A Constructive Characterisation of Circuits in the Simple $(2, 2)$-sparsity Matroid

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Abstract

We provide a constructive characterisation of circuits in the simple $(2, 2)$-sparsity matroid. A circuit is a simple graph $G = (V, E)$ with $|E| = 2|V| - 1$ where the number of edges induced by any $X \subsetneq V$ is at most $2|X| - 2$. Insisting on simplicity results in the Henneberg 2 operation being adequate only when the graph is sufficiently connected. Thus we introduce 3 different join operations to complete the characterisation. Extensions are discussed to when the sparsity matroid is connected and this is applied to the theory of frameworks on surfaces, to provide a conjectured characterisation of when frameworks on an infinite circular cylinder are generically globally rigid.

Keywords: (k, l)-tight, circuit, Henneberg 2 operation, rigidity matroid.

1. Introduction

For $k, l \in \mathbb{N}$ a multigraph $G = (V, E)$ is (k, l)-tight if $|E| = k|V| - l$ and for every subgraph $G' = (V', E')$ the inequality $|E'| \leq k|V'| - l$ holds. It is well known that the edge sets of such multigraphs induce matroids when $l < 2k$ [13, 22]; we denote these matroids as $M(k, l)$. These multigraphs can be decomposed into unions of trees and map graphs [15, 21, 23]; correspondingly the matroids are unions of cycle and bicycle matroids. (We direct the reader unfamiliar with matroids to [14] for a comprehensive introduction.)

There is an elegant recursive construction of the bases (maximal independent sets) in $M(k, l)$ due to Fekete and Szegő [3]. Their result is built on the construction of Tay [20] for $k = l$. A recursive characterisation of

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circuits (minimal dependent sets) in $M(k,k)$ can be found as a special case of a theorem of Frank and Szegő [4] on highly k-tree connected multigraphs.

These characterisations use generalisations of the Henneberg moves [8]. However each list of construction moves is insufficient if we restrict to (simple) graphs at each stage of the induction. When the (k,l)-tight graph is simple, they still induce a matroid and we denote it as $M^*(k,l)$. Recursive constructions for the bases of $M^*(2,l)$ $(l=2,1)$ can be found in [16, 17, 18]. In this paper we study circuits in $M^*(2,2)$.

From here on, for brevity, we define a *circuit* (resp. *multicircuit*) to be the graph (resp. multigraph) induced by a circuit in $M^*(2,2)$ (resp. $M(2,2)$) i.e. a graph (resp. multigraph) $G=(V,E)$ with $|E|=2|V|-1$ and for every proper subgraph $H=(V',E')\subset G$ we have $|E'|\leq 2|V'|-2$. Figure 1 gives three small examples of circuits. It is easy to see that circuits have minimum degree 3. Hence, throughout we will call a vertex of degree 3 a *node*. The *Henneberg* 2 *move* adds a node to a graph by subdividing an edge and connecting the new vertex to a third existing vertex. Other Henneberg moves will not be relevant here. In this paper we prove a constructive characterisation of all circuits in $M^*(2,2)$. See Figure 1 for the base graphs of the characterisation and Figure 2 for the join moves; both are formally defined in Subsection 1.4.

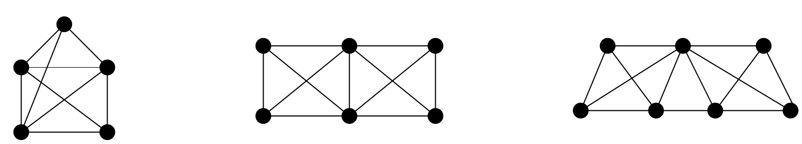

Figure 1: From left to right: the base graphs $K_5\setminus e$, $K_4\sqcup K_4$ and $K_4\veebar K_4$.

Theorem 1.1. *A graph G is a circuit in $M^*(2,2)$ if and only if G can be generated recursively from disjoint copies of base graphs by applying Henneberg* 2 *moves within connected components and taking* 1*-joins,* 2*-joins or* 3*-joins of different connected components.*

To prove Theorem 1.1 our main technical tool is Theorem 1.2 below. This theorem gives precise connectivity conditions that guarantee we can use the Henneberg 2 move. First we introduce some relevant terminology.

For the inverse Henneberg 2 operation, let $G=(V,E)$ be a graph and let G_v^{uw} denote the graph formed by removing a node v from G and adding the edge uw where $u,w\in N(v)$ (the neighbour set of v). Let G be a circuit and let v be a node in G. The pair of edges uv,wv is *admissible* if G_v^{uw} is

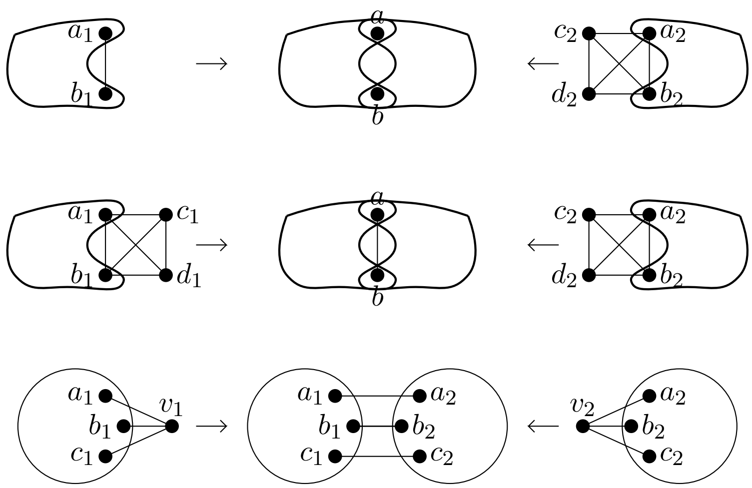

Figure 2: The 1-, 2- and 3-join operations forming $G_1 \oplus_i G_2$ from G_1 and G_2 for $i = 1, 2, 3$ respectively.

a circuit. A node v is *admissible* if there is $u, w \in N(v)$ such that uv, wv is admissible. Figures 3 and 4 illustrate admissibility.

By a *non-trivial k-edge cut* we mean a k-edge-cut in which the two components have at least two vertices. Since every circuit contains a degree 3 vertex, there always exist trivial 3-edge-cuts. Since we will primarily be considering non-trivial 3-edge cuts in 3-connected graphs we may assume the edges in any such cut are disjoint.

Theorem 1.2. *Let G be a 3-connected circuit in $M^*(2,2)$ with no non-trivial 3-edge cuts and $|V| \geq 6$. Then G has two admissible nodes.*

The second graph in Figure 3 gives an example showing the 3-connectivity assumption is necessary. Similarly, Figure 4 shows why we must assume there are no non-trivial 3-edge cuts.

1.1. Outline

In Section 2 we prove Theorem 1.2. We start with some elementary properties of circuits culminating in Lemma 2.5 where we establish two blocks to admissibility: (a) preserving simplicity and (b) preserving subgraph sparsity. The key novelty in Section 2 is in dealing with (a). Proposition 2.6 establishes the level of connectivity required to guarantee nodes not contained in copies of K_4. Combining this with Lemma 2.8 largely allows us

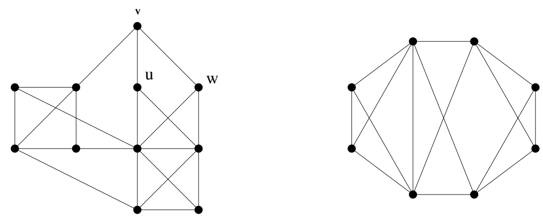

Figure 3: v is a non-admissible node in a 3-connected circuit with no non-trivial 3 edge-cuts. Choosing uw as the new edge creates a copy of $K_4 \sqcup K_4$ and not choosing uw leaves a degree 2 vertex. u and w are examples of admissible nodes. The second circuit contains no admissible nodes.

