Changes in Risk and Valuation of Options: A Unified Approach to Option Pricing Bounds*

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Abstract

In this paper we first prove a theorem which reveals how changes in risk affect option values. The theorem can be used to solve most problems in the theory of option pricing bounds with restrictions on probability distributions or risk preferences, given the prices of the underlying stock and multiple observed options. We then present analytical solutions to such problems subject to four interesting classes of probability distributions and four important classes of risk preferences, respectively.

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Introduction

In the literature on option pricing there are many studies that look at general rather than particular probability distributions or stochastic processes. For example, some of them investigate how changes in the underlying risk affect option prices.[1] Some reveal the properties of option prices under general probability distributions or stochastic processes.[2] Some calculate bounds on option prices (hereafter option bounds) given various moments of probability distributions.[3] Some derive option bounds using information on assets' performance measures.[4] Many others study option bounds for a given class of risk preferences.[5] The research we carry out in this paper is along similar lines. We start with the question how changes in the underlying probability density functions affect option prices. In the case where a change in the underlying risk is a mean-preserving spread, in particular, when the change in the risk neutral probability density function (hereafter PDF) is positive at the left end, has two changes of sign, and preserves the stock price, it is well known that the values of options will increase (see, for example, Franke et al. (1999)). Naturally, we may consider a slightly more general case where a change in the PDF has three changes of sign and preserves the values of the underlying stock and one option on the stock. How does this affect the values of other options? In an even more general

[1]See, for example, Merton (1973), Jagannathan (1984), Franke et al. (1999), Rasmusen (2007), and Huang (2012).

[2]See, for example, Merton (1973), Bergman et al. (1996), Frey and Sin (1999), and Kijima (2002).

[3]See, for example, Lo (1987), Grundy (1991), Bertsimas and Popescu (2002), and Schepper and Heijnen (2007).

[4]See, for example, Cochrane and Saa-Requejo (2000), Bernardo and Ledoit (2000), and Cerny (2003).

[5]See, for example, Perrakis and Ryan (1984), Levy (1985), and Ritchken (1985), Perrakis (1986), Ritchken and Kuo (1988, 1989), Basso and Pianca (1997), Mathur and Ritchken (2000), Constantinides and Zariphopoulou (1999, 2001), Constantinides and Perrakis (2002, 2007), and Constantinides et al. (2009).

case consider a change in the PDF which has $n+2$ changes of sign and preserves the values of the underlying stock and n options on the stock. How does such a change in risk affect the values of other options? In this paper we present an answer to the above question and show that, surprisingly, the answer to this question leads to a unified and convenient approach to option bounds, which can be used to solve most problems in the theory of option pricing bounds with restrictions on probability distributions or risk preferences, given the prices of the underlying stock and multiple observed options.

We then use the approach to derive analytical option pricing bounds, given the prices of the underlying stock and n observed options, when the probability distribution satisfies the following four different conditions respectively: (1) the PDF is bounded, (2) the PDF slightly deviates from a log-normal PDF, (3) the PDF is unimodal, and (4) the cumulative distribution function (hereafter CDF) is log-concave. We also derive optimal option pricing bounds, given the prices of the underlying stock and n observed options, when risk preferences satisfy the following four different conditions respectively: (1) risk preferences satisfy the second order stochastic dominance rule, (2) risk preferences satisfy the third or higher order stochastic dominance rule, (3) relative risk aversion is bounded, (4) absolute risk aversion is a decreasing function of wealth.

The importance of option bounds using information embedded in the prices of other options is highlighted by a recent important empirical study on the mispricing of S&P 500 index options by Constantinides et al. (2009). They show that there are widespread violations of second order stochastic dominance option bounds by one-month S&P 500 index call options. To take advantage of these opportunities, it is important to derive analytical solutions of option bounds given the prices of other options and then use these analytical solutions to establish arbitrage strategies. The theory presented in this paper will be very useful for this purpose.

The structure of the remaining paper is as follows: In Sections 1 we present a theorem on how changes in risk affect option values. In section 2 we use the above theorem to derive option bounds with restrictions on PDFs (CDFs). In section 3 we derive option bounds with restrictions on risk preferences. Section 4 concludes the paper.

1 Changes in Risk and Valuation of Options

We assume that there is a stock in an economy on which some option contracts are written. We only consider those European options with the same time to maturity t. The price of the stock at time t is denoted by S. The current time is assumed to be zero. Let $c(S, K)$ denote the time t payoff of an option with a strike price K. It is well known that given a risk-neutral probability measure (or an equivalent Martingale measure) which is induced by a CDF $Q(S)$, the forward price of an option with a strike price K is[6]

$$c_0(K) = \int_0^\infty c(S, K) dQ(S). \tag{1}$$

When there is a probability density function, then (1) becomes

$$c_0(K) = \int_0^\infty c(S, K) q(S) dS. \tag{2}$$

We first present the following lemma.

Lemma 1 *Given a CDF $Q(x)$ if $\int_0^\infty x dQ(x) < \infty$ then the following two equations are true.*

$$\int_0^\infty (1 - Q(x)) dx = \int_0^\infty x dQ(x). \tag{3}$$

$$\int_K^\infty (x - K) dQ(x) = \int_0^\infty x dQ(x) - K + \int_0^K Q(x) dx. \tag{4}$$

These two results are well known in statistics and finance; thus the proof is omitted for brevity.[7]

We are interested in the effect of changes in risk on the values of options. Let $V(S) = \hat{Q}(S) - Q(S)$ denote a change in the CDF, where $\hat{Q}(S)$ is the CDF after

[6]For simplicity we consider CDFs with a common support R_+, but the results in this section are obviously valid when the support is any subinterval of R_+.

[7]See, for example, Jagannathan (1984) or Huang (2012).

the risk change. If $Q(S)$ and $\hat{Q}(S)$ have PDFs $q(S)$ and $\hat{q}(S)$ respectively, then the corresponding change in the PDF is also denoted by $\nu(S) = \hat{q}(S) - q(S)$. We first give the following definition: given two functions $f(x)$ and $\hat{f}(x)$ defined on R_+, $\hat{f}(x) - f(x)$ is said to have n changes of sign if there exist $x_1, x_2, ..., x_n$, where $0 = x_0 < x_1 < x_2 < ... < x_n < x_{n+1} = \infty$, such that for all $i = 0, 1, ..., n$, $x \in (x_i, x_{i+1})$, $(-1)^i(\hat{f}(x) - f(x)) > 0$, or for all $i = 0, 1, ..., n$, $x \in (x_i, x_{i+1})$, $(-1)^i(\hat{f}(x) - f(x)) < 0$.[8] In this case $f(x)$ and $\hat{f}(x)$ are also said to have n crossings. We now present the following theorem.

Theorem 1 *Let K_1, ..., K_n be the strike prices of n options, where $0 = K_0 < K_1 < ... < K_n < K_{n+1} = \infty$. Any change in risk which satisfies either of the following two conditions strictly increases (decreases) the values of options with strike prices in (K_i, K_{i+1}) where $i \geq 0$ is an even (odd) integer.*

1. *The change in the CDF, $V(S) = \hat{Q}(S) - Q(S)$, preserves the values of the stock and n options, has $n+1$ sign changes, and is positive at the left end.*

2. *The change in the PDF, $\nu(S) = \hat{q}(S) - q(S)$, preserves the values of the stock and n options, has $n+2$ sign changes, and is positive at the left end.*

Proof: We first prove the first result of the theorem. As $\hat{Q}(S)$ and $Q(S)$ give the same prices of the underlying stock and options with strike prices $K_1, ..., K_n$, where $0 = K_0 < K_1 < ... < K_n < K_{n+1} = \infty$, from Lemma 1, we must have for all $i = 0, 1, ..., n+1$, $j = 1, ..., n+1$,

$$\int_0^{K_i} (\hat{Q}(S) - Q(S))dS = 0, \tag{5}$$

$$\int_{K_i}^{K_j} (\hat{Q}(S) - Q(S))dS = 0. \tag{6}$$

It follows that if $\hat{Q}(S)$ and $Q(S)$ does not cross in an interval (K_i, K_{i+1}) then they must be equal to each other in the entire interval. This implies that they

[8]This definition is obviously very narrow; however, it is just enough for the derivation of a unified approach to option bounds in this paper. When the two functions are PDFs, it is straightforward to extend the definition to the case where the inequalities hold almost surely.

must cross at least once in each interval (K_i, K_{i+1}), $i = 0, ..., n$. But as they cross $n+1$ times, they must cross exactly once in each of the $n+1$ intervals (K_i, K_{i+1}), $i = 0, ..., n$. Suppose that the $n+1$ crossings happen at $s_1, ..., s_{n+1}$, where $K_{i-1} < s_i < K_i$, $i = 1, ..., n+1$. Let $s_0 = 0$ and $s_{n+2} = \infty$. It is obvious that $\int_0^x (\hat{Q}(S) - Q(S))dS$ strictly increases (decreases) for $x \in (s_i, s_{i+1})$, where $i \in [0, n+1]$ is even (odd). This, together with the condition that $\int_0^{K_i} (\hat{Q}(S) - Q(S))dS = 0$, $i = 1, ..., n+1$, implies that $\int_0^x (\hat{Q}(S) - Q(S))dS$ changes sign exactly once at K_{i+1} in every interval $[s_i, s_{i+1})$, $i = 1, ..., n$, and has no sign change in $[0, s_1)$ or $[s_{n+1}, \infty)$. Hence $\int_0^x (\hat{Q}(S) - Q(S))dS$ changes sign exactly n times at $X = K_1, K_2, ..., K_n$ in the entire support. Moreover, as $V(S) = \hat{Q}(S) - Q(S)$ is positive at the left end of the support, it is straightforward that $\int_0^x (\hat{Q}(S) - Q(S))dS$ is positive at the left end. In the meantime from (4) we have $\int_K^\infty (S-K)d(\hat{Q}(S) - Q(S)) = \int_0^K (\hat{Q}(S) - Q(S))dS$. Thus we conclude that $\int_X^\infty (S - X)d(\hat{Q}(S) - Q(S))$ changes sign exactly n times at $X = K_1, K_2, ..., K_n$ in the entire support and is positive at the left end of the support, i.e., it strictly increases (decreases) the values of options with strike prices in (K_i, K_{i+1}) where $i \geq 0$ is an even (odd) integer.

