DISTANCE COLORING OF THE HEXAGONAL LATTICE

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Abstract

Motivated by the frequency assignment problem we study the d-distant coloring of the vertices of an infinite plane hexagonal lattice H. Let d be a positive integer. A d-distant coloring of the lattice H is a coloring of the vertices of H such that each pair of vertices distance at most d apart have different colors. The d-distant chromatic number of H, denoted $\chi_d(H)$, is the minimum number of colors needed for a d-distant coloring of H. We give the exact value of $\chi_d(H)$ for any d odd and estimations for any d even.

Keywords: distance coloring, distant chromatic number, hexagonal lattice of the plane, hexagonal tiling, hexagonal grid, radio channel frequency assignment.

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1. Introduction

By the distance $\mathrm{dist}_G(u, v)$ of two vertices u and v in a graph G we mean the number of edges of the shortest path joining them. Given a positive integer d, a *d-distant n-coloring* of a graph G is an assignment φ that assigns numbers from the set $\{1, 2, \ldots, n\}$ to the vertices of G in such a way that each pair of vertices within the distance d have assigned different colors. Then the *d-distant chromatic number* $\chi_d(G)$ of a graph G is the minimum number n such that there is a d-distant n-coloring of the graph G.

The d-distant coloring of graphs has been introduced by Wegner [14] (see also [9, 10]). Many articles, e.g. [1, 3, 4, 7], have studied a special case, the distant colorings with condition at distance 2. Several papers have been devoted to distant colorings during the last decade, see for example [2, 6, 7, 8, 12] and references therein.

Consider a natural partition of the plane to regular hexagons as a 3-regular graph, called a *hexagonal lattice*. The hexagonal lattice is a significant planar graph, one of the three regular-polygons-tilings of the plane. Given a positive integer d, how many colors are necessary to color the vertices of the hexagonal lattice so that any two vertices within distance d are of distinct color? This question naturaly arises from the *Frequency Assignment Problem* (see for example [5]), where the vertices of a graph correspond to transmitter locations and their colors represent frequencies of the channels. The transmitters that are "very close", cannot use the same frequency due to interference.

Let $[x]$ for any $x \in \mathbb{R}$ be an integer such that $x - \frac{1}{2} < [x] \le x + \frac{1}{2}$.

Several authors have studied situations when the transmitters are spread over the plane regularly, since the Cellular concept has been introduced [11]. Recently it has been shown by Fertin, Godard and Raspaud [2] that the d-distant chromatic number of the square lattice S is $\chi_d(S) = [\frac{1}{2}(d + 1)^2]$. Similar result has been discovered for the triangular lattice T by Ševčíková [13], where she proved that $\chi_d(T) = [\frac{3}{4}(d + 1)^2]$.

The main result of this paper is the following theorem:

Theorem 1.1. *The d-distant chromatic number χ_d of the hexagonal lattice H is*

(i) $\chi_d(H) = \left[\frac{3}{8}(d + 1)^2\right]$ *for every odd $d \ge 1$,*

(ii) $\chi_2(H) = 4$, $\chi_4(H) = 11$, $\chi_6(H) = 20$,

(iii) $\frac{3}{8}d^2 + \frac{3}{4}d + 2 \le \chi_d(H) \le \left[\frac{3}{8}(d + \frac{4}{3})^2\right]$ *for every even $d \ge 8$.*

We believe that the following is true

Conjecture 1.2. *For every even $d \geq 2$ there is*

$$\chi_d(H) = \left[\frac{3}{8} \left(d + \frac{4}{3} \right)^2 \right].$$

We will prove this theorem after introducing some necessary properties and system of coordinates in the hexagonal lattice.

The symbol H will be used for the hexagonal lattice, symbols V, E for the vertex-set and edge-set, respectively, of the lattice H, symbol $\mathrm{dist}(u, v)$ for the distance of any two vertices $u, v \in V$ and symbol χ_d for the d-distant chromatic number of a graph H in this article.

2. CHARACTERIZATION OF THE HEXAGONAL LATTICE

Notice that the hexagonal lattice H is a bipartite graph. Any circle of six vertices from V will be called an *eye*. Fix an arbitrary eye to be the *central eye* and denote R_1 its vertex-set. For any positive integer k define a *k-th ring* R_k as a subset of V such that the R_k contains a vertex $v \in V$ if and only if the distance between vertex v and the central eye satisfies $2k - 3 \leq \mathrm{dist}(v, R_1) \leq 2k - 2$ (note that the distance $\mathrm{dist}(v, R_1)$ is defined to be the minimum distance between v and any vertex belonging to the set R_1). Clearly, the system of sets R_k is a partition of the vertex-set of the hexagonal lattice H.

We similarly denote the following sets:

$$R_k^- = \{v \in V(H); \mathrm{dist}(v, R_1) = 2k - 3\} \text{ and}$$
$$R_k^+ = \{v \in V(H); \mathrm{dist}(v, R_1) = 2k - 2\}.$$

Thus, $R_k^- \cup R_k^+ = R_k$. Note, that $R_1^- = \emptyset$.