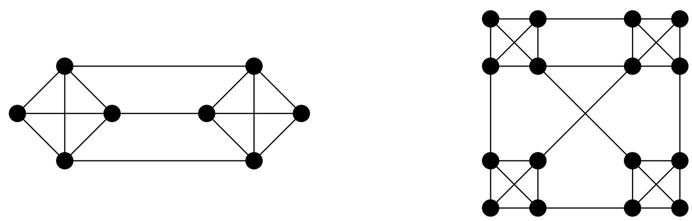

Figure 4: Two 3-connected circuits with no admissible nodes. Every node is in a copy of K_4 so any inverse Henneberg 2 move results in a multiple edge, while circuits are necessarily simple graphs.

to reduce to (b), which is considered in Subsection 2.3. This follows the method of [1] establishing structural results for circuits with non-admissible nodes. The proof of Theorem 1.2 is completed by deducing from Proposition 2.6 that a special subforest of nodes is non-empty and combining this with these structural results.

In Section 3 we consider circuits which are not sufficiently connected for Theorem 1.2 to apply. These are circuits with 2-vertex cuts or non-trivial 3-edge cuts for which we introduce the 1-, 2- and 3-join operations. There is one final technical point which we deal with in Section 4; there are 2-vertex cuts to which we cannot apply the inverse 2-join operation in a useful way. This happens precisely when the circuit takes the form $G_1 \oplus_2 G_2$, see Figure 2, and either G_1 or G_2 is isomorphic to K_4. In Section 4 we translate a circuit in $M^*(2,2)$ into a circuit in $M(2,2)$ in order to establish admissibility in circuits with no non-trivial 3-edge cuts where every 2-vertex cut has this special form. The results to this point prove that any circuit is either a base graph or can be reduced to smaller circuits using an inverse join operation or can be reduced to a smaller circuit using the inverse Henneberg 2 move. Combining this with the fact that these operations preserve the circuit property completes the proof of Theorem 1.1

In Section 5 we consider connectedness in $M^*(2,2)$ and obtain a precise analogue of [10, Theorem 3.2]. This is used to link our results to the unique realisation problem for frameworks in 3-dimensions supported on an infinite circular cylinder. We finish by conjecturing a combinatorial description of when such a realisation is unique, Conjecture 5.7, and outlining some extensions.

1.2. Motivation

The rigidity of frameworks on surfaces [11, 16] (particularly on a cylinder) provides geometric motivation for the study of $M^*(2,2)$. In particular the question of global rigidity - when a geometric realisation of a graph on a cylinder is unique (up to ambient motions). The corresponding question for frameworks in the plane was finally settled in 2005 by Jackson and Jordán [10], building upon results of Hendrickson [6], Connelly [2] and most relevantly to this paper, Berg and Jordán [1]. Berg and Jordán's contribution was a recursive characterisation of circuits in $M^*(2,3) = M(2,3)$. Circuits arise because they have the minimum number of edges (as a function of the number of vertices) possible for the realisation to be unique. We expect that our characterisation will be similarly useful in establishing a combinatorial description of global rigidity on the cylinder.

1.3. Comparing Constructions

While circuits in $M^*(2,l)$ $(l = 2,3)$ necessarily contain nodes there may be no node that is suitable for an inverse Henneberg 2 operation. This is the key reason why circuits are more challenging than bases. Berg and Jordán [1] showed that a circuit in $M^*(2,3)$ has a suitable node whenever the graph is 3-connected (compare Theorem 1.2). Thus the combination of the Henneberg 2 operation and the 2-sum operation [14], which glues two circuits together over a 2-vertex cut (contrast with the 1-,2- and 3-join operations), were sufficient to generate all such circuits.

In [4] it was shown that all circuits in $M(2,2)$ can be generated from a single loop using Henneberg 2 operations. Hence, without the insistence on simplicity, the graphs in Figure 4 can be reduced using the inverse Henneberg 2 move. With this insistence, they give examples of graphs for which multigraphs are required in the intermediate steps. Moreover repeated application of, say, 3-join operations on these examples give arbitrarily large circuits with no admissible nodes.

Since each of these examples contains a copy of K_4 it would be natural to consider a recursive operation in which a copy of K_4 was contracted to a single vertex, as used in [18] for bases in $M^*(2,2)$. However contracting a K_4 need not preserve simplicity and the inverse, extending a vertex into a K_4, need not preserve 2-connectivity (so by Lemma 2.3 does not preserve the circuit property).

Lastly, we comment that $M(2,3)$ provides a nice example of a matroid which is not closed under the 2-sum operation [19]. This is in contrast to the cycle matroid of a graph and hints at the added complexity of $(2,\ell)$-sparsity matroids.

1.4. Preliminaries

We finish the first section by giving formal definitions of some terms used in the introduction and by introducing some notation.

Let $K_4 \sqcup K_4$ denote the unique graph formed by two copies of K_4 intersecting in a single edge and let $K_4 \veebar K_4$ denote the unique graph formed from two copies of K_4 intersecting in a single vertex by adding any edge. We will say that $K_5 \setminus e$, $K_4 \sqcup K_4$ and $K_4 \veebar K_4$ are *base graphs*, see Figure 1.

Let G_1, G_2 be circuits such that G_1 contains an edge a_1b_1 and G_2 contains a two vertex cut a_2, b_2 within $K_4(a_2, b_2, c_2, d_2)$. A 1-*join operation* takes G_1 and G_2 and forms $G_1 \oplus_1 G_2$ by removing a_1b_1, c_2, d_2 and a_2b_2 and superimposing a_1, b_1 onto a_2, b_2 and calling the resulting vertices a, b. Secondly, let G_1, G_2 be circuits such that G_i contains a two vertex cut a_i, b_i with one component inducing $K_4(a_i, b_i, c_i, d_i)$. A 2-*join operation* takes G_1 and G_2 and forms $G_1 \oplus_2 G_2$ by removing c_i, d_i and superimposing a_1, b_1 onto a_2, b_2 and calling the resulting vertices a, b and keeping only one copy of the edge ab. Finally, let G_1, G_2 be circuits such that G_i contains a node v_i with $N(v_i) = \{a_i, b_i, c_i\}$. A 3-*join operation* takes G_1 and G_2 and forms $G_1 \oplus_3 G_2$ by deleting v_1, v_2 and adding edges a_1a_2, b_1b_2, c_1c_2.

In this paper graphs have no loops or multiple edges, multigraphs may have both. If $G = (V, E)$ is a graph with $v \in V$ then $d_G(v)$ denotes the degree of v in G and $N(v)$ denotes the neighbour set of v.

Define $f(H) = 2|V'| - |E'|$ for any $H = (V', E') \subseteq G$. For $X \subset V$ we let $i_G(X)$ denote the number of edges in the subgraph of G induced by X. We drop the subscript when the graph is clear from the context. If X and Y are disjoint subsets of the vertex set V of a given graph G, then we use $d(X, Y)$ to denote the number of edges from X to Y and $d(X) := d(X, V \setminus X)$.

2. Admissible Nodes

In this section we prove Theorem 1.2. First let us note the elementary 'inverse' of the theorem, whose proof we omit.

