correctly modeling time-varying volatility can still be quite important. Stochastic volatility has been mostly ignored in the literature on financial frictions though.

Our paper is related to the studies on the aggregate effects of uncertainty. Although this strand has been rapidly growing since the recent financial crisis (for instance, Bloom 2009, Alexopoulos and Cohen 2009, Bloom, Floetotto, Jaimovich, Saporta-Eksten and Terry 2012, Bachmann and Bayer 2011, Popescu and Smets 2010), there are only few studies on the effects of policy uncertainty.[2] Mumtaz and Zanetti (2013) estimate an SVAR model for the U.S. economy and show that an increase in the volatility of monetary policy leads to a fall in output growth. The authors also calibrate a simple DSGE model enriched with the time-varying standard deviation of monetary policy shocks in order to generate similar responses. Born and Peifer (2014) consider both fiscal and monetary uncertainty in a DSGE model and conclude that policy risk has an adverse effect on output. This result is also supported by Fernández-Villaverde, Guerrón-Quintana, Kuester and Rubio-Ramírez (2013) and Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez and Uribe (2011). Financial frictions are not incorporated in these models though.

The present study is one of the few that integrates volatility and financial frictions, which are two important issues emerging form the crisis, into a united framework to analyze macroeconomic dynamics. We briefly review this branch as follows. Dorofeenko, Lee and Salyer (2008) extend the Carlstrom and Fuerst (1997) agency cost model to study the effect of the volatility of firm's idiosyncratic productivity shocks and show that an increase in the uncertainty leads to a fall in investment supply. Christiano, Motto and Rostagno (2014) consider a so-called risk shock in an estimated DSGE model incorporating the Bernanke et al. (1999) financial frictions and find that an increase in this shock reduces consumption, investment and output. Moreover, they argue that this shock

[2]The influential paper of Bloom (2009) shows that jumps in uncertainty in response to major economic and political shocks cause firms to pause their investment and hiring leading to a fall in productivity growth and then output and employment. Alexopoulos and Cohen (2009) and Bloom et al. (2012) affirm that an increase in the uncertainty results in a sharp drop and slow recovery in GDP. In contrast, Bachmann and Bayer (2011) argue that uncertainty is unlikely to be a major quantitative source of business cycle fluctuations. Popescu and Smets (2010) report similar results with Bachmann and Bayer (2011).

plays the most important role in driving the U.S. business cycle over the 1985-2010 period. Arellano et al. (2010) build a model with heterogeneous firms and financial frictions and find that increases in uncertainty at the firm level cause a large increase in the dispersion of growth rates across firms and a contraction in economic activity. Gilchrist et al. (2014) consider a model with heterogeneous firms, partial investment, irreversible, nonconvex capital adjustment costs, and financial frictions in both the debt and equity markets. The authors document the negative effects of firm level uncertainty shocks on the economy and argue that credit spreads are an important channel through which uncertainty shocks affect the economy. Cesa-Bianchi and Fernandez-Corugedo (2014) investigate the impacts of two different types of uncertainty shocks: TFP and firm level uncertainty. They find that the latter has a greater impact on economic activity because it is greatly magnified by credit frictions. Finally, Bonciani and Van Roye (2013) consider the volatility of TFP and show that financial frictions amplify the effect of uncertainty on the economy.

Our work differs from the above papers in three important aspects. First, we are, to our best knowledge, the first to investigate the interaction between financial frictions and policy uncertainty. Second, the parameters of exogenous processes of volatility in our study are jointly estimated with other parameters of the model instead of being calibrated as common in this strand of the literature. Note that a few papers have used proxies for uncertainty shocks to estimate those parameters independently while calibrating other parameters of the model- an approach that differs from the one applied in this paper. Third, while the studies above are mainly confined to a certain type of uncertainty, we incorporate a variety of uncertainty shocks. With such a variety, our model provides a clearer picture about changes in the uncertainty in the U.S. by measuring the evolution of those volatility shocks. This kind of exercise has been conducted in models without financial frictions, for instance Justiniano and Primiceri (2008) and Fernández-Villaverde et al. (2010).

Regarding the estimation, likelihood-based inference is a useful tool to take DSGE models to the data (An and Schorfheide 2007). However, those models mostly do not imply a likelihood function that can be calculated numerically or analytically. Therefore, the model must be solved before it can be estimated. Linear approximation methods are very popular because they result in a linear state-space representation of the model whose likelihood can be obtained by the Kalman filter (An and Schorfheide 2007). Nevertheless, in a linearized version of our model, stochastic volatility would drop, canceling any possibility of studying its impacts on the real economy. We therefore have to solve the model to a higher-order approximation. This solution however leads to a non-linear state space representation so that the Kalman filter can not be utilized to evaluate the likelihood function. To overcome this issue, Fernández-Villaverde and Rubio-Ramírez (2007) propose to use the particle filter which performs sequential Monte Carlo estimation using a point mass representation of probability densities. Fernández-Villaverde et al. (2010) apply the method to estimate a DSGE model with stochastic volatility. Following these studies, we take advantages of the particle filter to evaluate the likelihood function in a maximum likelihood framework. We use U.S. data for the estimation.

Our results first show that an increase in the volatility of monetary policy shocks causes a contraction in consumption, investment, output and hours worked. The model is therefore successful in generating business-cycle co-movements among key macroeconomic variables (see Basu and Bundick (2012)). Moreover, this contractionary effect resembles the findings of Mumtaz and Zanetti (2013) and Born and Peifer (2014) who also study the impacts of monetary volatility shocks. Second, we find that financial frictions amplify the transmission of volatility shocks to the economy through the financial accelerator mechanism. Our counterfactual exercises indicate that an increase in the level of financial frictions leads to a greater fall in investment and output. This finding thus supports Gilchrist et al. (2014) and Bonciani and Van Roye (2013).

Third, in line with Born and Peifer (2014), we find that the pure effect of monetary uncertainty shocks is unlikely to play a considerable role in business cycle fluctuations. However, this does not imply that time-varying volatility is unimportant. The estimates of volatility shocks show that they were large during the 1970s and early 1980s but then have declined considerably since around the mid-1980s. Our results therefore suggest that the Great Moderation might be a consequence of a combination between both "good luck" and "good policy". Fourth, we document that the small impacts of a monetary volatility shock on economic activity is due to dampening general equilibrium effects in the model, mostly the stabilizing role of monetary policy. Our counterfactual experiment shows that the contraction in economic activity would be sizable if monetary policy became less responsive to current conditions.

The rest of the paper is organized as follows. Section 2 presents the baseline DSGE model. Section 3 shows the state-space representation of the model. In section 4, we present the estimates of model parameters and of structural and volatility shocks. Section 5 analyzes impulse response functions. Finally, section 6 concludes.

2 The DSGE Model

Our model is a cashless-limited closed-economy New Keynesian DSGE model that incorporates the financial-accelerator mechanism proposed by Bernanke et al. (1999). In this small-sized model of the economy, there are five agents: households, capital producers, entrepreneurs, retailers and policy authorities. Households make decisions on consumption and hours worked to maximize their utilities subject to their intertemporal budget constraints. Capital producers transform the investment component of output into new capital goods which replace depreciated capital and add to capital stock. Entrepreneurs produce wholesale goods. They borrow from financial intermediates to cover for the difference between the expenditure on new capital and their net worth. Because of imperfect information between entrepreneurs and lenders, the former faces an external finance premium that rises when their leverage increases. This is how financial frictions are incorporated into the model. Retailers are introduced to motivate sticky prices. They buy the wholesale goods from the entrepreneurs, transform them into differentiated goods, and set prices in the Calvo type. Finally, authorities conduct both monetary and fiscal policy. The nominal interest rate, which is supposed to be the only tool of monetary policy, follows a Taylor rule that responds to the deviations of inflation and output from their steady states. Regarding fiscal policy, government spending is financed by lump-sum taxes.

Although our main interest is on monetary policy innovations, we include technology innovations, investment specific technology innovations, and government spending innovations into the model to capture aggregate dynamics. All standard deviations are assumed to be time-varying following an AR(1) process. Consequently, there are four structural shocks and four volatility ones brought into the model, which makes the number of shocks driving the economy eight.

2.1 Households

The representative household chooses consumption C_t, the amount of risk-less bonds B_{t+1} and hours worked h_t to maximize the following lifetime utility function

$$E_t \sum_{k=0}^{\infty} \beta^k \left(log(C_{t+k} - \chi C_{t+k-1}) - \varpi \frac{h_{t+k}^{1+\vartheta}}{1+\vartheta} \right), \tag{2.1}$$

where $\beta \in (0,1)$ is the discount factor, χ controls habit persistence, ϖ controls the level of labor supply, and ϑ is the inverse of the Frisch elasticity. Moreover, C_t is the consumption index given by

$$C_t = \left(\int_0^1 C_t(i)^{1-\frac{1}{\zeta}} di \right)^{\frac{\zeta}{\zeta-1}}, \tag{2.2}$$

where ζ is the elasticity of substitution and $C_t(i)$ represents the quantity of good i consumed by the household in period t. We assume the existence of a continuum of goods represented by the interval [0,1].

Maximization of (2.1) is subject to a sequence of flow budget constraints given by

$$\frac{\int_0^1 P_t(i)C_t(i)di}{P_t} + \frac{B_{t+1}}{P_t} \leq \frac{R_{n,t-1}B_t}{P_t} + \frac{W_t h_t}{P_t} + \text{Transfers} + \text{Profits}, \tag{2.3}$$

where $P_t(i)$ is the price of good i, B_{t+1} is the amount of risk-less bonds held between period t and period $t+1$ which pay a nominal gross interest rate $R_{n,t}$ at maturity, and W_t is the wage rate. The household receives lump-sum transfers from the government and profits from firms. P_t is the aggregate price index given by

$$P_t = \left(\int_0^1 P_t(i)^{1-\zeta} di \right)^{\frac{1}{1-\zeta}}.$$

For each differentiated good i, the household must decide how to choose C_i to maximize (2.2) for any given level of expenditures $\int_0^1 P_t(i)C_t(i)di$. The first-order solution yields the set of demand equations for consumption

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\zeta} C_t,$$

for all $i \in [0,1]$. Thus,

$$\int_0^1 P_t(i)C_t(i)di = P_tC_t. \tag{2.4}$$

Substituting (2.4) into the budget constraint (2.3) results in

$$C_t + \frac{B_{t+1}}{P_t} \leq \frac{R_{n,t-1}B_t}{P_t} + \frac{W_t}{P_t}h_t + \text{Transfers} + \text{Profits}. \tag{2.5}$$

We then derive the first-order conditions for the household's problem as follows

$$\frac{1}{C_t - \chi C_{t-1}} - \beta E_t(\chi \frac{1}{C_{t+1} - \chi C_t}) = \lambda_t,$$

$$\lambda_t = \beta E_t(\lambda_{t+1}\frac{R_{n,t}}{\Pi_{t+1}}),$$

$$\varpi h_t^{\vartheta} = \lambda_t \frac{W_t}{P_t},$$

where λ_t is the Lagrangian multiplier associated with the budget constraint in (2.5).

