KW-sections for Vinberg's θ-groups of exceptional type

Paul Levy
paul.levy@epfl.ch

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Abstract

Let k be an algebraically closed field of characteristic not equal to 2 or 3, let G be a simple algebraic group of type F_4, G_2 or D_4 and let θ be a semisimple automorphism of G of finite order. In this paper we consider the θ-group (in the sense of Vinberg) associated to these choices; we classify the positive rank automorphisms via Kac diagrams and we describe the little Weyl group in each case. As a result we show that all θ-groups in types G_2, F_4 and D_4 have KW-sections, confirming a conjecture of Popov in these cases.

0 Introduction

Let G be a reductive algebraic group over the algebraically closed field k and let $\mathfrak{g} = \mathrm{Lie}(G)$. Let θ be a semisimple automorphism of G of order m, let $d\theta$ be the differential of θ and let ζ be a primitive m-th root of unity in k. (Thus if k is of positive characteristic p then $p \nmid m$.) There is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}(0) \oplus \ldots \oplus \mathfrak{g}(m-1) \text{ where } \mathfrak{g}(i) = \{x \in \mathfrak{g} \,|\, d\theta(x) = \zeta^i x\}$$

This is a $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$, that is $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$ $(i, j \in \mathbb{Z}/m\mathbb{Z})$. Let $G(0) = (G^\theta)^\circ$. Then $G(0)$ is reductive, $\mathrm{Lie}(G(0)) = \mathfrak{g}(0)$ and $G(0)$ stabilizes each component $\mathfrak{g}(i)$. In [24], Vinberg studied invariant-theoretic properties of the $G(0)$-representation $\mathfrak{g}(1)$. The central concept in [24] is that of a *Cartan subspace*, which is a subspace of $\mathfrak{g}(1)$ which is maximal subject to being commutative and consisting of semisimple elements. The principal results of [24] (for $k = \mathbb{C}$) are:

- any two Cartan subspaces of $\mathfrak{g}(1)$ are $G(0)$-conjugate and any semisimple element of $\mathfrak{g}(1)$ is contained in a Cartan subspace.

- the $G(0)$-orbit through $x \in \mathfrak{g}(1)$ is closed if and only if x is semisimple, and is unstable (that is, its closure contains 0) if and only if x is nilpotent.

- let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{g}(1)$ and let $W_\mathfrak{c} = N_{G(0)}(\mathfrak{c})/Z_{G(0)}(\mathfrak{c})$, the *little Weyl group*. Then we have a version of the Chevalley restriction theorem: the embedding $\mathfrak{c} \hookrightarrow \mathfrak{g}(1)$ induces an isomorphism $k[\mathfrak{g}(1)]^{G(0)} \to k[\mathfrak{c}]^{W_\mathfrak{c}}$.

- $W_\mathfrak{c}$ is a finite group generated by complex (often called pseudo-)reflections, hence $k[\mathfrak{c}]^{W_\mathfrak{c}}$ is a polynomial ring.

In the case of an involution, the decomposition $\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ is the symmetric space decomposition, much studied since the seminal paper of Kostant and Rallis [11]. (Many of the results of [11] were generalized to good positive characteristic by the author in [12].) While the theory of θ-groups can in some ways be thought of as an extension of the theory of symmetric spaces, there are certain differences of emphasis. Broadly speaking, one can say that the results here on geometry and orbits are weaker than for symmetric spaces, but the connection with groups generated by pseudoreflections is more interesting.

As outlined in [25, §8.8], for a particularly nice action of a reductive algebraic group G on a vector space V there may exist an affine linear subvariety $W \subset V$, called a Weierstrass section, such that restricting to W induces an isomorphism $k[V]^G \to k[W]$, i.e. such that $W \hookrightarrow V$ is a section for the quotient morphism $V \to V/\!\!/G = \mathrm{Spec}(k[V]^G)$. In [16] the more specific terminology of *Kostant-Weierstrass*, or *KW*-section was introduced for the case of $G(0)$ acting on $\mathfrak{g}(1)$, because of the similarity to Kostant's slice to the regular orbits in the adjoint representation. This has now become standard terminology in the theory of θ-groups. A long-standing conjecture of Popov [17] is the existence of a KW-section for any θ-group. In characteristic zero, this conjecture was proved by Panyushev in the cases when $G(0)$ is semisimple [15] and when $\mathfrak{g}(1)$ contains a regular nilpotent element of $\mathfrak{g}$ (the 'N-regular' case) [16]. Popov's conjecture is trivially true unless the invariants $k[\mathfrak{g}(1)]^{G(0)}$ are non-trivial, which holds if and only if the dimension of a Cartan subspace is positive; such cases are called *positive rank* θ-groups.

In [13], the results of [24] were extended to the case where k has positive characteristic p and G satisfies the *standard hypotheses*: (A) p is good for G, (B) the derived subgroup G' of G is simply-connected, and (C) there exists a non-degenerate G-equivariant symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$. (In fact, most of the results mentioned above hold for all $p > 2$; the standard hypotheses were required for the proof that the little Weyl group is generated by pseudo-reflections.) Moreover, an analysis of the little Weyl group and an extension and application of Panyushev's result on N-regular automorphisms revealed that KW-sections exist for all classical graded Lie algebras in zero or odd positive characteristic [13, Thm. 5.5]. Under the assumption of the standard hypotheses, the problem of the existence of a KW-section can be reduced to the case of a simple Lie algebra or $\mathfrak{gl}_n$, as indicated in [13, §3]. Thus [13, Thm. 5.5] reduces the proof of Popov's conjecture to the following cases: (i) G is of exceptional type and p is good; or (ii) G is simply-connected of type D_4, $p > 3$ and θ is an outer automorphism of G such that θ^3 is inner. Following Vinberg, we refer to all such cases as exceptional type θ-groups.

The automorphisms of finite order of a simple complex Lie algebra were classified by Kac [9]. In Sect. 2 we give a 'constructive' proof that Kac's classification extends to characteristic p, considering only automorphisms of order coprime to p. (The extension of Kac's results to positive characteristic was outlined by Serre [20]. A proof in the language of Tits buildings has recently been given in joint work of the author and Reeder, Yu and Gross [19]. We include another proof here in order to make this work as accessible as possible, and to describe explicit representatives of each conjugacy class of automorphisms.) Subsequently, we determine in Sect. 4 the positive rank exceptional θ-groups of types F_4, G_2 and D_4 and describe the corresponding little Weyl groups in Sect. 5. Our method is a continuation of the method used in [13] to determine the Weyl group for the classical graded Lie algebras in positive characteristic. In particular, given an automorphism θ let T be a θ-stable maximal torus such that $\mathrm{Lie}(T)$ contains a Cartan subspace. Then $\theta = \mathrm{Int}\, n_w$ where $n_w \in N_{\mathrm{Aut}\, G}(T)$, and $w = n_w T \in N_{\mathrm{Aut}\, G}(T)/T$ is either of order m, or is trivial (in which case θ is of zero rank). Using Carter's classification of conjugacy classes in the Weyl group, this approach gives us a relatively straightforward means to determine the positive rank automorphisms and their Weyl groups (see Tables 1-3). This classification allows us to show that all θ-groups in types G_2, F_4 or D_4 have KW-sections (see Thm. 5.11):

Theorem. *Any θ-group of type G_2, F_4 or D_4 in characteristic $\neq 2, 3$ admits a KW-section.*

Together with [13], this leaves only the θ-groups of type E to deal with. The existence of KW-sections for such θ-groups has been proved by somewhat different methods in the [19].

Notation.

Throughout, G will denote a simple (semisimple) algebraic group and $\mathfrak{g}$ its Lie algebra. All automorphisms of G are understood to be rational (i.e. algebraic group) automorphisms; we denote by $\operatorname{Aut} G$ the group of automorphisms of G, by $\operatorname{Int} G$ the group of inner automorphisms, and by $\operatorname{Ad} G$ the image of G in the adjoint representation. If T is a maximal torus of G and $\alpha \in \Phi(G, T)$ then we denote the corresponding coroot by $\alpha^\vee$, which we consider as a cocharacter $k^\times \to T$. If H is a subgroup of G then $N_G(H)$ (resp. $Z_G(H)$) will denote its normalizer (resp. centralizer) in G. Similar notation will be used with respect to subsets of G or $\mathfrak{g}$, with the exception that if $x \in G$ then the centralizer of x in $\mathfrak{g}$ is denoted $\mathfrak{g}^x$. The connected component of an algebraic group H is denoted H°; the derived subgroup of a connected algebraic group is denoted H'. We denote by μ_s the cyclic subgroup scheme of order s of the multiplicative group; we will generally assume that s is coprime to p, in which case one can identify μ_s with its set of k-points. Throughout the paper we will assume that m is coprime to the characteristic of k, if the latter is positive.

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1 Preparation

We continue with the basic set-up of the introduction. Here we recall some results and definitions which will be necessary in what follows.

Let Φ be an irreducible root system with basis $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Let $\hat{\alpha} = \sum_{i=1}^n m_i \alpha_i$ be the highest root with respect to Δ. Recall that p is *good* for G if $p > m_i$ for all i, $1 \leq i \leq n$; otherwise p is bad [22, I.4.3]. Specifically, p is bad if either: $p = 2$ and Φ is not of type A; $p = 3$ and Φ is of exceptional type, or; $p = 5$ and Φ is of type E_8. If Φ is a reducible root system, then p is good for Φ if and only if p is good for each irreducible component of Φ; p is good for the reductive group G if and only if it is good for the root system of G. If G is simple and of exceptional type and p is good for G, then G is separably isogenous to a group satisfying the standard hypotheses [22, I.5.3]. (In particular, if G is of type F_4 or G_2 then (B) is automatic and (C) is implied by (A).)

For the time being let T be any algebraic torus and let θ be an automorphism of T of finite order m. If θ is an involution then it is not difficult to see that there is a decomposition $T = T_+ \cdot T_-$, where $T_+ = \{t \in T \,|\, \theta(t) = t\}^\circ$ and $T_- = \{t \in T \,|\, \theta(t) = t^{-1}\}^\circ$. In [13] we introduced analogues of $T_\pm$ for an automorphism of arbitrary finite order m. (If the characteristic is $p < 0$, then we require $p \nmid m$.) Let $p_d(x)$ be the (monic) minimal polynomial of $e^{2\pi i/d}$ over $\mathbb{Q}$. Since p_d has coefficients in $\mathbb{Z}$, we can (and will) consider it as a polynomial in $\mathbb{F}_p[x]$ as well. If $p \nmid d$ then $p_d(x)$ has no repeated roots in k. For any polynomial

$h = \sum_{i=0}^n a_i x^i \in \mathbb{Z}[x]$, let $\overline{h}(\theta)$ denote the algebraic endomorphism $T \to T$, $t \mapsto \prod_{i=0}^n \theta^i(t^{a_i})$. Then the map $\mathbb{Z}[x] \to \operatorname{End} T$, $h \mapsto \overline{h}(\theta)$ is a homomorphism of rings, where the addition in $\operatorname{End} T$ is pointwise multiplication of endomorphisms, and the multiplication is composition.

Lemma 1.1 ([13, Lemma 1.10])**.** *For each $d|m$, let $T_d = \{t \in T \,|\, \overline{p_{m/d}}(\theta)(t) = e\}^\circ$. Then $T = \prod_{d|m} T_d$ and $T_{d_1} \cap T_{d_2}$ is finite for any distinct d_1, d_2. Moreover, $\mathfrak{t} = \sum_{i=0}^{m-1} \mathfrak{t}(i)$ where $\mathfrak{t}(i) = \{t \in \mathfrak{t} : d\theta(t) = \zeta^i t\}$ and $\operatorname{Lie}(T_d) = \sum_{(i,m)=d} \mathfrak{t}(i)$.*

Remark 1.2. An alternative description of the tori T_d is given in [3]. Since θ is an algebraic automorphism of T, it induces an automorphism θ^* of $Y(T) := \operatorname{Hom}(k^\times, T)$ given by $\theta^*(\lambda)(t) = \theta(\lambda(t))$. Then set $Y(T_d) = \ker p_d(\theta^*)$. This also allows us to define the tori T_d when m is divisible by p; however, in this case we do not have a direct sum decomposition $\mathfrak{t} = \oplus_{d|m} \operatorname{Lie}(T_d)$.

An immediate application of Lemma 1.1 is the following.

Lemma 1.3. *Let T be a maximal torus of G and let $w \in W = N_G(T)/T$ be of order m ($p \nmid m$) and have finitely many fixed points on T. Then any two representatives for w in $N_G(T)$ are T-conjugate, and have the same m-th power.*

Proof. Suppose $n_w \in N_G(T)$ is such that $w = n_w T$. Then $\operatorname{Int} n_w|_T$ is an automorphism of T of order m. We have $T = \prod_{i|m, i\neq m} T_i$ and therefore $tw(t)w^2(t)\ldots w^{m-1}(t) = 1$ for all $t \in T$. Thus $(n_w t)^m = n_w^m$ for all $t \in T$. Moreover, since $T_m = \{1\}$, the map $T \to T$, $t \mapsto tw(t^{-1})$ is surjective. In particular, if $t \in T$ then there exists $s \in T$ such that $tn_w = sn_w s^{-1}$. □

From now on θ will be an automorphism of G of order m. Fix a primitive m-th root ζ of 1 in k and let $\mathfrak{g}(i) = \{x \in \mathfrak{g} \,|\, d\theta(x) = \zeta^i x\}$. A θ-stable torus in G is **θ-split** if $T = T_1$, and is **θ-anisotropic** if $T_m = \{e\}$, that is, if θ has finitely many fixed points on T. We recall that a Cartan subspace of $\mathfrak{g}(1)$ is a maximal subspace which is commutative and consists of semisimple elements. We have the following results on Cartan subspaces.