Lemma 2.1. *Let G' be formed from G by a Henneberg 2 move and let G be a circuit. Then G' is a circuit.*

2.1. Basic Properties of Circuits

We begin by establishing some basic lemmas on circuits and then give a characterisation of admissibility.

Let $G = (V, E)$. We say that a subset $X \subset V$ is *critical* if $i(X) = 2|X| - 2$. The following is a simple analogue of [1, Lemma 2.3] and we omit the proof.

Lemma 2.2. *Let $G = (V, E)$ be a circuit and let $X, Y \subset V$ be critical such that $|X \cap Y| \geq 1$ and $|X \cup Y| \leq |V| - 1$. Then $X \cap Y$ and $X \cup Y$ are both critical, and $d(X \setminus Y, Y \setminus X) = 0$.*

Let $G = (V, E)$ be a circuit. For any critical set $X \subset V$, $G[X]$ is connected but need not be 2-connected.

Lemma 2.3. *Let G be a circuit. Then G is 2-connected and 3-edge-connected.*

Proof. Let $G = (V, E)$. Suppose there exists $v \in V$ such that $G \setminus v$ has a bipartition A, B with no edges from A to B.

$$\begin{aligned} 2|V| - 1 = |E| &= |E(A \cup v)| + |E(B \cup v)| \\ &\leq 2(|A| + 1) - 2 + 2(|B| + 1) - 2 \\ &= 2|V| - 2, \end{aligned}$$

a contradiction. This proves the first statement, the second is similar. □

The following is easy and similar to [1, Lemma 2.5]. We omit the proof.

Lemma 2.4. *Let $G = (V, E)$ be a circuit. Let $X \subset V$ be a critical set. Then $V \setminus X$ contains at least one node (in G).*

Our next lemma gives a criterion for admissibility.

Lemma 2.5. *Let G be a circuit, let v be a node in G with $N(v) = \{u, w, z\}$. Then uv, wv is not admissible if and only if either (a) $uw \in E$ or (b) there is a critical set $X \subset V$ with $u, w \in X$ and $v, z \notin X$.*

Proof. Suppose first that (b) holds. Then the inverse Henneberg 2 move creates a new edge uw implying $i(X) = 2|X| - 1$ and $X \subsetneq V$. Also if (a) holds then G_v^{uw} is not a simple graph.

Conversely, if uv, wv is not admissible and (a) fails there is $X \subset V(G_v^{uw})$ such that $G[X]$ is not $(2, 2)$-sparse. Then $|E(X)| \geq 2|X| - 1$. It follows that X is critical in G and $u, w \in X$. If $z \in X$ then $|E(X \cup v)| = |E(X)| + 3 = 2|X| - 2 + 3 = 2|X \cup v| - 1$, a contradiction. Thus $z \notin X$. □

Condition (b) in Lemma 2.5 leads us to strengthen the definition of critical as follows. Let $G = (V, E)$ be a circuit. For a node $v \in V$ with $N(v) = \{u, w, z\}$ we say that a critical set X is *v-critical* if $u, w \in X$ and $v, z \notin X$. If z is a node and such an X exists then an inverse Henneberg 2 move on uv, wv is not admissible. Here $V \setminus \{v, z\}$ is a *trivial* v-critical set on u and w. If X is a v-critical set on u and w for some node v with $N(v) = \{u, w, z\}$ and $d_G(z) \geq 4$ then X is *node-critical*. We will return to node-critical sets in Subsection 2.3.

2.2. Preserving Simplicity

Condition (a) in Lemma 2.5 is crucial in separating the problem at hand from the analogue in [1]. The following Proposition is the key step in bridging this difficulty.

Proposition 2.6. *Let $G = (V, E)$ be a 3-connected circuit with no non-trivial 3-edge-cuts and $|V| \geq 6$. Let $X_1, \dots, X_n$ be critical sets and let $Y = V \setminus \bigcup_{i=1}^{n} X_i$. Suppose that any one of the following conditions holds:*

1. $|Y| \geq 2$,
2. $\bigcup_{i=1}^{n} G[X_i]$ *is disconnected, or*
3. $X_1, \dots, X_n$ *induce copies of* K_4.

Then Y contains at least two nodes of G.

Proof. We prove 1 and 2 simultaneously. With vertices labelled $v_1, \dots, v_{|V|}$, since $|E| = 2|V| - 1$ we have

$$\sum_{i=1}^{|V|} (4 - d_G(v_i)) = 2.$$

Let $Z_1, \dots, Z_m$ be the connected components in $\bigcup_{i=1}^{n} G[X_i]$. In cases 1 and 2 Lemma 2.2 implies $X_i \cup X_j$ is critical and $d(X_i, X_j) = 0$ or $X_i \cap X_j = \emptyset$ for each $1 \leq i < j \leq n$. Now $i(Z_j) = 2|Z_j| - 2$ for each j. Thus

$$\sum_{i=1}^{|Z_j|} (4 - d_{G[Z_j]}(v_i)) = 4.$$

By assumption 1 or 2 $|V \setminus Z_j| \geq 2$ so there are at least 4 edges of the form xy with $x \in Z_j$, $y \in V \setminus Z_j$. This implies

$$\sum_{i=1}^{|Z_j|} (4 - d_G(u_i)) \leq 0$$

(with the vertices in Z_j labelled $u_1, \dots, u_{|Z_j|}$) for each j. Thus

$$\sum_{j=1}^{m}(\sum_{i=1}^{|Z_j|}(4 - d_G(u_i))) \leq 0.$$

Since the minimum degree in G is 3 comparing this with the first summation implies Y contains at least two nodes.

For 3 assume $X_1, \dots, X_n$ induce copies of K_4 and suppose $m = 1$ and $|Y| < 2$. (If $m > 1$ or $|Y| \geq 2$ then we can apply the previous cases.) Let $|Y| = 1$ then Z_1 is critical, $G[Z_1]$ is connected and every edge in $G[Z_1]$ is in a copy of K_4. Since every $A \subsetneq X_i$ with $|A| > 1$ satisfies $i(A) \leq 2|A| - 3$ we must have $X_1 \cap X_i = a$ for some i. If a is a cut-vertex in $G[Z_1]$ then we guarantee a cutpair in G which contradicts our assumptions so $m > 1$. However, if a is not a cut-vertex, there is a path in $G[Z_1]$ from any vertex in $X_1 \setminus a$ to any vertex in $X_i \setminus a$. Since $d(X_1, X_i) = 0$ the only way this may happen is if there is a set containing some $y_1 \in X_1 \setminus a$ and some $y_k \in X_i \setminus a$ which is not contained in $X_1 \cup X_i$. Let the path use vertices $y_1, y_2, \dots, y_k$ for some $k \geq 2$ and choose X' to be the union of all X_j's containing some y_j except X_1 and X_i. Then X' is critical. As $X_1 \cup X_i$ is critical this implies that $i(X' \cup X_1 \cup X_i) > 2|X' \cup X_1 \cup X_i| - 2$. Thus a must be a cut-vertex.

A similar argument applies when $Y = 0$; here $Z_1 = V$ and there is exactly one edge e not in a copy of K_4. As above we find a is a cut-vertex for $G \setminus e$ and hence a cut-pair exists in G. Therefore $m \geq 2$ and the result follows from 2. □

Let $V_3 = \{v \in V : v \text{ is a node}\}$. Let $V_3^* \subset V_3$ be the subgraph of nodes which are not contained in copies of K_4 (in G). Following [1] we call a node v with $d_{G[V_3^*]}(v) \leq 1$ a *leaf node*, with $d_{G[V_3^*]}(v) = 2$ a *series node* and with $d_{G[V_3^*]}(v) = 3$ a *branching node*. From Proposition 2.6 we can derive an analogue of [1, Lemma 2.1].