2.2 Capital Producers

Suppose that there is a single, representative, competitive capital producer who uses a portion of final goods purchased from retailers as investment goods I_t to produce capital goods. The production is subject to quadratic capital adjustment costs $S(.)$ and an investment-specific technology shock κ_t and it generates $e^{\kappa_t}(1 - S(\frac{I_t}{I_{t-1}}))I_t$ capital goods. These goods are sold at a real price Q_t per unit at the end of period t.

The adjustment cost function, similar to Smets and Wouters (2007) and Christiano, Eichenbaum and Evans (2005), is specified as

$$S\left(\frac{I_t}{I_{t-1}}\right) = \phi_s(\frac{I_t}{I_{t-1}} - 1)^2,$$

where ϕ_s is the adjustment parameter. Along the balanced growth path, $S(1) = S'(1) = 0$.

The investment specific technology shock is assumed to follow an AR(1) process

$$\kappa_t = \rho_\kappa \, \kappa_{t-1} + \sigma_\kappa e^{\sigma_{\kappa t}} \varepsilon_{\kappa t}, \quad \varepsilon_{\kappa t} \sim \mathcal{N}(0,1),$$

where $\sigma_{\kappa t}$ is the time-variant component of the standard deviation of investment specific technology shock $\varepsilon_{\kappa t}$. Its evolution is given by

$$\sigma_{\kappa t} = \rho_{\sigma_\kappa} \, \sigma_{\kappa t-1} + \eta_\kappa u_{\kappa t}, \quad u_{\kappa t} \sim \mathcal{N}(0,1).$$

The capital producer's optimization problem is to maximize its discounted profits with respect to I_t

$$E_t \sum_{k=0}^{\infty} \Lambda_{t,t+k}[Q_{t+k}e^{\kappa_{t+k}}(1 - S(X_{t+k}))I_{t+k} - I_{t+k}],$$

where $X_t = \frac{I_t}{I_{t-1}}$ and $\Lambda_{t,t+k}$ is the *real* stochastic discount factor over the interval $[t, t+k]$. The first-order condition for this problem is

$$Q_t e^{\kappa_t}(1 - S(X_t) - X_t S^{'}(X_t)) + E_t[\Lambda_{t,t+1}Q_{t+1}e^{\kappa_{t+1}}S'(X_{t+1})X_{t+1}^2] = 1.$$

The produced capital goods combine with the existing capital stock to generate new capital goods. In other words, the capital accumulation process is described by

$$K_t = (1 - \delta)K_{t-1} + e^{\kappa_t}(1 - S(X_t))I_t.$$

2.3 Entrepreneurs

Entrepreneurs manage the firms that produce the wholesale goods. This production uses labor and capital. While the former is supplied by both households and entrepreneurs, the latter is bought from capital producers. The entrepreneurs finance the expenditure on capital by entrepreneurial net worth (internal finance) and debts (external finance). In the latter, they face an external finance premium caused by the inability of lenders to monitor borrowers' actions or to share borrowers' information. In this way, financial market imperfections are introduced into the model.

The premium relies on the balance-sheet condition of the entrepreneurs. When their net worth declines, internal sources of funds are limited, forcing them to seek external sources by borrowing. However, the deterioration of their balance sheets causes the potential divergence between them (the borrowers) and the lenders to be greater, leading to an increase in agency costs. Consequently, the cost of external finance is pushed up resulting in a contraction of investment spending and then output.

The entrepreneurs are risk neutral. They are endowed with h_t^e units of entrepreneurial labor at the nominal entrepreneurial wage W_t^e in order to start off. Moreover, each of them is assumed to survive until the next period with a probability γ. This is to assure that they do not accumulate enough funds to finance their expenditures on capital only with their net worth. New entrepreneurs are allowed to enter to replace those exiting.

Production. The wholesale goods are produced according to a constant-return-to-scale technology

$$Y_t^W = e^{a_t}A(H_t)^{1-\alpha}K_{t-1}^{\alpha},$$

where K_{t-1} denotes the number of capital units, H_t is the labor input which is a composite of household labor h_t and entrepreneurial labor h_t^e, A is the level of technology which is normalized to one, and a_t is a shifter to the technology level which evolves as

$$a_t = \rho_a a_{t-1} + \sigma_a e^{\sigma_{at}}\,\varepsilon_{at}, \quad \varepsilon_{at} \sim \mathcal{N}(0, 1),$$

where σ_{at} is the time-variant component of the standard deviation of technology shock

ε_{at} and it follows an AR(1) process

$$\sigma_{at} = \rho_{\sigma_a}\sigma_{at-1} + \eta_a u_{at}, \quad u_{at} \sim \mathcal{N}(0,1).$$

For the labor input, h_t^e is assumed to be constant at one. In addition, the share of income going to the entrepreneurial labor is calibrated to be small (the order of 0.01), so that the modification of the production function does not have substantial effects on the results. The labor input H_t is written as follows

$$H_t = h_t^{\Omega}(h_t^e)^{1-\Omega}.$$

The demand for household and entrepreneurial labor is obtained by equating the marginal product of each type of labor to its corresponding cost

$$\frac{P_t^W}{P_t}\frac{(1-\alpha)\Omega Y_t^W}{h_t} = \frac{W_t}{P_t}, \tag{2.6}$$

$$\frac{P_t^W}{P_t}\frac{(1-\alpha)(1-\Omega)Y_t^W}{h_t^e} = \frac{W_t^e}{P_t}. \tag{2.7}$$

Meanwhile, the demand for capital of the entrepreneurs is considered below with the occurrence of financial frictions.

Financial frictions. At the end of time t, an entrepreneur borrows l_t equivalent to the difference between the expenditure on new capital $Q_t k_t$ and the net worth $n_{E,t}$

$$l_t = Q_t k_t - n_{E,t}.$$

The net worth accumulation $n_{E,t}$ is calculated by[3]

$$n_{E,t} = \psi_t R_{k,t} Q_{t-1} k_{t-1} - R_{l,t} l_{t-1},$$

where ψ_t is an idiosyncratic shock to the entrepreneur's return,[4] $R_{l,t}$ is the real loan rate set at time $t-1$, and $R_{k,t}$ is the real return on capital computed by

$$R_{k,t} = \frac{\alpha \frac{P_t^W}{P_t}\frac{Y_t^W}{K_{t-1}} + (1-\delta)Q_t}{Q_{t-1}}.$$

Note that the idiosyncratic shock is the private information of the entrepreneur. We follow Bernanke et al. (1999) to assume that ψ_t is distributed log-normally with positive support and its standard deviation is time-invariant. The distribution of ψ_t hence can be written as follows

$$log\,(\psi_t) \sim \mathcal{N}(-1/2\sigma_\psi^2, \sigma_\psi^2),$$

[3]Lower case variables denote the representative entrepreneur, while upper case variables introduced later denote the aggregate.

[4]The shock at $t+1$ is revealed at the end of period t right before investment decisions are made.

where σ_ψ is the standard deviation of the idiosyncratic shock ψ_t.

At time $t+1$, if the net worth $n_{E,t+1}$ becomes negative, the entrepreneur is bankrupt. In other words, the default occurs if the idiosyncratic shock falls below the cut-off value $\bar{\psi}_{t+1}$ given by

$$\overline{\psi}_{t+1} = \frac{R_{l,t+1} l_t}{R_{k,t+1} Q_t k_t}. \tag{2.8}$$

Otherwise, the entrepreneur makes the full payment of her loans, $R_{l,t+1} l_t$, to the lender.

Let f_{ψ_t} and ψ_{min} be the density function and the lower bound of ψ_t, respectively. Then, the probability of default at time $t+1$ is calculated by

$$F(\overline{\psi}_{t+1}) = \int_{\psi_{min}}^{\overline{\psi}_{t+1}} f(\psi) d\psi.$$

If default happens, the lender obtains the assets of the firm. However, it has to pay a proportion μ to observe the realized return. Therefore, the expected gross return on the loan of the lender is given by

$$E_t \left[(1 - F(\overline{\psi}_{t+1})) R_{l,t+1} l_t + (1-\mu) R_{k,t+1} Q_t k_t \int_{\psi_{min}}^{\overline{\psi}_{t+1}} \psi f(\psi) d\psi \right].$$

Substituting $R_{l,t+1} l_t$ by $\overline{\psi}_{t+1} R_{k,t+1} Q_t k_t$ (see (2.8)) yields

$$E_t \left[R_{k,t+1} Q_t k_t (\overline{\psi}_{t+1} (1 - F(\overline{\psi}_{t+1})) + (1-\mu) \int_{\psi_{min}}^{\overline{\psi}_{t+1}} \psi f(\psi) d\psi) \right]. \tag{2.9}$$

Define $\Gamma(\overline{\psi}_{t+1})$ as the share of entrepreneurial earnings accrued to the lender

$$\Gamma(\overline{\psi}_{t+1}) = \overline{\psi}_{t+1} (1 - F(\overline{\psi}_{t+1})) + G(\overline{\psi}_{t+1}), \tag{2.10}$$

where

$$G(\overline{\psi}_{t+1}) = \int_{\psi_{min}}^{\overline{\psi}_{t+1}} \psi f(\psi) d\psi. \tag{2.11}$$

For the optimal contract, the entrepreneur needs to find k_t and $\overline{\psi}_{t+1}$ to maximize her expected net earnings

$$E_t \left[(1 - \Gamma(\overline{\psi}_{t+1})) R_{k,t+1} Q_t k_t \right], \tag{2.12}$$

subject to

$$E_t \left[R_{k,t+1} Q_t k_t (\Gamma(\overline{\psi}_{t+1}) - \mu G(\overline{\psi}_{t+1})) \right] = E_t(R^{ex}_{t+1} l_t). \tag{2.13}$$

The constraint reflects the assumption that the lender is indifferent between the expected return from lending to the entrepreneur and the one from owning risk-free bonds. Then, using the Lagrange multiplier method, we obtain

$$E_t[R_{k,t+1}] = E_t[\iota(\overline{\psi}_{t+1}) R^{ex}_{t+1}],$$

where $\iota(\overline{\psi}_{t+1})$ is the *premium on external finance* given by

$$\iota(\overline{\psi}_{t+1}) = \frac{\Gamma'(\overline{\psi}_{t+1})}{(1-\Gamma(\overline{\psi}_{t+1}))(\Gamma'(\overline{\psi}_{t+1}) - \mu G'(\overline{\psi}_{t+1})) + \Gamma'(\overline{\psi}_{t+1})(\Gamma(\overline{\psi}_{t+1}) - \mu G(\overline{\psi}_{t+1}))}. \quad (2.14)$$

For the calculation of $\Gamma(.), \Gamma'(.), G(.)$ and $G'(.)$, see Appendix A.1.