Lemma 1.4. *(a) Any two Cartan subspaces of $\mathfrak{g}(1)$ are $G(0)$-conjugate, and any semisimple element of $\mathfrak{g}(1)$ is contained in a Cartan subspace.*

(b) Let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{g}(1)$. There is a θ-split torus T_1 in G such that $\mathfrak{c} \subset \operatorname{Lie}(T_1)$. We have $\dim T_1 = \dim \mathfrak{c} \cdot \varphi(m)$, where $\varphi(m)$ is the Euler number of m. (In particular, $\dim(\operatorname{Lie}(T_1) \cap \mathfrak{g}(i)) = \dim \mathfrak{c}$ for any i coprime to m.) If $p > 2$, then T_1 is unique.

(c) Let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{g}(1)$ and let $W_\mathfrak{c} = N_{G(0)}(\mathfrak{c})/Z_{G(0)}(\mathfrak{c})$, the little Weyl group. If $\operatorname{char} k \neq 2$, then the embedding $\mathfrak{c} \hookrightarrow \mathfrak{g}(1)$ induces an isomorphism $k[\mathfrak{g}(1)]^{G(0)} \to k[\mathfrak{c}]^{W_\mathfrak{c}}$.

(d) If $\operatorname{char} k = 0$ or if G satisfies the standard hypotheses, then $W_\mathfrak{c}$ is generated by pseudoreflections and $k[\mathfrak{c}]^{W_\mathfrak{c}}$ is a polynomial ring.

Proof. Part (a) is [24, Thm. 1], and [13, Thm. 2.5 and Cor. 2.6]. For (b), see [24, §3.1] and [13, Lemma 2.7]. Part (c) was [24, Th. 7], [13, Thm. 2.20]. Finally, part (d) is Thm. 8 in [24] and [13, Prop. 4.22]. □

We recall the following results of Steinberg [23, 7.5,9.16].

Lemma 1.5. *(a) If θ is a semisimple automorphism of G then there is a θ-stable Borel subgroup B of G and a θ-stable maximal torus T of B.*

(b) Let $\pi : \hat{G} \to G$ be the universal cover of the semisimple group G. Then there exists a unique automorphism $\hat{\theta}$ of $\hat{G}$ such that $\pi(\hat{\theta}(g)) = \theta(\pi(g))$ for all $g \in \hat{G}$. Moreover, if θ is of order m then so is $\hat{\theta}$.

Let G be a simple, simply-connected group and let $\mathfrak{g} = \mathrm{Lie}(G)$. Denote by $\mathrm{Aut}\, G$ (resp. $\mathrm{Aut}\,\mathfrak{g}$) the algebraic group of rational (resp. restricted Lie algebra) automorphisms of G (resp. $\mathfrak{g}$). In characteristic zero it is well-known that all automorphisms of $\mathfrak{g}$ arise as differentials of automorphisms of G, and the corresponding map $\mathrm{Aut}\, G \to \mathrm{Aut}\,\mathfrak{g}$ is bijective. This may fail to be true in small characteristic [7].

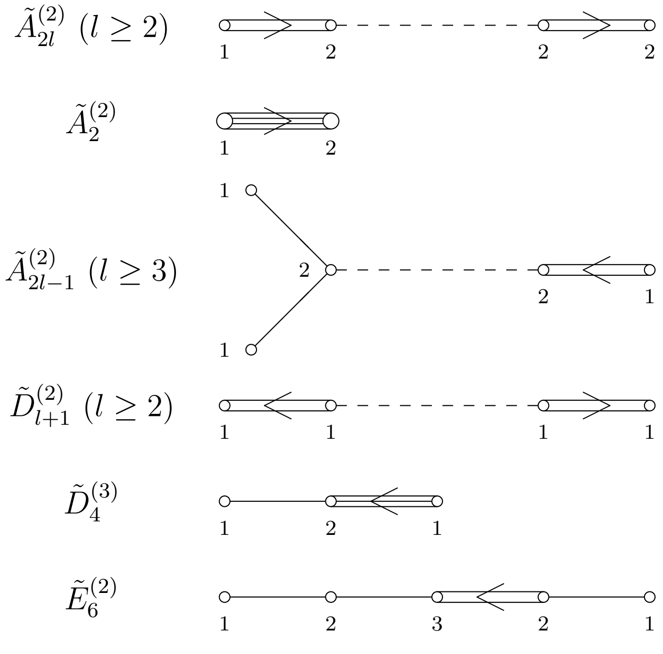
Lemma 1.6. *If $p > 2$ then differentiation $d : \mathrm{Aut}\, G \to \mathrm{Aut}\,\mathfrak{g}$ is bijective.*

Proof. If $G = \mathrm{SL}(n, k)$ then this was proved in [12, Lemma 1.4]. Otherwise the automorphism group of $\mathfrak{g}$ as an abstract Lie algebra is isomorphic to $\mathrm{Ad}\, G \rtimes \mathcal{D}$, where $\mathcal{D}$ is the group of 'graph automorphisms' of the Dynkin diagram of G [7, II. Table 1]. Furthermore, $\mathrm{Aut}\, G$ is isomorphic to $\mathrm{Int}\, G \rtimes \mathcal{D}$ where $\mathrm{Int}\, G$ denotes the group of all inner automorphisms [8, 27.4]; since $Z(G) = \ker \mathrm{Ad}$ is the intersection of the kernels of the roots on a fixed maximal torus, it follows that the map $d : \mathrm{Aut}\, G \to \mathrm{Aut}\,\mathfrak{g}$ is bijective. $\square$

2 Kac diagrams

Here we recall Kac's classification [9] of periodic (i.e. finite order) automorphisms of simple Lie algebras in characteristic zero, and give a 'constructive' proof that the classification extends to positive characteristic. (We consider only automorphisms of order coprime to the characteristic.) For more details on the classification, we recommend Chapters 6 to 8 of Kac's book [10]. As before, let G be simple and simply-connected with maximal torus T, let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a basis of the root system $\Phi = \Phi(G, T)$ and let $\tilde{\Delta} = \Delta \cup \{\alpha_0\}$ be a basis of the (untwisted) affine root system. Let $\{h_\alpha, e_\beta \,|\, \alpha \in \Delta, \beta \in \Phi\}$ be a Chevalley basis for G. Recall that $\mathrm{Aut}\, G/\, \mathrm{Int}\, G$ is finite. If θ is an automorphism of G then the *index* of θ is the order of θ modulo the inner automorphisms.

Let s be the index of some automorphism of G and let $\tilde{\Delta}^{(s)} = \{\beta_0, \ldots, \beta_l\}$ be a basis of the affine type root system associated to G and s. If $s = 1$ then $\tilde{\Delta}^{(s)} = \tilde{\Delta}$; for $s > 1$ the corresponding affine Dynkin diagram is as follows: (type $\tilde{A}_3^{(2)}$ is given by $\tilde{D}_3^{(2)}$):



(The diagrams for $\tilde{A}_{2n}^{(2)}$ are not given in their usual orientation. This choice of orientation is due to our 'type-free' choice of graph automorphism, see the discussion below.)

Let $\tilde{\Phi}^{(s)}$ be the affine root system with basis $\tilde{\Delta}^{(s)}$ and let $\delta = \sum_{i=0}^{l} m_i\beta_i$ be the smallest positive imaginary root in $\tilde{\Phi}^{(s)}$. Specifically, for $s = 1$: $m_0 = 1$ and for $1 \leq i \leq l$, m_i is the coefficient of α_i in the expression for the highest root $\hat{\alpha}$ in Φ; for the "twisted" cases $s > 1$ the numbers $m_0, \ldots, m_l$ are as indicated on the diagrams above. (Below we will fix a numbering of $\tilde{\Delta}^{(s)}$, $s > 1$. For the Dynkin diagrams of finite type we use Kac's numbering [10].)

Recall that a *Kac diagram* μ is simply a copy of $\tilde{\Delta}^{(s)}$ with non-negative integer weights $n_0, \ldots, n_l$ attached to each node. We denote by $\mu(\alpha_i)$ the integer n_i, therefore considering μ as a $\mathbb{Z}$-linear function from $\mathbb{Z}\tilde{\Delta}^{(s)}$ to $\mathbb{Z}$. The *order* of μ is $s(m_0n_0 + \ldots + m_ln_l)$. From now on, we assume that any Kac diagram μ is primitive, i.e., that the highest common factor of $n_0, \ldots, n_l$ is 1. To each Kac diagram of order m we can associate an automorphism (which we call a *Kac automorphism*) of $\mathfrak{g}$ of order m, as we shall explain below; Kac's theorem tells us that any automorphism of $\mathfrak{g}$ of order m is conjugate (possibly by an element of the outer automorphism group) to some Kac automorphism of order m. Moreover, two Kac automorphisms are conjugate in the outer automorphism group of $\mathfrak{g}$ if and only if they can be obtained from each other by symmetries of the affine Dynkin diagram $\tilde{\Delta}^{(s)}$.

From now on we assume that if $\operatorname{char} k = p > 0$ then m is coprime to p. Fix a primitive m-th root of unity ζ in k and let μ be a Kac diagram of order m on $\tilde{\Delta}^{(1)}$. The Kac automorphism θ_μ corresponding to μ is simply $\operatorname{Ad} t$, where $t \in T$ is such that $\alpha_i(t) = \zeta^{\mu(\alpha_i)}$ for $1 \leq i \leq r$. Moreover, it is then immediate that $\hat{\alpha}(t) = \zeta^{-\mu(\alpha_0)}$. In characteristic zero, it is also clear that the fixed point subalgebra $\mathfrak{g}^t$ is generated by $\mathfrak{t}$ and all $\mathfrak{g}_{\pm\alpha_i}$, where $\mu(\alpha_i) = 0$ ($0 \leq i \leq r$); in general $\mathfrak{g}^t$ is spanned by $\mathfrak{t}$ and all $\mathfrak{g}_\alpha$, where α is in the root subsystem of Φ generated by the α_i with $\mu(\alpha_i) = 0$.

Example 2.1. We describe all (classes of) automorphisms of order 3 of a Lie algebra of type F_4. Here any automorphism is inner. The affine Dynkin diagram is the dual of $\tilde{E}_6^{(2)}$, with α_0, α_1 and α_2 all long roots. The possible Kac diagrams of order 3 are 00100, 11000 and 10001. Thus there are three classes of automorphisms of order 3, with representatives $\operatorname{Int} t$, $\operatorname{Int} t'$, $\operatorname{Int} t''$ where $t, t', t'' \in T$ satisfy:

(i) $\alpha_1(t) = \alpha_3(t) = \alpha_4(t) = 1$, $\alpha_2(t) = \zeta$;
(ii) $\alpha_2(t') = \alpha_3(t') = \alpha_4(t') = 1$, $\alpha_1(t') = \zeta$;
(iii) $\alpha_1(t'') = \alpha_2(t'') = \alpha_3(t'') = 1$, $\alpha_4(t'') = \zeta$.

It then follows that $\hat{\alpha}(t) = 1$, $\hat{\alpha}(t') = \hat{\alpha}(t'') = \zeta^{-1}$. We note that $\mathfrak{g}^t \cong \mathfrak{sl}_3 \oplus \mathfrak{sl}_3$, $\mathfrak{g}^{t'} \cong \mathfrak{sp}_6 \oplus k$ and $\mathfrak{g}^{t''} \cong \mathfrak{so}_7 \oplus k$.

Now we consider Kac diagrams of twisted type. For each such diagram of order m we will construct an outer automorphism following Kac's construction in characteristic zero. To describe these automorphisms we have to specify certain elements $E_{\pm\beta_i}$, H_{β_i} in $\mathfrak{g}$, roughly corresponding to simple root elements in the Kac-Moody Lie algebra determined by $\tilde{\Delta}^{(s)}$. These elements (or rather, analogous "root subgroups" $U_{\pm\beta_i} \subset G$) will also allow us to describe the fixed points in $\mathfrak{g}$ for the action of an outer Kac automorphism.

For $\alpha \in \Phi$ let U_α be the root subgroup of G corresponding to α [8, 26.3]. There exists a unique isomorphism (of algebraic groups) $\epsilon_\alpha : \mathbb{G}_a \to U_\alpha$ such that $t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x)$ for all $t \in T$, $x \in k$ and $(d\epsilon_\alpha)_0(1) = e_\alpha$ (see for example [8, Thm. 26.3]). For a given index s let γ be a graph automorphism of the root system Φ of order s which preserves the basis Δ, and by abuse of notation let γ also denote the unique automorphism of G which satisfies

$\gamma(\epsilon_\alpha(1)) = \epsilon_{\gamma(\alpha)}(1)$ for $\alpha \in \pm\Delta$. (There is a unique such graph automorphism of Φ unless G is of type D_4 and $s = 3$; in this case there are two such γ, which are mutually inverse and conjugate under the action of the automorphism group of $\mathfrak{g}$. In this case let us fix the choice satisfying $\gamma(\alpha_1) = \alpha_3$.) If $s = 3$ then fix a primitive cube root of unity σ. Let $T(0) = (T^\gamma)^\circ$. Two roots $\alpha, \beta \in \Phi$ are in the same γ-orbit if and only if their restrictions to $T(0)$ are equal.