Proposition 2.1. *For any positive integer k, no two vertices from R_k^- are adjacent in H. Furthermore, one of the vertices adjacent to $v \in R_k^-$ belongs to R_{k-1}^+ and the other two belong to R_k^+.*

Proposition 2.2. *For $k \geq 2$, the set R_k^+ contains exactly six disjoint pairs of adjacent vertices (called corner vertices). Moreover, if $v \in R_k^+$ is a corner*

vertex, then one of the vertices adjacent to v belongs to R_k^-, one belongs to R_k^+ and the third vertex belongs to R_{k+1}^-. If $v \in R_k^+$ is not a corner vertex, then two of the vertices adjacent to v belong to R_k^- and the last one belongs to R_{k+1}^-.

From the two listed propositions we immediately get the following Lemma.

Lemma 2.3. *For any positive integer k is $|R_k^-| = 6(k-1)$ and $|R_k^+| = 6k$. Consequently, the number of vertices in the k-th ring is $|R_k| = 6(2k-1)$ and so the first m rings contain $\sum_{k=1}^m |R_k| = 6m^2$ vertices.*

3. System of Coordinates in the Hexagonal Lattice

Suppose we have a square lattice S with the usual Cartesian system of coordinates, i.e., such that the Euclidean distance of any two adjacent vertices is 1. Using the coordinates, any vertex v of this square lattice can be represented as a pair (a, b) and then clearly the (graph) distance of two vertices $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ in the square lattice is $\text{dist}_S(v_1, v_2) = |a_1 - a_2| + |b_1 - b_2|$. Removing all edges joining two vertices (a, b) and $(a, b+1)$ in this square lattice whenever $a + b$ is odd, we get a lattice with hexagonal cells. Then, the orthocenter's neighbours are vertices $(0, 1), (-1, 0), (1, 0)$, but vertex $(0, -1)$ is not adjacent to the orthocenter. We can use this system of coordinates for the vertices of the hexagonal lattice, because the change of the embedding of a graph does not influence graph's combinatorial properties.

Since the hexagonal lattice is a bipartite graph, there exist two disjoint sets V_0, V_1 of its vertices such that $V_0 \cup V_1 = V$ and each edge $uv \in E$ satisfies $u \in V_0$ and $v \in V_1$ or otherwise. Without loss of generality let orthocenter belong to V_0. Every vertex $v \in V_0$ is called to be of a *type* 0, similarly vertices in V_1 are called of a *type* 1. The type of a vertex v is denoted by $\tau(v)$.

Proposition 3.1. *Let $v = (a, b)$ be a vertex of the hexagonal lattice. Then the type of v is*

$$\tau(v) = ((a + b) \mod 2).$$

Lemma 3.2. *Let $v_1 = (a_1, b_1), v_2 = (a_2, b_2)$ be two vertices of the hexagonal lattice, assume that $b_1 \geq b_2$. Then the distance between v_1 and v_2 is*

$$\text{dist}(v_1, v_2) = \begin{cases} |a_1 - a_2| + |b_1 - b_2|, & \text{if } |a_1 - a_2| \geq |b_1 - b_2|; \\ 2|b_1 - b_2| - \tau(v_1) + \tau(v_2), & \text{if } |a_1 - a_2| < |b_1 - b_2|. \end{cases}$$

Proof. In the first case, every v_1v_2-path must contain at least $|a_1 - a_2|$ horizontal edges and at least $|b_1 - b_2|$ vertical edges. And a v_1v_2-path containing exactly $|a_1 - a_2| + |b_1 - b_2|$ edges clearly exists. Denoting $v_3 = (a_2 + (b_1 - b_2), b_1)$, we will investigate three possibilities for the second case:

α) if $\tau(v_1) = \tau(v_2)$: the distance is $\text{dist}(v_1, v_2) = \text{dist}(v_3, v_2) = |a_2 + (b_1 - b_2) - a_2| + |b_1 - b_2| = 2|b_1 - b_2|$;

β) if $\tau(v_1) = 0, \tau(v_2) = 1$: the distance is $\text{dist}(v_1, v_2) = \text{dist}(v_3, v_2) + 1 = |a_2 + (b_1 - b_2) - a_2| + |b_1 - b_2| + 1 = 2|b_1 - b_2| + 1$;

γ) if $\tau(v_1) = 1, \tau(v_2) = 0$: the distance is $\text{dist}(v_1, v_2) = \text{dist}(v_3, v_2) - 1 = |a_2 + (b_1 - b_2) - a_2| + |b_1 - b_2| - 1 = 2|b_1 - b_2| - 1$. ∎

Corollary 3.2.1. *Let $v_1 = (a_1, b_1), v_2 = (a_2, b_2)$ be two vertices of the hexagonal lattice, assume that $b_1 \geq b_2$. Then $\text{dist}(v_1, v_2) \geq 2|b_1 - b_2| - 1$.*

Corollary 3.2.2. *Let $v_1 = (a_1, b_1), v_2 = (a_2, b_2)$ be two vertices of the hexagonal lattice, assume that $b_1 \geq b_2$. Then $\text{dist}(v_1, v_2) \geq |a_1 - a_2|$.*

4. PROOF OF THE THEOREM

In terms of the definition of Heuvel, Leese and Shepherd [6], the n-colorings from Lemmas 4.1, 4.4, 4.7 and 4.10 below are colorings by arithmetic progression, i.e., there exist nonnegative integers a and b such that $\varphi(m_1, m_2) = am_1 + bm_2 \pmod{n}$ for all vertices $(m_1, m_2) \in V(H)$. Here φ is a d-distant n-coloring.