So far we have established the optimizing decision of a representative entrepreneur. We now assume that a fraction $1-\gamma$ of entrepreneurs exits at the end of period $t-1$ and they consume all their residual equities. Therefore, the aggregate net worth accumulating at the end of time t is calculated by

$$N_{E,t} = \gamma(1-\Gamma(\overline{\psi}_t))R_{k,t}Q_{t-1}K_{t-1} + \frac{W_{E,t}}{P_t}$$

and the consumption of the exiting entrepreneurs is

$$C_{E,t} = (1-\gamma)(1-\Gamma(\overline{\psi}_t))R_{k,t}Q_{t-1}K_{t-1}.$$

2.4 Retailers

In order to motivate sticky prices, two additional ingredients are added to the model. First, the retail sector is assumed to be monopolistically competitive. Second, there are costs of adjusting nominal prices.

Optimal Price Setting. Retailers purchase the wholesale goods from the entrepreneurs and transform them into differentiated goods according to

$$Y_t = \frac{Y_t^W}{\Delta_t}$$

where $Y_t = (\int_0^1 Y_t(i)^{1-\frac{1}{\zeta}} di)^{\frac{\zeta}{\zeta-1}}$ and $\Delta_t = \int_0^1 (\frac{P_t(i)}{P_t})^{-\zeta} di$ is the price dispersion. The retailers then set prices to optimize their expected profits. The setting is however constrained by the so-called Calvo-typed price rigidity (Calvo 1983). Specifically, each retailer can reoptimize her price in a given period with a constant probability $1-\xi$. The law of large number suggests that a fraction $1-\xi$ of firms re-optimize their prices at each period. The remaining retailers are assumed to adjust their prices based on the lagged inflation with a degree of indexation $\gamma \in [0,1]$ in order to capture the inertia observed in the response of inflation to a monetary policy shock (Woodford 2003).

Given a common real marginal cost MC_t to all retail firms, a new price $P_t^*(i)$ chosen by the retailer i in period t should maximize her discounted nominal profits given by

$$E_t \sum_{k=0}^{\infty} \xi^k D_{t,t+k} Y_{t+k}(i) \left[P_t^*(i) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma} - P_{t+k} MC_{t+k} \right],$$

subject to the sequence of demand constraints

$$Y_{t+k}(i) = \left(\frac{P_{t+k}(i)}{P_{t+k}}\right)^{-\zeta} Y_{t+k}.$$

Note that $D_{t,t+k} = \beta^k \frac{\lambda_{t+k}}{\lambda_t}$ is the *nominal* stochastic discount factor over the interval $[t, t+k]$ and $k = 0, 1, 2, ...$

The first order condition associated with the above problem has the form

$$E_t \sum_{k=0}^{\infty} \xi^k D_{t,t+k} Y_{t+k}(i) \left[P_t^*(i) \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma} - \mathcal{M}\, P_{t+k}\, MC_{t+k}\right] = 0, \tag{2.15}$$

where $\mathcal{M} \equiv \frac{\zeta}{\zeta-1}$ is the frictionless markup. We rearrange $D_{t,t+k}Y_{t+k}(i)$ as follows

$$\begin{aligned} D_{t,t+k}Y_{t+k}(i) &= \beta^k \frac{\lambda_{t+k}}{\lambda_t} \frac{P_t}{P_{t+k}} \left[\frac{P_t^*(i)}{P_{t+k}} \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma}\right]^{-\zeta} Y_{t+k} \\ &= \beta^k \frac{\lambda_{t+k}}{\lambda_t} \frac{P_t}{P_{t+k}} \left(\frac{P_t^*(i)}{P_{t+k}}\right)^{-\zeta} \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{-\zeta\gamma} Y_{t+k} \\ &= \beta^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{P_t^*(i)}{P_t}\right)^{-\zeta} \Pi_{t,t+k}^{\zeta-1} \Pi_{t-1,t+k-1}^{-\zeta\gamma} Y_{t+k}, \end{aligned}$$

where $\Pi_{t,t+k} = \frac{P_{t+k}}{P_t}$. By substituting this rearrangement into (2.15), then canceling out $(\frac{P_t^*(i)}{P_t})^{-\zeta}$ and multiplying by $\frac{\lambda_t}{P_t}$ in (2.15), we get

$$E_t \sum_{k=0}^{\infty} (\xi\beta)^k \lambda_{t+k} \Pi_{t,t+k}^{\zeta-1} \Pi_{t-1,t+k-1}^{-\zeta\gamma} Y_{t+k} \left[\frac{P_t^*(i)}{P_t} \Pi_{t-1,t+k-1}^{\gamma} - \mathcal{M}\, \Pi_{t,t+k}\, MC_{t+k}\right] = 0.$$

This is equivalent to

$$\frac{P_t^*(i)}{P_t} E_t \sum_{k=0}^{\infty} (\xi\beta)^k \lambda_{t+k} Y_{t+k} \overline{\Pi}_{t,t+k}^{\zeta-1} = \mathcal{M} E_t \sum_{k=0}^{\infty} (\xi\beta)^k \lambda_{t+k} Y_{t+k}\, \overline{\Pi}_{t,t+k}^{\zeta}\, MC_{t+k}, \tag{2.16}$$

where $\overline{\Pi}_t = \frac{\Pi_t}{\Pi_{t-1}^{\gamma}}$. We now define

$$H_t = E_t \sum_{k=0}^{\infty} (\xi\beta)^k \lambda_{t+k} Y_{t+k} \overline{\Pi}_{t,t+k}^{\zeta-1} \tag{2.17}$$

and

$$J_t = \mathcal{M} E_t \sum_{k=0}^{\infty} (\xi\beta)^k \lambda_{t+k} Y_{t+k}\, \overline{\Pi}_{t,t+k}^{\zeta}\, MC_{t+k}. \tag{2.18}$$

From (2.17) and (2.18), we derive

$$H_t - \xi\beta E_t[\overline{\Pi}_{t+1}^{\zeta-1} H_{t+1}] = \lambda_t Y_t$$

and

$$J_t - \xi\beta E_t[\overline{\Pi}_{t+1}^{\zeta} J_{t+1}] = \mathcal{M}\,\lambda_t\, MC_t Y_t.$$

Moreover, combining (2.16), (2.17), and (2.18) yields

$$\frac{P_t^*(i)}{P_t} = \frac{J_t}{H_t}. \tag{2.19}$$

Aggregate Price Level Dynamics. Equation (2.19) implies that all the retailers that are resetting their prices will choose an identical price which is P_t^*. The aggregate price level at time t therefore evolves according to

$$P_t = \left[(1-\xi)P_t^{*1-\zeta} + \xi\left(P_{t-1}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma}\right)^{1-\zeta}\right]^{\frac{1}{1-\zeta}}. \tag{2.20}$$

Dividing both side of (2.20) by P_t results in

$$1 = (1-\xi)\left(\frac{J_t}{H_t}\right)^{1-\zeta} + \xi\overline{\Pi}_t^{\zeta-1}.$$

2.5 The Central Bank

The model is closed by the presence of a central bank that sets the nominal interest rate according to a Taylor-type rule

$$\frac{R_{n,t}}{R_n} = \left(\frac{R_{n,t-1}}{R_n}\right)^{\rho_r}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\theta_\pi}\left(\frac{Y_t}{Y}\right)^{\theta_y}\right)^{(1-\rho_r)} e^{\sigma_m e^{\sigma_{mt}}\varepsilon_{mt}}, \quad \varepsilon_{mt} \sim \mathcal{N}(0,1),$$

where $\varepsilon_{m,t}$ is the monetary policy innovation whose time-varying component of the standard deviation σ_{mt} evolves according to an AR(1) process

$$\sigma_{mt} = \rho_{\sigma_m}\sigma_{mt-1} + \eta_m u_{mt}, \quad u_{mt} \sim \mathcal{N}(0,1).$$

In the Taylor rule, the first term on the right-hand-side $\frac{R_{n,t-1}}{R_n}$ represents the smoothing behavior of the central bank. The second term $\frac{\Pi_t}{\Pi}$ denotes the deviation of inflation from its steady level Π. The third term $\frac{Y_t}{Y}$ is the output gap which is the deviation of output from its balanced state Y.

2.6 Resource Constraint

The market for final goods clears in every period

$$Y_t = C_t + C_{E,t} + I_t + Ge^{g_t} + \mu G(\overline{\psi}_t) R_{k,t} Q_{t-1} K_{t-1}.$$

In that the government spending is financed by lump-sum taxes on the basis of a balanced budget. G is the steady state level of government spending which is influenced by an exogenous shock g_t following an AR(1) process

$$g_t = \rho_g g_{t-1} + \sigma_g e^{\sigma_{gt}} \varepsilon_{gt}, \quad \varepsilon_{gt} \sim \mathcal{N}(0,1),$$

where ε_{gt} is the government spending shock. The time-varying component of the standard deviation is σ_{gt} whose evolution is given by

$$\sigma_{gt} = \rho_{\sigma_g} \sigma_{gt-1} + \eta_g u_{gt}, \quad u_{gt} \sim \mathcal{N}(0,1).$$

3 State-Space Representation

3.1 State Transition Equations

The optimal decisions of households, capital producers, entrepreneurs, and retailers, the Taylor rule and the resource constraint form a non-linear rational expectations system. This system can not be estimated by likelihood-based approaches directly because the system does not imply a likelihood function that can be calculated numerically or analytically (Fernández-Villaverde and Rubio-Ramírez 2007). Therefore, we need to solve the model before estimating it.