For each $\alpha \in \Phi$ let (α) be the set of all $\langle\gamma\rangle$-conjugates of α and let $\mathfrak{g}_{(\alpha)} = \sum_{\beta\in(\alpha)} \mathfrak{g}_\beta$. Let $\overline{\alpha}$ be a positive root which is highest subject to $\mathfrak{g}_{(\alpha)} \not\subseteq \mathfrak{g}^{d\gamma}$. Specifically, we have: $\overline{\alpha} = \hat{\alpha}$ for type $\tilde{A}_{2n}^{(2)}$; $\overline{\alpha} = \sum_{i=1}^{2n-2} \alpha_i$ for type $\tilde{A}_{2n-1}^{(2)}$; $\overline{\alpha} = \sum_1^{n-1} \alpha_i$ in type $\tilde{D}_n^{(2)}$; $\overline{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ in type $\tilde{D}_4^{(3)}$; $\overline{\alpha} = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ in type $\tilde{E}_6^{(2)}$. Let β_0 be the $\langle\gamma\rangle$-conjugacy class of $-\overline{\alpha}$ and choose $E_{\pm\beta_0} \in \mathfrak{g}_{(\mp\overline{\alpha})}$ as follows: in case $\tilde{A}_{2n}^{(2)}$ let $E_{\pm\beta_0} = e_{\mp\overline{\alpha}}$; in cases $\tilde{A}_{2n-1}^{(2)}$, $\tilde{D}_n^{(2)}$ and $\tilde{E}_6^{(2)}$ let $E_{\pm\beta_0} = e_{\mp\overline{\alpha}} - \gamma(e_{\mp\overline{\alpha}})$; in case $\tilde{D}_4^{(3)}$ let $E_{\beta_0} = e_{-\overline{\alpha}} + \sigma^{-1} d\gamma(e_{-\overline{\alpha}}) + \sigma(d\gamma)^2(e_{-\overline{\alpha}})$ and $E_{-\beta_0} = e_{\overline{\alpha}} + \sigma d\gamma(e_{\overline{\alpha}}) + \sigma^{-1}(d\gamma)^2(e_{\overline{\alpha}})$. We label the node on the far left hand side (as displayed in the diagrams above; on the top in type $\tilde{A}_{2n-1}^{(2)}$) with β_0. As outlined above, we consider the roots β_i as certain characters of $T(0)$; in this respect β_0 is just $-\overline{\alpha}|_{T(0)}$. Note that $[E_{\beta_0}, E_{-\beta_0}] = H_{\beta_0} \in \mathfrak{t}^{d\gamma}$ satisfies $[H_{\beta_0}, E_{\pm\beta_0}] = \pm 2E_{\pm\beta_0}$.

The remaining nodes are all labelled by $\langle\gamma\rangle$-conjugacy classes in Δ. Specifically, we label the nodes $\beta_1, \ldots, \beta_l$ from left to right:

- $\beta_i = \{\alpha_i, \alpha_{2l+1-i}\}$ for each i, $1 \leq i \leq l$ in type $\tilde{A}_{2l}^{(2)}$, $l \geq 1$,
- $\beta_i = \{\alpha_i, \alpha_{2l-i}\}$ for $1 \leq i \leq l-1$ (β_1 appearing below β_0) and $\beta_l = \{\alpha_l\}$ in type $\tilde{A}_{2l-1}^{(2)}$,
- $\beta_i = \{\alpha_i\}$ for $1 \leq i \leq l-1$ and $\beta_l = \{\alpha_l, \alpha_{l+1}\}$ in type $\tilde{D}_{l+1}^{(2)}$,
- $\beta_1 = \{\alpha_1, \alpha_3, \alpha_4\}$ and $\beta_2 = \{\alpha_2\}$ in type $\tilde{D}_4^{(3)}$,
- $\beta_1 = \{\alpha_1, \alpha_6\}$, $\beta_2 = \{\alpha_3, \alpha_5\}$, $\beta_3 = \{\alpha_4\}$ and $\beta_4 = \{\alpha_2\}$ in type $\tilde{E}_6^{(2)}$.

We set $E_{\pm\beta_i} = \sum_{\alpha\in\beta_i} e_{\pm\alpha}$ for each i, $1 \leq i \leq l$ except for β_l in $\tilde{A}_{2l}^{(2)}$ in which case ($p \neq 2$ and) we set $E_{\beta_l} = e_{\alpha_l} + e_{\alpha_{l+1}}$, $E_{-\beta_l} = 2e_{-\alpha_l} + 2e_{-\alpha_{l+1}}$. By thinking of it as a γ-conjugacy class of elements of Φ, β_i defines a character on $T(0)$ such that $\operatorname{Ad} t(E_{\beta_i}) = \beta_i(t)E_{\beta_i}$ for any $t \in T(0)$. Moreover, setting $H_{\beta_i} = [E_{\beta_i}, E_{-\beta_i}]$ and letting $\langle\beta_i, \beta_j\rangle$ be the Cartan numbers determined by the twisted diagram, it is easy to establish the following commutation relations:

$$[E_{\beta_i}, E_{-\beta_j}] = \begin{cases} 0 & \text{if } i \neq j, \\ H_{\beta_i} & \text{if } i = j, \end{cases} \quad [H_{\beta_i}, E_{\pm\beta_j}] = \pm\langle\beta_i, \beta_j\rangle E_{\pm\beta_j}, \ [E_{\pm\beta_i}, E_{\pm\beta_j}] = 0 \text{ if } \langle\beta_i, \beta_j\rangle = 0$$

and $(\operatorname{ad} E_{\beta_i})^{1-\langle\beta_i,\beta_j\rangle}(E_{\beta_j}) = (\operatorname{ad} E_{-\beta_i})^{1-\langle\beta_i,\beta_j\rangle}(E_{-\beta_j}) = 0$ for $i \neq j$. In addition, $d\gamma(H_{\beta_i}) = H_{\beta_i}$ for all i and $d\gamma(E_{\pm\beta_i}) = E_{\pm\beta_i}$ for $1 \leq i \leq l$.

In characteristic zero, if I is a proper subset of $\tilde{\Delta}^{(s)}$ then the elements $E_{\pm\beta_i}$, $\beta_i \in I$ and $\mathfrak{t}^{d\gamma}$ together generate a reductive subalgebra of $\mathfrak{g}$ with root system given by the subdiagram of the twisted affine diagram which contains exactly the vertices corresponding to elements of I. However, in positive characteristic we have to be slightly more careful.

For $\beta_i \in \tilde{\Delta}^{(s)}$, either all roots $\alpha \in \beta_i$ are orthogonal and the subgroups U_α, $\alpha \in \beta_i$ commute, or $\beta_i = \{\alpha_l, \alpha_{l+1}\}$ in type $\tilde{A}_{2l}^{(2)}$. For $i \geq 1$, if the roots in β_i are orthogonal then let $\epsilon_{\pm\beta_i} : \mathbb{G}_a \to G$, $\epsilon_{\pm\beta_i}(x) = \prod_{\alpha\in\pm\beta_i} \epsilon_\alpha(x)$ and let $U_{\pm\beta_i} = \epsilon_{\pm\beta_i}(\mathbb{G}_a)$. Then $(d\epsilon_{\pm\beta_i})_0(1) = E_{\pm\beta_i}$. For the case β_l in type $\tilde{A}_{2l}^{(2)}$ we can construct $\epsilon_{\pm\beta_l}$ by restricting to the subgroup of type A_2 which contains $U_{\pm\alpha_l}$ and $U_{\pm\alpha_{l+1}}$. Specifically, $\epsilon_{\beta_l}(x) = \epsilon_{\alpha_l}(x/2)\epsilon_{\alpha_{l+1}}(x)\epsilon_{\alpha_l}(x/2)$ and $\epsilon_{-\beta_l}(x) = \epsilon_{-\alpha_l}(x)\epsilon_{-\alpha_{l+1}}(2x)\epsilon_{-\alpha_l}(x)$ define homomorphisms $\epsilon_{\pm\beta_l} : \mathbb{G}_a \to G^\gamma$ such that $(d\epsilon_{\pm\beta_l})_0(1) = E_{\pm\beta_l}$. Finally, either:

- $\overline{\alpha}$ is γ-stable, in which case we define $\epsilon_{\pm\beta_0}(x) = \epsilon_{\mp\overline{\alpha}}(x)$;

- or $s=2$ and $\gamma(\overline{\alpha}) \neq \alpha$, in which case we set $\epsilon_{\pm\beta_0}(x) = \epsilon_{\mp\overline{\alpha}}(x)\gamma(\epsilon_{\mp\overline{\alpha}}(-x))$;

- or $s=3$ and we set $\epsilon_{\pm\beta_0}(x) = \epsilon_{\mp\overline{\alpha}}(x)\gamma(\epsilon_{\mp\overline{\alpha}}(\sigma^{\mp 1}x))\gamma^{-1}(\epsilon_{\mp\overline{\alpha}}(\sigma^{\pm 1}x))$.

We have $\gamma(\epsilon_{\pm\beta_0}(x)) = \epsilon_{\pm\beta_0}(-x)$ if $s=2$, $\gamma(\epsilon_{\pm\beta_0})(x) = \epsilon_{\pm\beta_0}(\sigma^{\pm 1}x)$ if $s=3$ and $(d\epsilon_{\pm\beta_0})_0(1) = E_{\pm\beta_0}$. Let $U_{\pm\beta_0} = \epsilon_{\pm\beta_0}(\mathbb{G}_a)$.

With this set-up, let μ be a Kac diagram of twisted affine type, of order m. Fix a primitive m-th root of unity ζ; if $s=3$ then suppose further that $\zeta^{\frac{m}{3}} = \sigma$. There exists a unique $t \in T(0)$ such that $\beta_i(t) = \zeta^{\mu(\beta_i)}$ for $1 \leq i \leq l$. The Kac automorphism associated to μ is the automorphism $\theta_\mu = \operatorname{Int} t \circ \gamma$. In characteristic zero, $\mathfrak{g}^{d\theta_\mu}$ is the reductive subalgebra of $\mathfrak{g}$ generated by $\mathfrak{t}^{d\gamma}$ and the elements $E_{\pm\beta_i}$ where $\mu(\beta_i)=0$. In general, each of the subgroups $U_{\pm\beta_i}$ with $\mu(\beta_i)=0$ is contained in $G(0) := (G^{\theta_\mu})^\circ$. We recall [23, 8.1] that $G(0)$ is reductive. We claim that $G(0)$ is generated by $T(0)$ and the subgroups $U_{\pm\beta_i}$ with $\mu(\beta_i)=0$. To see this, we remark first of all that $\dim \mathfrak{g}(0)$ is independent of the characteristic. For if $\alpha \in \Phi$ then let $l(\alpha)$ be the number of γ-conjugates of α (here either 1, 2 or 3). Then $\mathfrak{g}_{(\alpha)} \cap \mathfrak{g}(0)$ is of dimension 1 if $d\theta_\mu^{l(\alpha)}(e_\alpha) = e_\alpha$ and of dimension 0 otherwise. Since this description is clearly independent of the characteristic of k, it follows that the same can be said of the dimension of $\mathfrak{g}(0)$.

Moreover, $(T^{\theta_\mu})^\circ = T(0)$ is regular in G by inspection, and hence is a maximal torus of $G(0)$. Let $\tilde{\Phi}_I$ be the root system with basis $\{\beta_i \in \tilde{\Delta}^{(s)} : \mu(\beta_i) = 0\}$. Then our assumption that p is coprime to the order of θ_μ implies that in all cases p is good for $\tilde{\Phi}_I$ except for one case in type $\tilde{D}_4^{(3)}$ (the Kac diagram 100) with $p=2$ and one case in type $\tilde{E}_6^{(2)}$ (the Kac diagram 10000) with $p=3$. If p is good for $\tilde{\Phi}_I$ then $E_{\pm\beta_i}$, $\mu(\beta_i)=0$ and $\mathfrak{t}^{d\gamma}$ generate a subalgebra of $\mathfrak{g}$ which is isomorphic to the Lie algebra of a reductive group with root system $\tilde{\Phi}_I$. Since the dimension is independent of the characteristic, this subalgebra must be all of $\mathfrak{g}(0)$. Now, since the subgroup of G generated by $T(0)$ and the $U_{\pm\beta_i}$, $\mu(\beta_i)$ is contained in $G(0)$, it must be reductive and have root system $\tilde{\Phi}_I$.

This leaves only the Kac diagrams 100 in type $\tilde{D}_4^{(3)}$ and 10000 in type $\tilde{E}_6^{(2)}$; in both cases the corresponding Kac automorphism is just γ. In the first case the roots $\alpha \in \Phi$ fall into two classes: (i) those such that $\gamma(e_\alpha) = e_\alpha$ (specifically $\pm\alpha \in \{\alpha_2, \sum_1^4 \alpha_i, \hat{\alpha}\}$), (ii) those such that $\gamma(\alpha) \neq \alpha$. It is thus easy to see that $\dim \mathfrak{g}_{(\alpha)} \cap \mathfrak{g}^{d\gamma} = 1$ for all α and, setting $H_{\beta_1} = h_{\alpha_1} + h_{\alpha_3} + h_{\alpha_4}$, $H_{\beta_2} = h_{\alpha_2}$, $E_{(\alpha)} = e_\alpha$ for α of type (i) and $E_{(\alpha)} = e_\alpha + \gamma(e_\alpha) + \gamma^{-1}(e_\alpha)$ for α of type (ii), the set $\{H_{\beta_1}, H_{\beta_2}, E_{(\alpha)} \,|\, \alpha \in \Phi\}$ is a basis for $\mathfrak{g}^{d\gamma}$ and satisfies the relations of a Chevalley basis for a Lie algebra of type G_2. Since $G(0)$ contains $T(0)$ and the subgroups $U_{\pm\beta_1}$, $U_{\pm\beta_2}$, this shows that $G(0)$ is generated by these subgroups and is semisimple of type G_2. In the second case, even though $p=3$ is not good for a root system of type F_4, it is nevertheless clear that the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{t}^{d\gamma}$ and the $E_{\pm\beta_i}$, $1 \leq i \leq 4$ is isomorphic to the Lie algebra of a semisimple group of type F_4 (since it contains all appropriate root subspaces for $T(0)$ as no root chain has length greater than 2). Thus by the same argument as in the preceding paragraph, $G(0)$ is generated by $T(0)$ and the subgroups $U_{\pm\beta_i}$, $1 \leq i \leq 4$.

