Cautiousness in the Small and in the Large

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Abstract

We characterize cautiousness, a downside risk aversion measure, using a simple portfolio problem in which agents invest in a stock, a risk-free bond, and an option on the stock. We present two different characterizations by answering the following two questions respectively: who buys the option? who buys more options per share of the stock? Our characterizations use a strong notion of an increase in skewness defined by Van Zwet (1964).

Keywords: cautiousness, downside risk aversion, demand for options, convex transformation of random variables, strong increases in skewness.

JEL codes: D81, G11.

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1 Introduction

Given a Von Neuman-Morgenstern utility of wealth function $u(w)$, an agent is downside risk averse if $u'''(w) > 0$. The intensity of downside risk aversion $P(w) = -u'''(w)/u''(w)$ or prudence, was introduced by Kimball (1990) and shown to determine the demand for precautionary saving. Chiu (2000) shows that $P(w)$ also determines the demand for self protection. Keenan and Snow (2010) discuss sufficient conditions for greater prudence to indicate greater downside risk aversion. An alternative measure for the intensity of downside risk aversion is studied by Modica and Scarsini (2005) and Crainich and Eekhoudt (2008). When risk is small, Modica and Scarsini show that $D(w) = u'''(w)/u'(w)$ measures premium for skewness, and Crainich and Eekhoudt show that it measures the pain associated with an increase in downside risk in monetary terms. Keenan and Snow (2002, 2009, 2012) suggest a further measure, the Schwarzian derivative $S(w) = D(w) - \frac{3}{2}u''^2(w)/u'^2(w)$, and characterize downside risk aversion by considering changes in risk that induce third-order mean-and-variance-preserving spreads in the utility distribution. Recently, Chiu (2010) points out that the current literature on skewness preference treats skewness largely as synonymous with the (unstandardized) third central moment, which may have caused the difficulty in getting comparative statics of downside risk aversion in the cases where risk is large, and raises the issue of skewness comparability. A strong notion of an increase in skewness (hereafter a strong increase in skewness), which has so-called strong skewness comparability, is given by Van Zwet (1964).[1] Van Zwet defines that a cumulative distribution function $F(x)$ is more skewed to

[1]See Chiu (2005) for an explanation of strong skewness comparability.

the right than $G(X)$ if $R(x) = F^{-1}(G(x))$ is convex, which results in a subset of increases in skewness. In this paper our characterizations of downside risk aversion use this notion of a strong increase in skewness.

We define the degree of downside risk aversion as $C(w) = (1/R(w))'$, where $R(w) = -u''(w)/u'(w)$ is the Pratt-Arrow measure of risk aversion. This measure of downside risk aversion was first introduced by Wilson (1968) who termed it cautiousness. In this paper we show that cautiousness characterizes preferences for a strong increase in skewness under Van Zwet's definition.

We consider the simplest possible scenario, where decision makers can buy or sell a single stock, a risk-free bond, and an option on the stock. We show that cautiousness determines the optimal position in the option in this simple portfolio problem. An option's payoff is a convex function of the underlying stock price; thus increasing positions in the option increases the convexity of a portfolio and results in a strong increase in skewness under Van Zwet's definition.[2] Using this simple portfolio problem, we present two different ways to characterize cautiousness as a measure of downside risk aversion by answering the following two questions respectively: who buys the option? who buys more options per share of the stock?

Our results here are related to the previous work of Leland (1980), Brennan and Solanki (1981), and Hara, Huang and Kuzmics (hereafter HHK) (2007). Leland shows that an agent with higher cautiousness is more likely to have a convex optimal payoff function which he regards as a proxy of port-

[2] For an explanation of an increase in skewness caused by a convex transformation of a random variable, see, for example, Van Zwet (1964) or Chiu (2010).

folio insurance. Brennan and Solanki obtain a similar result in a lognormal model where a risk-neutral valuation relationship holds for the valuation of options. In both studies results are obtained by comparing an agent's cautiousness with that of a representative agent whose characteristics are exogenously and arbitrarily defined. This undermines the rigor of their approach. HHK (2007) try to remedy this problem by endogenizing the representative agent; however, they find that in general this approach does not work.[3]

In a related paper on the effect of background risk, Franke, Stapleton and Subrahmanyam (hereafter FSS) (1998) also show that a convex payoff is optimal in a model where background risk increases the cautiousness of an investor with a HARA class utility function. HHK (2011) extend the above discussion about the effect of background risk on cautiousness to a more general class of utility functions.

Cautiousness has also been used in analyzing other problems. For example, Gollier (2001) discusses how an investor's cautiousness is related to the local convexity of her consumption rule.[4] In an earlier related study, Carroll and Kimball (1996) investigate the effect of uncertainty on the curvature of investors' consumption rules by examining its effect on their cautiousness. They show that if investors have HARA class utility functions then uncertainty will increase their cautiousness, which leads to concave optimal consumption rules. HHK (2007) show how heterogeneity in cautiousness af-

[3]They conclude that the results of Leland (1980) and Brennan and Solanki (1981) "are valid in a two-consumer economy, but do not generalize to an economy with a large number of consumers with diverse levels of relative risk aversion". For more explanations about this, see the discussions of Theorem 18 in HHK (2007).

[4]See Gollier (2001) page 207, Proposition 52.

fects consumers' portfolio strategies and the representative consumer's risk preferences. Gollier (2007) finds that cautiousness helps to explain the aggregation of heterogeneous beliefs. Gollier (2008) further shows that cautiousness plays an important role in understanding saving and portfolio choices with predictable changes in asset returns.

The structure of the remaining paper is as follows. In Section 2, we introduce the concept of being more cautious and the simple portfolio problem which underlies our analysis. In Section 3, we analyze the simple case where only small strong increases (decreases) in skewness are considered. In sections 4 and 5, we characterize cautiousness in the general case by answering the following two different questions respectively: (i) who buys the option? (ii) who buys more options per share of the stock? In Section 6, we give some numerical examples to illustrate the main results. The final section concludes the paper.

2 The Model

Assume there is a risk-free bond and a stock traded in the market. The risk-free interest rate is denoted by r, and the stock prices at time 0 and 1 are denoted by S_0 and S respectively. We assume that the distribution of the stock price S is continuous and its support, denoted by $I = [\underline{s}, \bar{s}]$, is a bounded subinterval of $[0, +\infty)$.[5] Although we assume that the stock price follows a continuous distribution, the results obtained in this paper can easily

[5]The boundedness of the support I is not required for Statement 1 in Theorem 1 to imply Statement 2, which can clearly be seen from the proof of the theorem.

be extended to the discrete case.

Assume there is an option written on the stock available in the market, which matures at time 1.[6] To avoid the trivial case where the option degenerates to a portfolio of the stock and the riskless bond, we assume that its strike price K is an interior point of the support, i.e., $K \in (\underline{s}, \bar{s})$. Denote the time 0 price and the time 1 payoff of the option by a_0 and $a(S)$ respectively.[7]

Consider an investor i who is a rational utility-maximizer and a price-taker.[8] Investor i's risk preferences are represented by a utility function $u_i(x)$. At time 0 she has initial wealth w_{0i}. Assume that at time 0 she buys x_i shares of the stock and y_i units of the option, and invests the rest of her wealth $(w_{0i} - x_i S_0 - y_i a_0)$ in the money market. Denote investor i's wealth at time 1 by $w_i(S; x_i, y_i)$. We have

$$w_i(S; x_i, y_i) = (w_{0i} - x_i S_0 - y_i a_0)(1 + r) + x_i S + y_i a(S). \tag{1}$$

For brevity we will often write $w_i(S; x_i, y_i)$ simply as $w_i(S)$. Note that, as $a(S)$ is continuous and piecewise infinitely differentiable, $w_i(S)$ is also continuous and piecewise infinitely differentiable.

