A Class of Linearly Constrained Nonlinear Optimization Problems with Corner Point Optimal Solutions and Applications in Finance

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Abstract

We identify a class of linearly constrained nonlinear optimization problems with corner point optimal solutions. These include some special polynomial fractional optimization problems with an objective function equal to the product of some power functions of positive linear functionals subtracting the sum of some power functions of positive linear functionals, divided by the sum of some power functions of positive linear functionals. The powers are required to be all positive integers, and the aggregate power of the product is required to be no larger than the lowest power in both of the two sums. The result has applications to some optimization problems under uncertainty, particularly in finance.

Key words: linear constraints, non-linear optimization, polynomial fractional optimization, corner point optimal solutions.

Introduction

It is well known that when the objective function and constraints of an optimization problem are all linear, the search for the optimal solution is greatly simplified so that we need only consider corner-point feasible (hereafter CPF) solutions. While we all know that this simplification, in general, does not work for a problem with a nonlinear objective function, we ask the question what conditions are needed to make it work. Surprisingly, little has been said about this in the literature. In this paper we try to fill this gap and identify a class of linearly constrained nonlinear optimization problems which have corner point optimal solutions.

As expectations are linear in probabilities, we can naturally find applications of the above result to some optimization problems under uncertainty. In particular, we present some examples to show how this result helps to solve some interesting problems in finance. For example, we use the above result to investigate the effects of background risk and wealth inequality on downside risk aversion.

This paper is related to the studies on linearly constrained nonlinear optimization problems in general. There is an extensive literature on this topic though its focus is primarily on numerical algorithms for this class of optimization problems.[1] In particular, this paper is related to the studies on linearly constrained polynomial and polynomial fractional optimization problems.[2] Moreover, this paper is related to the work of Charnes and Cooper (1962) who show that linear fractional optimization problems can

[1]There are well over 400 different solution algorithms in solving different kinds of linearly constrained optimization problems. See, for example, Kalantari (1985), Parpas et al. (2006), Zhang and Wang (2008), and Jeyakumar and Li (2011).

[2]See, for example, Jeyakumar and Li (2011).

be transformed to linear optimization problems.

The paper is also related to some studies on decision making under uncertainty, in particular, the recent advancements in the theory of downside risk aversion.[3] Moreover, the paper is also closely related to the work of Gollier and Kimball (1996) who develop a diffidence theorem which can solve a large set of problems related to the effects of uncertainty on preferences. The diffidence theorem deals with the situation where functions are all linear in probabilities while our result is useful in the non-linear situation.

1 Two Lemmas

In this section we establish two lemmas which are crucial to the derivation of our main results in the next section. Let $x = (x_1, ..., x_k) \in R^k$. Let $f_i(x)$ and $g_j(x)$ be linear functions of $x_1, ..., x_k$, $i = 1, ..., n$, $j = 1, ..., m$. Let $s_1, ..., s_n, t_1, ..., t_m$ be positive integers. We now consider the following inequality:

$$\sum_{i=1}^{n}[f_i(x)]^{s_i} \leq (<)\Pi_{i=1}^{m}[g_i(x)]^{t_i}. \tag{1}$$

Let $\bar{s} = \max_i\{s_i\}$ and $t = \sum_{i=1}^{m} t_i$. Following convention, let $C_s^{\bar{s}}$ and $P(\bar{s}, s)$ denote the number of s-combinations and the number of s-permutations from a set of $\bar{s}$ elements respectively. Let $\left(\begin{array}{c}\{1,...,\bar{s}\}\\ j_1,...,j_s\end{array}\right)$ and $\left[\begin{array}{c}\{1,...,\bar{s}\}\\ j_1,...,j_s\end{array}\right]$ denote the set of all s-combinations and the set of all s-permutations of $\{1, 2, ..., \bar{s}\}$ respectively, where $s \leq \bar{s}$. Let $\left[\begin{array}{c}\{1,...,\bar{s}\}\\ (j_1^1,...,j_{t_1}^1),...,(j_1^m,...,j_{t_m}^m)\end{array}\right]$ denote the set of all permutations of $t = \sum_{i=1}^{m} t_i$ numbers chosen from $\{1, 2, ..., \bar{s}\}$, where the order of the elements in each pair of the round brack-

[3]See the list of studies on downside risk aversion mentioned in Section 3.

ets does not matter. Let A be a non-empty subset of R^k. We first present the following lemma.

Lemma 1 *Assume* $\min\{s_1, ..., s_n\} \geq t = \sum_{i=1}^m t_i$, *and for all* $x \in A$, $f_i(x) \geq 0$, $g_j(x) \geq 0$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$. *Inequality (1) is true for every* $x \in A$ *if and only if the following inequality is true for all* $x^k \in A$, $k = 1, ..., \bar{s}$, *where* $\bar{s} = \max_i\{s_i\}$.[4]

$$\sum_{i=1}^{n} \frac{1}{C_{s_i}^{\bar{s}}} \sum_{\binom{\{1,...,\bar{s}\}}{j_1,...,j_{s_i}}} \Pi_{k=j_1}^{j_{s_i}} f_i(x^k)$$

$$\leq (<) \frac{t_1!...t_m!}{P(\bar{s}, t)} \sum_{\left[\substack{\{1,...,\bar{s}\} \\ (j_1^1,...,j_{t_1}^1),...,(j_1^m,...,j_{t_m}^m)}\right]} \Pi_{i=1}^m \Pi_{k=j_1^i}^{j_{t_i}^i} g_i(x^k). \quad (2)$$

Proof: See Appendix A.

Due to the generality of Lemma 1 in terms of the number of elements involved, its proof requires complicated calculations; in particular, it involves calculations of combinations, permutations, and their mixtures. But the idea of the proof can be explained using the following simple example where the number of elements involved is small. Consider the special case where $n = m = 2$ and $s_1 = s_2 = 2$, $t_1 = t_2 = 1$. In this case (1) becomes

$$(f_1(x))^2 + (f_2(x))^2 \leq (<) g_1(x)g_2(x), \quad (3)$$

and (2) becomes

$$f_1(x^1)f_1(x^2) + f_2(x^1)f_2(x^2) \leq (<) \frac{1}{2}[g_1(x^1)g_2(x^2) + g_1(x^2)g_2(x^1)] \quad (4)$$

[4]Throughout this paper, k in the expression x^k is not a power; it is a superscript instead.

If (4) is true for all $x^1, x^2 \in A$, then letting $x^1 = x^2 = x$, we conclude that (3) is true for all $x \in A$. Thus we need only show that the converse is true. To show this, as arithmetic mean is larger than geometric mean, the right hand side of (4) is larger than $\sqrt{g_1(x^1)g_2(x^1)}\sqrt{g_1(x^2)g_2(x^2)}$. Now applying (3), we obtain that the right hand side of (4) is (strictly) larger than $\sqrt{(f_1(x^1))^2 + (f_2(x^1))^2}\sqrt{(f_1(x^2))^2 + (f_2(x^2))^2}$. Now applying Cauchy's inequality we obtain that the above expression is larger than $f_1(x^1)f_1(x^2) + f_2(x^1)f_2(x^2)$. This proves that if (3) is true for all $x \in A$ then (4) is true for all $x^1, x^2 \in A$. This completes the proof of this special case.