A set system $\{F_i, i \in I\}$, where I is an index-set, is an *n-system of color classes* for a given graph G, or an *n-system* for brevity, if the two following conditions hold:

(i) $\{F_i, i \in I\}$ partitions the vertex-set of G,

(ii) there exists an injective assignment $\psi : I \rightarrow \{1, 2, \ldots, n\}$.

Letting $\varphi(v) = \psi(i)$ for any $i \in I$ and $v \in F_i$ we then get an induced n-coloring of the graph G.

To prove Theorem 1.1 we will consider four cases depending on the remainder of d modulo 4. In each case we introduce an n-system giving an upper bound for χ_d and then we show the number of colors that is necessary. By the statement $r = (a \mod m)$ we mean that r is the remainder of a divided by m. An expression $a \equiv b \pmod{m}$ denotes that a is congruent with b modulo m.

Case 1. $d = 4m - 1$ for some positive integer m

Lemma 4.1. *Let* $I = \{(i,j); i \in \{0, 1, 2, \ldots, 3m - 1\}$ *and* $j \in \{0, 1, 2, \ldots, 2m - 1\}\}$. *The set system*

$$F_{i,j} = \{(a,b); \ [a \equiv i \pmod{6m} \ and \ b \equiv j \pmod{2m}] \ or$$

$$[a \equiv i + 3m \pmod{6m} \ and \ b \equiv j + m \pmod{2m}]\}$$

for $(i,j) \in I$ *is a* $6m^2$*-system of color classes for the hexagonal lattice.*

Proof. (i) Let $(x,y) \in V$. If the remainder of x divided by $6m$ is less than $3m$, denote i, j in the following way: $i = (x \mod 6m)$ and $j = (y \mod 2m)$. Otherwise, denote $i = ((x - 3m) \mod 6m)$ and $j = ((y - m) \mod 2m)$. Then $(i,j) \in I$ and $(x,y) \in F_{i,j}$. The pair (i,j) is determined uniquely, and so this set system partitions the set V.

(ii) The assignment $\psi(i,j) = 2mi + j + 1$ for $(i,j) \in I$ is injective to the set $\{1, 2, \ldots, 6m^2\}$.   ■

Lemma 4.2. *There is a* $(4m - 1)$*-distant* $6m^2$*-coloring of the hexagonal lattice.*

Proof. We will prove that the $6m^2$-coloring induced by the set system in Lemma 4.1 is a $(4m - 1)$-distant coloring. Let (i,j) be arbitrary but fixed member of the index-set I from Lemma 4.1. Let $v_1, v_2 \in F_{i,j}$ be any two distinct vertices such that $v_1 = (a_1, b_1), v_2 = (a_2, b_2)$ and without loss of generality let assume $b_1 \geq b_2$. We have two cases:

Case (i) if $a_1 \equiv a_2 \pmod{6m}$
This implies that $b_1 \equiv b_2 \pmod{2m}$. If $a_1 \neq a_2$, then, using Corollary 3.2.2, $\text{dist}(v_1, v_2) \geq |a_2 - a_1| \geq 6m > 4m - 1$. For the possibility that

$a_1 = a_2$ realize that vertices v_1, v_2 are of the same type, because there exists a positive integer y such that $b_1 = b_2 + 2my$. Then by Lemma 3.2 $\text{dist}(v_1, v_2) = 2|b_2 - b_1| \geq 4m > 4m - 1$.

   Case (ii) if $a_1 \equiv a_2 - 3m \pmod{6m}$

Then $b_1 \equiv b_2 - m \pmod{2m}$, i.e., there exist an integer x and positive integer y such that $a_1 = a_2 - 3m + 6mx$ and $b_1 = b_2 - m + 2my$. When $x > 1$ or $x < 0$, from the Corollary 3.2.2 we have $\text{dist}(v_1, v_2) \geq |a_2 - a_1| \geq 9m > 4m - 1$. In case that $y > 1$ there is $|b_1 - b_2| \geq 3m$ and from the Corollary 3.2.1 there is $\text{dist}(v_1, v_2) \geq 2|b_2 - b_1| - 1 \geq 6m - 1 > 4m - 1$. The remaining possible values of x are 0 and 1, so $|a_2 - a_1| = 3m$, and $y = 1$, so $|b_2 - b_1| = m$. This means that $|a_2 - a_1| \geq |b_2 - b_1|$ and finally using the Lemma 3.2 we get that $\text{dist}(v_1, v_2) = |a_2 - a_1| + |b_2 - b_1| = 4m > 4m - 1$.