As regarded in the introduction, the most popular method in the literature is linearization because it leads to a linear state space representation of the model whose likelihood can be obtained by the Kalman filter (An and Schorfheide 2007). However, linearization is certainty-equivalent, which means that all volatility shocks will be dropped out, therefore canceling any chance of analyzing their impacts on the economy. In a second order approximation, volatility shocks enter as cross-products with the corresponding level shocks in the policy functions. In a third order approximation, volatility shocks play a role by themselves, thus allowing us to calculate the impulse response functions to a monetary volatility shock, while holding constant its level shock. This feature makes the third-order approximation very attractive, but it comes with high computational costs in the estimation, given that the particle filter is employed to obtain the likelihood function (see the computational issues of particle filters in Fernández-Villaverde and Rubio-Ramírez (2007)). In contrast, although the second-order approximation does not allow us to investigate independent effects of volatility shocks, it is sufficient to estimate the parameters of the model including those of stochastic processes, while having smaller computational costs than the third-order approximation does. Fernández-Villaverde et al. (2010) and Fernández-Villaverde and Rubio-Ramírez (2007) also estimate dynamic macroeconomic models with stochastic volatility based on their second-order approximations. Therefore, we first follow those papers to estimate a second-order approximation of our DSGE model. Given the estimates, we then solve the model to third-order approximation and compute the impulse response functions to a monetary volatility shock. By using such a strategy, we can take advantages of each method.

Let s_t be the vector of all variables of the model at time t with each variable expressed in terms of log deviation from its steady state. The system is driven by the vector of structural shocks $v_t = (\varepsilon_{\kappa t}, \varepsilon_{at}, \varepsilon_{gt}, \varepsilon_{mt})$ and by the vector of volatility shocks $w_t = (u_{\kappa t}, u_{at}, u_{gt}, u_{mt})$. The solution of the rational expectations system takes the form

$$\mathbf{s}_t = \Theta(\mathbf{s}_{t-1}, v_t, w_t; \Xi), \tag{3.1}$$

where Ξ is the vector of parameters in the model. Equation (3.1) represents the state transition equations in the state-space representation and it is non-linear.[5]The following part describes the measurement equations.

3.2 Measurement Equations

We assume that the time period t corresponds to one quarter. For the estimation, we use four data series including the Hodrick-Prescott output gap per capita , the log difference of the GDP deflator, the federal funds rate, and the Moody's seasoned data corporate bond yields, which are denoted by OUT_t, INF_t, INR_t, and CBY_t, respectively. Details on the sources and constructions of these time series are documented in Appendix A.2. All these series are demeanded and connected to the model variables by

$$INF_t = \hat{\Pi}_t + \sigma_{m\pi}\varepsilon_{m\pi,t}, \quad \varepsilon_{m\pi,t} \sim \mathcal{N}(0,1),$$

$$\begin{aligned} OUT_t &= \hat{Y}_t + \sigma_{my}\varepsilon_{my,t}, \quad \varepsilon_{my,t} \sim \mathcal{N}(0,1), \\ INR_t &= \hat{R}_{n,t} + \sigma_{mr_n}\varepsilon_{mr_n,t}, \quad \varepsilon_{mr_n,t} \sim \mathcal{N}(0,1), \\ CBY_t &= \hat{R}_{k,t} + \sigma_{mr_k}\varepsilon_{mr_k,t}, \quad \varepsilon_{mr_k,t} \sim \mathcal{N}(0,1), \end{aligned}$$

where $\varepsilon_{my,t}, \varepsilon_{m\pi,t}, \varepsilon_{mr_n,t}$, and $\varepsilon_{mr_k,t}$ are measurement errors and their standard deviations are $\sigma_{my}, \sigma_{m\pi}, \sigma_{mr_n}$ and σ_{mr_k}, respectively. The notation "^" above a variable denotes the log deviation of that variable form its steady state. These four measurement equations and the state transition equations in (3.1) establish the non-linear state-space representation of the model.

[5]For example, a second-order approximation is given by

$$\begin{aligned} \mathbf{s}_{j,t} = C_j &+ \sum_{i=1}^{J} \Theta_{j,i}^{(s)} s_{i,t-1} + \sum_{l=1}^{n} \Theta_{j,l}^{(v)} v_{l,t} + \sum_{l=1}^{n} \Theta_{j,k}^{(v2)} v_{l,t}^2 + \sum_{i=1}^{J}\sum_{l=1}^{J} \Theta_{j,il}^{(ss)} s_{i,t-1} s_{l,t-1} \\ &+ \sum_{i=1}^{J}\sum_{l=1}^{n} \Theta_{j,il}^{(sv)} s_{i,t-1} v_{l,t} + \sum_{l=1}^{n} \Theta_{j,ll}^{(vw)} v_{l,t} w_{l,t}. \end{aligned} \tag{3.2}$$

4 Estimation

In order to estimate the non-linear state space system described in the previous section, we follow Fernández-Villaverde and Rubio-Ramírez (2007) to use the particle filter to evaluate its likelihood function. Basically, the particle filter performs sequential Monte Carlo estimation using a point mass representation of probability densities to approximate the posterior density of the states and the likelihood function (see Appendix A.3 for the algorithm of the particle filter). As discussed above, we use the four quarterly U.S. time series for the estimation. Regarding the coverage of the sample, while including the post-2007 could be beneficial because of the increased observations, it will introduce extra problems originating from the recent crisis and its on-going consequences, among which is the zero-lower bound of the interest rate. Given the inherent complexity in the estimation of a higher-order approximated model, a more 'safe and sound' solution is to exclude the post-2007 period. Our sample therefore spans from $1959Q1$ to $2007Q1$. Advancing the model to include the recent crisis episode into consideration is a potential expansion of our work.

We summarize the procedure of the estimation in three steps. First, for given the values of parameters, we solve the non-linear rational expectations system by performing a second-order perturbation around the deterministic steady states (as in 3.1). Second, we construct the state-space representation of the model and apply the particle filter to it in order to evaluate the likelihood of the model. Finally, we use an maximum-likelihood algorithm to estimate parameters.

We are aware that obtaining the MLE is complicated because the shape of the likelihood function may be rugged and multimodal. In addition, the use of optimization algorithms based on derivatives is not applicable because the particle filter generates an approximation to the likelihood function that is not differentiable with respect to parameters. Instead, we follow Van Binsbergen, Fernández-Villaverde, Koijen and Rubio-Ramirez (2012) to use the covariance matrix adaption evolutionary strategy, whose aim is to cope with objective functions which are non-linear, non-convex, multimodal, as well as other difficult conditions, in order to obtain the maximum-likelihood estimates.

As customary when taking DSGE models to data, some parameters are fixed to values which are common in the existing literature or selected to satisfy some certain conditions in the steady state (for instance, Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez 2009, Smets and Wouters 2007, Justiniano and Primiceri 2008). This helps to reduce the numbers of parameters required to estimate, therefore lessening the computational issues. We discuss those fixed parameters in subsection 4.1. The estimates of unknown parameters are presented in subsection 4.2. Given the values of all parameters, we perform particle filtering to compute the posterior densities of structural and volatility shocks and then estimate their means. Combining these estimates over the sample shows us the evolution of structural and volatility shocks. They are presented in subsection 4.3.

4.1 Fixed Parameters

The standard parameters calibrated includes $\{\beta, \zeta, \alpha, \Omega, \vartheta, \phi_s, \chi, \varpi, \}$. The discount factor $\beta = 0.985$ is chosen to match the inverse of the average of risk-free rate observed in the U.S. The elasticity of substitution ζ is fixed at 10 which implies a 10% mark-up. The elasticity of capital to output $\alpha = 0.3$ reflects the share of national income that goes to capital. As mentioned previously, the share of income to entrepreneurial labor $(1-\alpha)(1-\Omega)$ is set to a very small number 0.01, which implies a value of 0.98 for Ω . The depreciation rate δ is assigned to 0.025, which is a common value in the literature of DSGE models on the U.S. economy. The inverse of the Frisch labor elasticity ϑ is set to 1.3 which pins down the Frisch elasticity to around 0.75 as suggested by Chetty, Guren, Manoli and Weber (2011). The adjustment cost $\phi_s = 4.5$ is similar to other estimates from DSGE models, for example, Fernández-Villaverde (2010). The habit persistence χ is set to 0.9 in order to reflect the observed sluggish response of consumption to shocks (Fernández-Villaverde et al. 2010). The steady state government spending to GDP ratio G/Y is fixed at 0.2 to match the U.S. data on average. The parameter controlling the level of labor supply ϖ is calibrated in such a way that generates a steady state level of hours worked $h = 0.35$.

We also calibrate three non-standard financial parameters including $\{\mu, \sigma_\psi, \sigma_E\}$. They are chosen to imply the three following conditions in the steady state:(i) a probability of default equal to 3%, (ii) a credit spread of 66.5 basis points which is consistent with the data over the sample, and (iii) a ratio of capital to net worth QK/N of 2. Specifically, the fraction of realized payoffs lost in bankruptcy μ is 0.0555, the existing rate of entrepreneurs γ is found to be 0.9708, and the steady state level of the variance of the idiosyncratic productivity variable σ_ψ is equal to 0.3388.

4.2 Parameter Estimates

Table 1 reports the estimates for the remaining 24 parameters. First, the degree of indexation γ is 0.2 implying a moderating inflation inertia. The price rigidity ξ is around 0.7 which suggests that the prices are reoptimized approximately once every three quarters. These values are common in the literature, see e.g., Smets and Wouters (2007). Regarding the estimates of policy parameters, the response to the deviation of inflation in the long run is about 1.560, which is close to the estimate of Christensen and Dib (2008) in a linearized DSGE model with financial frictions. In contrast, the interest rate does not appear to respond strongly to changes in the output gap. Given a 1% increase in the output gap, the interest rate only rises about 7 basis points. Smets and Wouters (2007) also document a very weak response to the output gap (0.09). Finally, the interest rate shows a moderating inertia with the smoothing parameter ρ_r around 0.6.