Investor i maximizes the expected utility of her time 1 wealth $w_i(S)$, that

[6] In case there are more than one option traded in the market, it is understood that only one of them is considered in the portfolio problem. This is in line with the approach used by Pratt (1964) and Arrow (1965) who consider only one risky asset in the portfolio characterization of risk aversion.

[7] The interest rate and the current prices of the stock and the option are all exogenous.

[8] We do not assume all investors are rational utility-maximizers or price takers.

is, she chooses x_i, y_i to solve the following problem:[9]

$$\max_{x_i, y_i} Eu_i(w_i(S)). \tag{2}$$

We obtain the first order conditions:

$$E[u_i'(w_i(S))(S - (1+r)S_0)] = 0, \quad \text{and} \quad E[u_i'(w_i(S))(a(S) - (1+r)a_0)] = 0,$$

which can be written as

$$\frac{E[u_i'(w_i(S))S]}{Eu_i'(w_i(S))} = (1+r)S_0, \quad \text{and} \quad \frac{E[u_i'(w_i(S))a(S)]}{Eu_i'(w_i(S))} = (1+r)a_0. \tag{3}$$

We assume that all utility functions are strictly increasing, strictly concave, and three times continuously differentiable. The strict concavity of the utility functions guarantees that a solution to (3) is a unique global maximum.[10]

Before we proceed to analyze the optimal solution, we first introduce some notation. Let $\phi_i(S) = u_i'(w_i(S))/Eu_i'(w_i(S))$. Then (3) can be written as

$$E[\phi_i(S)S] = (1+r)S_0, \quad \text{and} \quad E[\phi_i(S)a(S)] = (1+r)a_0. \tag{4}$$

Thus $\phi_i(S)$ can be regarded as investor i's individual pricing kernel, which she uses to price the stock and the option. As the investor has to take the market prices as given, her individual pricing kernel must price the stock and the option correctly; that is, an individual pricing kernel must be admissible with respect to the stock and the option.

To understand the characteristics of admissible pricing kernels, we may note that, as $w_i(S)$ is continuous and piecewise infinitely differentiable and

[9] We do not assume that all investors have homogeneous beliefs. Although her beliefs are not specified, investor i may not have the same beliefs as the market.

[10] See, for example, Cox and Huang (1991).

$u_i(w)$ is three times differentiable, $\phi_i(S) = u_i'(w_i(S))/Eu_i'(w_i(S))$ is also continuous and piecewise three times differentiable. In each of the two differentiable intervals separated by the option's strike price K, let $\delta_i(S)$ denote the negative derivative of the logarithm of investor i's individual pricing kernel, i.e., $\delta_i(S) = -\phi_i'(S)/\phi_i(S)$. From the definitions of $\phi_i(S)$ we have

$$\delta_i(S) = R_i(w_i(S))w_i'(S), \tag{5}$$

where $R_i(w)$ is investor i's absolute risk aversion. In each interval, as $w_i(S)$ is infinitely differentiable and $R_i(w)$ is twice differentiable, $\delta_i(S)$ is also twice differentiable; thus it is bounded in any bounded subinterval. Define $\delta_i(K) = \lim_{S\to K^+}\delta(S)$; then $\delta_i(S)$ is right continuous at $S = K$. As is well known, a bounded and almost everywhere continuous function is Riemann integrable; hence $\delta_i(S)$ is Riemann integrable. It follows that for any $S, a \in (\underline{s}, \bar{s})$, we have

$$\ln\frac{\phi_i(S)}{\phi_i(a)} = -\int_a^S \delta_i(x)dx. \tag{6}$$

We now finish this section with a lemma which shows a characteristic of admissible pricing kernels. This lemma will be used repeatedly later in the proofs of our main results in this paper.

Lemma 1 *Assume $\phi_i(S)$ and $\phi_j(S)$ are continuous. If they both price the stock correctly then they must cross at least twice unless for all S, $\phi_i(S) = \phi_j(S)$.*

Proof: See Appendix A.

3 Cautiousness in the Small

Cautiousness was first defined by Wilson (1968) based on another risk preference measure, the well-known Pratt-Arrow risk aversion. Given a utility function $u(w)$, Pratt (1964) and Arrow (1965) define the risk aversion measure $R(w) = -u''(w)/u'(w)$. Cautiousness $C(w)$ is defined as the rate of change of the inverse of this function, i.e., $C(w) = (1/R(w))'$.[11] Cautiousness is also closely related to another well-known risk preference measure, the measure of prudence. Prudence is defined by Kimball (1990) as $P(w) = -u'''(w)/u''(w)$. We have

$$(\frac{1}{R(w)})' = -\frac{(\ln R(w))'}{R(w)} = -\frac{(\ln(-u''(w)))' - (\ln u'(w))'}{R(w)} = \frac{P(w)}{R(w)} - 1.$$

Thus cautiousness is equivalent to the ratio of prudence to risk aversion minus one. Now we define a key concept in this paper.

Definition 1 *Investor i is said to be more cautious than investor j if for all w and v, $C_i(w) \geq C_j(v)$, where $C_i(w)$ and $C_j(v)$ are the cautiousness measures of investors i and j respectively.*[12]

The above concept gives an ordering of utility functions in terms of their cautiousness. Since HARA class utility functions have constant cautiousness, they can be ordered perfectly in this way.

We will charaterize cautiousness using the simple portfolio problem in the last section. We first consider the special case where positions in the

[11] Throughout the paper, we use R and C to denote risk aversion and cautiousness respectively.

[12] Throughout the paper, when we say for all w and v, $C_i(w) \geq C_j(v)$, we mean for all w and v in the natural domains of $u_i(w)$ and $u_j(v)$ respectively.

option are small, i.e., only small strong increases (decreases) in skewness are considered. Assume that there is an investor i whose optimal position in the option is zero, i.e., $y_i = 0$ and $w_i(S) = (w_{0i} - x_i S_0)(1 + r) + x_i S$. From (4), this implies that her optimal strategy is obtained when

$$E[\phi_i(S)S] = (1 + r)S_0, \quad \text{and} \quad E[\phi_i(S)a(S)] = (1 + r)a_0,$$

where $\phi_i(S) = \frac{u_i'(w_i(S))}{Eu_i'(w_i(S))}$. Consider another investor j who is strictly less cautious than her, i.e., for all w and v, $C_i(w) > C_j(v)$. Suppose that she does not consider investment in the option, that is, she only considers investment in the riskless bond and the stock. This implies that her optimal strategy is obtained when

$$E[\phi_j(S)S] = (1 + r)S_0,$$

where $\phi_j(S) = \frac{u_j'(w_j(S))}{Eu_j'(w_j(S))}$ and $w_j(S) = (w_{0j} - x_j S_0)(1 + r) + x_j S$. Note that investor j's pricing kernel $\phi_j(S)$ may not price the option correctly, as she did not consider the option in her optimal portfolio construction. We ask the following question: if she adds a small positive position in the option to his optimal portfolio, will this increase her expected utility? According to basic calculus, the answer depends on the sign of $\frac{d}{dy_j} Eu_j(w_j(S; x_j, y_j))|_{y_j=0}$, where $w_j(S; x_j, y_j) = (w_{0j} - x_j S_0 - y_j a_0)(1 + r) + x_j S + y_j a(S)$: if the sign is strictly positive (negative) then the answer is positive (negative). Some simple calculations show that this sign is equal to the sign of $E[\phi_j(S)a(S)] - (1 + r)a_0$. Hence if investor j's individual pricing kernel prices the option strictly higher (lower) than the market then, a small positive position in the option additional to the stock will strictly increase (decrease) her expected utility.