   We have got that any two distinct vertices from the same color class, i.e., colored with the same color, are at distance greater than $4m - 1$, hence the $6m^2$-coloring induced by the set system in Lemma 4.1 is indeed a $(4m - 1)$-distant $6m^2$-coloring. ∎

Lemma 4.3. *Any $(4m - 1)$-distant coloring of the hexagonal lattice uses at least $6m^2$ colors.*

Proof. Consider all vertices belonging the rings $R_1, R_2, \ldots, R_m$. The distance between any of them and the central eye is at most $2m - 2$. The greatest possible distance for two vertices within central eye is 3, so any two of the considered vertices are of distance at most $2 \cdot (2m - 2) + 3 = 4m - 1 = d$ and so all of them must be colored with distinct colors. Thus, by Lemma 2.3, at least $6m^2$ colors are necessary. ∎

   To summarize Case 1, $\chi_d = 6m^2 = \frac{3}{8}(d + 1)^2$ if $d = 4m - 1$.

   Case 2. $d = 4m + 1$ for some nonnegative integer m

Lemma 4.4. *Let $I = \{(i, j); i \in \{0, 1, 2, \ldots, 3m\} \text{ and } j \in \{0, 1, 2, \ldots, 2m\}\} \cup \{(3m + 1, j); j \in \{m, m + 1, m + 2, \ldots, 2m\}\}$. The set system*

$$F_{i,j} = \{(a, b); [a = i + s + (6m + 3)t \text{ and}$$
$$b = j + (2m + 1)s - t \text{ for some integers } s, t] \text{ or}$$
$$[a = i + s + (6m + 3)t + 3m + 2 \text{ and}$$
$$b = j + (2m + 1)s - t + m \text{ for some integers } s, t]\}$$

for $(i,j) \in I$ *is a* $(6m^2 + 6m + 2)$*-system of color classes for the hexagonal lattice.*

Proof. (i) Observe that there is a pattern of colors, which is replicated over the hexagonal lattice. Any two vertices (x, y) and $(x + 6m^2 + 6m + 2, y)$ are colored with the same color: Let (x, y) have a color (i, j) and let there exist integers s and t such that $x = i + s + (6m + 3)t$ and $y = j + (2m + 1)s - t$. Now $x + 6m^2 + 6m + 2 = i + s + (6m + 3)t + 6m^2 + 6m + 2 = i + s + (6m + 3)(t + m) + 3m + 2$ and $b = j + (2m + 1)s - (t + m) + m$. Since m is fixed, we can denote $t' = t + m$ and so we get that $x + 6m^2 + 6m + 2 = i + s + (6m + 3)t' + 3m + 2$ and $b = j + (2m + 1)s - t' + m$, which means that the vertex $(x + 6m^2 + 6m + 2, y)$ belongs to the same color set as (x, y). The remaining possibility for the vertex (x, y) that there exist integers s and t such that $x = i + s + (6m + 3)t + 3m + 2$ and $y = j + (2m + 1)s - t + m$, the proof is analogical. Similarly it holds that any two vertices (x, y) and $(x, y + 6m^2 + 6m + 2)$ are colored with the same color. Therefore, a "square" pattern containing $(6m^2 + 6m + 2)^2$ vertices is replicated throughout the lattice.

When we focus on any "row" (or "column") of this pattern, we can see that each of the $6m^2 + 6m + 2$ vertices belongs to different color set from our set system. A stable order of colors is repeated there. For example, it is easy to check that the vertices $(x, 0)$ for integers $x \in (0, 6m^2 + 6m + 1)$ belong to these respective color sets: $(0, 0), (1, 0), \ldots, (3m, 0), (0, m + 1), (1, m + 1), \ldots, (3m + 1, m + 1), \ldots, (0, m), (1, m), \ldots, (3m + 1, m)$. The same sequence is repeated in each row, just starting with a different element of the sequence. Thus, the set system $F_{i,j}$ partitions the vertex set V.

(ii) Let $\delta_{p,r}$ be Kronecker's Delta, i.e., a function which assigns $\delta_{p,r} = 1$, if $p = r$ and $\delta_{p,r} = 0$, if $p \neq r$. Let $\psi(i, j) = (2m + 1)i + j + 1 - m\delta_{3m+1,i}$ be for any $(i, j) \in I$. Since j gets values from the set $\{0, 1, 2, \ldots, 2m\}$, the assignment $\psi(i, j)$ is injective to the set $\{1, 2, \ldots, 6m^2 + 6m + 2\}$. ∎

Lemma 4.5. *There is a* $(4m + 1)$*-distant* $(6m^2 + 6m + 2)$*-coloring of the hexagonal lattice.*

Proof. We will prove that the $(6m^2 + 6m + 2)$-coloring induced by the set system from Lemma 4.4 is a $(4m+1)$-distant coloring. Let (i, j) be arbitrary but fixed member of the index-set I from Lemma 4.4. Let $v_1, v_2 \in F_{i,j}$ be any two distinct vertices such that $v_1 = (a_1, b_1), v_2 = (a_2, b_2)$ and without loss of generality let $b_1 \geq b_2$.