Turning to the stochastic processes of structural shocks, they appear to be very persistent with estimated $AR(1)$ coefficients equal to 0.962, 0.978, and 0.959 for the investment-specific technology, technology, and government spending process, respectively. For the time-invariant component of the standard deviations of structural shocks, the government shock has the largest value 0.04. The smallest figure is for the investment-specific shock roughly 0.0006.

Table 1: Parameters' estimates of the DSGE model

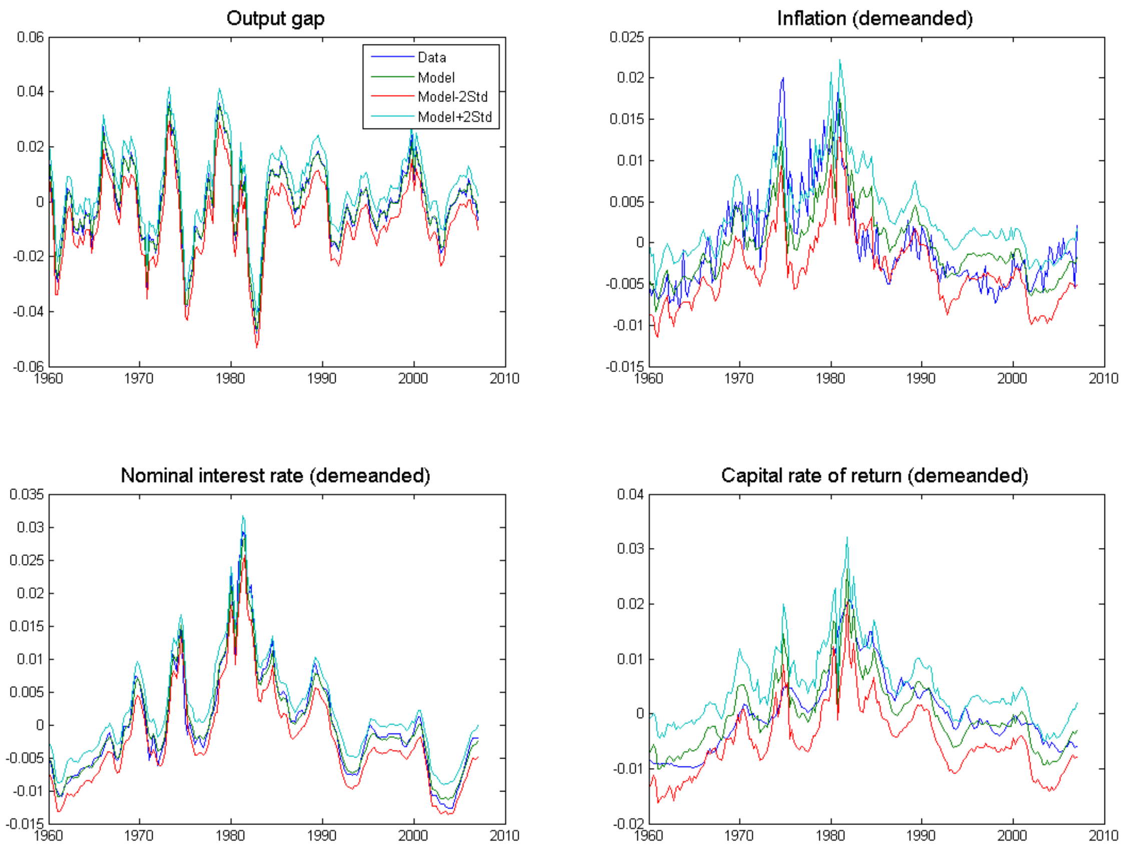
Parameter	Mean	S.E. ($\times 10^{-2}$)
Nominal rigidities parameters		
γ	0.203	0.308
ξ	0.708	0.162
Policy parameters		
ρ_r	0.597	0.189
θ_π	1.560	0.400
θ_y	0.069	0.114
Parameters of the stochastic process for structural shocks		
ρ_κ	0.962	0.101
ρ_a	0.978	0.092
ρ_g	0.959	0.159
σ_κ	0.060×10^{-2}	0.020
σ_a	0.881×10^{-2}	0.013
σ_m	0.188×10^{-2}	0.004
σ_g	4.130×10^{-2}	0.039
Parameters of the stochastic process for volatility shocks		
ρ_{σ_κ}	0.424	0.741
ρ_{σ_a}	0.980	0.685
ρ_{σ_m}	0.971	0.482
ρ_{σ_g}	0.624	0.651
η_κ	0.352	2.239
η_a	0.168	2.000
η_m	0.163	2.058
η_g	0.337	2.350
Parameters of measurement equations		
σ_{my}	0.338×10^{-2}	0.011
$\sigma_{m\pi}$	0.412×10^{-2}	0.012
σ_{mr_n}	0.154×10^{-2}	0.010
σ_{mr_k}	0.434×10^{-2}	0.012

Notes: The table shows the estimates of parameters in the baseline DSGE model in section 2.

Regarding the stochastic volatility processes, the standard deviation of the technology shock is the most persistent with an estimated AR(1) coefficient of 0.980, followed closely by the coefficient of the monetary shock, 0.971. The standard deviation of government shock is found to be fairly persistent with a coefficient of 0.624. Meanwhile, the corresponding value for the investment-specific technology shock is the least persistent, 0.424. For the standard deviations of volatility shocks, we find that those of investment-specific technology and government spending innovations are similar with a value of about 0.35 for each. Meanwhile, the corresponding values of monetary and teachnology innovations are close to each other, 0.17.

Finally, we find that the standard deviations of measurement noises are small, suggesting that the model captures the aggregate dynamics relatively well. To corroborate the statement, we plot the actual data and the data generated by the model (filtered

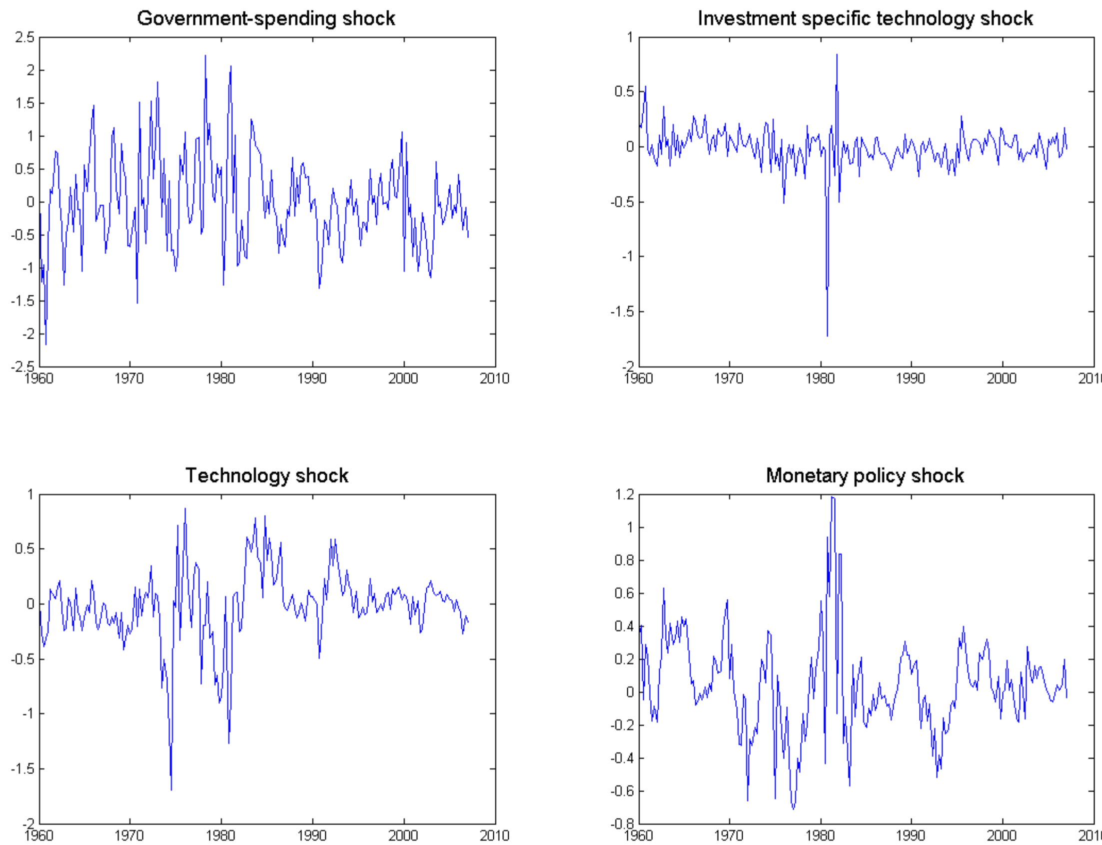
Figure 1: Model vs. Data



Notes: The graphs show the actual data and those generated by the baseline DSGE model with 2 S.D. bounds.

states) in Fig. 1. The top left graph shows that the model captures much of the dynamics of the real output gap *per capita*. The model value has a correlation of 99% with the data. The standard deviation of the former and the latter are of equal magnitude 0.015. The top right plot displays the actual data and the generated data for inflation. The correlation between them is 80% and their standard deviations are similar around 0.005. The bottom left graph depicts the actual observation and the one created by the model for the nominal interest rate, it appears that the model replicates the data very well with a correlation of 99% and a standard deviation of 0.008 for each. Finally, the actual data and the value produced by the model for the nominal rate of return on capital are displayed in the bottom right graph. Their correlation is 88% and they have similar standard deviations of 0.007. Based on these evidence, we conclude that the model is fairly successful in characterizing the properties of the economy.