We now prove the second result. Assume that the $n+2$ crossings between $\hat{q}(S)$ and $q(S)$ happen at $s_1, s_2, ..., s_{n+2}$, where $0 = s_0 < s_1 < ... < s_{n+2} < s_{n+3} = \infty$. It is obvious that $\int_0^x (\hat{q}(S) - q(S))dS$ strictly increases (decreases) for $x \in (s_i, s_{i+1})$, where i is even (odd). This, together with the condition that $\int_0^\infty (\hat{q}(S) - q(S))dS = 0$, implies that $\int_0^x (\hat{q}(S) - q(S))dS$ changes sign at most once in every interval $[s_i, s_{i+1})$, $i = 1, ..., n+1$ and it has no sign change in $[0, s_1)$ or $[s_{n+2}, \infty)$. Hence $V(S) = \hat{Q}(S) - Q(S)$ changes sign at most $n+1$ times in the entire support. But if $V(S) = \hat{Q}(S) - Q(S)$ changes sign n or fewer times in the entire support, from the first result, $V(S) = \hat{Q}(S) - Q(S)$ will not preserve the price of (at least) one of the options, which causes a contradiction. Thus $V(S) = \hat{Q}(S) - Q(S)$ must change sign exactly $n+1$ times in the entire support. Moreover, as $\nu(S) = \hat{q}(S) - q(S)$ is positive at the left end of the support, $\hat{Q}(x) - Q(x) = \int_0^x (\hat{q}(S) - q(S))dS$ is positive at the left end. Now applying the first result, the second result is proved. Q.E.D.

The second result of the above theorem can be extended to the case where changes in risk are caused by changes in the Radon-Nikodym derivative of the risk neutral probability measure with respect to the real world probability measure. Assume that the real world cumulative probability distribution function is $P(S)$. Given a CDF $Q(S)$, let $\phi(S)$ denote the Radon-Nikodym derivative, i.e., $\phi(S) = dQ(S)/dP(S)$. Function $\phi(S)$ is often called the pricing kernel. The second result of Theorem 1 remains true if we replace the PDF $q(S)$ with the pricing kernel $\phi(S)$ and replace the integration operator with the expectation operator under the real world probability measure. The proof is exactly the same.

Corollary 1 *Assume that a change in the pricing kernel preserves the prices of the stock and n options with strike prices K_1, ..., K_n, where $0 = K_0 < K_1 < ... < K_n < K_{n+1} = \infty$, and has $n+2$ changes of sign. If it is positive at the left end then it strictly increases (decreases) the values of options with strike prices in (K_i, K_{i+1}) where $i \geq 0$ is an even (odd) integer.*

2 Option Bounds with Restrictions on PDFs

The theorem derived in the last section has an interesting application in option pricing. We will show that it can be used to solve many complex problems in the theory of option bounds. As an important approach to option pricing, the theory of option bounds reveals important information about the range of possible values an option can have.[9]

A class of difficult problems in this approach are to derive optimal option bounds with various restrictions on risk neutral probability distributions or risk preferences given the prices of the underlying stock and some observed options. Theorem 1 suggests an interesting idea to solve such a complex problem of option bounds given the prices of n observed options. Assuming that we consider a

[9]See, for example, Jouini (2001) for explanations about the three main approaches to option pricing. See also a list of studies on option bounds mentioned in the introduction of this paper.

particular set of PDFs (CDFs), if we can find a PDF (CDF) in the closure of this set which crosses all these PDFs (CDFs) $n+2$ times ($n+1$ times), and preserves the values of the underlying stock and n observed options on the stock, then this PDF (CDF) can be used to calculate the optimal bounds on the prices of all options. At first thought, one may doubt the viability of this idea: is there such an optimal PDF (CDF)? How shall we find this optimal PDF (CDF) if there is one? Interestingly, we can show that the above theorem can not only help to prove the existence of such an optimal PDF (CDF) but also help to identify it.

Throughout the rest of the paper, we assume that the support of the CDFs is $[\underline{s}, \bar{s}] \subset [0, +\infty)$ and that as is in the last section, all options' strike prices are interior points of the above support. In this section, we present analytical solutions to some problems of option bounds with restrictions on PDFs (CDFs), given the prices of the underlying stock and n observed options with strike prices K_1, ..., and K_n respectively, where $n \geq 0$ and $\underline{s} = K_0 < K_1 < K_2 < ... < K_n < K_{n+1} = \bar{s}$. Given the prices of the stock and n options, a PDF (CDF) is said to be admissible if it prices the stock and n options correctly.

2.1 Option Bounds with Bounded PDFs

In this subsection, we consider the class of bounded PDFs and use Theorem 1 to identify the optimal PDFs which give optimal option bounds, given the prices of the underlying stock and n observed options.

Proposition 1 *Assume that the stock and options are all priced by a PDF which is bounded above by $\overline{q}$ and below by $\underline{q}$, where $\bar{q} > \underline{q} \geq 0$.*[10]

- *The optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible PDF $q_n^{**}(S)$ which is $(n+3)$-segment piecewise constant and has a constant value $\overline{q}$ at odd segments and a constant value $\underline{q}$ at even segments.*[11]

[10] This places constraints on the given prices of the stock and n options which can be worked out by applying this proposition to the cases where there are fewer than n observed options. The same can be said about similar assumptions in other propositions.

[11] Throughout the paper an n-segment piecewise constant (linear) function is defined in the

- *The optimal lower (upper) bound for options with strike prices in* (K_i, K_{i+1})*, where* i *is even (odd), is given by an admissible PDF* $q_n^*(S)$ *which is* $(n+3)$*-segment and piecewise constant and has a constant value* $\underline{q}$ *at odd segments and a constant value* $\overline{q}$ *at even segments.*

Proof: We need first prove the existence of the optimal PDFs specified in the proposition; this is done in the appendix. We now prove that the specified PDFs give option bounds. We first prove the case where $0 < \underline{q} < q(S) < \bar{q}$.

Note that $q_n^{**}(S)$ is $(n+3)$-segment and piecewise constant and has a constant value $\bar{q}$ at odd segments and a constant value $\underline{q}$ at even segments. Let the support of the ith segment of $q_n^{**}(S)$ be denoted by $[s_{n,i-1}^{**}, s_{n,i}^{**}]$, where $i = 1, ..., n+3$ and $\underline{s} = s_{n,0}^{**} \le s_{n,1}^{**} \le ... \le s_{n,n+3}^{**} = \bar{s}$. Given any admissible PDF $q(S)$ which satisfies $0 < \underline{q} < q(S) < \bar{q}$, it is straightforward that $q_n^{**}(S) - q(S)$ has at most $(n+2)$ changes of sign. But it must have at least $(n+2)$ changes of sign; otherwise, from Theorem 1, $q_n^{**}(S)$ and $q(S)$ cannot both price the underlying stock and the n options correctly. Thus $q_n^{**}(S) - q(S)$ has exactly $(n+2)$ changes of sign and is positive at the left end.[12] As both PDFs price the underlying stock and the n options correctly, from Theorem 1, an option with a strike price $X \in (K_i, K_{i+1})$, where i is even, has a higher value under the PDF $q_n^{**}(S)$ than under the PDF $q(S)$ while an option with a strike price $X \in (K_i, K_{i+1})$, where i is odd, has a lower value under the PDF $q_n^{**}(S)$ than under the PDF $q(S)$. Thus $q_n^{**}(S)$ gives an upper bound on the price of any option with a strike price $X \in (K_i, K_{i+1})$, where i is even, while it gives a lower bound on the price of any option with a strike price $X \in (K_i, K_{i+1})$, where i is odd. The proof for $q_n^*(S)$ is similar. Thus the result is proved for the case where $0 < \underline{q} < q(S) < \bar{q}$.

Now consider the case where $0 < \underline{q} \le q(S) \le \bar{q}$. For sufficiently small $\varepsilon > 0$, we have $0 < \underline{q} - \varepsilon < q(S) < \bar{q} + \varepsilon$. Let $q_n^{**}(S; \varepsilon)$ and $q_n^*(S; \varepsilon)$ denote the two

non-strict sense, i.e., the length of a segment is allowed to be zero. However, as is explained in Footnote 12, when the bounds are strict, i.e., $\underline{q} < q(S) < \bar{q}$, the piecewise constant PDFs in the proposition have the specified number of segments in the strict sense.