Note we have for $t = i, j$, $\delta_t(S) = -(\ln \phi_t(S))' = R_t(w_t(S))w_t'(S)$ and $(\frac{1}{\delta_i(S)} - \frac{1}{\delta_j(S)})' = C_i(w_i(S)) - C_j(w_j(S)) > 0$; thus $\delta_j(S)$ can cross $\delta_i(S)$ at most once from below. This, together with (6), implies that $\phi_j(S)$ can cross $\phi_i(S)$ at most twice. But according to Lemma 1, as they both price the stock correctly, they must cross at least twice; thus they cross exactly twice, and as $(\frac{1}{\delta_i(S)})' - (\frac{1}{\delta_j(S)})' > 0$, $\phi_j(S) - \phi_i(S)$ is negative at both ends of the support, i.e., there exist two points s_1 and s_2, where $\underline{s} < s_1 < s_2 < \bar{s}$, such that for $S \in (\underline{s}, s_1)$, $\phi_j(S) - \phi_i(S) < 0$; for $S \in (s_1, s_2)$, $\phi_j(S) - \phi_i(S) > 0$; for $S \in (s_2, \bar{s})$, $\phi_j(S) - \phi_i(S) < 0$. Let $L(S) = aS + b$ such that $L(s_1) = a(s_1)$ and $L(s_2) = a(s_2)$.[13] Then as $a(S)$ is convex, we have for $S \in (\underline{s}, s_1)$, $a(S) - L(S) \geq 0$; for $S \in (s_1, s_2)$, $a(S) - L(S) \leq 0$; $S \in (s_2, \bar{s})$, $a(S) - L(S) \geq 0$, and at least one of the three inequalities is strict.

Then we have

$$E[\phi_j(S)a(S)] - (1 + r)a_0 = E[(\phi_j(S) - \phi_i(S))a(S)] \tag{7}$$

$$= E[(\phi_j(S) - \phi_i(S))(a(S) - L(S))] < 0. \tag{8}$$

This implies that a small positive (negative) position in the option additional to the stock will decrease (increase) her expected utility.

As an option's payoff is a convex function of the underlying stock price, adding a positive position in the option transforms the original linear terminal wealth function into a convex function. A convex transformation of a random variable results in a strong increase in skewness under Van Zwet's (1964) definition.[14] In the above example, the prices of the stock and the

[13] $L(S)$ is obtained by connecting the two points $(s_1, a(s_1))$ and $(s_2, a(s_2))$ in the space of stock price and payoff.

[14] See Footnote 2.

option given in the market strike such a balance that, according to investor i's measure of cautiousness, she feels neither the need to buy the option to pursue a strong increase in skewness nor the need to sell the option to pursue a strong decrease in skewness. However, raising (lowering) the measure of cautiousness will upset the balance. An investor with a higher (lower) measure of cautiousness feels that adding a positive position in the option which leads to a strong increase in skewness will increase (decrease) her expected utility.

The above analysis explains the simple case where we only consider small strong increases (decreases) in skewness; however, when large strong increases (decreases) in skewness are considered, the situation is more complicated. In the rest of the paper, we carry out analyses of the general situation and characterize cautiousness.

4 Who Buys the Option?

To characterize the concept of cautiousness, we ask the question how increased cautiousness affects an agent's optimal portfolio strategy. Alternatively, consider the situation where two investors i and j have the same beliefs and face the same portfolio problem; we ask the following question: if investor i is more cautious than investor j, how is her optimal portfolio strategy compared with that of investor j?[15] Comparisons of optimal portfolio strategies can be done in different ways, which will lead to different

[15] It is obvious that an answer to the second question is also an answer to the first. Thus we need only present our results as answers to the second question.

characterizations of cautiousness. In this section we focus on the sign of the position in the option in an optimal portfolio strategy. We now present our first main result.

Theorem 1 *The following two statements are equivalent.*

1. *Investor i is more cautious than investor j.*

2. *Given any initial wealth, stock price, and option price such that there is a solution to problem (3) for both investors i and j, investor j holds a (strictly) positive position in the option only if investor i does so, i.e., $y_j \geq (>) 0$ implies $y_i \geq (>) 0$.*

Remark 1. Statement 2 of the theorem states that investor j has a (strictly) positive position in the option only if investor i does so. As has been already explained in the last section, adding a positive position in the option to a portfolio of a riskless bond and the underlying stock results in a strong increase in skewness. Thus the theorem implies that a more cautious investor is more likely to buy the option to pursue strong increases in skewness.

Remark 2. From the above proof, it is clear that if for all S, $C_i(w_i(S)) \geq C_j(w_j(S))$, then $y_j \geq 0$ implies $y_i \geq 0$. Moreover, if for all S, $C_i(w_i(S)) \geq C_j(w_j(S))$, and for at least some S the inequality is strict, then $y_j \geq 0$ implies $y_i > 0$.[16]

Remark 3. As was mentioned in the introduction, there are some studies in the literature which use cautiousness to explain investors' investment

[16]We need only note that the given condition implies that there is some S, $\delta_i(S) \neq \delta_j(S)$.

decision making. Among those studies Leland (1980) uses cautiousness to explain the convexity of an investor's optimal payoff function in a complete market. Theorem 1 is obviously different from Leland's results as it is about positions in an option while his results are about the convexity of an agent's optimal payoff function. Nevertheless, the two results are related: in the above theorem, positions in the option also determine the convexity of the optimal portfolio. However, even if we pursue this relationship, we must be aware that (i) as was mentioned in the introduction, Leland's results depend on the characteristics of the representative agent, which are exogenously and arbitrarily defined; (ii) Leland's results rely on the condition that investors' optimal payoff functions are monotonically increasing while Theorem 1 is valid whether the two investors' terminal wealth functions are increasing with the stock price or not.[17]

Remark 4. The above theorem gives an ordering of utility functions in terms of the motive to buy options. This ordering is perfect for HARA class utility functions as they all have constant cautiousness. Thus, if investor i and j have constant cautiousness C_i and C_j, i.e., they have HARA class utility functions, and $C_i > C_j$, then investor i will have a stronger motive to buy options. Moreover, for an exponential utility function, cautiousness is zero, while any utility function which displays decreasing absolute risk

[17]For example, in the case of homogeneous beliefs, Leland's results are derived from the equation $\frac{f_i''(x)}{f_i'(x)} = R(x)(C_i(f_i(x)) - C(x))$, where x is the aggregate wealth, $R(x)$ and $C(x)$ are the representative agent's risk aversion and cautiousness respectively, $f_i(x)$ and $C_i(f_i(x))$ are agent i's optimal payoff function and cautiousness along this function respectively. Thus $sign(f_i''(x))$ depends not only on $sign(C_i(f_i(x)) - C(x))$ but also on $sign(f_i'(x))$. See also page 657 in HHK (2007).

aversion has positive cautiousness and any utility function which displays increasing absolute risk aversion has negative cautiousness. Thus, according to the above theorem, any investor who has decreasing (increasing) absolute risk aversion always has a stronger (weaker) motive to buy options than an investor with an exponential utility function.