Figure 2:   Structural shocks



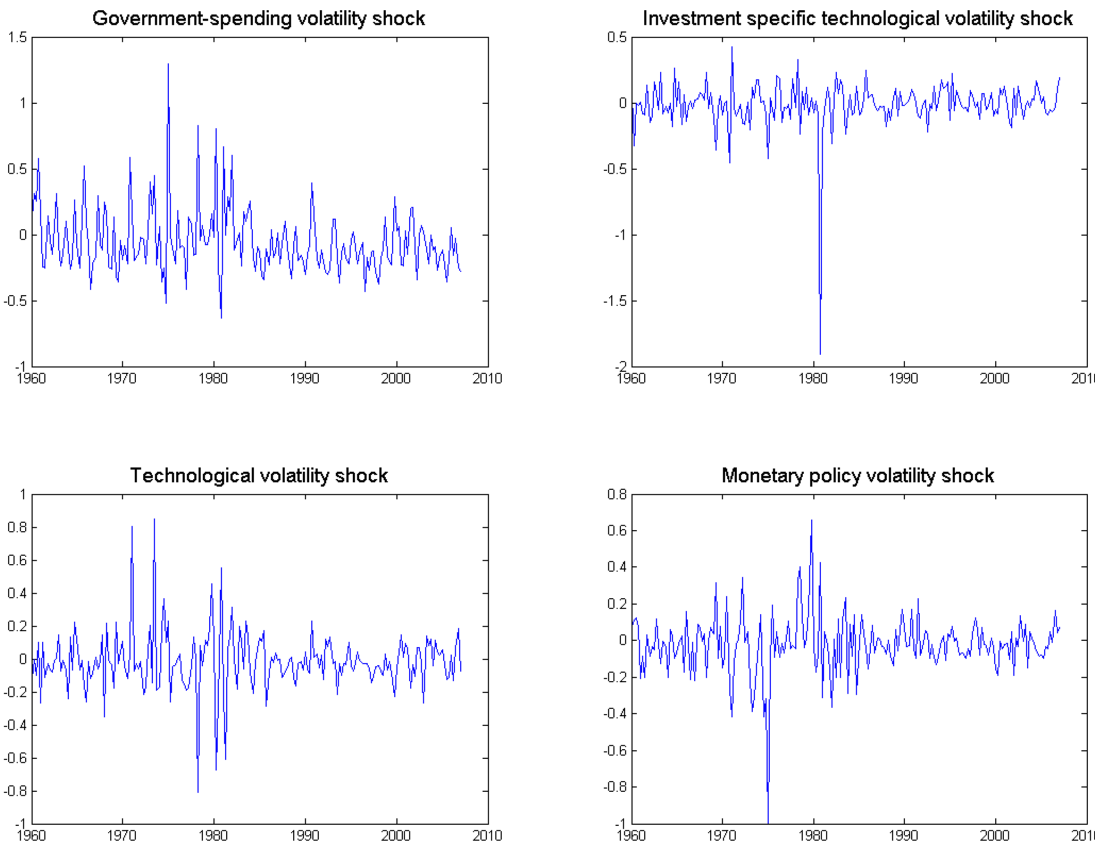
Notes: The graphs present the estimates of structural shocks. They are obtained by performing particle filtering to compute the posterior densities of the shocks given the values of parameters. Combining all the estimates of their means over the sample provides us the measures of the shocks.

4.3 The Evolution of Structural and Volatility Shocks

In this subsection, we present the estimates of the structural shocks and the volatility shocks of the model. This exercise has been done in models without financial frictions, for instance Fernández-Villaverde et al. (2010) and Justiniano and Primiceri (2008).

Figure 2 reports the evolution of structural shocks (ε_{mt}, ε_{at}, ε_{gt}, and $\varepsilon_{\kappa t}$). The figure shows that our model is successful in capturing striking features documented in the literature. First, there are two clear drops in the technology shocks in $1972-1974$ and $1980-1981$ and one substantial reduction in the investment-specific technology shocks in 1980 which are likely the consequences of the oil price shocks. Second, regarding the monetary policy shocks, our model shows large fluctuations in the first half of the 1980s which might be caused by fast changes in the policy by the Fed chairman Paul Volcker.

Figure 3: Volatility shocks



Notes: The graphs present the estimates of volatility shocks. They are obtained by performing particle filtering to compute the posterior densities of the shocks given the values of parameters. Combining all the estimates of their means over the sample provides us the measures of the shocks.

The volatility shocks are plotted in Figure 3. One common feature is that the shocks were higher in the 1970s and early half of the 1980s than in other periods. This result therefore asserts Blanchard and Simon (2001)'s observation that volatility had fallen in the 20th century with a temporal and surprising rise in the 1970s. Especially, the volatility shocks have substantially declined since the middle of the 1980s, around 1984. McConnell and Perez-Quiros (2000) and Kim and Nelson (1999) also document a decline in the volatility of U.S real GDP growth around this point in time. Stock and Watson (2002) consider 1984 as the start of the "Great Moderation" period in the U.S economy. Our results therefore suggest that the fall in the magnitude of shocks, especially that of volatility shocks, might have contributed to the stability during the Great Moderation period in the U.S., in accordance with Born and Peifer (2014) and Justiniano and Primiceri

(2008).

5 Impulse Response Functions

This section is devoted to investigate the impulse response functions (IRFs) generated by our model to a one-standard-deviation shock u_{mt}. There are two issues deserving discussion. First, recall that in the second order approximation the volatility shocks enter policy functions in the cross-product with the corresponding level shocks, e.g. $u_{mt}\varepsilon_{mt}$. This connection presents us from disentangling the impact of volatility shocks on the economy independently. To overcome this issue, we solve the model to the third-order approximation, given the parameters estimated in the previous section, because at that volatility shocks play a role by themselves, therefore allowing us to compute the IRFs to a second-moment shock of monetary policy while keeping its level shock unchanged.

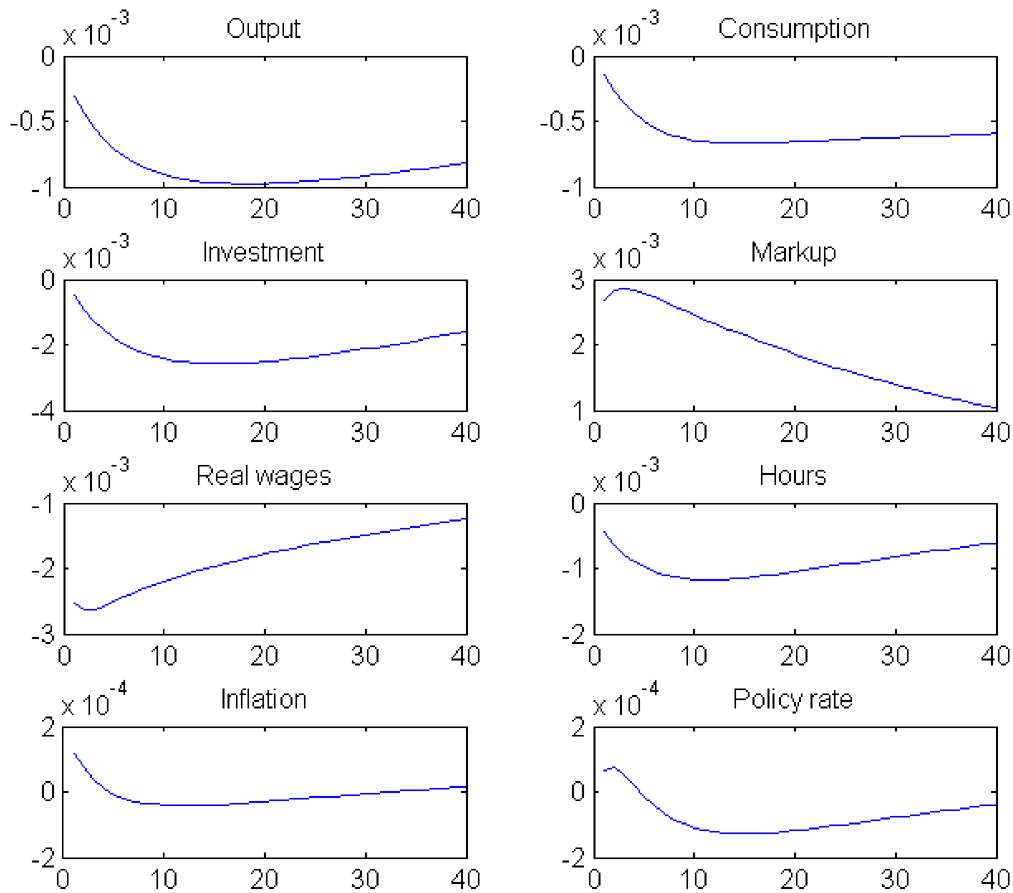
Second, the higher-order approximation of the model not only results in a nonlinear environment which makes the computation of IRFs somewhat complicated, but also makes the simulated paths of states and controls in the model move away from their state values. To deal with these issues, we follow the process proposed by Fernández-Villaverde et al. (2011) which calculates the IRFs as percentage deviations form their means, rather than their steady states.

Figure 4 plots the IRFs to a positive one-standard-deviation monetary volatility shock. This shock causes a prolonged contraction in economic activity: output, consumption, investment, real wages and hours fall. Our model is therefore successful in generating business-cycle co-movements in response to changes in the uncertainty of monetary policy. This feature is an important prerequisite for any shock that seeks to explain business cycle fluctuations because those co-movements are observed in the data (see Cesa-Bianchi and Fernandez-Corugedo 2014, Basu and Bundick 2012).

The principal transmission mechanism for monetary volatility shocks is in line with Basu and Bundick (2012). The uncertainty causes households to consume less, save more, and supply more hours for any given wage (precautionary behavior). An increased labor supply decreases wages, leading to a fall in marginal cost. The decline in marginal cost raises markups because prices adjust slowly due to the price rigidity. Consequently, the demand for household labor falls, which lowers the real wage earned by the representative household. Moreover, the decrease in labor demand reduces investment in the capital stock by entrepreneurs. Financial frictions amplify further the decrease in investment via the financial accelerator mechanism as will be analyzed below. The increase in inflation can be explained as a supply-shock-alike effect of the uncertainty because it lowers labor and capital demand. Policy rate, which follows a Taylor rule, rises in response to the increase in inflation. Then both inflation and the interest rate fall because of the contraction of economic activity.