Let Σ be the set of feasible solutions from some given linear constraints on x. Assume Σ is nonempty, bounded and closed. Let Σ_c be the set of corner-point feasible (hereafter CPF) solutions. We have the following lemma.

Lemma 2 *Inequality (2) is true for all $x^i \in \Sigma, i = 1, ..., \bar{s}$, if and only if it is true for all $x^i \in \Sigma_c, i = 1, ..., \bar{s}$, where $\bar{s} = \max_i\{s_i\}$.*

Proof: Let $f(x^1, ..., x^{\bar{s}})$ denote the following function

$$-\sum_{i=1}^{n} \frac{1}{C_{s_i}^{\bar{s}}} \sum_{\left(\begin{array}{c} \{1,...,\bar{s}\} \\ j_1,...,j_{s_i} \end{array} \right)} \Pi_{k=j_1}^{j_{s_i}} f_i(x^k)$$

$$+\frac{t_1!...t_m!}{P(\bar{s},t)} \sum_{\left[\begin{array}{c} \{1,...,\bar{s}\} \\ (j_1^1,...,j_{t_1}^1),...,(j_1^m,...,j_{t_m}^m) \end{array} \right]} \Pi_{i=1}^{m} \Pi_{k=j_1^i}^{j_{t_i}^i} g_i(x^k), \qquad (5)$$

where $t = \sum_{i=1}^m t_i$. As $f(x^1, ..., x^{\bar{s}})$ is continuous and Σ is bounded and closed, there must be an optimal solution to the following minimization problem:

$$\min_{x^1,...,x^{\bar{s}}\in\Sigma} f(x^1,...,x^{\bar{s}}).$$

We now assert that the optimal value can be achieved at a feasible solution $(x^{1*},...,x^{\bar{s}*})$ such that $x^{1*},...,x^{\bar{s}*}\in\Sigma_c$, i.e., $x^{1*},...,x^{\bar{s}*}$ are CPF solutions. This can be shown as follows. Let $(x^{1\circ},...,x^{\bar{s}\circ})$ be an optimal solution. Suppose $x^{i\circ}$ is not a CPF solution. Then we must be able to replace it with a CPF solution. To see this, consider the following minimization problem:

$$\min_{x^i\in\Sigma} f(x^{1\circ},...,x^{(i-1)\circ},x^i,x^{(i+1)\circ},...,x^{\bar{s}\circ}).$$

It is clear that $f(x^{1\circ},...,x^{(i-1)\circ},x^i,x^{(i+1)\circ},...,x^{\bar{s}\circ})$ is a linear function of x^i. Thus this minimization problem is a linear programming problem. Hence we must be able to achieve the optimal solution at a CPF solution $x^{i*}=(x_1^{i*},...,x_k^{i*})\in\Sigma_c$. This proves the lemma. Q.E.D.

2 Main Results

We are now ready to present our main theorem.

Theorem 1 *Assume* $\min\{s_1,...,s_n\}\geq\sum_{i=1}^m t_i$*, where* $n,m,s_1,...,s_n,t_1,...,t_m$ *are all positive integers, and for all* $x\in\Sigma$*,* $f_i(x)\geq 0$*,* $g_j(x)\geq 0$*,* $i=1,2,...,n$*,* $j=1,2,...,m$*. Then the following two statements are true.*

1. *Inequality (1) is true for every* $x\in\Sigma$*, if and only if it is true for every* $x\in\Sigma_c$*.*

2. *If there does not exist* $x\in\Sigma$ *such that* $f_1(x)=0$*,* $f_2(x)=0$*, ...,* $f_l(x)=0$*, and for all* $x\in\Sigma$*,* $\Pi_{i=1}^m(g_i(x))^{t_i}-\sum_{i=l+1}^n(f_i(x))^{s_i}\geq 0$*, then,* $\min_{x\in\Sigma}\Gamma(x)=\min_{x\in\Sigma_c}\Gamma(x)$*, where* $\Gamma(x)=\frac{\Pi_{i=1}^m(g_i(x))^{t_i}-\sum_{i=l+1}^n(f_i(x))^{s_i}}{\sum_{i=1}^l(f_i(x))^{s_i}}$*.*

Proof: We first prove Statement 1. Applying Lemma 1 (with $A = \Sigma$), we conclude that inequality (1) is true for every $x \in \Sigma$ if and only if inequality (2) is true for all $x^i \in \Sigma, i = 1, ..., \bar{s}$, where $\bar{s} = \max_i\{s_i\}$. Applying Lemma 2, we conclude that inequality (2) is true for all $x^i \in \Sigma, i = 1, ..., \bar{s}$, if and only if it is true for all $x^i \in \Sigma_c, i = 1, ..., \bar{s}$. The above two statements imply that inequality (1) is true for every $x \in \Sigma$ if and only if inequality (2) is true for all $x^i \in \Sigma_c, i = 1, ..., \bar{s}$. Now applying Lemma 1 again (with $A = \Sigma_c$), we obtain that inequality (2) is true for all $x^i \in \Sigma_c, i = 1, ..., \bar{s}$, if and only if inequality (1) is true for every $x \in \Sigma_c$. The last two statements immeadiately lead to the conclusion that inequality (1) is true for every $x \in \Sigma$ if and only if it is true for every $x \in \Sigma_c$. This proves the first statement.

To prove the second statement, note that as Σ is closed, for all $x \in \Sigma$, $f_i(x) \geq 0$, and there does not exist $x \in \Sigma$ such that $f_1(x) = 0$, ..., $f_l(x) = 0$, $\sum_{i=1}^{l}(f_i(x))^{s_i}$ must be bounded away from zero. As Σ is closed and bounded, this implies that $\min_{x\in\Sigma} \frac{\Pi_{i=1}^{m}(g_i(x))^{t_i} - \sum_{i=l+1}^{n}(f_i(x))^{s_i}}{\sum_{i=1}^{l}(f_i(x))^{s_i}} = \alpha \geq 0$ exists. Thus we have $\min_{x\in\Sigma_c} \frac{\Pi_{i=1}^{m}(g_i(x))^{t_i} - \sum_{i=l+1}^{n}(f_i(x))^{s_i}}{\sum_{i=1}^{l}(f_i(x))^{s_i}} \geq \alpha$. But this inequality cannot be strict; otherwise if $\min_{x\in\Sigma_c} \frac{\Pi_{i=1}^{m}(g_i(x))^{t_i} - \sum_{i=l+1}^{n}(f_i(x))^{s_i}}{\sum_{i=1}^{l}(f_i(x))^{s_i}} > \alpha$ then, for all $x \in \Sigma_c$,

$$\Pi_{i=1}^{m}(g_i(x))^{t_i} > \sum_{i=l+1}^{n}(f_i(x))^{s_i} + \alpha\sum_{i=1}^{l}(f_i(x))^{s_i}.$$

Applying the first statement, we conclude that for all $x \in \Sigma$, the above inequality is true, i.e., $\min_{x\in\Sigma} \frac{\Pi_{i=1}^{m}(g_i(x))^{t_i} - \sum_{i=l+1}^{n}(f_i(x))^{s_i}}{\sum_{i=1}^{l}(f_i(x))^{s_i}} > \alpha$, which causes a contradiction. This proves the second statement. Q.E.D.

The second statement of the above theorem shows that if the nonlinear objective function of the minimization problem has the required feature, then the search for the solution to the nonlinear minimization problem is greatly simplified such that we need only consider CPF solutions.