[12] This implies that $\underline{s} = s_{n,0}^{**} < s_{n,1}^{**} < ... < s_{n,n+3}^{**} = \bar{s}$.

optimal PDFs with the specified forms in the proposition corresponding to the above bounds on $q(S)$. Note that $q_n^{**}(S;\varepsilon)$ is $(n+3)$-segment and piecewise constant and has a constant value $\bar{q}+\varepsilon$ at odd segments and a constant value $\underline{q}-\varepsilon$ at even segments. Let the support of the ith segment of $q_n^{**}(S;\varepsilon)$ be denoted by $[s_{n,i-1}^{**}(\varepsilon), s_{n,i}^{**}(\varepsilon)]$, where $i=1,...,n+3$ and $\underline{s}=s_{n,0}^{**}(\varepsilon)\leq s_{n,1}^{**}(\varepsilon)\leq ... \leq s_{n,n+3}^{**}(\varepsilon)=\bar{s}$.[13] Let $\varepsilon\to 0$. As $(s_{n,1}^{**}(\varepsilon),...,s_{n,n+2}^{**}(\varepsilon))$ is a bounded sequence, there must exist a convergent subsequence $(s_{n,1}^{**}(\varepsilon_j),...,s_{n,n+2}^{**}(\varepsilon_j))$, where $\lim_{j\to\infty}\varepsilon_j=0$, such that $\lim_{j\to\infty}(s_{n,1}^{**}(\varepsilon_j),...,s_{n,n+2}^{**}(\varepsilon_j))=(s_{n,1}^{**},..., s_{n,n+2}^{**})$. Hence we obtain $\lim_{j\to\infty}q_n^{**}(S;\varepsilon_j)=q_n^{**}(S)=\bar{q}$, for all $S\in\cup_{0\leq 2i\leq n+2}(s_{n,2i}^{**},s_{n,2i+1}^{**})$; $\underline{q}$, for all $S\in\cup_{1<2i\leq n+3}(s_{n,2i-1}^{**},s_{n,2i}^{**})$, where $\underline{s}=s_{n,0}^{**}\leq s_{n,1}^{**}\leq ... \leq s_{n,n+3}^{**}=\bar{s}$. As it has already been proved above that for all arbitrarily small $\varepsilon>0$, $q_n^{**}(S;\varepsilon)$ gives an upper bound on the price of an option with any strike price $X\in(K_i,K_{i+1})$, where i is even, while it gives a lower bound on the price of an option with any strike price $X\in(K_i,K_{i+1})$, where i is odd, it follows that the limit $q_n^{**}(S)$ also gives an upper bound on the price of an option with any strike price $X\in(K_i,K_{i+1})$, where i is even, while it gives a lower bound on the price of an option with any strike price $X\in(K_i,K_{i+1})$, where i is odd. This proves the result for $q_n^{**}(S)$. The proof of the result for $q_n^{*}(S)$ is similar.

To prove the result in the case where $\underline{q}=0$, consider $q(S;\varepsilon)=\frac{q(S)+\varepsilon}{1+\varepsilon(\bar{s}-\underline{s})}$, where $\varepsilon>0$ is arbitrarily small. We have $0<\frac{\varepsilon}{1+\varepsilon(\bar{s}-\underline{s})}=\underline{q}(\varepsilon)\leq q(S;\varepsilon)\leq\bar{q}$ and $\int_a^{\bar{s}}q(S;\varepsilon)dS=1$. As is proved in the above argument, the results is valid for $q(S;\varepsilon)$ which is bounded below by $\underline{q}(\varepsilon)=\frac{\varepsilon}{1+\varepsilon(\bar{s}-\underline{s})}$ and above by $\bar{q}$. Using an argument similar to the above, when $\varepsilon\to 0$, we prove the result in the case where $\underline{q}=0$.

Finally, it is straightforward to see that the bounds given in the proposition are optimal as $q_n^{**}(S)$ and $q_n^{*}(S)$ are members of the set of PDFs which are bounded above by $\bar{q}$ and below by $\underline{q}$. Q.E.D.

[13] These inequalities are actually strict. See Footnote 12.

2.2 Option Bounds and the Black-Scholes Model

In this subsection, we use Theorem 1 to derive optimal option bounds when the situation slightly deviates from the Black-Scholes model. We consider a particular class of PDFs which are equal to a log-normal density function multiplied by functions whose elasticities with respect to the stock price are bounded.[14] Let $\pi(S)$ denote a log-normal probability density function.

Proposition 2 *Assume that the stock and all options are priced by a continuous and piecewise differentiable PDF $q(S)$ which satisfies the condition that the difference between the elasticity of $q(S)$ and that of $\pi(S)$ is bounded below by $\underline{\nu}$ and above by $\bar{\nu}$, i.e., $\underline{\nu} \leq \nu(S) = -\frac{d \ln \frac{q(S)}{\pi(S)}}{d \ln S} \leq \bar{\nu}$.*

- *The optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible PDF $q_n^{**}(S) = \alpha_0 f_n^{**}(S)\pi(S)$, where $\alpha_0 = \frac{1}{\int_{\underline{s}}^{\bar{s}} f_n^{**}(S)\pi(S)dS}$, and $f_n^{**}(S)$ satisfies the following two conditions: (i) it is continuous; (ii) its elasticity is $(n+2)$-segment and piecewise constant and has a constant value $\bar{\nu}$ at odd segments and a constant value $\underline{\nu}$ at even segments.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible PDF $q_n^{*}(S) = \alpha_0 f_n^{*}(S)\pi(S)$, where $\alpha_0 = \frac{1}{\int_{\underline{s}}^{\bar{s}} f_n^{*}(S)\pi(S)dS}$, and $f_n^{*}(S)$ satisfies the following two conditions: (i) it is continuous; (ii) its elasticity is $(n+2)$-segment and piecewise constant and has a constant value $\underline{\nu}$ at odd segments and a constant value $\bar{\nu}$ at even segments.*

Proof: The existence of the solutions specified in the proposition is guaranteed by the assumption that the stock and n options are priced by a continuous and piecewise differentiable PDF $q(S)$ which satisfies the condition that the difference between the elasticity of $q(S)$ and that of $\pi(S)$ is bounded below by $\underline{\nu}$ and above by $\bar{\nu}$. The proof is in the same spirit as that of Proposition 1, and

[14] The elasticity of a function $h(S)$ is defined as $\nu(S) = -\frac{d \ln h(S)}{d \ln S}$.

it is omitted for brevity. We now prove that the specified PDFs give option bounds as is stated in the proposition. We need only prove the case where the difference between the elasticity of $q(S)$ and that of $\pi(S)$ is strictly bounded by $\overline{\nu}$ and $\underline{\nu}$, i.e., $\underline{\nu} < \nu(S) = -\frac{d \ln \frac{q(S)}{\pi(S)}}{d \ln S} < \bar{\nu}$. Similar to Proposition 1, the case where the bounds are not strict is then proved by constructing a sequence of slightly loosened bounds $\overline{\nu} + \varepsilon$ and $\underline{\nu} - \varepsilon$ and taking the limit.

Let $\nu(S)$ and $\nu_n^{**}(S)$ denote the elasticities of $q(S)$ and $q_n^{**}(S)$ respectively. It is obvious that both $\nu_n^{**}(S) - \nu(S)$ has at most $n+1$ changes of sign. As $\nu(S)$ and $\nu_n^{**}(S)$ are both Riemann integrable and $q(S)$ and $q_n^{**}(S)$ are both continuous, we have for all $s, S \in (\underline{s}, \bar{s})$, $\ln \frac{q_n^{**}(S)}{q(S)} - \ln \frac{q_n^{**}(s)}{q(s)} = -\int_s^S (\nu_n^{**}(x) - \nu(x)) d \ln x$. This, together with the fact that $\nu_n^{**}(S) - \nu(S)$ has at most $n+1$ changes of sign, implies that $q_n^{**}(S) - q(S)$ can have at most $n+2$ changes of sign. But according to Theorem 1, $q_n^{**}(S) - q(S)$ must have at least $n+2$ changes of sign as $q_n^{**}(S) - q(S)$ preserves the values of the underlying stock and n options. Thus $q_n^{**}(S) - q(S)$ and $q_n^{*}(S) - q(S)$ must have exactly $n+2$ changes of sign. It is not difficult to see that $q_n^{**}(S) - q(S)$ must be positive at the left end while $q_n^{*}(S) - q(S)$ must be negative at the left end. Thus applying Theorem 1, we immediately conclude that $q_n^{**}(S)$ and $q_n^{*}(S)$ give option bounds as is stated in the proposition.

Moreover, it is straightforward that these option bounds are optimal as $q_n^{**}(S)$ and $q_n^{*}(S)$ are members of the set of PDFs which are bounded below by $\underline{q}$ and above by $\bar{q}$. Q.E.D.

2.3 Option Bounds under Unimodal PDFs

Most common probability distributions have a unimodal PDF, i.e., a PDF which is first increasing then decreasing. In this subsection we derive option bounds when risk neutral probability distributions are unimodal. We first consider the case where PDFs are not only unimodal but also bounded above.

Proposition 3 *Assume that the stock and options are all priced by a unimodal*

PDF which is bounded above by $\bar{q}$. First assume n is odd, and let $m = (n+1)/2$.

- *The optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible PDF $q_n^{**}(S)$ which is $(m+2)$-segment piecewise constant and unimodal with a maximum value of $\bar{q}$ and has a value zero in the last segment.*
- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible PDF $q_n^{*}(S)$ which is $(m+2)$-segment piecewise constant and unimodal with a maximum value of $\bar{q}$ and has a value zero in the first segment.*

Now assume n is even, and let $m = n/2$.