When we show that $|a_1 - a_2| > 4m+1$, the Corollary 3.2.2 implies $\operatorname{dist}(v_1, v_2)$ $\geq |a_1 - a_2| > 4m + 1$. And when we show that $|b_1 - b_2| > 2m + 1$, then the Corollary 3.2.1 implies $\operatorname{dist}(v_1, v_2) \geq 2|b_1 - b_2| - 1 > 4m + 1$. We have to analyse four cases:

Case (i). There exist integers x_1, x_2, y_1, y_2 such that

$$a_1 = i + x_1 + (6m + 3)y_1, \ b_1 = j + (2m + 1)x_1 - y_1,$$
$$a_2 = i + x_2 + (6m + 3)y_2, \ b_2 = j + (2m + 1)x_2 - y_2.$$

Hence there exist integers $x = x_1 - x_2, y = y_1 - y_2$ such that $a_1 - a_2 = x + (6m + 3)y$ and $b_1 - b_2 = (2m + 1)x - y$, and not both equal zero (otherwise $v_1 = v_2$). When $y = 0$, for $|x| > 1$ there is $|b_1 - b_2| > 2m + 1$. For the case that $|x| = 1$ we have $|b_1 - b_2| = 2m + 1$ and $|a_1 - a_2| = 1$, and the Lemma 3.2 follows $\operatorname{dist}(v_1, v_2) = 2|b_1 - b_2| = 4m + 2 > 4m + 1$.

Let now $y > 0$. For $x \geq 0$ there is $|a_1 - a_2| \geq 6m + 3 > 4m + 1$. Since $b_1 \geq b_2$, x cannot be negative. Finally, let $y < 0$. For $x \leq 0$ there is $|a_1 - a_2| \geq 6m + 3 > 4m + 1$. When $x > 0$ we have $|b_1 - b_2| > 2m + 1$.

Case (ii). There exist integers x_1, x_2, y_1, y_2 such that

$$a_1 = i + x_1 + (6m + 3)y_1 + 3m + 2, \ b_1 = j + (2m + 1)x_1 - y_1 + m,$$
$$a_2 = i + x_2 + (6m + 3)y_2 + 3m + 2, \ b_2 = j + (2m + 1)x_2 - y_2 + m.$$

But this case can be transformed to Case (i).

Case (iii). There exist integers x_1, x_2, y_1, y_2 such that

$$a_1 = i + x_1 + (6m + 3)y_1, \ b_1 = j + (2m + 1)x_1 - y_1,$$
$$a_2 = i + x_2 + (6m + 3)y_2 + 3m + 2, \ b_2 = j + (2m + 1)x_2 - y_2 + m.$$

Hence there are integers $x = x_1 - x_2, y = y_1 - y_2$ such that $a_1 - a_2 = x + (6m+3)y - 3m - 2$ and $b_1 - b_2 = (2m + 1)x - y - m$. For $x = y = 0$ we have $|a_1 - a_2| = 3m + 2$ and $|b_1 - b_2| = m$. Similarly, for $x = 1$ and $y = 0$ we have $|a_1 - a_2| = 3m + 1$ and $|b_1 - b_2| = m + 1$. Finally for $x = 1$ and $y = 1$ there is $|a_1 - a_2| = 3m + 2$ and $|b_1 - b_2| = m$. Thus in all three cases according to Lemma 3.2 there holds that $\operatorname{dist}(v_1, v_2) = |a_1 - a_2| + |b_1 - b_2| = 4m + 2 > 4m + 1$. The case $x = 0, y = 1$ cannot occur due to assumption that $b_1 \geq b_2$.

For $x \geq 2$ and $y = 1$ or $y = 0$ there is $|b_1 - b_2| \geq 3m + 1 > 2m + 1$. Assuming $x \geq 0, y \geq 0$ (except already investigated cases) we have $|a_1 - a_2| \geq 9m + 4 > 4m + 1$. The case $x < 0, y \geq 0$ cannot occur due to assumption that $b_1 \geq b_2$.

Let now $y \leq -1$. The subcase when $x \leq 1$ satisfies $|a_1 - a_2| \geq 9m + 4 > 4m + 1$. For the subcase $x \geq 2$ we have $|b_1 - b_2| \geq 3m + 3 > 2m + 1$.

Case (iv). There exist integers x_1, x_2, y_1, y_2 such that

$$a_1 = i + x_1 + (6m + 3)y_1 + 3m + 2, \ b_1 = j + (2m + 1)x_1 - y_1 + m,$$

$$a_2 = i + x_2 + (6m + 3)y_2, \ b_2 = j + (2m + 1)x_2 - y_2.$$

Hence there exist integers $x = x_1 - x_2 + 1, y = y_1 - y_2 + 1$ such that $a_1 - a_2 = x + (6m + 3)y - 3m - 2$ and $b_1 - b_2 = (2m + 1)x - y - m$. But this is the same as in Case (iii).