Remark 5. Furthermore, the theorem also implies the role of prudence in explaining the demand for options. According to Leland (1968) and Kimball (1990), an investor is prudent (imprudent) if her utility function has a positive (negative) third derivative. Consider the situation when one investor is prudent while another is imprudent. In this case, as cautiousness can be written as $C(w) = u'''(w)u'(w)/u''^2(w) - 1$, the first investor's cautiousness is larger than negative unity while the second investor's cautiousness is smaller than negative unity. According to Theorem 1, this implies that the first investor has a stronger motive to buy the option. Thus a prudent investor has a stronger motive to buy options than an imprudent investor.

The proof that Statement 2 of Theorem 1 implies Statement 1 can be found in Appendix B, and here we only show the proof that Statement 1 implies Statement 2. To prove this, we need the following two lemmas which are proved in Appendix A.

Lemma 2 *Assume $\phi_i(S)$ and $\phi_j(S)$ both price the stock correctly, and for all S, $C_i(w_i(S)) \geq C_j(w_j(S))$. If $y_i \leq 0$ and $y_j \geq 0$ then in the entire support, $\delta_i(S)$ crosses $\delta_j(S)$ once, and $\phi_i(S)$ crosses $\phi_j(S)$ twice, unless for all S, $\phi_i(S) = \phi_j(S)$.*

Lemma 3 *Assume that $\phi_i(S)$ and $\phi_j(S)$ both price the stock correctly and that $\delta_i(S)$ and $\delta_j(S)$ cross once. If for some S, $\delta_i(S) \neq \delta_j(S)$, then the two* *pricing kernels cannot both price the option correctly.*

With the help of the above two lemmas, we are now ready to prove that Statement 1 of Theorem 1 implies Statement 2.

Proof: By contradiction, suppose either $y_i \leq 0$ and $y_j > 0$ or $y_i < 0$ and $y_j \geq 0$. It is straightforward that there are some S, $\delta_i(S) \neq \delta_j(S)$.[18] In the meantime as $y_i \leq 0$, $y_j \geq 0$, investor i is more cautious than investor j, and both $\phi_i(S)$ and $\phi_j(S)$ price the stock correctly, from Lemma 2, $\delta_i(S)$ and $\delta_j(S)$ cross once, and $\phi_i(S)$ and $\phi_j(S)$ cross twice. Now applying Lemma 3, we conclude that the two pricing kernels $\phi_i(S)$ and $\phi_j(S)$ cannot both price the option correctly, which causes a contradiction. Q.E.D.

5 Who Buys More Options Per Share?

In the last section we focused on $sign(y_i)$, i.e., the sign of the position in the option in an investor's optimal portfolio strategy. In this section we focus on the ratio y_i/x_i, where x_i is the number of shares of the stock, which is the amount of options per share in an investor's optimal portfolio strategy. We show how an investor's level of cautiousness determines this ratio. We present the following result.

Theorem 2 *The following two conditions are equivalent.*

1. *Investor i is more cautious than investor j.*

[18] We need only note that under the given condition if for all $S \in (\underline{s}, K)$, $\delta_i(S) = \delta_j(S)$, then $\delta_i(K^+) \neq \delta_j(K^+)$.

2. *Given any initial wealth, stock price, and option price such that there is a solution to problem (3) for both investors i and j and $x_i x_j \neq 0$, if $x_i S + y_i a(S)$ is strictly monotone then $x_j > (<) 0$ implies $\frac{y_i}{x_i} \geq (\leq) \frac{y_j}{x_j}$; if $x_j S + y_j a(S)$ is strictly monotone then $x_i > (<) 0$ implies $\frac{y_i}{x_i} \geq (\leq) \frac{y_j}{x_j}$.*

Proof: See Appendix C.

Remark 1. In the case where $x_i x_j > 0$, the theorem tells us that a more cautious investor buys more options per share or sells fewer options per share. From Lemma 5 in Appendix C, increasing positive positions (reducing negative positions) in the option per share equates a convex transformation of the terminal wealth function, and according to Van Zwet (1964), this results in a strong increase in the skewness of the portfolio.[19] Thus the theorem implies that a more cautious investor pursue strong increases in skewness by trading the option.[20]

Remark 2. Similar to Remark 2 on Theorem 1, from the proof, it is clear that if for all S, $C_i(w_i(S)) \geq C_j(w_j(S))$, then Statement 2 is true. Moreover, if for all S, $C_i(w_i(S)) \geq C_j(w_j(S))$, and for at least some S the inequality is strict, then using the same proof and applying Remark 2 on Theorem 1, we can show that Statement 2 is true with strict inequalities, i.e., investor i buys strictly more options per share or sells strictly fewer options per share.

[19] See also Footnote 2.

[20] This becomes even clearer if we go through the proof of the theorem. As is shown in the proof, in the transformed problem, investor j's optimal portfolio has a negative position in the option $\hat{a}(\hat{S})$ while investor i's optimal portfolio has zero position in the option. This implies that investor j's terminal wealth is a concave function of investor i's terminal wealth, i.e., the difference between investor j's terminal wealth and investor i's terminal wealth is a strong decrease in skewness.

Remark 3. The condition that $x_iS+y_ia(S)$ or $x_jS+y_ja(S)$ is monotone is necessary for the conclusion in the theorem; this is shown in Section 6 using some numerical examples.[21] Also, note that this condition is equivalent to investor i's terminal wealth being a monotone function of the underlying stock price S. To understand this condition, consider the case where you have bought some units of a stock index. If you set up a normal portfolio insurance strategy using an option on the index, your terminal wealth will be a monotone increasing function of the index unless you over-insure your stock index. Thus, if you do not over-insure your stock index, the condition in the theorem will be satisfied. Consider another case where you have sold short some shares of a stock. If you buy some call options on the stock to cover this short position, your terminal wealth will be a monotone decreasing function of the stock price unless you over-cover your short position. Thus, if you do not over-cover your short position, the condition in the theorem will be satisfied.

6 Numerical Examples of Option Demand

In this section we present some numerical examples. These are designed to illustrate the conclusions of the theorems established above. Table 1 shows optimal stock and option demands given three different sets of (S_0, a_0) prices. In part a), $S_0 = 84$ and $a_0 = 3.00$. Marginal utility is of the HARA class with $u'(w) = (w + \alpha)^{-\gamma}$. For this utility function cautiousness is a constant with $C(w) = 1/\gamma$ and absolute risk aversion $R(w) = \gamma/(\alpha + w)$. Cautiousness is

[21] See the discussion at the end of Section 6.

shown for four different levels of γ in column 3 of the Table. Risk aversion is shown in column 4 (for $\alpha = 20$) and column 8 (for $\alpha = 70$). The first four rows of the table assume current wealth $w_0 = 100$ and the next four rows assume current wealth $w_0 = 200$. For all the examples we assume a 1-year horizon and an interest rate of 5%. The stock has a payoff with four states (120, 100, 80, 70) with equal probability. The option is a call option with a strike price of 100.