- *The optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible PDF $q_n^{**}(S)$ which is $(m+2)$-segment piecewise constant and unimodal with a maximum value of $\bar{q}$.*
- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible PDF $q_n^{*}(S)$ which is $(m+2)$-segment piecewise constant and unimodal with a maximum value of $\bar{q}$ and has a value zero in the first and last segments.*

Proof: The existence of the solutions specified in the proposition is guaranteed by the assumption that the stock and n options are priced by a unimodal PDF $q(S)$. The proof is in the same spirit as that of Proposition 1, and it is omitted for brevity. We now prove that the specified PDFs give option bounds as is stated in the proposition. We need only prove the case where $q(S)$ is first strictly increasing then strictly decreasing. Similar to Proposition 1, the non-strict case is then proved by constructing a sequence of strict cases converging to the non-strict case.

We only show the proof for the case where n is even; when n is odd, the proof is similar. As $q(S)$ is first strictly increasing then strictly decreasing and bounded above by $\bar{q}$ and $q_n^{**}(S)$ is $(m+2)$-segment piecewise constant and unimodal with a maximum value of $\bar{q}$, by observation, we will have the maximum number of crossings between $q_n^{**}(S)$ and $q(S)$ when $q(s)$ crosses $q_n^{**}(S)$ in the middle of each of the non-mode segments.[15] In this case we have a crossing at an interior point of the support of each of the $m+1$ non-mode segments, one crossing at either the left end or the right end of the support of each of the $m+1$ non-mode segments. Thus the maximum number of crossings between $q_n^{**}(S)$ and $q(S)$ is $2 \times m + 2 = n + 2$. But according to Theorem 1, $q_n^{**}(S) - q(S)$ must have at least $n+2$ changes of sign as $q_n^{**}(S) - q(S)$ preserves the values of the underlying stock and the n options. Thus $q_n^{**}(S) - q(S)$ must have exactly $n+2$ changes of sign. It is not difficult to see that $q_n^{**}(S) - q(S)$ must be positive at the left end. Thus applying Theorem 1, we immediately conclude that $q_n^{**}(S)$ give option bounds as is stated in the proposition. The result about $q_n^{*}(S)$ can be similarly proved.

Moreover, it is straightforward that these option bounds are optimal as $q_n^{**}(S)$ and $q_n^{*}(S)$ are members of the set of unimodal PDFs which are bounded above by $\bar{q}$. Q.E.D.

We now consider the case where PDFs are unimodal but not necessarily bounded above (or the upper bound on the PDFs is unknown).

Proposition 4 *Assume that the stock and options are all priced by a unimodal PDF. First assume n is odd, and let $m = (n+1)/2$.*

- *The optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible distribution which is the mixture of a single-atom discrete distribution and a continuous distribution with a PDF $q_n^{**}(S)$, where $q_n^{**}(S)$ is $(m+1)$-segment piecewise constant and unimodal and has a value zero in the last segment, and the single atom is located at either end of the mode segment.*

[15] The mode segment of an n-segment piecewise constant and unimodal PDF is the segment where the PDF attains its maximum value.

- *The optimal lower (upper) bound for options with strike prices in* (K_i, K_{i+1})*, where* i *is even (odd), is given by an admissible distribution which is the mixture of a single-atom discrete distribution and a continuous distribution with a PDF* $q_n^*(S)$*, where* $q_n^*(S)$ *is* $(m+1)$*-segment piecewise constant and unimodal and has a value zero in the first segment, and the single atom is located at either end of the mode segment.*

Now assume n *is even, and let* $m = n/2$*.*

- *The optimal upper (lower) bound for options with strike prices in* (K_i, K_{i+1})*, where* i *is even (odd), is given by an admissible distribution which is the mixture of a single-atom discrete distribution and a continuous distribution with a PDF* $q_n^{**}(S)$*, where* $q_n^{**}(S)$ *is* $(m+1)$*-segment piecewise constant and unimodal, and the single atom is located at either end of the mode segment.*

- *The optimal lower (upper) bound for options with strike prices in* (K_i, K_{i+1})*, where* i *is even (odd), is given by an admissible distribution which is the mixture of a single-atom discrete distribution and a continuous distribution with a PDF* $q_n^*(S)$*, where* $q_n^*(S)$ *is* $(m+2)$*-segment piecewise constant and unimodal and has a value zero in the first and last segments, and the single atom is located at either end of the mode segment.*

This result can be proved in a way similar to the proof of Proposition 3 by applying the first result of Theorem 1 instead of the second result. It can also be obtained as the limit case of Proposition 3 where $\bar{q} \to \infty$. Thus the proof is omitted for brevity.

2.4 Option Bounds under Log-concave CDFs

Many common probability distributions have log-concave CDFs.[16] For example, all probability distributions which have log-concave PDFs are in this class. Moreover, some non-log-concave distributions such as log-normal distributions

[16] Note that, a log-concave function is necessarily continuous.

also have log-concave CDFs. In this section, we derive option bounds when only log-concave CDFs are considered. We have the following result.

Proposition 5 *Assume that the stock and all options are priced by a CDF whose logarithm is piecewise differentiable and concave. First assume n is odd, and let $m = (n+1)/2$.*

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by a continuous and admissible CDF $Q_n^{**}(S)$ which satisfies the following condition: $\frac{d \ln Q_n^{**}(S)}{dS}$ is $(m+1)$-segment piecewise constant and decreasing and has a value zero on the last segment.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible CDF $Q_n^*(S)$ which satisfies the following two conditions: (i) $Q_n^*(S)$ is zero on the interval $[\underline{s}, s_{n,1}^*)$, where $s_{n,1}^* < \bar{s}$. (ii) For $S \in [s_{n,1}^*, \bar{s}]$, $Q_n^*(S)$ is continuous and $\frac{d \ln Q_n^*(S)}{dS}$ is m-segment piecewise constant and decreasing.*

Now assume n is even, and let $m = n/2$.

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by a continuous and admissible CDF $Q_n^{**}(S)$ which satisfies the following condition: $\frac{d \ln Q_n^{**}(S)}{dS}$ is $(m+1)$-segment piecewise constant and decreasing.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible CDF $Q_n^*(S)$ which satisfies the following two conditions: (i) $Q_n^*(S)$ is zero on the interval $[\underline{s}, s_{n,1}^*)$, where $s_{n,1}^* < \bar{s}$. (ii) For $S \in [s_n^*, \bar{s}]$, $Q_n^*(S)$ is continuous and $\frac{d \ln Q_n^*(S)}{dS}$ is $(m+1)$-segment piecewise constant and decreasing and has a value zero on the last segment.*

Proof: The existence of the solutions specified in the proposition is guaranteed by the assumption that the stock and n options are priced by a log-concave and piecewise differentiable CDF $Q(S)$. The proof is in the same spirit as that of Proposition 1, and it is omitted for brevity. We now prove that the specified CDFs give option bounds as is stated in the proposition. We need only prove the case where $Q(S)$ is strictly log-concave. Similar to Proposition 1, the non-strict case is then proved by constructing a sequence of strict cases converging to the non-strict case.

We only show the proof for the case where n is even; when n is odd, the proof is similar. As $\frac{d\ln Q(S)}{dS}$ is strictly decreasing and $\frac{d\ln Q_n^{**}(S)}{dS}$ is $(m+1)$-segment piecewise constant and decreasing, by observation, we may have one crossing between $\frac{d\ln Q_n^{**}(S)}{dS}$ and $\frac{d\ln Q(S)}{dS}$ at an interior point of the support of each of the $m+1$ segments and one crossing at the right end of the support of each of the first m segments. Thus the maximum number of crossings between $\frac{d\ln Q_n^{**}(S)}{dS}$ and $\frac{d\ln Q(S)}{dS}$ is $2\times m+1=n+1$. Suppose they do have $n+1$ crossings, which happen at s_1, ..., s_{n+1}, where $\underline{s} < s_1 < ... < s_{n+1} < \bar{s}$. As both $Q_n^{**}(S)$ and $Q(S)$ are continuous and piecewise differentiable and $Q_n^{**}(\bar{s}) = Q(\bar{s}) = 1$, we have[17]

$$-\ln\frac{Q_n^{**}(S)}{Q(S)} = \int_S^{\bar{s}}(\frac{d\ln Q_n^{**}(x)}{dx} - \frac{d\ln Q(x)}{dx})dx. \tag{7}$$

From this, it is clear that $Q_n^{**}(S)$ and $Q(S)$ do not cross in the interval $[s_{n,n+1}, \bar{s})$ and $\ln\frac{Q_n^{**}(S)}{Q(S)}$ is strictly monotone in each of the $n+1$ intervals $(\underline{s}, s_1), (s_1, s_2), ...,$ (s_n, s_{n+1}), i.e., $Q_n^{**}(S)$ and $Q(S)$ cross at most once in each of the $n+1$ intervals $[\underline{s}, s_1), [s_1, s_2), ..., [s_n, s_{n+1})$. Hence the maximum number of crossings between $Q_n^{**}(S)$ and $Q(S)$ is $n+1$. But according to Theorem 1, $Q_n^{**}(S)$ and $Q(S)$ must have at least $n+1$ crossings as $Q_n^{**}(S) - Q(S)$ preserves the values of the underlying stock and the n options. Thus $Q_n^{**}(S)$ and $Q(S)$ must have exactly $n+1$ crossings. Moreover, it is not difficult to see that $\frac{d\ln Q_n^{**}(S)}{dS} - \frac{d\ln Q(S)}{dS}$ must be strictly negative at the left end; hence from (7), $Q_n^{**}(S) - Q(S)$ must be strictly positive at the left end. Thus applying Theorem 1, we immediately conclude that $Q_n^{**}(S)$ give optimal option bounds as is stated in the proposition. The result about $Q_n^{*}(S)$ can be similarly proved.