Lemma 2.7. *Let $G = (V, E)$ be a 3-connected circuit with $|V| \geq 6$ and no non-trivial 3-edge cuts. Then $G[V_3^*]$ is a forest on at least two vertices.*

Proof. By Proposition 2.6 part 3 $|V_3^*| \geq 2$. Suppose $C \subset V_3^*$ induces a cycle. G is not a cycle so $\bar{C} := V \setminus C \neq \emptyset$. $|\bar{C}| > 1$ since G is not a wheel. Now

$$\begin{aligned} i(\bar{C}) &= 2|V| - 1 - i(C) - d(C, \bar{C}) = 2|V| - 1 - |C| - |C| \\ &= 2(|V| - |C|) - 1 = 2|\bar{C}| - 1, \end{aligned}$$

a contradiction. □

We take this opportunity to dispense with the case when the neighbour set of a node neither induces K_3 or induces a graph with no edges.

Lemma 2.8. *Let $G = (V, E)$ be a circuit containing a node v with $N(v) = \{w, u, z\}$. Suppose that either*

1. *G is 3-connected, $uz \notin E$ and $wz, wu \in E$ or*
2. *$uz, wu \notin E$ and $wz \in E$.*

Then v is admissible.

Proof. Since z, u is not a cutpair, $d_G(w) \geq 4$. Let $t \in N(w)$ and suppose v is not admissible. By Lemma 2.5 there exists a proper critical subset $X_{zu} \subset V$ containing z, u but not w, v. If $t \in X_{zu}$ then $i(X_{zu} \cup w) = 2|X_{zu} \cup w| - 1$, a contradiction as $v \notin X_{zu} \cup w$. If $t \notin X_{zu}$ then $X_{zu} \cup w$ is critical and $i(X_{zu} \cup w \cup v) = 2|X_{zu} \cup w \cup v| - 1$, a contradiction as $t \notin X_{zu} \cup w \cup v$. This proves 1.

Now assume for a contradiction that v is not admissible. By Lemma 2.5 there exists proper critical sets $X_{wu}, X_{uz} \subset V$. Note $d_G(z) \geq 4$ since $|N(z) \cap X_{uz}| \geq 2$ and similarly $d_G(w) \geq 4$. By Lemma 2.2 $X_{wu} \cup X_{uz}$ is critical so adding wz then v plus its three edges gives a contradiction. Thus at most one of the critical sets X_{wu} and X_{uz} can exist and 2 follows. □

Recall that Lemma 2.5 showed there are two blocks to admissibility; we must preserve simplicity and subgraph sparsity. Proposition 2.6 and Lemma 2.8 allow us to find nodes which we know will not violate simplicity. In the following subsection we consider subgraph sparsity.

2.3. Guaranteeing an admissible node

In this section we consider nodes whose 3 neighbours induce a null graph. For this we modify results from [1]. Then in the final subsection we will combine this analysis with Proposition 2.6 and Lemma 2.8 to deduce Theorem 1.2.

Lemma 2.9. *Let $G = (V, E)$ be a circuit with $|V| \geq 6$. Suppose v is a non-admissible node of G with $N(v) = \{x, y, z\}$ and none of xy, xz, yz present in E. Then there exists two v-critical sets X, Y such that $X \cup Y = V \setminus v$. Moreover we may choose X, Y such that $z \in X \cap Y$.*

Proof. Since v is non-admissible Lemma 2.5 implies there exist critical sets X on y, z, Y on x, z and Z on x, y. From Lemma 2.2 we deduce that $X \cup Y$ is critical and hence $X \cup Y = V \setminus v$, since $x, y, z \in X \cup Y$. □

The next lemma, an analogue of [1, Lemma 3.3] gives a crucial structural result about 3-connected circuits with no non-trivial 3-edge-cuts containing non-admissible nodes. Figure 5 illustrates this; see also the first graph in Figure 3 for an example of a non-admissible series node.

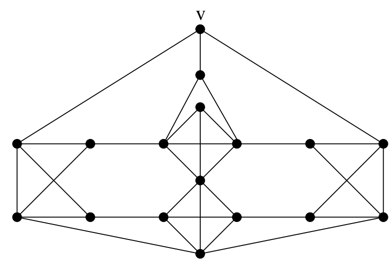

Figure 5: A 3-connected circuit with no non-trivial 3-edge-cuts. v is a non-admissible leaf node.

Lemma 2.10. *Let $G = (V, E)$ be a 3-connected circuit with no non-trivial 3-edge-cuts. Let $v \in V$ be a node with $N(v) = \{x, y, z\}$, $d_G(z) \geq 4$ and suppose no pair of neighbours of v defines an edge. Let X be a v-critical set on x, y. Furthermore suppose that either*

1. *there is a non-admissible series node $u \in V \setminus X \setminus v$ with no edges between its neighbours, precisely one neighbour w in X and w is a node, or*
2. *there is a non-admissible leaf node $t \in V \setminus X \setminus v$ with no edges between its neighbours.*

Then there is a node-critical set X' in G with $|X'| > |X|$ and $(X \cap V_3^*) \subseteq (X' \cap V_3^*)$.

Proof. First let $u \in V \setminus X \setminus v$ be a non-admissible series node with $N(u) = \{w, p, n\}$ and $d_G(w) = 3$. We may assume $d_G(p) = 3$ and $d_G(n) \geq 4$. Since u is non-admissible and $wp \notin E$ there exists a u-critical set Y on w and p by Lemma 2.5. By Lemma 2.7 $G[V_3^*]$ contains no cycles. Note $|Y| \geq 5$ since p, w are not in a copy of K_4. Now $X \cap Y$ contains w so $X' := X \cup Y \subseteq V \setminus u \setminus v$ is node-critical on u by Lemma 2.2. Also $p \notin X$ and $d_G(n) \geq 4$ so $|X'| \geq |X|$ and $(X \cap V_3^*) \subseteq (X' \cap V_3^*)$.

For the second part of the lemma let t be a non-admissible leaf node. Lemma 2.9 implies that there exist two t-critical sets Y_1 and Y_2 with $Y_1 \cup Y_2 = V \setminus t$ and if t has a neighbour r which is a node then we can also assume $r \in Y_1 \cap Y_2$. Note that Y_1 and Y_2 are node-critical and $|Y_1|, |Y_2| \geq 5$.

Now $x, y \in Y_1 \cup Y_2$ and Lemma 2.2 implies that $d(Y_1 \setminus Y_2, Y_2 \setminus Y_1) = 0$. Since also $Y_1 \cup Y_2 = V \setminus t$ and $t \notin X$ we know that $|X \cap Y_1| \geq 1$ or $|X \cap Y_2| \geq 1$. Without loss of generality assume $|X \cap Y_1| \geq 1$. $d(t, X) = 3$ implies $i(X \cup t) > 2|X \cup t| - 2$ so $d(t, X) \leq 2$. Moreover $d(t, X) \leq 1$ as if it were equal to 2 then $X \cup t$ is critical and the result follows.

