Large sample-to-sample fluctuations of the nonequilibrium critical current through mesoscopic Josephson junctions

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We present a theory for the nonequilibrium current in a mesoscopic Josephson junction which is coupled to a normal electron reservoir, and apply it to a chaotic junction. Large sample-to-sample fluctuations of the critical current I_c are found, with rms $I_c \simeq \sqrt{N} e\Delta/\hbar$, when the voltage difference eV between the electron reservoir and the junction exceeds the superconducting gap Δ and the number of modes N connecting the junction to the superconducting electrodes is large.

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Recently, there has been an increased interest in the nonequilibrium Josephson current in mesoscopic multiterminal superconductor-normal metal-superconductor (SNS) junctions. Nonequilibrium in the junction is created by quasiparticle injection from one or several normal electron reservoirs, connected to the normal part of the SNS junction. By controlling the voltage applied between the normal reservoirs and the SNS junction, it has been shown in recent experiments that the Josephson current can be suppressed,[1,2] reversed,[3] and in the case with injection from a superconducting reservoir, even enhanced.[4]

The microscopic mechanism for these effects, nonequilibrium population of the current-carrying Andreev levels, was discussed by van Wees *et al.*[5] already in 1991. Thereafter, the nonequilibrium Josephson current in various multiterminal geometries has been studied in both diffusive[6–8] and quantum ballistic[9,10] junctions. In Ref. 10 it was pointed out that the nonequilibrium Josephson current in ballistic SNS junctions cannot be described only in terms of the nonequilibrium population of Andreev levels: The Andreev levels also change properties when the SNS junction is connected to a normal reservoir, giving rise to a quantum interference addition to the Josephson current. This interference contribution, resulting from the difference between the scattering-state wave functions for injected electrons and holes, is only present in nonequilibrium and is a generic feature for all multiterminal mesoscopic SNS junctions. However, the ensemble average of this interference contribution is zero, and does thus not show up in approaches starting with ensemble averaged equations, e.g., the Usadel equation used for calculating the nonequilbrium Josephson current in diffusive junctions.[6–8]

In this paper we develop a general theory of the nonequilibrium Josephson current in three-terminal SNS junctions (see Fig. 1), within a scattering-matrix approach.[11] The theory is then applied to a chaotic junction, in the limit of weak coupling to the normal reservoir and at zero temperature. We find that the quantum-interference contribution gives rise to sample-to-sample fluctuations of the critical current I_c which are much larger than the equilibrium fluctuations:[11,12] For a large voltage V (with $eV \gtrsim \Delta$, the superconducting gap),

$$\mathrm{rms}\ \ I_c \equiv \sqrt{\langle I_c^2\rangle - \langle I_c\rangle^2} \simeq \langle I_c\rangle \simeq \sqrt{N}\,\frac{e\Delta}{\hbar}, \tag{1}$$

hence the fluctuations are of the order of the ensemble-averaged critical current itself. (Here N is the number of modes connecting the junction to each of the superconducting electrodes.) In this regime the current results from the quantum-interference contribution alone, and its statistics are dominated by fluctuations of wave functions. These are much larger than the fluctuations of transmission eigenvalues (which repell each other mutually) that characterize the equilibrium situation. Sample-to-sample fluctuations of this magnitude have never been predicted before. It should be possible to measure these fluctuations with some modifications of existing experimental setups.[2,12] For $eV \lesssim \Delta$ the critical current is of order $N(e\Delta/\hbar)$, with fluctuations of order $e\Delta/\hbar$.