Given these data, we solve equations (3) for the optimal stock and option demands. For $\alpha = 20$, these are shown in columns 5 and 6 respectively. For $\alpha = 70$, these are shown in columns 9 and 10 respectively. In part b) of the table the results are shown for a different set of prices, $S_0 = 85$ and $a_0 = 3.70$. Then, in part c) they are shown for $S_0 = 86$ and $a_0 = 4.50$.

Observing the results, first note that the relative option demand, y/x, is unaffected either by wealth w_0 or by the subsistence parameter α. For example, given $C = 2.00$ in part b), $y/x = 0.23$ for all combinations of w_0 and α. This illustrates a result of Rubinstein (1974) which shows that investors with the same constant cautiousness measure have an identical optimal risky portfolio. Looking at the column headed y, we observe that the option demand given $C = 0.25$ is never positive unless the demand given $C = 2.00$ is positive. Also, the option demand given $C = 2.00$ is only negative if the demand given $C = 0.25$ is negative. These results are consistent with Theorem 1.

Looking at the results in part a) or part b) it is tempting to conclude that the relative option demand y/x increases with C. However, the results in part c) of the Table show that this is not always the case. Given the prices

$S_0 = 86$ and $a_0 = 4.50$, the short position in the option increases with C. However, the relative position y/x decreases (from -1.65 to -1.75). Note that here the payoff $xS + ya(S)$ is not monotone. This case illustrates the need for the condition in Theorem 2.

7 Conclusions

In this paper we have characterized cautiousness, a downside risk aversion measure, using the simple portfolio problem with a risk-free bond, a stock, and an option on the stock. We establish that it is an investor's cautiousness that determines her demand for options. Unlike the current literature on skewness preference which treats skewness largely as synonymous with the (unstandardized) third central moment, our study uses the notion of a strong increase in skewness defined by Van Zwet (1964). This enables us to obtain monotone comparative statistics in the difficult cases where risk is large.

To some extent, our results provide a direct extension of Arrow (1965) and Pratt's (1964) portfolio characterization of risk aversion. They show that, given the choice between investing in a positive excess return risky asset and a risk-free asset, an agent has lower risk aversion than another agent *if and only if* she always invests more in the risky asset. Thus investment in the risky asset characterizes risk aversion. We show that, given the additional choice of investing in an option, an agent has higher cautiousness or downside risk aversion, (i) *if and only if* she is always more likely to buy the option, (ii) *if and only if* she always demands more options per share. Hence investment in the option characterizes cautiousness or downside risk aversion.

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Tables

Table 1: a) Stock and Option Demand (84, 3.00)

		γ	C	$R(w)$	x	y	y/x	$R(w)$	x	y	y/x
					α=20				α=70		
S_0	84	4	0.25	$\frac{4}{20+w}$	0.22	0.54	2.51	$\frac{4}{70+w}$	0.30	0.76	2.52
a_0	3.00	2	0.50	$\frac{2}{20+w}$	0.42	1.20	2.83	$\frac{2}{70+w}$	0.59	1.67	2.82
w_0	100	1	1.00	$\frac{1}{20+w}$	0.80	2.85	3.55	$\frac{1}{70+w}$	1.12	3.99	3.56
		0.5	2.00	$\frac{0.5}{20+w}$	1.36	7.48	5.50	$\frac{0.5}{70+w}$	1.91	10.48	5.50
S_0	84	4	0.25	$\frac{4}{20+w}$	0.40	1.00	2.52	$\frac{4}{70+w}$	0.49	1.22	2.52
a_0	3.00	2	0.50	$\frac{2}{20+w}$	0.78	2.20	2.83	$\frac{2}{70+w}$	0.95	2.69	2.84
w_0	200	1	1.00	$\frac{1}{20+w}$	1.47	5.24	3.55	$\frac{1}{70+w}$	1.80	6.37	3.55
		0.5	2.00	$\frac{0.5}{20+w}$	2.50	13.77	5.50	$\frac{0.5}{70+w}$	3.05	16.78	5.51

Table 1 a) shows the optimal stock and option demands given $(S_0, a_0) = (84, 3.00)$. Investors have HARA utility functions $u(w) = \frac{(w+\alpha)^{1-\gamma}}{1-\gamma}$ with $\alpha = 20,\ 70$.

Table 1 b) Stock and Option Demand (85, 3.70)

		γ	C	$R(w)$	x	y	y/x	$R(w)$	x	y	y/x
					α=20				α=70		
S_0	85	4	0.25	$\frac{4}{20+w}$	0.31	-0.06	-0.19	$\frac{4}{70+w}$	0.43	-0.08	-0.19
a_0	3.70	2	0.50	$\frac{2}{20+w}$	0.61	-0.08	-0.14	$\frac{2}{70+w}$	0.85	-0.11	-0.13
w_0	100	1	1.00	$\frac{1}{20+w}$	1.18	-0.03	-0.03	$\frac{1}{70+w}$	1.66	-0.05	-0.03
		0.5	2.00	$\frac{0.5}{20+w}$	2.21	0.50	0.23	$\frac{0.5}{70+w}$	3.08	0.71	0.23
S_0	85	4	0.25	$\frac{4}{20+w}$	0.56	-0.10	-0.19	$\frac{4}{70+w}$	0.69	-0.13	-0.19
a_0	3.70	2	0.50	$\frac{2}{20+w}$	1.12	-0.15	-0.14	$\frac{2}{70+w}$	1.36	-0.19	-0.14
w_0	200	1	1.00	$\frac{1}{20+w}$	2.18	-0.06	-0.03	$\frac{1}{70+w}$	2.65	-0.07	-0.03
		0.5	2.00	$\frac{0.5}{20+w}$	4.05	0.93	0.23	$\frac{0.5}{70+w}$	4.93	1.15	0.23

Table 1 b) shows the optimal stock and option demands given $(S_0, a_0) = (85, 3.70)$. Investors have HARA utility functions $u(w) = \frac{(w+\alpha)^{1-\gamma}}{1-\gamma}$ with $\alpha = 20,\ 70$.