In order to investigate the role of financial frictions, we compare the IRFs to a monetary volatility shock in the baseline economy in section 2 and in two different counterfactual ones which include: one with reduced financial frictions and the other with more pronounced financial frictions. These alternative cases are generated by modifying the value

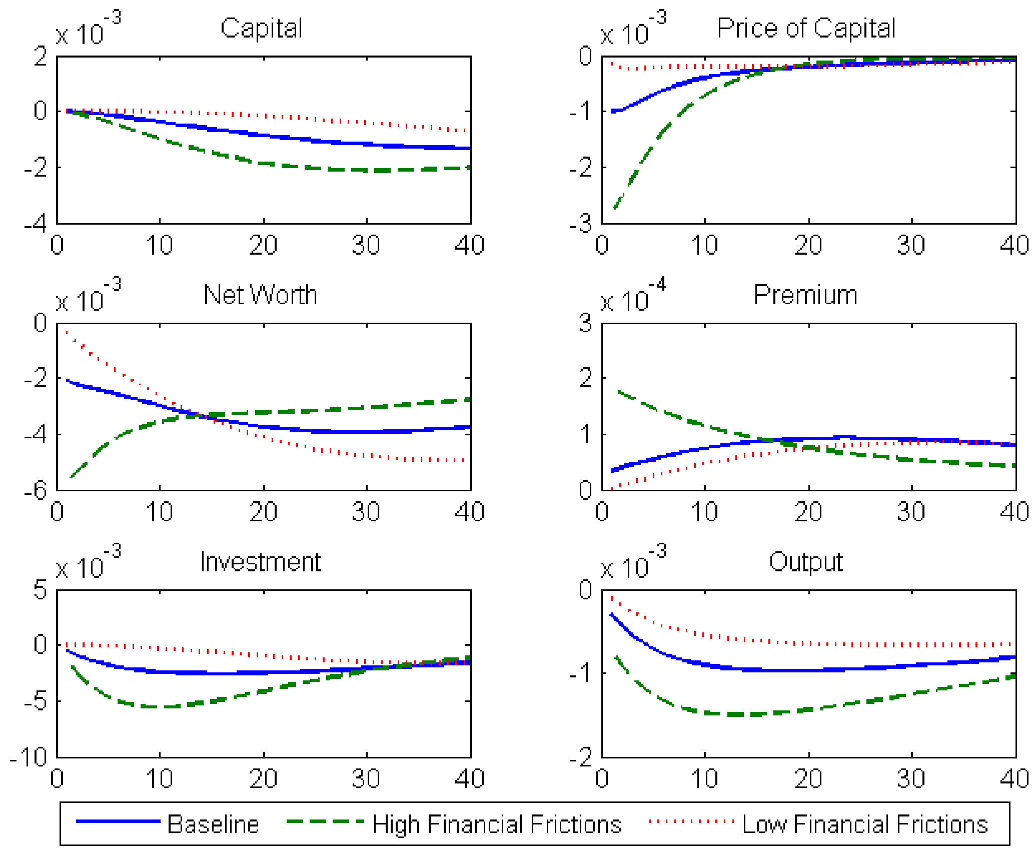
Figure 4: IRFs to a one S.D. monetary volatility shock



Notes: The graphs are expressed as percentage changes from their ergodic means.

of the monitoring cost parameter μ. The idea is that monitoring cost introduces a wedge in the lender's zero profit condition. Therefore, if the monitoring cost is higher, they require a higher return from lending, which in turn causes a greater external premium or, in other words, a more pronounced level of financial frictions. This intuition is captured by Equation (2.14). We present the IRFs in these models in Figure 5. The financial accelerator mechanism are magnified in both the baseline and the counterfactual models. The decline in capital demand caused by increased markups leads to a fall in its price, therefore decreasing firms' net worth. The fall in the net worth increases the external premium required by lenders, forcing down investment and output. More importantly, we note that a higher level of financial frictions lead to a greater premium, which decreases further investment. A kind of multiplier effect arises, since the fall in investment lowers the price of capital and net worth, therefore pushing down investment more substantially.

Figure 5: IRFs to a one S.D. monetary volatility shock- Effects of Financial Frictions



Notes: The graphs are expressed as percentage changes from their ergodic means. For low level of financial frictions: $\mu = 0.03$. For high level of financial frictions: $\mu = 0.1$

Consequently, the decline in output is larger when financial frictions are more pronounced.

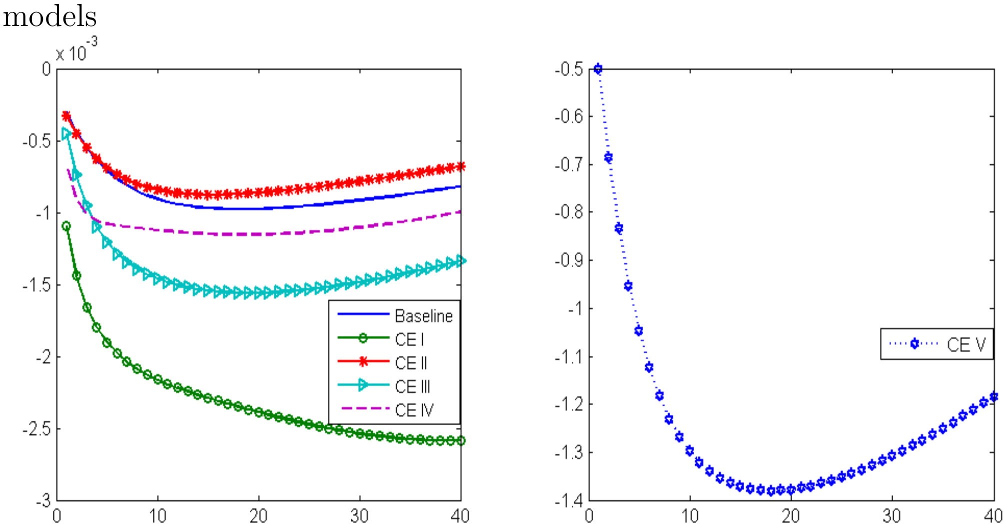
Nevertheless, we find that the overall effects of monetary volatility shock are small. This is consistent with previous studies on aggregate uncertainty including Born and Peifer (2014), Cesa-Bianchi and Fernandez-Corugedo (2014), Bonciani and Van Roye (2013), and Bachmann and Bayer (2011). According to Born and Peifer (2014), such relatively small effects of volatility shocks are due to dampening general equilibrium effects. In what follows, we conduct several counterfactual experiments (henceforth CE), described in Table 2, in order to understand those effects better. Specifically, in the first CE, the inverse of the Frisch labor elasticity ϑ is increased, which therefore makes labor supply less flexible in response to shocks. In the next three experiments, we decrease the values of Calvo parameter prices ξ, capital adjustment costs ϕ_s, and consumption habits λ, thus reducing the persistence in the model. The last experiment is about the counteracting reaction of monetary policy; specifically, we shut off the response of interest rate to output gap and considerably increase the smoothing parameter. These changes are expected to

Table 2: Counterfactual Experiments (CE)

Parameter	Descriptions	Baseline	CE I	CE II	CE III	CE IV	CE V
ϑ	Inverse of the Frisch elasticity	1.3	10	*	*	*	*
ξ	Calvo parameter prices	0.708	*	0.6	*	*	*
ϕ_s	Capital adjustment costs	4.5	*	*	0.5	*	*
λ	Consumption habits	0.9	*	*	*	0.6	*
θ_y	Taylor rule output gap	0.069	*	*	*	*	0
ρ_r	Interest smoothing	0.597	*	*	*	*	0.98

Notes: "*" means that the value of relevant parameter in the counterfactual model is the same with that in the baseline model. Other parameters not listed in the table are remained the same as in the baseline model.

Figure 6: IRFs of output to a one S.D. monetary volatility shock- Baseline and Counterfactual models

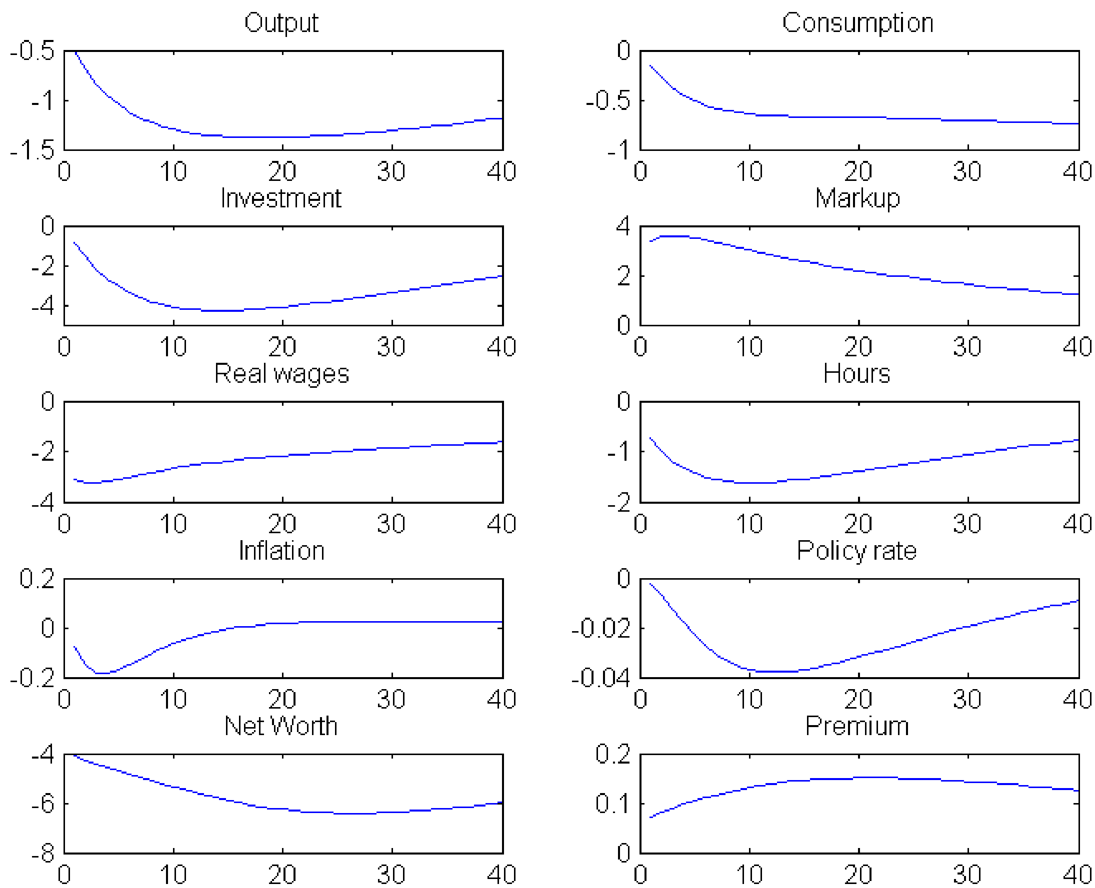


Notes: The graphs are expressed as percentage changes from their ergodic means. The left graph plots the IRFs of output in the baseline model (solid), the CE I (circle), the CE II (star), the CE III (triangle), the CE IV (dashed). The right graph shows the IRFs of output in the CE V (hexagram).

reduce the dampening general equilibrium effects, thus generating a bigger fall in the output in response to the volatility shock.