We will now sketch a proof that any automorphism of order m is $\operatorname{Aut} G$-conjugate to a Kac automorphism. For inner automorphisms, this result appeared in [20].

Lemma 2.2. *Let G be a simple algebraic group over the field k of characteristic p. Then any automorphism of G of finite order coprime to p is $\operatorname{Aut} G$-conjugate to a Kac automorphism.*

Proof. It will clearly suffice to prove the lemma for the case where G is of adjoint type. We first deal with inner automorphisms. Let T be a maximal torus of G; then any inner automorphism of G of order m is conjugate to $\operatorname{Int} t$ where $t \in T$ is such that $t^m = 1$. The automorphism $\operatorname{Int} t$ is defined uniquely by the series of integers $i_1, \ldots, i_r$ where $\alpha_j(t) = \zeta^{i_j}$. Hence the set of elements of $\operatorname{Int} T$ which have order divisible by m identifies naturally with the set of $\mathbb{Z}$-linear maps $\mathbb{Z}\Delta \to \mathbb{Z}/m\mathbb{Z}$. The action of W on this set is clearly independent of the characteristic of k. Thus (by Kac's theorem for $k = \mathbb{C}$), after replacing t by a W-conjugate we may assume that $i_0 = m - \sum_{j=1}^{r} i_j m_j \geq 0$. Then $\operatorname{Int} t$ is the Kac automorphism with coordinates $(i_0, \ldots, i_r)$.

Now, suppose θ is an outer automorphism of G. Since any semisimple automorphism of G fixes some Borel subgroup of G and a maximal torus contained in it [23, 7.5], θ is conjugate to an automorphism of the form $\operatorname{Int} t \circ \gamma$, where $t \in T$ and γ is one of the graph automorphisms described above. Let $T(0) = (T^\gamma)^\circ$ and let $T(1) = \{t \in T \mid \gamma(t) = t^{-1}\}^\circ$. Then T is the almost direct product of $T(0)$ and $T(1)$ by Lemma 1.1. Hence $t = t_0.t_1$ for some $t_0 \in T(0)$, $t_1 \in T(1)$. Thus there is some $s \in T(1)$ such that $s^2 = t_1$; then $\operatorname{Int} s^{-1} \circ \operatorname{Int} t \circ \gamma \circ \operatorname{Int} s = \operatorname{Int} ts^{-2} \circ \gamma = \operatorname{Int} t_0 \circ \gamma$. It follows that θ is conjugate to an automorphism of the form $\operatorname{Int} t \circ \gamma$ with $t \in T(0)$. We claim that $\operatorname{Int} t \circ \gamma$ and $\operatorname{Int} s \circ \gamma$, for $t, s \in T(0)$ are conjugate if and only if there exists some $g \in N_{G^\gamma}(T)$ such that $gtg^{-1} = s$. Since G is adjoint, $\operatorname{Int} t \circ \gamma$ and $\operatorname{Int} s \circ \gamma$ are conjugate if and only if there exists some $x \in G$ such that $xt\gamma(x^{-1}) = s$. Consider the Bruhat decomposition $x = unv$, where $u, v \in U$ and $n \in N_G(T)$. We have $xt = unt \cdot (t^{-1}vt)$ and $s\gamma(x) = (s\gamma(u)s^{-1}) \cdot s\gamma(n)\gamma(v)$, thus by uniqueness $nt = s\gamma(n)$. In particular, $nT \in W^\gamma$. Now, a direct check shows that each element of W^γ has a representative in G^γ (once again using adjointness of G), thus s and t are $N_{G^\gamma}(T)$-conjugate.

It follows that the $\operatorname{Int} G$-conjugacy classes of outer automorphisms of the form $\operatorname{Int} x \circ \gamma$ are in one-to-one correspondence with the W^γ-conjugacy classes in $T(0)$. But W^γ is independent of the ground field k and (since G has adjoint type) the elements of $T(0)$ are in natural one-to-one correspondence with the γ-invariant $\mathbb{Z}$-linear maps $\mathbb{Z}\Delta \to \mathbb{Z}/m\mathbb{Z}$. In other words, each $t \in T(0)$ of order m is uniquely determined by the set of values $\beta_1(t), \beta_2(t), \ldots, \beta_l(t)$ where $\beta_1, \ldots, \beta_l$ are the γ-conjugacy classes of roots defined previously. Now we can directly copy the argument in the first paragraph of this proof to show that $\theta = \operatorname{Int} t \circ \gamma$ is W^γ-conjugate to a Kac automorphism. $\square$

We note the following corollary of the above result for inner automorphisms. In good characteristic this was already known ([14, Prop. 30] or [18, Prop. 3.1]). Recall that a pseudo-Levi subgroup of G is a subgroup of the form $Z_G(x)^\circ$ for a semisimple element x of G. If I is a proper subset of $\tilde{\Delta}$ then we denote by L_I the subgroup of G generated by T and all $U_{\pm\alpha_i}$ with $\alpha_i \in I$.

Corollary 2.3. *The pseudo-Levi subgroups are the G-conjugates of subgroups of the form L_I, where I is a proper subset of the $\tilde{\Delta} = \{\alpha_0, \ldots, \alpha_l\}$ such that there is some $\alpha_i \in \tilde{\Delta} \setminus I$ with $p \nmid m_i$.*

(If p is good then there is no restriction here. The groups L_I which don't satisfy the conditions of Cor. 2.3 are the centralizers of non-smooth diagonalizable subgroups [20].)

For later use we make the following observation.

Lemma 2.4. *Suppose μ is a Kac diagram on $\tilde{\Delta}^{(s)}$ for a positive rank Kac automorphism θ. Then $\mu(\alpha) \in \{0, 1\}$ for each $\alpha \in \tilde{\Delta}^{(s)}$.*

Proof. Suppose $\mu(\alpha) > 1$ for some $\alpha \in \tilde{\Delta}^{(s)}$. Then $\Pi = \tilde{\Delta}^{(s)} \setminus \{\alpha\}$ is a union of finite type Dynkin diagrams. If μ corresponds to an inner automorphism then $\mathfrak{l} = \mathfrak{t} \oplus \sum_{\beta \in \mathbb{Z}\Pi} \mathfrak{g}_\beta$ is the Lie algebra of the subgroup L_Π of G, in the notation introduced above. Moreover, the subspace spanned by all $\mathfrak{g}_\beta$, where β is non-zero and has non-negative coefficients in the elements of Π, is contained in the Lie algebra $\mathfrak{u}_\Pi$ of a maximal unipotent subgroup of L_Π. In particular, $\mathfrak{g}(1) \subset \mathfrak{u}_\Pi$ and thus does not contain any semisimple elements.

Suppose therefore that θ is an outer automorphism of G. If $\mu(\beta_i) > 1$ for some i then let μ' be the Kac diagram with $\mu'(\beta_j) = \delta_{ij}$. There is a Kac automorphism ψ of G corresponding to μ'. Moreover, it follows from the description of Kac automorphisms ψ commutes with θ and that $\mathfrak{g}(1) \subset \mathfrak{g}^{d\psi}$. But $\theta|_{G^\psi}$ is a zero-rank (inner) automorphism by the same argument as above. □

3 Carter's classification of conjugacy classes

Let W be an irreducible Weyl group with natural complex representation V. Here we recall the set-up of Carter's classification [4] of the conjugacy classes in W. Any element w of W can be expressed as a product $w = w_1 w_2$, where $w_1^2 = w_2^2 = 1$ and $\{v \in V \mid w_1 \cdot v = -v\} \cap \{v \in V \mid w_2 \cdot v = -v\} = \{0\}$. Moreover, any involution $w' \in W$ can be expressed as a product of l reflections corresponding to orthogonal roots, where $l = \dim\{v \in V \mid w'(v) = -v\}$. Thus let I_1, I_2 be subsets of Φ with $\#(I_i) = l(w_i)$ such that $w_i = \prod_{\alpha \in I_i} s_\alpha$ $(i = 1, 2)$. Then this gives an expression for w as a product of reflections corresponding to $l_1 + l_2 = \dim V - \dim V^w$ roots, where $l_i = \dim\{v \in V \mid w_i(v) = -v\}$, $i = 1, 2$. Associate a graph Γ to w with one node for each root in $I_1 \coprod I_2$ and $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges between nodes corresponding to distinct roots $\alpha, \beta \in I_1 \cup I_2$. The graph Γ constructed in this way is the **admissible diagram** associated to w. Less formally, we will say that w has Carter type Γ. (In fact, it is possible to have two such expressions for w giving rise to two different diagrams; in [4] a unique choice of admissible diagram is fixed for each conjugacy class. On the other hand, non-conjugate elements can have the same admissible diagram. This problem only occurs in types D and E, where at most two conjugacy classes can have the same diagram X; these classes are distinguished with the notation X', X''. We refer to [4] for details.)

For example, if Γ is the Dynkin diagram for W then w is a Coxeter element of W, while if Γ is the trivial graph then w is the identity element. The irreducible admissible diagrams are classified and their characteristic polynomials determined in [4, Tables 2-3]. We will say that an admissible diagram Γ has order N if an element of the corresponding conjugacy class has order N, and it has *strict* order N if it has order N and its characteristic polynomial is divisible by a cyclotomic polynomial of order N. Let w be an element of W with admissible diagram Γ. We denote by Φ_1 the (closed) subsystem of Φ spanned by the elements of Γ, and by Φ_2 the set of roots $\alpha \in \Phi$ which are orthogonal to all roots in Φ_1. In [4, Tables 8-12], the conjugacy classes in the exceptional type Weyl groups are listed along with the corresponding admissible diagrams and the numbers of elements in each class (thus giving the order of the centralizer of an element in each class). Following Carter's notation, we will denote by $\tilde{A}_l$ a subsystem of type A_l which consists of short roots.

Example 3.1. Recall that any element of the symmetric group $S_n = W(A_{n-1})$ has a unique cycle type; then an admissible diagram of type A_{m-1} corresponds to an m-cycle in S_n. Similarly, conjugacy classes in $W(B_n)$ are determined by *signed cycle type*, so that the admissible diagram corresponding to a positive (resp. negative) m-cycle is a Coxeter graph of type A_{m-1} (resp. B_m).

We want to ensure that all of the relevant information from [4] can also be applied in positive characteristic, including the information about the characteristic polynomials of elements of W. The natural representation of W is its representation in a Cartan subalgebra of the corresponding complex simple Lie algebra. For $w \in W$, denote by $c_w(t)$ the characteristic polynomial of w in this natural representation. It follows from [4, Table 3] that $c_w(t) \in \mathbb{Z}[t]$.

Lemma 3.2. *Let* $\operatorname{char} k = p > 0$ *and let* G *be a simple group over* k *with root system* $\Phi = \Phi(G,T)$, *where* T *is a maximal torus of* G. *Then the minimal polynomial of* $w \in W$ *as an automorphism of* $\mathfrak{t} = \operatorname{Lie}(T)$ *is just the reduction modulo* p *of* $c_w(t)$.

Proof. Given a $\mathbb{Z}$-basis $\{\chi_1, \ldots, \chi_n\}$ of the lattice of cocharacters $Y(T)$ we can associate a matrix $A \in \mathrm{GL}(n,\mathbb{Z})$ to w by: $w(\chi_i) = \sum_j A_{ji}\chi_j$. Since $w(d\chi_i(1)) = \sum_j A_{ji} d\chi_j(1)$ and $\{d\chi_1(1), \ldots, d\chi_n(1)\}$ is a basis of $\mathfrak{t}$, clearly the characteristic polynomial of w as an element of $\mathrm{GL}(\mathfrak{t})$ is just the reduction modulo p of the characteristic polynomial of A. On the other hand, the characteristic polynomial of A is invariant under change of basis in $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, and in particular is the same if one replaces G by its universal cover. But now one can choose a basis for $Y(T)$ consisting of the $\alpha^\vee$ where α is a simple root. Since the matrix thus associated to w is clearly independent of the characteristic, its characteristic polynomial must be equal to $c_w(t)$. $\square$

Lemma 3.3. *Let* $G, T, \mathfrak{t}, \Phi, W, w, \Phi_1, \Phi_2$ *be as above and suppose that* w *has order* m, $p \nmid m$. *Let* $U = \{t \in \mathfrak{t} \mid w(t) = \zeta t\}$, *where* ζ *is a primitive* m*-th root of unity in* k. *Then any element of* $W(\Phi_2)$ *acts trivially on* U.