Table 1c) Stock and Option Demand (86, 4.50)

		γ	C	$R(w)$	x	y	y/x	$R(w)$	x	y	y/x
					α=20				α=70		
S_0	86	4	0.25	$\frac{4}{20+w}$	0.47	-0.78	-1.65	$\frac{4}{70+w}$	0.66	-1.09	-1.65
a_0	4.50	2	0.50	$\frac{2}{20+w}$	0.95	-1.58	-1.67	$\frac{2}{70+w}$	1.33	-2.22	-1.67
w_0	100	1	1.00	$\frac{1}{20+w}$	1.92	-3.25	-1.70	$\frac{1}{70+w}$	2.69	-4.56	-1.70
		0.5	2.00	$\frac{0.5}{20+w}$	3.85	-6.74	-1.75	$\frac{0.5}{70+w}$	5.39	-9.44	-1.75
S_0	86	4	0.25	$\frac{4}{20+w}$	0.87	-1.43	-1.65	$\frac{4}{70+w}$	1.05	-1.74	-1.65
a_0	4.50	2	0.50	$\frac{2}{20+w}$	1.75	-2.91	-1.67	$\frac{2}{70+w}$	2.12	-3.54	-1.67
w_0	200	1	1.00	$\frac{1}{20+w}$	3.53	-5.99	-1.70	$\frac{1}{70+w}$	4.30	-7.29	-1.70
		0.5	2.00	$\frac{0.5}{20+w}$	7.08	-12.41	-1.75	$\frac{0.5}{70+w}$	8.62	-15.09	-1.75

Table 1 c) shows the optimal stock and option demands given $(S_0, a_0) = (86, 4.50)$. Investors have HARA utility functions $u(w) = \frac{(w+\alpha)^{1-\gamma}}{1-\gamma}$ with $\alpha = 20,\ 70$.

Appendix A Proof of Lemmas 1, 2, and 3

A.1 Proof of Lemma 1

By contradiction, suppose $\phi_i(S)$ crosses $\phi_j(S)$ only once at a from above.[22] We have

$$E[(\phi_i(S) - \phi_j(S))S] = E[(\phi_i(S) - \phi_j(S))(S - a)].$$

Suppose $\phi_i(S)$ and $\phi_j(S)$ are not identical, i.e., there exists a point $b \in (\underline{s}, \bar{s})$ such that $\phi_i(b) \neq \phi_j(b)$. As both $\phi_i(S)$ and $\phi_j(S)$ are continuous at $S = b$, there must exist a neighborhood of b with positive probability mass such that for all S in this set, $\phi_i(S) \neq \phi_j(S)$. This, together with the fact that $\phi_i(S) - \phi_j(S)$ is non-negative when $S < a$ and non-positive when $S > a$, implies that $E[(\phi_i(S) - \phi_j(S))(S - a)] < 0$. Thus we obtain $E[(\phi_i(S) - \phi_j(S))S] < 0$. This inequality contradicts the assumption that both pricing kernels price the stock correctly. This completes the proof. Q.E.D.

A.2 Proof of Lemma 2

Proof: We first prove that if for all S, $C_i(w_i(S)) \geq C_j(w_j(S))$, then $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in each of the two intervals $(K, \bar{s})$ and $(\underline{s}, K)$. Consider $S < K$. As $\delta_t(S) = R_t(w_t(S))w_t'(S)$, $t = i, j$, if $w_i'(S) = 0$ or $w_j'(S) = 0$ then $\delta_i(S)$ cannot cross $\delta_j(S)$ in the interval $(\underline{s}, K)$. Moreover, if they have opposite signs then, they cannot cross either. Now suppose they are both strictly positive or negative. In this case, noting that for all $S < K$, $w_i''(S) = w_j''(S) = 0$, we have for all $S < K$, $(\frac{1}{\delta_t(S)})' = C_t(w_t(S))$, $t = i, j$, where we have used the definition of cautiousness, $C(x) = (\frac{1}{R(x)})'$. As for all S, $C_i(w_i(S)) \geq C_j(w_j(S))$, from the

[22]Note two pricing kernels must cross at least once because otherwise their expectations cannot both be unity.

above result we conclude that $\frac{1}{\delta_i(S)}$ can cross $\frac{1}{\delta_j(S)}$ at most once from below, which implies that $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in the interval $(\underline{s}, K)$. Similarly, we conclude that $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in the interval $(K, \bar{s})$. This proves the statement.

We now prove the lemma. Assume $\phi_i(S)$ and $\phi_j(S)$ are not identical. As $y_i \leq 0$ and ($y_j \geq 0$), at $S = K$, $w_i'(S)$ jumps down while $w_j'(S)$ jumps up. This implies that at $S = K$, $\delta_i(S) = R_i(w_i(S))w_i'(S)$ jumps down while $\delta_j(S) = R_j(w_j(S))w_j'(S)$ jumps up. In the meantime, as investor i is more cautious than investor j, from the statement we have just proved in the above paragraph, $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in the interval $(K, \bar{s})$ ($(\underline{s}, K)$). Combining the last two statements, we conclude that $\delta_i(S)$ can cross $\delta_j(S)$ at most once from above in the entire support. But according to Lemma 1, the two pricing kernels $\phi_i(S)$ and $\phi_j(S)$ must cross at least twice; from basic calculus and (6), this implies that $\delta_i(S)$ must cross $\delta_j(S)$ at least once. Thus $\delta_i(S)$ crosses $\delta_j(S)$ exactly once in the entire support. Now again from basic calculus and (6), this implies that $\phi_i(S)$ can cross $\phi_j(S)$ at most twice. Applying Lemma 1, we conclude that $\phi_i(S)$ crosses $\phi_j(S)$ exactly twice. Q.E.D.

A.3 Proof of Lemma 3

Proof: As is shown in the proof of Lemma 2, the given condition implies that $\phi_i(S)$ and $\phi_j(S)$ cross twice. Without loss of generality, suppose $\phi_i(S)$ crosses $\phi_j(S)$ first from above, then from below. First assume the two crossings both happen in one of the two intervals $(\underline{s}, K]$ and $[K, \bar{s})$. Without loss of generality, suppose they both happen in the interval $(\underline{s}, K]$. Note that if for all $S < K$, $\phi_i(S) = \phi_j(S)$ then as $\delta_i(S)$ and $\delta_j(S)$ cross once, from basic calculus and (6), $\phi_i(S)$ and $\phi_j(S)$ cross once, which contradicts the given condition. Thus we must have for some $S < K$,

$\phi_i(S) \neq \phi_j(S)$. This implies that for all $S \leq K$, $\phi_i(S) - \phi_j(S) \geq 0$, and for some $S \in (\underline{s}, K)$, $\phi_i(S) - \phi_j(S) > 0$. Because of the put-call parity, we can treat the option as a put, and it follows that $\phi_i(S)$ prices the option strictly higher than $\phi_j(S)$. Now assume the two crossings are not both contained in one of the two intervals $(\underline{s}, K]$ and $[K, \bar{s})$, i.e., there exist $s_1 \in (\underline{s}, K)$ and $s_2 \in (K, \bar{s})$ such that for all $S < s_1$ or $S > s_2$, $\phi_i(S) \geq \phi_j(S)$, for all $s_1 < S < s_2$, $\phi_i(S) \leq \phi_j(S)$, and for some S in each of the three intervals $(\underline{s}, s_1)$, (s_1, s_2), and $(s_2, \bar{s})$, $\phi_i(S) \neq \phi_j(S)$.

Now as in Section 3, construct a portfolio of the money instrument and the stock such that its payoff is equal to the payoff of the option at s_1 and s_2, and denote the payoff of the portfolio by $L(S)$. As $s_1 < K$ and $s_2 > K$, we must have $a(S) - L(S) > 0$, when $S < s_1$ or $S > s_2$; $a(S) - L(S) < 0$, when $s_1 < S < s_2$. Thus similar to (8) in Section 3, we have $E[(\phi_i(S) - \phi_j(S))a(S)] = E[(\phi_i(S) - \phi_j(S))(a(S) - L(S))] > 0$, i.e., the two pricing kernels $\phi_i(S)$ and $\phi_j(S)$ cannot both price the option correctly. Q.E.D.