We have got that any two distinct vertices from the same color class, i.e., colored with the same color, are at distance greater than $4m + 1$, hence the $(6m^2 + 6m + 2)$-coloring induced by the set system from Lemma 4.4 is indeed a $(4m + 1)$-distant $(6m^2 + 6m + 2)$-coloring. ∎

Lemma 4.6. *Any $(4m + 1)$-distant coloring of the hexagonal lattice uses at least $6m^2 + 6m + 2$ colors.*

Proof. Consider all vertices belonging to the rings $R_1, R_2, \ldots, R_m$ and the set R_{m+1}^-. Similarly to the proof of Lemma 4.3, one can see that at least $6m^2 + 6m$ colors are necessary. Let H' be the graph with vertex-set R_{m+1}^+ and where two vertices u, v are adjacent in H' if and only if they are at distance 2 in H. This graph has six components $C_1, C_2, \ldots, C_6$, each of which is a path with $m + 1$ vertices.

The distance in H between any two vertices of the $V(C_1)$ is at most $2m$, hence they must be colored with distinct colors. The effort to use only $6m^2 + 6m$ already used colors fails, because there are at most m vertices such that their colors may be from already used. Hence we need at least one new color. This is true for any component of H'. Clearly, there are two components C_i, C_j $(i, j \in \{1, 2, \ldots, 6\}, i \neq j)$ such that the distance between any two of their vertices is not more than $2m + 2m + 1$, thus their new colors must be distinct and we have already used $6m^2 + 6m + 2$ colors. ∎

To summarize Case 2, $\chi_d = 6m^2 + 6m + 2 = \frac{3}{8}(d + 1)^2 + \frac{1}{2}$ if $d = 4m + 1$.

Case 3. $d = 4m + 2$ for some nonnegative integer m-an upper bound.

Lemma 4.7. *Let $I = \{(i,j); i \in \{0,1,2,\ldots,6m+3\}, j \in \{0,1,2,\ldots, 2m+1\}$, and $i+j \equiv 0 \pmod{2}\}$. The set system*

$$F_{i,j} = \{(a,b); [a \equiv i \pmod{(6m+4)} \text{ and } b \equiv j \pmod{(2m+2)}] \text{ or}$$
$$[a \equiv i - 3m - 2 \pmod{(6m+4)} \text{ and } b \equiv j+m+1 \pmod{(2m+2)}]\}$$

for $(i,j) \in I$ is a $(6m^2 + 10m + 4)$-system of color classes for the hexagonal lattice.

Proof. (i) Let $(x,y) \in V$. If the sum $x+y$ is even, denote i, j in the following way: $i = (x \mod (6m+4))$ and $j = (y \mod (2m+2))$. Otherwise, denote $i = ((x+3m+2) \mod (6m+4))$ and $j = ((y-m-1) \mod (2m+2))$. Thus, $i+j$ is even, so $(i,j) \in I$ and $(x,y) \in F_{i,j}$. The pair (i,j) is determined uniquely, and so this set system partitions the set V.

(ii) It is easy to check that the assignment $\psi(i,j) = (m+1)i + [\frac{j+1}{2}]$ for $(i,j) \in I$ is injective to the set $\{1,2,\ldots,6m^2 + 10m + 4\}$. ∎

Lemma 4.8. *There is a $(4m+2)$-distant $(6m^2 + 10m + 4)$-coloring of the hexagonal lattice.*

Proof. If $m = 0$, any two vertices colored with the same color by the 4-coloring induced by the 4-system from Lemma 4.7 are at distance more than 2, hence this 4-coloring is 2-distant. Now, assume that $m > 0$.

We will prove that the $(6m^2 + 10m + 4)$-coloring induced by the set system from Lemma 4.7 is a $(4m+2)$-distant coloring. Let (i,j) be arbitrary but fixed member of the index-set I from Lemma 4.7. Let $v_1, v_2 \in F_{i,j}$ be any two distinct vertices such that $v_1 = (a_1, b_1), v_2 = (a_2, b_2)$ and, without loss of generality, $b_1 \geq b_2$. There are 4 cases:

Case (i) $\tau(v_1) = 0, \tau(v_2) = 0$

This means that $a_1 \equiv a_2 \pmod{(6m+4)}$ and $b_1 \equiv b_2 \pmod{(2m+2)}$, i.e., there are an integer x and a nonnegative integer y such that $a_1 = a_2 + x(6m+4), b_1 = b_2 + y(2m+2)$ and not both equal zero. When $y > 0$, then by the Corollary 3.2.1 $\text{dist}(v_1, v_2) \geq 2(b_1 - b_2) - 1 \geq 2(2m+2) - 1 = 4m+3 > 4m+2$. When $y = 0$ we have $b_1 = b_2$ and since $x \neq 0$, by Lemma 3.2 is then $\text{dist}(v_1, v_2) = |a_2 - a_1| = |x|(6m+4) \geq 6m+4 > 4m+2$.

Case (ii) $\tau(v_1) = 1, \tau(v_2) = 1$

This is the situation when it holds that $a_1 - 3m - 2 \equiv a_2 - 3m - 2 \pmod{6m+4}$ and that $b_1 + m + 1 \equiv b_2 + m + 1 \pmod{2m+2}$, which is the same situation as in Case (i).