[17] Both $\frac{d\ln Q_n^{**}(S)}{dS}$ and $\frac{d\ln Q(S)}{dS}$ are Riemann integrable on a bounded subinterval of $(\underline{s}, \bar{s})$.

Moreover, the option bounds given by $Q_n^{**}(S)$ are optimal as it is a member of the set of admissible PDFs whose logarithms are piecewise differentiable and concave. The option bounds given by $Q_n^*(S)$ are also optimal as it can be shown to be the limit of a sequence of such PDFs; however, the proof is omitted for brevity.[18] Q.E.D.

3 Option Bounds and Risk Preferences

In this section, we present analytical solutions to some problems of option bounds with restrictions on risk preferences, given the prices of the underlying stock and n observed options with strike prices K_1, ..., and K_n respectively, where $n \geq 0$ and $\underline{s} = K_0 < K_1 < K_2 < ... < K_n < K_{n+1} = \bar{s}$. As is at the end of the Section 1, we use $P(S)$ and $Q(S)$ denote the real-world and risk neutral CDFs respectively. Similar to the definition of an admissible PDF (CDF), given the prices of the stock and n options, a pricing kernel is said to be admissible if it prices the stock and n options correctly.

3.1 Second Order Stochastic Dominance Option Bounds

In this subsection, we derive the optimal second order stochastic dominance option bounds. It is well known that in a representative-agent model, second order stochastic dominance leads to a decreasing pricing kernel $\phi(S) \equiv \frac{dQ(S)}{dP(S)}$.[19] In this case we have $dQ(S) = \phi(S)dP(S)$, where $\phi(S)$ is a decreasing function.

Given the underlying stock price, Perrakis and Ryan (1984), Levy (1985), and Ritchken (1985) derived second order stochastic dominance option bounds. Ryan (2002, 2003) derived second order stochastic dominance option bounds,

[18] Similar to the last subsection, it can be shown that the solution in the proposition is the limit of the solution in the case where $\frac{d \ln Q(S)}{dS}$ is decreasing and bounded above by $\bar{\nu} \to \infty$. In the bounded cases, the optimality of the option bounds is easy to verify. The optimality of the limit case then follows from these bounded cases. The proof is available on request.

[19] See, for example, Perrakis and Ryan (1984) or Ritchken (1985).

given the underlying stock price and the price of one observed option. Given the current prices of the underlying stock and n observed options, we have the following result.

Proposition 6 *Assume that the second order stochastic dominance rule is valid, i.e., the stock and options are priced by a decreasing pricing kernel. First suppose n is odd, and let $m = (n+1)/2$.*

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible CDF $Q_n^{**}(S) = a_{n,0} + \int_{\underline{S}}^{S} \phi_n^{**}(x)dP(x)$, where $a_{n,0} \in [0,1)$ and $\phi_n^{**}(x)$ is $(m+1)$-segment piecewise constant and decreasing and has a constant value zero on the last segment.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible pricing kernel $\phi_n^*(S) \geq 0$ which is $(m+1)$-segment piecewise constant and decreasing.*

Now suppose n is even, and let $m = n/2$.

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible CDF $Q_n^{**}(S) = a_{n,0} + \int_{\underline{S}}^{S} \phi_n^{**}(x)dP(x)$, where $a_{n,0} \in [0,1)$ and $\phi_n^{**}(x) \geq 0$ is $(m+1)$-segment piecewise constant and decreasing.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible pricing kernel $\phi_n^*(S) \geq 0$ which is $(m+2)$-segment piecewise constant and decreasing and has a constant value zero on the last segment.*

Proof: The existence of the solutions specified in the proposition is guaranteed by the assumption that the stock and n options are priced by a decreasing pricing kernel $\phi(S)$. The proof is in the same spirit as that of Proposition 1, and it is omitted for brevity. We now prove that the specified pricing kernels and CDFs give option bounds as is stated in the proposition. We need only prove the case where $\phi(S)$ is strictly decreasing. Similar to Proposition 1, the non-strict case, is then proved by constructing a sequence of strict cases converging to the non-strict case.[20]

We only show the proof for the case where n is even; when n is odd, the proof is similar. As $\phi(S)$ is strictly decreasing and $\phi_n^{**}(x) \geq 0$ is $(m+1)$-segment piecewise constant and decreasing, it is clear that $\phi_n^{**}(x) - \phi(x)$ has at most $2 \times m + 1 = n + 1$ changes of sign. Suppose it does have $n+1$ changes of sign, which happen at s_1, ..., s_{n+1}, where $\underline{s} < s_1 < ... < s_{n+1} < \bar{s}$. Let $Q(S)$ be the CDF corresponding to $\phi(S)$, i.e., $Q(S) = \int_{\underline{s}}^{S} \phi(x)dP(x)$. We have $Q_n^{**}(S) - Q(S) = a_{n,0} + \int_{\underline{s}}^{S}(\phi_n^{**}(x) - \phi(x))dP(x)$, where $a_{n,0} \in [0, 1)$. The rest of the proof is the same as that of Proposition 5, and it is omitted for brevity.

Moreover, the option bounds given by $\phi_n^{*}(S)$ are optimal as it is a member of the set of decreasing and admissible pricing kernels. The option bounds given by $Q_n^{**}(S)$ are also optimal as it can be shown to be the limit of a sequence of CDFs corresponding to some decreasing and admissible pricing kernels; however, the proof is omitted for brevity. Q.E.D

It is not difficult to verify that in the case with no observed option, the above solution gives the second order stochastic dominance option bounds derived by Perrakis and Ryan (1984), Levy (1985), and Ritchken (1985), and in the case with one observed option, the above solution gives the option bounds derived by Ryan (2000, 2003).

3.2 Nth Order Stochastic Dominance Option Bounds

In this subsection, we derive the optimal Nth ($N \geq 3$) order stochastic dominance option bounds. Third or higher order stochastic dominance rules put restrictions on the monotonicity of the derivatives of the pricing kernel. As is shown by Ritchken and Kuo (1989), following the Nth order stochastic dominance rule, the pricing kernel $\phi(S)$ satisfies the following conditions: $\phi^{(i)}(S) \geq 0$, for even $i < N - 1$; $\phi^{(i)}(S) \leq 0$, for odd $i < N - 1$, and $(-1)^N \phi^{(N-2)}(S)$ is

[20]For example, let $\hat{\phi}(S; \varepsilon) \equiv (\varepsilon e^{-S} + \phi(S))/(\varepsilon E e^{-S} + 1)$, where $\varepsilon > 0$.

decreasing.[21] Given the current price of the underlying stock S_0 and the current prices of the n observed options c_0^1, ..., and c_0^n, we have the following result.

Proposition 7 *Assume that the Nth ($N \geq 3$) order stochastic dominance rule is valid, i.e., the stock and options are all priced by a pricing kernel $\phi(S) \geq 0$ which is $N-3$ times continuously differentiable and $N-2$ times piecewise differentiable and satisfies the condition that $(-1)^i \phi^{(i)}(S) \geq 0$, for $0 \leq i \leq N-2$, and $(-1)^N \phi^{(N-2)}(S)$ is decreasing in S. First suppose n is odd, and let $m = (n+1)/2$.*

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible CDF $Q_{N,n}^{**}(S) = a_{N,n,0} + \int_{\underline{s}}^{S} \phi_{N,n}^{**}(x)dP(x)$, where $a_{N,n,0} \in [0,1)$ and $\phi_{N,n}^{**}(x)$ satisfies the following condition: $(-1)^N \phi_{N,n}^{**(N-3)}(x)$ is continuous and $(m+1)$-segment piecewise linear with decreasing and positive slopes and $\phi_{N,n}^{**}(x)$ is zero on the last segment.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible pricing kernel $\phi_{N,n}^{*}(S)$ which satisfies the following condition: $(-1)^N \phi_{N,n}^{*(N-3)}(x)$ is continuous and $(m+1)$-segment piecewise linear with decreasing and positive slopes and $\phi_{N,n}^{*}(x)$ is positive and constant on the last segment.*

Now suppose n is even, and let $m = n/2$.

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible CDF $Q_{N,n}^{**}(S) = a_{N,n,0} + \int_{\underline{s}}^{S} \phi_{N,n}^{**}(x)dP(x)$, where $a_{N,n,0} \in [0,1)$ and $\phi_{N,n}^{**}(x)$ satisfies the following condition: $(-1)^N \phi_{N,n}^{**(N-3)}(x)$ is continuous and $(m+1)$-segment piecewise linear with decreasing and positive slopes and $\phi_{N,n}^{**}(x)$ is positive and constant on the last segment.*

[21] We use $\phi^{(i)}(S)$ to denote the ith derivative of $\phi(S)$, where $i \geq 0$ and $\phi^{(0)}(S) = \phi(S)$. We may also use primes to denote derivatives.

- *The optimal lower (upper) bound for options with strike prices in* (K_i, K_{i+1})*, where* i *is even (odd), is given by an admissible pricing kernel* $\phi^*_{N,n}(S) \geq 0$ *which satisfies the following condition:* $(-1)^N \phi^{*(N-3)}_{N,n}(x)$ *is continuous and* $(m+2)$*-segment piecewise linear with decreasing and positive slopes and* $\phi^*_{N,n}(x)$ *is zero on the last segment.*

Proof: Similar to Proposition 6, the proof of the existence of the solutions is omitted. We now prove that the specified pricing kernels and CDFs give option bounds as is stated in the proposition. We need only prove the case where $(-1)^N \phi^{(N-2)}(S)$ is strictly decreasing. Similar to Proposition 6, the non-strict case is then proved by constructing a sequence of strict cases converging to the non-strict case.