Now $|N(t) \cap X| \leq 1$. First suppose $|N(t) \cap X| = 0$. Lemma 2.2 implies that $X \cup Y_1$ is t-critical. Thus choosing $X' = X \cup Y_1$ completes the proof in this case. Now suppose $N(t) \cap X = \{s\}$. If $s \in Y_1$ then $N(t) \setminus (X \cup Y_1) \neq \emptyset$ (as $N(t) \nsubseteq Y_1$) and hence $X' = X \cup Y_1$ is node-critical and we are done. If $d_G(s) = 3$ then $s \in Y_1 \cap Y_2$ so we may assume $d_G(s) \geq 4$ and $s \notin Y_1$. Since $Y_1 \cup Y_2 = V \setminus t$ this gives $s \in Y_2$. $|X \cap Y_2| \geq 1$ so choose $X' = X \cup Y_2$ to complete the proof. □

Similarly to [1, Lemmas 3.5 and 3.6] we have the following two lemmas.

Lemma 2.11. *Let G be a 3-connected circuit with no non-trivial 3-edge-cuts and $|V| \geq 6$. Let $\mathcal{X} = \{X \subset V : X$ is a node-critical set in $G\}$. If $\mathcal{X} = \emptyset$ then G has two admissible nodes.*

Proof. By Lemma 2.7 V_3^* is a forest and $|V_3^*| \geq 2$. Since $\mathcal{X} = \emptyset$ the result follows from Lemmas 2.5 and 2.8. □

Lemma 2.12. *Let G be a 3-connected circuit with no non-trivial 3-edge-cuts and $|V| \geq 6$. Suppose v is an admissible node. Let $\mathcal{Y} = \{Y \subset V : v \in Y$, Y is a node-critical set in $G\}$. If $\mathcal{Y} = \emptyset$ then G has two admissible nodes.*

Proof. By Lemma 2.7 $|V_3^*| \geq 2$. Let $w \neq v$ be a leaf in $G[V_3^*]$ and suppose w is non-admissible. Either this contradict Lemma 2.8 or by Lemma 2.9 there exist node-critical sets X, Y with $X \cup Y = V \setminus w$, contradicting $\mathcal{Y} = \emptyset$. □

We remark that this final lemma is included to make the statement of Theorem 1.2 as strong as possible. It is sufficient, for the application in the proof of Theorem 1.1, to guarantee only one admissible node.

2.4. Proof of Theorem 1.2

We are now ready to prove that any sufficiently connected circuit contains admissible vertices.

Proof of Theorem 1.2. By Lemma 2.7 $G[V_3^*]$ is a forest and $|V_3^*| \geq 2$. By Lemma 2.8 we need consider only the case when there are no edges between the neighbours of every $a \in V_3^*$.

Let $\mathcal{X} = \{X \subset V : X$ is a node-critical set in $G\}$. If $\mathcal{X} = \emptyset$ we are done by Lemma 2.11. Otherwise let $X \in \mathcal{X}$ be maximal. Choose $t \in N(v)$ such that X is v-critical with $d_G(t) \geq 4$ and $t \notin X$. $X \cup v$ is critical and $|V \setminus X \setminus v| \geq 2$, otherwise $i(X \cup v \cup t) > 2|X \cup v \cup t| - 1$. By Lemma 2.4 $V \setminus X \setminus v$ contains a node.

Let $X = X_n$ and let $X_1, \dots, X_{n-1}$ be critical sets in G not contained in X such that every copy of K_4 is induced by some X_i and every X_i induces a copy of K_4. Then there are two cases. If $t \notin X_i$ for all i then $|Y| = |V \setminus \bigcup_{i=1}^n X_i| \geq 2$ so Proposition 2.6 part 1 implies there is a vertex not in $X \cup v$ which is a node not in a copy of K_4. Secondly if $t \in X_i$ for some i then $|X \cap X_i| \leq 1$ otherwise $i(X \cup X_i) > 2|X \cup X_i| - 2$. Moreover if $X \cap X_i = a$ then $X \cap X_i$ is critical so $d(X, X_i) = 0$ and $X \cup X_i \cup v = V$ implying a, v is a cut-pair for G. Hence $|X \cap X_i| = 0$ and $\bigcup_{i=1}^n G[X_i]$ is disconnected so Proposition 2.6 part 2 implies there is a vertex not in $X \cup v$ which is a node not in a copy of K_4.

Let $W^* := V_3^* \cap (V \setminus X \setminus v)$. $G[W^*]$ is a subforest of $G[V_3^*]$ on the vertex set W^*. By the preceding paragraph $|W^*| \geq 1$ so W contains a leaf u. Each vertex $z \in V \setminus X \setminus v \setminus t$ has at most one neighbour in X; otherwise $X \cup z$ is node-critical, contradicting the maximality of $|X|$. Therefore u is not a branching node of G.

Now if u is a leaf node then Lemma 2.10 part 2 and the maximality of $|X|$ imply that u is an admissible node. If u is a series node in G then, since u has at most one neighbour in X and u is a leaf in $G[W^*]$, it follows that it has precisely one neighbour y in X and y is a node. Thus Lemma 2.10 part 1 and the maximality of $|X|$ imply that u is an admissible node.

Finally let $\mathcal{Y} = \{Y \subset V : u \in Y,\ Y$ is a node-critical set in $G\}$. If $\mathcal{Y} = \emptyset$ the result follows from Lemma 2.12. Otherwise let $Y \in \mathcal{Y}$ be maximal, and argue similarly to the proof for $X \in \mathcal{X}$ to complete the proof. □

3. Joining Circuits

By Lemma 2.3 and Theorem 1.2, in order to prove Theorem 1.1, it remains to consider the generation of circuits with cutpairs or with non-trivial 3-edge cuts. In this section we introduce 3 new operations to do exactly this.

3.1. Circuits containing cut-pairs

We start by considering graphs that are not 3-connected. Let $K_n(a_1, \dots, a_n)$ denote the complete graph with vertex set $\{a_1, \dots, a_n\}$. Let $G = (V, E)$ be a circuit with a cutpair a, b and a bipartition A, B of $V \setminus \{a, b\}$. Since

$f(G) = 1$ and $f(H) \geq 2$ for all subgraphs there are two options: $ab \in E$ and $f(G[A \cup \{a,b\}]) = f(G[B \cup \{a,b\}]) = 2$ or $ab \notin E$ and $3 = f(G[A \cup \{a,b\}]) < f(G[B \cup \{a,b\}]) = 2$. This leads us to the 1- and 2-join operations. To refresh the readers memory we define the inverse operations.

Let G be as above and suppose $f(G[A \cup \{a,b\}]) < f(G[B \cup \{a,b\}])$. A 1-*separation* over the cutpair a, b forms disjoint graphs $G[A \cup \{a,b\}] \cup ab$ and $G[B \cup \{a,b\}] \cup K_4(a,b,c,d)$ where $c, d \notin B \cup \{a,b\}$. Also let $G = (V,E)$ be a circuit with a cutpair a, b with a bipartition A, B of $V \setminus \{a,b\}$ such that $f(G[A \cup \{a,b\}]) = f(G[B \cup \{a,b\}])$. A 2-*separation* over the cutpair a, b forms disjoint graphs $G[A \cup \{a,b\}] \cup K_4(a,b,c,d)$ and $G[B \cup \{a,b\}] \cup K_4(a,b,c,d)$ where $c, d \notin A \cup \{a,b\}$ or $B \cup \{a,b\}$.