Case (iii) $\tau(v_1) = 0, \tau(v_2) = 1$

We have $a_1 \equiv a_2 + 3m + 2 \pmod{6m+4}$ and $b_1 \equiv b_2 - m - 1 \pmod{2m+2}$, thus there are an integer x and a positive integer y such that $a_1 = a_2 + 3m + 2 + x(6m+4), b_1 = b_2 - m - 1 + y(2m+2)$. When $x > 0$ or $x < -1$, then by Corollary 3.2.2 $\mathrm{dist}(v_1, v_2) \geq |a_1 - a_2| \geq 9m + 6 > 4m + 2$. Having $x = -1$ or $x = 0$ there is $|a_2 - a_1| = 3m + 2$. If $y > 1$, using Corollary 3.2.1 we have $\mathrm{dist}(v_1, v_2) \geq 2(3m+3) - 1 > 4m + 2$. Finally if $y = 1$, we have $|b_1 - b_2| = m + 1$, and so by Lemma 3.2 we have $\mathrm{dist}(v_1, v_2) = 3m + 2 + m + 1 > 4m + 2$.

Case (iv) $\tau(v_1) = 1, \tau(v_2) = 0$

Now $a_1 + 3m + 2 \equiv a_2 \pmod{6m+4}$ and $b_1 - m - 1 \equiv b_2 \pmod{2m+2}$, so there are an integer x and a nonnegative integer y such that $a_1 = a_2 - 3m - 2 + x(6m+4), b_1 = b_2 + m + 1 + y(2m+2)$. By the substitution $\overline{x} = x - 1, \overline{y} = y + 1$ this case is transformed to Case (iii).

We have got that any two distinct vertices from the same color class, i.e., colored with the same color, are at distance greater than $4m + 2$, hence the $(6m^2 + 10m + 4)$-coloring induced by the set system from Lemma 4.7 is indeed a $(4m+2)$-distant $(6m^2 + 10m + 4)$-coloring. $\blacksquare$

To summarize Case 3, $\chi_d \leq 6m^2 + 10m + 4 = \frac{3}{8}\left(d + \frac{4}{3}\right)^2 - \frac{1}{6}$ if $d = 4m + 2$.

Case 4. $d = 4m$ for some positive integer m-an upper bound

Lemma 4.9. *Let $I = \{(i,j); i \in \{0, 1, 2, \ldots, 6m\}, j \in \{0, 1, 2, \ldots, 2m\}$, and $i + j \equiv 0 \pmod 2\}$. The set system*

$$
\begin{aligned}
F_{i,j} = \{(a,b); &[a = i + s - (6m+1)t \ and \\
&b = j + (2m+1)s + t \ for \ some \ integers \ s, t] \ or \\
&[a = i + s - (6m+1)t + 3m + 1 \ and \\
&b = j + (2m+1)s + t + m \ for \ some \ integers \ s, t]\}
\end{aligned}
$$

for $(i,j) \in I$ is a $(6m^2 + 4m + 1)$-system of color classes for the hexagonal lattice.

Proof. (i) Let $(x, y) \in V$. We need to find a pair $(i, j) \in I$ such that $(x, y) \in F_{i,j}$. Denote $x' = (x \mod (12m^2 + 8m + 2)), y' = (y \mod (12m^2 + 8m + 2))$. If we find a pair (i', j') such that $(x', y') \in F_{i',j'}$, then also $(x, y) \in F_{i',j'}$. Because $s' \equiv s \pmod{(12m^2 + 8m + 2)}$ and $t' \equiv t \pmod{(12m^2 + 8m + 2)}$ must hold. We assign the pair (i', j') to the vertex (x', y') uniquely (the proof that this is always possible is not difficult but technically pretentious, therefore we omit it). Hence the set system partitions the set V.

(ii) Assignment $\psi(i, j) = mi + [\frac{i+j+1}{2}]$ for $(i, j) \in I$ is injective to the set $\{1, 2, \ldots, 6m^2 + 4m + 1\}$. ∎

Lemma 4.10. *There is a $4m$-distant $(6m^2 + 4m + 1)$-coloring of the hexagonal lattice.*

Proof. The proof, that the $(6m^2 + 4m + 1)$-coloring induced by set system from Lemma 4.9 is a $4m$-distant coloring, is analogous to the proof of Lemma 4.8 and we omit it. ∎

Hence $\chi_d \leq 6m^2 + 4m + 1 = \frac{3}{8}\left(d + \frac{4}{3}\right)^2 + \frac{1}{3}$ for this case.

Lower bounds in Case 3 and Case 4.

The following lemma will be useful for our final result given in Lemma 4.12, which provides a lower bound on d-distant chromatic number for even d.

Lemma 4.11. *Let v_0 be any vertex of H and let i be a positive integer. Denote $V_i = \{v \in V; \mathrm{dist}(v, v_0) = i\}$. Then, $|V_i| = 3i$.*

Proof. We proceed by induction on distance i.

$1°$ The set V_1 contains just the neighbours of the vertex v_0, there are three of them. The set V_2 contains 6 vertices, which are neighbouring to the vertices of V_1 and different from v_0.