We only show the proof for the case where n is even; when n is odd, the proof is similar. As $(-1)^N \phi^{(N-2)}(S) \geq 0$ is strictly decreasing and $(-1)^N \phi^{**(N-2)}_{N,n}(x) \geq 0$ is $(m+1)$-segment piecewise constant and decreasing and is zero on the last segment, $(-1)^N [\phi^{**(N-2)}_{N,n}(x) - \phi^{(N-2)}(x)]$ can have at most $2m = n$ changes of sign. Suppose it does have n changes of sign, which happen at s_1, ..., s_n, where $\underline{s} < s_1 < ... < s_n < \bar{s}$. In the meantime, as both $\phi^{**(N-3)}_{N,n}(x)$ and $\phi^{(N-3)}(x)$ are continuous and both $\phi^{**(N-2)}_{N,n}(x)$ and $\phi^{(N-2)}(x)$ are Riemann integrable, we have for all $s, S \in (\underline{s}, \bar{s})$, $(-1)^N [(\phi^{**(N-3)}_{N,n}(S) - \phi^{(N-3)}(S)) - (\phi^{**(N-3)}_{N,n}(s) - \phi^{(N-3)}(s))] = \int_s^S (-1)^N (\phi^{**(N-2)}_{N,n}(x) - \phi^{(N-2)}(x))dx$. From the above two statements, it is clear that $(-1)^N [(\phi^{**(N-3)}_{N,n}(S) - \phi^{(N-3)}(S))$ can have at most one change of sign in each of the n intervals $[\underline{s}, s_1), [s_1, s_2), ..., [s_{n-1}, s_n)$. If $N > 3$, then $\phi^{**(N-3)}_{N,n}(S) = 0$, and $(-1)^N [\phi^{**(N-3)}_{N,n}(S) - \phi^{(N-3)}(S)]$ can have no change of sign in $[s_n, \bar{s})$. Hence if $N - 3 \geq 1$, $(-1)^N [\phi^{**(N-3)}_{N,n}(S) - \phi^{(N-3)}(S)]$ can have at most n changes of sign. Repeat the above argument, and we conclude that for all k, if $N - k \geq 1$, $(-1)^N [\phi^{**(N-k)}_{N,n}(S) - \phi^{(N-k)}(S)]$ can have at most n changes of sign. Thus $\phi^{**\prime}_{N,n}(S) - \phi'(S)$ can have at most n changes of sign. From basic calculus, this implies that $\phi^{**}_{N,n}(S) - \phi(S)$ can have at most $n+1$ changes of sign. The rest of proof is the same as in the proof of Proposition 6, hence omitted. Q.E.D.

It is not difficult to verify that when $N = 3$, in the special case where there is no observed option, the above solution gives the third order stochastic dominance option bounds derived by Ritchken and Kuo (1989). The verification is omitted for brevity.

3.3 Option Bounds and Bounded Risk Aversion

In this subsection, we derive optimal option bounds when investors are assumed to have bounded risk aversion. It is well known that in a representative-agent model, the pricing kernel $\phi(S) = -\frac{u''(S)}{u'(S)}$, where $u(x)$ is the representative agent's utility function. Let $R(S) = -\frac{u''(S)}{u'(S)}$ and $\gamma(S) = -S\frac{u''(S)}{u'(S)}$ denote the absolute and relative risk aversion measures of the representative agent. We have $R(S) = -\frac{\phi'(S)}{\phi(S)}$ and $\gamma(S) = -S\frac{\phi'(S)}{\phi(S)}$.[22] Thus if the representative agent's absolute (relative) risk aversion measure is bounded, then the function $R(S) = -\frac{\phi'(S)}{\phi(S)}$ (the function $\gamma(S) = -S\frac{\phi'(S)}{\phi(S)}$) is bounded. We have the following result.

Proposition 8 *Assume that the representative agent's utility function is strictly increasing, continuously differentiable, and piecewise twice differentiable and his relative risk aversion is bounded below by $\underline{\gamma}$ and above by $\bar{\gamma}$.*

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible, continuous, and piecewise differentiable pricing kernel $\phi_n^{**}(S)$ which satisfies the condition that $\gamma_n^{**}(S) = -S\frac{\phi_n^{**\prime}(S)}{\phi_n^{**}(S)}$ is $(n+2)$-segment piecewise constant and has a constant value $\bar{\gamma}$ on odd segments and a constant value $\underline{\gamma}$ on even segments.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible, continuous, and piecewise differentiable pricing kernel $\phi_n^{*}(S)$ which satisfies the condition that $\gamma_n^{*}(S) = -S\frac{\phi_n^{*\prime}(S)}{\phi_n^{*}(S)}$ is $(n+2)$-segment piecewise constant and has a constant value $\underline{\gamma}$ on odd segments and a constant value $\bar{\gamma}$ on even segments.*

[22] See, for example, Mathur and Ritchken (2000).

Proof: As the representative agent's utility function is strictly increasing, continuously differentiable, and piecewise twice differentiable, the stock and options are all priced by a strictly positive, continuous, and piecewise differentiable pricing kernel $\phi(S)$. Moreover, as the representative agent's relative risk aversion is bounded below by $\underline{\gamma}$ and above by $\bar{\gamma}$, the elasticity of the pricing kernel $\phi(S)$ is bounded below by $\underline{\gamma}$ and above by $\bar{\gamma}$, i.e., $\underline{\gamma} \leq \gamma(S) = -S\frac{\phi'(S)}{\phi(S)} \leq \bar{\gamma}$. The existence of the solutions specified in the proposition is guaranteed by the above conditions. The proof is in the same spirit as that of Proposition 1, and it is omitted for brevity. We now prove that the specified pricing kernels give option bounds as is stated in the proposition. We need only prove the case the elasticity of $\phi(S)$ is strictly bounded below by $\underline{\gamma}$ and above by $\bar{\gamma}$, i.e., $\underline{\gamma} < \gamma(S) < \bar{\gamma}$. Similar to Proposition 1, the case where $\underline{\gamma} \leq \gamma(S) \leq \bar{\gamma}$ is then proved by constructing a sequence of loosened bounds $\underline{\gamma} - \varepsilon$ and $\bar{\gamma} + \varepsilon$ and taking the limit when $\varepsilon \to 0$.

As $\underline{\gamma} < \gamma(S) < \bar{\gamma}$ and $\gamma_n^{**}(S)$ is $(n+2)$-segment piecewise constant and has a constant value $\bar{\gamma}$ on odd segments and a constant value $\underline{\gamma}$ on even segments, it is straightforward that $\gamma(S)$ and $\gamma_n^{**}(S)$ can cross at most $n+1$ times. From basic calculus, as both $\phi_n^{**}(S)$ and $\phi(S)$ are continuous and both $\gamma_n^{**}(x)/x$ and $\gamma(x)/x$ are Riemann integrable on a bounded interval, we have for all $s, S \in (\underline{s}, \bar{s})$, $\ln\frac{\phi_n^{**}(S)}{\phi(S)} - \ln\frac{\phi_n^{**}(s)}{\phi(s)} = \int_s^S (\gamma_n^{**}(x) - \gamma(x)) d\ln x$. From the above two statements, it is clear that $\phi_n^{**}(S)$ and $\phi(S)$ can cross at most $n+2$ times. The rest of the proof is the same as that of Proposition 2, and it is omitted for brevity.

Moreover, the option bounds given by $\phi_n^{**}(s)$ and $\phi_n^{*}(S)$ are optimal as both are members of the set of decreasing, continuous, piecewise differentiable, and admissible pricing kernels with a bounded elasticity. Q.E.D

3.4 DARA Option Bounds

Decreasing absolute risk aversion is a popular assumption in economics and finance.[23] In this subsection, we derive optimal option bounds under this as-

[23] See, for example, Kimball (1993).

sumption. From the last subsection, in a representative agent model, decreasing absolute risk aversion leads to a pricing kernel $\phi(S)$ which satisfies the condition that $R(S) = -\frac{\phi'(S)}{\phi(S)}$ is decreasing. We have the following result.

Proposition 9 *Assume that the representative agent's utility function is strictly increasing, concave, continuously differentiable, and piecewise twice differentiable and that his absolute risk aversion is decreasing. First suppose n is odd, and let $m = (n+1)/2$.*

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible CDF $Q_n^{**}(S) = a_{n,0} + \int_{\underline{S}}^{S} \phi_n^{**}(x)dP(x)$, where $a_{n,0} \in [0,1)$ and $\phi_n^{**}(x) > 0$ is continuous, decreasing, and piecewise differentiable and satisfies the condition that $R_n^{**}(S) = -S\frac{\phi_n^{**\prime}(S)}{\phi_n^{**}(S)}$ is m-segment piecewise constant and decreasing.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible pricing kernel $\phi_n^*(S) > 0$ which is continuous and piecewise differentiable and satisfies the condition that $R_n^*(S) = -S\frac{\phi_n^{*\prime}(S)}{\phi_n^*(S)}$ is $(m{+}1)$-segment piecewise constant and decreasing and has a value zero on the last segment.*

Now suppose n is even, and let $m = n/2$.