In Theorem 1 the exponents $s_1, ..., s_n, t_1, ..., t_m$ are required to be integers. This requirement can be relaxed for some cases. We have the following result.

Proposition 1 *Let $s \geq \sum_{i=1}^{m} t_i$, where s, and t_i, $i = 1, ...m$, are positive real numbers, and for all $x \in \Sigma$, $f_1(x) \geq 0$, $g_j(x) \geq 0$, $j = 1, 2, ..., m$. Then the following two statements are true.*

1. *The following inequality is true for every $x \in \Sigma$, if and only if it is true for every $x \in \Sigma_c$:*

$$[f_1(x)]^s \leq \Pi_{i=1}^{m} [g_i(x)]^{t_i}. \tag{6}$$

2. *If for all $x \in \Sigma$, $f_1(x) > 0$, then,*

$$\min_{x \in \Sigma} \frac{\Pi_{i=1}^{m} [g_i(x)]^{t_i}}{[f_1(x)]^s} = \min_{x \in \Sigma_c} \frac{\Pi_{i=1}^{m} [g_i(x)]^{t_i}}{[f_1(x)]^s}. \tag{7}$$

Proof: We only show the proof of the first statement. Then as in Theorem 1, the second statement is implied by the first statement.

As the necessity is obvious, we need only prove the sufficiency. From Theorem 1, it is straightforward that if $\frac{t_1}{s}, ..., \frac{t_m}{s}$ are rational, Proposition 1 is valid. Rewrite (6) as

$$f_1(x) \leq \Pi_{i=1}^{m} [g_i(x)]^{t_i/s}. \tag{8}$$

Suppose the above inequality is valid for all $x \in \Sigma_c$ while some of $\frac{t_1}{s}, ..., \frac{t_m}{s}$ are irrational. Without loss of generality, suppose $\frac{t_m}{s}$ is irrational. Construct a sequence $\{\delta_j | j = 1, 2, ...\}$, where $\delta_j > 0$, $j = 1, 2, ...$, and $\lim_{j \to \infty} \delta_j = 0$. We have, for all $x \in \Sigma_c$,

$$\frac{f_1(x)}{\Pi_{i=1}^{m-1} [g_i(x) + \delta_j]^{t_i/s}} < [g_m(x) + \delta_j]^{t_m/s}, \quad j = 1, 2, ...$$

We can construct a sequence of rational numbers $\{v_{mj}|v_{mj} > t_m/s, j = 1, 2, ...\}$ such that $\lim_{j\to\infty} v_{mj} = t_m/s$ and for all $x \in \Sigma_c$,[5]

$$\frac{f_1(x)}{\Pi_{i=1}^{m-1}[g_i(x)+\delta_j]^{t_i/s}} < (1+\delta_j)[g_m(x)+\delta_j]^{v_{mj}}, \quad j = 1, 2, ...$$

After this, if $\frac{t_{m-1}}{s}$ is irrational we can do the same as above. That is, with the same argument we can construct a sequence of rational numbers $\{v_{(m-1)j}|v_{(m-1)j} > t_{m-1}/s, j = 1, 2, ...\}$ such that $\lim_{j\to\infty} v_{(m-1)j} = t_{m-1}/s$ and for all $x \in \Sigma_c$, $j = 1, 2, ...,$

$$\frac{f_1(x)}{\Pi_{i=1}^{m-2}[g_i(x)+\delta_j]^{t_i/s}(1+\delta_j)[g_m(x)+\delta_j]^{v_{mj}}} < (1+\delta_j)[g_{m-1}(x)+\delta_j]^{v_{(m-1)j}}.$$

Hence by doing the same for all $i = 1, ..., m$, we can construct a sequence of rational numbers $\{v_{ij}|v_{ij} > t_i/s, i = 1, ..., l; j = 1, 2, ...\}$ such that $\lim_{j\to\infty} v_{ij} = t_i/s$ and for all $x \in \Sigma_c$, $j = 1, 2, ...,$

$$f_1(x) \le (1+\delta_j)^m \Pi_{i=1}^m [g_i(x+\delta_j)]^{v_{ij}}. \tag{9}$$

As Theorem 1 implies that the above inequality is valid for all $x \in \Sigma$, letting $j \to \infty$ in (9), we immediately conclude that (8) is valid for all $x \in \Sigma$. Q.E.D.

Proposition 1 can be further extended; we have the following result.

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[5]We need only require that $0 < v_{mj} - t_m/s < -\frac{\ln(1+\delta_j)}{\ln \delta_j}$; then, we have

$$\begin{aligned}
&(1+\delta_j)[g_m(x)+\delta_j]^{v_{mj}} - [g_m(x)+\delta_j]^{t_m/s} \\
&= [g_m(x)+\delta_j]^{t_m/s}[(1+\delta_j)[g_m(x)+\delta_j]^{v_{mj}-t_m/s} - 1] \\
&\ge [g_m(x)+\delta_j]^{t_m/s}[(1+\delta_j)\delta_j^{v_{mj}-t_m/s} - 1] \\
&\ge [g_m(x)+\delta_j]^{t_m/s}[e^{\ln(1+\delta_j)+(v_{mj}-t_m/s)\ln\delta_j} - 1] > 0.
\end{aligned}$$

Proposition 2 *For $\theta \in [a,b]$, assume $t(\theta) \geq 0$ is a continuous function and $H(\theta)$ is a cumulative probability distribution function.*[6] *Assume for all $x \in \Sigma$, $f_1(x) > 0$, for all $x \in \Sigma$ and all $\theta \in [a,b]$, $g(x,\theta) \equiv \nu_0(\theta) + \nu_1(\theta)x_1 + \ldots + \nu_k(\theta)x_k > 0$, where $\nu_i(\theta)$ is a continuous function of θ, $i = 0, 1, \ldots, k$. Then the following two statements are true.*

1. *If $s \geq \int_a^b t(\theta)dH(\theta)$ then, the following inequality is true for every $x \in \Sigma$, if and only if it is true for every $x \in \Sigma_c$:*

$$\int_a^b t(\theta) \ln g(x,\theta) dH(\theta) \geq s \ln f_1(x) + \alpha, \tag{10}$$

where α is a constant.

2. *If for all $x \in \Sigma$, $f_1(x) > 1$, and $\frac{\int_a^b t(\theta)\ln g(x,\theta)dH(\theta)-\alpha}{\ln f_1(x)} \geq \int_a^b t(\theta)dH(\theta)$, where α is a constant, then,*

$$\min_{x\in\Sigma} \frac{\int_a^b t(\theta) \ln g(x,\theta) dH(\theta) - \alpha}{\ln f_1(x)} = \min_{x\in\Sigma_c} \frac{\int_a^b t(\theta) \ln g(x,\theta) dH(\theta) - \alpha}{\ln f_1(x)}. \tag{11}$$

Proof: Similar to Proposition 1, we need only prove the first statement, as the second statement is implied by the first statement.

We need only prove the sufficiency. Note that as both $t(\theta)$ and $\ln g(x,\theta)$ are continuous functions of θ and $H(\theta)$ is an increasing and bounded function, the Riemann-Stieltjes integral $\int_a^b t(\theta) \ln g(x,\theta) dH(\theta)$ exists.