A model of the junction is presented in Fig. 1. A mesoscopic scatterer is connected to two superconducting leads via ballistic contacts, each supporting N transverse modes. The phase difference between the superconductors is ϕ. The scatterer is also connected to a normal reservoir via a contact with M modes, containing a tunnel barrier with transparency Γ. A voltage V is applied between the SNS junction and the normal reservoir. We assume that the resistance of the injection contact is the dominating resistance of the junction, such that the potential drops completely over the injection point. In order to preserve nonequilibrium, the strength of the tun-

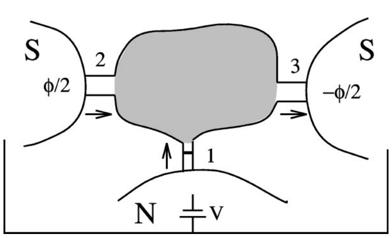

FIG. 1. Three-terminal SNS junction, consisting of a mesoscopic scatterer (gray shaded) connected to two superconducting reservoirs via contacts 2 and 3 and a normal reservoir via contact 1. The black bar in contact 1 indicates a tunnel barrier, the arrows the direction of positive current flow.

nel barrier Γ is, however, limited by the requirement that the dwell time of the injected quasiparticles $t_{\rm dwell}\propto 1/\Gamma$ must be smaller than the inelastic scattering time $t_{\rm inel}$ in the junction.

Under these conditions, the distribution of the quasiparticles in the junction is determined by the distributions $n^{\rm e(h)}=n_{\rm F}(E\mp eV)$ of electrons (holes) in the reservoir at energy E, where $n_{\rm F}=[1+\exp(E/kT)]^{-1}$. The current in contact $j=1,\,2,\,3$ can then be written as

$$I_j=\int_{-\infty}^{\infty}dE(i_j^{\rm e}n^{\rm e}+i_j^{\rm h}n^{\rm h}+i_j^{\rm s}n_{\rm F}), \qquad (2)$$

with $i_j^{\rm e(h)}$ the current density of the scattering states resulting from injected electron (hole) quasiparticles from the normal reservoir and $i_j^{\rm s}$ the total current density for quasiparticles injected from the superconductors ($i^{\rm s}=0$ for subgap energies $|E|<\Delta$).

The current $I=(I_2+I_3)/2$ flowing between the superconductors can be rewritten by using the current conservation for each energy $i_1^{\rm e,h}+i_2^{\rm e,h}=i_3^{\rm e,h}$ and the fact that no current is flowing in the injection lead in equilibrium, $i_1^{\rm e}+i_1^{\rm h}=0$. It takes then the form $I=I^{\rm neq}+I^{\rm eq}$, where the equilibrium current at $eV=0$ is given by $I^{\rm eq}=\int dE[i^{+}+(i_2^{\rm s}+i_3^{\rm s})/2]n_{\rm F}$, and

$$I^{\rm neq}=\int_{-\infty}^{\infty}dE\left[\frac{i^{+}}{2}(n^{\rm e}+n^{\rm h}-2n_{\rm F})+\frac{i^{-}}{2}(n^{\rm e}-n^{\rm h})\right]. \qquad (3)$$

Here the current densities $i^{+}=i_2^{\rm e}+i_2^{\rm h}=i_3^{\rm e}+i_3^{\rm h}$ and $i^{-}=(i_2^{\rm e}-i_2^{\rm h}+i_3^{\rm e}-i_3^{\rm h})/2$ are the sum and the difference of the current densities of the scattering states for injected electrons and holes. The contribution $\propto i^{+}$ to $I^{\rm neq}$ results from the nonequilibrium population of the Andreev levels, while the current $\propto i^{-}$ accounts for the quantum-interference contribution as well as for an asymmetric splitting of the injected current $I_1=\int dE(i_1^{\rm e}-i_1^{\rm h})(n^{\rm e}-n^{\rm h})/2$.

We will now express the current densities in terms of the scattering matrix S of injected quasiparticles from the reservoir.[11] The current densities are calculated most conveniently in the contacts $j=1,\,2,\,3$, where the wave functions are plane-wave solutions to the Bogoliubov-de Gennes equation. A wave incident on the scatterer from leads 2 and 3 is described by the $4N$ vector of wave function coefficients $c_{\rm in}=(c_2^{\rm e,+},c_3^{\rm e,-},c_2^{\rm h,-},c_3^{\rm h,+})$. The superscript $+(-)$ denotes a positive (negative) sign of the wave vector. Correspondingly the outgoing wave is given by $c_{\rm out}=(c_2^{\rm e,-},c_3^{\rm e,+},c_2^{\rm h,+},c_3^{\rm h,-})$. At the NS interfaces, Andreev reflection is described by the scattering matrix