Lemma 3.1. *Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be graphs such that G_1 contains an edge a_1b_1 and G_2 contains a two vertex cut a_2, b_2 within $K_4(a_2, b_2, c_2, d_2)$. Then the* 1*-join $G_1 \oplus_1 G_2 = G = (V,E)$ (merging $a_1 = a_2$ into a and $b_1 = b_2$ into b) is a circuit if and only if G_1 and G_2 are circuits.*

Proof. We have $V = (V_1 \setminus \{a_1, b_1\}) \cup (V_2 \setminus \{a_2, b_2, c_2, d_2\}) \cup \{a, b\}$ so

$$\begin{aligned} |E| = |E_1| - 1 + |E_2| - 6 &= 2|V_1| - 1 + 2|V_2| - 1 - 7 \\ &= 2(|V_1| + |V_2| - 4) - 1 = 2|V| - 1. \end{aligned}$$

Let $X \subset V$. Let $X_i = (V_i \cap X) \cup (\{a,b\} \cap X)$ and let $X_i' = (V_i \cap X) \cup (\{a_i, b_i\} \cap X)$. If X contains both a and b then

$$\begin{aligned} i_G(X) &= i_{G_1}(X_1') + i_{G_2}(X_2') - 2 \leq 2|X_1'| - 1 + 2|X_2'| - 2 - 2 \\ &= 2(|X_1'| + |X_2'|) - 5 = 2|X| - 1. \end{aligned}$$

where equality holds if and only if $X = V$. Similarly, if X contains at most one of a and b then $i_G(X) \leq 2|X| - 2$.

Conversely, suppose G_1 is not a circuit. Since $|E_1| = 2|V_1| - 1$ there exists X properly contained in $A \cup \{a,b\}$ with $i_{G_1}(X) = 2|X| - 1$. X contains a, b otherwise $X \subset V$. We have

$$\begin{aligned} i_G(X \cup B \cup \{a,b\}) &= 2|X| - 2 + 2|B \cup \{a,b\}| - 3 \\ &= 2(|X \setminus \{a,b\}| + |B \cup \{a,b\}| - 2) - 1, \end{aligned}$$

a contradiction.

Now suppose G_2 is not a circuit. Since $|E_2| = 2|V_2| - 1$ there exists X properly contained in $B \cup \{a,b,c,d\}$ with $i_{G_2}(X) = 2|X| - 1$. X contains

c, d otherwise X is a subset of V and thus X contains a, b. We have

$$\begin{aligned} i_G((X \setminus \{c,d\}) \cup A \cup \{a,b\}) &= 2|X \setminus \{c,d\}| - 2 + 2|A \cup \{a,b\}| - 2 - 1 \\ &= 2(|X \setminus \{c,d\}| + |A \cup \{a,b\}| - 2) - 1, \end{aligned}$$

a contradiction. □

Lemma 3.2. *Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be graphs such that G_i contains a two vertex cut a_i, b_i within $K_4(a_i, b_i, c_i, d_i)$. Then the 2-join $G_1 \oplus_2 G_2 = (V, E)$ (merging $a_1 = a_2$ into a and $b_1 = b_2$ into b) is a circuit if and only if G_1 and G_2 are circuits.*

Proof. We have $V = (V_1 \setminus \{a_1, b_1, c_1, d_1\}) \cup (V_2 \setminus \{a_2, b_2, c_2, d_2\}) \cup \{a, b\}$ so

$$\begin{aligned} |E| = |E_1| - 6 + |E_2| - 6 + 1 &= 2|V_1| - 1 + 2|V_2| - 1 - 11 \\ &= 2(|V_1| + |V_2| - 6) - 1 = 2|V| - 1. \end{aligned}$$

Let $X \subset V$. Let $X_i = (V_i \cap X) \cup (\{a, b\} \cap X)$ and let $X_i' = (V_i \cap X) \cup (\{a_i, b_i\} \cap X)$. If X contains both a and b then

$$\begin{aligned} i_G(X) &= i_{G_1}(X_1') + i_{G_2}(X_2') - 1 \leq 2|X_1'| - 2 + 2|X_2'| - 2 - 1 \\ &= 2(|X_1'| + |X_2'| - 2) - 1 = 2|X| - 1. \end{aligned}$$

where equality holds if and only if $X = V$. Similarly, if X contains at most one of a and b then $i_G(X) \leq 2|X| - 2$.

For the converse, by symmetry, it is enough to show that G_1 is a circuit.

Suppose G_1 is not a circuit. Since $|E_1| = 2|V_1| - 1$ there exists X properly contained in $A \cup \{a, b, c, d\}$ with $i_{G_1}(X) = 2|X| - 1$. X contains c, d otherwise X is a subgraph of G and thus X contains a and b. We have

$$\begin{aligned} i_G((X \setminus \{c,d\}) \cup (B \cup \{a,b\})) &= 2|X \setminus \{c,d\}| - 2 + 2|B \cup \{a,b\}| - 2 - 1 \\ &= 2(|X \setminus \{a,b\}| + |B \cup \{a,b\}| - 2) - 1, \end{aligned}$$

a contradiction. □

3.2. Circuits with 3-edge-cuts

We also require the 3-join operation. Let $G = (V, E)$ be a circuit with a non-trivial 3-edge-cut a_1a_2, b_1b_2, c_1c_2 with a bipartition A, B of V such that $f(G[A]) = f(G[B])$. A 3-*separation* over the cut a_1a_2, b_1b_2, c_1c_2 forms disjoint graphs $G[A] \cup v_1 \cup \{a_1v_1, b_1v_1, c_1v_1\}$ and $G[B] \cup v_2 \cup \{a_2v_2, b_2v_2, c_2v_2\}$.

Lemma 3.3. *Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be graphs. Then the 3-join $G = G_1 \oplus_3 G_2 = (V, E)$ (deleting $v_i \in V_i$ with $d_{G_i}(v_i) = 3$ and $N(v_i) = \{a_i, b_i, c_i\}$ for $i = 1, 2$ and adding a_1a_2, b_1b_2, c_1c_2) is a circuit if and only if G_1 and G_2 are circuits.*

Proof. We have $V = (V_1 \setminus v_1) \cup (V_2 \setminus v_2)$ so

$$\begin{aligned}|E| = |E_1| - 3 + |E_2| - 3 + 3 &= 2|V_1| - 1 + 2|V_2| - 1 - 3 \\ &= 2(|V_1| + |V_2| - 2) - 1 = 2|V| - 1.\end{aligned}$$

Let $X \subset V$. Let $X_i = (V_i \cap X)$. X contains at least one of a_i, b_i, c_i, otherwise $X \subset X_i$ and so $i(X) \leq 2|X| - 2$. Let $0 \leq t \leq 3$ be the number of edges in the subgraph induced by X from the set $\{a_1a_2, b_1b_2, c_1c_2\}$. Then

$$\begin{aligned}i_G(X) &= i_{G_1}(X_1) + i_{G_2}(X_2) + t \\ &\leq 2|X_1| - 2 + 2|X_2| - 2 + t \\ &\leq 2|X| - 1.\end{aligned}$$

where equality holds if and only if $X = V$; otherwise for some i, $X_i \subsetneq V_i$, $i(X_i) = 2|X_i| - 2$ and X_i contains a_i, b_i, c_i so adding back v_i contradicts G_i being a circuit.

For the converse, clearly $f(G[A]) = f(G[B]) = 2$. By symmetry it is enough to show that G_1 is a circuit.