Suppose (10) is valid for all $x \in \Sigma_c$. We construct a sequence of partitions of $[a,b]$: $P_i = \{\theta_{i1} = a, \theta_{i2}, \ldots, \theta_{i(j_i-1)}, \theta_{ij_i} = b\}$, $i = 1, 2, \ldots$, such that P_{i+1} is finer than P_i and $\lim_{i\to\infty} mesh(P_i) = 0$. As $t(\theta)$ is continuous, let $\theta^\circ_{ij} \in [\theta_{i(j-1)}, \theta_{ij}]$ be such that $t(\theta^\circ_{ij})(H(\theta_{ij}) - H(\theta_{i(j-1)})) = \int_{\theta_{i(j-1)}}^{\theta_{ij}} t(\theta)dH(\theta)$.

[6] In fact, $H(\theta)$ can be any increasing and bounded function.

Thus we have

$$\sum_{j=1}^{j_i} t(\theta_{ij}^{\circ})\Delta H(\theta_{ij}) = \int_a^b t(\theta)dH(\theta), i = 1, 2, ...,$$

$$\lim_{i\to\infty}\sum_{j=1}^{j_i} t(\theta_{ij}^{\circ})\ln g(x,\theta_{ij})\Delta H(\theta_{ij}) = \int_a^b t(\theta)\ln g(x,\theta)dH(\theta),$$

where $\Delta H(\theta_{ij}) = H(\theta_{ij}) - H(\theta_{i(j-1)})$. Let

$$\delta_i \equiv \max_{x\in\Sigma_c}|\sum_{j=1}^{j_i} t(\theta_{ij}^{\circ})\ln g(x,\theta_{ij})\Delta H(\theta_{ij}) - \int_a^b t(\theta)\ln g(x,\theta)dH(\theta)|,\ i = 1, 2, ...$$

From the definition of Riemann-Stieltjes integral, we have $\lim_{i\to\infty}\delta_i = 0$, and from (10), we also have for all $x \in \Sigma_c$,

$$\sum_{j=1}^{j_i} t(\theta_{ij}^{\circ})\ln g(x,\theta_{ij})\Delta H(\theta_{ij}) - s\ln f_1(x) - \alpha \geq -\delta_i. \tag{12}$$

Rewriting the last inequality, we obtain that for all $x \in \Sigma_c$,

$$\Pi_{j=1}^{j_i}[g(x,\theta_{ij})]^{t(\theta_{ij}^{\circ})\Delta H(\theta_{ij})} \geq e^{\alpha-\delta_i}[f_1(x)]^s, \quad i = 1, 2, ...$$

As $s \geq \int_a^b t(\theta)dH(\theta) = \sum_{j=1}^{j_i} t(\theta_{ij}^{\circ})\Delta H(\theta_{ij})$, applying Proposition 1, we immediately conclude that the above inequality is valid for all $x \in \Sigma$. This inequality is equivalent to (12). Letting $i \to \infty$ in (12), we conclude that (10) is valid for all $x \in \Sigma$. Q.E.D.

3 Applications

3.1 Optimization Under Uncertainty

The results obtained in the last section can be extended to optimization under uncertainty. Let Ω be the set of random variables that satisfy $Ew_i(\tilde{\epsilon}) = 0,\ i = 1, ..., \nu$, where $w_1(x), ..., w_\nu(x)$ are linearly independent functions.

Assume Ω is nonempty. Let $\Omega_{\nu+1}$ be the set of random variables with $(\nu+1)$-point distributions that satisfy $Ew_i(\tilde{\epsilon}) = 0$, $i = 1, ..., \nu$.

We only present the extension of Theorem 1. Proposition 1 and Proposition 2 can be extended in the same way.

Proposition 3 *Assume* $\min\{s_1, ..., s_n\} \geq \sum_{i=1}^{m} t_i$*, where* $n, m, s_1, ..., s_n, t_1, ..., t_m$ *are all positive integers. Given nonnegative functions* $u_i(x)$ *and* $v_j(x)$ *with expectations* $Eu_i(\tilde{\epsilon})$ *and* $Ev_i(\tilde{\epsilon})$ *well defined for all* $\tilde{\epsilon} \in \Omega$*, where* $i = 1, ..., n$*,* $j = 1, ..., m$*, the following two statements are true.*[7]

1. *The following inequality is true for all* $\tilde{\epsilon} \in \Omega$*, if and only if it is true for all* $\tilde{\epsilon} \in \Omega_{\nu+1}$*.*

$$\sum_{i=1}^{n} [Eu_i(\tilde{\epsilon})]^{s_i} \leq \Pi_{i=1}^{m} [Ev_i(\tilde{\epsilon})]^{t_i}. \tag{13}$$

2. *If there does not exist* $\tilde{\epsilon} \in \Omega$ *such that* $Eu_1(\tilde{\epsilon}) = 0$*, ...,* $Eu_l(\tilde{\epsilon}) = 0$*, and for all* $\tilde{\epsilon} \in \Omega$*,* $\Pi_{i=1}^{m}(v_i(\tilde{\epsilon}))^{t_i} - \sum_{i=l+1}^{n}(Eu_i(\tilde{\epsilon}))^{s_i} \geq 0$ *then,* $\min_{\tilde{\epsilon}\in\Omega} \Gamma(\tilde{\epsilon}) = \min_{\tilde{\epsilon}\in\Omega_{\nu+1}} \Gamma(\tilde{\epsilon})$*, where* $\Gamma(\tilde{\epsilon}) = \frac{\Pi_{i=1}^{m}[Ev_i(\tilde{\epsilon})]^{t_i} - \sum_{i=l+1}^{n}(Eu_i(\tilde{\epsilon}))^{s_i}}{\sum_{i=1}^{l}[Eu_i(\tilde{\epsilon})]^{s_i}}$.

The proof is almost the same as the proof in the linear case which can be found in any textbook on optimization under uncertainty; thus it is omitted for brevity. In the next section, we apply the above result to some interesting problems in finance.

3.2 Applications in Finance

In the recent literature on decision making under uncertainty, downside risk aversion has attracted considerable attention. Given a strictly increasing

[7] It is trivial to extend the above result to the case with inequality constraints. Thus it is omitted for brevity.

and concave utility function $u(x)$, three downside risk aversion measures are proposed in the literature, namely the prudence measure $P(x) = -\frac{u'''(x)}{u''(x)}$, $D(x) = \frac{u'''(x)}{u'(x)}$, and the Schwarzian derivative $S(x) = \frac{u'''(x)}{u'(x)} - \frac{3}{2}R^2(x)$, where $R(x) = -\frac{u''(x)}{u'(x)}$ is the Arrow-Pratt risk aversion measure.[8] The case of downside risk aversion is more complicated than risk aversion, and we show that the result obtained in the last subsection is very useful when we characterize utility functions regarding downside risk aversion.