$$S_{\rm A}=\alpha\begin{pmatrix}0 & r_{\rm A}\\ r_{\rm A}^{*} & 0\end{pmatrix},\quad r_{\rm A}=\begin{pmatrix}e^{i\phi/2} & 0\\ 0 & e^{-i\phi/2}\end{pmatrix}, \qquad (4)$$

such that $c_{\rm in}=S_{\rm A}c_{\rm out}$, with $\alpha=\exp[-i\arccos(E/\Delta)]$. The wave functions in the three contacts are then matched with help of the $(2N+M)\times(2N+M)$ scattering matrix S' of the normal region (including the tunnel barrier), with blocks (corresponding to contacts)

$$S'=\begin{pmatrix}r_{11} & t_{12} & t_{13}\\ t_{21} & r_{22} & t_{23}\\ t_{31} & t_{32} & r_{33}\end{pmatrix}. \qquad (5)$$

We introduce a nonunitary matrix $S_{\rm N}$, describing only the scattering between the contacts $j=2$ and 3,

$$S_{\rm N}=\begin{pmatrix}S_0(E) & 0\\ 0 & S_0^{*}(-E)\end{pmatrix},\quad S_0=\begin{pmatrix}r_{22} & t_{23}\\ t_{32} & r_{33}\end{pmatrix}, \qquad (6)$$

such that $c_{\rm out}=S_{\rm N}c_{\rm in}$, and matrices which involve also contact 1,

$$\mathcal{T}=\begin{pmatrix}t_{12}(E) & t_{13}(E) & 0 & 0\\ 0 & 0 & t_{12}^{*}(-E) & t_{13}^{*}(-E)\end{pmatrix},$$

$$\mathcal{T}'=\begin{pmatrix}t_{21}(E) & 0\\ t_{31}(E) & 0\\ 0 & t_{21}^{*}(-E)\\ 0 & t_{31}^{*}(-E)\end{pmatrix},\quad \mathcal{R}=\begin{pmatrix}r_{11}(E) & 0\\ 0 & r_{11}^{*}(-E)\end{pmatrix}.$$

The scattering matrix S for injected quasiparticles from the normal reservoir can be written as

$$S=\begin{pmatrix}r_{\rm ee} & r_{\rm he}\\ r_{\rm eh} & r_{\rm hh}\end{pmatrix}=\mathcal{R}+\mathcal{T}(S_{\rm A}^{\dagger}-S_{\rm N})^{-1}\mathcal{T}'. \qquad (7)$$

From these ingredients, the coefficients c can be calculated and the current densities in Eq. (2) are obtained from the quantum mechanical expression for current. The current densities i^{+} and i^{-} follow after some matrix algebra, and read (for subgap energies $|E|<\Delta$)

$$i^{+}(E)=\frac{2e}{ih}{\rm tr}\left(S^{\dagger}\frac{d}{d\phi}S\right),\quad i^{-}(E)=\frac{2e}{ih}{\rm tr}\left(S^{\dagger}\frac{d}{d\phi}S\tau_z\right), \qquad (8)$$

with $\tau_z={\rm diag}\,(1,-1)$. (The expression for i^{+} is well known.[13,14]) Equations (3) and (8) are our general results for the nonequilibrium Josephson current.

In general, the current flowing between the superconductors contains also the part of the injected current which is asymmetrically split between contacts 2 and 3.[15] This is not the case when the SNS junction is weakly coupled to the reservoir ($\Gamma\ll 1$), because the injected current is then negligible compared to the current flowing between the superconductors. It is, however, important to point out that the coupling strength Γ has a lower practical limit, since we still require that the inelastic relaxation time $t_{\rm inel}\gtrsim t_{\rm dwell}\propto 1/\Gamma$. The coupling strength also sets the time scale on which the nonequilibrium steady state is established, since this is of the order of the dwell time. The total energy transfered in establishing the steady state, $\propto t_{\rm dwell}\Gamma$, remains finite even for small Γ, as is demanded by general thermodynamic principles.