$2°$ Let $j \geq 3$ be an integer. Suppose that for all $i = 1, 2, \ldots, j$ it holds that $|V_i| = 3i$. Look at the set V_{j+1}. It contains all the neighbours of the vertices belonging to V_j such that does not belong to V_{j-1}. Each vertex belonging to V_j has 3 neighbours, so we have $3 \cdot 3j = 9j$ possible candidates. Note that for each vertex $v_x \in V_j$ there exist vertices $v_y, v_z \in V_j$ such that $\mathrm{dist}(v_x, v_y) = \mathrm{dist}(v_x, v_z) = 2$. Therefore, precisely $|V_j|$ vertices are counted twice; we are left with $9j - 3j = 6j$ candidates. Those still contain the vertices belonging to V_{j-1}, there are $3j - 3$ such vertices. Finally, we obtain $6j - (3j - 3) = 3j + 3$ vertices which belong to V_{j+1}. ∎

To avoid collision with the previous notation, consider here that $d = 2p$ for some integer $p \geq 2$.

Lemma 4.12. *Any $2p$-distant coloring of the hexagonal lattice requires at least $\frac{3}{2}p^2 + \frac{3}{2}p + 2$ colors.*

Proof. Let v_0 be any vertex of H and denote V_i for $i = 1, 2, \ldots, p$ as in Lemma 4.11. Consider all the vertices that belong to the sets $V_1, V_2, \ldots, V_p$ (denote their union as W), their number is $\sum_{i=1}^{p} 3i = 3\frac{p(p+1)}{2}$. Any vertex of W is at most at distance p from the vertex v_0, hence mutual distance of any pair of the vertices belonging to W is at most $2p = d$. This implies that all of them, including the vertex v_0, have to be colored by distinct colors, therefore at least $\frac{3}{2}p^2 + \frac{3}{2}p + 1$ colors are necessary for any $2p$-distant coloring.

Nevertheless, another color is enforced if we look at the set V_{p+1}. Let $v_1 \in V_{p+1}$ and $v_2 \in V_r$ such that $r \leq p-1$ (or $v_2 = v_0$). Since $\mathrm{dist}(v_1, v_2) \leq 2p$, these vertices cannot use the same color. In order to come to a contradiction, suppose that $\frac{3}{2}p^2 + \frac{3}{2}p + 1$ is a sufficient number of colors.

Thus, the vertices that belong to V_{p+1} can be colored only by colors used for the vertices in V_p. There are only $|V_p| = 3p$ of such colors, whereas we need to color $|V_{p+1}| = 3p + 3$ vertices. Observe that there cannot exist a color used at three (or more) vertices from the set V_{p+1}, as it must also color one vertex from W. By the pigeonhole principle there exist at least three colors such that each of them colors precisely one pair of vertices from the set V_{p+1} and one vertex from the set W. Moreover, we already know from the previous paragraph that the latter must belong to V_p.

If we try to find such a triple of vertices of distances greater than d, we will find it very restricted. A choice of a vertex from V_p which is not a corner (with the meaning as in Proposition 2.2) exhibits that at distance at least $d+1$ there are in V_{p+1} only vertices whose mutual distance is lower than d. So we have to consider a corner vertex from the set V_p. If we fix that corner vertex, there exist a unique pair of vertices belonging to V_{p+1} at pairwise distance greater than d. So we have a triple of vertices where the same color can be made use of. We can also find another such triple which does not coincide with the first triple. However, it is not possible to find one more triple, because the triples are enforced uniquely and their "shape" implies a coincidence.

Therefore, at least one vertex from the set V_{p+1} is left free and so, at least one more color is necessary. ∎

We conjecture that any $(4m + 2)$-distant coloring of the hexagonal lattice uses at least $6m^2 + 10m + 4$ colors.

This is true for the cases when $m = 0$ or 1. Consider any vertex and its three neighbours. Their mutual distances are not greater than 2, so they must be colored by four different colors. Thus, any 2-distant coloring of the hexagonal lattice uses at least 4 colors.

For $m \geq 1$ by Lemma 4.12 any $(4m+2)$-distant coloring of the hexagonal lattice uses at least $6m^2 + 9m + 5$ colors. This bound is lower than our conjectured value by $m - 1 = \frac{d}{4} - \frac{3}{2}$, that is, provides the same value for 6-distant coloring; uses one color less than hypothesized for 10-distant coloring; etc. However, using a computer routine that explores all possible colorings we can show that there is a subgraph of H containing 109 vertices such that it requires 47 different colors for 10-distant coloring, which is one more than predicts Lemma 4.12, whereas equals to the conjectured value.

Analogously, we conjecture that any $4m$-distant coloring of the hexagonal lattice requires at least $6m^2 + 4m + 1$ colors.

We proved in Lemma 4.12 that any $4m$-distant coloring of the hexagonal lattice uses at least $6m^2 + 3m + 2$ colors. This bound is lower than our conjectured value by $m - 1 = \frac{d}{4} - 1$, that is, provides the same value for 4-distant coloring, uses one color less than hypothesized for 8-distant coloring; etc. However, using a computer routine that explores all possible colorings we can show that there is a subgraph of H containing 109 vertices such that it requires 33 different colors for 8-distant coloring, which is one more than predicts Lemma 4.12, whereas equals to the hypothesized value.

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