3.2.1 Downside Risk Aversion and Background Risk

We now investigate the effect of background risk on downside risk aversion. Given a utility function $u(x)$, we ask the question under what conditions an independent fair background risk $\tilde{\epsilon}$ always increases the Schwarzian derivative $S(x) = \frac{u'''(x)}{u'(x)} - \frac{3}{2}(\frac{u''(x)}{u'(x)})^2$.[9] Let $\hat{u}(x) = Eu(x + \tilde{\epsilon})$, which is often called the derived utility function. The Schwarzian derivative of the derived utility function is $\hat{S}(x) = \frac{Eu'''(x+\tilde{\epsilon})}{Eu'(x+\tilde{\epsilon})} - \frac{3}{2}(\frac{Eu''(x+\tilde{\epsilon})}{Eu'(x+\tilde{\epsilon})})^2$. Thus our problem is to characterize utility functions which satisfy the following condition

$$E\tilde{\epsilon} = 0 \Rightarrow \frac{Eu'''(x + \tilde{\epsilon})}{Eu'(x + \tilde{\epsilon})} - \frac{3}{2}(\frac{Eu''(x + \tilde{\epsilon})}{Eu'(x + \tilde{\epsilon})})^2 \geq S(x).$$

The above condition is equivalent to

$$E\tilde{\epsilon} = 0 \Rightarrow S(x)(Eu'(x + \tilde{\epsilon}))^2 + \frac{3}{2}(Eu''(x + \tilde{\epsilon}))^2 \leq Eu'''(x + \tilde{\epsilon})Eu'(x + \tilde{\epsilon}).$$

If for all x, $S(x) \geq 0$, Proposition 3 is applicable to the above problem, where $\nu = 1$. Hence we immediately obtain the following result.

[8] See, for example, Kimball (1990), Keenan and Snow (2002, 2009, 2010), Modica and Scarsini (2005), and Crainich and Eeckhoudt (2008).

[9] The other two downside risk aversion measures of the derived utility function $\hat{u}(x)$, $\hat{P}(x) = -\frac{Eu'''(x+\tilde{\epsilon})}{Eu''(x+\tilde{\epsilon})}$ and $\hat{D}(x) = \frac{Eu'''(x+\tilde{\epsilon})}{Eu'(x+\tilde{\epsilon})}$, are linear fractionals and can be easily dealt with by applying Gollier and Kimball's (1996) Diffidence theorem.

Proposition 4 *Assume for all x, $S(x) = \frac{u'''(x)}{u'(x)} - \frac{3}{2}(\frac{u''(x)}{u'(x)})^2 \geq 0$. Then, the following two conditions are equivalent.*

- *$S(x)$ is increased by all fair risk $\tilde{\epsilon}$.*
- *$S(x)$ is increased by all binary fair risk $\tilde{\epsilon}$, i.e., for all $z \geq 0$ and $y \geq 0$*

$$S(x)[(u'(x-z)y + u'(x+y)z)^2 + \frac{3}{2}(u''(x-z)y + u''(x+y)z)^2] \leq [u'(x-z)y + u'(x+y)z][u'''(x-z)y + u'''(x+y)z]. \quad (14)$$

It is straightforward to extend the above result to an unfair background risk. Moreover, from the above result we can derive a sufficient condition (for both fair and unfair risk): $S' \leq 0$ and $S'' \geq 0$. The proof is omitted for brevity.

3.2.2 Cautiousness and Background Risk

Cautiousness is defined as the ratio of prudence to risk aversion minus one, which can be seen as a measure of downside risk aversion relative to risk aversion.[10] Given a utility function $u(x)$, assume $u'(x) < 0$, $u''(x) \neq 0$, and $u'''(x) \geq 0$. Its cautiousness can be written as $C(x) = \frac{u'(x)u'''(x)}{u''^2(x)} - 1$. This preference measure plays an important role in the theory of risk sharing.[11] Hara et al. (2011) investigate the effect of background risk on cautiousness. Their main result is a sufficient condition under which any fair background risk will increase cautiousness. Here we will use Proposition 3 to derive a necessary and sufficient condition. We have the following result.

Proposition 5 *The following two conditions are equivalent.*

[10]See, for example, Huang and Stapleton (2010).

[11]See Leland (1980), Hara et al. (2007), and Huang and Stapleton (2010).

- *For all fair risk* $\tilde{\epsilon}$

$$\frac{Eu'(x+\tilde{\epsilon})Eu'''(x+\tilde{\epsilon})}{[Eu''(x+\tilde{\epsilon})]^2} \geq \frac{u'(x)u'''(x)}{u''^2(x)}.$$

- *For all binary fair risk the above inequality is true, i.e., for all* $z \geq 0$ *and* $y \geq 0$

$$\begin{aligned}&[u'(x-z)y+u'(x+y)z][u'''(x-z)y+u'''(x+y)z]\\ &\geq \frac{u'(x)u'''(x)}{u''^2(x)}[u''(x-z)y+u''(x+y)z]^2. \qquad (15)\end{aligned}$$

Proof: Note that the inequality in the first condition is equivalent to

$$\frac{u'(x)u'''(x)}{u''^2(x)}[Eu''(x+\tilde{\epsilon})]^2 \leq Eu'(x+\tilde{\epsilon})Eu'''(x+\tilde{\epsilon}).$$

Thus the problem is to characterize utility functions ($u(x)$) which satisfy the following condition

$$E\tilde{\epsilon}=0 \Rightarrow \frac{u'(x)u'''(x)}{u''^2(x)}[Eu''(x+\tilde{\epsilon})]^2 \leq Eu'(x+\tilde{\epsilon})Eu'''(x+\tilde{\epsilon}).$$

Thus Proposition 3 is applicable to this case. Now applying Proposition 3, we are clear that the inequality in the first condition is true if and only if it is true for all binary fair risk. Q.E.D.

It is straightforward to extend the above result to an unfair background risk. Moreover, from the above result we can derive a sufficient condition (for both fair and unfair risk): $C(x) \geq -\frac{1}{2}$, $C'(x) \leq 0$, and $C''(x) \geq 0$. The proof is omitted for brevity.

3.2.3 Downside Risk Aversion and Wealth Inequality

We now investigate the effect of wealth inequality on the three downside risk aversion measures. We use a standard one-period Arrow-Debreu economy where all agents have the same beliefs and the same utility function $u(x)$ while they have different wealth. Assume that $u(x)$ is strictly increasing and concave and has a positive third derivative. Let different classes of agents be indexed by $\theta \in (0, \infty)$. The distribution of θ is characterized by a distribution function $F(\theta)$. Let z be the future wealth per capita. Let agent θ's sharing rule be $x(z, \theta)$. We have $E_\theta x(z, \theta) = z$, where E_θ denotes the expectation under the distribution function $F(\theta)$. It is well known that in this setup, an agent's sharing rule satisfies the following condition

$$\frac{\partial x(z, \theta)}{\partial z} = T(x(z, \theta))/T_e(z), \tag{16}$$

where $T_e(z)$ denotes the representative agent's risk tolerance and $T(x(z, \theta))$ is agent θ's risk tolerance along her optimal payoff function.[12] Moreover, the representative agent's risk tolerance is the mean of agents' risk tolerance:

$$T_e(z) = E_\theta T(x(z, \theta)). \tag{17}$$

We call an economy a two-class economy if θ has a two-point distribution. We now present the following result.

Proposition 6 *Assume that all agents have the same beliefs and the same utility function $u(x)$ while they have different wealth. The following three statements are true.*

1. *Prudence $P(x) = -\frac{u'''(x)}{u''(x)}$ is always increased by wealth inequality in all economies if and only if it is so in all two-class economies.*

2. *$D(x) = \frac{u'''(x)}{u'(x)}$ is increased by wealth inequality in all economies if and only if it is so in all two-class economies.*

[12] See, for example, Gollier (2001).