In this limit the matrix $S_{\rm N}=S_{\rm N0}+\Gamma\,\delta S_{\rm N}$ can be expanded to first order in Γ, where $S_{\rm N0}$ is unitary. The two current densities i^{+} and i^{-} have the same discrete spectrum of Andreev levels, given by the solutions E_n of $\det(1-S_A S_{N0})=0$, but different spectral weights. The current density i^+ reduces to the well-known expression for the closed junction,

$$i^+(E)=\sum_n I_n^+\,\delta(E-E_n),\quad I_n^+=\frac{2e}{\hbar}\frac{dE_n}{d\phi}. \tag{9}$$

The current density i^- can be found from the first-order perturbation theory in the tunnel-barrier transparency Γ,

$$i^-(E)=\sum_n I_n^-\,\delta(E-E_n),\quad I_n^-=R_n I_n^+, \tag{10a}$$

$$R_n=\frac{\mathrm{Re}(U^\dagger\sigma_z\,\delta S_N S_A U)_{nn}}{\mathrm{Re}(U^\dagger\,\delta S_N S_A U)_{nn}}, \tag{10b}$$

where $\sigma_z=\mathrm{diag}\,(1,1,-1,-1)$ and the unitary matrix U diagonalizes the unitary matrix product $S_A S_{N0}=U\mathrm{diag}(\lambda)U^\dagger$. One can show with help of the corresponding eigenvalue equation that the ratios $|R_n|\leqslant 1$. It should be pointed out that the matrix δS_N *cannot* be expressed in terms of the closed junction scattering matrix S_{N0}, i.e., the current density i^- depends manifestly on the properties of the contact between the normal reservoir and the SNS junction.

In order to investigate the mesoscopic fluctuations of the nonequilibrium current in more detail we now apply our theory to a chaotic SNS junction, in the limit of weak coupling to the normal reservoir.[14] The ergodic time is assumed to be much smaller than the dwell time and the inverse superconducting gap $\hbar/\Delta$. Here we only consider the simplest case, in which the dwell time in the normal scatterer (with the superconducting leads replaced by normal ones) $t_{\mathrm{dwell}}^{\mathrm{normal}}<\hbar/\Delta$. (Our main conclusions should apply also for the opposite case.) For such a junction we can neglect the energy dependence of S', which is then distributed with the so-called Poisson kernel $P(S')\propto|\det(1-\langle S'^\dagger\rangle S')|^{-(2N+M+1)}$, where $\langle S'\rangle$ is the ensemble-averaged scattering matrix.[16] (The magnetic field $B=0$, which gives a symmetric scattering matrix $S'=S'^T$.) Furthermore, the current density for energies outside the gap vanishes.[11] Using the energy symmetries $i^+(E)=-i^+(-E)$ and $i^-(E)=i^-(-E)$, the total current at zero temperature,

$$I=-\sum_{E_n>eV} I_n^+ + \sum_{E_n<eV} I_n^- \equiv I^+ + I^-, \tag{11}$$

can be written as a sum over the currents I_n^+ and I_n^- carried by the individual Andreev levels with positive energies E_n. Equation (11) provides a simple picture where in equilibrium all Andreev levels carry the currents I_n^+. Increasing the voltage, the Andreev levels one by one switch from I_n^+ to I_n^- when the voltage is passing through $eV=E_n$. At $eV\geqslant\Delta$, all levels carry the current I_n^-.

In terms of the transmission eigenvalues T_n of the matrix S_0, the Andreev bound-state energies are given by $E_n=\Delta(1-T_n\sin^2\phi/2)^{1/2}$, hence the relation[11]

$$I_n^+=-(e\Delta/2\hbar)T_n\sin\phi(1-T_n\sin^2\phi/2)^{-1/2}. \tag{12}$$

The statistical properties of the equilibrium current $I^{\mathrm{eq}}=-\sum_n I_n^+$ are known,[11] with $\langle I^{\mathrm{eq}}\rangle\simeq Ne\Delta/\hbar$ and rms $I^{\mathrm{eq}}\simeq e\Delta/\hbar$.