Proof. Since G is simply-connected, $\mathfrak{t}$ is spanned by elements $h_\alpha := d\alpha^\vee(1)$, with α in a basis of simple roots. Let $\mathfrak{t}_{\Phi_1}$ be the linear span of the h_α with $\alpha \in \Phi_1$ and let $c_w(t)$ denote the characteristic polynomial of w as above. By Lemma 1.4(b), the dimension of U is equal to the multiplicity of the cyclotomic polynomial $p_m(t)$ in $c_w(t)$. This multiplicity does not change on replacing the root system Φ by its subsystem Φ_1. Thus $U \subseteq \mathfrak{t}_{\Phi_1}$. But if $\beta \in \Phi_2$ then $s_\beta(h_\alpha) = h_\alpha$ for all $\alpha \in \Phi_1$, so any element of $W(\Phi_2)$ acts trivially on U. $\square$

We remark that a suitable generalization of Lemma 3.3 holds for any orthogonal subsystems Φ_1, Φ_2 and an element $w \in W(\Phi_1)$.

4 Determination of the positive rank automorphisms

In this section we give details of the calculations we use to determine the positive rank automorphisms. Assume that G is simple, simply-connected and of exceptional type, and that $\operatorname{char} k$ is either zero or a good prime for G. In particular we assume that $p > 3$, and therefore that all elements of a Weyl group of type G_2 or F_4 are semisimple. (We recall that $W(G_2)$ is a dihedral group of order 12, and $W(F_4)$ has order $1152 = 2^7.3^2$.) Let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{g}(1)$ and let T_1 be the unique maximal θ-split torus of G such that $\mathfrak{c} \subset \operatorname{Lie}(T_1)$ as described in Lemma 1.4(b). Recall that $Z_G(\mathfrak{c})^\circ = Z_G(T_1)$ [13, Rk. 2.8(c)]; since G is simply-connected and $\operatorname{char} k$ is not a torsion prime for G, $Z_G(\mathfrak{c})$ is connected and hence equals $Z_G(T_1)$ [22, II.3.19]. Let $T(0)$ be a maximal torus of $Z_{G(0)}(\mathfrak{c})^\circ$. Then $Z_G(T(0)) \cap Z_G(\mathfrak{c})$ is a θ-stable maximal torus of G [13, Lemma 4.1]. In this section we will fix $\mathfrak{c}$ and $T(0)$, and set $T = Z_G(T(0)) \cap Z_G(\mathfrak{c})$, $\mathfrak{t} = \mathrm{Lie}(T)$. In other words, we make the following key assumption on the θ-stable torus T:

$$\mathfrak{c} \subseteq \mathrm{Lie}(T) \text{ and } T(0) \text{ is a maximal torus of } Z_G(\mathfrak{c})^\theta \qquad (*)$$

Under this assumption, we make some straightforward observations.

Lemma 4.1. *Suppose θ is inner. Then $\theta = \mathrm{Int}\, n_w$, where $n_w \in N_G(T)$. Moreover, either $w = n_w T$ has order m, or $n_w \in T$ and θ is of zero rank.*

Proof. Since T is θ-stable, the first statement is obvious. But if n_w has order less than m, then $\{t \in \mathfrak{t} \,|\, d\theta(t) = \zeta t\}$ is trivial, and therefore by our choice of T, $T(0) = T$ and $n_w \in T$. $\square$

In fact, we can make a more precise statement than this.

Lemma 4.2. *If θ is of positive rank then $w = n_w T$ has Carter type $\Gamma^{(1)} \times \ldots \times \Gamma^{(l)}$, where the $\Gamma^{(i)}$, $1 \le i \le l$ are disjoint irreducible diagrams and at least one has strict order m.*

Proof. Let $\Phi^{(i)}$ be the smallest root subsystem of Φ containing all the elements of $\Gamma^{(i)}$, let $\mathfrak{t}^{(i)}$ be the vector subspace of $\mathfrak{t}$ spanned by all h_α with $\alpha \in \Phi^{(i)}$ and let $w = w^{(1)} \ldots w^{(l)}$, where $w^{(i)}$ is a product of reflections corresponding to the vertices of $\Gamma^{(i)}$. (Here we define $h_\alpha = d\alpha^\vee(1)$.) Since the roots in $\Gamma^{(1)} \cup \ldots \cup \Gamma^{(l)}$ are linearly independent ([4, §4]), and since $\mathrm{char}\, k$ is not a bad prime and G is simply-connected, it follows that $\mathfrak{t}$ contains the direct sum $\mathfrak{t}^{(1)} \oplus \ldots \oplus \mathfrak{t}^{(l)}$. Clearly w preserves each of the subspaces $\Phi^{(i)}$ and acts on $\Phi^{(i)}$ by $w^{(i)}$. Hence $\{t \in \mathfrak{t} \,|\, w(t) = \zeta t\}$ is spanned by the subspaces $\{t \in \mathfrak{t}^{(i)} \,|\, w^{(i)}(t) = \zeta t\}$. In particular, if none of the $w^{(i)}$ has strict order m then $\mathfrak{c}$ is trivial, hence so is w by our assumption on $T(0)$. $\square$

Lemma 4.2 excludes certain possibilities for w, such as an element of type $A_1 \times A_2$ (since here $m = 6$, but no irreducible subdiagram has order 6). Later we will see that in fact each of the $\Gamma^{(i)}$ must have strict order m. (This was already established for the classical case in [13]. For outer automorphisms the situation is more complicated.)

The next two lemmas are well-known, see for example [1, Ch. 9, §4, Ex.s 13,14]; we sketch proofs for completeness.

Lemma 4.3. *Let G be a simple and simply-connected group and let h be the Coxeter number of G. Assume the characteristic of the ground field is either zero or is coprime to h. If $n_w \in N_G(T)$ is such that $w = n_w T$ is a Coxeter element in W, then $\mathrm{Int}\, n_w$ is an automorphism of G of order h and the corresponding Kac diagram is the diagram with 1 on every node.*

Proof. First, we note that any two representatives of w are T-conjugate by Lemma 1.3. Let θ be the Kac automorphism with diagram which has 1 on every node. Then $\mathfrak{g}(1) = \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\hat{\alpha}}$ and $G(0) = T$. If x_0 (resp. x_α, $\alpha \in \Delta$) is the polynomial which sends an element of $\mathfrak{g}(1)$ to its coordinate of $f_{\hat{\alpha}}$ (resp. of e_α) then the polynomial $f = x_0 \prod_{\alpha \in \Delta} x_\alpha^{m_\alpha}$ in $k[\mathfrak{g}(1)]$ is T-equivariant, where $\hat{\alpha} = \sum_{\alpha \in \Delta} m_\alpha \alpha$. Thus θ is a positive rank automorphism of G. Now, inspection of [4, Table 3] reveals that the only admissible diagram for W which has strict order h is the diagram for w, i.e. the Dynkin diagram of G. Thus θ is conjugate to $\mathrm{Int}\, n_w$. $\square$

We will say that an automorphism which is conjugate to $\operatorname{Int} n_w$, where $n_w \in N_G(T)$ represents a Coxeter element in W, is a *Coxeter automorphism*. By Lemma 4.3, the Coxeter automorphisms form a single conjugacy class in $\operatorname{Int} G$.

Lemma 4.4. *Let $n_w \in N_G(T)$ be such that n_wT represents a Coxeter element in W. Assume as before that* $\operatorname{char} k$ *does not divide h. Then n_w^h is as follows:*

a) $G = \mathrm{SL}(n,k)$, then $h = n$ and $n_w^h = I$ if n is odd, $n_w^h = -I$ if n is even.

b) $G = \mathrm{Spin}(2n+1,k)$, then $h = 2n$ and $n_w^h = 1$ if $n \equiv 0$ or 3 modulo 4, while $n_w^h = \alpha_n^\vee(-1)$ otherwise.

c) $G = \mathrm{Sp}(2n,k)$, then $h = 2n$ and $n_w^h = -I$.

d) $G = \mathrm{Spin}(2n,k)$ then $h = 2(n-1)$ and $n_w^h = 1$ if $n \equiv 0$ or 1 modulo 4, while $n_w^h = \alpha_{n-1}^\vee(-1)\alpha_n^\vee(-1)$ otherwise.

e) G of type G_2 (resp. F_4, E_6, E_8) then $h = 6$ (resp. $h = 12$, $h = 12$, $h = 30$) and $n_w^h = 1$.

g) G of type E_7, then $h = 18$ and $n_w^h = \alpha_2^\vee(-1)\alpha_5^\vee(-1)\alpha_7^\vee(-1)$.

Proof. By Lemma 4.3, $n_w^h \in Z(G)$ in each case. Let ζ be a primitive h-th root of unity. Types A or C can be calculated directly. If G is of type G_2, F_4 or E_8 then $Z(G)$ is trivial. If G is of type B_n, then we use Lemma 4.3: $\operatorname{Int} n_w$ is conjugate to the Kac automorphism which sends each e_α, $\alpha \in \Delta$ to ζe_α, where ζ is a primitive h-th root of unity. But this automorphism is just $\operatorname{Int} t$, where $t = \prod_{i=1}^{n-1} \alpha_i^\vee(\zeta^{in-i(i-1)/2}) \cdot \alpha_n^\vee(\xi^{n(n+1)/2})$, ξ a square-root of ζ. Thus t^h is as described. The calculation for type D_n is similar: $\operatorname{Int} n_w$ is conjugate to $\operatorname{Int} t$, where $t = \prod_{i=0}^{n-2} \alpha_i^\vee(\zeta^{i(n-1)-i(i-1)/2}) \cdot \alpha_{n-1}^\vee(\xi^{n(n-1)/2})\alpha_n^\vee(\xi^{n(n-1)/2})$ and thus $t^h = 1$ if and only if $n \equiv 0$ or 1 modulo 4. If G is of type E_6 then let γ be an outer automorphism of G. Then γ permutes the two non-identity elements of $Z(G)$. On the other hand, $\operatorname{Int} \gamma(n_w) \in \mathcal{C}$ and thus $\gamma(n_w^{12}) = n_w^{12}$, whence $n_w^{12} = 1$. Finally, if G is of type E_7 then w is of order 18 and $\operatorname{Int} n_w$ is conjugate to $\operatorname{Int} t$, where $t = \alpha_1^\vee(\xi^{34})\alpha_2^\vee(\xi^{49})\alpha_3^\vee(\xi^{66})\alpha_4^\vee(\xi^{96})\alpha_5^\vee(\xi^{75})\alpha_6^\vee(\xi^{52})\alpha_7^\vee(\xi^{27})$, and thus $n_w^{18} = t^{18} = \alpha_2^\vee(-1)\alpha_5^\vee(-1)\alpha_7^\vee(-1)$. □

Recall that an automorphism θ is N-regular if $\mathfrak{g}(1)$ contains a regular nilpotent element of $\mathfrak{g}$, and that an automorphism θ of order m is of *maximal rank* if $\dim \mathfrak{c} \cdot \varphi(m) = \operatorname{rk} G$, where φ denotes the Euler number of m.

Lemma 4.5. *Let w be one of the Weyl group elements on the following list. Then all representatives n_w of w in $N_G(T)$ are T-conjugate and, modulo the centre of G, have the same order as w. The (unique) Kac diagram corresponding to* $\operatorname{Int} n_w$*, the order of w (and thus of* $\operatorname{Int} n_w$*) and the rank of* $\operatorname{Int} n_w$ *are as given below.*

a) G of type G_2: w of type G_2, A_2 or $A_1 \times \tilde{A}_1$. The respective Kac diagrams are: 111, 011 *and* 010*, the respective orders of w are 6, 3 and 2 and the respective ranks of* $\operatorname{Int} n_w$ *are 1, 1 and 2.*

b) G of type F_4: w of type F_4, $F_4(a_1)$, $D_4(a_1)$, $A_2 \times \tilde{A}_2$ and A_1^4. The corresponding Kac diagrams are: 11111, 10101, 10100, 00100 *and* 01000*, the respective orders of w are 12, 6, 4, 3 and 2 and the respective ranks are 1, 1, 2, 2, and 4.*

Each such automorphism is N-regular.

Proof. Each w listed in the Lemma is of maximal rank and is a power of the Coxeter element by inspection of [4, Table 3]. (In particular, the orders and ranks are as given in the lemma.) Thus all representatives of w are T-conjugate by Lemma 1.3. It remains to check that the Kac diagrams are as stated. For the Coxeter elements this is Lemma 4.4. Suppose on the other hand that l divides the Coxeter number. Then the l-th power of a Coxeter automorphism is conjugate to the automorphism which sends $e_{\pm\alpha}$ (α simple) to $\zeta^{\pm l}e_{\pm\alpha}$, where ζ is a primitive h-th root of unity (h the Coxeter number). (We note that since p is good it doesn't divide h in type G_2 or F_4.) In particular, $\mathfrak{g}(0)$ is conjugate to the subalgebra spanned by $\mathfrak{t}$ and all root subspaces $\mathfrak{g}_\alpha$ where α has height divisible by l. Thus, inspecting the possibilities for automorphisms of order h/l, it is straightforward to determine the Kac diagram in each case.

Type A_2 in G_2: here $l=2$ and hence $\mathfrak{g}(0)$ is conjugate to $\mathfrak{t}\oplus\mathfrak{g}_{\pm(2\alpha_1+\alpha_2)}$.

Type $A_1\times\tilde{A}_1$ in G_2: here $l=3$ and hence $\mathfrak{g}(0)$ is conjugate to $\mathfrak{t}\oplus\mathfrak{g}_{\pm(\alpha_1+\alpha_2)}\oplus\mathfrak{g}_{\pm(3\alpha_1+\alpha_2)}$.

Type $F_4(a_1)$ in F_4: we have $l=2$ and $\mathfrak{g}(0)$ is conjugate to $\mathfrak{t}\oplus\mathfrak{g}_{\pm 1221}\oplus\mathfrak{g}_{\pm 1122}$.