3. *Assume that for all x, the Schwarzian derivative $S(x) = \frac{u'''(x)}{u'(x)} - \frac{3}{2}(\frac{u''(x)}{u'(x)})^2 \geq 0$. Then, $S(x)$ is increased by wealth inequality in all economies if and only if it is so in all two-class economies.*

Proof: We first prove Statement 1. Differentiating both sides of (17) w.r.t z and noting that cautiousness is equal to the rate of change in risk tolerance, we obtain

$$C_e(z) = E_\theta[C(x(z,\theta))\frac{\partial x(z,\theta)}{\partial z}],$$

where $C_e(z)$ denotes the representative agent's cautiousness and $C(x(z,\theta))$ is agent θ's cautiousness along her optimal payoff function. As cautiousness is equal to the product of prudence and risk tolerance minus one, from the above equation we have

$$P_e(z)T_e(z) = E_\theta[P(x(z,\theta))T(x(z,\theta))\frac{\partial x(z,\theta)}{\partial z}],$$

where $P_e(z)$ denotes the representative agent's prudence respectively while $P(x(z,\theta))$ is agent θ's prudence. From the above equation and (16) we obtain

$$P_e(z) = \frac{E_\theta[P(x(z,\theta))T^2(x(z,\theta))]}{[E_\theta T(x(z,\theta))]^2}. \tag{18}$$

Thus in the first statement, the problem is to characterize utility functions which satisfy the following condition

$$E_\theta x(z,\theta) = z \Rightarrow P_e(z) = \frac{E_\theta[P(x(z,\theta))T^2(x(z,\theta))]}{[E_\theta T(x(z,\theta))]^2} \geq P(z).$$

This condition is equivalent to

$$E_\theta x(z,\theta) = z \Rightarrow E_\theta[P(x(z,\theta))T^2(x(z,\theta))] \geq P(z)[E_\theta T(x(z,\theta))]^2.$$

It is clear that Proposition 3 is applicable to this case where $\nu = 1$. Thus applying Proposition 3, we conclude that $P_e(z) \geq P(z)$ for all wealth allocations if and only if it is valid for all two-class economies where θ follows a two-point distribution. This proves the first result.

To prove the second result, noting that $D(x) = \frac{u'''(x)}{u'(x)} = P(x)R(x)$ and $T(x) = \frac{1}{R(x)}$, from (18), we have

$$D_e(z) = P_e(z)R_e(z) = \frac{E_\theta[P(x(z,\theta))R(x(z,\theta))T^3(x(z,\theta))]}{[E_\theta T(x(z,\theta))]^3}. \qquad (19)$$

Hence we obtain $D_e(z) = \frac{E_\theta[D(x(z,\theta))T^3(x(z,\theta))]}{[E_\theta T(x(z,\theta))]^3}$. The rest of the proof is very similar to the proof of the first statement; thus it is omitted for brevity.

To prove the third result, noting that the Schwarzian derivative $S(x) = \frac{u'''(x)}{u'(x)} - \frac{3}{2}(\frac{u''(x)}{u'(x)})^2 = P(x)R(x) - \frac{3}{2}R^2(x)$, from (17) and (19), $S_e(z) = P_e(z)R_e(z) - \frac{3}{2}R_e^2(x)$ is equal to

$$\frac{E_\theta[P(x(z,\theta))R(x(z,\theta))T^3(x(z,\theta))]}{[E_\theta T(x(z,\theta))]^3} - \frac{\frac{3}{2}}{[E_\theta T(x(z,\theta))]^2}.$$

Rewrite it as

$$\frac{E_\theta[(P(x(z,\theta))R(x(z,\theta)) - \frac{3}{2}R^2(x(z,\theta)))T^3(x(z,\theta))]}{[E_\theta T(x(z,\theta))]^3}. \qquad (20)$$

Henec we obtain $S_e(z) = \frac{E_\theta[S(x(z,\theta))T^3(x(z,\theta))]}{[E_\theta T(x(z,\theta))]^3}$. Again, the rest of the proof is very similar to the proof of the first statement, and it is omitted for brevity. Q.E.D.

From the above result, we can derive a sufficient condition for prudence to be always increased by wealth inequality: $(\frac{1}{P(x)})'' \leq 0$; a sufficient condition for $\frac{u'''(x)}{u'(x)}$ to be always increased by wealth inequality: $(\sqrt{\frac{u'(x)}{u'''(x)}})'' \leq 0$; a sufficient condition for the Schwarzian derivative $S(x)$ to be always increased by wealth inequality: $S(x) > 0$ and $(\sqrt{\frac{1}{S(x)}})'' \leq 0$. The proofs are omitted for brevity.

3.2.4 Other Examples

There are many other cases where the results obtained in the last secion are useful. We give the following examples.

1. Minising variance.

Given a random variable $\tilde{\epsilon}$, the variance of $f(\tilde{\epsilon})$, a function of $\tilde{\epsilon}$, has the form $Ef^2(\tilde{\epsilon}) - (Ef(\tilde{\epsilon}))^2$. Thus Proposition 3 is applicable to the problem of minimizing the variance of $f(\tilde{\epsilon})$ subject to linear constraints.

2. Maximizing Sharpe ratio.

Given an asset's future price S, the Sharpe ratio of a derivative with payoff $c(S)$ has the form $\frac{Ec(S)-r}{Ec^2(S)-(Ec(S))^2} = \frac{E(c(S)-r)}{Ec^2(S)-(Ec(S))^2}$, where r is the risk-free interest rate. Thus Proposition 3 is applicable to the problem of maximizing the Sharpe ratio of $c(S)$ subject to linear constraints (assuming $Ec(S) \geq r$).

3. Minimizing skewness and kurtosis.

Given a fair risk $\tilde{\epsilon}$, its skewness and kurtosis are equal to $\frac{E\tilde{\epsilon}^3}{(E\tilde{\epsilon}^2)^{\frac{3}{2}}}$ and $\frac{E\tilde{\epsilon}^4}{(E\tilde{\epsilon}^2)^2}$. Thus Proposition 3 is applicable to the problem of minimizing the skewness (if $E\tilde{\epsilon}^3 \geq 0$) and kurtosis subject to linear constraints.

4 Conclusion

In this paper, we have identified a class of linearly constrained nonlinear optimization problems with corner point optimal solutions. These include some special polynomial fractional optimization problems which have an objective function equal to the product of some power functions of positive linear functionals subtracting the sum of some power functions of positive linear functionals, divided by the sum of some power functions of positive linear functionals. The powers are required to be all positive integers, and the aggregate power of the product is required to be no larger than the lowest power of the two sums. This result has applications to many optimization problems under uncertainty, particulary in finance.

Appendix A Proof of Lemma 1

To show that if for all $x^1, ..., x^{\bar{s}} \in A$, (2) is true then, for all $x \in A$, (1) is true, given any $x \in A$, let $x^1 = ... = x^{\bar{s}} = x$ in (2); we immediately obtain (1). Thus we need only show that if for all $x \in A$, (1) is true then, for all $x^1, ..., x^{\bar{s}} \in A$, (2) is true.