- *Then the optimal upper (lower) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible CDF $Q_n^{**}(S) = a_{n,0} + \int_{\underline{S}}^{S} \phi_n^{**}(x)dP(x)$, where $a_{n,0} \in [0,1)$ and $\phi_n^{**}(x) > 0$ is continuous and piecewise differentiable and satisfies the condition that $R_n^{**}(S) = -S\frac{\phi_n^{**\prime}(S)}{\phi_n^{**}(S)}$ is $(m+1)$-segment piecewise constant and decreasing and has a value zero on the last segment.*

- *The optimal lower (upper) bound for options with strike prices in (K_i, K_{i+1}), where i is even (odd), is given by an admissible pricing kernel $\phi_n^*(S) > 0$ which is continuous, decreasing, and piecewise differentiable and satisfies the condition that $R_n^{**}(S) = -S\frac{\phi_n^{**\prime}(S)}{\phi_n^{**}(S)}$ is $(m+1)$-segment piecewise constant and decreasing.*

Proof: As the representative agent's utility function is strictly increasing, concave, continuously differentiable, and piecewise twice differentiable, the stock and options are all priced by a strictly positive, continuous, and piecewise differentiable pricing kernel $\phi(S)$. Moreover, as the representative agent's absolute risk aversion is decreasing, $R(S) = -\frac{\phi'(S)}{\phi(S)}$ is decreasing. The existence of the solutions specified in the proposition is guaranteed by the above conditions. The proof is in the same spirit as that of Proposition 1, and it is omitted for brevity. We now prove that the specified pricing kernels give option bounds as is stated in the proposition. We need only prove the case $R(S) = -\frac{\phi'(S)}{\phi(S)}$ is strictly decreasing. The non-strict case is then proved by constructing a sequence of strict cases converging to the non-strict case.

We only show the proof for the case where n is even; when n is odd, the proof is similar. As $R(S) = -\frac{\phi'(S)}{\phi(S)} \geq 0$ is strictly decreasing and $R_n^{**}(S) = -S\frac{\phi_n^{**\prime}(S)}{\phi_n^{**}(S)}$ is $(m+1)$-segment piecewise constant and decreasing and has a value zero on the last segment, it is clear that $R(S)$ and $R_n^{**}(S)$ can have at most $2 \times m = n$ times. From basic calculus, as both $\phi_n^{**}(S)$ and $\phi(S)$ are continuous and both $R_n^{**}(x)$ and $R(x)$ are Riemann integrable on a bounded interval, we have for all $s, S \in (\underline{s}, \bar{s})$, $\ln\frac{\phi_n^{**}(S)}{\phi(S)} - \ln\frac{\phi_n^{**}(s)}{\phi(s)} = \int_s^S (R_n^{**}(x) - R(x))dx$. From the above two statements, it is clear that $\phi_n^{**}(S)$ and $\phi(S)$ can cross at most $n+1$ times. The rest of the proof is the same as that of Proposition 6, and it is omitted for brevity. Q.E.D

It is straightforward to see that in the special where there is no observed option, the above solution gives the DARA option bounds derived by Mathur and Ritchken (2000).

4 Conclusions

In this paper we have developed a unified and convenient approach to option bounds, which can be used to solve most problems in the theory of option pricing bounds with restrictions on probability distributions or risk preferences, given the prices of the underlying stock and multiple observed options. Assuming that we consider a particular set of PDFs (CDFs), if we can find a PDF (CDF) in the closure of this set which crosses all these admissible PDFs (CDFs) $n+2$ times ($n+1$ times), and preserves the values of the underlying stock and n observed options on the stock, then this PDF (CDF) can be used to calculate the optimal bounds on the prices of all options. A similar idea also works when we calculate option bounds subject to a particular set of pricing kernels derived from a particular class of risk preferences. The theorem presented in this paper can not only help to prove the existence of such an optimal PDF, CDF, or pricing kernel but also help to identify it.

Using the approach, we have derived optimal option bounds with various restrictions on risk neutral probability distributions or risk preferences which are of particular interest, given the prices of the underlying stock and multiple observed options. However, because of limited space, detailed solutions of these option bounds are not calculated or analyzed. The analysis of these option bounds is left to follow-up studies which will cast some interesting insight into the effects of risk and risk preferences on option prices.

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Appendix 1 Existence of $q_n^{**}(S)$ and $q_n^*(S)$

We need prove the following result.

Lemma 2 *Let the support of the stock price distribution be* $[\underline{s}, \bar{s}] \subset [0, \infty)$. *Assume that the stock and* T *options with strike prices* $K_1, K_2, ..., K_T$, *where* $T \geq 0$ *and* $\underline{s} < K_1 < K_2 < ... < K_T < \bar{s}$, *are all priced by a PDF* $q(S)$ *which is bounded below by* $\underline{q}$ *and above by* $\bar{q}$, *where* $0 \leq \underline{q} < \bar{q}$. *Then, there always exist* $q_T^{**}(S)$ *and* $q_T^*(S)$ *which give bounds on option prices as stated in Proposition 1.*

We prove the lemma by induction. Let the forward prices of the underlying stock and n options with strike prices K_1, K_2, ..., K_n be S_0, c_0^1, c_0^2, ..., c_0^n respectively. When necessary we write $q_T^{**}(S)$ and $q_T^*(S)$ corresponding to $\underline{s}, \bar{s}, \underline{q}, \bar{q}, K_1, ..., K_T, S_0, c_0^1$, ..., and c_0^T explicitly as $q_T^{**}(S; \underline{s}, \bar{s}, \underline{q}, \bar{q}, K_1, ..., K_T, S_0, c_0^1, ..., c_0^T)$ and $q_T^*(S; \underline{s}, \bar{s}, \underline{q}, \bar{q}, K_1, ..., K_T, S_0, c_0^1, ..., c_0^T)$ respectively.

We first prove that when $T = 0$ the lemma is true. Let $q_{00}^{**}(S)$ be a probability density function such that $q_{00}^{**}(S) = \bar{q}$, for $S \in (\underline{s}, s_{0,0,1}^{**})$, and $q_{00}^{**}(S) = \underline{q}$, for $S \in (s_{0,0,1}^{**}, \bar{s})$. Let $q_{00}^*(S)$ be a probability density function such that $q_{00}^*(S) = \underline{q}$, for $S \in (\underline{s}, s_{0,0,1}^*)$, and $q_{00}^*(S) = \bar{q}$, for $S \in (s_{0,0,1}^*, \bar{s})$.[24] As $q_{00}^{**}(S)$ crosses any admissible PDF once from above, it must under-price the stock. Similarly, as $q_{00}^*(S)$ crosses any admissible PDF once from below, it must over-price the stock.

We now show the existence of $q_0^{**}(S)$. Given any $s_{0,1} \in [\underline{s}, s_{0,0,1}^{**}]$, let $\hat{q}(S; s_{0,1})$ be a probability density function such that $\hat{q}_0(S; s_{0,1}) = \bar{q}$, $S < s_{0,1}$; $\hat{q}_0(S; s_{0,1}) = \underline{q}$, $s_{0,1} < S < s_{0,2}$; $\hat{q}_0(S; s_{0,1}) = \bar{q}$, $S > s_{0,2}$.[25] As $\hat{q}(S; \underline{s}) = q_{00}^{**}(S)$ under-prices the stock while $\hat{q}(S; s_{0,0,1}^{**}) = q_{00}^*(S)$ over-prices the stock and $\int_{\underline{s}}^{\bar{s}} S\hat{q}_0(S; s_{0,1})dS$ is continuous with respect to $s_{0,1}$, from the well-known intermediate value theorem, we conclude that there exists $s_{0,1} \in [\underline{s}, s_{0,0,1}^{**}]$ such that $q_0^{**}(S) = \hat{q}(S; s_{0,1})$ is a probability density function which prices the stock

[24] It is not difficult to see that the existence of $q_{00}^{**}(S)$ and $q_{00}^*(S)$ is guaranteed by the existence of a PDF which is bounded below by $\underline{q}$ and above by $\bar{q}$.

[25] The existence of $q_{00}^{**}(S)$ guarantees the existence of $\hat{q}(S; s_{0,1})$ as $\hat{q}(S; s_{0,1})$ is obtained from $q_{00}^{**}(S)$ by moving some of the probability mass at the left end of the support to the right end.

correctly. This proves the existence of $q_0^{**}(S)$. The existence of $\phi_0^*(S)$ can be similarly proved. Finally, as $q_0^{**}(S)$ is a mean-preserving spread and $q_0^*(S)$ is a mean-preserving contraction, it is well known that $q_0^{**}(S)$ gives higher option prices than $q(S)$ while $q_0^*(S)$ gives lower option prices. This proves the case where $T = 0$.

Now arbitrarily given $n \geq 1$, we need prove that if the lemma is true for $T = n - 1$ then the lemma is true for $T = n$. We first prove the case where $\underline{q} > 0$.

Since for arbitrarily small $\varepsilon > 0$, for all $S \in (\underline{s}, \bar{s})$, $0 < \underline{q} - \varepsilon < q(S) < \underline{q} + \varepsilon$, as is assumed, there exist $q_{n-1}^*(S;\varepsilon) = q_{n-1}^*(S; \underline{s}, \bar{s}, \underline{q} - \varepsilon, \bar{q} + \varepsilon, K_1, ..., K_{n-1}, S_0, c_0^1, ..., c_0^{n-1})$ and $q_{n-1}^{**}(S;\varepsilon) = q_{n-1}^{**}(S; \underline{s}, \bar{s}, \underline{q} - \varepsilon, \bar{q} + \varepsilon, K_1, ..., K_{n-1}, S_0, c_0^1, ..., c_0^{n-1})$, where the PDF $q_{n-1}^*(S;\varepsilon)$ (the PDF $q_{n-1}^{**}(S;\varepsilon)$) is $(n+2)$-segment and piecewise constant, has a constant value $\underline{q} - \varepsilon$ at odd (even) segments and a constant value $\bar{q} + \varepsilon$ at even (odd) segments, and prices the stock and the first $n - 1$ observed options correctly.