Type $D_4(a_1)$ in F_4: since $l=3$, after conjugation we may assume $\mathfrak{g}(0)=\mathfrak{t}\oplus\mathfrak{g}_{\pm 1111}\oplus\mathfrak{g}_{\pm 1120}\oplus\mathfrak{g}_{\pm 0121}\oplus\mathfrak{g}_{\pm 1232}$. Here 1111 and 0121 generate a root system of type $\tilde{A}_2$, while 1120 is a long root, thus $\mathfrak{g}(0)$ is of type $\tilde{A}_2\times A_1$.

Type $A_2\times\tilde{A}_2$ in F_4: we have $l=4$, hence $\mathfrak{g}(0)$ is conjugate to $\mathfrak{t}\oplus\mathfrak{g}_{\pm 1110}\oplus\mathfrak{g}_{\pm 0120}\oplus\mathfrak{g}_{\pm 0111}\oplus\mathfrak{g}_{\pm 1221}\oplus\mathfrak{g}_{\pm 1122}\oplus\mathfrak{g}_{\pm 1242}$. Here 1110, 0111 generate a root subsystem of Φ of type $\tilde{A}_2$ while 0120, 1122 generate a root system of type A_2.

Type A_1^4 in F_4: finally, $l=6$ and hence $\mathfrak{g}(0)$ is conjugate to the subalgebra spanned by $\mathfrak{t}$ and all root spaces $\mathfrak{g}_\alpha$ where α is of even length. But the roots of even length have basis: 1100, 0011, 0110 and 1120, where the first three span a subsystem of type C_3 and 1120 is a long root.

It is now easily seen that the only possible Kac diagrams with coefficients 0 and 1 which could correspond to these automorphisms are those stated in the lemma. Finally, each such automorphism is N-regular because it is a power of a Coxeter automorphism, and Coxeter automorphisms are N-regular. □

In type G_2 the only remaining non-trivial conjugacy classes in W are involutions. Moreover, it is easy to see from Lemma 2.2 (or see, for example [21]) that there is a unique class of involutions of a simple group of type G_2 and hence Lemma 4.5 gives us a complete list of positive rank automorphisms. (See Table 1.)

Recall by Lemma 4.2 that any positive rank automorphism is of the form $\operatorname{Int} n_w$ where the admissible diagram of w contains at least one irreducible component of strict order m. In combination with Lemma 4.5, in type F_4 this only leaves elements of the Weyl group of the following types: A_2, $\tilde{A}_2$ (order 3), B_2, $B_2\times A_1$, A_3, $A_3\times\tilde{A}_1$ (order 4), C_3, B_3, $C_3\times A_1$, D_4 (order 6), B_4 (order 8).

Consider an admissible diagram $\Gamma=\Gamma'\cup\Gamma''$, where Γ' is a union of irreducible subdiagrams of strict order m, and Γ'' is a union of irreducible subdiagrams none of which has strict order m. Let Φ' be the smallest root subsystem of Φ containing all of the roots in Γ', and let Ψ be the set of roots in Φ which are orthogonal to all elements of Φ'. Let L (resp. M) be the subgroup of G generated by all root subgroups U_α, $\alpha\in\Phi'$ (resp. $\alpha\in\Psi$) and let S be the torus generated by all $\alpha^\vee(k^\times)$, $\alpha\in\Phi'$. Clearly S is a maximal torus of L and $Z_G(S)'=M$.

Let $w=w'w''$ be an element in the conjugacy class corresponding to Γ, where w' corresponds to Γ' and w'' corresponds to Γ''. Suppose $\theta=\operatorname{Int} n_w$, where $n_wT=w\in W$. Since T and S are θ-stable, it follows that L and M are θ-stable. Moreover, $w|_S=w'|_S$ and $(S^w)^\circ$ is trivial, thus there exists $n_{w'}\in N_L(S)$ such that $\theta|_S=\operatorname{Int} n_{w'}|_S$. Since $(\operatorname{Int} n_{w'}^{-1}\circ\theta)|_L$ acts trivially on S, we can replace $n_{w'}$ by an element of the form $n_{w'}s$, $s\in S$ such that $\theta|_L=\operatorname{Int} n_{w'}|_L$. Therefore $n_w=n_{w'}g$ for some $g\in Z_G(L)$. Recall the key assumption (*) that $\mathfrak{c}\subseteq\operatorname{Lie}(T)$ and $T(0)$ is a maximal torus of $Z_G(\mathfrak{c})^\theta$.

Lemma 4.6. *Under the assumption (*), we have $w'' = 1$, i.e. Γ is a union of irreducible subdiagrams of strict order m.*

Proof. Since S is a maximal torus of L and $Z_G(S) = TM$ we have $g \in Z_G(L) = C \cdot Z_M(L)$ where $C = \{t \in T \mid \alpha(t) = 1 \text{ for all } \alpha \in \Phi'\}$. Note that $C^\circ \cdot S = T$ and that w' acts trivially on C and has finitely many fixed points on S. Hence $C^\circ = (T^{w'})^\circ$. It follows that $C = C^\circ \cdot Z(L)$. Now any two representatives for w' in L are conjugate by Lemma 1.3 and thus, after replacing $n_{w'}$ by an appropriate element of $n_{w'}Z(L)$ if necessary, we may assume that $g \in C^\circ \cdot Z_M(L)$. Let $h \in L$ be such that $s = hn_{w'}h^{-1} \in S$. Then $hn_wh^{-1} = shgh^{-1} = sg$. Moreover, s and g commute and $(sg)^m = 1$, thus g is semisimple. We deduce that there exists $h' \in M$ such that $h'gh'^{-1} = s' \in C^\circ$. Thus n_w is G-conjugate to $ss' \in T$. But therefore n_w is conjugate to $h^{-1}ss'h = n_{w'}s'$. By the assumption of maximality of $T(0)$, we must have $w = w'$ in the first place. □

This result eliminates the cases $A_3 \times \tilde{A}_1$, $C_3 \times A_1$ and $B_2 \times A_1$ in type F_4 from our discussion. But it also eliminates the conjugacy class of type D_4 by the same argument, since D_4 has an alternative admissible diagram of type $B_3 \times \tilde{A}_1$. (This can easily be seen in the Weyl group of type B_4, in which an element of type D_4 is a product of a negative 3-cycle and a negative 1-cycle.)

To continue, we make a few observations in a similar vein to Lemma 4.5.

Lemma 4.7. *There exists a unique conjugacy class of positive rank automorphisms of order 8 in type F_4. Such automorphisms are of rank one; the corresponding Kac diagram is* 11101*.*

Proof. Let w be an element of W of type B_4. We observe that B_4 is the unique class in W of order 8 [4, Tables 3,8]. Moreover, any two representatives n_w, n_wt of w in $N_G(T)$ are T-conjugate (Lemma 1.3) and $n_w^4 = (n_wt)^4 = 1$ (Lemma 4.4), thus there is a unique conjugacy class of positive rank automorphisms of order 8. It remains to check that the Kac diagram is as indicated in the Lemma. Consider the B_4-type subsystem of Φ with basis $\{-\hat{\alpha} = \alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. Then $\operatorname{Int} n_w$ is conjugate to the automorphism of G which sends e_{α_i} to ζe_{α_i} for $i = 0, 1, 2, 3$, where ζ is a primitive 8-th root of unity. But for such an automorphism either $e_{\alpha_4} \mapsto \zeta^{-1} e_{\alpha_4}$ and thus $\mathfrak{g}(0) = \mathfrak{t} \oplus \mathfrak{g}_{\pm(\alpha_3+\alpha_4)}$, or $e_{\alpha_4} \mapsto \zeta^3 e_{\alpha_4}$, in which case $\mathfrak{g}(0) = \mathfrak{t} \oplus \mathfrak{g}_{\pm 1221}$. Since 11101 is the only Kac diagram of order 8 which has coefficients 1 and 0 and has fixed point subalgebra of type $\tilde{A}_1$, this proves the Lemma. □

Lemma 4.8. *There are two conjugacy classes of automorphisms of F_4 of order 6 and rank 1. They have Kac diagrams* 01010 *and* 11100 *and each element of the first (resp. second) class acts on some maximal torus as a Weyl group element of type C_3 (resp. B_3).*

Proof. We note first of all that there are 4 classes of automorphisms of order 6 with Kac coefficients 0 and 1: the Kac diagrams 10101, 11100, 01010 and 00011. In addition, we have seen above that 10101 is the unique class of automorphism of order 6 and rank two. Thus any automorphism of rank 6 and order 1 is conjugate to $\operatorname{Int} n_w$ where $w = n_wT \in W$ is either of type C_3 or of type B_3.

Let $\theta = \operatorname{Int} t$ be the Kac automorphism with diagram 01010 and consider the root subsystem Σ of Φ generated by $\alpha_3 + \alpha_4$, $\alpha_2 + \alpha_3$ and α_1. Then Σ is of type C_3 and if we let H be the (Levi) subgroup of G generated by T and the subgroups $U_{\pm\alpha}$ with $\alpha \in \Sigma$ then θ stabilizes H and is easily seen to be a Coxeter automorphism of H' (since $d\theta(e_\alpha) = \zeta e_\alpha$ for $\alpha = \alpha_3 + \alpha_4, \alpha_2 + \alpha_3, \alpha_1$). Thus t is H-conjugate to some $n_w \in N_G(T)$, where $w = n_wT$ is of type C_3.

Let $\theta = \operatorname{Int} t$ be the Kac automorphism for the Kac diagram 11100 and consider the root subsystem Σ of Φ generated by $-\hat{\alpha}$, α_1 and $\alpha_2+\alpha_3$. Then Σ is of type B_3 and $d\theta(e_\alpha) = \zeta e_\alpha$ for $\alpha = -\hat{\alpha}, \alpha_1, \alpha_2+\alpha_3$. Thus if we let H be the Levi subgroup of G generated by T and all α with $\alpha \in \Sigma$ then $\theta|_{H'}$ is a Coxeter automorphism. In particular, t is H-conjugate to some $n_w \in N_G(T)$, where $w = n_w T$ is of type B_3.

It remains to show that a Kac automorphism of type 00011 is of zero rank. Here $G(0)$ is the pseudo-Levi subgroup generated by T and U_α with $\alpha = \pm\alpha_2, \pm\alpha_1, \pm\hat{\alpha}$. In particular, $\dim G(0) = 16$. We have $\dim \mathfrak{g}(1) = 5$; thus it will suffice to show that there exists $e \in \mathfrak{g}(1)$ such that $\dim Z_{G(0)}(e) = 11$. Let $e = e_{\alpha_3} + e_{\alpha_4}$. Then $\mathfrak{z}_{\mathfrak{g}(0)}(e)$ is spanned by a two-dimensional subalgebra of $\mathfrak{t}$ and $\mathfrak{g}_\alpha$ with $\alpha = \pm\hat{\alpha}, \pm\alpha_1, \pm 1342, -\alpha_2, -(\alpha_1+\alpha_2), 1242$. In particular, $\dim Z_{G(0)}(e) \leq 11$, whence $\dim G(0)\cdot e \geq 5$. It follows that $G(0)\cdot e$ is dense in $\mathfrak{g}(1)$ and therefore θ is of zero rank. □

Corollary 4.9. *There are two conjugacy classes of automorphisms of F_4 of order 3 and rank 1. They have Kac diagrams* 11000 *and* 10001 *and each element of the first (resp. second) class acts on some maximal torus of G as a Weyl group element of type A_2 (resp. $\tilde{A}_2$).*

Proof. Apart from the (unique) rank 2 automorphism 00100, the two Kac automorphisms given in the corollary are the only remaining Kac automorphisms of order 3. Let θ be the square of the Kac automorphism with diagram 01010: then $\mathfrak{g}(0)$ is isomorphic to $\mathfrak{so}(7,k)\oplus k$, with basis of simple roots $\{\alpha_2, 1120, \alpha_4\}$. Thus θ is conjugate to a Kac automorphism with diagram 10001. The statement in this case now follows since θ must therefore have rank 1, and the square of an element of type C_3 is an element of type $\tilde{A}_2$. Now let θ be the square of the Kac automorphism with diagram 11100. Then $\mathfrak{g}(0)$ is isomorphic to $\mathfrak{sp}(6)\oplus k$, with basis of simple roots $\{\alpha_4, \alpha_3, 1220\}$. Thus, by the same reasoning, the Kac automorphism with diagram 11000 has rank 1 and corresponds to a Weyl group element of type A_2. □

Lemma 4.10. *There is one conjugacy class of automorphisms in type F_4 of order 4 and rank 1. The corresponding Kac diagram is* 01001 *and is represented by an element of W of type B_2.*