Suppose G_2 is not a circuit. Since $|E_1| = 2|V_2)| - 1$ there exists X properly contained in $A \cup v_1$ with $i_{G_1}(X) = 2|X| - 1$. X contains v_1, otherwise X is a subgraph of G, and thus contains a_1, b_1, c_1. We have

$$\begin{aligned}i_G((X \setminus v_1) \cup B) &= 2|X \setminus v_1| - 2 + 2|B| - 2 + 3 \\ &= 2(|X \setminus v_1| + |B|) - 1,\end{aligned}$$

a contradiction. □

4. A Recursive Construction of Circuits

It remains to deal with the case when every cutpair a, b in G with associated bipartition A, B is such that, at least one of the subgraphs induced by $A \cup \{a, b\}$ and $B \cup \{a, b\}$ is isomorphic to K_4. Here the 2-separation move results in a copy of G and a copy of $K_4 \sqcup K_4$. However we do not need a new recursive move to deal with this case. Consider a graph G with n cutpairs and each cutpair a_i, b_i with bipartition A_i, B_i leaves $G[A_i \cup \{a_i, b_i\}]$

isomorphic to $K_4(a_i, b_i, c_i, d_i)$. Now delete each c_i, d_i and all incident edges and add a second copy of each edge $a_i b_i$. We denote the resulting multigraph as $G^- = (V^-, E^-)$. G^- is a 3-connected *multicircuit*, see Figure 6. None of the a_i or b_i are nodes; if $d_G(a_i) = 3$ then $N(a_i) = \{b_i, x\}$ for some x but then b_i, x is a cutpair for G^- and hence for G. Thus every node in G^- has 3 distinct neighbours.

There is a node in a multicircuit in which an inverse Henneberg 2 move results in a multicircuit by Frank and Szegő [4, Theorem 1.10]. However we need the following stronger result which follows by the same proof as Theorem 1.2, noting that the simplicity assumption did not provide a simplification.

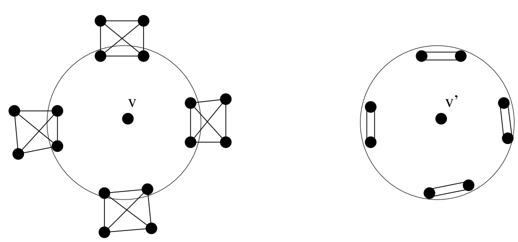

Figure 6: For every 2-vertex cut with one component a copy of K_4, replace each copy with a double edge. We show that if v' is an admissible node then so is v.

Proposition 4.1. *Let $G = (V, E)$ be a multigraph with $|V| \geq 6$. Let G be a 3-connected multicircuit with no non-trivial 3-edge-cuts in which every node has 3 distinct neighbours. Then G contains an allowable node.*

By *allowable* here we mean that there is an inverse Henneberg 2 move on a node that results in a multicircuit and that the new edge does not create a multiple edge. Thus if we can apply the proposition to find there is an allowable node in G^- then the corresponding node is admissible in G.

4.1. Proof of Theorem 1.1

We are now ready to prove our main result.

Proof of Theorem 1.1. By Lemmas 2.1, 3.1, 3.2 and 3.3 a connected graph built up recursively from disjoint copies of base graphs by 1-joins, 2-joins, 3-joins and Henneberg 2 moves is a circuit.

Conversely, since $K_5 \setminus e$ is the unique circuit on at most 5 vertices, by Theorem 1.2, we may apply an inverse Henneberg 2 move whenever G is

3-connected with no non-trivial 3-edge cuts. If G is 3-connected with a non-trivial 3-edge-cut then, by Lemma 3.3 we may apply a 3-separation to G resulting in smaller circuits.

If G is not 3-connected then there is a cutpair. Choose a cutpair a, b. If $ab \notin E$ then by Lemma 3.1 we can apply a 1-separation in such a way that the resulting graphs are circuits. Suppose then for every cutpair a, b, $ab \in E$ and suppose there is a choice of a, b such that $G[A \cup \{a, b\}]$ and $G[B \cup \{a, b\}]$ are not isomorphic to K_4. Then by Lemma 3.2 we can apply a 2-separation in such a way that the resulting graphs are circuits.

Now if every minimal choice of cutpair results in $G[A \cup \{a, b\}] \cong K_4(x_i, y_i, z_i, w_i)$ where x_i, y_i is the cutpair and the corresponding multigraph G^+, as above, has $|V^+| \geq 6$ then the result follows from Proposition 4.1.

It remains to check the cases when $|V^+| \leq 5$. If $|V^+| = 2$ then $G \cong K_4 \sqcup K_4$. If $|V^+| = 3$ then $G \cong K_4 \veebar K_4$. If $|V^+| = 4$ or $|V^+| = 5$ there are a small number of cases that are each easy to check (there is an admissible node or a separation to smaller circuits). □

5. Connected Matroids and Rigid Frameworks

In the remainder of the paper we consider potential applications of our results to frameworks on surfaces.

5.1. Rigidity on the cylinder

A *framework* (G, p) on the cylinder $S^1 \times \mathbb{R}$ in $\mathbb{R}^3$ is the combination of a graph G and a map $p : V \to S^1 \times \mathbb{R}$. We will focus only on when such frameworks are generic: there are no algebraic dependencies among the coordinates of the framework points that are not required by $\mathcal{M}$. The *cylinder rigidity matrix* $R_{S^1\times\mathbb{R}}(G, p)$ is the $(|E| + |V|) \times 3|V|$ matrix where the first $|E|$ rows correspond to the edges and the entries in the row for edge uv are 0 except in the column triples corresponding to u and v where the entries are $p(u) - p(v)$ and $p(v) - p(u)$ respectively. The final $|V|$ rows correspond to the vertices and the entries in the row for vertex i are zero except in the column triple corresponding to i where the entry is $N(p(i))$, the surface normal to the point $p(i)$. A framework (G, p) on $S^1 \times \mathbb{R}$ is *generic* if the only polynomial equations satisfied by the coordinates of p are those that define $S^1 \times \mathbb{R}$. Let $\mathcal{R}_{S^1\times\mathbb{R}}$ denote the *cylinder rigidity matroid*, that is the linear matroid induced by linear independence in the rows of $R_{S^1\times\mathbb{R}}(G, p)$ for generic p. A framework is *infinitesimally rigid* if its edge set has maximal rank in $\mathcal{R}_{S^1\times\mathbb{R}}$.

More detailed definitions may be found in [16], see also [5] for a detailed study of rigidity matroids.

Theorem 5.1 ([16])**.** *Let $G = (V, E)$ be a graph with $|V| \geq 4$ and let (G, p) be a generic framework in 3-dimensions constrained to $S^1 \times \mathbb{R}$. Then the matroids $\mathcal{R}_{S^1\times\mathbb{R}}$ and $M^*(2, 2)$ are isomorphic.*

Similarly if $\mathcal{R}_2$ denotes the rigidity matroid for generic frameworks in $\mathbb{R}^2$, then Laman's theorem [12] states $\mathcal{R}_2 \cong M(2, 3)$. We will need the following corollary to Theorem 5.1. A *redundantly rigid* framework (G, p) on $S^1 \times \mathbb{R}$ is a framework such that after deleting any single edge from G the rigidity matroid still has maximal rank.

Corollary 5.2. *Let $G = (V, E)$ and let p be generic. Then (G, p) is redundantly rigid on $S^1 \times \mathbb{R}$ if and only if (G, p) is infinitesimally rigid on $S^1 \times \mathbb{R}$ and every edge of G belongs to a $\mathcal{R}_{S^1\times\mathbb{R}}$-circuit.*

Remark 5.3. By Theorem 5.1 a generic framework (G, p) on $S^1 \times \mathbb{R}$ is rigid if and only if G contains a spanning $(2, 2)$-tight subgraph. However as $K_{3,6}$ illustrates (see also [10, Figure 6] for the plane case) extending Theorem 1.1 from circuits to 2-connected redundantly rigid graphs is non-trivial. For example $K_{3,6}$ is not a circuit so one of the operations must be an edge addition. The last move must be a Henneberg 2 move since $K_{3,6}$ is 3-connected with no non-trivial 3-edge cuts and minimal in the sense that removing any edge results in a graph $G = (V, E)$ with $|E| = 2|V| - 1$ that is not a circuit.