Given any $x_1, ..., x_{\bar{s}} \in A$, where $\bar{s} = \max_i\{s_i\}$, let $\{j_1, ..., j_t\}$ be a t-combination of $\{1, ..., \bar{s}\}$, where $t = \sum_{i=1}^m t_i$. There are $C_t^{\bar{s}}$ such combinations. Denote the summation in the right hand side of (2) by Ξ. The sum Ξ can be rewritten as

$$\sum_{\binom{\{1,...,\bar{s}\}}{j_1,...,j_t}} \; \sum_{\genfrac{[}{]}{0pt}{}{\{j_1,...,j_t\}}{(j_1^1,...,j_{t_1}^1),...,(j_1^m,...,j_{t_m}^m)}} \Pi_{i=1}^m \Pi_{k=j_1^i}^{j_{t_i}^i} g_i(x^k),$$

where $\sum_{i=1}^m t_i = t$ and $\genfrac{[}{]}{0pt}{}{\{j_1,...,j_t\}}{(j_1^1,...,j_{t_1}^1),...,(j_1^m,...,j_{t_m}^m)}$ denotes the set of all t-permutations of $\{j_1, ..., j_t\}$, where the order of the elements in each pair of the round brackets does not matter. Note that there are $\frac{t!}{t_1!...t_m!}$ such t-permutations, i.e., there are $\frac{t!}{t_1!...t_m!}$ items in the sum

$$\sum_{\genfrac{[}{]}{0pt}{}{\{j_1,...,j_t\}}{(j_1^1,...,j_{t_1}^1),...,(j_1^m,...,j_{t_m}^m)}} \Pi_{i=1}^m \Pi_{k=j_1^i}^{j_{t_i}^i} g_i(x^k).$$

Because the arithmetic mean of these $\frac{t!}{t_1!...t_m!}$ items is smaller than their geometric mean, we obtain

$$\Xi \geq \sum_{\binom{\{1,...,\bar{s}\}}{j_1,...,j_t}} \frac{t!}{t_1!...t_m!} [\Pi_{\genfrac{[}{]}{0pt}{}{\{j_1,...,j_t\}}{(j_1^1,...,j_{t_1}^1),...,(j_1^m,...,j_{t_m}^m)}} \Pi_{i=1}^m \Pi_{j=j_1^i}^{j_{t_i}^i} g_i(x^j)]^{\frac{t_1!...t_m!}{t!}}.$$

Let $\left[\begin{array}{c}\{j_1,...,j_t\}\\ j_1^1,...,j_{t_1}^1,...,j_1^m,...,j_{t_m}^m\end{array}\right]$ denote the set of all t-permutations of $\{j_1,...,j_t\}$. We write it as $\left[\begin{array}{c}\{j_1,...,j_t\}\\ [j_1^1,...,j_{t_1}^1],...,[j_1^m,...,j_{t_m}^m]\end{array}\right]$ to stress that now the order of the elements in each pair of the square brackets in the second row does matter. It is straightforward that every t-permutation in the set $\left[\begin{array}{c}\{j_1,...,j_t\}\\ (j_1^1,...,j_{t_1}^1),...,(j_1^m,...,j_{t_m}^m)\end{array}\right]$ is repeated $t_1!t_2!...t_m!$ times in the set $\left[\begin{array}{c}\{j_1,...,j_t\}\\ [j_1^1,...,j_{t_1}^1],...,[j_1^m,...,j_{t_m}^m]\end{array}\right]$. Hence from the preceding inequality, we have

$$\Xi \geq \sum_{\left(\begin{array}{c}\{1,...,\bar{s}\}\\ j_1,...,j_t\end{array}\right)} \frac{t!}{t_1!...t_m!}\left[\Pi_{\left[\begin{array}{c}\{j_1,...,j_t\}\\ [j_1^1,...,j_{t_1}^1],...,[j_1^m,...,j_{t_m}^m]\end{array}\right]} \Pi_{i=1}^m \Pi_{j=j_1^i}^{j_{t_i}^i} g_i(x^j)\right]^{\frac{1}{t!}}.$$

Now let $A_{k,i}$ denote the set of t-permutations of $\{j_1,...,j_t\}$ with j_k in the ith pair of square brackets. It is clear that the number of items in $A_{k,i}$ is equal to $t_i(t-1)!$ and that $\left[\begin{array}{c}\{j_1,...,j_t\}\\ [j_1^1,...,j_{t_1}^1],...,[j_1^m,...,j_{t_m}^m]\end{array}\right] = \cup_{k=1}^t \cup_{i=1}^m A_{k,i}$. It follows that in the product $\Pi_{\left[\begin{array}{c}\{j_1,...,j_t\}\\ [j_1^1,...,j_{t_1}^1],...,[j_1^m,...,j_{t_m}^m]\end{array}\right]} \Pi_{i=1}^m \Pi_{j=j_1^i}^{j_{t_i}^i} g_i(x^j)$, for each $k \in \{1,...,t\}$ and each $i \in \{1,...,l\}$, the factor $g_i(x^{j_k})$ appears $t_i(t-1)!$ times. This implies that

$$\Pi_{\left[\begin{array}{c}\{j_1,...,j_t\}\\ [j_1^1,...,j_{t_1}^1],...,[j_1^m,...,j_{t_m}^m]\end{array}\right]} \Pi_{i=1}^m \Pi_{j=j_1^i}^{j_{t_i}^i} g_i(x^j) = \Pi_{k=1}^t \Pi_{i=1}^m [g_i(x^{j_k})]^{t_i(t-1)!}.$$

Substituting this into the preceding inequality, we obtain

$$\Xi \geq \sum_{\binom{\{1,...,\bar{s}\}}{j_1,...,j_t}} \frac{t!}{t_1!...t_m!}[\Pi_{k=1}^t \Pi_{i=1}^m [g_i(x^{j_k})]^{t_i}]^{\frac{1}{t}}.$$

This together with (1) implies that

$$\Xi \geq \sum_{\binom{\{1,...,\bar{s}\}}{j_1,...,j_t}} \frac{t!}{t_1!...t_m!}\Pi_{k=1}^t [\sum_{i=1}^n [f_i(x^{j_k})]^{s_i}]^{\frac{1}{t}}.$$

Applying the generalized Hölder's inequality, from the above inequality, we obtain that[13]

$$\Xi \geq \sum_{\binom{\{1,...,\bar{s}\}}{j_1,...,j_t}} \frac{t!}{t_1!...t_m!}\sum_{i=1}^n \Pi_{k=1}^t [f_i(x^{j_k})]^{\frac{s_i}{t}}.$$

Rewrite it as

$$\Xi \geq \frac{t!}{t_1!...t_m!}\sum_{i=1}^n \Delta_i, \tag{21}$$

where

$$\Delta_i = \sum_{\binom{\{1,...,\bar{s}\}}{j_1,...,j_t}} \Pi_{k=1}^t [f_i(x^{j_k})]^{\frac{s_i}{t}}.$$

[13]From the generalized Hölder's inequality, we have $\Pi_{j=1}^t ||f_j||_{p_j} \geq ||\Pi_{j=1}^n f_j||_r$, where $r \in (0,\infty)$, $p_1, ..., p_t \in (0, \infty]$, and $\sum_{i=1}^t \frac{1}{p_i} = \frac{1}{r}$. Following convention, $||f_j||_{p_j} = \int_S |f_j|^{p_j} d\mu$. In the special case where $r = 1$, $p_1 = ... = p_t = t$, when $S = \{1, ..., n\}$, applying the counting measure, we obtain $\Pi_{j=1}^t (\sum_{i=1}^n a_{ij}^t)^{\frac{1}{t}} \geq \sum_{i=1}^n \Pi_{j=1}^t a_{ij}$, where for all i and j, $a_{ij} \geq 0$.