Proof. By Lemma 2.4 the Kac diagram corresponding to a positive rank automorphism can only have 0s and 1s on the nodes. Moreover, we have already seen in Lemma 4.5(b) that the Kac automorphism with diagram 10100 is the unique Kac automorphism of order 4 and rank 2. Hence there are only two other possibilities: 01001 and 00010. It is easy to see that the Kac automorphism with diagram 01001 is of positive rank since $e_{1100}+e_{0011}+f_{1122}$ is semisimple. Hence it is of rank 1. (By considering the root subsystem generated by 1100 and 0011, we can see that θ is conjugate to an automorphism of the form $\operatorname{Int} n_w$, where $w = n_w T$ is an element of W of type B_2.) It remains to prove that the Kac automorphism with diagram 00010 is of zero rank (first proved by Vinberg [24, §9]). In this case $\mathfrak{g}(0) \cong \mathfrak{sl}(4,k)\oplus\mathfrak{sl}(2,k)$ and $\mathfrak{g}(1)$ is isomorphic as a $\mathfrak{g}(0)$-module to the tensor product of the natural representation for $\mathfrak{sl}(4)$ with the natural representation for $\mathfrak{sl}(2)$. In particular, $\dim\mathfrak{g}(1) = 8$. If we let $e = e_{0010}+f_{1231}$ then it is easy to check that $\mathfrak{z}_{\mathfrak{g}(0)}(e)$ is spanned by h_{α_1}, e_{α_1}, f_{α_1}, $h_{\alpha_2}+h_{\alpha_3}+h_{\alpha_4}$, $f_{\hat{\alpha}}$, f_{1342}, $f_{1242}+f_{\alpha_4}$, $e_{1242}+e_{\alpha_4}$, f_{1100} and f_{α_2}, where the signs of f_{1242} and e_{1242} are chosen appropriately. Thus $\dim Z_{G(0)}(e) \leq 10$ and therefore $\dim G(0)\cdot e \geq 8$. Hence $G(0)\cdot e$ is dense in $\mathfrak{g}(1)$, and so $\mathfrak{g}(1)$ is of zero rank. □

This completes the determination of the positive rank automorphisms in type F_4. We will use these results on automorphisms in type F_4 to study the positive rank triality automorphisms in type D_4. In [24] all automorphisms of $\mathfrak{so}(2n,\mathbb{C})$ of the form $x \mapsto gxg^{-1}$,

$g \in \mathrm{O}(2n, k)$ were studied; this was generalized to (odd) positive characteristic in [13], and it was proved that all such θ-groups have a KW-section. If $n \geq 5$ then all automorphisms have this form; in type D_4 this leaves only the automorphisms of type $\tilde{D}_4^{(3)}$. Recall that the set of long roots in a root system of type F_4 is a root system of type D_4, and correspondingly $W(F_4)$ contains $W(D_4)$ as a normal subgroup of index 6. Let $\hat{G}$ be a simple algebraic group of type F_4, let T be a maximal torus of $\hat{G}$, let $\Phi = \Phi(\hat{G}, T)$ and let Φ_l be the set of long roots in Φ. If $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of Φ then $\{\alpha_2, \alpha_1, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\}$ is a basis of Φ_l. Note that $(\alpha_2 + 2\alpha_3)^\vee = \alpha_2^\vee + \alpha_3^\vee$ and $(\alpha_2 + 2\alpha_3 + 2\alpha_4)^\vee = \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee$, thus the subgroup G of $\hat{G}$ generated by T and all U_α with $\alpha \in \Phi_l$ is isomorphic to $\mathrm{Spin}(8, k)$. The subgroup H of $\hat{G}$ generated by G and $N_{\hat{G}}(T)$ normalizes G and the corresponding map $H \to \mathrm{Aut}\, G$ is surjective. (In fact H is isomorphic to the semidirect product of G by the symmetric group S_3.)

Lemma 4.11. *All Kac automorphisms of type $D_4^{(3)}$ which have Kac coefficients equal to 0 or 1 are of positive rank, with the exception of the Kac automorphism of order 9. These diagrams are:*

(a) 111, *which has rank 1 and order 12 and acts as a Coxeter element of F_4,*

(b) 101, *which has rank 2 and order 6 and corresponds to an element of type $F_4(a_1)$,*

(c) 001, *which has rank 2 and order 3 and corresponds to an element of type $A_2 \times \tilde{A}_2$,*

(d) 010, *which has rank 1 and order 6 and corresponds to an element of type C_3,*

(e) 100, *which has rank 1 and order 3 and corresponds to an element of type $\tilde{A}_2$ (this is the automorphism γ constructed in §2).*

Proof. Let $\hat{G}$, G and T be as above. By assumption on T, any automorphism of G is of the form $\mathrm{Int}\, n_w|_G$ for some $n_w \in N_{\hat{G}}(T)$. Recall by Lemma 2.4 that a Kac diagram which corresponds to a positive rank automorphism satisfies $h(\beta_i) \in \{0, 1\}$ for $i = 0, 1, 2$. Moreover, there are no elements of $W(F_4)$ of order 9 and hence the only Kac diagrams which can be of positive rank are the five diagrams listed in the Lemma. Clearly there is exactly one such Kac diagram of order 12. Since $n_w^{12} = 1$ if $n_w \in N_{\hat{G}}(T)$ represents a Coxeter element in $W(F_4)$, there exists at least one automorphism of rank 1 and order 12 and hence this automorphism has Kac diagram 111. Let us recall our description of the Kac automorphism corresponding to this class. Let ζ be a primitive 12-th root of unity. Then θ is the automorphism satisfying:

$$e_{\pm\alpha_1} \mapsto \zeta^{\pm 1} e_{\pm\alpha_3}, e_{\pm\alpha_2} \mapsto \zeta^{\pm 1} e_{\pm\alpha_2}, e_{\pm\alpha_3} \mapsto \zeta^{\pm 1} e_{\pm\alpha_4}, e_{\pm\alpha_4} \mapsto \zeta^{\pm 1} e_{\pm\alpha_1}$$

With this clarification, we can determine which Kac diagrams correspond to θ^2 and θ^4. (We note that θ^2 and θ^4 are necessarily of positive rank, since θ is.) Indeed, the fixed point subspace for θ^2 is spanned by h_2, $h_1 + h_3 + h_4$ and $e_{\pm(\alpha_1+\alpha_2)} + \zeta^{\pm 2} e_{\pm(\alpha_2+\alpha_3)} - \zeta^{\pm 4} e_{\pm(\alpha_2+\alpha_4)}$. Thus θ^2, which corresponds to an element of $W(F_4)$ of type $F_4(a_1)$, has Kac diagram 101. Similarly, the fixed point subspace for θ^4 is spanned by h_2, $h_1 + h_3 + h_4$, $e_{\pm\alpha_1} + \zeta^{\pm 4} e_{\pm\alpha_3} + \zeta^{\mp 4} e_{\pm\alpha_4}$, $e_{\pm(\alpha_1+\alpha_2)} + \zeta^{\pm 2} e_{\pm(\alpha_2+\alpha_3)} - \zeta^{\pm 4} e_{\pm(\alpha_2+\alpha_4)}$ and $e_{\pm(\alpha_1+\alpha_2+\alpha_3)} + e_{\pm(\alpha_2+\alpha_3+\alpha_4)} + e_{\pm(\alpha_1+\alpha_2+\alpha_4)}$. Thus the dimension of the fixed-point space is 6, and hence the corresponding Kac diagram is 100.

On the other hand, let $n_w \in N_{\hat{G}}(T)$ be such that $w = n_w T$ is an element of type C_3 (resp. $\tilde{A}_2$) and $\mathrm{Int}\, n_w$, as an automorphism of $\hat{G}$, is conjugate to a Kac automorphism with diagram 01010 (resp. 10001). Then $\mathrm{Int}\, n_w|_G$ has order 6 (resp. 3) and has rank at least 1 (since $\mathrm{Lie}(T) \cap \mathfrak{g}(1)$ is non-trivial); but $\mathrm{Int}\, n_w$ has rank 1 (as an automorphism of $\mathrm{Lie}(\hat{G})$) by Lemma 4.8 and Corollary 4.9. Thus $\mathrm{Int}\, n_w|_G$ is a rank one automorphism, which must also be of type $\tilde{D}_4^{(3)}$ since elements of $W(F_4)$ of type C_3 and $\tilde{A}_2$ are of order 3 modulo $W(D_4)$.

Hence there exist automorphisms of order 3 and 6 and of rank 1, which must correspond to the remaining two Kac diagrams as indicated in the diagram. □

5 Calculation of the Weyl group

It remains to calculate the little Weyl group for each of the automorphisms of the previous section. To begin we recall the following straightforward observation [13, Lemma 4.2]. We maintain the assumptions on θ, $\mathfrak{c}$, T from the previous section.

Lemma 5.1. *Let $\bar{W} = W^\theta / Z_{W^\theta}(\mathfrak{c})$. Then $\bar{W}$ acts on $\mathfrak{c}$ and $W_\mathfrak{c}$ is a subgroup of $\bar{W}$.*

We will see that for all θ-groups of type G_2, F_4 and $D_4^{(3)}$, $\bar{W} = W_\mathfrak{c}$. This is not true in general, see Rk. 5.10(c).

Lemma 5.2. *Let Γ be a product of irreducible admissible diagrams of strict order m and let $w \in W$ be an element of type Γ. Let Φ_1 be the smallest root subsystem of G containing all roots in Γ and let Φ_2 be the set of roots in Φ which are orthogonal to Φ_1. Let $\mathfrak{t}(1) = \{t \in \mathfrak{t} \,|\, w(t) = \zeta t\}$ and let $W_0 = \{w \in Z_W(w) \,|\, w|_{\mathfrak{t}(1)} = 1_{\mathfrak{t}(1)}\}$. Then W_0 contains $W(\Phi_2)$.*

Proof. This is clear from Lemma 3.3. □

From now on, fix w and let W_1 (resp. W_2) denote the subgroup of W generated by all s_α with $\alpha \in \Phi_1$ (resp. Φ_2). Then Lemma 5.2 shows that the order of $\bar{W}$ (and hence of $W_\mathfrak{c}$) divides $\#(Z_W(w))/\#(W_2)$. Use of this straightforward observation will allow us to identify $W_\mathfrak{c}$ for all the cases which concern us. Let T_i, $i|m$ be the subtori of T defined in Lemma 1.1. We will need the following lemma ([13, Lemma 4.3]).

Lemma 5.3. *Suppose G is simply-connected and let $T_m' = \prod_{i \neq m} T_i = \{t^{-1}\theta(t) \,|\, t \in T\}$. Suppose $\{t \in T_m = T(0) \,|\, t^m = 1\} \subset T_m'$. Then $W_\mathfrak{c} = \bar{W}$.*

Our first result is for maximal rank automorphisms.

Lemma 5.4. *Suppose θ is a maximal rank automorphism. Then W_0 is trivial. Thus $\bar{W} = W_\mathfrak{c} = W^\theta$.*

Proof. By assumption, G is simply-connected. But now $Z_G(\mathfrak{c})$ is connected [22, I.3.19], and therefore equals T. Thus $Z_{N_G(T)}(\mathfrak{c}) = Z_G(\mathfrak{c}) = T$. This shows that W_0 is trivial. Furthermore, $T_m' = T$ by assumption on θ, thus any element of $\bar{W}$ has a representative in $G(0)$ by Lemma 5.3. □

Finally, we have the following preparatory lemma, which appeared in [16] in characteristic zero, and was generalised to positive characteristic in [13, Prop. 5.3]. Recall that a *KW-section* for θ is an affine linear subvariety $\mathfrak{v} \subset \mathfrak{g}(1)$ such that the restriction of the categorical quotient $\pi : \mathfrak{g}(1) \to \mathfrak{g}(1)/\!\!/G(0)$ to $\mathfrak{v}$ is an isomorphism.

Lemma 5.5. *Suppose θ is an N-regular automorphism. Then $k[\mathfrak{t}]^W \to k[\mathfrak{c}]^{W_\mathfrak{c}}$ is surjective. In particular, if θ is inner then the degrees of the generators of $k[\mathfrak{c}]^{W_\mathfrak{c}}$ are simply those degrees of the invariants of $\mathfrak{g}$ which are divisible by m.*

Moreover, θ admits a KW-section.

We have the following application of Lemma 5.4. We identify automorphisms by Weyl group elements, that is, by w where θ is conjugate to $\operatorname{Int} n_w$ as described in Sect. 4. Let G_n denote the n-th group in the Shephard-Todd classification.

Lemma 5.6. *For the maximal rank automorphisms, the Weyl group is as described in the following list:*

(a) Type G_2. Coxeter element: $W_{\mathfrak{c}} = \mu_6$; A_2: $W_{\mathfrak{c}} = \mu_6$; $A_1 \times \tilde{A}_1$: $W_{\mathfrak{c}} = W(G_2)$.

(b) Type F_4. Coxeter element: $W_{\mathfrak{c}} = \mu_{12}$; B_4: $W_{\mathfrak{c}} = \mu_8$; $F_4(a_1)$ or $A_2 \times \tilde{A}_2$: $W_{\mathfrak{c}} = G_5$; $D_4(a_1)$: $W_{\mathfrak{c}} = G_8$; A_1^4: $W_{\mathfrak{c}} = W(F_4)$.

(c) Type $D_4^{(3)}$. F_4: $W_{\mathfrak{c}} = \mu_4$; $F_4(a_1)$ or $A_2 \times \tilde{A}_2$: $W_{\mathfrak{c}} = G_4$.

Proof. For the rank 1 inner automorphisms, this follows on reading off the orders of conjugacy classes (and hence centralizers) in [4]. Moreover, since the maximal rank involutions in type G_2 and F_4 lie in the centre of the Weyl group, it remains only to check type $\tilde{D}_4^{(3)}$ and classes $F_4(a_1)$, $A_2 \times \tilde{A}_2$ and $D_4(a_1)$ in type F_4.

Let W be a Weyl group of type F_4 and let W_l be the subgroup of W generated by all s_α with α a long root. Then W_l is isomorphic to the Weyl group of type D_4, and is a normal subgroup of W of index 6. For an element in class $F_4(a_1)$ or $A_2 \times \tilde{A}_2$, the centralizer in W has order 72 by [4]. Moreover, the square of an element in class $F_4(a_1)$ is an element in class $A_2 \times \tilde{A}_2$, thus the little Weyl group for these two cases is equal. There are six cosets of W_l in W and the quotient group is isomorphic to S_3. It follows that if w is an element of W which has order 3 modulo W_l then $Z_{W_l}(w)$ is a normal subgroup of $Z_W(w)$ of index 3. In particular, $Z_{W_l}(w)$ has order 4 (resp. 24) if w is a Coxeter element (resp. of type $F_4(a_1)$ or $A_2 \times \tilde{A}_2$). Thus it is clear that $Z_{W_l}(w) \cong \mu_4$ in the case where w is a Coxeter element of type F_4.