For $eV\geqslant\Delta$ the current is $I=\sum_n I_n^-=\sum_n R_n I_n^+$. The statistics of the ratios R_n follows from the construction of all perturbations δS_N which are compatible with a given S_{N0} (i.e., both matrices follow from the same scattering matrix of the open scatterer[16]). For $M=1$ such an analysis results in

$$R_n=(1-T_n)^{1/2}[(\sin^2\phi/2)^{-1}-T_n]^{-1/2}\sin\beta_n, \tag{13}$$

where the angles $\{\beta_n\}$ (parametrizing the coupling to the reservoir) are independent random numbers, uniformly distributed in the interval $[0,2\pi)$. As a consequence, for fixed phase difference ϕ the average current $\langle I\rangle=0$, and the fluctuations rms $I\simeq\sqrt{N}e\Delta/\hbar$ because I^- is a sum of N independently fluctuating numbers I_n^-. The precise value of the fluctuations can be calculated upon replacing the sum in $\langle I^2\rangle=\langle\sum_n(R_n I_n^+)^2\rangle$ (valid due to the independence of the β_n) by an integral over the transmission eigenvalues, with density $\rho(T)=N\pi^{-1}[T(1-T)]^{-1/2}$. This results in

$$\mathrm{rms}\ \ I=\frac{\sqrt{N}e\Delta}{2\hbar|\tan\phi/2|}\sqrt{\sin^2\frac{\phi}{2}+\frac{8\sin^2\phi/4}{\cos\phi/2}-3\frac{\sin^2\phi/2}{\cos\phi/2}}\quad \text{for}\ \ eV\geqslant\Delta, \tag{14}$$

which is parametrically larger than the equilibrium fluctuations when $N\gg 1$.

Another physical quantity of interest is the critical current I_c, the largest possible current for a given realization. Because of $I(\phi)=-I(-\phi)$ it sometimes makes sense to restrict the phase to $0<\phi<\pi$ and to consider the current which is largest in modulus; I_c can then be positive or negative.[3] (With this definition, the average critical current vanishes for $eV>\Delta$.) In the following, however, we maximize over $-\pi<\phi<\pi$, hence I_c is always positive, as it is obtained from the I/V characteristic in experiments. The ensemble-averaged critical current and its fluctuations (obtained from a numerical simulation of the random-matrix ensemble with $N=10$ and $M=1$) are shown in Fig. 2, as a function of applied voltage eV. The result is compared to the contribution of I^+ in Eq. (11) alone, which only takes the nonequilibrium population of the Andreev levels into account.

For $0\leqslant eV\lesssim 0.54\Delta$ the critical current is equal to its equilibrium value, because at the nonfluctuating critical phase[11] $\phi_c\simeq 2$ all bound-state energies $E_n>eV$ (in general the energies lie in the interval $[\Delta\cos\phi/2,\Delta]$). In the range $0.54\Delta\leqslant eV\lesssim 0.98\Delta$ the critical phase is determined by the condition $\cos\phi_c/2=eV/\Delta$ that the first Andreev bound state drops below eV, with only small fluctuations due to the high density of transmission eigenvalues $T_n\approx 1$. Hence the critical current is $I_c=I^{\mathrm{eq}}(\phi_c)$. In this regime the quantum-interference contribution I^- in Eq. (11) does not play any role because $\langle I^+\rangle\gg\mathrm{rms}\ I^-$. For a voltage $eV\approx 0.98\Delta$ very close to the gap, I^+ and I^- are both of order $\sqrt{N}e\Delta/\hbar$, and the critical current starts to deviate from what one would expect from a pure nonequilibrium population of the An-