5.2. $\mathcal{R}_{S^1\times\mathbb{R}}$-connected Graphs

Following [10], for $\mathcal{R}_{S^1\times\mathbb{R}} = (E, I)$, define a relation on E by saying $e, f \in E$ are related if $e = f$ or if there is a $\mathcal{R}_{S^1\times\mathbb{R}}$-circuit C with $e, f \in C$. We abuse notation slightly by referring to C as both the circuit in $\mathcal{R}_{S^1\times\mathbb{R}}$ and the graph induced by the circuit, i.e. the $\mathcal{R}_{S^1\times\mathbb{R}}$-circuit. This is an equivalence relation and the equivalence classes are the components of $\mathcal{R}_{S^1\times\mathbb{R}}$. If $\mathcal{R}_{S^1\times\mathbb{R}}$ has at least two elements and only one component then it is $\mathcal{R}_{S^1\times\mathbb{R}}$*-connected*. G is $\mathcal{R}_{S^1\times\mathbb{R}}$*-connected* if $\mathcal{R}_{S^1\times\mathbb{R}}$ is connected. The $\mathcal{R}_{S^1\times\mathbb{R}}$-components of G are the subgraphs of G induced by the components of $\mathcal{R}_{S^1\times\mathbb{R}}$.

Since bases in $M^*(2, 2)$ can contain cut-vertices while circuits cannot, to link redundantly rigid frameworks and $\mathcal{R}_{S^1\times\mathbb{R}}$-connected graphs requires 2-connectivity.

Theorem 5.4. *A graph G is 2-connected with a redundantly rigid realisation on $S^1 \times \mathbb{R}$ if and only if G is $\mathcal{R}_{S^1\times\mathbb{R}}$-connected.*

Proof. Suppose G is $\mathcal{R}_{S^1\times\mathbb{R}}$-connected. G is infinitesimally rigid since there is only one $\mathcal{R}_{S^1\times\mathbb{R}}$-connected component. $\mathcal{R}_{S^1\times\mathbb{R}}$ is connected so every edge is in a $\mathcal{R}_{S^1\times\mathbb{R}}$-circuit. Thus G has a redundantly rigid realisation by Corollary 5.2. Also Lemma 2.3 implies G is 2-connected.

Conversely let X be the set of $\mathcal{R}_{S^1\times\mathbb{R}}$-connected components of G and $\theta(X)$ the set of vertices of G belonging to two distinct elements of X. Let $d_X(v)$ denote the number of elements of X containing v. Let $r(G)$ denote the rank of the rigidity matroid $\mathcal{R}_{S^1\times\mathbb{R}}(G,p)$. Then

$$2|V|-2 = r(G) = \sum_{H\in X} r(H) = \sum_{H\in X}(2|V(H)|-2)$$

and

$$|V| = \sum_{H\in X}|V(H)| - \sum_{v\in\theta(X)}(d_X(v)-1).$$

This implies that $\sum_{v\in\theta(X)} d_X(v) < 2|X|$ so there exists $H\in X$ with $|V(H)\cap \theta(X)| \leq 1$. □

5.3. Global Rigidity

Definition 5.5. *A framework (G,p) on $S^1\times\mathbb{R}$ is* globally rigid *if every framework (G,q) which satisfies the (Euclidean 3-space) distance constraint equations $|p_i - p_j| = |q_i - q_j|$, for each edge ij where p_i, p_j, q_i, q_j are points on $S^1\times\mathbb{R}$ also satisfies $|p_i-p_j| = |q_i-q_j|$ for every pair of vertices i,j of G.*

We now recall the celebrated characterisation of generic global rigidity in the plane. This is due, in its various parts, to Connelly [2], Hendrickson [8] and Jackson and Jordán [10]. Giving a full 3-dimensional combinatorial characterisation remains a hard open problem.

Theorem 5.6. *Let $G=(V,E)$ with $|V|\geq 4$ and let p be generic. Then the following are equivalent:*

(1) *(G,p) is globally rigid in $\mathbb{R}^2$,*
(2) *G is 3-connected and (G,p) is redundantly rigid in the plane,*
(3) *G can be formed from disjoint copies of K_4 by Henneberg 2 moves and edge additions,*
(4) *G is 3-connected and $\mathcal{R}_2$-connected.*

The analysis in this paper leads us to make the following conjecture.

Conjecture 5.7. *Let $G = (V, E)$ with $|V| \geq 5$ and let p be generic for $S^1 \times \mathbb{R}$. The following are equivalent:*

1. *(G, p) is globally rigid on $S^1 \times \mathbb{R}$,*
2. *G is 2-connected and (G, p) is redundantly rigid on $S^1 \times \mathbb{R}$,*
3. *G can be formed from disjoint copies of $K_5 \setminus e, K_4 \sqcup K_4$ and $K_4 \veebar K_4$ by Henneberg 2 moves, 1-joins, 2-joins, 3-joins and edge additions,*
4. *G is $\mathcal{R}_{S^1 \times \mathbb{R}}$-connected.*

For $|V| \leq 4$, (G, p) is globally rigid on $S^1 \times \mathbb{R}$ if and only if G is a complete graph. Following the submission of this paper, (1) $\Rightarrow$ (2) has been confirmed in [11]. Theorem 5.4 shows the equivalence of (2) and (4).

6. Concluding Remarks

Our conjectured characterisation would provide a sufficient condition for global rigidity on the cylinder that fails somewhat trivially in the plane. Let G contain a spanning subgraph H which is a $\mathcal{R}_{S^1 \times \mathbb{R}}$-circuit and let p be generic for $S^1 \times \mathbb{R}$. Then Conjecture 5.7 implies that (G, p) is globally rigid on $S^1 \times \mathbb{R}$. Remark 5.3 illustrates why this does not characterise globally rigid frameworks on the cylinder.

The special case in which G has the minimum possible number of edges $2|V| - 1$ corresponding to [1, Theorem 6.1] conjectures that the generically globally rigid graphs on the cylinder are exactly the $\mathcal{R}_{S^1 \times \mathbb{R}}$-circuits. To prove the minimal case it remains to show that the Henneberg 2 and i-join moves preserve global rigidity.

The remaining combinatorial difficulty in Conjecture 5.7 is in showing that every $\mathcal{R}_{S^1 \times \mathbb{R}}$-connected graph can be generated using only the construction moves in Theorem 1.1. In the case of the plane this was done by Jackson and Jordán [10] who used the concept of an ear decomposition in a $\mathcal{R}_2$-connected graph. Such a theorem would complete the equivalence of $(2), (3)$ and (4).

Conjecture 5.7 would lead to an efficient algorithm for checking global rigidity. 2-connectedness can be checked in linear time [9] and redundant rigidity, via the pebble game [7], [13], can be checked in $O(|V|^2)$ time.

Finally we note that Theorems 1.2 and 1.1 do not easily extend to the case of circuits in $M^*(2, 1)$. A higher level of connectivity will be required to guarantee an admissible node when a node even exists. Moreover circuits in $M^*(2, 1)$ may contain cut-vertices and more elaborate i-join operations may be required. A characterisation of circuits in $M^*(2,1)$ would be a step towards proving the analogue of Conjecture 5.7 for frameworks on a surface of revolution [17], such as a cone [11].

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