Appendix B Proof of Theorem 1 (Second Half)

B.1 Lemma 4

Before we start to prove that Statement 2 implies Statement 1, consider the following explanation. In the rare case where the current prices of the stock and the option are equal to the risk neutral prices, a strictly risk averse investor will optimally hold zero investment in both the stock and the option. Thus if we use S_r and a_r to denote the risk neutral prices of the stock and the option respectively, when $(S_0, a_0) = (S_r, a_r)$, a solution to (3) is $(x_i, y_i) = (0, 0)$. We now show that for those (S_0, a_0) which are near (S_r, a_r), solutions to (4) exist too. We have the following lemma.

Lemma 4 *The following two statements are true.*

1. *There exists a neighborhood of* $(0,0)$*, B, such that for any* $(x_i, y_i) \in B$*, there exists* (S_0, a_0) *such that* (x_i, y_i) *is the solution to (4).*

2. *There exists a neighborhood of* (S_r, a_r)*, A, such that for any* $(S_0, a_0) \in A$*, a solution to (4) exists.*

Proof:

We first prove Statement 1 of this lemma. As the support of the stock price distribution is bounded the current prices of the stock and the option under the first stochastic dominance rule must be bounded. Let $\underline{S}$ and $\bar{S}$ be the lower and upper bounds of the stock price at time zero; let $\underline{a}$ and $\bar{a}$ be the lower and upper bounds of the option price at time zero.[23] Consider the problem in which given a pair of (x_i, y_i), we want to solve (4) for (S_0, a_0). Define a function $g: \ R_+^2 \rightarrow R_+^2$ as follows:

$$g(S_0, a_0) = \frac{1}{1+r}(E[\phi_i(S)S], E[\phi_i(S)a(S)]),$$

where as is defined, $\phi_i(S) = u_i'(w_i(S))/Eu_i'(w_i(S))$ and $w_i(S)$ is given by (1). Given any pair of (x_i, y_i) which is close enough to $(0,0)$, this function is well defined on $[\underline{S}, \bar{S}] \times [\underline{a}, \bar{a}]$. As utility functions are three times differentiable, $g(.)$ is obviously a continuous function from a non-empty, closed, bounded, convex set $[\underline{S}, \bar{S}] \times [\underline{a}, \bar{a}]$ to itself.[24] According to the well-known Brouwer's Fixed Point Theorem, there is always a fixed point. Thus a solution of (S_0, a_0) to (4) always exists. This proves the first statement of the lemma.

We now prove the second statement of the lemma. Define a function $f: \ R_+^2 \rightarrow R_+^2$ as follows. For a pair of stock price and option price (S_0, a_0), if there is a

[23] It is straightforward that $\underline{S} = \underline{s}/(1+r)$, $\overline{S} = \bar{s}/(1+r)$, $\underline{a} = \min_{x \in [\underline{s}, \bar{s}]} a(x)/(1+r)$, and $\overline{a} = \max_{x \in [\underline{s}, \bar{s}]} a(x)/(1+r)$.

[24] In a metric space sequential continuity and continuity are equivalent.

solution (x_i, y_i) to (4), then $f(S_0, a_0) = (x_i, y_i)$. Note as is well-known, because of the strict concavity of the utility function $u_i(w)$, the solution (x_i, y_i) is unique; thus the function is well defined. As utility functions are three times differentiable, $f(.)$ is obviously continuous.

From the first statement of the lemma we conclude that there is a neighborhood of $(0, 0)$, B, such that B is a set of images under function $f(.)$. Since $f(.)$ is continuous and B is open, the preimage of B is also open. Thus as $f(S_r, a_r) = (0, 0)$ there must exist a neighborhood of (S_r, a_r), A, such that for any $(S_0, a_0) \in A$, a solution to (4) exists. Q.E.D.

B.2 Proof of Theorem 1 (Second Half)

With the help of the above lemmas we can now start to prove that Statement 2 implies Statement 1. Note that if it is not true that for all w and v, $C_i(w) \geq C_j(v)$, then there must exist some w_0 and v_0 such that $C_i(w_0) < C_j(v_0)$. As all utility functions are assumed to be three times continuously differentiable, cautiousness is continuous; Thus there must be a neighborhood of w_0, A, a neighborhood of v_0, B, and a constant α, such that for all $w \in A$ and all $v \in B$, $C_i(w) < \alpha < C_j(v)$. If we can somehow make sure that investor i's terminal wealth is contained in A while investor j's terminal wealth is contained in B, then applying Remark 2 on the theorem in Section 4, we can show a situation where it happens that investor j optimally holds a long position in the option while i does not. This is the idea we use to prove that Statement 2 implies Statement 1.

We need only show that if it is not true that for all w and v, $C_i(w) \geq C_j(v)$ then there is a set of w_{i0}, w_{j0}, S_0, and a_0 such that investor j optimally holds a long position in the option while i does not.

Applying the first statement of Lemma 4, we conclude that there is a series: $\{(x_i^n, 0)|n = 1, 2, ...\}$, where x_i^n is strictly decreasing in n, $\lim_{n\to\infty} x_i^n = 0$, and for all n, $(x_i^n, 0)$ is the solution to (4) corresponding to $(S_0, a_0) = (S_{0n}, a_{0n})$. Obviously we have $\lim_{n\to\infty} S_{0n} = S_r$ and $\lim_{n\to\infty} a_{0n} = a_r$.

According to the second statement of Lemma 4, there exists a neighborhood of (S_r, a_r), A, such that for any $(S_0, a_0) \in A$, the solution to problem (3) exists. Without loss of generality assume for all n, $(S_{0n}, a_{0n}) \in A$. This implies that given the series $\{(S_{0n}, a_{0n})|n = 1, 2, ...\}$, there exist a series of solutions $\{(x_{jn}, y_{jn})|n = 1, 2, ...\}$ to problem (3) for investor j. Since $\lim_{n\to\infty}(S_{0n}, a_{0n}) = (S_r, a_r)$ from the continuity of the solutions we have $\lim_{n\to\infty}(x_{jn}, y_{jn}) = (0, 0)$.

As is pointed out at the beginning of this proof, if it is not true that for all w and v, $C_i(w) \geq C_j(v)$, from the continuity of $C_i(w)$ and $C_j(v)$, there must be w_0, v_0, A, which is a neighborhood of w_0, and B, which is a neighborhood of v_0, such that for all $w \in A$ and all $v \in B$, $C_i(w) < C_j(v)$. Let $w_{i0} = w_0/(1 + r)$ and $w_{j0} = v_0/(1+r)$.[25] Use $w_{in}(S)$ to denote investor i's terminal wealth corresponding to trading strategy $(x_i, y_i) = (x_{in}, y_{in})$, where $y_{in} = 0$, which is defined in Equation (1). Then, since the support of the stock price distribution, $[a, b]$, is bounded, there must exist $N > 0$ such that for all $n > N$, we have that for all $S \in [a, b]$, $w_{in}(S) \in A$ and $w_{jn}(S) \in B$.