We claim that if w is of type $A_2 \times \tilde{A}_2$ then $Z_{W_l}(w)$ is isomorphic to G_4. Indeed, $\#(Z_{W_l}(w)) = 24$, and thus the only other possibilities are $G(6,3,2)$, $G(12,12,2)$ or a product of two cyclic groups [5]. (We require coprimeness of the characteristic here, which is automatic since we assume $\operatorname{char} k = 0$ or $\operatorname{char} k > 3$.) But if w is the fourth power of a Coxeter element w_0 then $w_0^3 \in Z_{W_l}(w)$ is non-central and thus $Z_{W_l}(w)$ cannot be commutative. If we write w as $w_1 w_2 = w_2 w_1$, where w_1 is an element of type A_2 and w_2 is an element of type $\tilde{A}_2$ then we can construct a basis $\{c_1, c_2\}$ for $\mathfrak{c}$ such that $w_i(c_j) = \zeta^{\delta_{ij}} c_j$. Moreover, $w_1 \in W_l$ and hence there exists an element of $Z_{W_l}(w)$ with characteristic polynomial $(t-\zeta)(t-1)$ as an automorphism of $\mathfrak{c}$. Since there is no such element of $G(6,3,2)$ or $G(12,12,2)$, $Z_{W_l}(w)$ is isomorphic to G_4. This proves the remaining cases in type $D_4^{(3)}$. But now $Z_W(w)$ is a non-commutative pseudoreflection group of rank 2 which has polynomial generators of degree 6 and 12 by Lemma 5.5, and thus is either $G(6,1,2)$, $G(12,4,2)$ or G_5. Since $Z_W(w)$ contains G_4 as a normal subgroup, it follows that it is isomorphic to G_5.

For an element of type $D_4(a_1)$, the centralizer in W has order 96 and has degrees 8 and 12 by Lemma 5.5. Thus the only possibilities for $Z_W(w)$ are $G(12,3,2)$, G_8, G_{13} or $\mu_8 \times \mu_{12}$ [5]. Since W_l is a normal subgroup of W, $\bar{W}$ contains a normal subgroup which is isomorphic to $G(4,2,2)$. This rules out $\mu_8 \times \mu_{12}$ and $G(12,3,2)$ since for example if ξ is a primitive 12-th root of unity

$$\begin{pmatrix} \xi^2 & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi^{-2} & 0 \\ 0 & \xi^{-1} \end{pmatrix} = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and hence $G(4,2,2)$ is not normal in $G(12,3,2)$. Moreover, by [6, p. 395], G_{13} contains no reflections of order 4. Thus $\bar{W}$ is equal to G_8. $\square$

For $w \in W$, let Φ_1 be the smallest root subsystem of Φ containing all roots corresponding to vertices of the admissible diagram for w, let Φ_2 be the set of roots in Φ which are orthogonal to all elements of Φ_1 and let W_i be the subgroup of W generated by all s_α with $\alpha \in \Phi_i$ $(i = 1, 2)$. Let L_1 be the Levi subgroup of G generated by T and all U_α with $\alpha \in \Phi_1$. (See the discussion in the paragraph preceding Lemma 4.6.) Then L_1 is θ-stable and $\mathfrak{c} \subset \mathrm{Lie}(L_1')$, hence one can also consider the little Weyl group in L_1, which is naturally a subgroup of $W_\mathfrak{c}$.

Lemma 5.7. *For the following automorphisms in type F_4, $\bar{W} = W_\mathfrak{c}$ is equal to the little Weyl group one obtains on restricting to the subgroup L_1. We have:*
(a) $W_\mathfrak{c} = \mu_6$ if w is of type C_3 or B_3,
(b) $W_\mathfrak{c} = \mu_4$ if w is of type B_2.
(c) $W_\mathfrak{c} = \mu_2$ if w is of type $\tilde{A}_1$.

Proof. This is a straightforward observation of the orders of the centralizer and the subgroup W_2 (see [4, Table 8]). If w is of type C_3 or B_3 then Φ_2 has a basis consisting of one element and hence $W_2 = W(\Phi_2)$ has order 2. But the centralizer of w has order 12, thus the order of $\bar{W}$ divides 6. If w is of type B_2 then Φ_2 is isomorphic to B_2 (hence W_2 has order 8) and the centralizer of w has order 32, thus the order of $\bar{W}$ divides 4. Finally, if w is of type $\tilde{A}_1$ then $\Phi_2 = B_3$ and hence the order of W_2 is 48. Since the order of the centralizer is 96 by [4], this shows that $\bar{W} = \mu_2$. ☐

We remark that in other types, $W_\mathfrak{c}$ may not be equal to the group one obtains on restricting to L_1.

Lemma 5.8. *If $\theta = \mathrm{Int}\, n_w$ is an automorphism in type F_4 such that w is of type A_2 or $\tilde{A}_2$ then $\bar{W} = W_\mathfrak{c} = \mu_6$. In fact, there exists a θ-stable semisimple subgroup L of G of type B_3 (if w is of type A_2) or type C_3 (if w is of type $\tilde{A}_2$) such that $\mathfrak{c} \subset \mathrm{Lie}(L)$, each element of $W_\mathfrak{c}$ has a representative in $L(0)$, and $\theta|_L$ is N-regular.*

Proof. This is immediate since if w is of type A_2 (resp. $\tilde{A}_2$) then θ is the square of an automorphism corresponding to a Weyl group element of type B_3 (resp. C_3). ☐

Lemma 5.9. *Let θ be an automorphism of type $\tilde{D}_4^{(3)}$ with Kac diagram* 010 *or* 100. *Then $W_\mathfrak{c} = \bar{W} = \mu_2$.*

Proof. We noted in the proof of Lemma 5.6 that if $w \in W = W(F_4)$ has order 3 modulo $W_l = W(D_4)$ then $Z_{W_l}(w)$ has index 3 in $Z_W(w)$. Thus in both cases here the group $\bar{W}$ of Lemma 5.1 is isomorphic to μ_2. It therefore remains only to prove that the non-trivial element of $\bar{W}$ has a representative in $G(0)$.

Let w be an element of W of type C_3. Then $(T^w)^\circ$ is of dimension 1 and is generated by the coroot of a long root element β. The idea here is that we can centralize by $\beta^\vee(k^\times)$ and obtain a Levi subgroup of G whose Lie algebra contains $\mathfrak{c}$, and which has a little Weyl group isomorphic to μ_2. It is easy to see that $Z_G(\beta^\vee(k^\times))$ is of type $A_1 \times A_1 \times A_1$. Moreover, w and w^2 permute the 3 subgroups of type A_1 transitively. (If, for example, we take $w = s_2 s_3 s_4$ in $W(F_4)$, then Φ_l has basis $\{\alpha_2, \alpha_1, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\}$; $\beta = \hat{\alpha}$, the longest root in F_4, and hence the roots in Φ_l which are orthogonal to β are $\pm\alpha_2$, $\pm(\alpha_2 + 2\alpha_3)$ and $\pm(\alpha_2 + 2\alpha_3 + 2\alpha_4)$; furthermore, $w(\alpha_2) = \alpha_2 + 2\alpha_3$, $w(\alpha_2 + 2\alpha_3) = \alpha_2 + 2\alpha_3 + 2\alpha_4$ and $w(\alpha_2 + 2\alpha_3 + 2\alpha_4) = -\alpha_2$.) Setting $L = Z_G(\beta^\vee(k^\times))'$, it is thus easy to see that $\mathfrak{c} \subset \mathrm{Lie}(L)$ and that $N_{L(0)}(\mathfrak{c})/Z_{L(0)}(\mathfrak{c}) \cong \mu_2$. (Here the Kac automorphism with diagram 010 restricts to

Table 1: Positive rank automorphisms in type G_2

Kac diagram	m	w	r	$W_{\mathfrak{c}}$	L
111	6	G_2	1	μ_6	N-reg.
011	3	A_2	1	μ_6	N-reg.
010	2	$A_1 \times \tilde{A}_1$	2	$W(G_2)$	N-reg.

an automorphism of L of the form $(g_1, g_2, g_3) \mapsto ({}^t g_3^{-1}, g_1, g_2)$; the Kac automorphism with diagram 100 is conjugate to the square of the Kac automorphism with diagram 010.) Thus $\bar{W} = W_{\mathfrak{c}} = \mu_2$ in either case. $\square$

Remark 5.10. (a) In the case where the ground field has characteristic zero and $G(0)$ is semisimple (arbitrary G), the rank and little Weyl group were determined by Vinberg in [24]. It was shown in [24, Prop. 18] that $G(0)$ is semisimple if and only if the corresponding Kac diagram has exactly one non-zero entry, which is equal to 1. Our calculations for $W_{\mathfrak{c}}$ agree with [24] in these cases.

(b) Let $G_Z^\theta = \{g \in G \,|\, g^{-1}\theta(g) \in Z(G)\}$ and let $W_{\mathfrak{c}}^Z = N_{G_Z^\theta}(\mathfrak{c})/Z_{G_Z^\theta}(\mathfrak{c})$. Then $W_{\mathfrak{c}}^Z$ is a subgroup of $\bar{W}$. Recall that θ is *saturated* if $W_{\mathfrak{c}} = W_{\mathfrak{c}}^Z$. It is immediate that all automorphisms in type F_4 and G_2 are saturated since in both cases the centre is trivial. Moreover, it is not difficult to show without using our classification that any automorphism of type $\tilde{D}_4^{(3)}$ is saturated.

(c) There is a strong relationship between Vinberg's construction of $W_{\mathfrak{c}}$ and work of Broué and Malle constructing certain pseudo-reflection groups in finite groups of (exceptional) Lie type [2]. In general, the group constructed by Broué and Malle corresponds to our $\bar{W}$. It is possible for $W_{\mathfrak{c}}$ to be a proper subgroup of $\bar{W}$. For example, if $\theta = \operatorname{Int} n_w$ in type E_6, where w is an element of type $D_4(a_1)$, then $W_{\mathfrak{c}}$ is either G_8 or $G(4,1,2)$.

(d) In Table 3 we have indicated the action of $\theta|_L$ for the cases above using the automorphism $\tau : \mathrm{SL}(2,k)^3 \to \mathrm{SL}(2,k)^3$, $(g_1, g_2, g_3) \mapsto ({}^t(g_3)^{-1}, g_1, g_2)$.

Theorem 5.11. *Any θ-group of type G_2, F_4 or $D_4^{(3)}$ has a KW-section.*

Proof. To prove this we will observe that there exists a θ-stable reductive subgroup L of G such that $\mathfrak{c} \subset \mathrm{Lie}(L)$, $N_{L(0)}(\mathfrak{c})/Z_{L(0)}(\mathfrak{c}) = W_{\mathfrak{c}}$ and $\theta|_L$ is N-regular. Then we can apply Lemma 5.5. Indeed, L is simply the group L_1 (see the discussion before Lemma 4.6) in all cases except automorphisms of type A_2 or $\tilde{A}_2$ in type F_4 or those of type C_3 or $\tilde{A}_2$ in type $D_4^{(3)}$. In case A_2 (resp. $\tilde{A}_2$) in type F_4 we can reduce to a group of type B_3 (resp. C_3) by (the proof of) Lemma 5.8. In cases C_3 and $\tilde{A}_2$ in type $D_4^{(3)}$ we can reduce to a subgroup of G isomorphic to $\mathrm{SL}(2,k)^3$ as indicated in the proof of Lemma 5.9. $\square$

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Table 2: Positive rank automorphisms in type F_4

Kac diagram	m	w	r	$W_{\mathfrak{c}}$	L	$\theta\|_L$
11111	12	F_4	1	μ_{12}	N-reg.	
11101	8	B_4	1	μ_8	SO(9)	Coxeter
10101	6	$F_4(a_1)$	2	G_5	N-reg.	
01010	6	C_3	1	μ_6	Sp(6)	Coxeter
11100	6	B_3	1	μ_6	Spin(7)	Coxeter
10100	4	$D_4(a_1)$	2	G_8	N-reg.	
01001	4	B_2	1	μ_4	Spin(5)	Coxeter
00100	3	$A_2 \times \tilde{A}_2$	2	G_5	N-reg.	
11000	3	A_2	1	μ_6	Spin(7)	positive 3-cycle
10001	3	$\tilde{A}_2$	1	μ_6	Sp(6)	positive 3-cycle
01000	2	A_1^4	4	$W(F_4)$	N-reg.	
00001	2	$\tilde{A}_1$	1	μ_2	short SL(2)	Coxeter

Table 3: Positive rank automorphisms in type $D_4^{(3)}$

Kac diagram	m	w	r	$W_{\mathfrak{c}}$	L	$\theta\|_L$
111	12	F_4	1	μ_{12}	N-reg.	
101	6	$F_4(a_1)$	2	G_4	N-reg.	
010	6	C_3	1	μ_2	$\mathrm{SL}(2)^3$	τ
001	3	$A_2 \times \tilde{A}_2$	2	G_4	N-reg.	
100	3	$\tilde{A}_2$	1	μ_2	$\mathrm{SL}(2)^3$	τ^2

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