Figure 6 presents the impulse response functions of output in the baseline and counterfactual models to a monetary volatility shock. The left graph compares the response in the baseline model and those in the first four counterfactual experiments (I to IV). As it can be seen, output falls more in most of counterfactual experiments than in the baseline model. Especially, the CE I, in which the inverse of the Frisch labor elasticity is increased, triples the negative output response (after 40 quarters). Although the size of responses are still relatively small.

Figure 7: IRFs to a one S.D. monetary volatility shock: Counterfactual experiment V



Notes: The graphs are expressed as percentage changes from their ergodic means.

However, this is not the case in the counterfactual experiment V which is shown in the right graph of the same figure. A one-standard-deviation monetary volatility shock causes a large and persistent decline in output. Specifically, output drops immediately by 0.5% after the shock, falls as great as 1.5%, reaching to the lowest point, after 20 quarters, and then slowly returns to its mean. This implies that monetary policy plays the most important role in deciding the size of the effects of uncertainty. A similar result is documented by Born and Peifer (2014). In the baseline model, with positive response to output gap and a small value of interest smoothing, the response of the central bank is more aggressive and quicker to offset the negative shock, therefore mitigating the potential impacts of uncertainty. In the experiment V, we however force the response to output down to zero and give more weight to past interest rates. This means that the current economic conditions affect the nominal interest rate less than its past values. Figure 7 plots the IRFs of output and other variables to a monetary volatility shock in the counterfactual experiment V. The transmission mechanism of the shock is similar to what we discussed in the baseline model with increased markups and greater premium. However, the sluggish response of monetary policy exacerbates the contraction. As it can be seen, the nominal interest rate falls as a result of the reduction in inflation caused by the contraction in the economy (because the response to output gap is fixed at zero), but the decrease of inflation is much stronger than that of the nominal interest rate, leading to an increase in the real interest rate. Consequently, investment decreases further, which is again amplified by the occurrence by financial frictions in the model. Eventually, investment falls by more than 4% after 10 quarters, resulting in a substantial decline in output.

The finding that the more sluggish monetary policy the more substantial the effects of monetary volatility shocks on the economy might have a very important implication regarding the zero-lower bound in the nominal interest rate, although our current model does not explicitly account for it. In such a situation, the nominal interest rate is clearly independent to current conditions and substantially, if not completely, depends on its last values. We can think about it as a Taylor rule in which the interest smoothing approaches one. This limits the ability of the nominal interest rate to mitigate negative shocks to the economy, therefore resulting in a more negative effects on economic activity. Basu and Bundick (2012) consider the uncertainty of TFP shocks and argue that the uncertainty has larger effects in the zero-lower bound. A similar result is documented by Fernández-Villaverde et al. (2013) who consider fiscal uncertainty.

6 Conclusion

The paper attempts to investigate the role of financial frictions in the transmission of monetary volatility shocks on output. In order to do that, we employed the particle filter to estimate a non-linear DSGE model incorporating the financial frictions *à* la Bernanke et al. (1999) and allowing for stochastic volatility of structural shocks. The results show that our model captures aggregate dynamics relatively well. We also find that the magnitude of volatility shocks was large during 1970s and early 1980s, but has declined considerably since the mid-1980s, around 1984. Therefore, the fall in the magnitude of shocks might have contributed to the Great Moderation in the U.S.

We also show that an increase in the volatility of monetary policy causes a contraction in economic activity: output, consumption, investment, hours, and real wages fall.

In addition, we argue that financial frictions amplify the effects of monetary volatility shocks via the financial accelerator mechanism. Nevertheless, the effects of a monetary volatility shock are relatively small, which is due to the dampening general equilibrium effects, especially the counteracting response of monetary policy to the shock. Our results document that the impacts of monetary volatility shocks would be substantial if the monetary policy was restrained to response to current conditions in the economy.

Our work does not examine the impact of monetary volatility shocks under environments in which there is a zero-lower bound in the nominal interest rate or unconventional monetary policies. Advancing the model to address these issues is an interesting and important expansion which we would like to consider in the future research.

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A Appendix

A.1 Choice of density function for ψ_t

Then, one can draw that $E_t(\psi_{t+1}) = 1$. Some other outputs can be calculated including

$$F(\psi_t) = \Phi(z_t)$$

$$\begin{aligned} G(\psi_t) &= \int_0^{\psi_t} \psi f(\psi) d\psi \\ &= 1 - \int_{\psi_t}^{\infty} \psi f(\psi) d\psi \\ &= 1 - \Phi(\sigma_\psi e^{\sigma_{\psi t}} - z_t) \\ &= \Phi(z_t - \sigma_\psi e^{\sigma_{\psi t}}) \end{aligned}$$

$$\Gamma(\psi_t) = \psi_t(1 - \Phi(z_t)) + \Phi(z_t - \sigma_\psi e^{\sigma_{\psi t}})$$

$$G^{'}(\psi_t) = \psi_t \mathrm{f}(\psi_t)$$

$$\Gamma^{'}(\psi_t) = 1 - F(\psi_t)$$

where $z_t = (\frac{log\,(\psi_t) + 0.5\sigma_\psi^2 e^{2\sigma_{\psi t}}}{\sigma_\psi e^{\sigma_{\psi t}}})$, f($\psi$) is the p.d.f of ψ, and $\Phi(.)$ is the standard normal c.d.f.

A.2 Data Sources and Construction

The original time series' sources are summarized as follows

- RGDP: Real Domestic Product, Billions of chained (2005) dollars, Seasonally adjusted at annual rates, Bureau of Economic Analysis Table 1.1.6, line 1
- GDPDEF: Gross Domestic Product: Implicit Price Deflator (GDPDEF), Index 2009 = 100, Quarterly, Seasonally Adjusted, Federal Reserve Economic Data
- LNU00000000Q: Labor force status: Civilian noninstitutional population; Bureau of Labor Statistics
- LNS10000000Q: Labor force status: Civilian noninstitutional population; Bureau of Labor Statistics (Before 1976: LNU00000000Q)
- LNSindex: LNS10000000Q(2005 : 2) = 1
- FFR: Federal Funds Rate; Federal Reserve Bank of St. Louis
- BAA: Moody's seasoned Baa corporate bond yields; Federal Reserve Bank of St. Louis

The four observable data used in the estimation are constructed as below

- $ROUT_t = LN\left(\frac{RGDP_t}{LNSindex_t}\right)$

- $OUT_t = ROUT_t - \overline{ROUT}_t$ in which $\overline{ROUT}_t$ is the potential output per capital filtered by the Hodrick-Prescott method.
- $INP_t = LN\left(\frac{GDPDEF_t}{GDPDEF_{t-1}}\right)_{demeanded}$
- $INR_t = FFR_t/400_{demeanded}$
- $CBY_t = BAA_t/400_{demeanded}$

A.3 Particle Filter Algorithm

The model considered above belongs to a larger class of non-linear and/or non-normal dynamic macroeconomic models which can be written generally in the following state-space system. First, the law of motion for the state vector $\mathbf{x}_t$ is given by

$$\mathbf{x}_t = h(\mathbf{x}_{t-1}, \mathbf{w}_t; \boldsymbol{\Xi}) \tag{A.1}$$

where $\mathbf{w}_t$ is a random vector of innovations, in our specific case $\mathbf{w}_t$ includes structural and volatility shocks, with dimension n_w and Ξ is the vector of parameters of the model. Second, the set of observables denoted by $\mathbf{z}_t$ are connected to the state variables $\mathbf{x}_t$ by the measurement equation

$$\mathbf{z}_t = g(\mathbf{x}_t, \mathbf{v}_t; \Xi) \tag{A.2}$$

where $\mathbf{v}_t$ is a random vector of measurement errors. To be convenient, we assume independence between $\mathbf{v}_t$ and $\mathbf{w}_t$. The functions h and g come from the equations that characterize the behavior of the model. The particle filter algorithm is presented below.
Particle Filter Algorithm

- *Initialization* $t = 0$
  Draw N particles $\mathbf{x}_0^{(i)}$, $i = 1, 2, .., N$, from $p(\mathbf{x}_0; \Xi)$ and let $\pi_0^{(i)} = \frac{1}{N}$ for all i.
- *Propagation*
  Draw N particles $\mathbf{x}_t^{(i)}$, $i = 1, 2, .., N$, from $p(\mathbf{x}_t|\hat{\mathbf{x}}_{t-1}^{(i)}; \Xi)$.
- *Importance weights*
  Evaluate the importance weights $\pi_t^{(i)}$, $i = 1, 2, .., N$

$$\pi_t^{(i)} = \pi_{t-1}^{(i)} p(\mathbf{z}_t|\mathbf{x}_t^{(i)}; \Xi)$$

- *Log-Likelihood Contribution*

$$log\, L_t = log\, L_{t-1} + log(\sum_{i=1}^{N} \pi_t^{(i)})$$

- *Normalization*
  Normalize the importance weights $\pi_t^{(i)}$, $i = 1, 2, .., N$

$$\tilde{\pi}_t^{(i)} = \frac{\pi_t^{(i)}}{\sum_{i=1}^{N} \pi_t^{(i)}}$$

- *Resampling step*
  We use the systematic resampling algorithm to generate a new set of particles $\{\hat{\mathbf{x}}_t^{(j)}\}_{j=1}^N$ by resampling (with replacement) from the existing particles $\{\mathbf{x}_t^{(i)}\}_{i=1}^N$ with probability $\{\hat{\pi}_t^{(i)}\}_{i=1}^N$.

- *Propagation*
  Set $t = t + 1$ and go to Step 2: Propagation

Systematic Resampling Algorithm

- *Construction of the cumulative sum of weights (CSW)*
  Let $c_1 = \tilde{\pi}_t^1$ and define $c_i = c_{i-1} + \tilde{\pi}_t^{(i)}$ for $i = 2, ..., N$

- *Resampling step*
  Generate a starting point from a uniform distribution: $u_1 \sim U[0, N^{-1}]$ and define $u_j = u_1 + N^{-1}(j-1)$ for $j = 2, ..., N$. For each $j = 1, ..., N$, find $i = 1, ..., N$ to satisfy

$$c(i-1) \leq u(j) \leq c(i)$$

  - Assign sample: $\mathbf{s}_k^{(j)^*} = \mathbf{x}_k^{(i)}$
  - Assign weight: $\pi_t^{(j)} = N^{-1}$