For every t-combination $(j_1, ..., j_t)$ of $\{1, ..., \bar{s}\}$ we do the following operation. We choose $(s_i - t)$ numbers $j_{t+1}, ..., j_{s_i}$ from $\{1, ..., \bar{s}\} - \{j_1, ..., j_t\}$. There are $C_{s_i-t}^{\bar{s}-t}$ different such choices in total. Now consider all t-combinations of $(j_1, ..., j_{s_i})$. There are $C_t^{s_i}$ such t-combinations in total. For every such a t-combination $(k_1, ..., k_t)$ of $\{j_1, ..., j_{s_i}\}$, we add a term $[\Pi_{j=k_1}^{k_t}[f_i(x^j)]^{s_i}]^{\frac{1}{t}}$ to Δ_i. After the operation for all t-combinations of $\{1, ..., \bar{s}\}$, Δ_i becomes Δ_i' as follows:

$$\Delta_i' = \sum_{\left(\substack{\{1,...,\bar{s}\} \\ j_1,...,j_t}\right)} \sum_{\left(\substack{\{1,...,\bar{s}\}-\{j_1,...,j_t\} \\ j_{t+1},...,j_{s_i}}\right)} \sum_{\left(\substack{\{j_1,...,j_{s_i}\} \\ k_1,...,k_t}\right)} \Pi_{j=1}^{t}[f_i(x^{k_j})]^{\frac{s_i}{t}}.$$

The total number of terms in Δ_i' is $C_{s_i-t}^{\bar{s}-t} C_t^{s_i}$ times the original number of terms in Δ_i while every added term in Δ_i' is actually one of the original terms in Δ_i. Note as the operation is symmetric w.r.t all the original terms, we can conclude that each original term is repeated by $C_{s_i-t}^{\bar{s}-t} C_t^{s_i}$ times after the operation.

In other words, if we allow all original terms to be repeated $C_{s_i-t}^{\bar{s}-t} C_t^{s_i}$ times and put them in a set, then this set of terms can be divided into groups of $C_t^{s_i}$ terms, and in every group all terms consist of the same s_i elements, say $\{[f_i(x^{j_1})]^{\frac{s_i}{t}}, ..., [f_i(x^{j_{s_i}})]^{\frac{s_i}{t}}\}$, chosen from $\{[f_i(x^1)]^{\frac{s_i}{t}}, ..., [f_i(x^{\bar{s}})]^{\frac{s_i}{t}}\}$, while each of the s_i elements appears exactly $C_{t-1}^{s_i-1}$ times in the group. Such a group of terms is said to be generated by the s_i elements. It is obvious that the total number of such groups is

$$\frac{C_{s_i-t}^{\bar{s}-t} C_t^{s_i}}{C_t^{s_i}} C_t^{\bar{s}} = C_{s_i-t}^{\bar{s}-t} C_t^{\bar{s}}.$$

For every such a group, there is a s_i-combination $(j_1, ..., j_{s_i})$ of $\{1, ..., \bar{s}\}$ which corresponds to the s_i elements $\{[f_i(x^{j_1})]^{\frac{s_i}{t}}, ..., [f_i(x^{j_{s_i}})]^{\frac{s_i}{t}}\}$ which generate the group. Let Θ be the set of all these s_i-combinations of $\{1, ..., \bar{s}\}$

corresponding to each of the groups. As there are $C_{s_i-t}^{\bar{s}-t}C_t^{\bar{s}}$ such groups, there must be $C_{s_i-t}^{\bar{s}-t}C_t^{\bar{s}}$ such s_i-combinations in Θ. Now for every such a group generated by s_i elements $\{[f_i(x^{j_1})]^{\frac{s_i}{t}}, ..., [f_i(x^{j_{s_i}})]^{\frac{s_i}{t}}\}$, we apply the result that the geometric mean of all the terms in the group is smaller than their arithmetic mean. Noting that there are $C_t^{s_i}$ terms in the group and each of the s_i elements appears exactly $C_{t-1}^{s_i-1}$ times in the group, we obtain that the sum of the group is larger than

$$C_t^{s_i}\Pi_{k=1}^{s_i}\{[f_i(x^{j_k})]^{\frac{s_i}{t}C_{t-1}^{s_i-1}}\}^{\frac{1}{C_t^{s_i}}} = C_t^{s_i}\Pi_{k=1}^{s_i}f_i(x^{j_k}).$$

Thus the sum of all the $C_{s_i-t}^{\bar{s}-t}C_t^{\bar{s}}$ groups is larger than $C_t^{s_i}\sum_{(j_1,...,j_{s_i})\in\Theta}\Pi_{k=1}^{s_i}f_i(x^{j_k})$, where Θ is the set of s_i-combinations of $\{1, ..., \bar{s}\}$ resulted from the above operation. It is obvious that $\binom{\{1, ..., \bar{s}\}}{j_1, ..., j_t} \subset \Theta$. Moreover, as the operation is symetric w.r.t the $\bar{s}$ elements $\{1, ..., \bar{s}\}$, each s_i-combination of $\{1, ..., \bar{s}\}$ must be repeated by the same times. Furthermore, as there are $C_{s_i-t}^{\bar{s}-t}C_t^{\bar{s}}$ elements in Θ while there are in total $C_{s_i}^{\bar{s}}$ different s_i-combinations of $\{1, ..., \bar{s}\}$, each s_i-combination of $\{1, ..., \bar{s}\}$ must be repeated by $\frac{C_{s_i-t}^{\bar{s}-t}C_t^{\bar{s}}}{C_{s_i}^{\bar{s}}}$ times in Θ. It follows that

$$\Delta_i' \geq \frac{C_{s_i-t}^{\bar{s}-t}C_t^{\bar{s}}}{C_{s_i}^{\bar{s}}}C_t^{s_i}\sum_{\binom{\{1, ..., \bar{s}\}}{j_1, ..., j_{s_i}}}\Pi_{k=1}^{s_i}f_i(x^{j_k}).$$

However, as Δ_i' is obtained by repeating every term in Δ_i by $C_{s_i-t}^{\bar{s}-t}C_t^{s_i}$ times, we obtain

$$\Delta_i = \frac{\Delta_i'}{C_{s_i-t}^{\bar{s}-t}C_t^{s_i}} \geq \frac{C_t^{\bar{s}}}{C_{s_i}^{\bar{s}}}\sum_{\binom{\{1, ..., \bar{s}\}}{j_1, ..., j_{s_i}}}\Pi_{k=1}^{s_i}f_i(x^{j_k}).$$

Substituting the above result into (21), we have

$$\Xi \geq \frac{t!}{t_1!...t_m!} \sum_{i=1}^{n} \frac{C_t^{\bar{s}}}{C_{s_i}^{\bar{s}}} \sum_{\binom{\{1,...,\bar{s}\}}{j_1,...,j_{s_i}}} \Pi_{k=1}^{s_i} f_i(x^{j_k}).$$

Substituting this into the right hand side of (2), we immediately obtain that (2) is true for $x_1, ..., x_{\bar{s}} \in A$. As $x_1, ..., x_{\bar{s}} \in A$ are arbitrarily given, we conclude that (2) is true for all $x_1, ..., x_{\bar{s}} \in A$. Q.E.D.

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