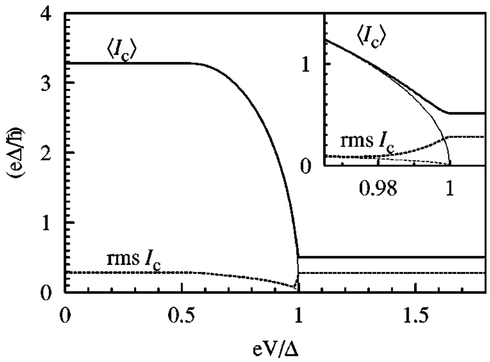

FIG. 2. Ensemble-averaged critical current $\langle I_c \rangle$ (solid thick line) and the fluctuations rms I_c (dashed thick line) as a function of voltage V between the normal reservoir and the junction. The thin lines are the result with $I^- = 0$ in Eq. (11). The junction has $N = 10$ modes to each of the superconducting electrodes and $M = 1$ mode to the normal reservoir. Inset: the voltage range $0.965\Delta < eV < 1.01\Delta$. ($10^3$ random matrices S' have been generated).

dreev levels. (For increasing N the cross-over voltage $eV \rightarrow \Delta$.) In parallel the fluctuations of the critical current increase. The critical current remains constant for $eV \geqslant \Delta$, where it is given solely by I^-.

The critical current for $eV \geqslant \Delta$ and its fluctuations as a function of junction modes N are shown in the upper panel of Fig. 3. The mean critical current is $\langle I_c \rangle \simeq 0.16\sqrt{N} e\Delta/\hbar$. The fluctuations are of the same order, rms $I_c \simeq 0.1\sqrt{N} e\Delta/\hbar$, which is by a factor of about $\sqrt{N}/3$ larger than the equilibrium fluctuations. Hence the N dependence in Eq. (14) carries over to the average critical current and its fluctuations.

Finally let us consider the dependence of the critical current on the number of injection modes M. This number is significant because the current I^- depends manifestly on the coupling of the reservoir to the junction [see Eq. (10)], in contrast to the current I^+ which only depends on properties of the decoupled junction. The lower panel of Fig. 3 shows that the critical current and its fluctuations at $eV \geqslant \Delta$ are suppressed when M is increased. The functional dependence is approximately $\propto M^{-1/3}$. The curves flatten out when M becomes larger than the total number $2N$ of modes connected to the superconductors. Thus, for an experimental observation of the large fluctuations predicted above, an injection contact with few modes is favorable.

In conclusion, we have studied the nonequilibrium Josephson current in a mesoscopic SNS junction connected to a normal electron reservoir. It is found that the current can be expressed in terms of the scattering matrix for the quasiparticles injected from the normal reservoir, Eqs. (3) and (8). As an application we considered the nonequilibrium current in a chaotic Josephson junction at zero temperature, weakly coupled to the normal reservoir. It is found that the fluctuations of the critical current for a voltage $eV \geqslant \Delta$ are of order rms $I_c \simeq \sqrt{N}\Delta e/\hbar$, which is of the same order as the mean critical current itself, and much larger than the equilibrium fluctuations (of order $\Delta e/\hbar$).

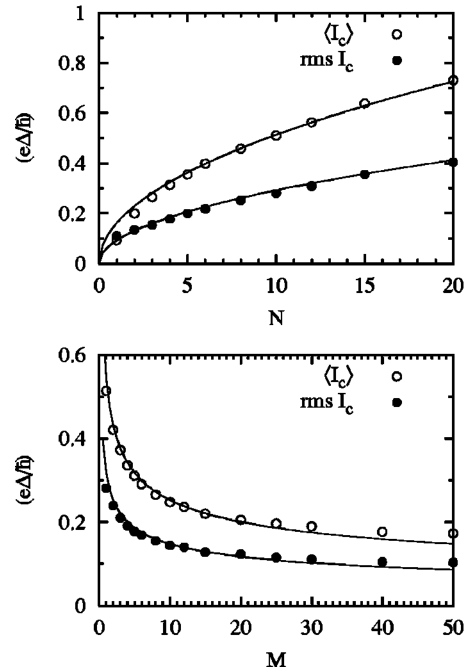

FIG. 3. Ensemble-averaged critical current $\langle I_c \rangle$ (open circles) and the fluctuations rms I_c (full circles) as a function of the number of junction modes N for a single injection mode $M = 1$ (upper panel) and the number of injection modes M for $N = 10$ junction modes (lower panel). The curves are $\propto N^{1/2}$ (upper panel) and $\propto M^{-1/3}$ (lower panel).

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