Let the support of the ith segment of $q_{n-1}^*(S;\varepsilon)$ be denoted by $[s_{n-1,i-1}^*(\varepsilon), s_{n-1,i}^*(\varepsilon)]$, where $i = 1, ..., n+3$ and $\underline{s} = s_{n-1,0}^*(\varepsilon) < ... < s_{n-1,n+3}^*(\varepsilon) = \bar{s}$.[26] We assert that given any $a \in [\underline{s}, s_{n-1,1}^*(\varepsilon)]$, there exists a PDF $\varphi_n^*(S;\varepsilon,a)$ which is $(n+3)$-segment piecewise constant with the support of the first segment being $[s_0, a]$, has a constant value $\underline{q} - \varepsilon$ at odd segments and a constant value $\bar{q} + \varepsilon$ at even segments, and prices the stock and the first $n - 1$ observed options correctly. This is proved as follows.

First, $\varphi_n^*(S;\varepsilon,a)|_{a=s_{n-1,1}^*(\varepsilon)} = q_{n-1}^*(S;\varepsilon)$ and $\varphi_n^*(S;\varepsilon,a)|_{a=\underline{s}} = q_{n-1}^{**}(S;\varepsilon)$ are two desired PDFs where in the first case the last of the $n + 3$ segments has a length of zero while in the second case the first of the $n + 3$ segments has a length of zero. Now given any $a \in (\underline{s}, s_{n-1,1}^*(\varepsilon))$, consider the case where the stock price distributions are defined on $[a, \bar{s}]$.[27] For all $S \in (a, \bar{s})$,

[26] See Footnote 12 for an explanation of these strict inequalities.

[27] We must have $K_1 > s_{n-1,1}^*(\varepsilon)$; otherwise, the forward price of the put option with a strike price K_1 will be $\int_{\underline{s}}^{K_1}(K_1 - S)(\underline{q} - \varepsilon)dS < \int_{\underline{s}}^{K_1}(K_1 - S)\underline{q}dS \leq \int_{\underline{s}}^{K_1}(K_1 - S)q(S)dS$ which causes a contradiction. It follows that $K_1 > a$.

let $\hat{q}(S;\varepsilon,a) = \alpha q^*_{n-1}(S;\underline{s},\bar{s},\underline{q}-\varepsilon,\bar{q}+\varepsilon,K_1,...,K_{n-1},\ S_0,c^1_0,...,c^{n-1}_0)$, where $\alpha = \frac{1}{1-(\underline{q}-\varepsilon)(a-\underline{s})} > 0$. We have $\int_a^{\bar{s}} \hat{q}(S;\varepsilon,a)dS = 1$. Let $\hat{S}_0 = \int_a^{\bar{s}} \hat{q}(S;\varepsilon,a))dS$ and $\hat{c}^i_0 = \int_a^{\bar{s}} c^i(S)\hat{q}(S;\varepsilon,a)dS$, $i = 1,...,n-1$.

As the forward prices $\hat{S}_0$, $\hat{c}^i_0$, $i = 1,...,n-1$, are given by the risk neutral density $\hat{q}(S;\varepsilon,a)$, which is bounded below by $\alpha(\underline{q}-\varepsilon)$ and above by $\alpha(\bar{q}+\varepsilon)$, according to the previously made assumption, we must have

$$q^{**}_{n-1}(S;a,\bar{s},K_1,...,K_{n-1},\alpha(\underline{q}-\varepsilon),\alpha(\bar{q}+\varepsilon),\hat{S}_0,\hat{c}^1_0,...,\hat{c}^{n-1}_0),$$

which is defined on $[a,\bar{s}]$ and gives the same prices of the stock and the first $n-1$ options as $\hat{q}(S;\varepsilon,a)$ does. Now let $\varphi^*_n(S;\varepsilon,a)$ be defined as follows:

$$\left\{ \begin{array}{ll} \frac{1}{\alpha}q^{**}_{n-1}(S;a,\bar{s},\alpha(\underline{q}-\varepsilon),\alpha(\bar{q}+\varepsilon),K_1,...,K_{n-1},\hat{S}_0,\hat{c}^1_0,...,\hat{c}^{n-1}_0), & S \geq a \\ \underline{q}-\varepsilon, & S < a, \end{array} \right.$$

where $\alpha = \frac{1}{1-(\underline{q}-\varepsilon)(a-\underline{s})}$. It is straightforward that this PDF is $(n+3)$-segment and piecewise constant and has a constant value $\underline{q}-\varepsilon$ at odd (even) segments and a constant value $\bar{q}+\varepsilon$ at even (odd) segments with the support of its first segment being $[\underline{s},a]$. Moreover, some simple calculations show that[28]

$$\int_{\underline{s}}^{\bar{s}} \varphi^*_n(S;\varepsilon,a)dS = 1,$$
$$\int_{\underline{s}}^{\bar{s}} \varphi^*_n(S;\varepsilon,a)dS = S_0,$$
$$\int_{\underline{s}}^{\bar{s}} c^i(S)\varphi^*_n(S;\varepsilon,a)dS = c^i_0, \quad i = 1,...,n-1.$$

Hence we conclude that for all $a \in [\underline{s}, s^*_{n-1,1}(\varepsilon)]$, there exists a PDF $\varphi^*_n(S;\varepsilon,a)$ which is $(n+3)$-segment piecewise constant, has a constant value $\underline{q}-\varepsilon$ at odd segments and a constant value $\bar{q}+\varepsilon$ at even segments with the support of its first segment being $[\underline{s},a]$, and prices the underlying stock and the $n-1$ options correctly. Moreover, as was mentioned earlier, when $a = s^*_{n-1,1}(\varepsilon)$, $\varphi^*_n(S;\varepsilon,a) = q^*_{n-1}(S;\varepsilon)$; when $a = \underline{s}$, $\varphi^*_n(S;\varepsilon,a) = q^{**}_{n-1}(S;\varepsilon)$. As these two PDFs give lower and upper bounds on the price of the nth option and $\int_{\underline{s}}^{\bar{s}} c(S,K_n)\varphi^*_n(S;\varepsilon,a)dS$

[28] This is not difficult to see as $q(S) = \frac{1}{\alpha}\hat{q}(S;\varepsilon,a)$, if $S \geq a$; $\underline{q}-\varepsilon$, if $S < a$.

is continuous with respect to a, from the well-known intermediate value theorem, we must have a PDF $\varphi_n^*(S;\varepsilon,a^*)$, where $a^* \in [\underline{s}, s^*_{n-1,1}(\varepsilon)]$. Note that, $\varphi_n^*(S;\varepsilon,a^*)$ is $(n+3)$-segment piecewise constant, has a constant value $\underline{q}-\varepsilon$ at odd segments and a constant value $\bar{q}+\varepsilon$ at even segments, and prices the stock and the first n observed options correctly; thus we have proved the existence of $q_n^*(S;\varepsilon) = q_n^*(S;\underline{s},\bar{s},\underline{q}-\varepsilon,\bar{q}+\varepsilon,K_1,...,K_n,S_0,c_0^1,...,c_0^n) = \varphi_n^*(S;\varepsilon,a^*)$. Similarly, we can prove the existence of $q_n^{**}(S;\varepsilon) = q_n^{**}(S;\underline{s},\bar{s},\underline{q}-\varepsilon,\bar{q}+\varepsilon,K_1,...,K_n,S_0,c_0^1,...,c_0^n)$. As is shown in the first half of the proof of Proposition 1 in Section 2.1, since $\underline{q}-\varepsilon < q(S) < \bar{q}+\varepsilon$, the above two PDFs give bounds on all options as stated in the proposition. Letting $\varepsilon \to 0$, we obtain $q_n^*(S) = q_n^*(S;\underline{s},\bar{s},\underline{q},\bar{q},K_1,...,K_n,S_0,c_0^1,\ ...,c_0^n)$ and $q_n^{**}(S) = q_n^{**}(S;\underline{s},\bar{s},\underline{q},\bar{q},K_1,...,K_n,S_0,c_0^1,...,c_0^n)$ which give bounds on all options as stated in the proposition.[29]

To prove the case where $\underline{q}=0$, as in the first half of the proof of Proposition 1 in Section 2.1, consider $q(S;\varepsilon) = \frac{q(S)+\varepsilon}{1+\varepsilon(\bar{s}-\underline{s})}$, where $\varepsilon > 0$ is arbitrarily small. We have $0 < \frac{\varepsilon}{1+\varepsilon(\bar{s}-\underline{s})} = \underline{q}(\varepsilon) \leq q(S;\varepsilon) \leq \bar{q}$ and $\int_{\underline{s}}^{\bar{s}} q(S;\varepsilon)dS = 1$. As is proved in the above argument, the results is valid for $q(S;\varepsilon)$ which is bounded below by $\underline{q}(\varepsilon) = \frac{\varepsilon}{1+\varepsilon(\bar{s}-\underline{s})}$ and above by $\bar{q}$. When $\varepsilon \to 0$, using an argument similar to the first half of the proof of Proposition 1 in Section 2.1 for the existence of a converging subsequence, we prove the result in the case where $\underline{q}=0$. Q.E.D.

[29] We need only have a convergent subsequence of $q_n^*(S;\varepsilon)$ and $q_n^{**}(S;\varepsilon)$ when $\varepsilon \to 0$. The argument for the existence of such a convergent subsequence is the same as in the first half of the proof of Proposition 1 in Section 2.1.