This implies that for all $S \in [a, b]$, $C_i(w_{in}(S)) < \alpha < C_j(w_{jn}(S))$. Now applying Remark 2 on the theorem in Section 4, for $n > N$ we must have $y_{jn} > 0$. Thus we have a situation where investor j holds a (strictly) positive position in the option, but investor i does not do so. This completes the proof. Q.E.D.

[25]Negative initial wealth will be avoided if we require positive terminal wealth. This does not have any effect on the proof.

Appendix C Proof of Theorem 2

To prove the theorem, we need the following lemma.

Lemma 5 *Assume for all S, $1 + ba'(S) > 0$, where b is a constant. Let $\hat{S} = S + ba(S) = h(S)$ and $\hat{a}(\hat{S}) = a(h^{-1}(\hat{S}))$. Then $\hat{a}(\hat{S})$ is a positive fraction of an option on $\hat{S}$ with strike price $\hat{K} = K + ba(K)$, which is the same type of option as $a(S)$.*

Proof: Let $\hat{K} = K + ba(K)$. As $\hat{S} = S + ba(S) = h(S)$ and $\hat{a}(\hat{S}) = a(h^{-1}(\hat{S}))$, in either interval $\hat{S} < \hat{K}$ or interval $\hat{S} > \hat{K}$, we have $\frac{d\hat{a}(\hat{S})}{d\hat{S}} = \frac{a'(h^{-1}(\hat{S}))}{h'(h^{-1}(\hat{S}))}$. Simplifying it, we obtain

$$\frac{d\hat{a}(\hat{S})}{d\hat{S}} = \frac{a'(S)}{1 + ba'(S)}, \tag{9}$$

where $S = h^{-1}(\hat{S})$. As $a(S)$ is the payoff of an option with strike price K, we have $a'(S) = \alpha$ for $S < K$ and $a'(S) = \beta$ for $S > K$, where $\alpha < \beta$. For a call option, $\alpha = 0$ and $\beta = 1$. In this case, we have for $\hat{S} < \hat{K}$, $\frac{d\hat{a}(\hat{S})}{d\hat{S}} = \frac{\alpha}{1+b\alpha} = 0$; for $\hat{S} > \hat{K}$, $\frac{d\hat{a}(\hat{S})}{d\hat{S}} = \frac{\beta}{1+b\beta} = \frac{1}{1+b}$. Thus $\hat{a}(\hat{S})$ is $\frac{1}{1+b}$ of the payoff of a call on $\hat{S}$ with strike price $\hat{K}$. For a put option, $\alpha = -1$ and $\beta = 0$. For a put option, $\alpha = -1$ and $\beta = 0$. In this case, we have for $\hat{S} < \hat{K}$, $\frac{d\hat{a}(\hat{S})}{d\hat{S}} = \frac{\alpha}{1+b\alpha} = -\frac{1}{1-b}$; for $\hat{S} > \hat{K}$, $\frac{d\hat{a}(\hat{S})}{d\hat{S}} = \frac{\beta}{1+b\beta} = 0$. Thus $\hat{a}(\hat{S})$ is $\frac{1}{1-b}$ of the payoff of a put on $\hat{S}$ with strike price $\hat{K}$. In both cases, $\hat{a}(\hat{S})$ is a positive fraction of an option on $\hat{S}$ with strike price $\hat{K}$, which is the same type of option as $a(S)$. Q.E.D.

With the help of the above lemma, we now prove the theorem. We first prove that the first statement implies the second statement. Assume $x_i \neq 0$ and $x_j \neq 0$. Let $\tilde{y}_i = y_i/x_i$ and $\tilde{y}_j = y_j/x_j$. Suppose $S + \tilde{y}_i a(S)$ is strictly monotone. Let $\hat{S} = h(S) = S + \tilde{y}_i a(S)$. As $h(S)$ is strictly monotone, it follows that $S = h^{-1}(\hat{S})$. Let $\hat{a}(\hat{S}) = a(h^{-1}(\hat{S}))$.

From Lemma 5, $\hat{a}(\hat{S})$ is a positive fraction of an option on $\hat{S}$ with strike price $\hat{K} = K + ba(K)$, which is the same type of option as $a(S)$. Thus the original investment problem with stock S and option $a(S)$ is transformed into a new investment problem with stock $\hat{S}$ and a positive fraction of an option $\hat{a}(\hat{S})$. From (1) in the original problem, investor i's terminal wealth is

$$w_i(S; x_i, \tilde{y}_i) = (w_{0i} - x_i(S_0 + \tilde{y}_i a_0))(1 + r) + x_i(S + \tilde{y}_i a(S)),$$

and investor j's terminal wealth is

$$w_j(S; x_j, \tilde{y}_j) = (w_{0i} - x_j(S_0 + \tilde{y}_j a_0))(1 + r) + x_j(S + \tilde{y}_j a(S)).$$

If we let $\hat{S}_0 = S_0 + \tilde{y}_i a_0$ and $\hat{a}_0 = a_0$, then in the transformed problem investor i's terminal wealth is $w_i(\hat{S}; x_i, 0) = (w_{0i} - x_i \hat{S}_0)(1+r) + x_i \hat{S}$, and investor j's terminal wealth is

$$w_j(\hat{S}; x_j, \tilde{y}_j - \tilde{y}_i) = (w_{0i} - x_j(\hat{S}_0 + (\tilde{y}_j - \tilde{y}_i)\hat{a}_0))(1 + r) + x_j(\hat{S} + (\tilde{y}_j - \tilde{y}_i)\hat{a}(\hat{S})).$$

From the above two equations, we can clearly see that in the transformed problem investor i has x_i shares of the stock $\hat{S}$ and zero position in the option on $\hat{S}$ with strike price $\hat{K}$ in her optimal portfolio while investor j's optimal positions in the stock $\hat{S}$ and the option on $\hat{S}$ are x_j and $x_j(\tilde{y}_j - \tilde{y}_i)$ multiplied by a positive fraction respectively. Now assume investor i is more cautious than investor j. Applying Theorem 1 to the transformed problem, as investor i is more cautious than j, we immediately conclude that we must have $x_j(\tilde{y}_j - \tilde{y}_i) \leq 0$. Thus if $x_j > (<)0$, $\tilde{y}_i \geq (\leq)\tilde{y}_j$.

The proof for the case where $S + \tilde{y}_j a(S)$ is strictly monotone is similar. This proves that the first statement implies the second statement.

The proof of the converse is similar to the proof of Theorem 1. Without loss of generality, assume the option is a put. By contradiction, suppose that it is not true that for all w and v, $C_i(w) \geq C_j(v)$. As is shown in the second half of the proof of Theorem 1, in the special case set up there, we have a situation where $y_j > 0$ while $x_i > 0$ and $y_i = 0$. As the option is a put, we must have $x_j > 0$; otherwise, if $x_j \leq 0$, then as $y_j > 0$, $w_j(S)$ is decreasing. But as $x_i > 0$ and $y_i = 0$, $w_i(S)$ is strictly increasing. This implies that $\phi_i(S)$ is strictly decreasing while $\phi_j(S)$ is increasing, and they cannot both price the stock correctly. Now we have a situation where $x_i S + y_i a_i(S)$ is strictly monotone, $x_i > 0$, $x_j > 0$, and $0 = \frac{y_i}{x_i} < \frac{y_j}{x_j}$. This completes the proof. Q.E.D.